# Dirac operators, gauge systems and quantisation

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# Contents

# Chapter 0

# Introduction

This thesis is concerned with mathematical aspects of *gauge theories* and the role of *Dirac operators* in such theories.

Gauge theories form an important class of theories in physics, the most important example being the Standard Model of elementary particle physics, from now on simply called the Standard Model. Basically, a gauge theory is a physical theory for which the dynamics is invariant under some group of *local* symmetries, also called the *gauge group*. The dynamics is deduced from a *Lagrangian density*, i.e. a functional depending on either the position or velocity coordinates as in classical mechanics, or (particle) fields as in classical field theory. The symmetry operations in a gauge theory are also known as *gauge transformations*.

Mathematically, a classical gauge theory on a spacetime $M$ is represented by a principal $G$-bundle $\mathtt{P} \to M$, where $G$ is a Lie group. The *fermionic* particle fields are sections of specific associated vector bundles of $\mathtt{P}$. The connections on $\mathtt{P}$ are interpreted as *gauge potentials* and, after quantisation, these describe the *gauge bosons*. The gauge group is given by bundle automorphisms of the bundle $\mathtt{P}$ over the identity map $M \to M$. The gauge group acts on the gauge potentials and on the particle fields in a natural way. The Lagrangian density is then a gauge-invariant functional depending on the particle fields, the gauge potentials and the metric.

Let us illustrate the notion of a gauge theory by looking at an example that is part of the Standard Model, namely Quantum Chromo Dynamics (QCD). Assume that $\mathtt{P}$ is a principal $SU(3)$-bundle over, let's say, $\mathbb{R}^4$ or $S^4$, and let $\mathtt{E}$ be the complex vector bundle with fibre $\mathbb{C}^3$ associated to $\mathtt{P}$ by the fundamental representation of $SU(3)$ on $\mathbb{C}^3$. Write $\mathtt{S}$ for the *spinor bundle* and $\nabla^S$ for the corresponding *spin connection*. Consider the *fermionic Lagrangian density*

$$\mathcal{L}_f := \langle \psi, D^{\mathtt{E}} \psi \rangle, \tag{1}$$

where $D^{\mathtt{E}} = -i\gamma^\mu(\nabla^{\mathtt{S}}_\mu \otimes 1 + 1 \otimes \nabla^{\mathtt{E}}_\mu)$ is a generalised *Dirac operator*, $\psi$ is a section of $\mathtt{E}$, also called a *quark field*, and $\langle\cdot,\cdot\rangle$ is the fibre-wise inner product on $\mathtt{E}$. To give an idea of what gauge transformations do, we provide a local description of the Lagrangian density. First of all, the quark field $\psi$ is locally a function

$$M \to \mathtt{S}_x \otimes \mathbb{C}^3,$$

where $\mathtt{S}_x$ is the typical fibre of $\mathtt{S}$ and $\mathbb{C}^3$ is the typical fibre of $\mathtt{E}$. Locally, the connection $\nabla^{\mathtt{E}}$ is of the form $d + A$, where $A$ is a $\mathfrak{su}(3)$-valued 1-form. The local form of the Lagrangian density is then

$$\mathcal{L} := -i\overline{\psi}\gamma^{\mu}(\nabla^{\mathtt{S}}_{\mu} \otimes 1 + 1 \otimes (\partial_{\mu} + A_{\mu}))\psi,$$

where $\nabla^{\mathtt{S}}_{\mu}$ denotes the spin connection and each $A_{\mu}$ is an element in the Lie algebra $\mathfrak{su}(3)$ acting in the fundamental representation on $\mathbb{C}^3$. The field $A_{\mu}$ is called the *gauge potential*. Locally, a gauge transformation is a point-wise change of orthonormal basis for the vector space $\mathbb{C}^3$, and as such it can be interpreted as an $SU(3)$-valued function $u$ on $M$, acting in the fundamental representation on $\mathbb{C}^3$. It replaces the field $\psi$ by $u\psi$ and replaces $A_{\mu}$ by $uA_{\mu}u^{-1} + u\partial_{\mu}u^{-1}$, since $A_{\mu}$ is a connection 1-form. From its local expression, one can deduce that $\mathcal{L}$ is invariant under local $SU(3)$-gauge transformations. That is, the Lagrangian density does not depend on the choice of basis of the fibre of $\mathtt{E}$. Of course, we already knew this from the global expression of the Lagrangian density, see Equation (1).

Now that we have introduced the notion of a classical gauge theory, let us focus on two important questions regarding gauge theories that mathematical physicists try to answer. In both, Dirac operators play an important role.

1. Let us assume that spacetime is flat. The Standard Model is a quantum field theory, though the above mathematical framework deals with *classical* gauge theories. To obtain a quantum version of the classical description of gauge theories, one needs to find a Lorentz- and gauge-invariant way to quantise the classical field theory into a (renormalisable) quantum field theory. Quantising non-abelian gauge theories, and in particular Yang-Mills theory, is one of the hardest problems in (mathematical) physics at the moment, partly due to the constraints and partly because of the difficulties of constructing interacting quantum field theories in dimension 4. To date there is no mathematically rigorous way to do so. The situation is greatly simplified, however, if the continuum spacetime is replaced by a finite lattice. The advantage of these *lattice gauge theories* is that the gauge group is a finite-dimensional *Lie group* and that the *state space* is a finite-dimensional manifold. In ordinary field theories the gauge group and the state space are usually infinite dimensional. Because of their finite-dimensionality we refer to (finite) lattice gauge theories as gauge *systems*, to distinguish them from the much more complicated gauge theories on continuous spacetimes.

   In this thesis, we consider the quantisation of the gauge system consisting of a *cotangent bundle of a compact connected Lie group*. In this case, the action of the Lie group $G$ on itself by conjugation induces an action of $G$ on its own cotangent bundle $T^*G$. If one considers a single plaquette in two-dimensional lattice gauge theory, then the state space is precisely the cotangent bundle of the structure group of the gauge theory, and the gauge symmetries are implemented by the action of the structure group on its own cotangent bundle, as described above [29]. Inspired by techniques from noncommutative geometry [19] quantisation is then performed by taking the kernel of the Dolbeault-Dirac operator corresponding to some Kähler structure on the cotangent bundle. We refer to Definition 1.4.3 for a more precise definition.

In gauge theories, *states* that are related through the application of gauge transformations are physically indistinguishable and the directions of the orbits of these gauge transformations therefore correspond to non-physical degrees of freedom. To obtain a state space with only physical degrees of freedom, one needs to remove these gauge symmetries. This can be done in two ways. First, one can quantise the full space including its gauge symmetries to obtain a quantisation that also incorporates the gauge symmetries, and then apply a reduction procedure to remove the latter. Second, one may remove the gauge symmetries on the classical side and then quantise the ensuing *classical reduced space*. It is often easier to remove the gauge symmetries after quantisation (so-called *reduction after quantisation*) than to quantise the classical reduced space (so-called *quantisation after reduction*), because the classical reduced spaces are usually highly *singular*. Nevertheless, once a quantisation of the classical reduced space has been defined, its quantisation should yield the same answer as reduction after quantisation. If that happens, we say that *quantisation commutes with reduction*. In Chapters 2 to 4, we prove that quantisation commutes with reduction for the previously mentioned example of cotangent bundles of compact connected Lie groups.

2. The Standard Model incorporates the electroweak and strong nuclear force, but the gravitational force is described by General Relativity. Many mathematical physicists are looking for a single mathematical framework that unifies the Standard Model and General Relativity. Whereas the Standard Model is a quantum field theory, General Relativity is a purely geometrical, classical field theory. This is one of the biggest discrepancies between both theories, and to unify General Relativity with the Standard Model one arguably first requires a quantum version of General Relativity.

   *Noncommutative geometry* [19] does not provide a solution to the quantisation problem of General Relativity, but it does catch both classical Yang-Mills theory and classical gravity on a compact Riemannian spin manifold into one mathematical framework, namely the framework of spectral triples (i.e. *almost-commutative manifolds*). A major role here is played by a suitable generalised Dirac operator, as the dynamics can be obtained from it in the form of the *spectral action principle* [14]. The action is 'spectral' in the sense that it is equal to the sum of the eigenvalues of a generalised Dirac operator, up to some 'energy scale'. Using heat kernel techniques, the spectral action can be expanded asymptotically with respect to this energy scale. This expansion produces the Einstein-Hilbert action *plus* the Yang-Mills action, thereby obtaining both the gravitational part and the gauge part of the theory from a single operator. Even the full Standard Model can be obtained in this way [15]. In *loc. cit.* the gauge theories are globally trivial in the sense that the principal fibre bundle is a globally trivial bundle. In Part II we extend the definition of almost-commutative manifolds to include also the description of globally nontrivial gauge theories.

## Outline of this thesis

### Chapter 1

Section 1.1 contains some preliminaries. There we discuss the structure of the quotient of non-free proper actions of a Lie group on a smooth manifold. For non-free actions the quotient space has no natural manifold structure in general. From a classical mechanical point of view this is problematic, since the smooth structure of the configuration space is needed to define derivatives and thence, in particular, time evolution. Fortunately, even though the quotient is not a manifold, it can be partitioned into smooth manifolds in a very nice way. This partition is called a *stratification*. One important feature of this stratification is the existence of a dense, open subset, which is called the *principal stratum* of the quotient. This principal stratum is one of the main objects of interest in this thesis.

In Section 1.2 we give a brief introduction to symplectic geometry and its relation to classical mechanics. If the symplectic manifold $M$ has a group of symmetries $G$, which may or may not be be gauge symmetries, then the removal of these gauge symmetries is known as Marsden-Weinstein reduction. If the group is not discrete, the Marsden-Weinstein quotient is different from the ordinary quotient $M/G$. We pay special attention to the Marsden-Weinstein quotient in the case of non-free proper actions, in which case the Marsden-Weinstein quotient is in general not a smooth manifold. Nonetheless, it can be stratified by smooth and (in this case) symplectic, manifolds. This fact is known as the symplectic stratification theorem [82]. Once again, there is a dense, open stratum, still called the *principal stratum.*[1]

Section 1.3 is devoted to the basics of Kähler geometry. Our notion of quantisation on Kähler manifolds, which is *Dolbeault-Dirac quantisation*, is formulated

[1]Stratified symplectic spaces are not the same thing as *symplectic decompositions.* In the latter a *regular Poisson manifold* is decomposed into so-called *symplectic leaves.*

in terms of the Chern connection on some hermitian holomorphic line bundle over this Kähler manifold. We briefly discuss these notions and we particularly study the hermitian holomorphic structure of the so-called *canonical line bundle*. The canonical line bundle will play an important role when we consider the Dolbeault-Dirac quantisation of compact connected Lie groups in Chapter 2.

In Section 1.4 we give a short historical overview of geometric quantisation and introduce our definition of Dolbeault-Dirac quantisation. If there is a group of symmetries acting on the Kähler manifold, then, as we will show, the quantisation carries a natural group action as well.

## Chapter 2

We consider the cotangent bundle of a compact connected Lie group. As a cotangent bundle of an ordinary manifold it has a natural symplectic structure. Following Hall [40], we then put a complex structure on this cotangent bundle, which turns it into a Kähler manifold. This Kähler structure is used to define the Dolbeault-Dirac operator. If the curvature of the canonical line bundle is *semi-negative*, that is, if, on holomorphic coordinate patches, the curvature 2-form of the canonical line bundle is of the form

$$R = \sum_{kl} R_{kl} dz^k \wedge d\bar{z}^l,$$

where $R_{kl}$ is a *negative semi-definite* matrix, we show that the Dolbeault-Dirac quantisation is unitarily isomorphic to the square-integrable functions on the Lie group, in a natural way. Moreover, this isomorphism is equivariant if the action of the Lie group on its own cotangent bundle is induced by left- or inverse right-multiplication of the group on itself.

We conclude this chapter by showing that the curvature of the canonical line bundle on $T^*SU(2)$ is semi-negative, so that the results in this chapter apply in particular to the cotangent bundle of $SU(2)$.

## Chapter 3

In Chapter 3 we focus on the singular nature of Marsden-Weinstein quotients. We study the special case in which the Weyl group acts on the cotangent bundle of a maximal torus of a compact connected Lie group. We define the Dolbeault-Dirac quantisation of the corresponding singular Marsden-Weinstein quotient to be the Dolbeault-Dirac quantisation of its principal stratum and show that this definition is feasible in the sense that it makes quantisation commute with reduction. The main idea is that the quantisation of the inverse image of the principal stratum under the projection map is the same as the quantisation of the full cotangent bundle. We could therefore simply ignore all the points outside the inverse image of the principal stratum. The action of the Weyl group is free on the inverse image of this principal stratum, and because the Weyl group is finite, it is not difficult to prove that quantisation commutes with reduction. In Section 3.4 we carry these ideas a bit further.

## Chapter 4

In Chapter 4 we once again consider a compact connected Lie group and the action of this group on its cotangent bundle that is induced by the action of the Lie group on itself by conjugation. The main result is that quantisation commutes with reduction if the canonical line bundle on $T^*G$ is semi-negative. In Chapter 2 we have already seen that in this case Dolbeault-Dirac quantisation of the cotangent bundle yields the Hilbert space of square-integrable functions on the Lie group itself. Applying quantum reduction yields the Hilbert space of square-integrable functions on a fixed maximal torus of the Lie group invariant under the Weyl group.

The Marsden-Weinstein reduced space is in general not a manifold, but a symplectic stratified space. We define the quantisation of the reduced space to be the Dolbeault-Dirac quantisation of its principal stratum. This Marsden-Weinstein reduced space is closely related to the quotient of the cotangent bundle of a maximal torus under the action of the Weyl-group. In fact, these quotients are naturally homeomorphic, and their stratifications are identified under this homeomorphism if the Lie group is simply connected. We do not know if the stratifications in question are identified when $G$ is not simply connected, and neither do we know if the principal strata of both stratifications are identified. Therefore, we first prove that the Dolbeault-Dirac quantisation of both principal strata are equal. Using the results of Chapter 3, we can subsequently show that the Dolbeault-Dirac quantisation of the Marsden-Weinstein quotient is also naturally unitarily isomorphic to the Hilbert space of square-integrable Weyl-group invariant functions on a maximal torus. Consequently, quantisation commutes with reduction if the canonical line bundle on the cotangent bundle of the Lie group is semi-negative.

## Chapters 5 to 8

In the second part of this thesis we study classical gauge theories in the framework of Connes' noncommutative geometry [19]. As mentioned above, within this framework, the special case of a (globally trivial) almost-commutative manifold has been shown to describe a (classical) gauge theory over a Riemannian spin manifold, which ultimately led to a description of the full Standard Model of elementary particle physics, including the Higgs mechanism and neutrino mixing [15]. These gauge theories are, by construction, topologically trivial (in the sense that the corresponding principal bundles are globally trivial bundles). In this chapter the framework is adapted in order to allow for globally nontrivial gauge theories as well. Such a generalisation has previously been obtained only for the special case of Yang-Mills theory [11].

The work in this chapter has been carried out in collaboration with Koen van den Dungen (Australian National University and University of Wollongong), and is based on [10].

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# PartI

## Quantisation commutes with reduction for cotangent bundles of compact connected Lie groups

As a first step towards studying the singular case, the quantisation-commutes-with-reduction problem is studied for the co-adjoint action of a compact connected Lie group on its own cotangent bundle. Quantisation is defined as the kernel of a twisted Dirac operator. This quantisation procedure that we propose yields a Hilbert space, which can naturally be identified with the square-integrable functions on the Lie group if the canonical bundle on the cotangent bundle is semi-negative.

Classically, the Marsden-Weinstein quotient is a symplectic stratified space. We formulate a quantisation procedure for this quotient that only depends on the principal stratum. To show that this definition is feasible, we subsequently prove that it leads to a quantisation-commutes-with-reduction result for a non-empty class of compact connected Lie groups, including $SU(2)$.

# Chapter 1

# Preliminaries

The preliminaries mainly concern geometric structures such as symplectic geometry and Kähler geometry, but we also devote one section to reviewing some basic results on stratifications coming from proper Lie group actions. We fix some conventions and notation here as well. We conclude with a short history overview of quantisation.

## 1.1 Proper group actions

If a Lie group $G$ acts smoothly on a manifold $G$, then the quotient $M/G$ is a smooth manifold, provided that $G$ acts properly and freely. Moreover, in that case the projection map $\pi : M \to M/G$ is a principal $G$-bundle over the base manifold $M/G$. If the action of $M$ is not free, then the quotient space $M/G$ need no longer carry a natural manifold structure. If the action is still proper, the group action can be used to decompose the quotient space as a disjoint union of smooth manifolds that are glued together in a very nice way. This decomposition is an example of what is called a *stratified space*. Non-free proper group actions occur frequently in gauge systems and the understanding of the stratification of the classical reduced space is of great importance. Therefore, we review some basic results of stratifications by proper group actions here. We restrict ourselves to recalling only those facts that we actually need later on, even though there is much more that can be said about these kind of stratifications. A detailed account on the subject can be found in for instance [28, 77].

A map $f : X \to Y$ between topological spaces is called ***proper*** if the pre-image under $f$ of any compact subset $K \subset Y$ is again compact. If $Y$ is locally compact Hausdorff, then a proper map $X \to Y$ is always closed.

**Definition 1.1.1.** A (continuous) group action $\Phi : G \times M \to M$ of a locally compact group $G$ on a locally compact Hausdorff space $X$ is called a ***proper***

***action***, or $G$ is said to ***act properly*** on $X$, if the map $\Theta : G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is proper.

**Example 1.1.2.**

1. If $G$ is compact, then any continuous action of $G$ on a topological space $X$ is proper.
2. If $G$ acts properly on a compact space $X$, it follows that $\Theta^{-1}(X \times X) = G \times X$ is compact. Therefore, an action of a locally compact group on a compact space $X$ is proper if and only if the group is compact.

**Lemma 1.1.3.** *Let $G$ be a locally compact group acting properly on a locally compact topological space $X$. Then $X/G$ is Hausdorff, provided that $X$ is Hausdorff.*

*Proof.* First of all, note that the projection map $\pi$ is open. Indeed, for any open $U$ in $X$, its saturation $\pi^{-1}(\pi(U)) = \cup_{g \in G} gU$ is open, since it is a union of open sets. A quotient space for which the projection map is open is Hausdorff if and only if the graph of the equivalence relation $R = \{(gx, x) \mid x \in X, g \in G\}$ is a closed subset of $X \times X$. Let now $\Theta : G \times X \to X \times X$ be the proper map $(g, x) \mapsto (gx, x)$. Since $X \times X$ is locally compact and Hausdorff, the map $\Theta$ is necessarily closed. Because $R = \Theta(G \times X)$, $R$ is a closed subset. □

We are mainly interested in smooth, proper actions of *Lie groups* on *smooth manifolds*, that is, we only consider group actions for which the map $\Phi : G \times M \to M$ is smooth. *Our manifolds are always assumed to be smooth, Hausdorff and second countable.* By Lemma 1.1.3, if $M$ is second countable and Hausdorff, so is its quotient $M/G$ (the quotient is second countable because the projection map is open). Some basic properties of proper Lie group actions on manifolds are stated in the following proposition. The ***isotropy*** or ***stabiliser group*** $G_x$ of an element $x \in M$ is defined to be the group $G_x = \{g \in G \mid g \cdot x = x\}$.

**Proposition 1.1.4.** *If $\Phi : G \times M \to M$ is a proper (and smooth) action of a Lie group $G$ on a manifold $M$, then:*

1. *For any $x \in M$, the isotropy group $G_x$ is compact.*
2. *If the action is free, then $M/G$ has a unique smooth manifold structure such that $\pi : M \to M/G$ is a surjective submersion. The quotient map $\pi : M \to M/G$ defines a smooth left-principal $G$-bundle.*
3. *If all isotropy groups are conjugate to a single subgroup $H \subset G$, then $M/G$ is a smooth manifold and the projection $\pi : M \to M/G$ is a locally trivial fibre bundle with structure group $N(H)/H$ and fibre $G/H$.*

*Proof.* If $\Theta : G \times M \to M \times M$ denotes the map $(g, x) \mapsto (gx, x)$, then $\Theta^{-1}(x, x) = G_x \times \{x\}$. Since $\{(x, x)\}$ is compact, properness of the action implies that $G_x$ is compact. The second and third statements are well known. A proof of the second can be found in [1, Proposition 4.23] and the third is [1, Exercise 4.1M]. □

If the group action is proper but *non-free* and has non-conjugate isotropy groups, then in general the quotient $M/G$ has no natural manifold structure. However, if $M$ is partitioned into submanifolds in such a way that each piece only consists of points that have conjugate isotropy groups, then Proposition 1.1.4 implies that the projections of these pieces constitute a partition into smooth manifolds of $M/G$ . These thoughts lead to the following definitions.

**Definition 1.1.5** (See e.g. [75, 77])**.** For an action $\Phi : G \times M \to M$ the sets

$$\begin{aligned} M_H &= \{x \in M \mid G_x = H\}, \\ M_{(H)} &= \{x \in M \mid G_x = gHg^{-1} \text{ for some } g \in G\}, \\ M^H &= \{x \in M \mid G_x \supset H\} \end{aligned}$$

are called the ***H-isotropy type submanifold***, the ***(H)-orbit type submanifold*** and the ***H-fixed point type submanifold*** respectively.

If $x \in M_H$, then the notation $M_H^x$ is used to indicate the connected component of $M_H$ containing $x$. Similarly, if $x \in M_{(H)}$ ($x \in M^H$), then $M_{(H)}^x$ ($M_x^H$) indicates the connected component of $M_{(H)}$ ($M^H$) containing $x$.

We will see later (*cf.* Proposition 1.1.9) that the connected components of $M_H$, $M^H$ or $M_{(H)}$ are submanifolds of $M$. Despite their names, the subsets $M_H$, $M^H$ and $M_{(H)}$ themselves are in general not submanifolds, because the dimension of different connected components may vary. This is shown by the following example, which has been taken from [82] (see also [75, 2.4.9]).

**Example 1.1.6.** Let $S^5 \subset \mathbb{C}^3$ be given by

$$\{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid |z_0|^2 + |z_1|^2 + |z_2|^2 = 1\}.$$

The circle $S^1$ acts freely and properly on $S^5$ as $e^{i\theta} \cdot (z_1, z_2, z_3) = (e^{i\theta} z_0, e^{i\theta} z_1, e^{i\theta} z_2)$. The quotient space is the complex projective plane $\mathbb{CP}^2$ which is a smooth Hausdorff manifold. On $\mathbb{CP}^2$ one can define a $S^1$ action by

$$e^{i\phi} \cdot [z_0, z_1, z_2] = [e^{i\phi} z_0, z_1, z_2].$$

The submanifold $(\mathbb{CP}^2)_{S^1} = (\mathbb{CP}^2)^{S^1}$ consists of the point $[1, 0, 0]$ as well as $[0, z_1, z_2]$, $(z_1, z_2 \in \mathbb{C})$.

To show that the (connected components of) $M_H$, $M_{(H)}$ and $M^H$ are indeed submanifolds, one uses the existence of *tubes* for proper Lie group actions.

**Definition 1.1.7.** Let $M$ be a manifold on which a Lie group $G$ acts properly. Let $x \in M$. A ***tube*** around the orbit $G \cdot x$ is a $G$-equivariant diffeomorphism

$$\phi : G \times_{G_x} T \to U,$$

where $U$ is a $G$-invariant neighbourhood of $G \cdot m$ and $T$ is a manifold on which $G_x$ acts. Since $G_x$ is compact, the canonical action of $G$ on $G \times_{G_x} T$ is proper.

The following theorem is also known as the Tube Theorem.

**Theorem 1.1.8** ([75], Theorem 2.3.38)**.** *Let $M$ be a manifold upon which $G$ acts properly and let $x \in M$. Then there exists a tube $G \times_{G_x} T$, where $T$ is an open $G_x$-invariant neighbourhood of $0$ in the $G_x$-equivariant vector space $T_xM/T_x(G \cdot m)$.*

We are now ready to prove that the connected components of the isotropy-type, orbit-type, and fixed-type manifolds are indeed submanifolds. We shall sketch the proof given in [75, Theorem 2.4.7].

**Proposition 1.1.9.** *Let $G$ be a Lie group acting properly on a manifold $M$. Let $H$ be an isotropy group for this action. The connected components of the sets $M_{(H)}$, $M_H$ are locally closed embedded submanifolds of $M$. If $H$ is any closed subgroup of $G$, then the connected components of $M^H$ are closed submanifolds of $M$.*

*Proof.* This is a consequence of the Tube Theorem. For $x \in M$ with $H = G_x$, there exists a tube $\phi : G \times_H T$ around the orbit $G \cdot m$. Then, by the $G$-equivariance of the map $\phi$, we obtain

$$\phi^{-1}(U \cap M_H) = (G \times_H T)_H = N_G(H) \times_H T^H,$$
$$\phi^{-1}(U \cap M_{(H)}) = (G \times_H T)_{(H)} = G \times_H T^H.$$

Since the action of $H$ on both $G \times T^H$ and $N_G(H) \times T^H$ is free and proper, the spaces $N_G(H) \times_H T^H$ and $G \times_H T^H$ are closed submanifolds of $G \times_H T$. Therefore, the connected components of $M_H$ and $M_{(H)}$ are locally closed embedded submanifolds of $M$.

We now consider the space $M^H$. The set $M^H$, and hence its connected components, are clearly closed in $M$. To prove that the connected components of $M^H$ are submanifolds of $M$, we restrict the group action of $G$ on $M$ to $H$. If $K \subset H$, write $\tilde{M}_K$ for the isotropy type submanifolds for the action of $H$ on $M$. Then $M^H = \tilde{M}_H$, and by the previous paragraph the connected components of $\tilde{M}_H = M^H$ are embedded submanifolds of $M$. □

The next proposition will be important when we consider stratifications of symplectic group actions.

**Proposition 1.1.10.** *Suppose that $G$ acts properly on a manifold $M$. For each $x \in M$, one has $(T_xM)^{G_x} = T_x(M_{G_x})$. Similarly, if $x \in M^H$ for some subgroup $H \subset G$, then $T_xM^H = (T_xM)^H$.*

*Proof.* A proof of the first claim can be found in [75]. Again, the second claim follows from the first one by restricting the action of $G$ on $M$ to the subgroup $H$, as in Proposition 1.1.9. □

If $G$ is not connected, the submanifolds $M^x_{(H)}$ are not necessarily $G$-invariant. An alternative approach to divide $M_{(H)}$ into submanifolds is taken by Duistermaat and Kolk in [28] who write $x \approx y$ if there exists a $G$-invariant diffeomorphism from an open neighbourhood of $x$ onto a $G$-invariant open neighbourhood of $y$, and subsequently define

$$M_x^{\approx} = \{y \in M \mid y \approx x\}.$$

The spaces $M_x^{\approx}$ are called the ***local action types*** of $M$. Each $M_x^{\approx}$ is a locally closed $G$-invariant embedded submanifold of $M$ and each local action type $M_x^{\approx}$ is an open and closed subset of $M_{(G_x)}$ (see [28, Theorem 2.6.7]). Each space $M_x^{\approx}$ is therefore equal to a union of connected components of $M_{(G_x)}$. Indeed, since each local orbit type $M_x^{\approx}$ is open and closed in its corresponding orbit type $M_{(G_x)}$, the intersection $M_{(G_x)}^y \cap M_x^{\approx}$ is open and closed in $M_{(G_x)}^y$. For each $y_k$ the space $M_{(G_x)}^{y_k}$ is maximal connected in $M_{(G_x)}$, so it is maximal connected in the subspace $M_x^{\approx}$, too. Consequently, the partition of $M$ into connected components of the orbit type strata coincides with the partition of $M$ into the connected components of the local action types. These local action type submanifolds are no longer connected. However, each local action type is a $G$-invariant submanifold of $M$.

By Proposition 1.1.4 the quotient $M_x^{\approx}/G$ is naturally a smooth manifold, so that the projections of the connected components of the orbit-type submanifolds constitute a partition of $M/G$ into manifolds. This partition has the following properties (among many others).

**Theorem 1.1.11.** *Let $G$ be a proper action on a manifold $M$. Write $\{S_i\}$ for the set of connected components of the orbit type submanifolds in $M$ or for the set of their projections in $M/G$, then $\{S_i\}$ has the following properties:*

1. *$\{S_i\}$ is a locally finite partition of $M$ or $M/G$ into manifolds (embedded submanifolds in the case of $M$) and each piece $S_i$ is a locally closed subset;*
2. *If $S_i \cap \overline{S}_j \neq \emptyset$, then $S_i \subset \overline{S}_j$ (**frontier condition**).*

*Proof.* For a proof of these statements, which is another application of the Tube Theorem, see [77, Section 4.3] or [28, Section 2.7]. □

**Definition 1.1.12.** A locally finite partition $\mathcal{S} := \{S_i\}$ of a paracompact Hausdorff topological space $X$ into locally closed subspaces such that $\mathcal{S}$ satisfies the frontier condition, is called a ***decomposition*** of $X$, and $(X, \mathcal{S})$ is said to be a ***decomposed space*** (*cf.* [77, Definition 1.1.1]). The elements $S_i$ are known as ***pieces*** or ***strata***.

**Remark 1.1.13.** Decomposed spaces are examples of so-called *stratified spaces* and the decomposition is an example of a so-called *stratification.* A ***stratification*** of a topological space $X$ is a map that to each point $x \in X$ associates the set germ $S(x)$ of a closed subset of $X$ in such a way that for every $x \in X$ there is a neighbourhood $U$ of $x$ and a decomposition $\mathcal{S}$ of $U$ such that for all $y \in U$ the set germ $S(y)$ coincides with the set germ of the piece of $\mathcal{S}$ that contains $y$. The pair $(X, \mathcal{S})$ is then called a ***stratified space***. If $(X, \mathcal{S})$ is a decomposed space, then it is a stratified space if to each point $y$ one assigns the germ of the piece it is sitting in. For more details on stratified spaces, see for instance [77, Section 1.2].

The above decompositions of $M$ and $M/G$ by (connected components of) orbit-type manifolds satisfy many additional properties. The interested reader should consult [77, Chapter 1,2,4] for more details (see also [28, Chapter 2]).

We conclude this section with the following proposition, which mentions two of these additional properties. The first says that smaller strata are always of lower dimension.

First we introduce the following notation: let $H$, $H'$ be two closed subgroups in $G$. We say that $H$ and $H'$ are conjugate in $G$ if there exists an element $g \in G$ such that $gHg^{-1} = H'$. The ***conjugacy class*** of a closed subgroup $H$ in $G$ is denoted by

$$(H) = \{H' \subset G \mid H' = gHg^{-1}, g \in G\}.$$

We can define a partial ordering on the set of conjugacy classes of *compact* subgroups of $G$ by saying that $(H') \leq (H)$ if and only if $H$ is conjugate to a subgroup of $H'$. The anti-symmetry of $\leq$ follows from the fact that for any compact subgroup $H$ of $G$, the inclusion $gHg^{-1} \subset H$ implies that $gHg^{-1} = H$ ([75, Lemma 2.1.14]). In particular, as each isotropy group for a proper action is compact, the relation $\leq$ defines a partial ordering on the conjugacy classes of isotropy groups.

**Proposition 1.1.14** ([28, Proposition 2.7.2 and Corollary 2.8.6])**.** *Let $G$ be a proper action on a smooth manifold $M$ and write $\pi : M \to M/G$ for the canonical projection. Let $x \in M_{(H)}$. Then the following statements hold:*

1. *For all $y$ sufficiently close to $x$, one has $(H) \leq (G_y)$. Moreover, for all $y$ that are sufficiently close to $x$ but do not sit in $M^x_{(H)}$, the inequalities*

$$\begin{aligned} dim\ M^y_{(G_y)} &> dim\ M^x_{(H)} \\ dim\ \pi(M^y_{(G_y)}) &> dim\ \pi(M^x_{(H)}) \end{aligned}$$

   *hold. Thus, dim $S_j <$ dim $S_i$ if $S_j < S_i$, where $S_i$, $S_j$ are pieces of the stratification of either $M$ or $M/G$.*

2. *If $M/G$ is connected, then there exists a unique conjugacy class of subgroups $(H_0)$ such that $M_{(H_0)}$ is open and dense in $M$. Moreover, $\pi(M_{(H_0)})$ is open, dense and connected in $M/G$.*

The unique orbit-type submanifold $M_{(H_0)}$ that is open and dense in $M$ is called the ***principal stratum*** of $M$, also denoted by $M_{princ}$.

## 1.2 Symplectic geometry

Symplectic geometry is the mathematical language of classical Hamiltonian mechanics. In this section we give a short overview of the most basic notions. More information on symplectic geometry and its relation to classical mechanics can be found in [1, 36].

**Definition 1.2.1.** A ***symplectic structure*** on a manifold $M$ (of dimension $2n$) is a non-degenerate closed 2-form $\omega$ on $M$. If $M$ is endowed with a symplectic structure $\omega$, then $M$ is said to be a ***symplectic manifold*** denoted by $(M, \omega)$.

Let $(M, \omega)$ be a symplectic manifold. Because $\omega$ is non-degenerate, the top form

$$\varepsilon = (-1)^n \frac{1}{n!} \omega^n$$

is nowhere vanishing. We use $\varepsilon$ to endow $M$ with an orientation. The corresponding measure on $M$ is called the ***Liouville measure***, also denoted by $\varepsilon$.

**Example 1.2.2.** The basic example of a symplectic manifold is the cotangent bundle $T^*N$ of an ordinary manifold $N$. The ***fundamental*** 1***-form*** $\theta$ on $T^*N$ is defined as

$$\theta_{\alpha_q}(v_{\alpha_q}) = \alpha_q(T_{\alpha_q}\pi v_{\alpha_q}), \quad (\alpha_q \in T_q^*N, v_{\alpha_q} \in T_{\alpha_q}T^*N).$$

The 2-form

$$\omega = d\theta$$

is the ***canonical symplectic structure*** on $T^*N$. The non-degeneracy of $\omega$ follows from its local expression: choose local coordinates $(q^i)$ for $N$ and set $p_i(\alpha_q) = \alpha_q(\frac{\partial}{\partial q^i}|_q)$. The functions $\{q^1, \ldots, q^n, p_1, \ldots p_n\}$ provide local coordinates for $T^*N$. With respect to these coordinates the fundamental 1-form $\theta$ is of the form

$$\theta = \sum_{k=1}^{n} p_k dq^k,$$

where $n$ denotes the dimension of $N$. Consequently, the local expression for the symplectic form $\omega = d\theta$ is

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq^k.$$

The coordinates $\{q^1, \ldots, q^n, p_1, \ldots p_n\}$ are also called ***standard*** (or ***Darboux***) coordinates for $T^*N$.

In classical mechanics without constraints, the manifold $N$ of space coordinates is usually called the ***configuration space***. Newton's second law is a second order differential equation (in time), so next to the initial values of the positions, the initial values of the the velocities or momenta need to be known as well. These momenta are precisely the coordinates $p_i$ for $T^*N$ (hence the notation). The actual state of the physical system is therefore determined by a point in $T^*N$ which is therefore also known as ***phase space***, though the term ***state space*** also occurs frequently in the literature. In this sense, symplectic manifolds are generalised phase spaces.

Let $(M, \omega)$ be any symplectic manifold. The non-degeneracy of the symplectic 2-form $\omega$ determines an isomorphism

$$\flat_\omega : TM \to T^*M, \quad X \mapsto i_X\omega = \omega(X, \cdot),$$

whose inverse is denoted by $\#_\omega : T^*M \to TM$. If $f \in C^\infty(M, \mathbb{R})$ is any smooth function on $M$, one assigns a vector field $X_f$ to $f$ by the equation

$$i_{X_f}\omega = -df, \quad \text{or equivalently, } X_f = -(df)^{\#_\omega}.$$

The vector field $X_f$ is said to be the ***Hamiltonian vector field*** of $f$. In general, for any manifold and any vector bundle $E \to M$, the map $i_{e_0} : \Lambda^k(E_x) \to \Lambda^{k-1}(E_x)$ is defined as

$$(i_{e_0}\alpha_x)(e_1, \dots e_{k-1}) = \alpha_x(e_0, e_1, \dots, e_{k-1}), \quad (e_0, \dots e_k \in E_x), \tag{1.1}$$

and it is known as the ***interior product***. For every $s \in \Gamma^\infty(E)$, Equation (1.1) induces a $C^\infty(M)$-linear map

$$i_{s_0} : \Gamma^\infty(\Lambda^k E^*) \to \Gamma^\infty(\Lambda^{k-1}E^*), \quad (i_{s_0}\alpha)(s_1, \dots s_{k-1}) = \alpha(s_0, s_1, \dots, s_{k-1}).$$

In physics, time evolution of a classical system is determined by a real-valued function $H \in C^\infty(M)$, called the ***Hamiltonian***. Namely, the time evolution generated by $H$ corresponds to the (parametrised) integral curves of the Hamiltonian vector field $X_H$. That is, if $x_0 \in M$ is an initial state and $\gamma : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$ is an integral curve of $X_H$ such that $\gamma(0) = x_0$, then the state at time $t \in (-\varepsilon, \varepsilon)$ is $\gamma(t) \in M$.

Note that the partial differential equations for the integral curves of $X_H$ are first-order time derivatives, so we only need the point $x_0$ as our initial data. This is the fundamental advantage of passing from the configuration space $N$ to the phase space $T^*N$. In the standard coordinates for $T^*N$, the Hamiltonian vector field $X_H$ is given by

$$X_H = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q_k}\frac{\partial}{\partial p^k},$$

and its integral curves satisfy the system of partial differential equations

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q^k},$$

which are the so-called ***Hamilton equations***.

In quantisation-commutes-with-reduction problems, the phase space is usually invariant under some group of symmetries. 'The physics' is supposedly invariant under these transformations and therefore these symmetries do not correspond to actual, physical, degrees of freedom. The removal of these symmetries from the classical phase space is called ***classical reduction***. For symplectic manifolds this reduction is known as Marsden-Weinstein reduction, which we now briefly explain.

Note first that any $G$-action on $M$ determines a map $\mathfrak{g} \to \mathcal{X}(M)$ by

$$Y \mapsto Y_M, \quad Y_M(x) = \frac{d}{dt} \exp(tY)x|_{t=0},$$

where $\mathfrak{g}$ is the Lie algebra of $G$.

**Definition 1.2.3.** A ***symplectic action*** of a Lie group $G$ on a symplectic manifold $(M, \omega)$ is a (smooth) $G$-action $\Phi : G \times M \to M$ such that

$$\Phi_g^* \omega = \omega,$$

for all $g \in G$.

A symplectic action is called ***Hamiltonian*** if there exists a map $j : M \to \mathfrak{g}^*$ such that for all $Y \in \mathfrak{g}$ the equation

$$X_{j(Y)} = Y_M$$

holds, where $j(Y) \in C^\infty(M, \mathbb{R})$ is defined as $j(Y)(x) = \langle j(x), Y \rangle$, and where $X_{j(Y)}$ is its associated Hamiltonian vector field. The map $j$ is called a ***moment map*** for the action of $G$ on $M$.

If in addition the map $j$ satisfies

$$j \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ j, \quad (g \in G),$$

then the action of $G$ on $M$ is called ***strongly*** Hamiltonian.

**Example 1.2.4.** If $\Phi : G \times M \to M$ is a $G$-action on a manifold $M$, then this action carries over to an action on the cotangent bundle $T^*M$. An element $g \in G$ acts on $\alpha_x \in T^*_x N$ as

$$g(\alpha_x) := (\Phi^*_{g^{-1}})\alpha_x \in T^*_{gx} N.$$

This map is automatically symplectic because $\Phi_g^*\theta = \theta$ for all $g \in G$:

$$\begin{aligned}(\Phi_g^*\theta_{\alpha_x})(v_{g^{-1}\alpha_x}) &= \theta_{\alpha_x}(\Phi_{g,*}v_{g^{-1}\alpha_x}) = \alpha_x(\pi_*(\Phi_{g,*}v_{g^{-1}\alpha_x})) = \alpha_x((\pi\circ\Phi_g)_*v_{g^{-1}\alpha_x}) \\ &= \alpha_x((\Phi_g\circ\pi)_*v_{g^{-1}\alpha_x}) = (\Phi_g^*\alpha_x)(\pi_*v_{g^{-1}\alpha_x}) = \theta_{\Phi_g^*\alpha_x}(v_{g^{-1}\alpha_x}) \\ &= \theta_{g^{-1}\alpha_x}(v_{g^{-1}\alpha_x}).\end{aligned}$$

The moment map is given by

$$j(Y) = i_{Y_M}\theta, \quad (Y \in \mathfrak{g}), \tag{1.2}$$

or, more explicitly, by

$$\begin{aligned}j(Y)(\alpha_x) &= \alpha_x\left(\pi_*(Y_M)_{\alpha_x}\right) = \alpha_x\left(\frac{d}{dt}|_{t=0}\pi\exp(tY)\alpha_x\right) \\ &= \alpha_x\left(\frac{d}{dt}|_{t=0}\exp(tY)\pi\alpha_x\right) = \alpha_x\left(\frac{d}{dt}|_{t=0}\exp(tY)x\right) = \alpha_x(Y_N(x)).\end{aligned}$$

Indeed, it follows from equation Equation (1.2) and from the fact that the group action is symplectic that

$$d(j(Y)) = d(i_{Y_M}\theta) = -i_{Y_M}d\theta + L_{Y_M}\theta = -i_{Y_M}\omega.$$

Moreover, the moment map $j$ satisfies

$$\begin{aligned}\langle j(g\alpha_x), Y\rangle &= \langle j(\Phi_{g^{-1}}^*\alpha_x), Y\rangle = j(Y)(\Phi_{g^{-1}}^*\alpha_x) \\ &= (\Phi_{g^{-1}}^*\alpha_x)((Y_N)_{gx}) = \alpha_x\left[\Phi_{g^{-1}*}(Y_N)_{gx}\right] \\ &= \alpha_x\left[\frac{d}{dt}|_{t=0}g^{-1}\exp(tY)gx\right] = \alpha_x\left[(\mathrm{Ad}_{g^{-1}}Y)_N\right] \\ &= j(\mathrm{Ad}_{g^{-1}}Y)(\alpha_x) = \langle j(\alpha_x), \mathrm{Ad}_{g^{-1}}Y\rangle \\ &= \langle \mathrm{Ad}_{g^{-1}}^* j(\alpha_x), Y\rangle,\end{aligned}$$

for all $Y \in \mathfrak{g}$, so that the $G$-action on $T^*M$ is strongly Hamiltonian.

Let $G$ be a group of symmetries of the symplectic manifold $(M, \omega)$, acting in strongly Hamiltonian fashion. The classical reduction of this system is known as ***Marsden-Weinstein reduction*** or ***symplectic reduction***.

Let $\phi : M \to N$ be a smooth map between manifolds. An element $y \in N$ is called a ***regular value*** of $\phi$ if for each $x \in \phi^{-1}(y)$, the differential $T_x\phi : T_xM \to T_yN$ is surjective.

**Theorem 1.2.5** (Marsden-Weinstein reduction [69])**.** *Suppose that $(M, \omega)$ is a symplectic manifold upon which a Lie group $G$ acts in a strongly Hamiltonian fashion with moment map $j$. Let $\mu \in \mathfrak{g}^*$ and assume that $\mu \in \mathfrak{g}^*$ is a regular value* *of $j$, and suppose that the isotropy group $G_\mu := \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\,\mu = \mu\}$ acts freely and properly on $j^{-1}(\mu)$. Then $j^{-1}(\mu)/G_\mu$ carries a unique symplectic structure $\omega_\mu$ such that*

$$\pi_\mu^*\omega_\mu = \iota_\mu^*\omega,$$

*where $\pi_\mu : j^{-1}(\mu) \to j^{-1}(\mu)/G_\mu$ is the projection map and $\iota_\mu : j^{-1}(\mu) \hookrightarrow M$ is the inclusion.*

The assumption that $\mu$ is a regular value of $j$ ensures that $j^{-1}(\mu)$ is a submanifold of $M$. Freeness and properness of the action of $G_\mu$ on $j^{-1}(\mu)$ is sufficient for $j^{-1}(\mu)/G_\mu$ to have a unique manifold structure as well. We are especially interested in the Marsden-Weinstein reduction at $\mu = 0 \in \mathfrak{g}^*$, in which case $G_\mu = G$.

When the action is non-free, but still proper, $0 \in \mathfrak{g}^*$ is usually not a regular value.In that case the spaces $j^{-1}(\mu)$ and $j^{-1}(\mu)/G_\mu$ have no natural manifold structure. However, just as in the case of ordinary quotients of proper group actions, the Marsden-Weinstein quotient can be stratified by smooth manifolds. Moreover, the strata of the Marsden-Weinstein quotient are even symplectic manifolds. The full statement, which is due to Sjamaar and Lerman [82], is as follows (with definitions and notation borrowed from [75, Chapter 8]).

Define

$$M_{x,\mu}^{(H)} := (j^{-1}(\mu) \cap G_\mu M_H^x)/G_\mu.$$

The set $j^{-1}(\mu)\cap G_\mu M_H^x$ is a submanifold of $M$ and the quotient $M_{x,\mu}^{(H)}$ has a unique differentiable structure such that the projection

$$\pi_{x,\mu}^{(H)} : j^{-1}(\mu) \cap G_\mu M_H^x \to M_{x,\mu}^{(H)}$$

is a surjective submersion. Write $\iota_{x,\mu}^{(H)}$ for the inclusion $j^{-1}(\mu) \cap M_{(H)}^x \hookrightarrow M$.

**Theorem 1.2.6** (Symplectic stratification theorem, [81, 82]). *Let $(M,\omega)$ be a symplectic manifold. Assume that a Lie group $G$ acts properly on $M$ in strongly Hamiltonian fashion, with moment map $j$. Then the space $M_{x,\mu}^{(H)}$ carries a unique symplectic structure $\omega_{x,\mu}^{(H)}$ such that*

$$\pi_{x,\mu}^{(H)*}\omega_{x,\mu}^{(H)} = \iota_{x,\mu}^{(H)*}\omega.$$

*Moreover, the set $\{M_{x,\mu}^{(H)}\}$ constitutes a stratification of $M_\mu := j^{-1}(\mu)/G_\mu$ into disjoint symplectic strata, i.e. $M_\mu = \sqcup M_{x,\mu}^{(H)}$.*

*Furthermore, each connected component of $M_\mu$ contains a unique connected, open and dense stratum.*

See also [75, Chapter 8] for more details on the symplectic stratification theorem. We also use the notation $M//G$ to denote the Marsden-Weinstein quotient $j^{-1}(0)/G$.

In Chapters 3 and 4 we will study the symplectic reductions of cotangent bundles of compact connected Lie groups when the Lie group actions on their own cotangent bundles are induced by the actions of these Lie groups on themselves by conjugation.

## 1.3 Kähler geometry

In this section we explain the basic properties of Kähler manifolds and hermitian holomorphic line bundles over Kähler manifolds. We start, however, by examining the more general class of almost complex manifolds, because our formulation of quantisation of a symplectic manifold requires only a (compatible) almost complex structure. We then restrict our attention to those almost complex manifolds that are actually complex, and in particular to Kähler manifolds. For any compact, connected Lie group $G$ the symplectic manifold $T^*G$ will be endowed with a natural Kähler structure and our proofs on the quantisation of $T^*G$ rely heavily on this Kähler structure. The material in this section can also be found in many books on complex geometry or Kähler geometry, e.g. [3, 51].

### 1.3.1 Almost complex manifolds

In this section we describe the basics of almost complex manifolds.

**Definition 1.3.1.** An ***almost complex structure*** $J$ on a *real* vector bundle $E \to M$ is a (smooth) vector bundle homomorphism $E \to E$ such that $J^2 = -1$. A manifold $M$ that carries an almost complex structure $J$ on its tangent bundle $TM$ is called an ***almost complex manifold***.

If $J : V \to V$ is a real-linear map on a real vector space $V$ such that $J^2 = -1$, then $V$ can be turned into a complex vector space by defining

$$\lambda \cdot v = \mathrm{Re}\ \lambda \cdot v + \mathrm{Im}\ \lambda \cdot Jv.$$

If $J$ is an almost complex structure on a vector bundle $E \to M$, then the fibre-wise operators $J_x : E_x \to E_x$, $(x \in M)$, satisfy $J_x^2 = -1$, so that the fibres $E_x$ can be turned into complex vector spaces. This turns $E$ into a complex vector bundle. Therefore, if a vector bundle $E \to M$ carries an almost complex structure, the real dimension of its fibres is necessarily even. We also write $(E, J)$ for the complex bundle $E$, where the complex vector space structure in the fibres is given by $J$.

**Lemma 1.3.2.** *Let $M$ be an almost complex manifold, and let $g$ be a $J$-invariant Riemannian metric on $M$, i.e. $g(JX, JY) = g(X, Y)$ for all $X, Y$ in the same fibre. Then the formula*

$$h(X, Y) = g(X, Y) - ig(X, JY), \quad (X, Y \in T_xM, x \in M)$$

*determines a hermitian structure on $(TM, J)$ (anti-linear in the first variable), and $\omega(X, Y) = g(X, JY)$ is a $J$-invariant non-degenerate 2-form.*

*Conversely, given a hermitian metric $h$ on $(TM, J)$, then*

$$g(X, Y) = \frac{1}{2}\left(h(X, Y) + \overline{h(X, Y)}\right)$$

*defines a $J$-invariant Riemannian metric. The corresponding $J$-invariant non-degenerate 2-form is equal to $\omega = -\frac{1}{2i}\left(h(X, Y) - \overline{h(X, Y)}\right)$. We therefore also say that $g = Re\ h$ and $\omega = -Im\ h$.*

Lemma 1.3.2 says that an hermitian structure on the tangent bundle of an almost complex manifold is already determined by its real or imaginary part, the real part being a $J$-invariant Riemannian metric, the imaginary part being a $J$-invariant non-degenerate 2-form.

**Definition 1.3.3.** An almost complex manifold $(M, J)$ is called an ***almost hermitian manifold*** if $M$ carries a $J$-invariant Riemannian metric. According to Lemma 1.3.2 this is equivalent to the existence of a hermitian structure on $(TM, J)$. The corresponding 2-form $\omega$ is called the ***fundamental form***.

We say that a 2-form $\omega$ and an almost complex structure $J$ are ***compatible*** if $g(X, Y) = \omega(JX, Y)$ determines a Riemannian metric.

Let $(M, g)$ be a Riemannian manifold. For each $x \in M$ the metric $g$ induces a linear isomorphism between $T_xM$ and $T_x^*M$ by the rule

$$\flat : T_xM \to T_x^*M, \quad X^\flat(Y) = g(X, Y), (X, Y \in T_xM).$$

The isomorphism $\flat$ satisfies

$$Y^\flat(JZ) = g(Y, JZ) = -g(JY, Z) = -(JY)^\flat(Z) = (-JY)^\flat(Z),$$

where $Y, Z \in T_xM$. So, if we define

$$J\alpha := -\alpha \circ J, \quad (\alpha \in T_x^*M)$$

then $J$ commutes with $\flat$. The inverse of $\flat$ is denoted by $\#$.

As $J^2 = -1$, it cannot be diagonalised as a real linear map on $TM$. Let us therefore consider its complex-linear extension to $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$, the ***complexified tangent bundle*** of $M$. The almost complex structure $J_x$ *can* be diagonalised as a map on $T_xM \otimes \mathbb{C}$, its eigenvalues being $\pm i$. The eigenspaces of $J$ in $T_xM \otimes \mathbb{C}$ belonging to the eigenvalues $\pm i$ are given by

$$T_x^{(1,0)}M := \left\{ \frac{1}{2}(X - iJX) \;\middle|\; X \in T_xM \right\},$$
$$T_x^{(0,1)}M := \left\{ \frac{1}{2}(X + iJX) \;\middle|\; X \in T_xM \right\},$$

respectively. The maps

$$T_xM \to T_x^{(1,0)}M, \quad X \mapsto \frac{1}{2}(X - iJX), \tag{1.3}$$

and

$$T_xM \to T_x^{(0,1)}M, \quad X \mapsto \frac{1}{2}(X + iJX), \tag{1.4}$$

are linear and anti-linear isomorphisms, respectively. The unions $TM^{(1,0)} := \cup_x T_x^{(1,0)}M$ and $T^{(0,1)}M := \cup_x T_x^{(0,1)}M$ are complex subbundles of $T_{\mathbb{C}}M$, and

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M.$$

From now on, we always identify $T^{(1,0)}M$ with $TM$ through the map of Equation (1.3).

The following results are immediate from the above definitions.

**Lemma 1.3.4.** *Let $M$ be an almost-complex manifold.*

1. *The decomposition $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ induces a decomposition*

$$T^*_{\mathbb{C}}M = T^{*(1,0)}M \oplus T^{*(0,1)}M,$$

*where $T^{*(1,0)}M$ and $T^{*(0,1)}M$ are the dual bundles of $T^{(1,0)}M$ and $T^{(0,1)}M$, respectively.*

2. *The complex bundle of $k$-form $\Lambda^k T^*_{\mathbb{C}}M$ decomposes as*

$$\Lambda^k T^*_{\mathbb{C}}M = \oplus_{r+s=k} \Lambda^{(r,s)}T^*M,$$

*where $\Lambda^{(r,s)}T^*M = \Lambda^r(T^{*(1,0)}M) \otimes \Lambda^s(T^{*(0,1)}M)$. We use the notation $\pi^{(r,s)}$ for the projection maps $\Lambda^k T^*_{\mathbb{C}}M \to \Lambda^{(r,s)}T^*M$ (with respect to the above decomposition).*

Sections of $\Lambda^{(r,s)}T^*M$ are also called $(r,s)$***-forms*** and we write $\Omega^{(r,s)}(M)$ for the space of all smooth sections of $\Lambda^{(r,s)}T^*M$.

**Remark 1.3.5.** Elements in $T_x^{*(1,0)}M$ are of the form $\alpha + iJ\alpha$, ($\alpha \in T_x^*M$), since, by definition, $J$-acts on a covector $\alpha$ as $(J\alpha)(X) = -\alpha(JX)$. This definition is unfortunate in the sense that the $J$-action on $T^{*(1,0)}M$ corresponds to multiplication with $-i$ instead of $i$. We would like to stress that complex multiplication on the fibres of $T^{*(1,0)}M$ is determined by multiplication with $i$ (rather than by the application of $J$).

If $(M, \omega)$ is a symplectic manifold and $J$ is a compatible almost-complex structure, then there are two natural measures on $M$. First, there is the Liouville measure $\varepsilon$, and second, there is the Riemannian measure $\mu_g$ that comes from the Riemannian metric $g(X, Y) = \omega(JX, Y)$. The following proposition shows that both measures are equal.

**Proposition 1.3.6.** *If $(M, \omega)$ is symplectic with compatible almost structure $J$, then the Liouville and Riemannian measure agree.*

*Proof.* We prove that both measures are equal on each oriented chart. On such a chart the symplectic structure is of the form $\omega = \sum_{i,j} \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$. On the one hand, the volume form $\varepsilon$ is equal to

$$\varepsilon = \frac{(-1)^n}{n!}\omega^n = (-1)^n \mathrm{Pf}(\omega_{ij}) dx^1 \wedge \cdots \wedge dx^{2n}.$$

Here, $(-1)^n \mathrm{Pf}(\omega_{ij})$ is always positive because the chosen chart preserves orientation.

On the other hand, as $g(X, Y) = \omega(JX, Y)$, we see that on the same chart, when we write each vector $Z$ as $Z = \sum_i Z_i \frac{\partial}{\partial i}$,

$$g(X, Y) = \omega(JX, Y) = (JX)_i \omega_{ij} Y_j = X_k J_{ik} \omega_{ij} Y_j = X_k (J^T \omega)_{kj} Y_j.$$

The map $J$ is a linear transformation of $T_x M$. Since $\det J^2 = 1$, we see that $\det J = \pm 1$ (independent of the chosen basis). Consequently, with respect to the local coordinates $(x^i)$ of the chart, we have

$$\det(g_{ij}) = \det((J^T \det \omega)_{ij}) = \det J \det(\omega_{ij}) = \pm \det(\omega_{ij}).$$

Hence $\sqrt{|\det g_{ij}|} = \sqrt{|\det(\omega_{ij})|} = \sqrt{|\mathrm{Pf}(\omega_{ij})^2|} = (-1)^n \mathrm{Pf}(\omega_{ij})$. Thus, on oriented charts the Riemannian measure is

$$\sqrt{|\det g_{ij}|} dx^1 \wedge \cdots \wedge dx^n = (-1)^n \mathrm{Pf}(\omega_{ij}) dx^1 \wedge \cdots \wedge dx^n = \varepsilon.$$

□

### 1.3.2 Complex manifolds

We now discuss the notion of *complex manifolds*.

**Definition 1.3.7.** An almost complex structure $J$ on a manifold $M$ is a ***complex structure*** if there exist local coordinates $\{(x^k, y^k)\}_{i=1}^n$ of $M$ such that $J$ is of the form

$$J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}, \quad (1 \leq k \leq n). \tag{1.5}$$

Similarly, an almost hermitian manifold is called a ***hermitian manifold*** if the almost complex structure is a complex structure.

The coordinates $\{z^k = x^k + iy^k\}$ provide complex coordinates for the manifold $M$. The transition functions between two such coordinate charts are holomorphic maps, which turns $M$ into a complex manifold. Conversely, if $M$ is a complex manifold, then Equation (1.5) defines a complex structure on $M$. Thus, complex manifolds and manifolds carrying a complex structure are the same thing.

If $M$ is a complex manifold, define (*cf.* Equations (1.3) and (1.4))

$$\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \overline{z}^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right),$$

and their dual covectors

$$dz^k := dx^k + idy^k, \quad d\overline{z}^k := dx^k - idy^k.$$

The exterior derivative on $M$ is then locally of the form

$$d = \sum_{k=1}^n dx^k \frac{\partial}{\partial x^k} + dy^k \frac{\partial}{\partial y^k} = \sum_{k=1}^n dz^k \frac{\partial}{\partial z^k} + d\overline{z}^k \frac{\partial}{\partial \overline{z}^k}.$$

Under a holomorphic change of coordinates $w^i = w^i(z^1, \ldots z^n)$, the expression $\sum_k dz^k \frac{\partial}{\partial z^k}$ transforms to $\sum_k dw^k \frac{\partial}{\partial w^k}$ and similarly, $\sum_k d\overline{z}^k \frac{\partial}{\partial \overline{z}^k}$ transforms to $\sum_k d\overline{w}^k \frac{\partial}{\partial \overline{w}^k}$. In other words, the operator $d$ decomposes globally into two separate operators

$$d = \partial + \overline{\partial},$$

and $\partial = \pi^{(r+1,s)} \circ d$ and $\overline{\partial} = \pi^{(r,s+1)} \circ d$ on $\Lambda^{(r,s)}T^*M$. With respect to some holomorphic coordinates $\{z^k\}_{k=1}^n$, we have $\partial = \sum_k dz^k \frac{\partial}{\partial z^k}$ and $\overline{\partial} = d\overline{z}^k \frac{\partial}{\partial \overline{z}^k}$. The operator $\overline{\partial}$ is called the ***Dolbeault operator***.

Let $\alpha \in \Omega^{(r,s)}(M)$ be given, then

$$d^2\alpha = \partial^2\alpha + (\partial\overline{\partial} + \overline{\partial}\partial)\alpha + \overline{\partial}^2\alpha.$$

The terms on the right-hand side take values in $\Omega^{(r+2,s)}(M)$, $\Omega^{(r+1,s+1)}(M)$ and $\Omega^{(r,s+2)}(M)$, respectively. Therefore, because $d^2 = 0$, each of these terms is zero. On complex manifolds we therefore have the relations

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

In the rest of this section we study hermitian holomorphic vector bundles over complex manifolds.

**Definition 1.3.8.** Let $M$ be a complex manifold. A ***holomorphic vector bundle*** $\pi : E \to M$ is a complex vector bundle $E$, such that $E$ is a complex manifold, the projection map $\pi$ is holomorphic, and $E$ is locally trivial with biholomorphic local trivialisations. A section $M \to E$ is called ***holomorphic*** if it is holomorphic as a map $M \to E$.

A holomorphic vector bundle $E$ is said to be ***hermitian*** if $E$ carries a smooth hermitian metric.

Let $E \to M$ be a holomorphic bundle over $M$. We define a map

$$\overline{\partial}^E : \Omega^{(r,s)}(M,E) \to \Omega^{(r,s+1)}(M,E)$$

by defining it with respect to local *holomorphic* frames $(Z_1, \dots, Z_k)$ of $E$ as

$$\overline{\partial}^E \left( \sum_{j=1}^{k} \alpha^j \otimes Z_j \right) = \sum_{j=1}^{k} \overline{\partial}(\alpha^j) \otimes Z_j, \quad (\alpha_j \in \Omega^{(r,s)}(M,E)). \tag{1.6}$$

This definition of $\overline{\partial}^E$ is independent of the choice of local holomorphic frame.

**Example 1.3.9.** If $M$ is a complex manifold, then all changes of coordinates are holomorphic, so $TM$ is a holomorphic vector bundle in a natural way. The map $TM \ni X \to \frac{1}{2}(X - iJX) \in T^{(1,0)}M$ is a complex-linear isomorphism, turning the bundle $T^{(1,0)}M$ into a holomorphic vector bundle as well. One can prove that a vector field $X \in \Gamma^\infty(M,TM)$ is holomorphic if and only if $[X, JY] = J[X, Y]$ for all $Y \in \Gamma^\infty(M,TM)$ (see [3]).

If $M$ is a hermitian manifold, then $TM$ is a hermitian holomorphic vector bundle.

If $\nabla$ is a connection on a holomorphic vector bundle $E$, then $\nabla$ can be decomposed as

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)},$$

where $\nabla^{(1,0)} = \pi^{(1,0)} \circ \nabla$ and $\nabla^{(0,1)} = \pi^{(0,1)} \circ \nabla$.

Hermitian holomorphic vector bundles have a canonical connection.

**Lemma 1.3.10.** *Let $E \to M$ be a hermitian holomorphic vector bundle over a complex manifold $M$. There exists a unique hermitian connection $\nabla^E$ on $E$ such that $(\nabla^E)^{(0,1)} = \overline{\partial}^E$. Moreover, if $(Z_1, \ldots, Z_k)$ is a local holomorphic frame and $\theta$ is the matrix of 1-forms defined by $\nabla^E Z_i = \theta_{ji} Z_j$, then*

$$\theta = \theta^{(1,0)} = h^{-1}\partial h,$$

*where $h = (h_{ij}) = (h(Z_i, Z_j))$ denotes the matrix of the hermitian structure $h$ with respect to the holomorphic frame $(Z_1, \ldots, Z_k)$. In particular, $\theta$ is a $(1,0)$-form.*

*Proof.* Assume that such a $\nabla^E$ exists. Let $(Z_1, \ldots Z_k)$ be a local holomorphic frame and let $\theta_{ji}$ be the connection 1-form given by $\nabla^E Z_i = \theta_{ji} Z_j$. The condition $(\nabla^E)^{(0,1)} = \overline{\partial}$ implies that $\theta$ is of type $(1,0)$. Since the connection is hermitian, we obtain

$$\begin{aligned} dh_{ij} &= dh(Z_i, Z_j) = h(\theta_{ki} Z_k, Z_j) + h(Z_i, \theta_{kj} Z_k) = \overline{\theta_{ki}} h_{kj} + \theta_{kj} h_{ik} \\ &= \overline{\theta_{ki} h_{jk}} + \theta_{kj} h_{ik}. \end{aligned}$$

It follows that $\partial h_{ij} = h_{ik}\theta_{kj}$, or equivalently, $\theta = h^{-1}\partial h$. Thus, $h$ determines $\nabla^E$ uniquely.

Existence of $\nabla^E$ is proved by showing that the locally defined forms $\theta = h^{-1}\partial h$ transform correctly under changes of holomorphic frames. □

**Definition 1.3.11.** The (unique) connection $\nabla^E$ in Lemma 1.3.10 is called the ***Chern connection*** on $E$. It depends on both the holomorphic and the hermitian structure on the bundle $E$.

The Chern connection $\nabla^E$ determines the holomorphic structure on a hermitian vector bundle $E$, in the sense that a local section $s$ of $E$ is holomorphic if and only if $(\nabla^E)^{(0,1)} s = 0$.

The dual $E^*$ of a hermitian vector bundle $E$ is conjugate-linearly isomorphic to $E$: an element $T \in E^*_x$ is sent to the unique element $h(T)$ in $E_x$ such that $T(X) = h(h(T), X)$ for all $X \in E_x$. If $\nabla^E$ is a hermitian connection on $E$, then $\alpha \mapsto [\nabla^E, \alpha]$ is the corresponding connection on $E^*$. Now,

$$[\nabla^E, \alpha](s) = d(\alpha s) - \alpha(\nabla^E s) = d\,[h(h(\alpha), s)] - h(h(\alpha), \nabla^E s) = h(\nabla^E h(\alpha), s),$$

where we have used that $\nabla^E$ is a hermitian connection. Furthermore, $E^*$ can be endowed with a holomorphic structure by defining a local section $\alpha : U \to E^*$ to be holomorphic if and only if $\alpha(s)$ is holomorphic for all local holomorphic sections $s : U' \to E$ with $U' \subset U$. This is a well-defined holomorphic structure on $E^*$.

The Chern connection behaves well under vector bundle operations:

**Proposition 1.3.12.** *Let $(E_1, h_1)$, $(E_2, h_2)$, $(E, h)$ be hermitian holomorphic vector bundles over $M$. Denote their Chern connections by $\nabla_1$, $\nabla_2$ and $\nabla$, respectively. Then*

1. *the Chern connection on* $E_1 \oplus E_2$ *is given by* $\nabla_1 \oplus \nabla_2$ *(with respect to the hermitian form* $h_{1\oplus 2}\left[(s_1,s_2),(t_1,t_2)\right] = h_1(s_1,t_1) + h_2(s_2,t_2)$*);*

2. *the Chern connection on* $E_1 \otimes E_2$ *is given by* $\nabla_{1\otimes 2} := \nabla_1 \otimes 1 + 1 \otimes \nabla_2$ *(with respect to the hermitian form* $h_{1\otimes 2}\left(s_1 \otimes s_2, t_1 \otimes t_2\right) = h_1(s_1,t_1)h_2(s_2,t_2)$*);*

3. *the Chern connection on* $E^*$ *is given by* $(\nabla^* T)(s) = d(Ts) - (T\nabla s)$ *(with respect to the hermitian form* $h_*(T,S) = h(h(S),h(T))$*).*

**Proposition 1.3.13.** *Let* $E \to M$ *be a holomorphic vector bundle with hermitian metric* $h$ *and Chern connection* $\nabla^E$*. If* $(Z_1,\ldots,Z_k)$ *is a local holomorphic frame of* $E$*, then the curvature form* $R^E$ *of* $\nabla^E$ *is equal to* $R^E = \overline{\partial}\theta$ *with respect to this frame. Here,* $\theta$ *is as in Lemma 1.3.10. In particular,* $R^E$ *is of type* $(1,1)$*.*

*Proof.* With respect to the holomorphic frame $(Z_1,\ldots Z_k)$ the curvature $R^E$ is given by $R^E = d\theta + \theta\wedge\theta = (\partial\theta + \theta\wedge\theta) + \overline{\partial}\theta$, where we have separated the $(2,0)$-part and the $(1,1)$-part. Now,

$$\partial\theta = \partial(h^{-1}\partial h) = \partial h^{-1}\wedge\partial h = -h^{-1}(\partial h)h^{-1}\wedge\partial h = -\theta\wedge\theta,$$

so that the $(2,0)$-part of $R^E$ is equal to 0. Thus, $R^E = \overline{\partial}\theta$. □

Let $V$, $W$ be vector spaces over $\mathbb{R}$ or $\mathbb{C}$. If $A : V \to W$ is a linear map, then the ***transpose*** $A^T : W^* \to V^*$ is defined as $A^T(\alpha)(v) = \alpha(Av)$. Let $h$ be an (hermitian) inner product on $V$. Choose a basis $(e_i)$ and let $h$ also denote the matrix $(h_{ij})$ with entries $h_{ij} = h(e_i,e_j)$. With respect to the basis $(e_i)$, the inner product $h(v,w)$ is equal to $v^*hw$, where in the latter expression $v$ and $w$ are considered column vectors with respect to the chosen basis. Let $\alpha$ be an element of $V^*$ and consider it as a column vector $(\alpha_1,\ldots\alpha_n)^T$ with respect to the dual basis $\{e_i^*\}$. Then

$$\alpha^T\cdot v = \alpha(v) = h(h(\alpha),v) = h(\alpha)^*\cdot h\cdot v, \qquad \text{for all } v\in V,$$

so that $\alpha^T = h(\alpha)^*\cdot h$. Now,

$$h(\alpha,\beta) = h(\alpha,\beta)^T = \left(h(\beta)^* h h(\alpha)\right)^T = \left(\beta^T h^{-1} h h^{-1}\overline{\alpha}\right)^T = \alpha^*(h^{-1})^T\beta,$$

where we have used that $h = h^*$ and $\alpha^T = h(\alpha)^*\cdot h$. Hence, if $h$ is the matrix of a hermitian inner product with respect to some basis, then the matrix of the inner product on $V^*$ with respect to the corresponding dual basis is equal to $(h^{-1})^T$. We are now ready to prove the following fact.

**Proposition 1.3.14.** *Let* $E$ *be a hermitian holomorphic vector bundle with curvature* $R^E$*. Then* $R^{E^*} = -(R^E)^T$*.*

*Proof.* Let $(Z_i)$ be a local holomorphic frame for $E$ and let $h$ be the matrix of the hermitian structure with respect to the frame $(Z_i)$. The dual frame of $(Z_i)$ is a holomorphic frame of $E^*$. By the remarks preceding this Proposition and by Proposition 1.3.13, the curvature of $E^*$ is equal to

$$R^{E^*} = \overline{\partial}\left(h^T \partial (h^{-1})^T\right) = \left(\overline{\partial}\left(\partial(h^{-1})h\right)\right)^T = -\left(\overline{\partial}\left(h^{-1}\partial h\right)\right)^T,$$

which is precisely $-(R^E)^T$. □

### 1.3.3 Kähler manifolds

Kähler manifolds are special cases of complex manifolds. We recall the definition of a Kähler manifold and apply the results of Section 1.3.2 to (exterior) powers of the tangent and cotangent bundle.

**Definition 1.3.15.** A hermitian manifold $M$ is called a ***Kähler manifold*** if the corresponding fundamental form $\omega$ is symplectic.

Recall that on a hermitian manifold $M$ the Riemannian metric $g$ and the fundamental form $\omega$ are related by $g(X,Y) = \omega(JX,Y)$. This is reflected by their forms with respect to holomorphic charts.

**Lemma 1.3.16.** *If $M$ is a Kähler manifold and $(U,z)$ is a holomorphic chart, then*

$$g|_U = \sum_{k,l} g_{k\bar{l}} dz^k \vee d\overline{z}^l, \quad \omega|_U = -i \sum_{k,l} g_{k\bar{l}} dz^k \wedge d\overline{z}^l,$$

*where $g_{k\bar{l}} = g(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \overline{z}^l})$ is a hermitian matrix in the sense that $g_{k\bar{l}} = \overline{g_{l\bar{k}}}$. Here, $dz^k \vee d\overline{z}^l := dz^k \otimes d\overline{z}^l + d\overline{z}^l \otimes dz^k$.*

Since $M$ is a complex manifold, the tangent bundle $TM$ is a holomorphic vector bundle over $M$ (see also Example 1.3.9). Recall that the symplectic structure and the Riemannian metric determine a hermitian structure on $TM$ by the formula

$$h(X,Y) = g(X,Y) - i\omega(X,Y), \quad (X,Y \in TM),$$

which is anti-linear in the first variable. Since the tangent bundle $TM$ of a Kähler manifold $M$ is hermitian holomorphic, it carries, besides the Levi-Civita connection, another natural connection, namely the Chern connection. The following Theorem states that both connections are equal if and only if $(TM, J, g)$ defines a Kähler structure on $M$.

**Theorem 1.3.17.** *Let $(M,J)$ be a complex manifold with compatible metric $g$. Denote the Levi-Civita connection by $\nabla^g$ and the Chern connection on $TM$ by $\nabla$. The following conditions are equivalent:*

1. *$M$ is Kähler;*
2. *$d\omega = 0$, (i.e. the non-degenerate form $\omega$ is closed);*
3. *$\nabla^g J = 0$;*
4. *$\nabla^g$ preserves the complex bundles $T^{(1,0)}M$ and $T^{(0,1)}M$;*
5. *The Chern connection $\nabla$ of the hermitian metric $h$ on $TM$ is equal to the Levi-Civita connection $\nabla^g$.*

On $T_{\mathbb{C}}M$ we consider the hermitian structure

$$(Z, W) \mapsto g(\overline{Z}, W), \quad ((Z, W) \in T_{\mathbb{C}}M \times_M T_{\mathbb{C}}M)$$

One can check that under the isomorphism $X \mapsto \frac{1}{2}(X - iJX)$ between $TM$ and $T^{(1,0)}M$, the equality

$$g(\bar{\cdot}, \cdot) = \tfrac{1}{2}h(\cdot, \cdot)$$

or, more precisely,

$$g\left(\frac{X + iJX}{2}, \frac{Y - iJY}{2}\right) = \frac{1}{2}h(X, Y)$$

holds. The dual bundle of $T^{(1,0)}M$ is equal to $T^{*(1,0)}M$. Being a dual bundle of a hermitian holomorphic vector bundle, $T^{*(1,0)}M$ has a natural holomorphic structure, hermitian structure and a corresponding Chern connection.

**Lemma 1.3.18.** *Let $M$ be a Kähler manifold.*

1. *The induced hermitian structure $h$ on $T^*_{\mathbb{C}}M$ is*

$$h(\alpha, \beta) = g(\overline{\alpha}, \beta),$$

*where $g$ denotes the complex-linear extension of the (inverse) Riemannian metric $g$ on $T^*M$ to $T^*_{\mathbb{C}}M$.*

2. *The Chern-connection on $T^{*(1,0)}M$ is given by the restriction of the Levi-Civita-connection $\nabla^g$ on $T^*_{\mathbb{C}}M$.*

If $g$ is a (real/complex) bilinear form on a real/complex vector space $V$, then $g$ is extended to $T^kV := V^{\otimes k}$ as

$$\widetilde{g}(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k) = g(v_1, w_1) \cdots g(v_k, w_k). \tag{1.7}$$

The bilinear form $g$ on $\Lambda^k V$ is defined as

$$g = \frac{1}{k!}\widetilde{g}.$$

**Remark 1.3.19.** Suppose $g$ is a real bilinear form on a real vector space $V$. The corresponding complex bilinear form on $V \otimes \mathbb{C}$ induces a complex bilinear form on $T^k(V \otimes \mathbb{C})$ according to Equation (1.7). On the other hand, Equation (1.7) also determines a real bilinear form on $T^kV$, which can then be extended by complex-linearity to a complex bilinear form on $(T^kV) \otimes \mathbb{C} \cong T^k(V \otimes \mathbb{C})$. Both bilinear forms on $T^k(V \otimes \mathbb{C})$ are equal as they are equal on the real subspace $T^kV$.

We endow the bundles $\Lambda^k T^*_{\mathbb{C}} M$ with the hermitian inner product

$$h(\alpha, \beta) := g(\overline{\alpha}, \beta) = \frac{1}{k!}\widetilde{g}(\overline{\alpha}, \beta), \tag{1.8}$$

where $\alpha, \beta \in \Lambda^k T^*_{\mathbb{C}} M$ are in the same fibre. The subbundles $\Lambda^{(r,s)}T^*M \subset \Lambda^k T^*_{\mathbb{C}} M$, $(r + s = k)$ are mutually orthogonal with respect to this hermitian structure on $\Lambda^k T^*_{\mathbb{C}} M$. Together, these hermitian structures determine an hermitian structure on $\Lambda^\bullet T^*_{\mathbb{C}} M = \Lambda^{(\bullet,\bullet)}T^*M$.

The following subbundle of $\Lambda^\bullet T^*_{\mathbb{C}} M$ plays a crucial role in Dolbeault-Dirac quantisation.

**Definition 1.3.20.** Let $M$ be a Kähler manifold. The bundle $\mathbb{E}$ is the hermitian vector bundle $\Lambda^{(0,\bullet)}T^*M$ with hermitian structure

$$h(\alpha, \beta) := g(\overline{\alpha}, \beta),$$

where $\alpha, \beta \in \mathbb{E}$ are contained in the same fibre.

**Remark 1.3.21.** The hermitian structures on $T_{\mathbb{C}}M$, $T^*_{\mathbb{C}}M$, $\Lambda^{(0,\bullet)}T^*M$ are obtained from the Riemannian metric $g$ by point-wise operations. These hermitian structures can therefore also be defined on arbitrary almost hermitian manifolds. In particular, Definition 1.3.20 extends to almost hermitian manifolds.

Another important bundle is the *canonical line bundle*.

**Definition 1.3.22.** Let $M$ be a Kähler manifold. The hermitian holomorphic line bundle $K := \Lambda^{(n,0)}(T^*M) = \Lambda^n T^{*(1,0)}M$ is called the ***canonical line bundle*** of $M$. The holomorphic structure is induced by the holomorphic structure on $T^{*(1,0)}M$ and the hermitian structure on $K$ is given by Equation (1.8).

For later use we give the following alternative form of the hermitian structure on $K$. If $(M, g)$ is a $2n$-dimensional oriented Riemannian manifold with volume form $\mu$, then the Hodge $*$-operator $\Omega^\bullet_{\mathbb{R}}(M) \to \Omega^{2n-\bullet}_{\mathbb{R}}(M)$ is the *invertible* $C^\infty(M, \mathbb{R})$-linear operator that is defined on $\Lambda^k T^*M$ as

$$\alpha \wedge *\beta = g(\alpha, \beta)\mu, \quad (\alpha, \beta \in \Lambda^k T^*M). \tag{1.9}$$

It satisfies $*^2 = (-1)^{(2n-k)k}$ on $k$-forms. If we extend both $g$ and $*$ to $\Omega^\bullet_{\mathbb{C}}(M)$ by complex-(bi)linearity, then Equation (1.9) remains valid when $\alpha, \beta \in (\Lambda^k T^*M) \otimes \mathbb{C}$. This leads to the following result.

**Lemma 1.3.23.** *Let $M$ be Kähler manifold. The hermitian structure $h(\alpha, \beta) = g(\overline{\alpha}, \beta)$ on $K$ satisfies*

$$(-1)^{n(n-1)/2} i^k g(\overline{\alpha}, \beta)\mu = \overline{\alpha} \wedge \beta,$$

*where $\mu$ is the Riemannian measure on $M$ (which is equal to the Liouville measure $\varepsilon$).*

*Proof.* Let $\alpha, \beta \in \Omega^{(n,0)}(T^*M)$. Since $g(\beta', \beta) = 0$ if $\beta' \notin \Omega^{(0,n)}(M)$, $*\beta \in \Omega^{(n,0)}(M)$. Consequently,

$$g(\overline{\alpha}, \beta)\mu = c\overline{\alpha} \wedge \beta$$

for some constant $c$ that squares to $(-1)^n$. It remains to determine $c$. Let $x \in M$. The value of $c$ can be determined that by requiring that $c\overline{\alpha}_x \wedge \alpha_x$ is a positive multiple of the Liouville form $\varepsilon$ if $\alpha_x \neq 0$.

Let $z = (z^i)$ be holomorphic coordinates around $x \in M$, such that

$$g_x\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) = g_x\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right) = \delta_{kl}, \quad g_x\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l}\right) = 0.$$

Then $\varepsilon_x = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$. If $\alpha = dz^1 \wedge \cdots \wedge dz^n$, then

$$c\overline{\alpha}_x \wedge \alpha_x = cd\overline{z}^1 \wedge \cdots \wedge d\overline{z}^n \wedge dz^1 \wedge \cdots \wedge dz^n.$$

Interchanging forms, after $n(n-1)/2$ steps one arrives at

$$c(-1)^{n(n-1)/2} d\overline{z}^1 \wedge dz^1 \wedge \cdots \wedge d\overline{z}^n \wedge dz^n,$$

which is equal to

$$c(-1)^{n(n-1)/2}(2i)^n dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n.$$

So, $(-1)^{n(n-1)/2} i^n c = 1$. □

## 1.4 Introduction to quantisation

We give a brief overview of the history of quantisation and the quantisation-commutes-with-reduction principle. We also explain our approach to the equivariant quantisation of $T^*G$, where the $G$-action on $T^*G$ is the pull-back of the $G$-action on itself by conjugation.

### 1.4.1 A short history of quantisation

In physics, quantisation is the transition of a classical mechanical system to a "corresponding" quantum mechanical system. A classical mechanical system is defined by a *phase space*. For our purpose we restrict ourselves to symplectic manifolds $(M, \omega)$ but one could also consider the more general class of Poisson manifolds. The phase space consists of all physical *pure* states of the system (pure in the sense that pure states reflect maximal information on the state, as opposed to *mixed states*, which are probability measures on phase space).

To probe the phase space, one considers so-called *observables*, e.g. (kinetic) energy, (angular) momentum, position, etc. These are represented by real-valued functions on the phase space. The value of an observable $f \in C^\infty(M, \mathbb{R})$ in a pure state $x \in M$ is simply the evaluation of $f$ at that point, i.e. $f(x)$.

The symplectic structure $\omega$ induces a (real) bi-linear anti-symmetric bracket $\{\cdot, \cdot\}$ on the algebra of observables $C^\infty(M, \mathbb{R})$, the so-called *Poisson bracket*:

$$\{f, g\} := \omega(X_g, X_f) = X_g(f), \quad (f, g \in C^\infty(M, \mathbb{R})).$$

The *dynamics* or *time evolution* of the classical system is determined by a *Hamiltonian*, which is a real-valued function $H$ on $\mathbb{R}$. More specifically, the time evolution is given by the integral curves of the Hamiltonian vector field associated to $H$. The time evolution of an observable $f \in C^\infty(M, \mathbb{R})$ can then be shown to be the solution of the differential equation given by

$$\frac{df}{dt} = X_H(f) = \omega(X_H, X_f) = \{f, H\}.$$

Quantum mechanically, the situation is quite different. The phase space is replaced by a Hilbert space, but the time evolution is still induced by a single observable, which is again called the *Hamiltonian* (of the quantum mechanical system). Roughly speaking, quantisation of the classical system then means that one constructs a certain Hilbert space and a quantum mechanical Hamiltonian from the corresponding classical data.

Before we explain how the Hilbert space may be constructed, let us say what we mean by 'quantisation commutes with reduction'. Assume that some Lie group $G$ acts properly and symplectically on a symplectic manifold $(M, \omega)$. By an *equivariant quantisation* of $(M, \omega)$ we mean that the $G$-action on $M$ determines a unitary $G$-action on the Hilbert space $\mathcal{Q}(M, \omega)$.

Suppose we are given equivariant quantisation maps $\mathcal{Q}$ for the symplectic manifold $(M, \omega)$ and its, possibly singular, Marsden-Weinstein reduction $M_0 = j^{-1}(0)/G$. We then say that *quantisation commutes with reduction* (for $0 \in \mathfrak{g}^*$) if the diagram

$$\begin{CD} (M,\omega) @>{\mathcal{Q}}>> \mathcal{Q}(M,\omega) \\ @V{()//G}VV @VV{()^G}V \\ (M_0,\omega_0) @>{\mathcal{Q}}>> \mathcal{Q}(M_0,\omega_0) \end{CD}$$

commutes, up to unitary isomorphism. The left downward arrow is the (singular) Marsden-Weinstein reduction of $(M,\omega)$ at 0. The right downward arrow, which is the reduction at the quantum side, sends $\mathcal{Q}(M,\omega)$ to its subspace $\mathcal{Q}(M,\omega)^G$ of $G$-invariant elements. The horizontal arrows are called *quantisation maps* and the vertical arrows are called *reduction maps*. This definition of quantisation-commutes-with-reduction was introduced in the paper [33] by Guillemin and Sternberg. Commutativity of the above diagram for specific symplectic manifolds $(M,\omega)$ and Lie groups $G$ is therefore also known as the *Guillemin-Sternberg conjecture*.

The earliest approaches to the construction of a (quantum) Hilbert space $\mathcal{H} = \mathcal{Q}(M,\omega)$ from a symplectic manifold $(M,\omega)$ come from geometric quantisation [59, 84] (see also [92]). Let $(M,\omega)$ be a symplectic manifold. We call $(M,\omega)$ ***pre-quantisable*** if there exists a hermitian vector bundle $L$ over $M$ endowed with a hermitian connection $\nabla^L$ such that $(\nabla^L)^2 = 2\pi i\omega$. The pair $(L,\nabla^L)$ is called a ***pre-quantisation*** for $(M,\omega)$. If $M$ is a $G$-manifold, then $L$ is supposed to be an equivariant line bundle and $\nabla^L$ is supposed to be $G$-equivariant.

If $M$ is a *compact* Kähler manifold and $L$ a hermitian holomorphic line bundle such that $(\nabla^L)^2 = 2\pi i\omega$, where $\nabla^L$ denotes the Chern connection, then the quantisation of $M$ is defined to be the vector space of all holomorphic sections of $L$. If a *compact* Lie group $G$ that acts on $M$ in a strongly Hamiltonian fashion also preserves the complex structure $J$, then Guillemin and Sternberg [33] proved that quantisation commutes with reduction in the above sense: $\dim(\mathcal{Q}(M,\omega))^G = \dim(\mathcal{Q}(M_0,\omega_0))$.

Later, geometric quantisation was redefined as the index of a suitable Dirac operator (see e.g. [83]). More precisely, given a $G$-equivariant pre-quantisable compact symplectic manifold $(M,\omega)$, one picks a $G$-invariant, compatible, almost complex structure $J$ on $M$ and uses $J$ to define a spin$^c$ Dirac operator on $M$. One then considers the index of the generalised Dirac operator $D$ on $M$ obtained by twisting with a pre-quantum line bundle $L$ (actually, one takes the index of $D_+$). This index is a formal difference of finite-dimensional $G$-representations. A 'quantisation commutes with reduction' result for compact actions on compact manifolds in this setting was proved by Meinrenken in [71] and many others under the assumption that 0 is a regular value of the momentum map. In [72], the case where 0 is not a regular value, was studied. If $M$ is Kähler and $L \otimes K^*$ is *positive*, where $K^*$ denotes the *anti-canonical line bundle*, then both definitions of quantisation coincide by Kodaira's vanishing theorem [58].

In more recent work [44] the quantisation-commutes-with-reduction conjecture has been studied for cocompact Hamiltonian group actions on pre-quantisable symplectic manifolds. The unreduced symplectic space is no longer assumed to be compact, so that the index of the Dirac operator is in general not finite. Instead, one assigns a class in $K_0(C^*(G))$ to the Dirac operator (where $K_0(C^*(G))$ is the $K_0$-group of the group $C^*$-algebra of $G$) by the so-called assembly map [4, 74], which extends the definition of the index to the non-compact case. (For compact groups the ring $K_0(C^*(G))$ may be identified with the representation ring of $G$ (*cf.* [5, Remark 30]).) In [44] it is proved that on this new definition for the quantisation map, under some additional assumptions, quantisation commutes with reduction for cocompact actions. These results were further generalised, using different methods, in [70], where a more general setting of cocompact actions are studied. Also, the approach of [44] fits nicely into the framework of [64] (see also [63]), where quantisation commutes with reduction is interpreted as a special case of the functoriality of a quantisation functor that maps into the category of $KK$.

More recently, in [68, 76] the quantisation-commutes-with-reduction principle has been studied for $G$-actions on arbitrary non-compact manifolds with proper momentum map. By the properness of the momentum map, the quantisation of $(M, \omega)$ can be interpreted as an element of the generalised representation ring of $G$. In [45] the 'quantisation commutes with reduction' problem is considered for so-called *tame actions*, under the assumption that the reduction at the 0-orbit of $\mathfrak{g}^*$ is compact. Their quantisation maps takes values in $K^0(C_r^*(G))$, the $K$-homology group of the reduced group $C^*$-algebra, so that the quantisation map is again interpreted in a $KK$-theoretical way.

In Part I we study the quantisation-commutes-with-reduction problem for the co-adjoint action of a compact, connected Lie groups on its own cotangent bundle. By the co-adjoint action we mean that $G$ acts on $T^*G$ by the inverse pull-back of the action of $G$ on itself by conjugation. The corresponding momentum map is not proper and neither is the reduction at 0 compact, so that the just mentioned methods cannot be applied. In fact, as we shall see, the multiplicity of the trivial representation in the equivariant quantisation of $T^*G$ is infinite. Moreover, as the co-adjoint action is non-free, the Marsden-Weinstein quotient (at $0 \in \mathfrak{g}$) is singular and is an example of a symplectic stratified space (see Theorem 1.2.6 or [82]). See [65] for work on the quantisation of singular Marsden-Weinstein quotients.

### 1.4.2 Dolbeault-Dirac quantisation

In this section we define our notion of Dolbeault-quantisation. Let $(M, \omega)$ be a symplectic manifold. Suppose that $(M, \omega)$ is endowed with a compatible almost complex structure. The class of such manifolds is the largest for which we can define Dolbeault-Dirac quantisation.

**Definition 1.4.1.** Let $(M, \omega)$ be a symplectic manifold and let $J$ be a compatible almost-complex structure. The ***Dolbeault-Dirac operator*** is defined as the symmetric first-order differential operator on $\Gamma_c^\infty(M, \Lambda^{(0,\bullet)}T^*M)$ given by

$$\sqrt{2}\left(\pi^{(0,\bullet)} \circ d + (\pi^{(0,\bullet)} \circ d)^*\right),$$

where the adjoint is taken with respect to the inner product

$$\langle s_1, s_2 \rangle = \int_M \langle s_1(x), s_2(x) \rangle \varepsilon, \quad (s_1, s_2 \in \Gamma_c^\infty(M, \Lambda^{(0,\bullet)}T^*M))$$

where $\varepsilon$ denotes the Liouville measure on $M$, and the hermitian structure on $\Lambda^{(0,\bullet)}T^*M$ is obtained by extending the Riemannian metric to a hermitian form on $T_{\mathbb{C}}M$ as in Definition 1.3.20. The map $\pi^{(0,\bullet)}$ denotes the projection $\Lambda^\bullet(T^*_{\mathbb{C}}M) \to \Lambda^{(0,\bullet)}T^*M = \Lambda^\bullet(T^{*(0,1)}M)$.

If $L$ is a hermitian line bundle with hermitian connection $\nabla^L$, then the ***twisted Dolbeault-Dirac operator*** $D^L$ is defined as the symmetric first-order differential operator on $\Gamma_c^\infty(M, \Lambda^{(0,\bullet)}T^*M \otimes L)$ given by

$$D^L = \sqrt{2}\left(\pi^{(0,\bullet)}(d \otimes 1 + (-1)^{\deg} \otimes \nabla^L) + (\pi^{(0,\bullet)}(d \otimes 1 + (-1)^{\deg} \otimes \nabla^L))^*\right),$$

where deg is the map that to each form assigns its degree.

**Remark 1.4.2.** Note that by Proposition 1.3.6 the Liouville measure is equal to the Riemannian measure, so that the (twisted) Dolbeault-Dirac operator is essentially self-adjoint if $M$ is geodesically complete.

If we speak of a Dolbeault-Dirac operator or any other differential operator on a manifold $M$, then, unless specified otherwise, **their domain is always taken to be** $\Gamma_c^\infty(M, E)$ where $E$ is the vector bundle on which the differential operator acts.

We mainly consider Kähler manifolds. In that case, the Dolbeault-Dirac operator is simply equal to

$$\sqrt{2}\left(\overline{\partial} + \overline{\partial}^*\right),$$

and similarly, the twisted Dolbeault-Dirac operator $D^L$ is equal to

$$D^L = \sqrt{2}\left(\overline{\partial}^L + (\overline{\partial}^L)^*\right),$$

where $\overline{\partial}^L$ is the first-order differential operator on $\Lambda^{(0,\bullet)}T^*M \otimes L$ given by

$$\overline{\partial}^L(\alpha \wedge s) := \overline{\partial}\alpha \otimes s + (-1)^{|\alpha|}\alpha \otimes (\nabla^L)^{(0,1)}s,$$

which is precisely the Dolbeault-operator $\overline{\partial}^L$ of Equation (1.6). The hermitian connection $\nabla^L$ determines a holomorphic structure on $L$ by declaring a local section $s \in \Gamma^\infty(U, L)$ to be holomorphic if and only if $\nabla^{(0,1)}s = 0$ (see [34, Proposition 6.30]). With respect to this holomorphic structure, the connection $\nabla^L$ is the Chern connection. The untwisted or ordinary Dolbeault-Dirac operator is a special case of a twisted Dolbeault-Dirac operator, where $L$ is the trivial hermitian holomorphic vector bundle.

The bundle $\Lambda^{(0,\bullet)}T^*M$ on $M$ decomposes into an even and an odd part as $\Lambda^{(0,even)}T^*M \oplus \Lambda^{(0,odd)}T^*M$. With respect to this decomposition the twisted Dolbeault-Dirac operator is an odd operator, and hence it is of the form

$$D^L = \begin{pmatrix} 0 & D^L_- \\ D^L_+ & 0 \end{pmatrix},$$

where $D_+$ maps $\Lambda^{(0,even)}T^*M$ to $\Lambda^{(0,odd)}T^*M$. If $M$ is compact, then Dolbeault-Dirac quantisation is defined as the *index* of $\overline{D}^L_+$ [71]. On non-compact manifolds the kernel of $\overline{D}^L_+$ and $\overline{D}^L_-$ can fail to be finite-dimensional so that an index cannot be defined. In this thesis we mainly consider cotangent bundles of compact connect Lie groups, which are non-compact manifolds. The corresponding twisted Dolbeault-Dirac operators have infinite-dimensional kernel, so that an index cannot be defined. In these cases, one needs to work with a different definition of Dolbeault-Dirac quantisation. For the manifolds $M$ of interest in Part I of this thesis, which are all Kähler manifolds, we use the following definition:

**Definition 1.4.3.** Let $L$ be a hermitian holomorphic line bundle with Chern connection $\nabla^L$ such that $(\nabla^L)^2 = 2\pi i\omega$, where $\omega$ denotes the symplectic structure on $M$, and let $D^L$ be the corresponding twisted Dolbeault-Dirac operator. Then ***Dolbeault-Dirac quantisation*** of the Kähler manifold $M$ is defined as the *Hilbert space*

$$\mathcal{Q}^L_{DD}(M) := \ker \overline{D}^L, \tag{1.10}$$

where the bar denotes the closure of the operator.

If furthermore, $M$ has a (fixed) spin structure with corresponding Dirac operator $\not{D}$, then the ***spin quantisation*** is defined analogously as the *Hilbert space*

$$\mathcal{Q}^L_S(M) := \ker(\overline{\not{D}}^L).$$

In all situations of interest, the Kähler manifold is geodesically complete, so that $D^L$ is essentially self-adjoint [17]. However, $\ker(\overline{D}^L)$ will be an infinite-dimensional Hilbert space, so that Equation (1.10) does not lead to an interpretation of quantisation as an index. Also, when we consider a group $G$ of symmetries acting appropriately on $M$, so that the quantisations carry natural $G$-actions, then the multiplicities of some of the irreducible representations will have infinite multiplicity, ruling out an interpretation of $\ker(\overline{D}^L)$ as an element of some generalised representation ring of $G$. In this thesis, the Dolbeault-Dirac quantisation is therefore considered as a *Hilbert space*, possibly with a unitary $G$-action.

We emphasise that Definition 1.4.3 is not supposed to replace the usual definition of Dolbeault-Dirac quantisation in the cases where the latter is defined. We only use Definition 1.4.3 for the examples in this thesis. For the manifolds of which we actually compute the Dolbeault-Dirac quantisation, the kernel of $D^L_-$ turns out to be trivial, so that in these cases Definition 1.4.3 is quite close to the usual definition of Dolbeault-Dirac quantisation. The main difference is then that we consider the Dolbeault-Dirac quantisation as a $G$-Hilbert space and not as an element in some $KK$-group or generalised representation ring.

# Chapter 2

# Quantisation of the cotangent bundle

The quantisation of cotangent bundles of compact connected Lie groups is interesting from both a physical and mathematical point of view. First of all, in physics, and more specifically, in lattice gauge theory, the cotangent bundle of such a Lie group appears as the phase space of a single plaquette, consisting of four vertices and four bonds [29]. Of course, the quantisation of $T^*G$ is expected to be $L^2(G)$, the Hilbert space of square-integrable functions on the configuration space $G$. From a mathematical viewpoint the quantisation of $T^*G$ is related to representation theory.

In this chapter, we endow $T^*G$ with a $G \times G$-invariant Kähler structure and explicitly construct the equivariant Dolbeault-Dirac and spin quantisations of $T^*G$ (see Definition 1.4.3). We show that for both quantisations the resulting Hilbert space is $G \times G$-equivariantly isomorphic to $L^2(G)$ in some natural way, provided that the canonical line bundle on $T^*G$ is semi-negative.

The quantisation of $T^*G$ has also been studied by Hall in [40], and the material in this chapter is strongly related to his work. The main difference is the definition of the quantisation: Hall uses a holomorphic polarisation, whereas we use a twisted Dolbeault-Dirac operator or a spin Dirac operator. If the canonical line bundle on $T^*G$ is semi-negative, then our definition of quantisation coincides with Hall's.

In the upcoming chapters we prove that quantisation commutes with reduction for the action of $G$ on $T^*G$ that is induced by the action of $G$ on itself by conjugation. This action is obtained from the $G \times G$-action on $T^*G$ by restricting the $G \times G$-action to the diagonal of $G \times G$.

**In the remainder of Part I the group $G$ is always assumed to be a compact connected Lie group** .

## 2.1 Kähler structure on the cotangent bundle of compact connected Lie groups

The cotangent bundle of $T^*G$ has a natural symplectic structure $\omega$ (*cf.* Example 1.2.2). In this section we compute the explicit form of $\omega$ and recall from [39] how $T^*G$ can be endowed with a (natural) Kähler structure by identifying $T^*G$ with the complexification of $G$.

Let $g \in G$ be given. We write $L_g$, $R_g$ for the left, respectively, right multiplication of $g$ on $G$, i.e..

$$L_h : G \to G, \quad g \mapsto hg, \quad R_h : G \to G, \quad g \mapsto gh.$$

The map $g \mapsto L_g$ defines a left-action of $G$ on itself and we also refer to this action as ***the left action*** of $G$ on itself. Similarly, the map $g \mapsto R_g$ defines a right action of $G$ on itself, in the sense that $R_{g_1 g_2} = R_{g_2} R_{g_1}$, and once again this right action is simply referred to as ***the right action*** of $G$ on itself. Since left and right multiplication on $G$ commute, we obtain a left action of $G \times G$ on $G$ by

$$(g_1, g_2) \mapsto L_{g_1} R_{g_2^{-1}}, \quad \text{or} \quad (g_1, g_2) \mapsto (h \mapsto g_1 h g_2^{-1}) \quad (h, g_1, g_2 \in G),$$

By push-forward the actions $L$ and $R$ on $G$ determine a left, respectively, right action of $G$ on the tangent bundle $TG$:

$$L_{g*} X_h = T_h L_g (X_h) \in T_{gh}, \quad R_{g*} X_h = T_h R_g (X_h) \in T_{hg} G, \quad (X_h \in T_h G).$$

Similarly, the actions $L$ and $R$ determine a $G \times G$-action on $T^*G$ by pull-back:

$$L^*_{g^{-1}} \alpha_h = (T_{gh} L_{g^{-1}})^* \alpha_h \in T_{gh} G, \quad R^*_g \alpha_h = (T_{hg^{-1}} R_g)^* \alpha_h \in T_{hg^{-1}} G,$$

where $\alpha_h \in T^*_h G$. These group actions turn $TG$ and $T^*G$ into $G \times G$-equivariant vector bundles. Moreover, by Example 1.2.4 the action of $G \times G$ on $T^*G$ is automatically symplectic and strongly Hamiltonian, when the moment map is defined as in Example 1.2.4.

In order to identify $T^*G$ with its complexification $G^{\mathbb{C}}$, we first construct a diffeomorphism $T^*G \to G \times \mathfrak{g}$. First, the cotangent bundle $T^*G$ is diffeomorphic to $G \times \mathfrak{g}^*$ via left-translation. More precisely, a covector $\alpha_g \in T^*G$ is sent to $G \times \mathfrak{g}^*$ through the map

$$\alpha_g \mapsto (g, (T_e L_g)^* \alpha_g) \in G \times T^*_e G = G \times \mathfrak{g}^*.$$

**Remark 2.1.1.** Note that

$$((T_e L_g)^* \alpha_g)(T_g L_{g^{-1}} X_g) = \alpha_g (T_e L_g \circ T_g L_{g^{-1}} X_g) = \alpha_g (X_g)$$

for all $\alpha_g \in T^*_g G$, $X_g \in T_g G$. So for each point $g \in G$, the pairing map $T_g G \times T^*_g G \to \mathbb{R}$ corresponds to the pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ if $T_g G$ and $T^*_g G$ are identified via left-translation with respectively $\mathfrak{g}$ and $\mathfrak{g}^*$.

Choose an Ad $G$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$. Using this inner product we can construct a natural linear isomorphism $\flat : \mathfrak{g} \to \mathfrak{g}^*$ by the relation

$$X^\flat(Y) = \langle X, Y\rangle_{\mathfrak{g}},$$

so that $T^*G$ is diffeomorphic to $G \times \mathfrak{g}$. In what follows we identify $T^*G$ with $G \times \mathfrak{g}$, as the latter space is much more convenient to work with. We also use left-trivialisation to identify the tangent spaces of $G \times \mathfrak{g}^*$ and $G \times \mathfrak{g}$ with $\mathfrak{g} \times \mathfrak{g}^*$ and $\mathfrak{g} \times \mathfrak{g}$, respectively. The next Lemma describes the canonical symplectic form on $G \times \mathfrak{g}$.

**Lemma 2.1.2.** *1. The symplectic structure on $G \times \mathfrak{g}^*$ coming from the pull-back of the canonical symplectic structure on $T^*G$ is equal to*

$$\omega_{(g,\nu)}((X_1,\xi_1),(X_2,\xi_2)) = \xi_1(X_2) - \xi_2(X_1) - \nu([X_1,X_2]),$$

*where $g \in G$, $X_{1,2} \in \mathfrak{g}$, $\nu, \xi_{1,2} \in \mathfrak{g}^*$. Furthermore, the fundamental 1-form $\theta$ is equal to*

$$\theta_{(g,\nu)}(X,\xi) = \nu(X),$$

*where $g \in G$, $X \in \mathfrak{g}$, $\nu, \xi \in \mathfrak{g}^*$.*

*2. The corresponding symplectic structure on $G \times \mathfrak{g}$ is equal to*

$$\omega_{(g,Y)}((X_1,Z_1),(X_2,Z_2)) = \langle X_2, Z_1\rangle_{\mathfrak{g}} - \langle X_1, Z_2\rangle_{\mathfrak{g}} - \langle Y, [X_1,X_2]\rangle_{\mathfrak{g}}, \quad (2.1)$$

*where $g \in G$, $Y, X_{1,2}, Z_{1,2} \in \mathfrak{g}$. Furthermore, the fundamental 1-form $\theta$ is equal to*

$$\theta_{(g,Y)}(X,Z) = \langle Y, X\rangle, \quad (2.2)$$

*where $g \in G$, $Y, X, Z \in \mathfrak{g}$.*

*Proof.* This follows from a straightforward calculation. Details can be found in e.g. [75] (note that their definition of the canonical symplectic structure on cotangent bundles differs from ours by a minus sign).

□

Note that Equation (2.1) depends on the inner product on $\mathfrak{g}$. This is, of course, not surprising as the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ is defined with respect to this inner product. The choice of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ will not matter, as long as it is chosen to be Ad $G$-invariant.

The left and right $G$-action on $G \times \mathfrak{g}$ are described in the following Lemma, whose proof we omit.

**Lemma 2.1.3.** *The left $G$-action $\Phi^L$ and the right $G$-action $\Phi^R$ on $G \times \mathfrak{g}^*$ are given by*

$$\Phi^L_h(g,\xi) = (hg,\xi), \quad \Phi^R_{h^{-1}}(g,\xi) = (gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\,\xi), \quad (g,h \in G, \xi \in \mathfrak{g}^*),$$

*respectively. Similarly, the left $G$- action $\Phi^L$ and the right $G$-action $\Phi^R$ on $G \times \mathfrak{g}$ are given by*

$$\Phi^L_h(g,Y) = (hg,Y), \quad \Phi^R_{h^{-1}}(g,Y) = (gh^{-1}, \mathrm{Ad}_h\,Y), \quad (g,h \in G, Y \in \mathfrak{g}),$$

*respectively.*

We now follow [39, 40] and put a Kähler structure on $G \times \mathfrak{g}$ (and hence on $T^*G$) by identifying $G \times \mathfrak{g}$ with the complexification of $G$, and by subsequently pulling back the complex structure on $G^{\mathbb{C}}$ to $G \times \mathfrak{g}$.

**Definition 2.1.4.** Let $G$ be a compact connected Lie group. A ***complexification of*** $G$ is a connected, complex Lie group satisfying the following properties:

1. The group $G^{\mathbb{C}}$ contains $G$ as a closed subgroup.
2. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is equal to $\mathfrak{g} + i\mathfrak{g}$.
3. Every homomorphism $G \to H$, where $H$ is a complex group, extends to a holomorphic homomorphism of $G^{\mathbb{C}} \to H$.

If $G$ is a compact connected group, then the complexification $G^{\mathbb{C}}$ always exists and is unique up to isomorphism (see [46, Chapter XVII.5]).

It follows from the first and second condition in Definition 2.1.4 that the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is equal to $\mathfrak{g}+i\mathfrak{g} = \mathfrak{g}\times\mathfrak{g}$, where the first component in $\mathfrak{g}\times\mathfrak{g}$ corresponds to the Lie algebra of $G$ (as a closed subgroup of $G^{\mathbb{C}}$), and the second corresponds to $i\mathfrak{g}$. Furthermore, multiplication in $G^{\mathbb{C}}$ is holomorphic. Therefore, under left-trivialisation of the tangent bundle of $G^{\mathbb{C}}$, the complex structure $J^{G^{\mathbb{C}}} : T_gG^{\mathbb{C}} \to T_gG^{\mathbb{C}}$, as a map from $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$ to itself, is simply

$$J^{\mathbb{C}}_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for any $g \in G^{\mathbb{C}}$. The map $\Phi : G \times \mathfrak{g} \to G^{\mathbb{C}}$ given by

$$\Phi : (g,Y) \mapsto g\exp(iY), \quad (g \in G, Y \in \mathfrak{g})$$

is a diffeomorphism (see [37, Proof of Lemma 12]). Since

$$\Phi(\Phi^L_h(g,Y)) = hg\exp(iY)$$

and

$$\Phi(\Phi^R_{h^{-1}}(g,Y)) = gh^{-1}\exp(i\mathrm{Ad}_hY) = gh^{-1}\mathrm{Ad}_h\left(\exp(iY)\right) = g\exp(iY)h^{-1},$$

and as multiplication in $G^{\mathbb{C}}$ is holomorphic, the induced complex structure on $G\times\mathfrak{g}$ is invariant under the action of $G \times G$ on $G \times \mathfrak{g}$ (which is given by $\Phi^L \times (\Phi^R)^{-1}$).

**Theorem 2.1.5** (Hall, [39, 40]). *The canonical symplectic structure $\omega$ and the complex structure $J$ on $T^*G$ induced by the diffeomorphism $\Phi : T^*G \to G^{\mathbb{C}}$ are compatible. That is, $(T^*G, \omega, J)$ is a Kähler manifold.*

We refer to this Kähler structure as the ***standard Kähler structure*** on $T^*G$. The standard Kähler structure on $T^*G$ is invariant under the action of $G \times G$ in the sense that $\omega$, $J$, and hence also $g$, are all invariant.

In order to simplify the calculations we work exclusively with $G \times \mathfrak{g}$ and $G^{\mathbb{C}}$. We repeatedly switch from $G \times \mathfrak{g}$ to $G^{\mathbb{C}}$, as the complex structure is easy to handle on $G^{\mathbb{C}}$, whereas $G \times \mathfrak{g}$ allows for more convenient coordinates. To be able to pass vector fields or forms from one space to the other, we need to compute the differential of $\Phi : G \times \mathfrak{g} \to G^{\mathbb{C}}$. This computation has been carried out by Hall in [39]. For completeness we repeat the precise form here.

First, the tangent spaces of $G \times \mathfrak{g}$ are identified with $\mathfrak{g} \times \mathfrak{g}$ using left-translation on $G$, and, similarly, the tangent spaces of $G^{\mathbb{C}}$ are identified with $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus i\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}$ using left-translation on $G^{\mathbb{C}}$. The differential of $\Phi$ at the point $(g, Y)$ is then nothing but a linear endomorphism of $\mathfrak{g} \times \mathfrak{g}$.

**Lemma 2.1.6** (Hall, [39]). *The differential $T_{(g,Y)}\Phi$ of $\Phi$ at the point $(g, Y)$ is equal to*

$$T_{(g,Y)}\Phi = \begin{pmatrix} \cos \operatorname{ad} Y & \frac{1-\cos \operatorname{ad} Y}{\operatorname{ad} Y} \\ -\sin \operatorname{ad} Y & \frac{\sin \operatorname{ad} Y}{\operatorname{ad} Y} \end{pmatrix}, \tag{2.3}$$

*as a real-linear transformation of $\mathfrak{g} \times \mathfrak{g}$.*

As a first application of this Lemma, let us show that the function $(g, Y) \mapsto |Y|^2$ is a ***Kähler potential*** for the symplectic form $\omega$, that is,

$$\omega = -i\partial\overline{\partial}|Y|^2.$$

We base our computation on [40]. Let $\{e_k\}$ be an orthonormal basis of $\mathfrak{g}$ for the Ad $G$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. All forms on $G \times \mathfrak{g}$ are $C^\infty(M)$-linear combinations of the ("horizontal") left-invariant forms $\{\alpha_k\}$, where $\alpha_k(e_G) = e_k^*$ in $\mathfrak{g}^*$, and the ("vertical") forms $\{dy_k\}$, where $(y_k)_k$ are the coordinates on $\mathfrak{g}$ with respect to the orthonormal basis $\{e_1, \dots e_n\}$ of $\mathfrak{g}$. Similarly, we choose left-invariant 1-forms $\{\eta_k\}$ on $G^{\mathbb{C}}$ such that $\eta_k(e_{G^{\mathbb{C}}}) = (e_k^*, 0)$ for all $k \in \{1, \dots n\}$. Note that our definition of $J$ on forms is such that $J\eta(e_{G^{\mathbb{C}}}) = (0, e_k^*)$. We now compute

$$\overline{\partial}|Y|^2 = \pi^{(0,1)} d|Y|^2 = \pi^{(0,1)} \left( \sum_{k=1}^{n} 2y_k dy_k \right).$$

So, with respect to the basis $\{\alpha_k, dy_k\}$, $d|Y|^2 = (0, 2Y)$. We can now apply Lemma 2.1.6 to transfer this form to $G^{\mathbb{C}}$. To do so, we actually need to apply

$(T_{(\Phi(x),\Phi(Y))}\Phi^{-1})^*$, where $*$ denotes transpose. One can check that

$$(T_{(\Phi(x),\Phi(Y))}\Phi)^{-1} = \left(\frac{\sin \mathrm{ad} Y}{\mathrm{ad} Y}\mathbb{1}\right)^{-1} \begin{pmatrix} \frac{\sin \mathrm{ad} Y}{\mathrm{ad} Y} & \frac{\cos \mathrm{ad} Y - 1}{\mathrm{ad} Y} \\ \sin \mathrm{ad} Y & \cos \mathrm{ad} Y \end{pmatrix}.$$

Since $\{e_k\}$ was chosen to be an orthonormal basis of the Ad-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, it follows that $(\mathrm{ad}\, Y)^* = -\mathrm{ad}\, Y$ in this basis. So, with respect to the bases $\{\alpha_k, dy_k\}$ and $\{\eta_k, J\eta_k\}$, we obtain

$$(T_{(g,Y)}\Phi)^* = \begin{pmatrix} \cos \mathrm{ad}\, Y & \sin \mathrm{ad}\, Y \\ \frac{\cos \mathrm{ad}\, Y - 1}{\mathrm{ad}\, Y} & \frac{\sin \mathrm{ad}\, Y}{\mathrm{ad}\, Y} \end{pmatrix},$$

$$(T_{(\Phi(x),\Phi(Y))}\Phi^{-1})^* = \left(\frac{\sin \mathrm{ad}\, Y}{\mathrm{ad}\, Y}\right)^{-1} \begin{pmatrix} \frac{\sin \mathrm{ad}\, Y}{\mathrm{ad}\, Y} & -\sin \mathrm{ad}\, Y \\ \frac{1-\cos \mathrm{ad}\, Y}{\mathrm{ad}\, Y} & \cos \mathrm{ad}\, Y \end{pmatrix}.$$

Because $\mathrm{ad}\, Y$ acts trivially on $Y$, the form $\overline{\partial}|Y|^2$ on $G^{\mathbb{C}}$ is equal to

$$\begin{aligned} \overline{\partial}|Y|^2 &= \pi^{(0,1)} \left(\sum_{k=1}^{n} 2y_k J\eta_k\right) = \pi^{(0,1)} \sum_{k=1}^{n} \left(-iy_k(\eta_k + iJ\eta_k) + iy_k(\eta_k - iJ\eta_k)\right) \\ &= \sum_{k=1}^{n} \left(iy_k\eta_k + y_k J\eta_k\right). \end{aligned}$$

Pulling back this form to $G \times \mathfrak{g}$ we obtain

$$\overline{\partial}|Y|^2 = \sum_{k=1}^{n} \left(iy_k\alpha_k + y_k dy_k\right),$$

and thus

$$-i\partial\overline{\partial}|Y|^2 = -id\overline{\partial}|Y|^2 = d\left(\sum_{k=1}^{n} y_k\alpha_k\right) = d\theta = \omega.$$

Here we have used that

$$\theta_{(g,Y)}(Z) = \langle Y, Z\rangle_{\mathfrak{g}} = \sum_{k=1}^{n} y_k z_k = \sum_{k=1}^{n} y_k \alpha_k(Z)$$

for any $Z = \sum_{k=1}^{n} z_k e_k$ in $T_{(g,Y)}(G \times \mathfrak{g})$ (see Equation (2.2)).

## 2.2 Geodesic completeness

In Section 2.1 we have put a Kähler structure $(\omega, J)$ on the cotangent bundle $T^*G$ for any compact connected Lie group $G$. When $G$ is non-abelian, the Riemannian metric

$$g(X, Y) = \omega(JX, Y) \tag{2.4}$$

corresponding to this Kähler structure is in general different from the product metric on $T^*G \cong G \times \mathfrak{g}$ (*cf.* Equations (2.1) and (2.3)). Later on, when we consider Dirac operators on $T^*G$, we need to know that $T^*G$ is geodesically complete so that the Dirac operators are essentially self-adjoint. Since the metric $g$ on $T^*G$ is not the ordinary product metric, it is not clear that $(T^*G, g)$ is geodesically complete, so we prove it here. We do this by showing that $(T^*G, g)$ has another property that, by the Hopf-Rinow theorem, is equivalent to the geodesic completeness of $(T^*G, g)$.

Let us now recall the Hopf-Rinow Theorem.

**Theorem 2.2.1** (Hopf-Rinow Theorem). *Let $(M, g)$ be a connected Riemannian manifold. The following statements are equivalent:*

1. *$(M, g)$ is geodesically complete.*
2. *$(M, g)$ is complete as a metric space.*
3. *The balls $B_r(p) = \{x \in M \mid \delta(p, x) \leq r\}$, are compact for all $p \in M$, $r \geq 0$.*
4. *There exists a function $f \in C^\infty(M, \mathbb{R})$ such that the sets*

$$M_c = \{x \in M \mid f(x) < c\}, \quad (c \in \mathbb{R}),$$

*are relatively compact, and $\|df\|_g \leq C$ for $C > 0$.*

*Proof.* The equivalence of the first three statements can be found in [53]. The equivalence of statements (3) and (4) can be found in [7]. □

In the following Theorem we use the notation introduced below Lemma 2.1.6.

**Theorem 2.2.2.** *Let $G$ be a compact connected Lie group and endow $T^*G$ with the standard Kähler structure. The Riemannian manifold $(T^*G, g)$, where $g$ is the Riemannian metric defined in Equation* (2.4)*, is geodesically complete.*

*Proof.* By Theorem 2.2.1 is suffices to show that there exists a function $f$ such that $\|df\|_g \leq C$ for some $C > 0$ and $M_c = \{(x, Y) \in G \times \mathfrak{g} \mid f(x, Y) < c\}$ is relatively compact for all $c \in \mathbb{R}$. Take the function $f : (x, Y) \mapsto \log(1 + |Y|^2)$, where $|Y|$ is the norm of $Y \in \mathfrak{g}$ with respect to the chosen Ad-invariant inner product on $\mathfrak{g}$. First of all, it is clear that

$$\log(1 + |Y|^2) < c \iff |Y|^2 < e^c - 1,$$

for all $c \in \mathbb{R}$. So, $M_c$ is empty if $c \leq 0$ and $M_c = \{(x, Y) \in G \times \mathfrak{g} \mid |Y|^2 < e^c - 1\} \neq \emptyset$ if $c > 0$. Hence, $M_c$ is relatively compact for all $c \in \mathbb{R}$.

We now show that $\|df\|_g \leq 2$. Recall that $\|df\|_g^2 = g((df)^\#, (df)^\#)$, where $(df)^\#$ is the unique vector field that satisfies $df(W) = g((df)^\#, W)$ for all vector fields $W$. Let $\alpha_k, dy_k$, $\eta_k$ and $J\eta_k$ be as in the paragraphs below Lemma 2.1.6.

Using $d(\psi \circ s)(m) = \psi'(s(m))ds(m)$ for any $s \in C^\infty(M, \mathbb{R})$ and $\psi : \mathbb{R} \to \mathbb{R}$, we obtain

$$df = d(\log(1 + |Y|^2)) = \frac{1}{1 + |Y|^2} d(1 + |Y|^2) = \frac{1}{1 + |Y|^2} \sum_{k=1}^{n} 2y_k dy_k.$$

We first calculate $Jdf$. To this end, note that $(1 + |Y|^2)df = \sum_{k=1}^n 2y_k dy_k$ corresponds (at the point $(x, Y)$ in $G \times \mathfrak{g}$) to $(0, 2Y)$ in the basis $\{\alpha_k, dy^k\}$. The complex structure $J$ on $T_{(g,Y)}(G \times \mathfrak{g})$ is

$$(T_{(x,Y)}\Phi)^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (T_{(\Phi(x),\Phi(Y))}\Phi^{-1})^*,$$

where $*$ denotes the transpose. Recall that with respect to the basis $\{\alpha_k, dy^k\}$ and $\{\eta_k, J\eta_k\}$ we have,

$$(T_{(x,Y)}\Phi)^* = \begin{pmatrix} \cos \operatorname{ad} Y & \sin \operatorname{ad} Y \\ \frac{\cos \operatorname{ad} Y - 1}{\operatorname{ad} Y} & \frac{\sin \operatorname{ad} Y}{\operatorname{ad} Y} \end{pmatrix},$$

and

$$(T_{(\Phi(x),\Phi(Y))}\Phi^{-1})^* = \left(\frac{\sin \operatorname{ad} Y}{\operatorname{ad} Y}\right)^{-1} \begin{pmatrix} \frac{\sin \operatorname{ad} Y}{\operatorname{ad} Y} & -\sin \operatorname{ad} Y \\ \frac{1 - \cos \operatorname{ad} Y}{\operatorname{ad} Y} & \cos \operatorname{ad} Y \end{pmatrix},$$

where $\operatorname{ad} Y$ is expressed in the orthonormal basis $\{e_k\}$ of $\mathfrak{g}$.

Now, since $\operatorname{ad} Y$ acts trivially on $Y$,

$$\begin{aligned}(T\Phi)^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (T\Phi^{-1})^*(0, 2Y) &= (T\Phi)^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (0, 2Y) \\ &= (T\Phi)^*(-2Y, 0) = (-2Y, 0).\end{aligned}$$

Let us write $Z := Z(x, Y) = (Z_1(x, Y), Z_2(x, Y)) =: (Z_1, Z_2)$ for a vector field on $G \times \mathfrak{g}$. On the one hand, we have

$$(1 + |Y|^2)(Jdf)(Z) = -2\langle Y, Z_1\rangle_{\mathfrak{g}}. \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}(1 + |Y|^2)g((Jdf)^\#, Z) &= (1 + |Y|^2)\omega(J(Jdf)^\#, Z) \\ &= -(1 + |Y|^2)\omega(df^\#, Z).\end{aligned} \tag{2.6}$$

Using that (see Equation (2.1))

$$\omega_{(x,Y)}((X_1, X_2), (Z_1, Z_2)) = \langle X_2, Z_1\rangle_{\mathfrak{g}} - \langle X_1, Z_2\rangle_{\mathfrak{g}} - \langle Y, [X_1, Z_1]\ \rangle_{\mathfrak{g}}$$

and equating Equations (2.5), (2.6), we obtain

$$2\langle Y, Z_1\rangle_{\mathfrak{g}} = (1+|Y|^2)\left(\langle (df)_2^{\#}, Z_1\rangle_{\mathfrak{g}} - \langle (df)_1^{\#}, Z_2\rangle_{\mathfrak{g}} - \langle Y, [(df)_1^{\#}, Z_1]\rangle_{\mathfrak{g}}\right).$$

This equality holds for all $Z$ if and only if $(df)^{\#}(x,Y) = \left(0, \frac{2Y}{1+|Y|^2}\right)$. An argument similar to the one we used to determine $Jdf$ shows that $(1+|Y|^2)J(df)^{\#} = (-2Y, 0)$. Hence,

$$\begin{aligned} g(df, df) &= g(df^{\#}, df^{\#}) = \omega(J(df)^{\#}, df^{\#}) = (1+|Y|^2)^{-2}\omega((-2Y,0),(0,2Y)) \\ &= (1+|Y|^2)^{-2}4|Y|^2 \leq 4. \end{aligned}$$

So, we have proved that $g(df, df)$ is bounded and $M_c$ is relatively compact for all $c \in \mathbb{R}$. By Theorem 2.2.1, the Riemannian manifold $(T^*G, g)$ is geodesically complete.

□

## 2.3 Kodaira's vanishing theorem

The main result of this section is to extend Kodaira's vanishing argument on compact Kähler manifolds (see e.g [6]) to non-compact, geodesically complete, Kähler manifolds. More specifically, we show that if $K^* \otimes L$ is a positive line bundle over $M$ (*cf.* Definition 2.3.7 below), where $K$ denotes the canonical line bundle and $L$ denotes a hermitian holomorphic line bundle, then the kernel of the closure $\overline{D}^L$ of the twisted Dolbeault-Dirac $D^L$ operator on $M$ is contained in the smooth $(0,0)$-forms, i.e. the smooth functions.

We say that an unbounded operator $T$ is ***positive*** if it is self-adjoint and $\langle x, Tx\rangle \geq 0$ for all $x \in \text{Dom } T$. For such a positive operator $T$ there exists a unique (self-adjoint) positive operator $T^{\frac{1}{2}}$ such that $(T^{\frac{1}{2}})^2 = T$ (see [79, Proposition 5.13]). The following Lemma says that the positivity condition on a self-adjoint operator needs only be checked on a dense subset of the domain (with respect to the graph norm). Its proof is standard.

**Lemma 2.3.1.** *Assume that $T : \mathcal{H} \to \mathcal{H}$ is an unbounded operator with dense domain Dom $T$ such that*

$$\langle x, Tx\rangle \geq 0, \quad \textit{for all } x \in \textit{Dom } T. \tag{2.7}$$

*Let $\overline{T}$ denote its closure. Then*

$$\langle x, \overline{T}x\rangle \geq 0,$$

*for all $x \in$ Dom $\overline{T}$.*

If $T$ is an essentially self-adjoint operator satisfying Equation (2.7), then $\overline{T} = T^*$ is positive by Lemma 2.3.1. For this reason, we say that $T$ is ***essentially positive*** if $T$ is essentially self-adjoint and satisfies Equation (2.7). We emphasise that the term 'positive operator' always means a positive, *self-adjoint* operator.

**Proposition 2.3.2.** *Let $S$, $T$ be essentially positive operators, both defined on the same dense domain $\mathcal{D} \subset \mathcal{H}$. Consider the densely defined operator $C := \overline{S+T}$. Then Dom $C \subset$ Dom $\overline{S}^{\frac{1}{2}} \cap$ Dom $\overline{T}^{\frac{1}{2}}$. Moreover, for all $x \in$ Dom $C$ we have*

$$\langle Cx, x\rangle = \langle \overline{S}^{\frac{1}{2}} x, \overline{S}^{\frac{1}{2}} x\rangle + \langle \overline{T}^{\frac{1}{2}} x, \overline{T}^{\frac{1}{2}} x\rangle. \tag{2.8}$$

*Proof.* Assume that $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{D}$ such that $x_n \to x$ and $Cx_n \to z =: Cx$ in $\mathcal{H}$. We show that $(\overline{S}^{\frac{1}{2}} x_n)_n$ is a Cauchy sequence in $\mathcal{H}$. First,

$$\langle Cy, y\rangle = \langle Sy, y\rangle + \langle Ty, y\rangle = \langle \overline{S}^{\frac{1}{2}} y, \overline{S}^{\frac{1}{2}} y\rangle + \langle \overline{T}^{\frac{1}{2}} y, \overline{T}^{\frac{1}{2}} y\rangle \geq \langle \overline{S}^{\frac{1}{2}} y, \overline{S}^{\frac{1}{2}} y\rangle,$$

for all $y \in \mathcal{D}$. In particular,

$$\begin{aligned}\langle \overline{S}^{\frac{1}{2}}(x_n - x_m), \overline{S}^{\frac{1}{2}}(x_n - x_m)\rangle &\leq \langle C(x_n - x_m), x_n - x_m\rangle \\ &\leq \|Cx_n - Cx_m\|\|x_n - x_m\|,\end{aligned}$$

and the right hand side goes to zero as $n, m$ go to infinity, since both $(x_n)_n$ and $(Cx_n)_n$ are Cauchy sequences. In particular, $(\overline{S}^{\frac{1}{2}} x_n)_n$ is a Cauchy sequence, and hence $x \in \text{Dom } \overline{S}^{\frac{1}{2}}$ with $\overline{S}^{\frac{1}{2}} x = \lim_n \overline{S}^{\frac{1}{2}} x_n$. Similarly, $x \in \text{Dom } \overline{T}^{\frac{1}{2}}$.

We already know that Equation (2.8) is valid on $\mathcal{D}$. Let $x \in \text{Dom } C$ be as above. By the previous paragraph $\overline{S}^{\frac{1}{2}} x_n \to \overline{S}^{\frac{1}{2}} x$ and $\overline{T}^{\frac{1}{2}} x_n \to \overline{T} x$. Thus,

$$\begin{aligned}\langle Cx, x\rangle = \lim_n \langle Cx_n, x_n\rangle &= \lim_n \langle \overline{S}^{\frac{1}{2}} x_n, \overline{S}^{\frac{1}{2}} x_n\rangle + \langle \overline{T}^{\frac{1}{2}} x_n, \overline{T}^{\frac{1}{2}} x_n\rangle \\ &= \langle \overline{S}^{\frac{1}{2}} x, \overline{S}^{\frac{1}{2}} x\rangle + \langle \overline{T}^{\frac{1}{2}} x, \overline{T}^{\frac{1}{2}} x\rangle,\end{aligned}$$

which proves that Equation (2.8) holds for all $x \in \text{Dom } C$. □

**Corollary 2.3.3.** *Let $S, T$ be essentially positive operators on the same dense domain $\mathcal{D} \subset \mathcal{H}$. Consider the densely defined operator $C = \overline{S+T}$. Then* $\ker C \subset \ker \overline{S} \cap \ker \overline{T}$.

*Proof.* If $x \in \ker C$, then $x \in \text{Dom } \overline{S}^{\frac{1}{2}} \cap \text{Dom } \overline{T}^{\frac{1}{2}}$ by Proposition 2.3.2 and

$$\langle Cx, x\rangle = \langle \overline{S}^{\frac{1}{2}} x, \overline{S}^{\frac{1}{2}} x\rangle + \langle \overline{T}^{\frac{1}{2}} x, \overline{T}^{\frac{1}{2}} x\rangle = 0.$$

Therefore, $x \in \ker \overline{S}^{\frac{1}{2}} \cap \overline{T}^{\frac{1}{2}} = \ker \overline{S} \cap \ker \overline{T}$ as $\ker \tilde{T} = \ker \tilde{T}^2$ for any self-adjoint operator $\tilde{T}$. □

The following Lemma is just a matter of linear algebra. For completeness we give the proof anyway. Let $V$ be a finite-dimensional inner product space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A linear operator $A$ on $V$ naturally extends to a derivation $\widetilde{A}$ of degree 0 on the tensor algebra $TV$ by defining it on the simple tensors as

$$\widetilde{A}(v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} v_1 \otimes \cdots \otimes Av_j \otimes \cdots \otimes v_k,$$
$$\widetilde{A}(1) = 0,$$

where $1 \in T^0V = \mathbb{K}$ denotes the identity element of $TV$.

A self-adjoint operator $A : V \to V$ is called ***positive-definite*** if $\langle v, Av\rangle > 0$ for all non-zero $v \in V$. This is equivalent to saying that $A$ is a positive, invertible operator on $V$.

**Lemma 2.3.4.** *If $A$ is a positive-definite (self-adjoint) linear operator on a finite-dimensional inner product space $(V, \langle,\rangle)$ and $\widetilde{A}$ is its extension to $TV$ as a derivation, then* $\ker \widetilde{A} = T^0V$. *Moreover, $\widetilde{A}$ is a positive-definite self-adjoint operator on $\oplus_{k\geq 1} T^kV$.*

*Proof.* It is clear that $T^0V$ is contained in the kernel of $A$. Write $v = \oplus_k v_k$ for an arbitrary element $v \in TV$, where $v_k \in T^kV$. Now, since $\widetilde{A}$ is of degree 0, the element $v \in TV$ is mapped to zero if and only if $\widetilde{A}v_k = 0$ for each $k \in \mathbb{N}$. It therefore suffices to show that for each $k \geq 1$ the kernel of $\widetilde{A}$ as a linear transformation $T^kV \to T^kV$ is trivial.

Positive-definiteness of $A$ as a self-adjoint linear transformation on $V$ is equivalent to the property that $(v, w) \mapsto \langle v, Aw\rangle$ defines an inner product on $V$. Consequently, for $1 \leq l \leq k$ the sesquilinear (or bilinear form if $\mathbb{K} = \mathbb{R}$) form

$$\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k\rangle_l^k := \langle v_1, w_1\rangle \cdots \langle v_{l-1}, w_{l-1}\rangle\langle v_l, Aw_l\rangle\langle v_{l+1}, w_{l+1}\rangle \cdots \langle v_k, w_k\rangle$$

defines an inner product on $T^kV$. Let $v = \sum_{\alpha=1}^m v_1^\alpha \otimes \cdots \otimes v_k^\alpha$, $w = \sum_{\beta=1}^p w_1^\beta \otimes \cdots \otimes w_k^\beta$ be given elements of $T^kV$. Then

$$\begin{aligned}
&\langle \sum_{\alpha=1}^m v_1^\alpha \otimes \cdots \otimes v_k^\alpha, \widetilde{A}\left(\sum_{\beta=1}^p w_1^\beta \otimes \cdots \otimes w_k^\beta\right)\rangle \\
&= \sum_{l=1}^k \sum_{\alpha,\beta} \langle v_1^\alpha, w_1^\beta\rangle \cdots \langle v_{l-1}^\alpha, w_{l-1}^\beta\rangle\langle v_l^\alpha, Aw_l^\beta\rangle\langle v_{l+1}^\alpha, w_{l+1}^\beta\rangle \cdots \langle v_k^\alpha, w_k^\beta\rangle \\
&= \sum_{l=1}^k \sum_{\alpha,\beta} \langle v_1^\alpha \otimes \cdots \otimes v_k^\alpha, w_1^\beta \otimes \cdots \otimes w_k^\beta\rangle_l^k = \sum_{l=1}^k \langle v, w\rangle_l^k.
\end{aligned}$$

Since $\widetilde{A}$ is also self-adjoint on $T^kV$, the form $\langle\cdot, \widetilde{A}(\cdot)\rangle$ defines an inner product on $T^kV$ for each $k \geq 1$ as well. In particular, for each $k \geq 1$: $\langle v, \widetilde{A}v\rangle > 0$ for all non-zero $v \in T^kV$, and hence $\ker \widetilde{A} \cap T^kV = \{0\}$. The Lemma now follows from the discussion in the first paragraph of the proof. □

**Remark 2.3.5.** If $V, W$ are two finite-dimensional inner product spaces and $A : V \to V$ is a positive-definite linear transformation, then, as in the proof of Lemma 2.3.4, the form on $T^kV \otimes L$ ($k \geq 1$) defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \langle v_1, \widetilde{A}v_2\rangle\langle w_1, w_2\rangle, \quad v_1, v_2 \in T^kV, w_1, w_2 \in W,$$

is an inner product on $T^k \otimes W$. So, $\widetilde{A} \otimes 1$ is a positive-definite operator on $T^kV \otimes W$ and so $\ker(\widetilde{A} \otimes 1) \cap (T^kV \otimes W) = \{0\}$ for all $k \geq 1$.

We now turn to (possibly unbounded) zeroth-order differential operators. The next proposition shows that any symmetric zeroth-order differential operator is essentially self-adjoint.

**Proposition 2.3.6.** *Let $E \to M$ be a hermitian vector bundle over an arbitrary oriented Riemannian manifold $M$. Consider the Hilbert space $L^2(M, E)$, where the measure on $M$ is the Riemannian measure. If $R$ is a smooth vector bundle homomorphism such that $R_x \in End(E_x)$ is symmetric for each $x \in M$, then $R$ is essentially self-adjoint on the domain $\Gamma_c^\infty(M, E)$. The closure $\overline{R}$ is positive if $R_x$ acts fibrewise by positive operators.*

*Proof.* Since $R_x$ is symmetric for each $x \in M$, $R$ is a symmetric zeroth-order differential operator on $\Gamma_c^\infty(M, E)$. We prove that $\overline{R} = R^*$ as an operator on $L^2(M, E)$.

The domain of $R^*$ is equal to

$$\text{Dom } R^* = \{s \in L^2(M, E) \mid Rs \in L^2(M, E)\}, \quad \text{and} \quad R^*s = Rs,$$

where $R$ acts on $s$ as a zeroth-order differential operator. Indeed, since $R$ is a zeroth-order differential operator, the equality

$$\langle s, Rt\rangle = \int_M \langle s(x), R(x)t(x)\rangle_x dx = \int_M \langle R(x)s(x), t(x)\rangle_x dx,$$

holds for all $s \in L^2(M, E)$, $t \in \text{Dom } R$. In particular, $s \in \text{Dom } R^*$ if and only if $x \mapsto R_x s_x \in L^2(M, E)$ and $R^*s = Rs$.

The domain of $R^{**}$ is defined as

$$\text{Dom } R^{**} = \{s \in L^2(M, E) \mid \exists C > 0 : \langle s, R^*t\rangle \leq C\|t\|_{L^2} \text{ for all } t \in \text{Dom } R^*\}.$$

If $s \in \text{Dom } R^*$, then

$$\langle s, R^*t\rangle = \langle s, Rt\rangle = \langle Rs, t\rangle \leq \|Rs\|_{L^2}\|t\|_{L^2},$$

for all $t \in \text{Dom } R^*$. Here, $R$ is again viewed as a zeroth-order differential operator. Consequently, $s \in \text{Dom } R^{**}$, and so $R^* \subset R^{**} = \overline{R}$. Because $R$ is symmetric, $\overline{R} \subset R^*$, and hence $\overline{R} = R^*$. In other words, $R$ is essentially self-adjoint.

□

To formulate Kodaira's vanishing theorem, we need the notion of a positive line bundle (*cf.* [6]).

**Definition 2.3.7.** A hermitian holomorphic line bundle $L$ over $M$ is said to be ***(semi-)positive*** if its curvature $R^L$ is of the form

$$R^L = \sum_{k,l} R_{kl} dz^k \wedge d\overline{z}^l, \tag{2.9}$$

where $(R_{kl})$, which is always hermitian, is a positive (semi-)definite matrix at each point. Similarly, a hermitian holomorphic line bundle $L$ over $M$ is said to be ***(semi-)negative*** if $L^*$, which has curvature $-R^L$, is (semi-)positive.

**Remark 2.3.8.** Note that positive (semi-)definiteness of the matrix $(R_{kl})$ at a point $x$ is equivalent to saying that the sesquilinear form $(v_1, v_2) \mapsto \overline{R_x(v_1, \overline{v}_2)}$ defines a positive (semi-)definite hermitian form on $T^{(1,0)}M$. Hence, equivalently, $L$ is a (semi-)positive line bundle if $R^L(x)$ defines a positive (semi-)definite hermitian form on $T_x^{(1,0)}M$ for each point $x \in M$. This condition on $R^L$ can be checked point-wise and moreover, it can be checked with respect to an arbitrary frame of $T_x^{(1,0)}M$. Indeed, let $\{W^p\}$ be another frame for $T_x^{*(1,0)}M$ and let $\{\overline{W^p}\}$ be the corresponding frame for $T_x^{*(0,1)}M$. The bases $\{dz^k\}$ and $\{W^p\}$ are related through a complex-linear invertible map $A$: $dz^k = \sum_p A_{pk}W^p$, and so $d\overline{z}^k = \sum_p \overline{A_{pk}} \cdot \overline{W^p}$. By inserting these expressions in Equation (2.9), we obtain

$$\begin{aligned}\sum_{k,l,p,q} R_{kl}A_{pk}\overline{A_{ql}}W^p \wedge \overline{W^q} &= \sum_{k,l,p,q} A_{pk}R_{kl}(A^*)_{lq}W^p \wedge \overline{W^q} \\ &= \sum_{p,q}(ARA^*)_{pq}W^p \wedge \overline{W^q} =: \sum_{p,q} \widetilde{R}_{pq}W^p \wedge \overline{W^q}.\end{aligned}$$

Thus, $R_{kl}$ is positive (semi-)definite if and only if $\widetilde{R}_{pq}$ is positive (semi-)definite. That is, positive (semi-)definiteness can be checked with respect to arbitrary frames. In particular, Definition 2.3.7 makes sense.

In view of the above Remark we therefore also say that $R^L$ is ***(semi-)positive at*** $x$ if $R^L(x)$ defines a positive (semi-)definite hermitian form on $T_x^{(1,0)}M$, and we denote this by $R^L(x) > 0$ ($R^L(x) \geq 0$). If $R^L(x)$ is positive (semi-)definite at each point $x$, then $R^L$ is simply said to be ***(semi-)positive*** and this is denoted by $R > 0$ ($R \geq 0$). Similar definitions are introduced for the (semi-)negative case.

The curvature $R^L$ is always closed. This is a consequence of the fact that $R^L$ if locally of the form

$$R^L = \overline{\partial}\theta = \overline{\partial}(h^{-1}\partial h) = \overline{\partial}\partial \log h,$$

where $h = h(s,s)$ and $s$ a local nowhere vanishing section. Now use

$$d\overline{\partial}\partial = (\partial + \overline{\partial})\overline{\partial}\partial = -\partial^2\overline{\partial} + \overline{\partial}^2\partial = 0.$$

Moreover, the 2-form $\omega = \frac{1}{2\pi i}R^L$ is then locally of the form

$$\omega = -\frac{i}{2\pi}\sum_{k,l} R_{kl}dz^k \wedge d\overline{z}^l,$$

with $R_{kl}$ positive definite. Therefore, if $L$ is positive, then $\omega = \frac{1}{2\pi i}R^L$ is a symplectic form and $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ is a Riemannian metric. Indeed, by Lemma 1.3.16 the symmetric 2-tensor $g$ is locally of the form

$$g = \frac{1}{2\pi}\sum_{k,l} R_{kl}dz^k \vee d\overline{z}^l,$$

with $(R_{kl})$ positive-definite at each point. Hence $g$ is a complex-linear extension of a Riemannian metric on $TM$. Conversely, if $\omega = \frac{1}{2\pi i}R^L$ is a symplectic structure such that $\omega(J\cdot,\cdot)$ defines a Riemannian metric, then $L$ is positive. This relates our definition of a positive definite line bundle, which is taken from [6], to the definition in [27].

Having explicitly proved the necessary prerequisites, we now state a Kodaira's vanishing theorem for geodesically complete Kähler manifolds. The corresponding result for the compact case can be found in [6, Proposition 3.72] and [27, Proposition 6.1]. The theorem is a vanishing theorem because it states that the kernel of $\overline{D}^L$ contains no forms of non-zero degree. On compact manifolds this is equivalent to the vanishing of the higher cohomology groups of $L$ (see Remark 2.3.10 below).

**Theorem 2.3.9** (Kodaira's vanishing theorem)**.** *Let $M$ be a geodesically complete Kähler manifold. If $K^* \otimes L$ is a positive line bundle, then*

$$\ker \overline{D}^L \subset \Gamma^\infty(M, L),$$

*where $D^L$ denotes the twisted Dolbeault-Dirac operator. Or in other words,* $\ker \overline{D}^L$ *is concentrated in degree* 0*.*

*Proof.* Since $D^L$ is an elliptic operator on $M$, the kernel of $\overline{D}^L$ is contained in the smooth sections of $\Lambda^{(0,\bullet)}T^*M \otimes L$ (*cf.* [42, Proposition 10.4.8]). We show that all the sections in the kernel are necessarily $(0,0)$-forms. To do so, we first recall the ***Bochner-Kodaira formula*** (see for instance [6, Proposition 3.71]) which says that on a Kähler manifold the square of $D^L$ on $\Gamma_c^\infty(M, \Lambda^{(0,\bullet)}T^*M \otimes L)$ is equal to

$$\frac{1}{2}(D^L)^2 = \Delta^{(0,\bullet)} + \sum_{k,l} e(\overline{\xi}_k) i(\overline{Z}_j) R^{K^*\otimes L}(Z_j, \overline{Z}_k),$$

where $\{\overline{\xi}_k\}$ is any unitary frame of $T^{*(0,1)}M$ with dual frame $\{\overline{Z}_k\}$ in $T^{(0,1)}M$. Denote the second (zeroth-order differential) operator by $R$. As $M$ is geodesically complete, the operators $\Delta^{(0,\bullet)}$ and $(D^L)^2$ are essentially self-adjoint on the domain $\Gamma_c^\infty(M, \Lambda^{(0,\bullet)}T^*M \otimes L)$. By Proposition 2.3.6 the operator $R$ is essentially self-adjoint, too. Since

$$\int_M \langle \Delta^{(0,\bullet)} s, s\rangle dx = \int_M \langle \nabla^{(0,1)} s, \nabla^{(0,1)} s\rangle dx, \quad s \in \Omega_c^{(0,\bullet)}(M, L),$$

the closure of $\Delta^{(0,\bullet)}$ is a positive self-adjoint operator by Lemma 2.3.1.

Note that $R$ acts trivially on $L$, so that we can simply regard $R$ as a morphism of the bundle $\Lambda^{(0,\bullet)}T^*M$. Since $K^* \otimes L$ is positive, the bundle endomorphism

$$R = \sum_{k,l} e(\overline{\xi}_k) i(\overline{Z}_j) R^{K^*\otimes L}(Z_j, \overline{Z}_k)$$

acts by invertible, positive complex-linear operators on the fibres of $T^{*(0,1)}M$. Indeed, with respect to the unitary frame $\{\overline{\xi}_k\}$, the matrix of $R$ on $T^{*(0,1)}M$ is precisely $R(Z_j, \overline{Z}_k)$ which is positive definite by assumption. If $\omega_1, \omega_2$ are forms of degree $(0, |\omega_1|)$ and $(0, |\omega_2|)$, respectively, the action of $R$ on $\omega_1 \wedge \omega_2$ is

$$\begin{aligned} R(\omega_1 \wedge \omega_2) &= \sum_{j,k} R^{K^*\otimes L}(Z_j, \overline{Z}_k) e(\overline{\xi}_k) \Big( i(\overline{Z}_j)\omega_1 \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge i(\overline{Z}_j)\omega_2 \Big) \\ &= \sum_{j,k} R^{K^*\otimes L}(Z_j, \overline{Z}_k) \big( e(\overline{\xi}_k) i(\overline{Z}_j)\omega_1 \wedge \omega_2 + \omega_1 \wedge e(\overline{\xi}_k) i(\overline{Z}_j)\omega_2 \big) \\ &= R\omega_1 \wedge \omega_2 + \omega_1 \wedge R\omega_2. \end{aligned}$$

This means that $R$ acts (fibrewise) as a derivation of degree 0. We can now apply Lemma 2.3.4: let $A$ be the restriction of $R_x$ to $V = T_x^{*(0,1)}M$. Then, with the notation of Lemma 2.3.4, $R_x$ is the restriction of $\widetilde{A}$ on $T(T_x^{*(0,1)}M)$ to $\Lambda^\bullet(T_x^{*(0,1)}M)$. Hence $R$ acts as a positive operator on the fibres of $\Lambda^\bullet(T^{*(0,1)}M) \otimes L$ (*cf.* Remark 2.3.5). By Proposition 2.3.6 the operator $R$ is essentially positive.

The operators $(\overline{D}^L)^2$ and $\overline{(D^L)^2}$ are both self-adjoint extensions of the essentially self-adjoint operator $(D^L)^2$, hence they are equal. Thus, $s \in \ker \overline{D}^L$ if and only if $s \in \ker \overline{(D^L)^2}$. Since both $\Delta^{(0,\bullet)}$ and $R$ are essentially positive on $\Gamma_c^\infty(M, \Lambda^\bullet(T^{*(0,1)}M) \otimes L)$, an element $s \in \ker \overline{D}^L \subset \Gamma^\infty(M, \Lambda^\bullet(T^{*(0,1)}M) \otimes L)$ is also in $\ker R$ by Corollary 2.3.3, i.e. $Rs = 0$. As $s$ is smooth, so is $Rs$. Thus, $Rs = 0$ if and only if $R_x s(x) = 0$ for all $x \in M$. Another application of Lemma 2.3.4 and Remark 2.3.5 shows that $s(x) \in (\Lambda^0 T^{*(0,1)}M \otimes L)_x$ for each $x$. Consequently, $s \in \Gamma^\infty(M, L)$. □

**Remark 2.3.10.** On compact manifolds the index of the Dolbeault-Dirac operator is related to the cohomology of $L$ as follows. First, we note that

$$\ker \overline{D}^L = \ker \overline{\partial}^L \cap \ker(\overline{\partial}^L)^*.$$

Here, we keep writing $\overline{\partial}^L$ for the closure of $\overline{\partial}^L$. Consequently, if $\overline{D}_k^L$, $\overline{\partial}_k^L$ denote the restrictions of $\overline{D}^L$ and $\overline{\partial}^L$ to $L^2(M, \Lambda^k(T^{*(0,1)}M) \otimes L)$, then

$$\begin{aligned}\ker \overline{D}_k^L &= \ker \overline{\partial}_k^L \cap \ker(\overline{\partial}_k^L)^* = \ker \overline{\partial}_k^L \cap (\mathrm{im}\overline{\partial}_{k-1}^L)^\perp \\ &\cong \ker \overline{\partial}_k^L / \overline{\mathrm{im}\,\overline{\partial}_{k-1}^L} = \ker \overline{\partial}_k^L / \operatorname{im} \overline{\partial}_{k-1}^L.\end{aligned}$$

In the final step we have used the fact that on compact manifolds the image of $\overline{\partial}_{k-1}^L$ is closed. This follows from the fact that $\overline{\partial}^L$ is elliptic, so that it is a Fredholm operator on compact manifolds. Hence, the kernel of $\overline{D}^L$ contains no forms of non-zero degree if and only if the higher cohomology groups of $L$ vanish.

On non-compact manifolds, the (twisted) Dolbeault-operator $\overline{\partial}^L$ fails in general to be Fredholm and the above argument does not apply.

## 2.4 Quantisation of Lie group cotangent bundles with semi-negative canonical line bundle

In this section we study both the Dolbeault-Dirac quantisation and the spin quantisation of $T^*G$ for the following class of compact connected Lie groups.

**Definition 2.4.1.** The class $\mathcal{C}_K$ consists of all compact connected Lie groups for which the canonical line bundle on $G^{\mathbb{C}}$ is semi-negative (*cf.* Definition 2.3.7).

The following Proposition shows that $\mathcal{C}_K$ is not empty. Because the proof is quite lengthy, we postpone it to Section 2.A.

**Proposition 2.4.2.** $SU(2) \in \mathcal{C}_K$.

**Remark 2.4.3.** We do not know how big the class $\mathcal{C}_K$ is and we conjecture that the class $\mathcal{C}_K$ is even equal to the class of all compact connected Lie groups.

In this section we prove that, if $G \in \mathcal{C}_K$, then the Dolbeault-Dirac and spin quantisation of $T^*G$ are $G\times G$-equivariantly and unitarily isomorphic to $L^2(G, dg)$, where $dg$ denotes the Haar measure on $G$. The proof consists of two steps. First, we use Kodaira's vanishing theorem Theorem 2.3.9 to identify the Dolbeault-Dirac and the spin quantisation of $T^*G$ ($G \in \mathcal{C}_K$) with (different) spaces of square-integrable holomorphic functions. These spaces coincide with the quantisations of $T^*G$ in [40]. However, in *loc. cit.* these spaces are obtained by different methods: Hall uses geometric quantisation with a holomorphic polarisation and we use the kernel of a twisted Dolbeault-Dirac operator. Second, a result from the same paper [40] and from [37] shows that both of these quantisations of $T^*G$ are equivariantly and unitarily isomorphic with $L^2(G, dg)$.

For the moment, let $G$ be an arbitrary compact connected Lie group. Define a smooth function $\phi : G \times \mathfrak{g} \to \mathbb{R}$ by

$$\phi(g, Y) = \pi|Y|^2.$$

Recall that the pre-quantum line bundle $L$ on $T^*G$ is the trivial hermitian line bundle endowed with the connection $\nabla^L = d+2\pi i\theta$, where $\theta$ denotes the symplectic potential. We take the $G\times G$-action on $L$ to be trivial. Since $\theta$ is a $G\times G$-invariant 1-form, the connection $\nabla^L$ is $G \times G$-invariant, too.

**Proposition 2.4.4.** *Let $\alpha \in \Gamma^\infty(T^*G, \Lambda^\bullet(T^{*(0,1)}M) \otimes L)$ be a smooth section. Then $\overline{\partial}^L\alpha = 0$ if and only if $\overline{\partial}(e^\phi\alpha) = 0$, with $\phi$ as above.*

*Proof.* We show that $\overline{\partial}\phi = 2\pi i(\pi^{(0,1)}\theta)$. This would imply that

$$\overline{\partial}(e^\phi\alpha) = e^\phi \left(\overline{\partial}\alpha + (\overline{\partial}\phi) \wedge \alpha\right) = e^\phi \left(\overline{\partial} + 2\pi i\pi^{(0,1)}\theta\right) \alpha = e^\phi\overline{\partial}^L\alpha,$$

so that $\overline{\partial}^L\alpha = 0$ if and only if $\overline{\partial}(e^\phi\alpha) = 0$.

With respect to the basis $\{\alpha_k, dy_k\}$ (see Theorem 2.2.2 or the remarks below Lemma 2.1.6) the form $d\phi$ is equal to $2\pi y_k dy^k$, which we simply write as $(0, 2\pi Y)$. Since $\operatorname{ad} Y$ acts trivially on $Y$, we compute, in the same way as we did below Lemma 2.1.6,

$$\overline{\partial}\phi = (0, \pi Y) + (i\pi Y, 0) = (i\pi Y, \pi Y).$$

On the other hand, $\theta_{(g,Y)}(Z_1, Z_2) = \langle Y, Z_1\rangle_{\mathfrak{g}} = \sum_{k=1}^n y_k z_k$, when $Z_1 = \sum_{k=1}^n z_k e_k$. Consequently, $\theta_{(g,Y)} = (Y, 0)$ in the basis $\{\alpha_k, dy^k\}$. Again, the same argument as before gives

$$\pi^{(0,1)}(\theta_{g,Y}) = \frac{1}{2}\left((\theta_{g,Y}) - iJ(\theta_{g,Y})\right) = \frac{1}{2}\left((Y, 0) - (0, iY)\right) = \frac{1}{2}(Y, -iY).$$

Multiplying by $2\pi i$ gives $2\pi i\pi^{(0,1)}\theta = (i\pi Y, \pi Y)$, which is equal to $\overline{\partial}\phi$. □

If the bundle $L$ is endowed with the unique holomorphic structure for which the section $e^{-\phi}$ is holomorphic, then $\nabla^L = d + 2\pi i\theta$ is the unique Chern connection on $L$. Since $(\nabla^L)^2 = 2\pi i\omega$, the line bundle $L$ is positive.

Since the twisted Dolbeault-Dirac operator $D^L$ is elliptic, the kernel of $\overline{D}^L = (D^L)^*$ consists of smooth sections. So, to determine the kernel of $(D^L)^*$ it is important to know how $(D^L)^*$ acts on smooth sections. The following Lemma shows that for any symmetric differential operator $D$, the action of $D^*$ on smooth sections coincides with the action of $D$ as a differential operator.

**Lemma 2.4.5.** *Let $D$ be a symmetric differential operator on a hermitian vector bundle $E$ over a Riemannian manifold $M$ (with domain $\Gamma_c^\infty(M, E)$). If $s \in \Gamma^\infty(M, E) \cap$ Dom $D^*$, then*

$$D^* s = Ds. \tag{2.10}$$

*Proof.* Suppose that $s \in \Gamma^\infty(M, E) \cap \text{Dom } D^*$. Let $t \in \Gamma_c^\infty(M, E)$ be given and let $\psi \in C_c^\infty(M)$ be such that $0 \le \psi \le 1$ and $\psi \equiv 1$ in a neighbourhood $U$ of supp $t$. Then $\langle s, Dt\rangle = \langle \psi s, Dt\rangle$, since $D$ is a local operator. The section $\psi s$ is compactly supported and smooth, so that by symmetry of $D$

$$\langle s, Dt\rangle = \langle \psi s, Dt\rangle = \langle D(\psi s), t\rangle = \int_M \langle D(\psi s)(x), t(x)\rangle_x dx,$$

where $D$ acts on $\psi s$ as a differential operator. Now, $\langle D(\psi s)(x), t(x)\rangle_x$ is zero outside supp $t$ and the differential operator $D$ commutes with $\psi$ on the neighbourhood $U$, since $\psi \equiv 1$ there. Consequently,

$$\langle D(\psi s)(x), t(x)\rangle_x = \langle \psi(x) Ds(x), t(x)\rangle_x$$

for all $x \in M$, and so

$$\begin{aligned}\langle s, Dt\rangle &= \int_M \langle D(\psi s)(x), t(x)\rangle_x dx = \int_M \langle \psi(x)(Ds)(x), t(x)\rangle_x dx = \langle \psi Ds, t\rangle \\ &= \langle Ds, t\rangle,\end{aligned}$$

where $D$ acts as a differential operator on $s$. Since this equality holds for all $t \in \Gamma_c^\infty(M, E)$, this proves Equation (2.10). □

We are now ready to determine the Dolbeault-Dirac quantisation of $T^*G$ for Lie groups $G$ in the class $\mathcal{C}_K$ of Definition 2.4.1.

**Theorem 2.4.6.** *Let the canonical line bundle $K$ on $T^*G$ be semi-negative. Then*

$$\ker \overline{D}^L \cong \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon),$$

*where the prefix $\mathcal{H}$ indicates that only holomorphic square-integrable functions are considered. Moreover, the $G \times G$-action on $\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon)$ is the usual one.*

*Proof.* If $K$ is semi-negative, then $K^*$ is semi-positive. Since the total curvature of a tensor product of two line bundles is the sum of the curvatures of the individual line bundles, we see that $K^* \otimes L$ is positive. According to Theorem 2.3.9 the kernel of $\overline{D}^L$ is contained in $\Gamma^\infty(T^*G, L)$. Since $T^*G$ is geodesically complete, we have $\overline{D}^L = (D^L)^*$. Therefore, by Lemma 2.4.5,

$$\ker \overline{D}^L = \ker(D^L)^* = \{s \in \Gamma^\infty(T^*G, L) \cap L^2(T^*G, L) \mid D^L s = 0\}.$$

Now, $D^L s = 0$ for a smooth section $s$ of $L$, if and only if $\overline{\partial}^L s = 0$, or equivalently, $s$ is a holomorphic section of $L$. The holomorphic sections of $L$ are of the form $s = fe^{-\phi}$, with $\phi = \pi|Y|^2$ and $f$ a holomorphic function on $T^*G$. We then obtain that

$$\begin{aligned} \ker \overline{D}^L &= \left\{ fe^{-\phi} \;\middle|\; f \text{ holomorphic and } \int_M |f|^2 e^{-2\pi|Y|^2}\varepsilon < \infty \right\} \\ &\cong \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon). \end{aligned}$$

Since $e^{-\phi}$ is $G \times G$-invariant, the last isomorphism intertwines the $G \times G$-actions on both spaces. □

We now turn our attention to the spin quantisation of $T^*G$. It is shown in e.g. [27, Chapter 6] that for a fixed spin structure on any Kähler manifold $M$, the spinor bundle $S$ is isomorphic to the bundle $\Lambda^\bullet(T^{*(0,1)}M) \otimes K_{\frac{1}{2}}$, where $K_{\frac{1}{2}}$ is the line bundle of half-forms corresponding to the chosen spin structure. It satisfies $K_{\frac{1}{2}} \otimes K_{\frac{1}{2}} = K$ and so $R(K_{\frac{1}{2}}) = \frac{1}{2}R(K)$. Moreover, the spin Dirac operator on $S$ coincides with the twisted Dolbeault-Dirac operator on $\Lambda^\bullet(T^{*(0,1)}M) \otimes K_{\frac{1}{2}}$.

The canonical line bundle on $T^*G$, with $G$ a compact connected Lie group, is trivial. Let $\{\beta_i\}_{i=1}^n$ be a linearly independent system of left-invariant holomorphic $(1,0)$-forms on $G^{\mathbb{C}}$. Then $\beta := \beta_1 \wedge \cdots \wedge \beta_n$ is a left $G^{\mathbb{C}}$-invariant holomorphic trivialising section of $K$. The section $\beta$ is also invariant under the right action of $G$ on $G^{\mathbb{C}}$. To see this, note that

$$(T_h R_g)^* \beta(hg) = \det(\mathrm{Ad}^*_{g^{-1}})\beta(h), \quad (h \in G^{\mathbb{C}}, g \in G),$$

where $\mathrm{Ad}^*_{g^{-1}}$ is viewed as a real-linear map $\mathfrak{g}^* \to \mathfrak{g}^*$. The function

$$G \to \mathbb{R}^\times, \quad g \mapsto \det(\mathrm{Ad}^*_{g^{-1}})$$

is a group homomorphism. Since $G$ is compact and connected, the image of this map is a compact and connected subgroup of $\mathbb{R}^\times$, hence the image is $\{1\}$. Consequently, $\beta$ is also invariant under the right action of $G$ on $G^{\mathbb{C}}$. Since $\beta$ is both left- and right- invariant, it is invariant under the action of $G \times G$ on $G^{\mathbb{C}}$.

By Lemma 1.3.23 the inner product $h(\beta, \beta)$ satisfies

$$(-1)^{n(n-1)/2} i^n h(\beta, \beta)\varepsilon = \overline{\beta} \wedge \beta. \tag{2.11}$$

Since $\overline{\beta} \wedge \beta$ is a left-invariant $(n, n)$-form on $G^{\mathbb{C}}$, the measure $h(\beta, \beta)\varepsilon$ is proportional to the Haar measure on $G^{\mathbb{C}}$. By [39, Lemma 5], we obtain

$$h(\beta, \beta) = c^2 \eta^2, \tag{2.12}$$

where $c > 0$ is a constant and $\eta : G^{\mathbb{C}} \to \mathbb{R}$ depends only on $Y \in \mathfrak{g}$ (if $G^{\mathbb{C}}$ is identified with $G \times \mathfrak{g}$) and is given by the unique Ad $G$-invariant extension of

$$\eta(Y) = \prod_{\alpha \in R^+} \frac{\sinh(\alpha(Y))}{\alpha(Y)}, \quad (Y \in \mathfrak{t}), \tag{2.13}$$

where $\mathfrak{t}$ is some maximal abelian subalgebra of $\mathfrak{g}$, and $R^+$ a set of positive, real roots. We renormalise $\beta$ such that $c = 1$.

Now choose $K_{\frac{1}{2}}$ to be the trivial holomorphic line bundle with trivialising holomorphic section $\alpha$ that satisfies $\alpha^2 := \alpha \otimes \alpha = \beta$ and with hermitian structure determined by

$$h(\alpha, \alpha) = h(\beta, \beta)^{\frac{1}{2}} = \eta,$$

so that $R^{K_{\frac{1}{2}}} = \frac{1}{2} R^K$. The group action on $K_{\frac{1}{2}}$ is such that $\alpha$ is $G \times G$-invariant.

**Proposition 2.4.7.** *If the canonical line bundle $K$ on $T^*G$ is semi-negative, then*

$$\ker \overline{\not{D}}^L \cong \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2} \eta\varepsilon),$$

*where the prefix $\mathcal{H}$ indicates that only holomorphic square-integrable functions are considered. Moreover, this isomorphism intertwines the pertinent $G \times G$-actions on* $\ker \overline{\not{D}}^L$ *and* $\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon)$.

*Proof.* The twisted operator $\not{D}^L$ is equal to the twisted Dolbeault-Dirac operator on $\Lambda^\bullet(T^{*(0,1)}M) \otimes (K_{\frac{1}{2}} \otimes L)$ (see e.g [27, 30]). If the canonical line bundle $K$ is semi-negative, then $K^* \otimes (K_{\frac{1}{2}} \otimes L)$ is a positive line bundle so that by Theorem 2.3.9 the kernel of $\overline{\not{D}}^L$ is contained in $\Gamma^\infty(M, K_{\frac{1}{2}} \otimes L)$, and as in Theorem 2.4.6 this kernel is equal to

$$\ker \overline{\not{D}}^L = \mathcal{H}L^2(T^*G, K_{\frac{1}{2}} \otimes L) \cong \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon),$$

where the last isomorphism is given by the map

$$\mathcal{H}L^2(T^*G, K_{\frac{1}{2}} \otimes L) \xrightarrow{\cong} \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon), \quad f e^{-\pi|Y|^2}\alpha \mapsto f. \tag{2.14}$$

This is an isomorphism, because the trivialising section $\alpha$ of $K_{\frac{1}{2}}$ satisfies

$$h(\alpha, \alpha) = \eta.$$

Because $\alpha$ and $e^{-\pi|Y|^2}$ are invariant under $G \times G$, the isomorphism of Equation (2.14) is $G \times G$-equivariant. □

**Remark 2.4.8.** The spaces $\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon)$ and $\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon)$ were first obtained as quantisations of $T^*G$ by Hall (see e.g. [40]) for any compact connected Lie group $G$. The first one through geometric quantisation using a holomorphic polarisation, the second one by adding a half-form correction to this. For $G \in \mathcal{C}_K$, we found these same Hilbert spaces as kernels of Dirac operators.

Because we have found the same Hilbert spaces as in [40], we can now proceed as in *loc. cit.* to show that both the Dolbeault-Dirac and the above spin quantisation of $T^*G$ are $G \times G$-equivariantly isomorphic to $L^2(G, dg)$, where $dg$ denotes the Haar measure, in a natural way. On $L^2(G, dg)$ we consider the $G \times G$-action

$$((h_1, h_2) \cdot f)(g) = f(h_1^{-1} g h_2), \quad g, h_1, h_2 \in G, f \in L^2(G, dg).$$

**Proposition 2.4.9** ([37, Theorem 10] and [40, Theorem 2.6])**.** *Let $G$ be a compact connected Lie group. There exists a unitary isomorphism*

$$\Pi^* : \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon) \to L^2(G, dg),$$

*intertwining the pertinent $G \times G$-actions.*

*Similarly, there exists a unitary isomorphism*

$$\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon) \to L^2(G, dg),$$

*intertwining the pertinent $G \times G$-action.*

*Proof.* The construction of the isomorphism $\Pi^*$ is made explicit in [40, Theorem 2.6]; $\Pi^*$ is just a constant times the inverse Segal-Bargmann transform for $G$ ([38]). One can deduce that $\Pi^*$ is $G \times G$-equivariant by its explicit form as given in [40, Theorem 2.6, Equation (2)].

For the second isomorphism we apply [37, Theorem 10] where unitary isomorphisms $L^2(G, dg) \to \mathcal{H}L^2(G^{\mathbb{C}}, \nu)$ are constructed for a specific class of $G$-bi-invariant measures $\nu$ on $G^{\mathbb{C}}$. To verify that this theorem applies to the $G \times G$-invariant measure $\nu := e^{-2\pi|Y|^2/h}\varepsilon$, we need to check:

1. $\nu$ is given by a density with respect to the Haar measure $d\mu$ on $G^{\mathbb{C}}$ that is locally bounded away from zero, and

2. for each irreducible representation $\pi$ of $G$, analytically continued to $G^{\mathbb{C}}$, the expression

$$\int_{G^{\mathbb{C}}} \|\pi(t)^{-1}\|^2 d\nu_t$$

is finite.

By [39, Lemma 5], $d\mu = \eta^2\varepsilon$, so that $\nu = e^{-2\pi|Y|^2}\varepsilon = e^{-2\pi|Y|^2}/\eta^2 d\mu$. The function $e^{-2\pi|Y|^2}/\eta^2$ is a smooth strictly positive function, in particular it is locally bounded away from zero. Furthermore, since any irreducible representation of $G$ is finite-dimensional and hence can be assumed to be unitary, we have

$$\begin{aligned}\int_{G^{\mathbb{C}}} \|\pi(t)^{-1}\|^2 e^{-2\pi|Y|^2}\varepsilon &= \int_{G\times\mathfrak{g}} \|\pi(ge^{iY})^{-1}\|^2 e^{-2\pi|Y|^2} dgdY \\ &= \int_{G\times\mathfrak{g}} \|\pi(g^{-1})\pi(e^{iY})^{-1}\|^2 e^{-2\pi|Y|^2} dgdY \\ &= |G| \int_{\mathfrak{g}} \|\pi(e^{iY})^{-1}\|^2 e^{-2\pi|Y|^2} dY,\end{aligned}$$

where we used that on $G \times \mathfrak{g}$ the Liouville measure is equal to $dgdY$, when $dg$, the Haar measure on $G$, and $dY$, the Lebesgue measure on $\mathfrak{g}$, are appropriately normalised (see [39, Lemma 4]). Writing $\overline{\pi}$ for the induced representation of the Lie algebra, we obtain

$$\begin{aligned}\int_{G^{\mathbb{C}}} \|\pi(t)^{-1}\|^2 e^{-2\pi|Y|^2}\varepsilon &= |G| \int_{\mathfrak{g}} \|e^{-i\overline{\pi}(Y)}\|^2 e^{-2\pi|Y|^2} dY \\ &\leq |G| \int_{\mathfrak{g}} e^{2\|\overline{\pi}(Y)\|} e^{-2\pi|Y|^2} dY \\ &\leq |G| \int_{\mathfrak{g}} e^{C|Y|} e^{-2\pi|Y|^2} dY,\end{aligned}$$

for some constant $C > 0$. This last integral is finite, as can be seen by selecting an orthonormal basis for the inner product on $\mathfrak{g}$. Thus, the measure $\nu$ satisfies the Conditions (1) and (2), so we can apply [37, Theorem 10]. If we now apply this with $U \equiv 1$, then the induced unitary map $L^2(G, dg) \to \mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon)$ is

$$(C_\phi f)(t) = \int_G f(x)\phi(x^{-1}t)dg, \quad (f \in L^2(G, dg)).$$

Here, the entire function $\phi : G^{\mathbb{C}} \to \mathbb{C}$ is given by

$$\phi(t) = \sum_{\pi \in \hat{G}} \frac{\dim V_\pi}{\sqrt{\sigma(\pi)}} \operatorname{Tr}(\pi(t^{-1})),$$

where the sum is over all irreducible representations of $G$, each of which is extended holomorphically to an irreducible representation of $G^{\mathbb{C}}$, and where

$$\sigma(\pi) = \frac{1}{\dim V_\pi} \int_{G^{\mathbb{C}}} \|\pi(t^{-1})\|^2 e^{-2\pi|Y|^2} \varepsilon.$$

From these expressions one can deduce that $C_\phi$ is a $G \times G$-invariant map.

□

Combining Theorem 2.4.6 and Propositions 2.4.7 and 2.4.9 we obtain:

**Theorem 2.4.10.** *Let $G$ be a compact connected Lie group and endow $T^*G$ with its standard Kähler structure. Let $(L, \nabla^L)$ be the above pre-quantisation. If the canonical line bundle $K$ on $T^*G$ is semi-negative, then the Dolbeault-Dirac quantisation of $T^*G$ is equal to*

$$\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\varepsilon),$$

*which is unitarily and $G \times G$-equivariantly isomorphic to $L^2(G, dg)$ via the isomorphism of Proposition 2.4.9.*

*Similarly, if the canonical line bundle $K$ on $T^*G$ is semi-negative, then the spin quantisation of $T^*G$ is equal to*

$$\mathcal{H}L^2(T^*G, e^{-2\pi|Y|^2}\eta\varepsilon),$$

*which is unitarily and $G \times G$-equivariantly isomorphic to $L^2(G, dg)$ via the isomorphism of Proposition 2.4.9.*

## 2.A Example: $SU(2)$

In this chapter we determined the Dolbeault-Dirac and spin quantisation of $T^*G$ for $G \in \mathcal{C}_K$. We now show that the statements are not empty, by showing that $SU(2)$ is an element of $\mathcal{C}_K$.

In general, if $s$ is a nowhere-vanishing local holomorphic section of a hermitian holomorphic line bundle $L$ with hermitian structure $h$, then by Proposition 1.3.13, the curvature of $L$ is equal to

$$R^L = \overline{\partial}\theta = \overline{\partial}\left(h^{-1}\partial h\right) = \overline{\partial}\partial \log h = -\partial\overline{\partial} \log h,$$

where $\theta$ denotes the connection $(1,0)$-form with respect to the local holomorphic frame $s$, and $h = h(s,s)$. The canonical line bundle $K$ on $G^{\mathbb{C}}$ is trivial, and $\beta = \beta_1 \wedge \cdots \wedge \beta_n$, where $\{\beta_i\}$ is a linearly independent set of left-invariant holomorphic $(1,0)$-forms, is a trivialising holomorphic section. The positive function $h(\beta, \beta)$ is equal to $c\eta^2$ by Equation (2.12), where $c$ is a positive constant and where $\eta$ is defined by Equation (2.13). Consequently (see the paragraph below Remark 2.3.8 for the definition of a semi-negative curvature form), we obtain

$$G \text{ is in } \mathcal{C}_K \text{ if and only if } \overline{\partial}\partial \log \eta \text{ is semi-negative on } T^*G. \tag{2.15}$$

Real-valued, smooth functions $f$ on a Kähler manifold $M$ for which $\overline{\partial}\partial f$ is semi-negative are called ***plurisubharmonic***. A function $f : M \to \mathbb{R}$ is plurisubharmonic if and only if on each local holomorphic chart the matrix

$$\Delta^z_{kl} := \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}$$

is positive semi-definite, as is clear from the local formula

$$\overline{\partial}\partial f = -\sum \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} dz^i \wedge d\overline{z}^j.$$

The matrix $\Delta^z_{kl}$ is known as the ***complex Hessian*** of $f$ with respect to the holomorphic coordinates $(z_1, \ldots, z_n)$.

**Remark 2.A.1.** As for the ordinary (or real Hessian) of a real-valued function $f$, the complex Hessian is positive semi-definite at those points where $f$ attains a local minimum. In fact, the complex Hessian is positive semi-definite if the ordinary Hessian is. To see this, assume that the ordinary Hessian is positive semi-definite. Then there exist real $n \times n$-matrices $\alpha$, $\beta$, $\gamma$ such that

$$\begin{pmatrix} \frac{\partial^2 \phi}{\partial x_i \partial x_j} & \frac{\partial^2 \phi}{\partial x_i \partial y_j} \\ \frac{\partial^2 \phi}{\partial y_i \partial x_j} & \frac{\partial^2 \phi}{\partial y_i \partial y_j} \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}^*,$$

where $\alpha$, $\beta$ can be assumed to be symmetric. Expanding the entries of the complex Hessian into ordinary derivatives, we obtain

$$4\frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} = \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{\partial^2 \phi}{\partial y_i \partial y_j} \right) + i \left( \frac{\partial^2 \phi}{\partial x_i \partial y_j} - \frac{\partial^2 \phi}{\partial y_i \partial x_j} \right),$$

so that

$$\begin{aligned} 4\left( \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} \right) &= \alpha^2 + \gamma\gamma^* + \gamma^*\gamma + \beta^2 + i\left(\alpha\gamma + \gamma\beta - \gamma^*\alpha - \beta\gamma^*\right) \\ &= (\alpha - i\gamma^*)(\alpha + i\gamma) + (\beta + i\gamma)(\beta - i\gamma^*), \end{aligned}$$

which is equivalent to the complex Hessian being positive semi-definite.

It is a direct consequence of Remark 2.A.1 that $\overline{\partial}\partial f$ is semi-negative at those points, where $f$ attains a (non-strict) local minimum.

By Equation (2.15) we need to prove the following proposition.

**Proposition 2.A.2.** *The form $\overline{\partial}\partial \log \eta$ is semi-negative on $T^*SU(2)$, with $\eta$ as given in Equation* (2.13).

**Remark 2.A.3.** There is another way to see that the curvature of the canonical line bundle on $G^{\mathbb{C}}$ is equal to $2\overline{\partial}\partial \log \eta$. In general, for any Kähler manifold $M$, it is well known that in holomorphic coordinates the curvature of the canonical line bundle is given by [3]

$$R^K = \partial\overline{\partial} \log \det(g_{kl}), \tag{2.16}$$

where $(g_{kl})$ is the matrix determined by

$$g = \sum_{k,l} g_{kl} dz^k \vee d\overline{z}^l.$$

Actually, the holomorphic frame $\{dz^k\}$ of $T^{*(1,0)}M$ can be replaced by any other holomorphic frame $\{W^k\}$ when computing $R^K$ using Equation (2.16). If $dz^k = A_{pk}W^p$ for some holomorphic matrix $A = (A_{pk})$, then

$$g = \sum_{k,l} g_{kl} dz^k \vee d\overline{z}^l = \sum_{k,l} (AgA^*)_{pq} W^p \vee \overline{W}^q.$$

Consequently,

$$\begin{aligned}
\partial\overline{\partial} \log(\det(AgA^*)_{pq}) &= \partial\overline{\partial} \log |\det(A)|^2 \det(g_{kl}) \\
&= \partial\overline{\partial} \log |\det(A)|^2 + \partial\overline{\partial} \log \det(g_{kl}) \\
&= \partial\overline{\partial} \log \det(g_{kl}) = R^K,
\end{aligned}$$

where we have used that $\partial\overline{\partial} \log |f|^2 = 0$ for any nowhere-vanishing, local holomorphic function $f$.

Let $\{W^k\}$ be linearly independent left-invariant holomorphic $(1,0)$-forms on $G^{\mathbb{C}}$, and write

$$\omega = -i \sum_{k,l} g_{kl} W^k \wedge \overline{W}^l.$$

The Liouville measure is defined as

$$\begin{aligned}
\varepsilon = \frac{(-1)^n}{n!}\omega^n &= \frac{i^n}{n!}\left(\sum_{k,l} g_{kl} W^k \wedge \overline{W}^l\right)^n \\
&= \frac{i^n}{n!}\left(n! \sum_{\sigma\in S_n} (-1)^\sigma g_{1\sigma(i_1)}\cdots g_{n\sigma(i_n)} W^1 \wedge \overline{W}^1 \wedge \cdots \wedge W^n \wedge \overline{W}^n\right) \\
&= i^n(-1)^{n(n+1)/2} \det(g_{kl}) \overline{W}^1 \wedge \cdots \wedge \overline{W}^n \wedge W^1 \wedge \cdots \wedge W^n.
\end{aligned}$$

Recall that $\eta$ satisfies (*cf.* Equations (2.11) and (2.12))

$$bc^2\eta^2\varepsilon = \overline{W}^1 \wedge \cdots \wedge \overline{W}^n \wedge W^1 \wedge \cdots \wedge W^n,$$

where $b = i^n(-1)^{n(n-1)/2}$ and $c > 0$ a constant, so that

$$\det(g_{kl}) = \frac{1}{i^n(-1)^{n(n+1)/2} bc^2\eta^2} = \frac{1}{i^{2n}(-1)^{n(n+1)/2}(-1)^{n(n-1)/2}c^2\eta^2} = \frac{1}{c^2\eta^2}.$$

By Equation (2.16),

$$R^K = \partial\overline{\partial}\log\det(g_{kl}) = \partial\overline{\partial}\log\frac{1}{c^2\eta^2} = 2\overline{\partial}\partial\log\eta.$$

### 2.A.1 Proof of Proposition 2.A.2

We now give the proof of Proposition 2.A.2. This is achieved by many explicit calculations, which are not too difficult to handle in the case when the group $G$ is equal to $SU(2)$. The proof is quite lengthy and probably much more complicated than necessary.

Recall that our goal is to show that $\overline{\partial}\partial\log\eta$ is semi-negative, where $\eta(g, Y) = \eta(Y)$ is the unique Ad $G$-invariant function on $\mathfrak{g}$ that is equal to

$$\eta(Y) = \prod_{\alpha\in R^+} \frac{\sinh(\alpha(Y))}{\alpha(Y)},$$

in some maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, where $R^+$ is a set of real, positive roots.

Suppose, from now on, that $G = SU(2)$. Then there is only one positive root. Consequently, $\alpha(Y) = \pm c|Y|$ on $\mathfrak{t}$ for some fixed non-zero constant $c$. However, since the power series for

$$\widetilde{\eta} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{\sinh x}{x}$$

contains only even powers of $x$, the restriction of $\eta$ to $\mathfrak{t}$ is equal to

$$\eta(Y) = \frac{\sinh(c|Y|)}{c|Y|}, \tag{2.17}$$

where $c$ can be assumed to be positive. Because both $\eta$ and $|\cdot|$ are Ad $G$-invariant, Equation (2.17) is valid on all of $G \times \mathfrak{g}$.

**Lemma 2.A.4.** *When $G = SU(2)$, the function $Y \mapsto |Y|$ is plurisubharmonic on $G \times \mathfrak{g} \setminus (G \times \{0\})$.*

**Remark 2.A.5.** Note that $|\cdot|$ is an Ad-invariant norm on $\mathfrak{g}$, and hence it is uniquely determined by its restriction to a maximal torus of $\mathfrak{g}$. Since a maximal torus in $\mathfrak{g}$ is 1-dimensional, all Ad-invariant norms on $\mathfrak{g}$ differ only by a positive constant. We can therefore assume that the norm $|\cdot|$ on $Y$ is induced by the Ad-invariant inner product

$$\langle Y_1, Y_2 \rangle = -\frac{1}{2} \operatorname{Tr}(\operatorname{ad} Y_1 \operatorname{ad} Y_2), \tag{2.18}$$

which is, up to the factor $-\frac{1}{2}$, equal to the Killing form on $\mathfrak{g}$. The matrices

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

form a basis for $\mathfrak{g}$. They satisfy the relations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{2.19}$$

The matrix of $\operatorname{ad} Y$ with respect to the basis $\{e_1, e_2, e_3\}$ is therefore given by

$$\begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix},$$

so that the basis $\{e_1, e_2, e_3\}$ is orthonormal with respect to the inner product (2.18).

For the proof of Lemma 2.A.4 we need to know the action of $\cos \operatorname{ad}(y_1 e_1)$ and $\sin \operatorname{ad}(y_1 e_1)$, $(y_1 \in \mathbb{R})$ on $\mathfrak{g}$. We express these maps as matrices with respect to the basis $\{e_1, e_2, e_3\}$. The linear map $(\operatorname{ad} y_1 e_1)$ is represented by the matrix

$$\operatorname{ad} y_1 e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_1 \\ 0 & y_1 & 0 \end{pmatrix},$$

and its square is given by

$$(\operatorname{ad} y_1 e_1)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -y_1^2 & 0 \\ 0 & 0 & -y_1^2 \end{pmatrix}.$$

Consequently,

$$\begin{aligned}\cos(\operatorname{ad} y_1 e_1) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_1 \\ 0 & y_1 & 0 \end{pmatrix}^{2k} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_1^2 & 0 \\ 0 & 0 & y_1^2 \end{pmatrix}^{k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh y_1 & 0 \\ 0 & 0 & \cosh y_1 \end{pmatrix}, \end{aligned} \tag{2.20}$$

and

$$\sin(\operatorname{ad} y_1 e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sinh y_1 \\ 0 & \sinh y_1 & 0 \end{pmatrix}. \tag{2.21}$$

Also,

$$\frac{\sin(\operatorname{ad} y_1 e_1)}{\operatorname{ad} y_1 e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sinh y_1}{y_1} & 0 \\ 0 & 0 & \frac{\sinh y_1}{y_1} \end{pmatrix},$$

and

$$\frac{1 - \cos(\operatorname{ad} y_1 e_1)}{\operatorname{ad}(y_1 e_1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1-\cosh y_1}{y_1} \\ 0 & -\frac{1-\cosh y_1}{y_1} & 0 \end{pmatrix}.$$

As before, for each $i = 1, 2, 3$, $X_i$ denotes the left-invariant vector field on $G$ satisfying $X_i(e_G) = e_i$, and $\alpha_i$ is the left-invariant 1-form on $G$ satisfying $\alpha_i(X_j) = \delta_{ij}$ for all $j$. Then, by the Koszul-formula,

$$\begin{aligned} d\alpha_1(X_1, X_2) &= X_1(\alpha_1(X_2)) - X_2(\alpha_1(X_1)) - \alpha_1([X_1, X_2]) = -\alpha_1([X_1, X_2]) \\ &= -\alpha_1(X_3) = 0, \end{aligned}$$

and similarly,

$$d\alpha_1(X_1, X_3) = 0, \quad d\alpha_1(X_2, X_3) = -\alpha_1([X_2, X_3]) = -\alpha_1(X_1) = -1.$$

Therefore, $d\alpha_1 = -\alpha_2 \wedge \alpha_3$. Because the commutation relations in Equation (2.19) are cyclic, we conclude that

$$d\alpha_1 = -\alpha_2 \wedge \alpha_3, \quad d\alpha_2 = -\alpha_3 \wedge \alpha_1, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2.$$

Finally, under left-trivialisation the tangent spaces of $G^{\mathbb{C}}$ can be identified with the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus J\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}$ with its natural complex structure. As the tangent spaces of $G \times \mathfrak{g}$ can also be identified with $\mathfrak{g} \oplus \mathfrak{g}$ under left-trivialisation of $G$, the derivative $T_{(g,Y)}\Phi$ at a point $(x, Y) \in G \times \mathfrak{g}$ is a real-linear map from $\mathfrak{g} \times \mathfrak{g}$ onto itself. It is given by

$$T\Phi_{(g,Y)} = \begin{pmatrix} \cos \operatorname{ad} Y & \frac{1-\cos \operatorname{ad} Y}{\operatorname{ad} Y} \\ -\sin \operatorname{ad} Y & \frac{\sin \operatorname{ad} Y}{\operatorname{ad} Y} \end{pmatrix}.$$

Denote by $\eta_k$ the left-invariant 1-forms on $G^{\mathbb{C}}$ for which $\eta_k(e_{G^{\mathbb{C}}}) = (e_k^*, 0) \in \mathfrak{g}^* \times \mathfrak{g}^* = T_{e_{G^{\mathbb{C}}}} G^{\mathbb{C}}$, $(k = 1, 2, 3)$. As before, $J\eta_k(e_{G^{\mathbb{C}}}) = (0, e_k^*)$.

*Proof of Lemma 2.A.4.* We determine the matrix $F_{ij}$ of $\overline{\partial}\partial|Y|$ with respect to the frame $\{\eta_k + iJ\eta_k\}$ of $T^{*(1,0)}M$. First of all, we rewrite $\overline{\partial}\partial|Y|$ on $G \times \mathfrak{g} \setminus (G \times \{0\})$ as

$$\begin{aligned}
\overline{\partial}\partial|Y| = \overline{\partial}(\pi^{(1,0)} d|Y|) &= \sum_k \overline{\partial}(\pi^{(1,0)} \frac{1}{|Y|} y_k dy_k) \\
&= \overline{\partial}\left(\frac{1}{2|Y|}\partial|Y|^2\right) \\
&= \frac{1}{2|Y|}\overline{\partial}\partial|Y|^2 - \frac{1}{2|Y|^3}\sum_k(\pi^{(0,1)} y_k dy_k) \wedge \partial|Y|^2 \\
&= \frac{1}{2|Y|}\overline{\partial}\partial|Y|^2 - \frac{1}{2|Y|^3}\left(\frac{1}{2}\overline{\partial}|Y|^2\right) \wedge \partial|Y|^2 \\
&= \frac{1}{2|Y|}\overline{\partial}\partial|Y|^2 + \frac{1}{4|Y|^3}\partial|Y|^2 \wedge \overline{\partial}|Y|^2.
\end{aligned}$$

That is,

$$4|Y|^3\overline{\partial}\partial|Y| = 2|Y|^2\overline{\partial}\partial|Y|^2 + \partial|Y|^2 \wedge \overline{\partial}|Y|^2.$$

In other words, since $4|Y|^3$ is positive on $G \times \mathfrak{g} \setminus (G \times \{0\})$, the matrix of $\overline{\partial}\partial|Y|$ is negative semi-definite at a point if and only if the matrix of $2|Y|^2\overline{\partial}\partial|Y|^2 + \partial|Y|^2 \wedge \overline{\partial}|Y|^2$ is.

Since the function $(x, Y) \to |Y|$ and the complex structure $J$ are both $G \times G$-invariant, so is $4|Y|^3\overline{\partial}\partial|Y|$. It therefore suffices to show that $4|Y|^3\overline{\partial}\partial|Y|$ is negative semi-definite on $\{e\} \times \mathfrak{t}$, where $\mathfrak{t}$ is a maximal torus in $\mathfrak{g}$. We choose the maximal torus $\mathfrak{t}$ of $\mathfrak{g}$ to be the span of $e_1$. We finish the proof by showing that $2|Y|^2\overline{\partial}\partial|Y|^2 + \partial|Y|^2 \wedge \overline{\partial}|Y|^2$ is negative semi-definite on $\{e\} \times (\mathfrak{t} \setminus \{0\})$.

We first determine how $\overline{\partial}\partial|Y|^2$ can be expressed with respect to the basis of 1-forms $\{\eta_k, J\eta_k\}_{k=1}^3$ at a point $g\exp(iy_1e_1)$ in $G^{\mathbb{C}}$. To do so, we will frequently move between $G \times \mathfrak{g}$ and $G^{\mathbb{C}}$. Recall that the map $(T_{(g\exp(iY))}\Phi^{-1})^*$ is given by the matrix

$$(T\Phi^{-1})^*_{(g,Y)} := (T_{g\exp(iY)}\Phi^{-1})^* = \frac{\operatorname{ad}Y}{\sin\operatorname{ad}Y}\begin{pmatrix} \frac{\sin\operatorname{ad}Y}{\operatorname{ad}Y} & -\sin\operatorname{ad}Y \\ \frac{1-\cos\operatorname{ad}Y}{\operatorname{ad}Y} & \cos\operatorname{ad}Y \end{pmatrix},$$

with respect to the bases $\{\alpha_1, \alpha_2, \alpha_3, dy_1, dy_2, dy_3\}$ and $\{\eta_1, \eta_2, \eta_3, J\eta_1, J\eta_2, J\eta_3\}$ (here we have only used that $\{e_1, e_2, e_3\}$ is an orthonormal basis for the $\operatorname{Ad}G$-invariant inner product on $\mathfrak{g}$). Now,

$$\overline{\partial}\partial|Y|^2 = d(\partial|Y|^2) = d(\pi^{(1,0)}2y_k dy_k).$$

With respect to the basis $\{\alpha_k, dy^k\}$, the form $2y_k dy_k$ is equal to the vector $(0, 2Y)$ at the point $(g, Y)$. Since $\operatorname{ad}Y$ maps $Y$ to 0, the form $d(\pi^{(1,0)}2y_k dy_k)$ on $G \times \mathfrak{g}$ is mapped by $(T\Phi^{-1})^*_{(g,Y)}$ to the form

$$\begin{aligned}\sum_{k=1}^3 d\left(\pi^{(1,0)}2y_k J\eta_k\right) &= \sum_{k=1}^3 d\pi^{(1,0)}\left(2y_k\frac{1}{2i}\left((\eta_k + iJ\eta_k) - (\eta_k - iJ\eta_k)\right)\right) \\ &= -i\sum_{k=1}^3 d(y_k(\eta_k + iJ\eta_k))\end{aligned}$$

on $G^{\mathbb{C}}$. Transferring this last expression back to $G \times \mathfrak{g}$ via $(T_{(g,Y)}\Phi)^*$, we obtain

$$-i\sum_{k=1}^3 d(y_k\alpha_k + iy_k dy_k) = -i\sum_{k=1}^3 d(y_k\alpha_k) = -\sum_{k=1}^3 idy_k \wedge \alpha_k - \sum_{k=1}^3 iy_k d\alpha_k.$$

We now compute the pullback of $-\sum_{k=1}^3 idy_k \wedge \alpha_k - \sum_{k=1}^3 iy_k d\alpha_k$ to $G^{\mathbb{C}}$ to a form on $G^{\mathbb{C}}$. Since this cannot be done as easily as the previous times, we now assume that $(g, Y) = (g, y_1e_1)$ with $y_1 \neq 0$. At these points,

$$\begin{aligned}(T\Phi^{-1})^*_{(g,y_1e_1)} &= \frac{\operatorname{ad}(y_1e_1)}{\sin\operatorname{ad}(y_1e_1)}\begin{pmatrix} \frac{\sin\operatorname{ad}(y_1e_1)}{\operatorname{ad}(y_1e_1)} & -\sin\operatorname{ad}(y_1e_1) \\ \frac{1-\cos\operatorname{ad}(y_1e_1)}{\operatorname{ad}(y_1e_1)} & \cos\operatorname{ad}(y_1e_1)\end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 1 & 0 & -y_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1-\cosh y_1}{\sinh y_1} & 0 & \frac{y_1\cosh y_1}{\sinh y_1} & 0 \\ 0 & -\frac{1-\cosh y_1}{\sinh y_1} & 0 & 0 & 0 & \frac{y_1\cosh y_1}{\sinh y_1}\end{pmatrix},\end{aligned}$$

where $\operatorname{ad} y_1 e_1$ has been expressed as a matrix with respect to the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ with the help of Equation (2.20), Equation (2.21) and the other equations below these. So, if $Y = y_1 e_1$, then

$$\begin{aligned}
\alpha_1 &\mapsto \eta_1; \\
\alpha_2 &\mapsto \eta_2 - \frac{1-\cosh y_1}{\sinh y_1} J\eta_3; \\
\alpha_3 &\mapsto \eta_3 + \frac{1-\cosh y_1}{\sinh y_1} J\eta_2; \\
dy_1 &\mapsto J\eta_1; \\
dy_2 &\mapsto -y_1\eta_3 + \frac{y_1\cosh y_1}{\sinh y_1} J\eta_2; \\
dy_3 &\mapsto y_1\eta_2 + \frac{y_1\cosh y_1}{\sinh y_1} J\eta_3.
\end{aligned}$$

We can now express $\overline{\partial}\partial|Y|^2$ in wedge products of $\eta_k$'s and $J\eta_l$'s. The expression for $dy_1 \wedge \alpha_1$ becomes

$$dy_1 \wedge \alpha_1 = -\eta_1 \wedge J\eta_1.$$

The expression for $dy_2 \wedge \alpha_2$ is

$$\begin{aligned}
dy_2 \wedge \alpha_2 &= \left(-y_1\eta_3 + \frac{y_1\cosh y_1}{\sinh y_1} J\eta_2\right) \wedge \left(\eta_2 - \frac{1-\cosh y_1}{\sinh y_1} J\eta_3\right) \\
&= y_1\eta_2 \wedge \eta_3 + \frac{y_1(1-\cosh y_1)}{\sinh y_1}\eta_3 \wedge J\eta_3 \\
&\quad - \frac{y_1\cosh y_1}{\sinh y_1}\eta_2 \wedge J\eta_2 - \frac{y_1\cosh y_1(1-\cosh y_1)}{\sinh^2 y_1} J\eta_2 \wedge J\eta_3,
\end{aligned}$$

and the one for $dy_3 \wedge \alpha_3$ is

$$\begin{aligned}
dy_3 \wedge \alpha_3 &= \left(y_1\eta_2 + \frac{y_1\cosh y_1}{\sinh y_1} J\eta_3\right) \wedge \left(\eta_3 + \frac{1-\cosh y_1}{\sinh y_1} J\eta_2\right) \\
&= y_1\eta_2 \wedge \eta_3 - \frac{y_1\cosh y_1}{\sinh y_1}\eta_3 \wedge J\eta_3 \\
&\quad + \frac{y_1(1-\cosh y_1)}{\sinh y_1}\eta_2 \wedge J\eta_2 - \frac{y_1\cosh y_1(1-\cosh y_1)}{\sinh^2 y_1} J\eta_2 \wedge J\eta_3.
\end{aligned}$$

Finally, the expression for $y_1 d\alpha_1$ is

$$\begin{aligned} y_1 d\alpha_1 = -y_1\alpha_2\wedge\alpha_3 &= -y_1\left(\eta_2 - \frac{1-\cosh y_1}{\sinh y_1}J\eta_3\right)\wedge\left(\eta_3 + \frac{1-\cosh y_1}{\sinh y_1}J\eta_2\right) \\ &= -y_1\eta_2\wedge\eta_3 - \frac{y_1(1-\cosh y_1)}{\sinh y_1}\eta_3\wedge J\eta_3 \\ &\quad - \frac{y_1(1-\cosh y_1)}{\sinh y_1}\eta_2\wedge J\eta_2 - y_1\left(\frac{1-\cosh y_1}{\sinh y_1}\right)^2 J\eta_2\wedge J\eta_3. \end{aligned}$$

Note that $y_2 = y_3 = 0$ at the point $y_1 e_1$, so that the terms $y_2 d\alpha_2$ and $y_3 d\alpha_3$ are both zero. Thus, we obtain

$$\begin{aligned} &\sum_{k=1}^{3} dy_k\wedge\alpha_k + \sum_{k=1}^{3} y_k d\alpha_k \\ &\quad = -\eta_1\wedge J\eta_1 + y_1\eta_2\wedge\eta_3 - \frac{y_1\cosh y_1}{\sinh y_1}\eta_3\wedge J\eta_3 \\ &\quad - \frac{y_1\cosh y_1}{\sinh y_1}\eta_2\wedge J\eta_2 + y_1 J\eta_2\wedge J\eta_3 \\ &\quad = -\frac{i}{2}\left(\eta_1 + iJ\eta_1\right)\wedge\left(\eta_1 - iJ\eta_1\right) + \frac{y_1}{2}\left(\eta_2 + iJ\eta_2\right)\wedge\left(\eta_3 - iJ\eta_3\right) \\ &\quad - \frac{y_1}{2}\left(\eta_3 + iJ\eta_3\right)\wedge\left(\eta_2 - iJ\eta_2\right) - \frac{iy_1\cosh y_1}{2\sinh y_1}\left(\eta_2 + iJ\eta_2\right)\wedge\left(\eta_2 - iJ\eta_2\right) \\ &\quad - \frac{iy_1\cosh y_1}{2\sinh y_1}\left(\eta_3 + iJ\eta_3\right)\wedge\left(\eta_3 - iJ\eta_3\right), \end{aligned}$$

where we have used the identity

$$\begin{aligned} &\frac{2y_1\cosh y_1(1-\cosh y_1)}{\sinh^2 y_1} + y_1\left(\frac{1-\cosh y_1}{\sinh y_1}\right)^2 \\ &\qquad = y_1\left(\frac{2\cosh y_1 - 2\cosh^2 y_1 + 1 + \cosh y_1^2 - 2\cosh y_1}{\sinh^2 y_1}\right) \\ &\qquad = y_1\frac{1-\cosh^2 y_1}{\sinh^2 y_1} = -y_1. \end{aligned}$$

If $W^k = \eta_k + iJ\eta_k$ for $k = 1, 2, 3$, then

$$\begin{aligned} \overline{\partial}\partial|Y|^2 &= -i\sum_{k=1}^{3}\left(dy_k\wedge\alpha_k + y_k d\alpha_k\right) \\ &= -\frac{1}{2}\left(W^1\wedge\overline{W^1} + \frac{y_1\cosh y_1}{\sinh y_1}W^2\wedge\overline{W^2} + \frac{y_1\cosh y_1}{\sinh y_1}W^3\wedge\overline{W^3}\right) \\ &\quad - \frac{i}{2}\left(y_1 W^2\wedge\overline{W^3} - y_1 W^3\wedge\overline{W^2}\right). \end{aligned}$$

The matrix of $\overline{\partial}\partial|Y|^2$ with respect to the frame $\{W^k\}$ , which we denote by $F = (F_{kl})$, is therefore equal to

$$F = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{y_1\cosh y_1}{\sinh y_1} & iy_1 \\ 0 & -iy_1 & \frac{y_1\cosh y_1}{\sinh y_1} \end{pmatrix}.$$

Let us turn our attention to $\partial|Y|^2 \wedge \overline{\partial}|Y|^2$. A simple verification shows that

$$\begin{aligned}
\partial|Y|^2 \wedge \overline{\partial}|Y|^2 &= 4\sum_{k,l}\left(\pi^{(1,0)}y_k dy^k\right)\wedge\left(\pi^{(0,1)}y_l dy^l\right) \\
&= 4\sum_{k,l}\left(\pi^{(1,0)}y_k J\eta_k\right)\wedge\left(\pi^{(0,1)}y_l J\eta_l\right) \\
&= 4\sum_{k,l} y_k y_l\left[\pi^{(1,0)}\frac{1}{2i}\left((\eta_k + iJ\eta_k) - (\eta_k - iJ\eta_k)\right)\right] \\
&\quad \wedge\left[\pi^{(0,1)}\frac{1}{2i}\left((\eta_l + iJ\eta_l) - (\eta_l - iJ\eta_l)\right)\right] \\
&= \sum_{k,l} y_k y_l W^k \wedge \overline{W^l} = y_1^2 W^1 \wedge \overline{W^1}.
\end{aligned}$$

At the point $(g, y_1e_1)$, the matrix $\widetilde{F} = (\widetilde{F}_{kl})$ of $\partial|Y|^2 \wedge \overline{\partial}|Y|^2$ with respect to the frame $W^k$ is therefore equal to

$$\widetilde{F} = \begin{pmatrix} y_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $H = (H_{kl})$ of $2|Y|^2\overline{\partial}\partial|Y|^2 + \partial|Y|^2 \wedge \overline{\partial}|Y|^2$ is then equal to

$$\begin{aligned}
H(x, y_1e_1)_{kl} = 2y_1^2F_{kl} + \widetilde{F}_{kl} &= -\begin{pmatrix} y_1^2 & 0 & 0 \\ 0 & \frac{y_1^3\cosh y_1}{\sinh y_1} & iy_1^3 \\ 0 & -iy_1^3 & \frac{y_1^3\cosh y_1}{\sinh y_1} \end{pmatrix} + \begin{pmatrix} y_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{y_1^3\cosh y_1}{\sinh y_1} & -iy_1^3 \\ 0 & iy_1^3 & -\frac{y_1^3\cosh y_1}{\sinh y_1} \end{pmatrix}.
\end{aligned}$$

This matrix is semi-negative if the eigenvalues of the right-corner $2\times 2$-submatrix are nonpositive, or equivalently, if the eigenvalues of the same matrix divided by $y_1^2$ are nonpositive. To compute the eigenvalues of the latter matrix, we need to solve the quadratic equation

$$\left(\lambda + \frac{y_1\cosh y_1}{\sinh y_1}\right)^2 - y_1^2 = \lambda^2 + 2\frac{y_1\cosh y_1}{\sinh y_1}\lambda + \left(\frac{y_1^2\cosh^2 y_1}{\sinh^2 y_1} - y_1^2\right) = 0$$

for $\lambda$. The solutions are given by

$$\lambda = -\frac{y_1 \cosh y_1}{\sinh y_1} \pm \sqrt{\frac{y_1^2 \cosh^2 y_1}{\sinh^2 y_1} - \left(\frac{y_1^2 \cosh^2 y_1}{\sinh^2 y_1} - y_1^2\right)} = -\frac{y_1 \cosh y_1}{\sinh y_1} \pm |y_1|,$$

or equivalently,

$$\lambda = y_1 \left(-\frac{\cosh y_1}{\sinh y_1} \pm 1\right),$$

which is always negative when $y_1 \neq 0$. □

**Remark 2.A.6.**

1. Actually, the proof can be shortened a lot after the images of $\alpha_k$ and $dy_k$ under $(\Phi_*^{-1})^*$ have been computed. Indeed, it is immediate from those expressions that the matrix $F$ is of the form

$$F = -\frac{1}{2}\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A$}} \\ 0 & & \end{array}\right),$$

where $A$ is a $2 \times 2$-matrix. However, since $\omega = i\overline{\partial}\partial|Y|^2$, we already know that $(F_{kl})$ is negative-definite. Consequently, $A$ is a positive-definite matrix. The matrix $(\widetilde{F}_{kl})$ was easily calculated, so we obtain that $(H_{kl})$ is of the form

$$H = -\frac{y_1^2}{2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A$}} \\ 0 & & \end{array}\right).$$

Because $A$ is positive-definite, the eigenvalues of $H$ are nonpositive. Conversely, the calculation of $(F_{kl})$ in the proof of Lemma 2.A.4 gives a direct proof for the fact that $T^*G$, with the above symplectic and complex structure, is a Kähler manifold.

2. Notice that the complex Hessian for the function $(g, Y) \mapsto |Y|$ has zero determinant at $T^*G \setminus G$. This is in agreement with the results in [35].

3. Let

$$g = \sum_{k,l} g_{kl} W^k \vee \overline{W^l}$$

denote the Riemannian metric of a Kähler manifold $M$ with respect to a local holomorphic frame $\{W^k\}$ of $T^{*(1,0)}M$. The curvature of the canonical line bundle is then equal to (see e.g. [3])

$$R^K = \partial\overline{\partial}\log(\det g_{kl}).$$

Since $\omega = i\overline{\partial}\partial|Y|^2$, it follows from Lemma 1.3.16 that with respect to the holomorphic frame $\{\eta_k + iJ\eta_k\}$, we have

$$\det g_{kl}(g, y_1 e_1) = \det \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{y_1 \cosh y_1}{\sinh y_1} & iy_1 \\ 0 & -iy_1 & \frac{y_1 \cosh y_1}{\sinh y_1} \end{pmatrix} = \frac{y_1^2}{2^3}\left(\frac{\cosh^2 y_1}{\sinh^2 y_1} - 1\right)$$
$$= \frac{y_1^2(\cosh^2 y_1 - \sinh^2 y_1)}{2^3 \sinh^2 y_1} = \frac{y_1^2}{2^3 \sinh^2 y_1} = \frac{1}{2^3 \eta(y_1 e_1)^2},$$

for all $(g, y_1 e_1) \in G \times \mathfrak{g}$. One can show that, with respect to this frame, the determinant function $(g, Y) \mapsto \det g_{kl}(g, Y)$ is $G \times G$-invariant (use that the determinant of $\mathrm{Ad}_g : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is equal to 1 for all $g \in G$). Consequently, $\det g_{kl}(g, Y) = \frac{1}{2^3 \eta(Y)^2}$ for all points $(g, Y) \in G \times \mathfrak{g}$. So we have explicitly verified that $R^K = 2\overline{\partial}\partial \log \eta$.

**Lemma 2.A.7.** *The function* $\log \circ \widetilde{\eta} : \mathbb{R} \to \mathbb{R}$*, where* $\widetilde{\eta} : \mathbb{R} \to \mathbb{R}$ *is given by* $t \mapsto \frac{\sinh ct}{ct}$*, is convex when* $c > 0$*.*

*Proof.* We can assume that $c = 1$. For any smooth function $\phi : \mathbb{R} \to \mathbb{R}^{>0}$, the function $\log \circ \phi$ is convex if and only if

$$\phi(t)\phi''(t) - \phi'(t)^2 \geq 0 \quad \text{for all } t \in \mathbb{R}.$$

The first and second derivative of $\widetilde{\eta}$ are equal to

$$\widetilde{\eta}'(t) = \frac{\cosh t}{t} - \frac{\sinh t}{t^2}, \quad \widetilde{\eta}''(t) = \frac{\sinh t}{t} - \frac{2\cosh t}{t^2} + \frac{2\sinh t}{t^3}.$$

The expression $\widetilde{\eta}(t)\widetilde{\eta}''(t) - \widetilde{\eta}'(t)^2$ is then equal to

$$\begin{aligned} \widetilde{\eta}(t)\widetilde{\eta}''(t) - \widetilde{\eta}'(t)^2 &= \frac{\sinh^2 t}{t^2} - \frac{2\cosh t \sinh t}{t^3} + \frac{2\sinh^2 t}{t^4} \\ &\quad - \frac{\cosh^2 t}{t^2} + \frac{2\cosh t \sinh t}{t^3} - \frac{\sinh^2 t}{t^4} \\ &= \frac{\sinh^2 t - \cosh^2 t}{t^2} + \frac{\sinh^2 t}{t^4} = \frac{\sinh^2 t - t^2}{t^4} > 0, \end{aligned}$$

for all $t \in \mathbb{R} \setminus \{0\}$. Since $\widetilde{\eta}$ is smooth, $\widetilde{\eta}(t)\widetilde{\eta}''(t) - \widetilde{\eta}'(t)^2 \geq 0$ for all $t \in \mathbb{R}$. Hence $\log \widetilde{\eta}$ is convex. ☐

**Proposition 2.A.8.** *The function* $(g, Y) \mapsto \log\left(\frac{\sinh(c|Y|)}{c|Y|}\right)$ *is plurisubharmonic on* $T^*G \setminus G$*, when* $G = SU(2)$*.*

*Proof.* We know from Lemma 2.A.4 that the function $f : (x, Y) \mapsto |Y|$ is plurisubharmonic on $T^*G \setminus G$, i.e. with respect to holomorphic coordinates on $T^*G \setminus G$:

$$\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} \geq 0.$$

Let us denote the function $t \mapsto \log\left(\frac{\sinh ct}{ct}\right)$ by $\phi$. Then

$$\begin{aligned} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log\left(\frac{\sinh c|Y|}{c|Y|}\right) &= \frac{\partial^2}{\partial z_i \partial \overline{z}_j}(\phi \circ f)(Y) = \frac{\partial}{\partial z_i}\left(\phi'(f(Y))\frac{\partial f}{\partial \overline{z}_j}(Y)\right) \\ &= \phi''(f(Y))\frac{\partial f}{\partial z_i}\frac{\partial f}{\partial \overline{z}_j} + \phi'(f(Y))\frac{\partial f}{\partial z_i \partial \overline{z}_j}. \end{aligned} \tag{2.22}$$

Now the second term determines a positive semi-definite matrix because $\phi' > 0$ on Im $f = \mathbb{R}^{>0}$ and $f$ is plurisubharmonic on $T^*G \backslash G$ by Lemma 2.A.4. The first term determines a positive semi-definite matrix, because $\phi$ is convex by Lemma 2.A.7, and $A = (A_{ij}) = (\frac{\partial f}{\partial z_i}\frac{\partial f}{\partial \overline{z}_j})$ is positive semi-definite. Positive semi-definiteness of $A$ can be deduced by noting that $A = vv^*$, where $v$ is the vector $v = (v_i) = (\frac{\partial f}{\partial z_i})$. Thus, Equation (2.22) implies that the matrix $\left(\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log\left(\frac{\sinh c|Y|}{c|Y|}\right)\right)$ is a sum of two positive semi-definite matrices, and is therefore positive semi-definite. Equivalently, the function $(g, Y) \mapsto \log\left(\frac{\sinh c|Y|}{c|Y|}\right)$ is plurisubharmonic on $T^*G \setminus G$. □

**Remark 2.A.9.** The above argument is identical to the proof of a more general result: if $M$ is a complex manifold, $f : M \to \mathbb{R}$ plurisubharmonic, and $\phi : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and convex, then $\phi \circ f$ is plurisubharmonic as well. Note that although the function $\phi$ of Proposition 2.A.8 is not monotonically increasing on $\mathbb{R}$ (it is a non-constant even function), it is monotonically increasing on the range of $f$.

We now show that $\overline{\partial}\partial \log \circ \eta$ is semi-negative at $G \subset T^*G$, where $G \subset T^*G$ as its zero section. This will conclude the proof of Proposition 2.A.2, as we have then shown that $\overline{\partial}\partial \log \circ \eta$ is semi-negative at every point of $T^*G$.

**Lemma 2.A.10.** *The form* $\overline{\partial}\partial \log \circ \eta$ *is semi-negative on* $G \subset T^*G$.

*Proof.* The function $\eta$, and hence $\log \eta$, takes it minimum values precisely at those points of $T^*G$ that belong to $G$. In general, if $f$ is any real function on an open subset $\Omega \subset \mathbb{C}^n$ and $f$ takes a (not necessarily strict) local minimum at the point $w$, then (*c.f.* Remark 2.A.1)

$$\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(w) \geq 0.$$

In particular, $\overline{\partial}\partial \log \circ \eta$ is semi-negative on $G \subset T^*G$. □

*Alternative proof.* Let $(x, 0)$ be a point of $G \subset T^*G$ and let $(U, z)$ be a holomorphic chart around $(x, 0)$. Since $\left(\frac{\partial^2 \log \eta}{\partial z_i \partial \overline{z}_j}\right)$ is positive semi-definite on $U \cap (T^*G \setminus G)$ and since the cone of positive semi-definite matrices is closed, the matrix $\left(\frac{\partial^2 \log \eta}{\partial z_i \partial \overline{z}_j}\right)$ is positive semi-definite on all of $U$. □

**Remark 2.A.11.** We would like to know if the canonical line bundle $K$ on $G^{\mathbb{C}}$ is semi-negative for any compact connected Lie group $G$. This would require

$$\log \eta(Y) = \log\left(\prod_{\alpha \in R^+} \frac{\sinh(\alpha(Y))}{\alpha(Y)}\right) = \sum_{\alpha \in R^+} \log\left(\frac{\sinh(\alpha(Y))}{\alpha(Y)}\right),$$

to be plurisubharmonic. One could be tempted to look at the individual terms in the sum over the positive roots, and to show that these are plurisubharmonic. However, the individual terms are not invariant under the Weyl group, and therefore they cannot be extended to Ad $G$-invariant functions on $\mathfrak{g}$. In the future I hope to be able to determine for which groups $\log \eta$ is plurisubharmonic and I conjecture that this might be true for all compact, connect Lie groups.

Since $SU(2) \in \mathcal{C}_K$, we can use Theorem 2.4.10 to describe the Dolbeault-Dirac and spin quantisation of $T^*SU(2)$.

**Theorem 2.A.12.** *The Dolbeault-Dirac quantisation of $T^*SU(2)$ with its standard Kähler structure is equal to*

$$\mathcal{H}L^2(T^*SU(2), e^{-2\pi|Y|^2}\varepsilon),$$

*which is unitarily and $SU(2) \times SU(2)$-equivariantly isomorphic to $L^2(SU(2))$ via the isomorphism of Proposition 2.4.9.*

*Similarly, the spin quantisation of $T^*SU(2)$ with its standard Kähler structure is equal to*

$$\mathcal{H}L^2(T^*SU(2), e^{-2\pi|Y|^2}\eta\varepsilon),$$

*which is unitarily and $SU(2) \times SU(2)$-equivariantly isomorphic to $L^2(SU(2))$ via the isomorphism of Proposition 2.4.9.*

# Chapter 3

# The cotangent bundle of a maximal torus

As we will see in Chapter 4 the Marsden-Weinstein quotient of the cotangent bundle $T^*G$ of a compact Lie group $G$ is related to the quotient of the cotangent bundle $T^*\mathbf{T}$ of a maximal torus $\mathbf{T}$ by the action of the Weyl group $W(G, \mathbf{T})$. Because the action of the Weyl group on $T^*\mathbf{T}$ is not free, the quotient $T^*\mathbf{T}/W(G, \mathbf{T})$ is singular. We define the quantisation of $T^*\mathbf{T}/W(G, \mathbf{T})$ as the Dolbeault-Dirac quantisation of the principal stratum and prove that, with that definition, quantisation commutes with reduction.

The proof can be divided into two steps. In the first step (Section 3.2) we show that the Dolbeault-Dirac quantisation of the principal stratum of $T^*\mathbf{T}$ is equal to the Dolbeault-Dirac quantisation of the whole space $T^*\mathbf{T}$. The main idea is that $T^*\mathbf{T} \setminus (T^*\mathbf{T})_{princ}$ is a finite union of closed embedded submanifolds of codimension $\geq 2$ and that essential self-adjointness of the twisted Dolbeault-Dirac operator is not affected by the removal of these closed submanifolds. This latter fact follows from a result on Sobolev spaces on $\mathbb{R}^n$, which we prove in Section 3.1 (see Theorem 3.1.3). In the second step (Section 3.3) we show that the quantisation of the singular quotient is isomorphic to the $W(G, \mathbf{T})$-invariant part of the quantisation of $(T^*\mathbf{T})_{princ}$.

We conclude this chapter with a discussion in Section 3.4. In particular, we show that the results of Section 3.2 are valid in many other situations.

## 3.1 Some Sobolev theorems

In this section we prove some well-known results that are mainly used in Section 3.2. Let us fix some notation first. We set $D_j = -i\partial_j$ and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, where $\alpha \in \mathbb{N}_0$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. The *length* of $\alpha$ is $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

By $\mathbb{R}^{n-k} \subset \mathbb{R}^n$, $(1 \leq k \leq n)$ we always mean the $k$-codimensional submanifold

$$\{(x_1, \ldots, x_n) \mid x_{n-k+1} = \cdots = x_n = 0\}.$$

Let $s \in \mathbb{R}$ be a real number and let $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions on $\mathbb{R}^n$. The ***Sobolev space*** $H^s(\mathbb{R}^n)$ is defined by (see *e.g.* [32])

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s(\mathcal{F}u)(\xi) \in L^2(\mathbb{R}^n)\},$$

where $\mathcal{F}$ denotes the Fourier transform and $\langle\xi\rangle = \sqrt{1 + \|\xi\|^2}$. The Sobolev space $H^s(\mathbb{R}^n)$ is a Hilbert space under the inner product

$$\langle u, v\rangle_s = \int_{\mathbb{R}^n} (\mathcal{F}u)(\xi)(\mathcal{F}v)(\xi)\langle\xi\rangle^{2s} d\xi.$$

It is well known that the space $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for each $s \in \mathbb{R}$ ([32, Lemma 6.10]). Moreover, for each real number $s > \frac{k}{2}$ there is a continuous, surjective map

$$\gamma_0 : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^{n-k}), (\gamma_0 f)(x_1, \ldots x_{n-k}) = f(x_1, \ldots x_{n-k}, 0, \ldots, 0),$$

initially defined on $C_c^\infty(\mathbb{R}^n)$ and then extended to all of $H^s(\mathbb{R}^n)$ by continuity (see for instance [88, Theorem 2.9.4, page 223]). The kernel of $\gamma_0$ contains $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$, so that $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is not dense in $H^s(\mathbb{R}^n)$ if $s > \frac{k}{2}$.

If $s \leq \frac{k}{2}$, the space $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ *is* dense in $H^s(\mathbb{R}^n)$, as we now show. The next theorem, which is a combination of [47, Theorem 2.3.4], and [32, Example 6.10], deals with the maximal codimension $k = n$.

**Lemma 3.1.1.** *The space $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $H^s(\mathbb{R}^n)$ if and only if $s \leq \frac{n}{2}$.*

*Proof.* It remains to show the 'if'-part. Assume that $s \leq \frac{n}{2}$. We show that any continuous anti-linear functional on $H^s(\mathbb{R}^n)$ vanishing on $C_c^\infty(\mathbb{R}^n \setminus \{0\})$, vanishes on $H^s(\mathbb{R}^n)$. Let $u$ be an anti-linear functional on $H^s(\mathbb{R}^n)$. Then $u$ can be assumed to be an element of $H^{-s}(\mathbb{R}^n)$, because $H^{-s}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ is the (anti-)dual of $H^s(\mathbb{R}^n)$ (see [32, Theorem 6.15]). As an anti-linear functional on $H^s(\mathbb{R}^n)$, $u$ is of the form

$$\phi \mapsto u(\overline{\phi}), \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

If $u$ vanishes on $C_c^\infty(\mathbb{R}^n \setminus \{0\})$, then $u$ is a distribution with support contained in $\{0\}$. Therefore (see [47, Theorem 2.3.4]),

$$u = \sum_{|\alpha| \leq k'} c_\alpha D^\alpha \delta,$$

where $\delta$ denotes the Dirac-distribution, $c_\alpha \in \mathbb{C}$ and $k' \in \mathbb{N}_0$. By definition, $\mathcal{F}(H^{-s}(\mathbb{R}^n)) = \{v \in L^1_{loc}(\mathbb{R}^n) \mid \langle\xi\rangle^{-s} v \in L^2(\mathbb{R}^n)\}$. In particular,

$$u \in H^{-s}(\mathbb{R}^n) \iff \sum_{|\alpha|\leq k'} c_\alpha \langle\xi\rangle^{-s}\xi^\alpha \in L^2(\mathbb{R}^n).$$

This right-hand side holds if and only if $c_\alpha = 0$ for each $\alpha$ satisfying $2s - 2|\alpha| \leq n$. Since $s \leq \frac{n}{2}$, $c_\alpha = 0$ for all $\alpha$. Hence, $u = 0$. □

We now prove a similar result for arbitrary codimension. Let $f_1 \in C_c^\infty(\mathbb{R}^{n_1})$, $f_2 \in C_c^\infty(\mathbb{R}^{n_2})$ and write $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$. Then $f_1 \otimes f_2 \in C_c^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined as

$$(f_1 \otimes f_2)(x) = f_1(x_1) f_2(x_2). \tag{3.1}$$

Define $C_c^\infty(\mathbb{R}^{n_1}) \otimes C_c^\infty(\mathbb{R}^{n_2}) \subset C_c^\infty(\mathbb{R}^{n_1+n_2})$ to be the space spanned by all functions of the form of Equation (3.1). This subspace is dense in $C_c^\infty(\mathbb{R}^{n_1+n_2})$ with respect to the usual inductive topology (see [87, Theorem 39.2]).

**Lemma 3.1.2.** *Let $s > 0$. Then*

$$\|f_1 \otimes f_2\|_{H^s(\mathbb{R}^{n_1+n_2})} \leq \|f_1\|_{H^s(\mathbb{R}^{n_1})} \|f_2\|_{H^s(\mathbb{R}^{n_2})}.$$

*for all $f_1 \in C_c^\infty(\mathbb{R}^{n_1})$, $f_2 \in C_c^\infty(\mathbb{R}^{n_2})$.*

*Proof.* Let $f_1 \in C_c^\infty(\mathbb{R}^{n_1})$, $f_2 \in C_c^\infty(\mathbb{R}^{n_2})$ be given. Write $x = (x_1, x_2)$. Since

$$1 + \|x\|^2 = 1 + \|x_1\|^2 + \|x_2\|^2 \leq (1 + \|x_1\|^2)(1 + \|x_2\|^2),$$

we have $\langle x\rangle^s \leq \langle x_1\rangle^s \langle x_2\rangle^s$ for any $s > 0$. Furthermore, $\mathcal{F}(f_1 \otimes f_2) = \mathcal{F}_1 f_1 \otimes \mathcal{F}_2 f_2 = \mathcal{F}_1 f_1 \mathcal{F}_2 f_2$. We now compute that

$$\begin{aligned}
\|f_1 \otimes f_2\|_{H^s(\mathbb{R}^n)} &= \|\langle\xi\rangle^s \mathcal{F}(f_1 \otimes f_2)\|_{L^2} = \|\langle\xi\rangle^s \mathcal{F}_1 f_1 \mathcal{F}_2 f_2)\|_{L^2} \\
&\leq \|\langle\xi_1\rangle^s \mathcal{F}_1 f_1 \langle\xi_2\rangle^s \mathcal{F}_2 f_2)\|_{L^2} = \|\langle\xi_1\rangle^s \mathcal{F}_1 f_1\|_{L^2} \|\langle\xi_2\rangle^s \mathcal{F}_2 f_2\|_{L^2} \\
&= \|f_1\|_{H^s(\mathbb{R}^{n_1})} \|f_2\|_{H^s(\mathbb{R}^{n_2})} < \infty.
\end{aligned}$$

□

**Theorem 3.1.3.** *The space $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is dense in $H^s(\mathbb{R}^n)$ if and only if $s \leq \frac{k}{2}$ (where, for $n = k$, $\mathbb{R}^0 = \{0\} \subset \mathbb{R}^n$).*

*Proof.* Again, it remains to prove the "if"-part. First, fix the codimension $k$ and the number $s \leq \frac{k}{2}$. We show by induction that $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is dense in $H^s(\mathbb{R}^n)$ for all $n$. The case $n = k$ (the maximal codimension) is precisely Lemma 3.1.1.

Let us suppose that the statement is true for $n \geq k$ and consider the series of inclusions

$$C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k}) \to C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^{n+1}) \to H^s(\mathbb{R}^{n+1}).$$

First, $C_c^\infty(\mathbb{R}^{n+1})$ is dense in $H^s(\mathbb{R}^{n+1})$ in the $H^s(\mathbb{R}^{n+1})$-norm. Next, $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n)$ is dense in $C_c^\infty(\mathbb{R}^{n+1})$ with respect to the $H^s(\mathbb{R}^{n+1})$-norm, because it is dense in $C_c^\infty(\mathbb{R}^{n+1})$ with respect to the the inductive topology, which is finer than the $H^s(\mathbb{R}^{n+1})$-norm topology. By the induction assumption, $C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is dense in $C_c^\infty(\mathbb{R}^n)$ in the $H^s(\mathbb{R}^n)$-norm. Consequently, by Lemma 3.1.2, $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is dense in $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n)$ in the $H^s(\mathbb{R}^{n+1})$-norm. Thus, $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is a dense subset of $H^s(\mathbb{R}^{n+1})$.

The image of $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-k})$ is contained in $C_c^\infty(\mathbb{R}^{n+1} \setminus \mathbb{R}^{n+1-k})$, so that $C_c^\infty(\mathbb{R}^{n+1} \setminus \mathbb{R}^{n+1-k})$ is dense in $H^s(\mathbb{R}^{n+1})$ as well.

□

**Remark 3.1.4.** Theorem 3.1.3 is probably well known among experts. As I was not able to find such a result in the literature in this full generality, I included the proof here for completeness.

I found a result similar to Theorem 3.1.3 in [67]. There, Theorem 3.1.3 is proved for $k = 1$. Compared with Lemma 3.1.1, one could say that that result deals with the lowest possible non-zero codimension, whereas Lemma 3.1.1 deals with the highest possible codimension.

There is a direct relation between the Sobolev spaces $H^m(\mathbb{R}^n)$ and the domains of elliptic differential operators of order $m$ with constant coefficients. For instance, in [32, Theorem 6.24] it is proved that on $\mathbb{R}^n$ the (minimal(=maximal)) domain of an elliptic differential operator $D$ of order $m$ with constant coefficients is precisely equal to $H^m(\mathbb{R}^n)$, with the graph-norm of $D$ equivalent to the Sobolev norm. With almost exactly the same proof, this result extends to elliptic differential operators of order $m$ on (necessarily trivial) hermitian vector bundles over $M$.

Let $E \to \mathbb{R}^n$ be a hermitian vector bundle of rank $m$. We decompose $L^2(\mathbb{R}^n, E)$ as a direct sum of $m$-copies of $L^2(\mathbb{R}^n)$. With respect to this decomposition we introduce the norm $\|\cdot\|_1$ on $\Gamma_c^\infty(\mathbb{R}^n, E)$, which is defined by

$$\|(f_1, \dots, f_m)\|_1^2 = \sum_{j=1}^{m} \|f_j\|_{H^1(\mathbb{R}^n)}^2.$$

**Proposition 3.1.5.** *Let $E \to \mathbb{R}^n$ be a hermitian vector bundle of rank $m$ and identify $L^2(\mathbb{R}^n, E)$ with the direct sum of $m$ copies of $L^2(\mathbb{R}^n)$. Let $D = \sum_k A_l(-i\partial_l)$ be a first-order elliptic differential operator on $L^2(\mathbb{R}^n)^m$ with constant coefficients. Then the graph norm of $D$ on $\Gamma_c^\infty(\mathbb{R}^n, E)$ is equivalent to the $\|\cdot\|_1$-norm on $L^2(\mathbb{R}^n, E)$.*

*Proof.* By assumption the operator $D$ is $D = \sum_k A_l(-i\partial_l)$, where the $A_l$'s are constant $m \times m$-matrices. Let us write $\hat{s}$ for the Fourier transform of $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$.

By definition, for every $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$:

$$\begin{aligned}\|Ds\|_{L^2} &= \|\sum_l A_l \partial_l s\|_{L^2} \leq \sum_l \|A_l \xi_l \hat{s}\|_{L^2} \leq \sum_l \max_j\{\|A_j\|\}\|\xi_l \hat{s}\|_{L^2} \\ &\leq m \max_j\{\|A_j\|\}\|\langle\xi\rangle\hat{s}\|_{L^2} \leq C' \sum_{l=1}^m \|s_l\|_{H^1(\mathbb{R}^n)} \\ &\leq C''\|s\|_1,\end{aligned}$$

for some positive constants $C', C''$. Here we have used the Plancherel identity, the estimates $\langle\xi\rangle = \sqrt{1+\|\xi\|^2} \geq \|\xi\| \geq |\xi_l|$, and the fact that for any norm $\|\cdot\|$ on a vector space $V$ the norms $(v_1, \ldots, v_m) \mapsto \sum_{l=1}^m \|v_l\|$ and $(v_1, \ldots, v_m) \mapsto (\sum_{l=1}^m \|v_l\|^2)^{\frac{1}{2}}$ on $V^m$ are equivalent. Altogether, we have showed that there exists a constant $C > 0$ such that $\|s\|_D \leq C\|s\|_1$ for all $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$.

To show that there exists a constant $c > 0$ such that $c\|s\|_1 \leq \|s\|_D$ for all $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$, we use ellipticity of $D$. If $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$, then

$$\|Ds\|_{L^2}^2 = \sum_{k,l}\langle A_k(-i\partial_k)s, A_l(-i\partial_l)s\rangle = \sum_{k,l}\langle A_k\xi_k\hat{s}, A_l\xi_l\hat{s}\rangle = \langle\hat{s}, \sum_{k,l} A_k^* A_l \xi_k \xi_l \hat{s}\rangle,$$

where we used the Plancherel identity in the second step. The hermitian matrix $\sum_{k,l} A_k^* A_l \xi_k \xi_l \in \mathrm{End}(E_\xi)$ is positive and, by ellipticity of $D$, it is invertible for each $\xi \neq 0$. The function $\lambda$ on $S = \{\xi \in \mathbb{R}^n \mid \|\xi\| = 1\}$ that sends $\sum_{k,l} A_k^* A_l \xi_k \xi_l \in \mathrm{End}(E_\xi)$ to its lowest eigenvalue is a continuous function into the positive real numbers. Since $S$ is compact, the function $\lambda$ attains its positive minimum value somewhere on $S$. Hence there exists a positive number $c' > 0$ such that, for all $\xi \in S$,

$$c' \sum_{k,l} A_k^* A_l \xi_k \xi_l \geq \mathrm{id}_{E_\xi} = \|\xi\|^2 \mathrm{id}_{E_\xi}$$

as operators in $\mathrm{End}(E_\xi)$. Because both sides of the inequality are homogeneous of order 2 in $\xi$, the inequality

$$c' \sum_{k,l} A_k^* A_l \xi_k \xi_l \geq \|\xi\|^2 \mathrm{id}_{E_\xi}$$

holds for all $\xi \in \mathbb{R}^n$. Using this inequality, we now obtain

$$
\begin{aligned}
\|s\|_1^2 = \|\langle\xi\rangle\hat{s}\|_{L^2}^2 &= \int\limits_{\mathbb{R}^n} \langle \hat{s}(\xi), (1+\|\xi\|^2)\hat{s}(\xi)\rangle_\xi d\xi \\
&= \|s\|_{L^2}^2 + \int\limits_{\mathbb{R}^n} \langle \hat{s}(\xi), \|\xi\|^2\hat{s}(\xi)\rangle_\xi d\xi \\
&\leq \|s\|_{L^2}^2 + c' \int\limits_{\mathbb{R}^n} \langle \hat{s}(\xi), \sum_{k,l} A_k^* A_l \xi_k \xi_l \hat{s}(\xi)\rangle_\xi d\xi \\
&= \|s\|_{L^2}^2 + c'\|Ds\|_{L^2}^2 \leq \frac{1}{c}\|s\|_D^2,
\end{aligned}
$$

for some constant $c > 0$.

Thus, we have obtained constants $c, C > 0$ such that

$$
c\|s\|_1 \leq \|s\|_D \leq C\|s\|_1,
$$

for all $s \in \Gamma_c^\infty(\mathbb{R}^n, E)$, and so both norms are equivalent. □

Endow $\mathbb{R}^{2n}$ with its standard Kähler structure. The bundle $\Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n}$ is identified with the direct sum of $2^n$-copies of $L^2(\mathbb{R}^{2n})$ by considering the unitary frame

$$
\left\{ \frac{1}{\sqrt{2^k}} d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_k} \;\middle|\; 1 \leq i_1 < i_2 < \cdots < i_k, 0 \leq k \leq n \right\}.
$$

With respect to this unitary frame the Dolbeault-Dirac operator has constant coefficients, and so Proposition 3.1.5 can be applied to this operator. Theorem 3.1.3 now implies the following:

**Lemma 3.1.6.** *Let $2 \leq k \leq 2n$ be a natural number. The subspace*

$$
\Gamma_c^\infty(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-k}, \Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n})
$$

*is dense in $\Gamma_c^\infty(\mathbb{R}^{2n}, \Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n})$ with respect to the graph norm of the Dolbeault-Dirac operator on $\mathbb{R}^{2n}$.*

The following Theorem is an immediate consequence of the Lemma.

**Theorem 3.1.7.** *Let $2 \leq k \leq 2n$ be a natural number. The closure of the Dolbeault-Dirac operator on the domain $\Gamma_c^\infty(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-k}, \Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n})$ is equal to the closure of the Dolbeault-Dirac operator on the domain $\Gamma_c^\infty(\mathbb{R}^{2n}, \Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n})$. Consequently, the Dolbeault-Dirac operator is essentially self-adjoint on the domain $\Gamma_c^\infty(\mathbb{R}^{2n} \setminus \mathbb{R}^{2n-k}, \Lambda^{(0,\bullet)}T^*\mathbb{R}^{2n})$.*

**Example 3.1.8.** Let $\Delta_1 \setminus \{0\}$ be the open punctured disk of radius 1 in $\mathbb{C}$. Consider the differential operator $\partial_z = \partial_x - i\partial_y$ on $\mathbb{C}$. This is a first-order elliptic differential operator with constant coefficients. Since $\{0\} \subset \mathbb{C}$ is of codimension 2, $C_c^\infty(\Delta_1 \setminus \{0\})$ is dense in $C_c^\infty(\Delta_1)$ in the $\partial_z$-graph norm by Theorem 3.1.3 and Proposition 3.1.5 (see also Lemma 3.2.7 below). Therefore, the adjoint of $(\partial_z, C_c^\infty(\Delta_1 \setminus \{0\}))$ coincides with the adjoint of $(\partial_z, C_c^\infty(\Delta_1))$ in $L^2(\Delta_1 \setminus \{0\}) = L^2(\Delta_1)$. Consequently, the kernels of both adjoint operators coincide, which implies that any holomorphic $L^2$-function on $\Delta_1 \setminus \{0\}$ can be extended to a holomorphic function on $\Delta_1$.

This can also be proved by direct means. Let $f$ be a holomorphic function in $L^2(\Delta_1 \setminus \{0\})$, and consider its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad (a_n \in \mathbb{C}).$$

Now change to polar coordinates ($z = re^{i\phi}$) to obtain

$$f(re^{i\phi}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\phi}, \quad (a_n \in \mathbb{C}).$$

Using the orthogonality relations from Fourier theory, we obtain

$$\int_{\Delta_1} |f(z)|^2 dz = \int_0^1 \int_0^{2\pi} r|f(re^{i\phi})|^2 d\phi dr = 2\pi \sum_{n=-\infty}^{\infty} \int_0^1 r^{2n+1} |a_n|^2 dr.$$

If $f$ is square-integrable on $\Delta_1 \setminus \{0\}$, then each term in the sum on the right-hand side is finite. Therefore, $a_n = 0$ for all $n < 0$. Hence, $f$ can be extended to a holomorphic function on $\Delta_1$.

## 3.2 Quantisation of $T^*\mathbf{T}$ and its principal stratum

Let $G$ be a compact connected Lie group and let $\mathbf{T}$ be a maximal torus in $G$. Given $\mathbf{T} \subset G$, we denote the Weyl group by $W(G, \mathbf{T})$. In this section we study the orbit-type stratification of $\mathbf{T} \times \mathfrak{t}$ for the Weyl group action on $\mathbf{T} \times \mathfrak{t}$. We prove that the Dolbeault-Dirac quantisation of the principal stratum coincides with the Dolbeault-Dirac quantisation of $\mathbf{T} \times \mathfrak{t}$. In the rest of this chapter the group $G$ is a compact connected Lie group.

### 3.2.1 Weyl group action on maximal torus

A subgroup $\mathbf{T} \subset G$ is said to be a ***torus*** if it is a connected abelian Lie subgroup of $G$. Being a compact connected abelian Lie group, $\mathbf{T}$ is necessarily isomorphic to $\mathbb{T}^k$ for some $k \in \mathbb{N}$. A torus $\mathbf{T}$ is said to be ***maximal*** if for any other torus $\mathbf{T}' \subset G$ the inclusion $\mathbf{T} \subset \mathbf{T}'$ implies that $\mathbf{T} = \mathbf{T}'$.

The ***Weyl group*** $W(G, \mathbf{T})$ , or simply $W$ when $G$ and $\mathbf{T}$ are understood, is defined to be the group

$$W(G, \mathbf{T}) = N_G(\mathbf{T})/C_G(\mathbf{T}),$$

where

$$N_G(\mathbf{T}) = \{g \in G \mid g\mathbf{T}g^{-1} = \mathbf{T}\}, \quad C_G(\mathbf{T}) = \{g \in G \mid gt = tg \text{ for all } t \in \mathbf{T}\},$$

are, respectively, the ***normaliser*** and ***centraliser*** of $\mathbf{T}$ within $G$. If $\mathbf{T}$ is a maximal torus, then $C_G(\mathbf{T}) = \mathbf{T}$, so that in that case $W(G, \mathbf{T}) = N_G(\mathbf{T})/\mathbf{T}$ (see e.g. [56, Corollary 4.52]). **In the remainder of this chapter $\mathbf{T}$ always denotes a *maximal torus* of a compact connected Lie group $G$.**

The following statements concerning maximal tori are well known (see for instance [56, IV.5 and Proposition 4.53] or [28]).

**Proposition 3.2.1.** *Let $G$ be a compact connected Lie group.*

1. *Any two maximal tori of $G$ are conjugate. (The dimension of a maximal torus within $G$ is also called the* **rank** *of the Lie group.)*
2. *If $\boldsymbol{T}$ is a maximal torus, then any element of $G$ is conjugate to an element in $\boldsymbol{T}$.*
3. *Two elements of $\boldsymbol{T}$ are conjugate within $G$ if and only if they are conjugate through $W(G, \boldsymbol{T})$. Moreover, if $G$ acts on itself by conjugation, then*

   $$G/\operatorname{Ad} G \cong \boldsymbol{T}/W(G, \boldsymbol{T}),$$

   *including topologies.*
4. *The Weyl group $W(G, \boldsymbol{T})$ is finite and acts effectively on $\boldsymbol{T}$ and $\boldsymbol{T}_{\{e\}}$ is open and dense in $\boldsymbol{T}$.*

The group $W(G, \mathbf{T})$ acts on $\mathbf{T}$ by homomorphisms. If $w = \operatorname{Ad}_g : \mathbf{T} \to \mathbf{T}$ ($g \in N_G(\mathbf{T})$), the corresponding action on the Lie algebra $\mathfrak{t}$ is

$$T_e w : \mathfrak{t} \to \mathfrak{t}, \quad w(X) = \operatorname{Ad}_g X.$$

For any Lie group homomorphism $\Phi : H_1 \to H_2$, the equation $\Phi(\exp(X)) = \exp(T_e\Phi X)$ holds for all $X \in \mathfrak{h}_1$. In particular, the exponential map $\mathfrak{t} \to \mathbf{T}$ intertwines the actions of the Weyl group on $\mathfrak{t}$ and $\mathbf{T}$.

Let $\langle\cdot,\cdot\rangle$ be a $W(G, \mathbf{T})$-invariant inner product on $\mathfrak{t}$ and suppose that $U := \{X \in \mathfrak{g} \mid |X| < \varepsilon\}$ is an open neighbourhood of $0 \in \mathfrak{g}$, with $\varepsilon > 0$ such that the restriction of $\exp : \mathfrak{g} \to G$ to $U$ is a diffeomorphism onto an open subset $V$ of

$\mathbf{T}$. The inverse $\exp^{-1} : V \to U$ is a coordinate chart for $\mathbf{T}$ around the identity $e$. The composition $\exp^{-1} \circ L_{g^{-1}} : gV \to U$ is a coordinate chart around an arbitrary point $g \in \mathbf{T}$. As $\mathbf{T}$ is abelian, $(T_X \exp)Y = T_e L_{\exp(X)} Y$. Consequently, the tangent map of the composition $L_g \circ \exp$ is

$$T_X(L_g \circ \exp) = (T_{\exp X} L_g) \circ T_e L_{\exp(X)} = T_e L_{g \exp(X)}. \tag{3.2}$$

If the tangent space of each point in $\mathbf{T}$ is identified with $\mathfrak{t}$ through left-translation, then the tangent map of $L_g \circ \exp$ as a map $\mathfrak{t} \to \mathfrak{t}$ is just the identity map. Since $w \circ L_g = L_{wg} w$, we obtain $(T_g w) \circ (T_e L_g) = (T_e L_{wg}) \circ (T_e w)$, so that the action of $W(G, \mathbf{T})$ on $\mathbf{T} \times \mathfrak{t}$ is

$$w(g, Y) = (wg, wY), \quad (w \in W(G, \mathbf{T}), (g, Y) \in \mathbf{T} \times \mathfrak{t}), \tag{3.3}$$

when $T\mathbf{T}$ is identified with $T \times \mathfrak{t}$ through left-translation.

### 3.2.2 Weyl group action on $T^*\mathbf{T}$

Let us apply the results of Section 2.1 to the compact connected Lie group $\mathbf{T}$ to endow $\mathbf{T} \times \mathfrak{t}$ with a Kähler structure. The cotangent bundle $T^*\mathbf{T}$ can be identified with $\mathbf{T} \times \mathfrak{t}$ by first left-trivialising the cotangent bundle and then applying the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ induced by the $W(G, \mathbf{T})$-invariant inner product $\langle \cdot, \cdot \rangle$. Note that the restriction of any $\operatorname{Ad} G$-invariant product on $\mathfrak{g}$ to $\mathfrak{t}$ is automatically $W(G, \mathbf{T})$-invariant. The tangent spaces of $\mathbf{T} \times \mathfrak{t}$ are in turn identified with $\mathfrak{t} \times \mathfrak{t}$, again by left-trivialising the tangent bundle of $\mathbf{T}$.

The action of $W(G, \mathbf{T})$ on $T^*\mathbf{T}$ on $\mathbf{T} \times \mathfrak{t}$ under these identifications is precisely Equation (3.3). Furthermore, the canonical symplectic structure on $\mathbf{T} \times \mathfrak{t}$ is

$$\omega_{(g,Y)}((X_1, X_2); (Z_1, Z_2)) = \langle X_2, Z_1 \rangle - \langle X_1, Z_2 \rangle,$$

for all $(g, Y) \in \mathbf{T} \times \mathfrak{t}$, $X_1, X_2, Z_1, Z_2 \in \mathfrak{t}$. By the invariance of the inner product $\langle \cdot, \cdot \rangle$, the symplectic structure $\omega$ is also $W(G, \mathbf{T})$-invariant.

The complex structure $J$ on $\mathbf{T} \times \mathfrak{t}$ is equal to

$$J_{(g,Y)}(X_1, X_2) = (-X_2, X_1), \tag{3.4}$$

which is easily seen to be $W(G, \mathbf{T})$-invariant, too. From these formulae one sees immediately that $J$ and $\omega$ determine a $W(G, \mathbf{T})$-equivariant Kähler structure on $\mathbf{T} \times \mathfrak{t}$.

To carry out the analysis for the Dolbeault-Dirac operator, we construct a finite atlas for $\mathbf{T} \times \mathfrak{t}$ as follows. First of all, by Equation (3.2) and the remarks below that equation, the tangent map of the diffeomorphism $(\exp^{-1} \circ L_{g^{-1}}) \times \mathrm{id} : gV \times \mathfrak{t} \to U \times \mathfrak{t}$ is equal to the identity map $\mathfrak{t} \times \mathfrak{t} \to \mathfrak{t} \times \mathfrak{t}$. Next, choose an orthonormal basis $\{e_i\}$ for the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}$ and use this basis to obtain coordinates for $\mathfrak{t}$.

This yields a chart on $\mathbf{T} \times \mathfrak{t}$ with domain $gV \times \mathfrak{t}$, such that with respect to these coordinates the symplectic structure on $gV \times \mathfrak{t}$ takes the standard form

$$\omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Furthermore, the complex structure on $gV \times \mathfrak{t}$ is mapped to the standard one on $U \times \mathfrak{t}$. Thus, the charts $(\exp^{-1} \circ L_{g^{-1}}) \times \mathrm{id}$, $(g \in G)$ form an atlas of $\mathbf{T} \times \mathfrak{g}$, such that each chart identifies the Kähler structure on $\mathbf{T} \times \mathfrak{t}$ with the standard Kähler structure on $\mathbb{R}^{2n}$.[1] Consequently, the Dolbeault-operator on $gV \times \mathfrak{t}$ corresponds to the ordinary Dolbeault-Dirac operator on $U \times \mathfrak{t}$, at least as far as forms with compact support contained in $gV \times \mathfrak{t}$ are concerned. Since $\mathbf{T}$ is compact, $\mathbf{T} \times \mathfrak{t}$ can be covered by a finite number of such charts.

### 3.2.3 The stratification of $T^*\mathbf{T}$

We now want to analyse the stratification of $\mathbf{T} \times \mathfrak{t}$. We start by proving a result for general proper actions of discrete groups on manifolds.

**Lemma 3.2.2.** *If $\Gamma$ is a discrete group acting properly on a manifold $M$, then $M^x_{(\Gamma_x)} = M^x_{\Gamma_x}$ for each $x \in M$. Consequently, the strata of $M$ are the connected components of $M_H$. In particular, if $M$ is symplectic and $\Gamma$ acts symplectically, then all lower-dimensional strata have even codimension $\geq 2$.*

*Proof.* Being a connected component of $M_{\Gamma_x}$, $M^x_{\Gamma_x}$ is closed in $M_{\Gamma_x}$; since $M_{\Gamma_x}$ is closed in $M_{(\Gamma_x)}$, the subset $M^x_{\Gamma_x}$ is closed in $M_{(\Gamma_x)}$, too. In particular, $M^x_{\Gamma_x}$ is closed in $M^x_{(\Gamma_x)}$. Moreover, $M^x_{\Gamma_x}$ and $M^x_{(\Gamma_x)}$ are both embedded submanifolds of $M$, so that $M^x_{\Gamma_x}$ is an embedded submanifold of $M^x_{(\Gamma_x)}$. By discreteness of the group $\Gamma$, the dimensions of $M^x_{\Gamma_x}$ and $M^x_{(\Gamma_x)}$ are equal, so that $M^x_{\Gamma_x}$ is an open, embedded submanifold of $M^x_{(\Gamma_x)}$. Thus, $M^x_{\Gamma_x}$ is a non-empty open and closed subset of the connected set $M^x_{(\Gamma_x)}$, and therefore $M^x_{\Gamma_x} = M^x_{(\Gamma_x)}$. □

In general, if a group $G$ acts properly on two manifolds $M_1$ and $M_2$, then the diagonal action of $G$ on $M_1 \times M_2$ is also proper. Since $g(x_1, x_2) = (gx_1, gx_2) = (x_1, x_2)$ if and only if $gx_1 = x_1$ and $gx_2 = x_2$, the isotropy group of $(x_1, x_2)$ is $G_{x_1} \cap G_{x_2}$. Suppose that $G$ is discrete (or abelian) and that the partitions of $M_1$ and $M_2$ into connected components of isotropy type manifolds are both finite, i.e. $M_1$ and $M_2$ are covered by finitely many submanifolds of the form $(M_1)^{x_1}_{G_{x_1}}$ and $(M_2)^{x_2}_{G_{x_2}}$, respectively. Now, let $M^{(x_1,x_2)}_{G_{x_1} \cap G_{x_2}}$ be a stratum of $M_1 \times M_2$. Then

$$(M_1)^{x_1}_{G_{x_1}} \times (M_2)^{x_2}_{G_{x_2}} \subset M^{(x_1,x_2)}_{G_{x_1} \cap G_{x_2}}.$$

[1] This also proves that $\mathbf{T} \times \mathfrak{t}$ is a Kähler manifold.

Since

$$\{(M_1)^{x_1}_{G_{x_1}} \times (M_2)^{x_2}_{G_{x_2}} \mid x_1 \in X_1, x_2 \in X_2\}$$

is already a finite partition of $M_1 \times M_2$, the partition $\{M^{(x_1,x_2)}_{G_{x_1} \cap G_{x_2}} \mid x_1 \in X_1, x_2 \in X_2\}$ of $M_1 \times M_2$ is also finite.

**Proposition 3.2.3.** *The stratification of $\boldsymbol{T} \times \mathfrak{t}$ that is determined by the action of $W(G, \boldsymbol{T})$, is finite, i.e. there are only finitely many strata.*

*Proof.* According to Lemma 3.2.2, the strata of $\mathbf{T} \times \mathfrak{t}$ are given by the connected components of the isotropy type manifolds. Both the $W(G, \mathbf{T})$-action on $\mathbf{T}$ and the $W(G, \mathbf{T})$-action on $\mathfrak{t}$ only have finitely many connected components of isotropy type manifolds, the first because $\mathbf{T}$ is compact, the second because $W(G, \mathbf{T})$ acts linearly on $\mathfrak{t}$. By the remarks preceding this proposition, also the stratification of $\mathbf{T} \times \mathfrak{t}$ into connected components of isotropy type manifolds is finite. □

**Example 3.2.4.** Suppose $G = SU(N)$ with $N \geq 2$. A maximal torus $\mathbf{T}$ in $SU(N)$ is given by the subgroup of diagonal matrices in $SU(N)$, i.e. all matrices of the form $\mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_N})$ such that $\Pi_{k=1}^N e^{i\phi_k} = 1$. The corresponding Lie algebra $\mathfrak{t}$ consists of all matrices of the form $\mathrm{diag}(i\lambda_1, \ldots, i\lambda_N)$, $(\lambda_k \in \mathbb{R})$ such that $\sum_k \lambda_k = 0$. The Weyl group is isomorphic to the permutation group $S_N$ and it acts on $\mathbf{T} \times \mathfrak{t}$ by permuting the elements on the diagonal.

The isotropy subgroup of an element $(\mathrm{diag}(e^{i\phi_k}), \mathrm{diag}(\lambda_k))$ is generated by all permutations $\sigma_{k_1,k_2}$ such that $e^{i\phi_{k_1}} = e^{i\phi_{k_2}}$ *and* $\lambda_{k_1} = \lambda_{k_2}$. So, if $\sigma_{k_1,k_2}$ is in some isotropy group $W_1 \neq \{e\}$ and $(g, Y) \in (\mathbf{T} \times \mathfrak{t})_{W_1}$, then for both $g$ and $Y$ the $(k_1, k_1)$-entry equals the $(k_2, k_2)$-entry. Consequently, the codimensions of the singular strata are at least 2. Note that an element $(\mathrm{diag}(e^{i\phi_k}), \mathrm{diag}(\lambda_k))$ is in the principal stratum if and only if $\lambda_{k_1} \neq \lambda_{k_2}$ when $e^{\phi_{k_1}} = e^{\phi_{k_2}}$, $(k_1 \neq k_2)$. The principal stratum is therefore bigger than the product of the principal strata on $\mathbf{T}$ and $\mathfrak{t}$, which is equal to

$$\left\{(\mathrm{diag}(e^{i\phi_k}), \mathrm{diag}(i\lambda_k)) \mid e^{i\phi_{k_1}} \neq e^{i\phi_{k_2}} \text{ and } \lambda_{k_1} \neq \lambda_{k_2} \text{ if } k_1 \neq k_2\right\}.$$

If $N = 2$, then $\mathbf{T} \times \mathfrak{t}$ is a cylinder $\mathbb{T} \times \mathbb{R}$ with two singular points $(\pm e, 0)$.

It follows from Proposition 3.2.1 that the stratum $(\mathbf{T} \times \mathfrak{t})_{\{e\}}$ is open and dense in $\mathbf{T} \times \mathfrak{t}$: it is the principal stratum $(\mathbf{T} \times \mathfrak{t})_{princ}$ of the $W(G, \mathbf{T})$-action on $\mathbf{T} \times \mathfrak{t}$. We now show that $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ with respect to the graph norm of the Dolbeault-Dirac operator.

**Remark 3.2.5.** By Lemma 3.2.2 and Proposition 3.2.3, $\mathbf{T} \times \mathfrak{t} \setminus (\mathbf{T} \times \mathfrak{t})_{princ}$ is a finite union of submanifolds, each of which has codimension $\geq 2$. We will apply Theorem 3.1.7 to show that $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \Lambda^{(0,\bullet)}T^*M)$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \Lambda^{(0,\bullet)}T^*M)$ with respect to the graph norm of the Dolbeault-Dirac operator.

However, in the proofs we need to work with closed submanifolds. Therefore, we shift our focus to the submanifolds $(\mathbf{T}\times\mathfrak{t})_x^{W_1}$ ($x\in\mathbf{T}\times\mathfrak{t}$, $W_1\subset W(G,\mathbf{T})$), which are *closed* symplectic submanifolds. Lemma 3.2.2 and Proposition 3.2.3 are not needed in the upcoming analysis. They are only proved here for later reference.

The complement of $(\mathbf{T}\times\mathfrak{t})_{princ}$ is equal to

$$P=\bigcup_{W_1\neq\{e\}\text{ subgroup of }W(G,\mathbf{T})}(\mathbf{T}\times\mathfrak{t})^{W_1},$$

where the union is finite because $W(G,\mathbf{T})$ is a finite group. Let $W_1\subset W(G,\mathbf{T})$ be a subgroup. The space $\mathbf{T}^{W_1}$ is a closed Lie subgroup of $\mathbf{T}$ with Lie algebra equal to $\mathfrak{t}^{W_1}$. Indeed, it is clear that $\mathbf{T}^{W_1}$ is a closed subgroup. Now, let $V\subset\mathbf{T}$ and $U\subset\mathfrak{t}$ be as in Section 3.2.1 and suppose that $g\in\mathbf{T}^{W_1}$. Because $W(G,\mathbf{T})$ acts on $\mathbf{T}$ by homomorphisms, the open subset $gV$ is $W_1$-invariant and $L_g^{-1}$ maps $\mathbf{T}^{W_1}\cap gV=(gV)^{W_1}$ onto $V^{W_1}$, which is diffeomorphic to $U^{W_1}\subset U$. Consequently, $\mathbf{T}^{W_1}$ is a submanifold of $\mathbf{T}$ and so $\mathbf{T}^{W_1}$ is a closed Lie subgroup.

Note that, since the action of $W(\mathbf{T},G)$ is diagonal, $(g,Y)\in(\mathbf{T}\times\mathfrak{t})^{W_1}$ if and only if $g\in\mathbf{T}^{W_1}$ and $Y\in\mathfrak{t}^{W_1}$. Therefore, $(\mathbf{T}\times\mathfrak{t})^{W_1}=\mathbf{T}^{W_1}\times\mathfrak{t}^{W_1}$ is a submanifold of $\mathbf{T}\times\mathfrak{t}$. If $W_1\neq\{e\}$, the dimension of $(\mathbf{T}\times\mathfrak{t})^{W_1}$ is smaller than the dimension of $\mathbf{T}\times\mathfrak{t}$. Because $(\mathbf{T}\times\mathfrak{t})^{W_1}$ is symplectic for each subgroup $W_1$, the complement $P$ of $(\mathbf{T}\times\mathfrak{t})_{princ}$ is a finite union of closed embedded submanifolds, each of which has codimension $\geq 2$.

**Remark 3.2.6.** To simplify notation in the proofs, unless specified otherwise, the symbol $\mathbb{E}$ is used to denote the bundle $\Lambda^{(0,\bullet)}T^*M$ and the letter $D$ denotes the Dolbeault-Dirac operator, no matter what the underlying manifold is. Which manifold is meant, should be clear from the context. For instance, we always mention the manifold when we consider the space of sections $\Gamma_c^\infty(M,\mathbb{E})$. Moreover, if $O\subset M$ is an open submanifold, then the notation $\Gamma_c^\infty(O,\mathbb{E})$ is used to denote the space of all sections $M\to\mathbb{E}$ that have their support in $O$. This does not contradict the notation of $\Gamma_c^\infty(O,\mathbb{E})$ being the space of sections $O\to\mathbb{E}$, where $\mathbb{E}$ is now considered as a bundle over $O$, since both spaces can be identified (using extension by zero). If confusion about the underlying manifold can arise, the Dolbeault-operator on $M$ is denoted by $D_M$. Recall that the domain of $D_M$ is always supposed to be $\Gamma_c^\infty(M,\mathbb{E})$.

The following Lemma will be used in Proposition 3.2.8.

**Lemma 3.2.7.** *Suppose that $(s_m)_m$ is a sequence in $\Gamma_c^\infty(M,E)$ such that $s_m\to s$ in $\Gamma_c^\infty(M,E)$ with respect to the graph norm of a first-order differential operator $D$ on a hermitian vector bundle $E$, and suppose that $\psi:M\to[0,1]$ is a compactly supported smooth function such that $\psi\equiv 1$ on supp $s$. Then $\psi s_m\to s$ with respect to the graph norm of $D$.*

*Proof.* We verify that

$$\begin{aligned}\|s-\psi s_m\|_D^2 &= \|\psi(s-s_m)\|_D^2 = \|\psi(s-s_m)\|^2 + \|D(\psi(s-s_m))\|^2 \\ &\le \|s-s_m\|^2 + \|[D,\psi](s-s_m)+\psi D(s-s_m)\|^2 \\ &\le \|s-s_m\|^2 + \left(\|[D,\psi]\|\|s-s_m\| + \|D(s-s_m)\|\right)^2,\end{aligned}$$

which goes to zero as $m\to\infty$, because $[D,\psi]$ is a bounded operator and $s_m\to s$ with respect to $\|\cdot\|_D$. □

Denote the dimension of $\mathbf{T}$ by $n$.

**Proposition 3.2.8.** *Let $W(G,\boldsymbol{T})$ be the action of the Weyl group on $\boldsymbol{T}\times\mathfrak{t}$ and let $W_1\neq\{e\}$ be a subgroup of $W(G,\boldsymbol{T})$. Denote the Dolbeault-Dirac operator on $\boldsymbol{T}\times\mathfrak{t}$ by $D$. Then $\Gamma_c^\infty(\boldsymbol{T}\times\mathfrak{t}\setminus(\boldsymbol{T}\times\mathfrak{t})^{W_1},\mathbb{E})$ is dense in $\Gamma_c^\infty(\boldsymbol{T}\times\mathfrak{t},\mathbb{E})$ with respect to the graph norm of $D$.*

*Proof.* Because $\mathbf{T}^{W_1}$ is compact, one can choose finitely many $g_l\in\mathbf{T}^{W_1}$ such that $\{g_lV\times\mathfrak{t}\}$ covers $(\mathbf{T}\times\mathfrak{t})^{W_1}=\mathbf{T}^{W_1}\times\mathfrak{t}^{W_1}$. The set $\mathcal{U}:=\{g_lV\times\mathfrak{t}\}\cup\{\mathbf{T}\times\mathfrak{t}\setminus(\mathbf{T}\times\mathfrak{t})^{W_1}\}$ is a finite open cover of $\mathbf{T}\times\mathfrak{t}$.

Consider the diffeomorphism

$$(g_lV\times\mathfrak{t})^{W_1}\xrightarrow{\cong}U^{W_1}\times\mathfrak{t}^{W_1} \tag{3.5}$$

obtained by restricting the chart $(\exp^{-1}\circ L_{g_l^{-1}})\times\mathrm{id}: g_lV\times\mathfrak{t}\to U\times\mathfrak{t}$ for some fixed $l$. Set $O:=U\times\mathfrak{t}$. Pick an orthonormal basis for the inner product on $\mathfrak{t}$ such that the first $n-k$ (where $k\geq 1$) basis vectors span the subspace $\mathfrak{t}^{W_1}$. Then the subset $(g_lV\times\mathfrak{t})^{W_1}$ is described by the first $n-k$ 'space' coordinates, and the first $n-k$ 'momenta' coordinates under the chart of Equation (3.5). (This also provides a direct proof of the fact that the strata are symplectic.) After reordering the coordinates, we obtain a coordinate chart for $g_lV\times\mathfrak{t}$ such that $(g_lV\times\mathfrak{t})^{W_1}$ is mapped onto $\{(x_1,\dots x_{2n})\in O\mid x_{2n-2k+1}=\cdots=x_{2n}=0\}=O^{W_1}$.

Let $f\in\Gamma_c^\infty(O,\mathbb{E})$. When $O$ is considered an open subset of $\mathbb{R}^{2n}$, $f$ can be extended (by zero) to a section in $\Gamma_c^\infty(\mathbb{R}^{2n},\mathbb{E})$. Recall that the chart $(\exp^{-1}\circ L_{g^{-1}})\times$ id maps the Kähler structure on $\mathbf{T}\times\mathfrak{t}$ to the standard Kähler structure on $U\times\mathfrak{t}$. Therefore, the operator $D$ on $\Gamma_c^\infty(O,\mathbb{E})$ is just the restriction of the ordinary Dolbeault-Dirac operator $\widetilde{D}$ on $\mathbb{R}^{2n}$ to $O$. Note that for compactly supported sections on $O$ the graph norm with respect to $D$ is the same as the graph norm with respect to $\widetilde{D}$, as the operator $\widetilde{D}$ is local.

By Theorem 3.1.7, a section $s\in\Gamma_c^\infty(O,\mathbb{E})$ can be approximated in the graph norm of $\widetilde{D}$ by a sequence $(s_m)_m\in\Gamma_c^\infty(\mathbb{R}^{2n}\setminus\mathbb{R}^{2n-2k},\mathbb{E})$. Let $\psi:\mathbb{R}^{2n}\to[0,1]$ be a smooth function with compact support contained in $O$ such that $\psi\equiv 1$ on supp $s$. By Lemma 3.2.7, $\psi s_m\to s$ in the graph norm of $\widetilde{D}$. But

$$\mathrm{supp}(\psi s_m)\subset\mathrm{supp}(\psi)\cap\mathrm{supp}(s_m)\subset O\cap(\mathbb{R}^{2n}\setminus\mathbb{R}^{2n-2k}),$$

so that $\psi s_m \in \Gamma_c^\infty(O \setminus O^{W_1}, \mathbb{E})$ and $\psi s_m \to s$ in the graph norm of $D$. We have now proved that $\Gamma_c^\infty(g_l V \times \mathfrak{t} \setminus (g_l V \times \mathfrak{t})^{W_1}, \mathbb{E})$ is dense in $\Gamma_c^\infty(g_l V \times \mathfrak{t}, \mathbb{E})$ for arbitrary $l$.

Suppose now that $s \in \Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$. Let $\{\psi_{U_i}\}_{U_i \in \mathcal{U}}$ be a partition of unity subordinate to the finite cover $\mathcal{U}$. The supports of the $\psi_{U_i}$'s are not compact, but the support of each $\psi_{U_i} s$ is, as $\operatorname{supp}(\psi_{U_i} s) \subset \operatorname{supp}(\psi_{U_i}) \cap \operatorname{supp}(s)$ is a closed subset of the compact set $\operatorname{supp}(s)$. Moreover, $\operatorname{supp}(\psi_{U_i} s)$ is contained in $U_i$. By the previous paragraphs, each $\psi_{U_i} s$ can be approximated in the graph norm of $D$ by a sequence $(s_{m,i})_m \in \Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ such that $\operatorname{supp}(s_{m,i}) \subset U_i \setminus (U_i \cap (\mathbf{T} \times \mathfrak{t})^{W_1})$.

Then $\sum_i s_{m,i} \to \sum_i \psi_{U_i} s = s$. Since the sum is finite, $\operatorname{supp}\left(\sum_i s_{m,i}\right)$ is compact and

$$\operatorname{supp}\,(\sum_i s_{m,i}) \subset \cup_i (U_i \setminus (U_i \cap (\mathbf{T} \times \mathfrak{t})^{W_1})) = (\mathbf{T} \times \mathfrak{t}) \setminus (\mathbf{T} \times \mathfrak{t})^{W_1}$$

for each $m$. Thus, $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t}) \setminus (\mathbf{T} \times \mathfrak{t})^{W_1}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$. □

The following lemma will be applied to $P$ to prove that $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ with respect to the graph norm of $D$.

**Lemma 3.2.9.** *Let $M$ be an oriented Riemannian manifold and let $D$ be a first-order differential operator on a hermitian vector bundle $E$ over $M$. Let $A_{1,2}$ be two closed subsets of $M$ such that $\Gamma_c^\infty(M \setminus A_i, E)$, ($i = 1, 2$), is dense in $\Gamma_c^\infty(M, E)$ with respect to the graph norm of $D$. Then $\Gamma_c^\infty(M \setminus (A_1 \cup A_2), E)$ is dense in $\Gamma_c^\infty(M, E)$ in this graph norm as well.*

*Proof.* It suffices to show that $\Gamma_c^\infty(M \setminus (A_1 \cup A_2), E)$ is dense in $\Gamma_c^\infty(M \setminus A_1, E)$. Let $s \in \Gamma_c^\infty(M \setminus A_1, E)$ be given. Since $\Gamma_c^\infty(M \setminus A_2, E)$ is dense in $\Gamma_c^\infty(M, E)$ with respect to the graph norm of $D$, there exists a sequence $s_m \in \Gamma_c^\infty(M \setminus A_2, E)$ such that $s_m \to s$ in this norm. Now, let $\psi : M \to [0, 1]$ be a function with compact support contained in $M \setminus A_1$ and which is equal to 1 on supp $s$. By Lemma 3.2.7 $\psi s_m \to s$ in the graph norm of $D$. But

$$\operatorname{supp}\,(\psi s_m) \subset \operatorname{supp}\,\psi \cap \operatorname{supp}\,s_m \subset (M \setminus A_1) \cap (M \setminus A_2) = M \setminus (A_1 \cup A_2)$$

for each $m$. □

**Proposition 3.2.10.** *Let $(\boldsymbol{T} \times \mathfrak{t})_{princ}$ be the principal stratum of $\boldsymbol{T} \times \mathfrak{t}$. Then $\Gamma_c^\infty((\boldsymbol{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\boldsymbol{T} \times \mathfrak{t}, \mathbb{E})$ with respect to the graph norm of $D$. Moreover, $D$ is essentially self-adjoint on the domain $\Gamma_c^\infty((\boldsymbol{T} \times \mathfrak{t})_{princ}, \mathbb{E})$*

*Proof.* The union defining $P$ is finite. Proposition 3.2.8 and Lemma 3.2.9 now imply that $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t}) \setminus P, \mathbb{E}) = \Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ in the graph norm of $D$.

Concerning essential self-adjointness, first note that $D$ is essentially self-adjoint on $\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$ as $\mathbf{T}\times\mathfrak{t}$ is geodesically complete and $D$ has finite propagation speed (see [42, Proposition 10.2.11]). Since $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$ with respect to the graph norm of $D$, $D$ is also essentially self-adjoint on $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$. □

Proposition 3.2.10 deals with the ordinary Dolbeault-Dirac operator on $\mathbf{T}\times\mathfrak{t}$. However, quantisation is defined in terms of the *twisted* Dolbeault-Dirac operator. Recall that for cotangent bundles the twisting line bundle is $L$ is the trivial hermitian line bundle with hermitian connection $\nabla^L = d+2\pi i\theta$, where $\theta$ is the fundamental 1-form. Therefore, the twisted Dolbeault-Dirac operator differs from the untwisted one by a zeroth-order differential operator only. The following theorem is precisely Proposition 3.2.10 for the twisted Dolbeault-Dirac operator $D^L$.

**Proposition 3.2.11.** *The domain $\Gamma_c^\infty((\boldsymbol{T}\times\mathfrak{t})_{princ},\mathbb{E}\otimes L)$ is dense in $\Gamma_c^\infty(\boldsymbol{T}\times\mathfrak{t},\mathbb{E}\otimes L)$ in the graph norm of $D^L$, with $L$ and $D^L$ as above. In particular, the twisted Dolbeault-Dirac operator $D^L$ is essentially self-adjoint on the domain $\Gamma_c^\infty((\boldsymbol{T}\times\mathfrak{t})_{princ},\mathbb{E}\otimes L)$.*

*Proof.* Since $L$ is trivial with the standard hermitian structure, one can identify $\mathbb{E}\otimes L$ with $\mathbb{E}$ as hermitian vector bundles. As differential operators on $\mathbb{E}$ the difference $D^L - D$ is of order zero. In particular, $D^L$ has the same principal symbol as $D$, so that $D^L$ still has finite propagation speed and is therefore still essentially self-adjoint on $\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$. We show that $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$ in the graph norm of $D^L$. For simplicity we denote $D^L = D+B$, where $B$ is an element of $\Gamma^\infty(\mathbf{T}\times\mathfrak{t},\mathrm{End}(\mathbb{E}))$.

Suppose that $s\in\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$. By Proposition 3.2.10 there exists a sequence $(s_m)_m$ in $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$ such that $s_m\to s$ in the graph norm of $D$. According to Lemma 3.2.7 one can assume that there exists a compact set $K$ such that $(\cup_m \mathrm{supp}\ s_m)\cup \mathrm{supp}\ s\subset K$. Consequently,

$$\begin{aligned}\|(D+B)(s_m-s)\| &\le \|D(s_m-s)\| + \|B(s_m-s)\|\\ &\le \|D(s_m-s)\| + \sup_{x\in K}\{\|B(x)\|\}\|s_m-s\|,\end{aligned}$$

which approaches 0 as $m$ goes to infinity. So $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T}\times\mathfrak{t},\mathbb{E})$ in the graph norm of $D^L = D+B$. □

Proposition 3.2.11 implies that the Dolbeault-Dirac quantisation of $(T\times\mathfrak{t})_{princ}$ can be naturally identified with the Dolbeault-Dirac quantisation of $T\times\mathfrak{t}$. Let $D^L_{princ}$ denote the Dolbeault-Dirac operator on $(\mathbf{T}\times\mathfrak{t})_{princ}$ (on the domain $\Gamma_c^\infty((\mathbf{T}\times\mathfrak{t})_{princ},\mathbb{E})$).

**Theorem 3.2.12.** *The Dolbeault-Dirac quantisations of $(\boldsymbol{T}\times\mathfrak{t})_{princ}$ and $\boldsymbol{T}\times\mathfrak{t}$ are the same.*

*Proof.* Consider the embedding

$$u : \Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E} \otimes L) \to \Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E} \otimes L).$$

By Proposition 3.2.11, $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ with respect to the $D^L$-graph norm. Since this graph norm dominates the $L^2$-norm, $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E})$ is dense in $L^2(\mathbf{T} \times \mathfrak{t}, \mathbb{E})$ and therefore the isometry $u$ extends to a unitary isomorphism

$$u : L^2((\mathbf{T} \times \mathfrak{t})_{princ}, \mathbb{E}) \to L^2(\mathbf{T} \times \mathfrak{t}, \mathbb{E}).$$

This isomorphism identifies the closures of $D^L_{princ}$ and $D^L$ by another application of Proposition 3.2.11. In particular, the kernels of these closures are isomorphic. □

## 3.3 The Dolbeault-Dirac operator on the quotient

In the previous section we considered the action of the Weyl group $W(G, \mathbf{T})$ on the cotangent bundle $T^*\mathbf{T}$ of a maximal torus $\mathbf{T}$ in a compact connected Lie group $G$. We have shown that the Dolbeault-Dirac quantisation of $(\mathbf{T} \times \mathfrak{t})_{princ}$ is equal to the Dolbeault-Dirac quantisation of $\mathbf{T} \times \mathfrak{t}$. In this section we show that quantisation commutes with reduction for the Weyl group action on $\mathbf{T} \times \mathfrak{t}$ if the quantisation of the singular quotient $(\mathbf{T} \times \mathfrak{t})/W(G, \mathbf{T})$ is defined as the Dolbeault-Dirac quantisation of the principal stratum $(\mathbf{T} \times \mathfrak{t})_{princ}/W(G, \mathbf{T})$. Since the Dolbeault-Dirac quantisation of $(\mathbf{T} \times \mathfrak{t})_{princ}$ is equal to the Dolbeault-Dirac quantisation of $\mathbf{T} \times \mathfrak{t}$ by Theorem 3.2.12, we can restrict ourselves to the principal stratum $(\mathbf{T} \times \mathfrak{t})_{princ}$, on which $W(G, \mathbf{T})$ acts freely, and prove that quantisation commutes with reduction for any finite group acting symplectically and freely on a pre-quantisable symplectic manifold that carries a compatible, invariant, almost complex structure.

The proof of the following Lemma is straightforward.

**Lemma 3.3.1.** *Let $D : Dom\ D \to \mathcal{H}$ be a closable operator, and denote the closure of $D$ by $\overline{D}$. Let $p \in \mathcal{B}(\mathcal{H})$ be a projection such that $p(Dom\ D) \subset Dom\ D$ and $pD = Dp$. Then $D$ restricts to a densely defined closable operator $D_p : p(Dom\ D) \to p\mathcal{H}$ on $p\mathcal{H}$.*

*Moreover, $p\overline{D} = \overline{D}p$ on $\mathcal{H}$ and $cl(D_p) = \overline{D}|_{p\mathcal{H}}$ on $p\mathcal{H}$, where cl denotes 'closure'. If $D$ is essentially self-adjoint, then so is $D_p$.*

If a Lie group acts on an oriented Riemannian manifold, we always assume that the action preserves the metric as well as the orientation.

**Proposition 3.3.2.** *Let $\Gamma$ be a* finite *group acting on an arbitrary oriented Riemannian manifold $M$. Suppose that $D$ is a symmetric $\Gamma$-invariant differential operator on a $\Gamma$-equivariant hermitian vector bundle $E$. Then $\overline{D}|_{L^2(M,E)^\Gamma} =$*

*$cl(D|_{L^2(M,E)^\Gamma})$. Moreover, if $D$ is essentially self-adjoint on $\Gamma_c^\infty(M,E)$, then $D|_{L^2(M,E)^\Gamma}$ is essentially self-adjoint on $\Gamma_c^\infty(M,E)^\Gamma$.*

*Proof.* The inclusion of $L^2(M,E)^\Gamma$ into $L^2(M,E)$ has a left-inverse $p$, given by

$$ps(x) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g(s(g^{-1}x)), \quad (x \in M).$$

The map $p$ is easily verified to be a projection onto $L^2(M,E)^\Gamma$. Since $D$ is assumed to be $\Gamma$-invariant, it commutes with the projection $p$. Furthermore, $p(\Gamma_c^\infty(M,E)) = \Gamma_c^\infty(M,E)^\Gamma$. Now apply Lemma 3.3.1. □

Suppose that $\Gamma$ is a finite group acting *freely* on $M$ and let $\pi : M \to M/\Gamma$ be the quotient map. Let $E$ be a $\Gamma$-equivariant hermitian vector bundle over $M$. It can be verified that $E/\Gamma \to M/\Gamma$ with projection map $[e_x] \mapsto [x]$, is again a hermitian vector bundle with hermitian structure

$$\langle [e_1],[e_2]\rangle_{[x]} = \langle \tau_x^{-1}[e_1], \tau_x^{-1}[e_2]\rangle_x, \quad ([e_1],[e_2] \in (E/\Gamma)_{[x]}),$$

which is independent of the choice of the representative $x$ of $[x]$, since $\Gamma$ preserves the hermitian structure. Here, we have used that, by freeness of the action of $\Gamma$ on $M$, the quotient $E \to E/\Gamma$ restricts to an isomorphism $\tau_x := E_x \to E_{[x]}$ for each $x \in M$.

The bundle $E$ is naturally isomorphic to the pull-back bundle $\pi^*(E/\Gamma)$ through the isomorphism

$$E \ni e \mapsto (\pi(e),[e]) \in M \times_{M/\Gamma} E/\Gamma = \pi^*(E/\Gamma),$$

with inverse

$$M \times_{M/\Gamma} E/\Gamma \ni (x,[e]) \mapsto \tau_x^{-1}[e] \in E_x.$$

This isomorphism of vector bundles is an isomorphism of $\Gamma$-*equivariant* vector bundles if $\Gamma$ acts on $\pi^*(E/\Gamma)$ as $g(x,[e]) = (gx,[e])$. It is an isomorphism of equivariant hermitian vector bundles over $M$ if $\pi^*(E/\Gamma)$ is endowed with the hermitian structure

$$\langle (x,[e_1]),(x,[e_2])\rangle = \langle [e_1],[e_2]\rangle \quad ([e_1],[e_2] \in (E/\Gamma)_{[x]}).$$

The map $\Gamma^\infty(M,E)^\Gamma \to \Gamma^\infty(M/\Gamma, E/\Gamma)$ given by

$$s^\Gamma \mapsto \widetilde{s}, \quad \widetilde{s}([x]) = [s^\Gamma(x)], \quad (x \in M),$$

is well defined and defines an isomorphism of $C^\infty(M)^\Gamma$-modules: the section $s^\Gamma \in \Gamma^\infty(M,E)^\Gamma$ is obtained from $\widetilde{s}$ by the equation

$$s^\Gamma(x) = \tau_x^{-1}\widetilde{s}([x]), \quad (x \in M).$$

Under the isomorphism $E \cong \pi^*(E/\Gamma)$, the section $s^\Gamma$ corresponds to the pull-back section $\pi^*\widetilde{s}$. By finiteness of the group $\Gamma$, the correspondence $s^\Gamma \leftrightarrow \widetilde{s}$ identifies $\Gamma_c^\infty(M,E)^\Gamma$ with $\Gamma_c^\infty(M/\Gamma, E/\Gamma)$.

If $E$ is a vector bundle, then let $T^kE$ denote the $k$-th tensor power of $E$. Let $\sigma^\Gamma : \Gamma^\infty(T^k(E)) \to \Gamma^\infty(T^l(E))$ be a $\Gamma$-equivariant map. As the map $E_x \to E_{[x]}$ is surjective, one can define

$$\widetilde{\sigma}_{[x]}(\tilde{e}_1, \ldots, \tilde{e}_k) = \left[\sigma_x\left(\tau_x^{-1}\tilde{e}_1, \ldots, \tau_x^{-1}\tilde{e}_k\right)\ \right], \tag{3.6}$$

for all $\tilde{e}_1, \ldots, \tilde{e}_k \in E_{[x]}$. This expression is well defined by the $G$-invariance of $\sigma$. Indeed, if one replaces $x$ by $gx$, then

$$\begin{aligned}\widetilde{\sigma}_{[gx]}(\tilde{e}_1, \ldots, \tilde{e}_k) &= \left[\sigma_{gx}\left(\tau_{gx}^{-1}\tilde{e}_1, \ldots, \tau_{gx}^{-1}\tilde{e}_k\right)\ \right] = \left[\sigma_{gx}\left(g\tau_x^{-1}\tilde{e}_1, \ldots, g\tau_x^{-1}\tilde{e}_k\right)\right] \\ &= \left[g\left(\sigma_x\left(\tau_x^{-1}\tilde{e}_1, \ldots, \tau_x^{-1}\tilde{e}_k\right)\right)\right] = \left[\sigma_x\left(\tau_x^{-1}\tilde{e}_1, \ldots, \tau_x^{-1}\tilde{e}_k\right)\right].\end{aligned}$$

Suppose from now on that $\Gamma$ is a finite group acting symplectically and *freely* on a symplectic manifold $(M,\omega)$ that carries a compatible, invariant, almost complex structure and equivariant pre-quantisation $(L, \nabla^L)$. We apply the above discussion to the following examples.

**Example 3.3.3.** Consider the tangent bundle $E = TM$. The differential of the projection map $\pi : M \to M/\Gamma$ induces a natural homomorphism of vector bundles $TM/\Gamma \to T(M/\Gamma)$ over $M/\Gamma$. Smoothness of this map follows from smoothness of the differential $TM \to T(M/\Gamma)$ and from the fact that $TM \to TM/\Gamma$ is a surjective submersion. By finiteness of $\Gamma$, the map $TM/\Gamma \to T(M/\Gamma)$ is also bijective and it is therefore an isomorphism of vector bundles.

Under this identification of $TM/\Gamma$ with $T(M/\Gamma)$ we obtain

$$\tau_x^{-1} = (\pi_*^{-1})_x : T_{[x]}(M/\Gamma) \to T_xM$$

and so

$$(\pi^*\widetilde{\omega})_x(X_1, X_2) = \widetilde{\omega}_{\pi(x)}(\pi_*X_1, \pi_*X_2) = \omega(X_1, X_2), \quad (X_1, X_2 \in T_xM),$$

where $\omega$ is the symplectic form on $M$ and $\widetilde{\omega}$ is defined by Equation (3.6). Thus, $\widetilde{\omega}$ is the usual quotient symplectic form on $M/\Gamma$. Similarly, the Riemannian metric $g$ on $M$ determines a Riemannian metric $\widetilde{g}$ on $M/\Gamma$ and the almost complex structure $J$ determines an almost complex structure $\widetilde{J} : T(M/\Gamma) \to T(M/\Gamma)$. One can verify that $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = \widetilde{\omega}(\widetilde{J}X, \widetilde{Y})$ for all $\widetilde{X}, \widetilde{Y} \in T(M/\Gamma)$.

As $\Gamma$ is discrete, the map $\pi : M \to M/\Gamma$ is locally invertible. Since $\tau_x^{-1} = (\pi_*^{-1})_x$, it follows from Equation (3.6) that under these local diffeomorphisms the structures $\widetilde{\omega}, \widetilde{J}$ and $\widetilde{g}$ are identified with $\omega$, $J$, and $g$, respectively. In particular, if $J$ is a complex structure, so is $\widetilde{J}$.

**Example 3.3.4.** Another example of interest is the $\Gamma$-equivariant hermitian line bundle $L \to M$ with $\Gamma$-invariant hermitian connection $\nabla$ satisfying $(\nabla^L)^2 = 2\pi i\omega$. There exists a unique connection $\widetilde{\nabla}$ on $L/\Gamma$ such that

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{s} = \widetilde{\nabla_{X^\Gamma} s^\Gamma}, \quad \widetilde{s} \in \Gamma^\infty(M/\Gamma, L/\Gamma), \widetilde{X} \in \Gamma^\infty(M/\Gamma, T(M/\Gamma)).$$

Indeed, by equivariance of $\nabla^L$:

$$g(\nabla^L_{X^\Gamma} s^\Gamma) = \nabla^L_{gX^\Gamma}(gs^\Gamma) = \nabla^L_{X^\Gamma} s^\Gamma,$$

so that $\nabla^L_{X^\Gamma} s^\Gamma$ is a $\Gamma$-invariant section and $\widetilde{\nabla_{X^\Gamma} s^\Gamma}$ is well defined. Note that $\pi^*\widetilde{\nabla} = \nabla^L$ when $L$ is identified with $\pi^*(L/\Gamma)$. We also write $\nabla^{L/\Gamma}$ for $\widetilde{\nabla}$.

Because $\pi : M \to M/\Gamma$ is a local diffeomorphism and $\pi^*(L/\Gamma) = L$ as $\Gamma$-equivariant hermitian vector bundles, as well as $\pi^*(\nabla^{L/\Gamma}) = \nabla^L$, it follows that $\nabla^{L/\Gamma}$ is hermitian and satisfies $(\nabla^{L/\Gamma})^2 = 2\pi i\widetilde{\omega}$.

We now show that the bijective map

$$\Gamma^\infty_c(M, \mathbb{E} \otimes L)^\Gamma \to \Gamma^\infty_c(M/\Gamma, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma), \quad s^\Gamma \mapsto \widetilde{s}$$

intertwines the twisted Dolbeault-Dirac operators in question. Here we have identified $(\mathbb{E} \otimes L)/\Gamma$ with $\mathbb{E}/\Gamma \otimes L/\Gamma \cong \mathbb{E}_{M/\Gamma} \otimes L/\Gamma$ in the natural way. As the Dolbeault-Dirac operator is a local operator, it is sufficient to show this locally. We therefore cover $M/\Gamma$ by open subsets $U$ such that $\pi^{-1}(U)$ is a disjoint union of $|\Gamma|$ open subsets $U_i$, each of which is mapped diffeomorphically onto $U$ under the projection map $\pi$. Let $U \subset M/\Gamma$ be an open subset with this property. On the one hand, we identify each $\Gamma^\infty_c(U_i, \mathbb{E} \otimes L)$ with $\Gamma^\infty_c(\pi^{-1}(U), \mathbb{E} \otimes L)^\Gamma$ by $\Gamma$-invariant extension. This identification intertwines the action of the twisted Dolbeault-Dirac operators on $\Gamma^\infty_c(U_i, \mathbb{E} \otimes L)$ and $\Gamma^\infty_c(\pi^{-1}(U), \mathbb{E} \otimes L)^\Gamma$. On the other hand, the diffeomorphism $\pi|_{U_i} : U_i \to U$ provides an identification of $\Gamma^\infty_c(U_i, \mathbb{E} \otimes L)$ with $\Gamma^\infty_c(U, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma)$. As $\pi|_{U_i}$ maps $(g, J, \omega)$ to $(\widetilde{g}, \widetilde{J}, \widetilde{\omega})$ and identifies $(L, \nabla^L)$ with $(L/\Gamma, \nabla^{L/\Gamma})$, the twisted Dolbeault-Dirac operators on $\Gamma^\infty_c(U_i, \mathbb{E} \otimes L)$ and $\Gamma^\infty_c(U, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma)$ are intertwined by this identification.

Combining these two identifications we find that the map

$$\Gamma^\infty_c(\pi^{-1}(U), \mathbb{E} \otimes L) \to \Gamma^\infty_c(U, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma), \quad s^\Gamma \to \widetilde{s}$$

satisfies

$$\widetilde{D^L s^\Gamma} = D^{L/\Gamma}\widetilde{s}, \tag{3.7}$$

for all $s^\Gamma \in \Gamma^\infty_c(\pi^{-1}(U), \mathbb{E} \otimes L)$.

One shows that Equation (3.7) holds for arbitrary $\widetilde{s} \in \Gamma^\infty_c(M/\Gamma, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma)$ by covering supp $\widetilde{s}$ with a finite number of open subsets $V_l$ such that $\pi^{-1}(V_l) \cong$

$V_l \times \Gamma$ for each $l$. Then take the open cover $\{V_l\} \cup \{M \setminus \operatorname{supp} \widetilde{s}\}$ of $M$ and let $\{\widetilde{\psi}_l\} \cup \{\widetilde{\psi}_{M\setminus \operatorname{supp} \widetilde{s}}\}$ be a partition of unity subordinate to this cover. The argument of the previous paragraph now applies to each $\widetilde{\psi}_{V_l}\widetilde{s}$, whereas $\widetilde{\psi}_{M\setminus \operatorname{supp} \widetilde{s}}\widetilde{s} = 0$.

**Remark 3.3.5.** Equation (3.7) can also be found in [43, Chapter 10] for, not necessarily finite, discrete $\Gamma$ .

Let $\varepsilon$ and $\widetilde{\varepsilon}$ be the Liouville measures on $M$ and $M/\Gamma$, respectively. Since $\pi : M \to M/\Gamma$ is a $|\Gamma|$-fold covering map and $\pi^*\widetilde{\omega} = \omega$, we find that

$$\int_M (\pi^* \widetilde{f})\varepsilon = |\Gamma| \int_{M/\Gamma} \widetilde{f}\widetilde{\varepsilon},$$

for all $\widetilde{f} \in C^\infty(M/\Gamma)$. Therefore, for any $\Gamma$-equivariant hermitian vector bundle $E \to M$, the map

$$u : s^\Gamma \mapsto |\Gamma|^{\frac{1}{2}}\widetilde{s}, \quad s^\Gamma \in \Gamma_c^\infty(M,E)^\Gamma$$

is a unitary map $\Gamma_c^\infty(M,E)^\Gamma \to \Gamma_c^\infty(M/\Gamma, E/\Gamma)$. Taking $E = \mathbb{E} \otimes L$ and using Proposition 3.3.2 and Equation (3.7), we obtain the following result.

**Theorem 3.3.6.** *Let $\Gamma$ be a finite group acting symplectically and* freely *on a symplectic manifold $(M,\omega)$ that carries a compatible, invariant, almost complex structure and equivariant pre-quantisation $(L, \nabla^L)$. The map*

$$u : L^2(M, \mathbb{E} \otimes L)^\Gamma \to L^2(M/\Gamma, \mathbb{E}_{M/\Gamma} \otimes L/\Gamma), \quad s^\Gamma \mapsto |\Gamma|^{\frac{1}{2}}\widetilde{s}$$

*is a unitary isomorphism that intertwines the Dolbeault-Dirac operators $D^L$ and $D^{L/\Gamma}$. Moreover,*

$$(\ker \overline{D}^L_+)^\Gamma = \ker((\overline{D}^L_+)|_{\mathcal{H}^\Gamma}) = \ker cl((D^L_+)|_{\mathcal{H}^\Gamma}) \overset{u}{\cong} \ker \overline{D}^{L/\Gamma}_+,$$
$$(\ker \overline{D}^L_-)^\Gamma = \ker((\overline{D}^L_-)|_{\mathcal{H}^\Gamma}) = \ker cl((D^L_-)|_{\mathcal{H}^\Gamma}) \overset{u}{\cong} \ker \overline{D}^{L/\Gamma}_-,$$

*where we have written $\mathcal{H} = L^2(M, \mathbb{E} \otimes L)$.*

We now return to the action of $W(G,\mathbf{T})$ on $\mathbf{T}\times\mathfrak{t}$. Let $L$ be the trivial hermitian line bundle over $\mathbf{T} \times \mathfrak{t}$ with hermitian connection $\nabla^L = d + 2\pi i\theta$, where $\theta$ is the fundamental 1-form. Suppose that $W(G,\mathbf{T})$ acts on $L$ by sending $((g,Y),\lambda) \in L = (\mathbf{T}\times\mathfrak{t})\times\mathbb{C}$ to $((wg,wY),\lambda)$. Then $(L,\nabla^L)$ is a $W(G,\mathbf{T})$-equivariant pre-quantum line bundle.

**Proposition 3.3.7.** *Let $G$ be a compact connected Lie group and let $\boldsymbol{T}$ be a maximal torus. Consider the action of the Weyl group $W(G,\boldsymbol{T})$ on the Kähler manifold $\boldsymbol{T} \times \mathfrak{t}$. Then the Dolbeault-Dirac quantisation of $\boldsymbol{T} \times \mathfrak{t}$ is $W(G,\boldsymbol{T})$-equivariantly and unitarily isomorphic to $L^2(\boldsymbol{T})$ via the isomorphism of Proposition 2.4.9.*

*Proof.* Because the Kähler structure on $\mathbf{T}\times\mathfrak{t}$ is locally equal to the Kähler structure of $\mathbb{C}^n$, the curvature of the canonical line bundle is zero; in particular, its curvature is semi-negative. From the proof of Theorem 2.4.6 we then know that

$$\begin{aligned}\ker \overline{D}^L &= \left\{ f e^{-\pi|Y|^2} \;\middle|\; f \text{ holomorphic}, \int_M |f|^2 e^{-2\pi|Y|^2}\varepsilon < \infty \right\} \\ &\cong \mathcal{H}L^2(T^*\mathbf{T}, e^{-2\pi|Y|^2}\varepsilon).\end{aligned}$$

Since $(g,Y)\mapsto e^{-\pi|Y|^2}$ is invariant under the Weyl group, the above isomorphism is equivariant for the Weyl group actions.

It remains to check that the isomorphism $\mathcal{H}L^2(T^*\mathbf{T}, e^{-2\pi|Y|^2}\varepsilon) \cong L^2(\mathbf{T})$ of Proposition 2.4.9 or [37, Theorem 10] is $W(G,\mathbf{T})$-equivariant. The explicit form of the inverse $C_\phi$ of this isomorphism, as presented in [37, Theorem 10] (*cf.* Proposition 2.4.9) with $U\equiv 1$, is

$$(C_\phi f)(g) = \int_{\mathbf{T}} f(x)\phi(x^{-1}g)dx, \quad (f\in L^2(\mathbf{T})).$$

Here. the entire function $\phi : \mathbf{T}^{\mathbb{C}} \to \mathbb{C}$ is given by

$$\phi(g) = \sum_{\pi\in\hat{\mathbf{T}}} \frac{1}{\sqrt{\sigma(\pi)}}\operatorname{Tr}(\pi(g^{-1})),$$

where the sum is over all the irreducible representations of $\mathbf{T}$, each of which is extended holomorphically to an irreducible representation of $\mathbf{T}^{\mathbb{C}}$, and

$$\sigma(\pi) = \int_{\mathbf{T}^{\mathbb{C}}} \|\pi(g^{-1})\|^2 e^{-2\pi|Y|^2}\varepsilon.$$

Let $w\in W(G,\mathbf{T})$ be given. We show that $C_\phi(f\circ w) = (C_\phi f)\circ w$. Using the fact that $w\in W(G,\mathbf{T})$ acts on $\mathbf{T}^{\mathbb{C}}$ by homomorphisms, we obtain

$$(C_\phi f)(wg) = \int_{\mathbf{T}} f(x)\phi(x^{-1}wg)dx = \int_{\mathbf{T}} f(x)\phi(w(w^{-1}(x^{-1})g))dx.$$

We claim that $\phi(wh) = \phi(h)$ for all $w\in W(G,\mathbf{T})$ and $h\in\mathbf{T}^{\mathbb{C}}$. From the invariance of $e^{-2\pi|Y|^2}\varepsilon$ under $W(G,\mathbf{T})$ we see that

$$\sigma(\pi\circ w) = \int_{\mathbf{T}^{\mathbb{C}}} \|\pi(w\cdot g^{-1})\|^2 e^{-2\pi|Y|^2}\varepsilon = \int_{\mathbf{T}^{\mathbb{C}}} \|\pi(g^{-1})\|^2 e^{-2\pi|Y|^2}\varepsilon = \sigma(\pi),$$

so that

$$\begin{aligned}\phi(wh) &= \sum_{\pi\in\hat{\mathbf{T}}} \frac{1}{\sqrt{\sigma(\pi)}}\operatorname{Tr}(\pi(wh^{-1})) = \sum_{\pi\in\hat{\mathbf{T}}} \frac{1}{\sqrt{\sigma(\pi\circ w)}}\operatorname{Tr}(\pi(wh^{-1}))\\ &= \sum_{\pi\in\hat{\mathbf{T}}} \frac{1}{\sqrt{\sigma(\pi)}}\operatorname{Tr}(\pi(h^{-1})) = \phi(h),\end{aligned}$$

where in the the second-last step we have used the fact that $\pi \mapsto \pi \circ w$ maps $\hat{\mathbf{T}}$ bijectively onto itself. Consequently,

$$(C_\phi f)(wg) = \int_{\mathbf{T}} f(x)\phi(w^{-1}(x^{-1})g)dx.$$

The Haar measure on $G$ is invariant under $W(G,\mathbf{T})$, because $W(G,\mathbf{T})$ acts by automorphisms. Therefore,

$$(C_\phi f)(wg) = \int_{\mathbf{T}} f(x)\phi(w^{-1}(x^{-1})g)dx = \int_{\mathbf{T}} f(w\cdot x)\phi(x^{-1}g)dx,$$

or in other words $C_\phi(f\circ w) = (C_\phi f)\circ w$. Thus, $C_\phi$ is a $W(G,\mathbf{T})$-equivariant isomorphism, so that its inverse establishes an equivariant isomorphism

$$\mathcal{Q}_{DD}(\mathbf{T}\times\mathfrak{t}) = \mathcal{H}L^2(T^*\mathbf{T}, e^{-2\pi|Y|^2}\varepsilon) \overset{\cong}{\to} L^2(\mathbf{T}).$$

□

**Theorem 3.3.8.** *Let $G$ be a compact connected Lie group and let $\boldsymbol{T}$ be a maximal torus. Consider the action of the Weyl group $W(G,\boldsymbol{T})$ on the Kähler manifold $\boldsymbol{T}\times\mathfrak{t}$. Define the Dolbeault-Dirac quantisation of the singular quotient to be the Dolbeault-Dirac quantisation of its principal stratum. Then the Dolbeault-Dirac quantisation of $(\boldsymbol{T}\times\mathfrak{t})/W(G,\boldsymbol{T})$ is naturally isomorphic to $L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}$ via the isomorphisms of Theorem 3.3.6 and Proposition 3.3.7. In particular, quantisation commutes with reduction.*

*Proof.* Combining Theorems 3.2.12 and 3.3.6 and Proposition 3.3.7, we obtain the following unitary isomorphisms

$$\begin{aligned}\mathcal{Q}_{DD}((\mathbf{T}\times\mathfrak{t})/W(G,\mathbf{T})) &:= \ker \overline{D}^{L/\Gamma}_{princ} \cong (\ker \overline{D}^{L}_{princ})^{W(G,\mathbf{T})}\\ &= (\ker \overline{D}^{L})^{W(G,\mathbf{T})} = \mathcal{Q}_{DD}(\mathbf{T}\times\mathfrak{t})^{W(G,\mathbf{T})}\\ &\cong L^2(\mathbf{T})^{W(G,\mathbf{T})},\end{aligned}$$

where $D^{L/\Gamma}_{princ}$ denotes the twisted Dolbeault-Dirac operator on the principal stratum of $(\mathbf{T}\times\mathfrak{t})/W(G,\mathbf{T})$ and $D^{L}_{princ}$ denotes the twisted Dolbeault-Dirac operator on $(\mathbf{T}\times\mathfrak{t})_{princ}$. □

We have proved that quantisation commutes with reduction if the singular quotient of $T^*\mathbf{T}/W(G,\mathbf{T})$ is quantised by taking the Dolbeault-Dirac quantisation of its principal stratum. The main point is not that quantisation after reduction and reduction after quantisation yield isomorphic Hilbert spaces - because any two infinite-dimensional separable Hilbert spaces are isomorphic - but the way this isomorphism is constructed. The crucial step was Proposition 3.2.11 which shows that the closure of the Dolbeault-Dirac operator, and hence the Dolbeault-Dirac quantisation, is not affected by the omission of the singular strata on $\mathbf{T} \times \mathfrak{t}$. In Chapter 4 we will apply similar ideas to the singular quotient $T^*G// \operatorname{Ad} G$, where the group action is not discrete. There, we will also relate the symplectic stratified structure of $T^*\mathbf{T}/W(G,\mathbf{T})$ to the symplectic stratified structure of $T^*G// \operatorname{Ad} G$.

**Remark 3.3.9.** Let us make a few comments about the spin quantisation, too. The curvature of the canonical line bundle on $T^*\mathbf{T}$ is 0, since the Kähler structure on $T^*\mathbf{T}$ is locally isomorphic to the Kähler structure on $\mathbb{C}^n$. Therefore, the Dolbeault-Dirac quantisation is isomorphic to the spin quantisation for $T^*\mathbf{T}$.

When we also consider the $W(G,\mathbf{T})$-action on the canonical bundle, then the left-invariant holomorphic $(n,0)$-form $\beta_1 \wedge \cdots \wedge \beta_n$, where each $\beta_k$ is a left-invariant $(1,0)$-form on $\mathbf{T}^{\mathbb{C}}$, is not $W(G,\mathbf{T})$-invariant. Indeed, an element $w \in W(G,\mathbf{T})$ sends $\beta_1 \wedge \cdots \wedge \beta_n$ to $\det(w)\beta_1 \wedge \cdots \wedge \beta_n$, where $\det_{\mathfrak{t}}(w) = \det_{\mathfrak{t}_{\mathbb{C}}}(w)$ is the determinant of the action of $w$ as a real-linear map on $\mathfrak{t}$, or equivalently, the determinant of its complex linear extension to $\mathfrak{t}_{\mathbb{C}}$, considered as a complex-linear map. This determinant is $\pm 1$ depending on whether $w$ is a rotation or a reflection of $\mathfrak{t}$. Now the question is if there exists an equivariant half-form bundle whose square is the equivariant canonical line bundle, and if $T^*\mathbf{T}_{princ}/W(G,\mathbf{T})$ is a spin manifold or not.

## 3.4 Principal strata on compact manifolds

In Section 3.2 we showed that the Dolbeault-Dirac quantisation of $(\mathbf{T} \times \mathfrak{t})_{princ}$ is equal to the Dolbeault-Dirac quantisation of $\mathbf{T} \times \mathfrak{t}$. In this section, we prove that this result also holds for arbitrary abelian or discrete groups and compact manifold $M$.

So, let $(M,\omega)$ be a compact, symplectic manifold and let $G$ be a finite group or a compact abelian Lie group acting symplectically on $M$ such that $M/G$ is connected. Suppose that $M$ carries a $G$-invariant, compatible, almost complex structure and suppose that $L$ is an equivariant hermitian line bundle with an equivariant hermitian connection $\nabla^L$ satisfying $\nabla^2 = 2\pi i\omega$. Write $D^L$ for the corresponding twisted Dolbeault-Dirac operator. Compactness of $M$ ensures that there are only finitely many strata. Moreover, for both abelian and discrete groups the strata of $M$ are given by the connected components of the isotropy type submanifolds, i.e. the strata are of the form $M^x_{G_x} = M^x_{(G_x)}$, where $x \in M$. For abelian

Lie groups this is clear, for discrete groups this was proved in Lemma 3.2.2.

The following lemma shows that $M \setminus M_{princ}$ is also a finite (but not necessarily disjoint) union of connected components of fixed point type manifolds with codimension $\geq 2$.

**Lemma 3.4.1.** *Let $(H_0)$ be the unique conjugacy class of subgroups such that $M_{princ} = M_{(H_0)}$ is open and dense in $M$. Define the finite set*

$$S = \{M^x_{G_x} \mid (G_x) \neq (H_0), x \in M\}.$$

*Then*

$$P = \bigcup_{M^x_H \in S} M^H_x.$$

*is the complement of $M_{princ}$ in $M$.*

*Proof.* We show that

$$\bigcup_{M^x_H \in S} M^x_H = \bigcup_{M^x_H \in S} M^H_x.$$

Since $M^x_H \subset M^H_x$, the left-hand side is clearly contained in the right-hand side. Conversely, if $y \in M^H_x$ for some isotropy group $H$ for which $(H) \neq (H_0)$, then $H \subset G^y$, so that $(G_y) \neq (H_0)$. Hence, $M^y_{G_y} \in S$ and so $y$ is contained in the left-hand side. The left-hand side is, almost by definition, equal to the complement of $M_{princ}$.

□

According to Lemma 3.4.1 the complement of the principal stratum in $M$ is a *finite* union of closed embedded submanifolds, each of even codimension $\geq 2$. If $\Gamma^\infty_c(M \setminus M^H_x, \mathbb{E} \otimes L)$ is dense in $\Gamma^\infty_c(M, \mathbb{E} \otimes L)$ with respect to the graph norm of $D^L$ for each $M^H_x$ occurring in the union of $P$, then, as in Proposition 3.2.10, Lemma 3.2.9 implies that $\Gamma^\infty_c(M_{princ}, \mathbb{E} \otimes L)$ is dense in $\Gamma^\infty_c(M, \mathbb{E} \otimes L)$. By Theorem 3.2.12 the Dolbeault-Dirac quantisations of $M_{princ}$ and $M$ are equal. Note that by compactness of $M$, the operators $D^L_{princ}$ and $D^L$ are always essentially self-adjoint.

**Remark 3.4.2.** By Proposition 3.2.3 the action of $W(G, \mathbf{T})$ on $\mathbf{T} \times \mathfrak{t}$, where $G$ is any compact connected Lie group and $\mathbf{T}$ a maximal torus in $G$, has finitely many strata. In particular, the proof of Lemma 3.4.1 applies to that situation, too. However, in Section 3.2 we could prove directly that the set

$$\{(\mathbf{T} \times \mathfrak{t})^H_x \mid x \in M, H \text{ isotropy group}\}$$

is already finite.

It remains to show that $\Gamma_c^\infty(M \setminus M_x^H, \mathbb{E} \otimes L)$ is dense in $\Gamma_c^\infty(M, \mathbb{E} \otimes L)$ with respect to the graph norm of $D^L$. The following proposition proves this for symplectic actions of compact Lie groups on compact manifolds.

**Proposition 3.4.3.** *Let $G$ be an arbitrary compact Lie group and let $M_x^H$ be given with $H$ an isotropy group such that $(H) \neq (H_0)$. Then $\Gamma_c^\infty(M \setminus M_x^H, \mathbb{E} \otimes L)$ is dense in $\Gamma^\infty(M, \mathbb{E} \otimes L)$ with respect to the graph norm of $D^L$.*

*Proof.* Let $\{(U_i, \kappa_i)\}_{i=1}^k$ be a finite atlas of $M$ onto $\mathbb{R}^n$, where each chart either has empty intersection with $M_x^H$, or maps $M_x^H$ onto $\mathbb{R}^{n-k} \subset \mathbb{R}^n$. Let $\{\psi_i\}_{i=1}^k$ be a partition of unity subordinate to $\{U_i\}$. On $\Gamma^\infty(M, \mathbb{E} \otimes L)$ one defines the norm $\|\cdot\|_1$ as

$$\|s\|_1^2 := \sum_{i=1}^k \|(\psi_i u) \circ \kappa_i\|^2_{H^1(\mathbb{R}^n, \mathbb{E}\otimes L)};$$

let $H^1(M, \mathbb{E} \otimes L)$ be the completion of $\Gamma^\infty(M, \mathbb{E} \otimes L)$ with respect to this norm. The norm $\|\cdot\|_1$ depends on the choice of the charts and on the choice of partition of unity, but different choices lead to equivalent norms, so that $H^1(M, \mathbb{E} \otimes L)$ is well defined. It is well known that the inclusion $H^1(M, \mathbb{E} \otimes L) \to L^2(M, \mathbb{E} \otimes L)$ is compact, and that $\Gamma^\infty(M, \mathbb{E} \otimes L)$ extends to a continuous map $H^1(M, \mathbb{E} \otimes L) \to L^2(M, \mathbb{E} \otimes L)$. In particular, there exists a constant $C > 0$ such that

$$\|D^L s\| \leq C\|s\|_1.$$

On the other hand, by the Gårding inequality for elliptic first-order differential operators on compact manifolds, there exists $c > 0$ such that

$$\|s\| + \|D^L s\| \geq c\|s\|_1$$

for all $s \in H^1(M, \mathbb{E} \otimes L)$. Consequently, the $D^L$-graph norm is equivalent to the norm $\|\cdot\|_1$.

So it suffices to show that $\Gamma_c^\infty(M \setminus M_x^H, \mathbb{E} \otimes L)$ is dense in $\Gamma^\infty(M, \mathbb{E} \otimes L)$ in the $\|\cdot\|_1$-norm. Let $s \in \Gamma^\infty(M, \mathbb{E} \otimes L)$ be given. Now, if $s \in \Gamma^\infty(M, \mathbb{E} \otimes L)$, then, by Theorem 3.1.3, $\psi_i s$ can be approximated by a sequence $(f_{i,m})_m$ in $\Gamma_c^\infty(U_i \setminus M_x^H, \mathbb{E} \otimes L)$ (recall that $M_x^H$ is of codimension at least 2). The sequence $(\sum_{i=1}^k f_{i,m})_m$ lies in $\Gamma_c^\infty(M \setminus M_x^H, \mathbb{E} \otimes L)$ and approximates $\sum_{i=1}^k \psi_i s = s$ in the norm $\|\cdot\|_1$. □

**Remark 3.4.4.** Proposition 3.4.3 also holds when $M_x^H$ is replaced by any closed submanifold $N$ of codimension $\geq 2$ and $D$ by any other first-order elliptic differential operator. In particular, if $M$ is a Kähler manifold and $s$ is a section of the holomorphic line bundle $L$ such that $s$ is holomorphic on $M \setminus N$, then $s$ can be extended (uniquely) to a holomorphic section of $L$ on $M$ .

Thus, we have proved the following analogue of Theorem 3.2.12.

**Theorem 3.4.5.** *Let $G$ be a finite group or a compact, abelian Lie group and suppose that $G$ acts symplectically on a compact symplectic manifold $(M, \omega)$ such that $M/G$ is connected. If $J$ is a $G$-invariant, compatible, almost complex structure and if $(L, \nabla^L)$ is an equivariant pre-quantisation line bundle, then the subspace $\Gamma_c^\infty(M_{princ}, \mathbb{E} \otimes L)$ is dense in $\Gamma^\infty(M, \mathbb{E} \otimes L)$ in the graph norm of $D^L$. Consequently, the Dolbeault-Dirac quantisations of $M$ and of $M_{princ}$ coincide.*

If $G$ is an arbitrary compact group acting symplectically on a compact, symplectic manifold $M$, then the partition $M_H^x$ is no longer locally finite, let alone finite. The same applies to the associated cover of Lemma 3.4.1 by connected components of fixed point type submanifolds. Therefore, the arguments of Section 3.4 cannot be applied to general symplectic actions by compact Lie groups.

If the group $G$ is now discrete or abelian, but $M$ non-compact, there are complications of another nature. In Proposition 3.4.3 we could cover $\mathbf{T} \times \mathfrak{t}$ by a finite subset of open neighbourhoods such that each neighbourhood was locally diffeomorphic (as a Kähler manifold) to $\mathbb{C}^N$. On arbitrary non-compact manifolds the Kähler structure can be much wilder and $D^L$ might have non-constant coefficients on charts, so that it is hard to say how 'sensitive' $D^L$ is to the removal of an embedded submanifold of codimension $\geq 2$.

# Chapter 4

# Quantisation commutes with reduction

Let $G$ be a compact connected Lie group and consider the action of $G$ on $T^*G$ induced by the action of $G$ on itself by conjugation. Due to the non-freeness of the action of $G$ on $T^*G$, the Marsden-Weinstein quotient is in general not a manifold, but merely a symplectic stratified space. By Theorem 1.2.6 the symplectic stratified quotient always has a dense and open stratum, the so-called principal stratum.

In Section 4.1, we consider *quantisation after reduction*. We first explicitly determine the stratified structure of the Marsden-Weinstein quotient and we show how it is related to the stratified structure on $\mathbf{T} \times \mathfrak{t}$ that is obtained from the $W(G, \mathbf{T})$-action (see Chapter 3). Here, $\mathbf{T}$ denotes a maximal torus in $G$ and $\mathfrak{t}$ denotes its Lie algebra. As in Chapter 3, we define the quantisation of the Marsden-Weinstein quotient to be the Dolbeault-Dirac quantisation of its principal stratum. We prove that quantisation after reduction yields the Hilbert space $L^2(\mathbf{T})^{W(G,\mathbf{T})}$, without any requirement on the Lie group $G$ except that it be compact and connected.

In Section 4.2, we consider the *reduction after quantisation* procedure for $T^*G$ when $G \in \mathcal{C}_K$. This procedure also yields the Hilbert space $L^2(\mathbf{T})^{W(G,\mathbf{T})}$, so that quantisation commutes with reduction if $G \in \mathcal{C}_K$.

We mention that the quantisation of the Marsden-Weinstein quotient $T^*G//G$ is also analysed in [48, 49]. Here the aim of the authors is to construct a costratification on the quantum Hilbert space that resembles the stratified structure on the classical side.

## 4.1 Quantisation after reduction

Let $G$ be a compact connected Lie group. As in Section 2.1, we choose an Ad $G$-invariant inner product on $\mathfrak{g}$ and use this inner product and left-trivialisation to identify $T^*G$ with $G \times \mathfrak{g}$. Calculations similar to those in [1, Section 4.4] lead to the following result.

**Lemma 4.1.1.** *The moment map $j : G \times \mathfrak{g} \to \mathfrak{g}^*$, considered as a map to $\mathfrak{g}$, is equal to*

$$j(g, Y) = \operatorname{Ad}_g Y - Y \in \mathfrak{g},$$

*where $g \in G$, $Y \in \mathfrak{g}$.*

Having computed an explicit formula for the moment map, we now determine the strata of the Marsden-Weinstein quotient $j^{-1}(0)/G$. We show that it is sufficient to know how these strata partition $\mathbf{T} \times \mathfrak{t}$, where $\mathbf{T}$ is a maximal torus in $G$ and $\mathfrak{t}$ its Lie algebra. Let us fix a maximal torus $\mathbf{T}$ of $G$ once and for all. The isotropy group of an element $g \in G$ under the action of $G$ on itself by conjugation is simply the *centraliser* $\mathcal{Z}_G(g)$. Similarly, we denote $\mathcal{Z}_G(Y)$ for the isotropy group of $Y \in \mathfrak{g}$ under the adjoint action of $G$ on $\mathfrak{g}$, and we also refer to $\mathcal{Z}_G(Y)$ as the ***centraliser*** of $Y$.

**Lemma 4.1.2.** *Let $(g, Y) \in j^{-1}(0) \subset G \times \mathfrak{g}$ be arbitrary. The orbit of $(g, Y)$ (under the action of $G$) contains an element of $\boldsymbol{T} \times \mathfrak{t}$. Moreover, $j^{-1}(0)/G$ is homeomorphic to $(\boldsymbol{T} \times \mathfrak{t})/W(G, \boldsymbol{T})$.*

*Proof.* Let $(g, Y) \in j^{-1}(0)$ be arbitrary. Consider the one-dimensional Lie subalgebra $\mathfrak{h}_Y \subset \mathfrak{g}$ generated by $Y$. The Lie group $\exp(\mathfrak{h}_Y)$ is a torus in $\mathfrak{g}$. Since $(g, Y) \in j^{-1}(0)$, we obtain

$$g \exp(tY) g^{-1} = \exp(\operatorname{Ad}_g tY) = \exp(tY), \quad (t \in \mathbb{R}),$$

so that $g$ centralises the torus $\exp(\mathfrak{h}_Y)$. Consequently, there is a maximal torus in $G$ that contains both $\exp(\mathfrak{h}_Y)$ and $g$ (see [56, Theorem 4.50]). Since any two maximal tori are conjugate in $G$, there exists an $h \in G$ such that $\mathbf{T}$ contains both $hgh^{-1}$ and $\exp(\operatorname{Ad}_h \mathfrak{h}_Y)$. In particular, the path $t \mapsto \exp(t \operatorname{Ad}_h Y)$ lies in $\mathbf{T}$, so that $\operatorname{Ad}_h Y \in \mathfrak{t}$. Thus, $(hgh^{-1}, \operatorname{Ad}_h Y)$ lies in $\mathbf{T} \times \mathfrak{t}$.

Clearly, $\mathbf{T} \times \mathfrak{t} \subset j^{-1}(0)$ and this inclusion induces a continuous map

$$\iota : (\mathbf{T} \times \mathfrak{t})/W(G, \mathbf{T}) \to j^{-1}(0)/G.$$

By the previous paragraph $\iota$ is surjective. To show that it is injective, we proceed in the same way as in [56, Proposition 4.53], where it is shown that $G/\operatorname{Ad} G \cong \mathbf{T}/W(G, \mathbf{T})$. Suppose that $(g, Y), (g', Y') \in \mathbf{T} \times \mathfrak{t}$ are in the same $G$-orbit. That is, there exists $h \in G$ such that $(g', Y') = (hgh^{-1}, \mathrm{Ad}_h Y)$. Consider the closed Lie subgroup $\mathcal{Z}_G(g, Y) = \mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$, i.e. the closed Lie subgroup of $G$ consisting of all elements that centralise both $g$ and $Y$. Its Lie algebra is equal to

$$\mathcal{Z}_{\mathfrak{g}}(g, Y) := \{X \in \mathfrak{g} \mid \mathrm{Ad}_g X = X \text{and } [X, Y] = 0\},$$

and $\mathcal{Z}_{\mathfrak{g}}(g, Y)$ contains $\mathfrak{t}$. It also contains $\mathrm{Ad}_h^{-1} \mathfrak{t}$: pick $X \in \mathfrak{t}$, then

$$\mathrm{Ad}_g \mathrm{Ad}_{h^{-1}} X = \mathrm{Ad}_{h^{-1}} \mathrm{Ad}_{hgh^{-1}} X = \mathrm{Ad}_{h^{-1}} X,$$

where we used the fact that $hg^{-1}h \in \mathbf{T}$. Similarly,

$$[\mathrm{Ad}_{h^{-1}} X, Y] = \mathrm{Ad}_{h^{-1}}[X, \mathrm{Ad}_h Y] = 0.$$

Both $\mathfrak{t}$ and $\mathrm{Ad}_{h^{-1}} \mathfrak{t}$ are maximal abelian subalgebras in $\mathcal{Z}_{\mathfrak{g}}(g, Y)$. Hence, there exists an element $k$ in the identity component of $\mathcal{Z}_G(g, Y)$ such that $\mathrm{Ad}(kh^{-1})\mathfrak{t} = \mathfrak{t}$. Consequently, $kh^{-1} \in N_G(\mathbf{T})$, and

$$kh^{-1} \cdot (hgh^{-1}, \mathrm{Ad}_h Y) = k \cdot (g, Y) = (g, Y),$$

so that $(g, Y)$ and $(g', Y')$ are in the same $W(G, \mathbf{T})$ orbit. Thus, $\iota$ is a continuous bijection.

To prove that $\iota$ is a homeomorphism we show that it is closed. For $\iota$ to be closed it is sufficient that the map $(\mathbf{T} \times \mathfrak{t}) \to j^{-1}(0)/G$ is closed. Since $\mathbf{T} \times \mathfrak{t}$ is a closed subset of $j^{-1}(0)$, it is in turn sufficient to show that the projection $j^{-1}(0) \to j^{-1}(0)/G$ is closed, which is what we do now. By compactness of $G$, the map

$$\Phi : G \times j^{-1}(0) \to j^{-1}(0), \quad (g, x) \mapsto gx$$

is proper. Every proper map into a locally compact Hausdorff space is closed, and so in particular $\Phi$ is a closed map. Therefore, if $C$ is a closed set of $j^{-1}(0)$, then $GC = \Phi(G \times C)$ is closed. Thus, the quotient map $j^{-1}(0) \to j^{-1}(0)/G$ is a closed map. Consequently, the continuous bijection $\iota : (\mathbf{T} \times \mathfrak{t})/W(G, \mathbf{T}) \to j^{-1}(0)/G$ is a closed map, hence a homeomorphism. □

**Remark 4.1.3.** The observation that $(G \times \mathfrak{g})// \mathrm{Ad}\, G$ is homeomorphic to $(\mathbf{T} \times \mathfrak{t})/W(G, \mathbf{T})$ is not new (see for instance [48]).

Recall from Theorem 1.2.6 that for a strongly Hamiltonian, proper group action on a symplectic manifold $M$, the reduced space $j^{-1}(0)/G$ is stratified by the strata

$$M_{x,0}^{(H)} = (j^{-1}(0) \cap GM_H^x)/G,$$

where $H \subset G$ is an isotropy group for the $G$-action on $M$ and $x \in M_H$. Define

$$N_G(H)^x = \{g \in G \mid g \cdot M_H^x = M_H^x\}.$$

Then

$$M_{x,0}^{(H)} = (j^{-1}(0) \cap M_H^x)/(N_G(H)^x/H). \tag{4.1}$$

We now use Equation (4.1) to determine the strata of $j^{-1}(0)/G$ in the case of the $G$-action on $T^*G$. By Lemma 4.1.2 it is sufficient to consider only those submanifolds $(G \times \mathfrak{g})_H^{(g,Y)}$ that have non-empty intersection with $\mathbf{T} \times \mathfrak{t}$. But if $(g, Y) \in (\mathbf{T} \times \mathfrak{t}) \cap (G \times \mathfrak{g})_H^{(g,Y)}$, then necessarily $H \supset \mathbf{T}$. Conversely, if $H \supset \mathbf{T}$, then $(G \times \mathfrak{g})_H \subset (\mathbf{T} \times \mathfrak{t})$, because $(G \times \mathfrak{g})^{\mathbf{T}} = \mathbf{T} \times \mathfrak{t}$ . This latter fact can be seen as follows: if $(g, Y) \in (G \times \mathfrak{g})^{\mathbf{T}}$, then $\mathbf{T}$ centralises both $g$ and $Y$. In particular, there exists a maximal torus containing both $g$ and $\mathbf{T}$. Then $g \in \mathbf{T}$, because $\mathbf{T}$ is already a maximal torus. Similarly, $Y \in \mathfrak{t}$. Thus, we have proved:

**Proposition 4.1.4.** *1. If $H \supset \boldsymbol{T}$, then*

$$j^{-1}(0) \cap (G \times \mathfrak{g})_H^{(g,Y)} = (G \times \mathfrak{g})_H^{(g,Y)} \subset \boldsymbol{T} \times \mathfrak{t}.$$

*2. Each stratum of $j^{-1}(0)/G$ is of the form*

$$(G \times \mathfrak{g})_H^{(g,Y)}/(N_G(H)^{(g,Y)}/H)$$

*with $H \supset \boldsymbol{T}$.*

Proposition 4.1.4 basically says that the symplectic stratification is obtained by partitioning $\mathbf{T} \times \mathfrak{t}$ into the connected components of $(G \times \mathfrak{g})_H$ for $H \supset \mathbf{T}$. We now further analyse this partition of $\mathbf{T} \times \mathfrak{t}$.

Consider the action of $G$ on itself by conjugation and let $\mathbf{T}$ be a maximal torus. The principal stratum of $G$ is equal to

$$G_{princ} = G_{(\mathbf{T})} = \{g \in G \mid \mathcal{Z}_G(g) \text{ is a maximal torus }\}.$$

Similarly, the principal stratum of the adjoint action of $G$ on $\mathfrak{g}$ is equal to

$$\mathfrak{g}_{princ} = \mathfrak{t}_{(\mathbf{T})} = \{X \in \mathfrak{g} \mid \mathcal{Z}_G(X) \text{ is a maximal torus }\}.$$

Proofs of these facts may be found in [28, Theorem 3.7.1 and Corollary 3.3.2]. In particular, $G_{princ}$ and $\mathfrak{g}_{princ}$ are open and dense in $G$ and $\mathfrak{g}$, respectively. Since $G/\operatorname{Ad} G \cong \mathbf{T}/W$ and the projection $G \to G/\operatorname{Ad} G$ is open, the subset $G_{princ} \cap \mathbf{T} = G_{\mathbf{T}}$ is open and dense in $\mathbf{T}$. Similarly, $\mathfrak{g}_{\mathbf{T}}$ is open and dense in $\mathfrak{t}$.

The principal stratum of $G \times \mathfrak{g}$ is $(G \times \mathfrak{g})_{\{e\}}$, but this stratum does not intersect $j^{-1}(0)$, because each element in $j^{-1}(0)$ is at least fixed by some maximal torus of $G$. Instead, there is the following candidate for the principal stratum on $\mathbf{T} \times \mathfrak{t}$.

**Proposition 4.1.5.** *The space $(G \times \mathfrak{g})_{\boldsymbol{T}} \subset \boldsymbol{T} \times \mathfrak{t}$ is an open and dense submanifold of $\boldsymbol{T} \times \mathfrak{t}$.*

*Proof.* The submanifold $(G \times \mathfrak{g})_{\mathbf{T}}$ is non-empty, since both $G_{\mathbf{T}}$ and $\mathfrak{g}_{\mathbf{T}}$ are non-empty. Let $(g, Y) \in (G \times \mathfrak{g})_{\mathbf{T}}$ be given. By Proposition 1.1.10, the dimension of $(G \times \mathfrak{g})_{\mathbf{T}}^{(g,Y)}$ is equal to the dimension of $(G \times \mathfrak{g})_{(g,Y)}^{\mathbf{T}} = \mathbf{T} \times \mathfrak{t}$. In particular, $(G \times \mathfrak{g})_{\mathbf{T}}$ is an open submanifold of $\mathbf{T} \times \mathfrak{t}$.

To see that $(G \times \mathfrak{g})_{\mathbf{T}}$ is dense in $\mathbf{T} \times \mathfrak{t}$, note that $G_{\mathbf{T}} \times \mathfrak{g}_{\mathbf{T}}$ is contained in $(G \times \mathfrak{g})_{\mathbf{T}}$ and that $G_{\mathbf{T}} \times \mathfrak{g}_{\mathbf{T}}$ is dense in $\mathbf{T} \times \mathfrak{t}$ by the observations preceding this proposition. □

The quotient $(G \times \mathfrak{g})_{\mathbf{T}}/W(G, \mathbf{T})$ is the principal stratum of $j^{-1}(0)/G$. We show how the Kähler structure and pre-quantisation on $G \times \mathfrak{g}$ induce a Kähler structure and pre-quantisation on this principal stratum, respectively.

**Proposition 4.1.6.** *Provided that the inner product on* $\mathfrak{t}$ *is taken to be the restriction of the* $\operatorname{Ad} G$*-invariant inner product on* $\mathfrak{g}$*, the restriction of the Kähler structure and the pre-quantisation on* $G \times \mathfrak{g}$ *to* $\boldsymbol{T} \times \mathfrak{t}$ *coincide with the* $W(G, \boldsymbol{T})$*-invariant Kähler structure and the* $W(G, \boldsymbol{T})$*-equivariant pre-quantisation on* $\boldsymbol{T} \times \mathfrak{t}$ *of Section 3.2.2, respectively.*

*Proof.* One can see that the complex structures coincide by considering the explicit formula for the differential of $\Phi : G \times \mathfrak{g} \to G^{\mathbb{C}}$ (see Equation (2.3)) which is used to transfer the complex structure on $G^{\mathbb{C}}$ to $G \times \mathfrak{g}$. The matrix $(T_{(g,Y)}\Phi)|_{\mathfrak{t}\times\mathfrak{t}}$ is just the identity matrix if $(g, Y) \in \mathbf{T} \times \mathfrak{t}$. The formula for the induced complex structure on $\mathbf{T} \times \mathfrak{t}$ then coincides with Equation (3.4).

Comparison of the expression for $\omega$ on $G \times \mathfrak{g}$ in Section 2.1 with the formula for the symplectic structure on $\mathbf{T} \times \mathfrak{t}$ in Section 3.2.2 shows that the pull-back of the symplectic structure on $G \times \mathfrak{g}$ produces the correct symplectic structure on $\mathbf{T} \times \mathfrak{t}$, provided that the inner product on $\mathfrak{t}$ is the restriction of the Ad $G$-invariant inner product on $\mathfrak{g}$.

If $\iota : \mathbf{T} \times \mathfrak{t} \to G \times \mathfrak{g}$ denotes the inclusion, then the induced line bundle $\iota^* L \to \mathbf{T} \times \mathfrak{t}$ is isomorphic to the bundle $(\mathbf{T} \times \mathfrak{t}) \times \mathbb{C}$ with trivial hermitian structure on the fibres and the obvious $W(G, \mathbf{T})$-action. Moreover, the connection $\nabla^L = d + 2\pi i\theta$ on $L \to G \times \mathfrak{g}$ pulls back to the $W(G, \mathbf{T})$-invariant connection $\iota^*\nabla^L = d + 2\pi i\theta$ on $\iota^* L \to \mathbf{T} \times \mathfrak{t}$, where $\theta$ now denotes the fundamental 1-form on $\mathbf{T} \times \mathfrak{t}$. This can be seen by recalling that $\theta(g, Y)(Z_1, Z_2) = \langle Y, Z_1 \rangle$ on both $\mathbf{T} \times \mathfrak{t}$ and $G \times \mathfrak{g}$. □

Because the induced Kähler structure and pre-quantisation on $(G \times \mathfrak{g})_{\mathbf{T}}$ are $W(G, \mathbf{T})$-invariant and $W(G, \mathbf{T})$-equivariant, respectively, they descend to a Kähler structure and pre-quantisation on $(G \times \mathfrak{g})_{\mathbf{T}}/W(G, \mathbf{T})$ as in Section 3.3. We are now in a position to state the following definition.

**Definition 4.1.7.** The ***Dolbeault-Dirac quantisation*** of $T^*G// \operatorname{Ad} G$ is defined to be the Dolbeault-Dirac quantisation of the principal stratum $(G \times \mathfrak{g})_{\mathbf{T}}/W(G, \mathbf{T})$ (with Kähler structure and pre-quantisation as above).

We prove that $\Gamma_c^\infty((G \times \mathfrak{g})_{\mathbf{T}}, \mathbb{E} \otimes L)$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E} \otimes L)$ in the graph norm of $D^L$. We proceed in a similar way as in Section 3.2, where we proved that $\Gamma_c^\infty((\mathbf{T} \times \mathfrak{t})_{\{e_{W(G,\mathbf{T})}\}}, \mathbb{E} \otimes L)$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E} \otimes L)$ in the graph norm of $D^L$, where $(\mathbf{T} \times \mathfrak{t})_{\{e_{W(G,\mathbf{T})}\}}$ is the open and dense submanifold of $\mathbf{T} \times \mathfrak{t}$ consisting of all points that have trivial isotropy group with respect to the action of $W(G, \mathbf{T})$ (*cf.* Proposition 3.2.11).

**Remark 4.1.8.** Note that we have constructed two possibly different partitions on $\mathbf{T} \times \mathfrak{t}$. On the one hand, we have the partition of $\mathbf{T} \times \mathfrak{t}$ into the connected components of all $(G \times \mathfrak{g})_H$ with $H \subset G$ containing $\mathbf{T}$. On the other hand, we have the partition of $\mathbf{T} \times \mathfrak{t}$ into the connected components of the isotropy type manifolds of the $W(G, \mathbf{T})$-action that we considered in Section 3.2.2. It is still an open question if both partitions coincide for any compact connected Lie group. In Remark 4.1.13 below we show that both partitions do coincide at least when $G$ is simply connected.

Neither do we know if $(G \times \mathfrak{g})_{\mathbf{T}}$ and $(\mathbf{T} \times \mathfrak{t})_{e_{W(G,\mathbf{T})}}$ coincide for general compact connected Lie groups. However, the inclusion $(G \times \mathfrak{g})_{\mathbf{T}} \subset (\mathbf{T} \times \mathfrak{t})_{e_{W(G,\mathbf{T})}}$ always holds.

If $(G \times \mathfrak{g})_{\mathbf{T}}$ is a proper subset of $(\mathbf{T} \times \mathfrak{t})_{e_{W(G,\mathbf{T})}}$, then we cannot simply apply Proposition 3.2.11 to conclude that $\Gamma_c^\infty((G \times \mathfrak{g})_{\mathbf{T}}, \mathbb{E} \otimes L)$ is dense in $\Gamma_c^\infty(\mathbf{T} \times \mathfrak{t}, \mathbb{E} \otimes L)$ in the graph norm of $D^L$. Therefore, we provide a different proof for this fact, which goes through for any compact connected Lie group $G$. The line of reasoning is the same as in Section 3.2.

We start by determining the complement of $(G \times \mathfrak{g})_{\mathbf{T}}$ in $\mathbf{T} \times \mathfrak{t}$.

**Lemma 4.1.9.** *There are only finitely many isotropy groups for the $G$-action on $G \times \mathfrak{g}$ that contain $\mathbf{T}$.*

*Proof.* If $H$ is the isotropy subgroup of $(g, Y)$, then $H = \mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$. If $H$ contains $\mathbf{T}$, then clearly both $\mathcal{Z}_G(g)$ and $\mathcal{Z}_G(Y)$ contain $\mathbf{T}$ as well. It therefore suffices to show that both the action of $G$ on itself by conjugation and the adjoint action of $G$ on $\mathfrak{g}$ only have finitely many different isotropy subgroups containing $\mathbf{T}$.

Since $G$ is a compact connected group there are only finitely many conjugacy classes of isotropy subgroups. Let $S = \{(H_i)\}$ be this finite set. Without loss of generality we can assume that $\mathbf{T} \cap G_{H_i}$ is non-empty for each $H_i$, so that $H_i \supset \mathbf{T}$ and $G_{H_i} \subset \mathbf{T}$ for each $H_i$. Pick an element $(H_j) \in S$ and suppose that $g \in \mathbf{T} \cap G_{(H_j)}$. Since $g \in G_{(H_j)}$ there exist $g_j \in G_{H_j} \subset \mathbf{T}$ and $h \in G$ such that $h g_j h^{-1} = g$. Since both $g_j$ and $g$ are in the maximal torus $\mathbf{T}$, there exists $n \in N_G(\mathbf{T})$ such that $n g_j n^{-1} = g$. Consequently, $\mathcal{Z}_G(g) = n H_j n^{-1}$. So, for each $x \in \mathbf{T}$, the isotropy group $\mathcal{Z}_G(x)$ is of the form $n H_i n^{-1}$ for some $H_i$ and some $n \in N_G(\mathbf{T})$. Because the Weyl group is finite of order $|W|$, the number of different subgroups that occur as an isotropy group of an element in $\mathbf{T}$ is at most $\#S \cdot |W|$.

A similar argument shows that the adjoint action of $G$ on $\mathfrak{g}$ only has finitely many isotropy groups containing $\mathbf{T}$. □

**Remark 4.1.10.** In [12, Corollary 1, pp. 316] one can find a proof for the stronger assertion that the number of closed subgroups of $G$ containing a given maximal torus $\mathbf{T} \subset G$, is finite.

We now consider the complement of $(\mathbf{G} \times \mathfrak{g})_{\mathbf{T}}$ in $\mathbf{T} \times \mathfrak{t}$. This complement is given by the union

$$P := \cup_{H \supsetneq \mathbf{T}, \text{isotropy group}} (G \times \mathfrak{g})^H.$$

By Lemma 4.1.9 this union is *finite*. The connected components of $(G \times \mathfrak{g})^H$ are closed submanifolds of $\mathbf{T} \times \mathfrak{t}$. If one of these connected components were open as well, then it would be a non-empty, closed and open subset of the connected space $\mathbf{T} \times \mathfrak{t}$ and therefore it must be equal to $\mathbf{T} \times \mathfrak{t}$. This is impossible if $H \supsetneq \mathbf{T}$, because $(G \times \mathfrak{g})_{\mathbf{T}}$ is non-empty. Consequently, the connected components of $(G \times \mathfrak{g})^H$ are of lower dimension than $(G \times \mathfrak{g})^{\mathbf{T}}$ if $H \supsetneq \mathbf{T}$. Since these connected components are symplectic manifolds, their codimensions as submanifolds of the symplectic manifold $\mathbf{T} \times \mathfrak{t}$ are at least 2.

We now proceed as in Section 3.2.3. Choose $\varepsilon > 0$ such that the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from the open $G$-invariant neighbourhood $U = \{Y \in \mathfrak{g} \mid |Y| < \varepsilon\}$ onto an open neighbourhood $V \subset G$ of $e_G$. Let $H \supsetneq \mathbf{T}$ be an isotropy group for the action of $G$ on $G \times \mathfrak{g}$. For each $(g, Y) \in (G \times \mathfrak{g})^H = G^H \times \mathfrak{g}^H$, we consider the chart

$$(\exp^{-1} \circ L_g^{-1}) \times \mathrm{id} : gV \times \mathfrak{g} \to U \times \mathfrak{g}.$$

Because $g$ is fixed under $H$, this diffeomorphism intertwines the $H$-action. Consequently, $\exp^{-1} \circ L_g^{-1}$ maps $(gV \times \mathfrak{g}) \cap (G \times \mathfrak{g})^H$ onto

$$U^H \times \mathfrak{g}^H \subset \mathfrak{t} \times \mathfrak{t} \subset \mathfrak{g} \times \mathfrak{g}.$$

Choose an orthonormal basis $\{e_i\}_{i=1}^t$ of $\mathfrak{t}$ such that $\{e_1, \ldots, e_{t-l}\}$ spans $\mathfrak{g}^H \subset \mathfrak{t}$, where $l \geq 1$. By reordering the coordinates induced by the orthonormal basis $\{e_1, \ldots, e_t\}$, the subset $(G \times \mathfrak{g})^H$ is mapped onto

$$\{(x_1, \ldots, x_{2t}) \in (U \cap \mathfrak{t}) \times \mathfrak{t} \mid x_{2t-2l+1} = \cdots x_{2t} = 0\} \subset \mathfrak{t} \times \mathfrak{t}$$

under the coordinate chart

$$(\exp^{-1} \circ L_g^{-1}) \times \mathrm{id} : (gV \cap \mathbf{T}) \times \mathfrak{t} \to (U \cap \mathfrak{t}) \times \mathfrak{t}$$

for $\mathbf{T} \times \mathfrak{t}$. Under this chart the Kähler structure on $\mathbf{T} \times \mathfrak{t}$ corresponds to the standard Kähler structure on $(U \cap \mathfrak{t}) \times \mathfrak{t} \subset \mathbb{R}^{2t}$. One can now proceed as in Section 3.2.3 from Proposition 3.2.8 onward, to obtain the following analogue of Proposition 3.2.11 and Theorem 3.2.12.

**Proposition 4.1.11.** *The domain $\Gamma_c^\infty((G\times\mathfrak{g})_{\boldsymbol{T}}, \mathbb{E}\otimes L)$ is dense in $\Gamma_c^\infty(\boldsymbol{T}\times\mathfrak{t}, \mathbb{E}\otimes L)$ in the graph norm of $D^L$. In particular, the twisted Dolbeault-Dirac operator $D^L$ is essentially self-adjoint on the domain $\Gamma_c^\infty((G\times\mathfrak{g})_{\boldsymbol{T}}, \mathbb{E}\otimes L)$ and the Dolbeault-Dirac quantisation of $(G\times\mathfrak{g})_{\boldsymbol{T}}$ is equal to the Dolbeault-Dirac quantisation of $\boldsymbol{T}\times\mathfrak{t}$.*

As a consequence we obtain the following Theorem.

**Theorem 4.1.12.** *If the Dolbeault-Dirac quantisation of $j^{-1}(0)/G$ is defined to be the Dolbeault-Dirac quantisation of the principal stratum, then the isomorphisms of Theorem 3.3.6 and Proposition 3.3.7 determine an isomorphism*

$$\mathcal{Q}_{DD}(j^{-1}(0)/G) \cong L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}.$$

*Proof.* The projection of the connected submanifold $(G\times\mathfrak{g})_{\mathbf{T}}$ is open, dense and connected in $j^{-1}(0)/G$, so it is the principal stratum of $j^{-1}(0)/G$. By definition, it is

$$(G\times\mathfrak{g})_{\mathbf{T}}/(N_G(\mathbf{T})/\mathbf{T}) \cong (G\times\mathfrak{g})_{\mathbf{T}}/W(G,\mathbf{T}).$$

The Theorem is now a consequence of Proposition 4.1.11, of Theorem 3.3.6 and of the fact that the Dolbeault-Dirac quantisation of $\mathbf{T}\times\mathfrak{t}$ is $W(G,\mathbf{T})$-equivariantly isomorphic to $L^2(\mathbf{T})$ by Proposition 3.3.7. ☐

**Remark 4.1.13.**

1. If the partition of $\mathbf{T}\times\mathfrak{t}$ into the connected components of $(G\times\mathfrak{g})_H^{(g,Y)}$ with $H\supset\mathbf{T}$ coincides with the stratification of $\mathbf{T}\times\mathfrak{g}$ into connected components of isotropy type manifolds with respect to the $W(G,\mathbf{T})$-action on $\mathbf{T}\times\mathfrak{t}$, then the first part of Theorem 4.1.12 follows immediately from Section 3.2. In fact, it is already sufficient that the principal strata coincide.

2. It remains an open question if both partitions always coincide. If we partition by isotropy type manifolds, rather than by their connected components, the partitions are not always equal. For instance, if $G=SO(3)$, then there exist two different elements in a maximal torus $\mathbf{T}\cong U(1)$ of $SO(3)$ that are fixed under the Weyl-group but have different isotropy group for the action of $SO(3)$ on itself by conjugation. This phenomenon persists if one considers the $SO(3)$-action on $SO(3)\times\mathfrak{so}(3)$. However, both partitions do agree up to connected components.

If $G$ is simply connected, the situation is much better. In that case, the partitions into isotropy type submanifolds do coincide, so that the above phenomenon in the example of $SO(3)$ can only occur for non-simply connected Lie groups.

**Proposition 4.1.14.** *If $G$ is a simply connected group, then the isotropy type stratifications of $T^*G//G$ and $(\boldsymbol{T}\times\mathfrak{t})/W(G,\boldsymbol{T})$ coincide.*[1]

[1] I am grateful to Reyer Sjamaar for explaining me a possible proof.

*Proof.* Before we prove the claim that the partitions into isotropy type submanifolds do coincide when $G$ is simply connected, we first recall some facts for general compact connected Lie groups. If $H \subset G$ is a closed, connected Lie subgroup containing $\mathbf{T}$, then $\mathbf{T}$ is also a maximal torus in the compact connected Lie group $H$. Therefore, there is a corresponding root system $R(H,\mathbf{T})$ which is contained in $R(G,\mathbf{T})$. The map $H \mapsto R(H,\mathbf{T})$ is a bijection between closed, connected Lie subgroups of $G$ containing $\mathbf{T}$ and the closed symmetric subsets of $R(G,\mathbf{T})$ (see [12, Proposition 12, pp. 316]). Here, $P \subset R(G,\mathbf{T})$ is called symmetric if $-P = P$ and closed if $(P+P) \cap R(G,\mathbf{T}) \subset P$. The corresponding group $N(H)/H$ is the Weyl group $W(H,\mathbf{T})$ and it is a subgroup of $W(G,\mathbf{T})$. The Weyl group $W(H,\mathbf{T})$ is generated by the reflections $r_\alpha$, $\alpha \in R(H,\mathbf{T})$. In fact, all reflections in $W(H,\mathbf{T})$ are of this form (see [50, Proposition 1.14]). Consequently, the map $H \mapsto W(H,\mathbf{T})$ is an injective map into the set of subgroups of $W(G,\mathbf{T})$.

Now, if $G$ is simply connected, the crucial property is that $\mathcal{Z}_G(g)$ is connected for every $g \in G$ ([12, Theorem 1 or Corollary 1, pp. 329]). Proposition 14 on pp. 317 *loc. cit.* then shows that the isotropy group $W_g \subset W(G,\mathbf{T})$ is equal to the group $N_{\mathcal{Z}_G(g)}(\mathbf{T})/\mathbf{T}$, where $\mathbf{T}$ is regarded as a maximal torus in the compact connected group $\mathcal{Z}_G(g)$. By the previous paragraph any two connected, closed subgroups $H_1$, $H_2$ containing $\mathbf{T}$ are equal if and only if $N_{H_1}(\mathbf{T})/\mathbf{T} = N_{H_2}(\mathbf{T})/\mathbf{T}$ and so $W_g = W_{g'}$ if and only if $\mathcal{Z}_G(g) = \mathcal{Z}_G(g')$. Similarly, for any compact, not necessarily simply connected, connected Lie group $G$ the centraliser $\mathcal{Z}_G(Y)$ is connected and we have $W_Y = N_{\mathcal{Z}_G(Y)}(\mathbf{T})/\mathbf{T}$ and $W_Y = W_{Y'}$ if and only if $\mathcal{Z}_G(Y) = \mathcal{Z}_G(Y')$. This proves that the natural isomorphism $G/\operatorname{Ad} G \to \mathbf{T}/W(G,\mathbf{T})$ identifies the stratifications if $G$ *is simply connected*, whereas the natural isomorphism $\mathfrak{g}/\operatorname{Ad} G \to \mathfrak{t}/W(G,\mathbf{T})$ *always* identifies the stratifications, i.e. even if $G$ is not simply connected.

Now consider $G \times \mathfrak{g}$ for simply-connected $G$ and suppose that $(g,Y) \in \mathbf{T} \times \mathfrak{t}$. We show that $G_{(g,Y)} = \mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$ is connected. Note that

$$\mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y) = \{h \in \mathcal{Z}_G(g) \mid \operatorname{Ad}_h Y = Y\} = \mathcal{Z}_{\mathcal{Z}_G(g)}(Y),$$

where we consider $Y$ as an element in the Lie algebra of $\mathcal{Z}_G(g)$. By simply-connectedness of $G$, the group $\mathcal{Z}_G(g)$ is connected. Therefore, $\mathcal{Z}_G(g)$ is a compact connected Lie group and so $\mathcal{Z}_{\mathcal{Z}_G(g)}(Y)$ is connected. Thus, $G_{(g,Y)} = \mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$ is connected.

On the one hand the isotropy group of $(g,Y)$ for the $W(G,\mathbf{T})$-action is

$$\begin{aligned} N_{\mathcal{Z}_G(g)}(\mathbf{T})/\mathbf{T} \cap N_{\mathcal{Z}_G(Y)}(\mathbf{T})/\mathbf{T} &= (N_{\mathcal{Z}_G(g)}(\mathbf{T}) \cap N_{\mathcal{Z}_G(Y)}(\mathbf{T}))/\mathbf{T} \\ &= N_{\mathcal{Z}(g) \cap \mathcal{Z}_G(Y)}(\mathbf{T})/\mathbf{T}, \end{aligned}$$

and on the other hand the isotropy group for the $G$-action is $\mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$. Because for each $(g,Y) \in \mathbf{T} \times \mathfrak{t}$ the subgroup $\mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)$ is closed, connected, and contains $\mathbf{T}$, we have

$$N_{\mathcal{Z}_G(g) \cap \mathcal{Z}_G(Y)}(\mathbf{T}) = N_{\mathcal{Z}(g') \cap \mathcal{Z}(Y')}(\mathbf{T}) \iff \mathcal{Z}(g) \cap \mathcal{Z}(Y) = \mathcal{Z}(g') \cap \mathcal{Z}(Y').$$

Or in other words, the isotropy groups $W_{(g,Y)}$ and $W_{(g',Y')}$ are equal if and only if the isotropy groups $G_{(g,Y)}$ and $G_{(g',Y')}$ are equal. This proves that the isotropy type stratifications of $T^*G//G$ and $(\mathbf{T} \times \mathfrak{t})/W(G,\mathbf{T})$ coincide when $G$ is simply connected. □

## 4.2 Reduction after quantisation

Let $G$ be an element of $\mathcal{C}_K$ and consider $T^*G$ with the action of $G$ on $T^*G$ induced by the action of $G$ on itself by conjugation. We showed in Theorem 2.4.10 that the Dolbeault-Dirac quantisation of $T^*G$ is equivariantly isomorphic to $L^2(G, dg)$, where $dg$ denotes the Haar measure on $G$. In the previous section we determined the reduced space $j^{-1}(0)/G$ and its quantisation, the so-called quantisation after reduction. To show that quantisation commutes with reduction, we need to perform reduction after quantisation and show that the resulting Hilbert space is unitarily isomorphic to the quantisation of $j^{-1}(0)/G$, which is $L^2(\mathbf{T})^{W(G,\mathbf{T})}$ by Theorem 4.1.12. As we saw in Section 1.4.1, the reduction procedure (at value 0 of the moment map) at the classical side corresponds to taking the $G$-invariant part of the quantum Hilbert space at the quantum side.

The following Lemma is the Weyl integration formula. This formula immediately implies that reduction after quantisation yields $L^2(\mathbf{T})^{W(G,\mathbf{T})}$ if $G \in \mathcal{C}_K$.

**Lemma 4.2.1.** *Let $G$ be a compact connected Lie group and $\boldsymbol{T}$ a maximal torus. Write $\delta : \boldsymbol{T} \to \mathbb{C}$ for the Weyl-denominator function. Then there exists $c > 0$ such that*

$$f \mapsto c|\delta| \cdot f|_{\boldsymbol{T}}$$

*defines a unitary isomorphism from $L^2(G, dg)^{\mathrm{Ad}\, G}$ onto $L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}$.*

*Proof.* See e.g. [28, Corollary 3.14.2]. □

**Proposition 4.2.2.** *If $G \in \mathcal{C}_K$, reduction after quantisation yields the Hilbert space $L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}$.*

We are now ready to summarise the chapters in this part of the thesis into the following theorem.

**Theorem 4.2.3.** *Let $G$ be a compact connected Lie group. Consider the action of $G$ on $T^*G$ induced by the action of $G$ on itself by conjugation. Endow $T^*G$ with its standard Kähler structure. Let $(L, \nabla^L)$ be the equivariant pre-quantisation given by the trivial hermitian line bundle $L = T^*G \times \mathbb{C}$ with equivariant hermitian connection $\nabla^L = d + 2\pi i\theta$. Then there is an induced Kähler structure and pre-quantisation line bundle on the principal stratum of $j^{-1}(0)/G$. Define the quantisation of $T^*G$ to be the Dolbeault-Dirac quantisation and define the quantisation of $j^{-1}(0)/G$ to be the Dolbeault-Dirac quantisation of its principal stratum. Fix a maximal torus $\boldsymbol{T}$ of $G$. Then:*

1. *the reduced space* $j^{-1}(0)/G$ *is homeomorphic to* $(\boldsymbol{T} \times \mathfrak{t})/W(G, \boldsymbol{T})$*;*

2. *the quantisation of* $j^{-1}(0)/G$ *is unitarily isomorphic to* $L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}$*.*

   *Moreover, if* $G \in \mathcal{C}_K$*, i.e. the canonical line bundle on* $T^*G$ *is semi-negative, then:*

3. *the Dolbeault-Dirac quantisation of* $T^*G$ *is* $G$*-equivariantly, unitarily isomorphic to* $L^2(G, dg)$*, where* $G$ *acts on* $L^2(G, dg)$ *as* $(g, f) \mapsto \mathrm{Ad}^*_{g^{-1}} f = f \circ \mathrm{Ad}_{g^{-1}}$*.*

4. *reduction after quantisation is unitarily isomorphic to* $L^2(\boldsymbol{T})^{W(G,\boldsymbol{T})}$*.*

*Here, all isomorphisms arise in a natural way.*

*Consequently, Dolbeault-Dirac quantisation commutes with reduction if* $G \in \mathcal{C}_K$*. In particular, Dolbeault-Dirac quantisation commutes with reduction if* $G = SU(2)$*.*

*Proof.* The first and second statement are precisely the statements of Lemma 4.1.2 and Theorem 4.1.12, respectively. The third and fourth statement are contained in Theorem 2.4.10 and Proposition 4.2.2 and in Section 2.A it was proved that $SU(2) \in \mathcal{C}_K$. □

## 4.3 Outlook

For cotangent bundles of compact connected Lie groups, we could not define Dolbeault-Dirac quantisation as an element of some KK-group or generalised representation ring. However, if $G \in \mathcal{C}_K$, Definition 1.4.3, which is then quite close to the original definition of Dolbeault-Dirac quantisation because $\ker D^L_- = 0$, yields the $G \times G$-equivariant Hilbert space $L^2(G, dg)$. One of the next steps is to look for a general framework, like KK-theory or some generalised representation ring, where this Dolbeault-Dirac quantisation can be interpreted, even in the case when we restrict the $G \times G$-action to the diagonal $\Delta(G \times G)$.

The quantisation of the singular Marsden-Weinstein quotient $T^*G// \mathrm{Ad}\, G$ was defined to be the Dolbeault-Dirac quantisation of its principal stratum. We have seen that this definition leads to a quantisation-commutes-with-reduction result if $G \in \mathcal{C}_K$. This raises the question if quantisation always commutes with reduction if one defines in this way the Dolbeault-Dirac quantisation of the singular Marsden-Weinstein quotient for any proper, strongly Hamiltonian $G$-action on a, let's say compact or, more generally, geodesically complete, connected Kähler manifold.

The quantisation of singular quotients using Dolbeault-Dirac operators was also studied in [72], in the case of a compact group acting in a strongly Hamiltonian fashion on a *compact* symplectic orbifold. There, the singular quotient is quantised by first constructing a *desingularisation* of the singular quotient and subsequently taking the index of a Dolbeault-Dirac operator on this desingularised space.

If the singular quotient could be quantised by taking the Dolbeault-Dirac quantisation of its principal stratum, then the corresponding twisted Dolbeault-Dirac operator on the, possibly non-compact, principal stratum should probably be self-adjoint and have well-defined index. Moreover, its index should then agree with the index of the Dolbeault-Dirac operator on the desingularised space in *loc. cit.* It is part of future research to find an answer to these questions.

There are also other open questions concerning the case $T^*G$. We list of some of them here.

1. Does the stratification of the Marsden-Weinstein quotient $j^{-1}(0)/G$ always coincide with the stratified structure of $\mathbf{T} \times \mathfrak{t}/W(G, \mathbf{T})$ (where the latter is stratified by the projections of the connected components of the orbit type submanifolds)? If not, what about the principal strata?

2. Is any compact connected Lie group an element of $\mathcal{C}_K$? If not, which compact connected Lie groups are?

3. How does the Dolbeault-Dirac quantisation of $(T^*G, \omega)$ depend on the chosen complex structure? Or, is the canonical line bundle semi-negative for all complex structures compatible with the symplectic structure $\omega$?

# PartII

# Globally non-trivial gauge theories as almost-commutative manifolds

Within the framework of Connes' noncommutative geometry, we define and study globally non-trivial (sometimes also called topologically non-trivial) almost-commutative manifolds. In particular, we focus on those almost-commutative manifolds that lead to a description of a (classical) gauge theory on the underlying base manifold. Such an almost-commutative manifold is described in terms of a 'principal module', which we build from a principal fibre bundle and a finite spectral triple. We also define the purely algebraic notion of 'gauge modules', and show that this yields a proper subclass of the principal modules. We describe how a principal module leads to the description of a gauge theory, and we provide two basic yet illustrative examples.

The following work is based on a pre-print [10] written by the author and Koen van den Dungen (Australian National University and University of Wollongong).

# Chapter 5

# Preliminaries

## 5.1 Introduction

The framework of Connes' noncommutative geometry [19] provides a generalisation of ordinary Riemannian manifolds to noncommutative manifolds. Within this framework, the special case of a (globally trivial) almost-commutative manifold has been shown to describe a (classical) gauge theory over a Riemannian spin manifold, which ultimately led to a description of the full Standard Model of high energy physics, including the Higgs mechanism and neutrino mixing [15].

The gauge theories mentioned above are, by construction, topologically trivial (in the sense that the corresponding principal bundles are globally trivial bundles). The aim is to adapt the framework in order to allow for globally non-trivial gauge theories as well. Such a generalisation has previously been obtained for the special case of Yang-Mills theory [11].

Let us briefly recall how a description of a gauge theory is obtained from an almost-commutative manifold in the globally trivial case (for a more detailed introduction we refer to e.g. [89]). We start with a smooth compact 4-dimensional Riemannian spin manifold $M$, which can be described in terms of a (real, even) spectral triple $(C^\infty(M), L^2(\mathtt{S}), \not{D}, \gamma_5, J_M)$, where $\not{D}$ is the Dirac operator on the spinor bundle $\mathtt{S} \to M$, $\gamma_5$ is the grading operator and $J_M$ is charge conjugation [21]. If we take a real even finite spectral triple $(A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$, one can consider the product triple

$$M \times F := \big(C^\infty(M, A_F), L^2(\mathtt{S}) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\big). \tag{5.1}$$

Write $\mathcal{U}(\mathcal{A})$ for the group of unitary elements of a $*$-algebra $\mathcal{A}$. For a real spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J)$, we define its gauge group as

$$\mathcal{G}(T) := \big\{uJuJ^* \mid u \in \mathcal{U}(\mathcal{A})\big\} \cong \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A}_J), \tag{5.2}$$

where $\mathcal{A}_J$ is the central subalgebra of $\mathcal{A}$ consisting of all elements $a \in \mathcal{A}$ for which $aJ = Ja^*$. Now suppose we have a real even finite spectral triple $F = (A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ with gauge group $G_F = \mathcal{G}(F)$. Then the product triple $M \times F$ defined above has gauge group $\mathcal{G}(M \times F) \cong C^\infty(M, G_F)$ (at least when $M$ is simply connected[1]), which coincides with the 'classical' notion of the gauge group of the (globally trivial) principal $G_F$-bundle $\mathtt{P} = M \times G_F$. One can show that the inner fluctuations of the operator $\not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F$ yield gauge fields (i.e. connection forms on the principal bundle $\mathtt{P}$) as well as scalar fields (interpreted as Higgs fields in the Standard Model). Finally, the spectral action principle [14] yields a (gauge-invariant) Lagrangian from the data of the triple $M \times F$.

This part of the thesis is organised as follows. We start in this chapter by gathering some preliminary material that will be useful later on. In Chapter 6 we then describe the generalisation of the product triples $M \times F$ to (in general globally non-trivial) almost-commutative manifolds. We show that these almost-commutative manifolds are naturally given by the internal Kasparov product of an *internal space* $I$ (replacing the finite spectral triple $F$) with the underlying manifold $M$.

While every globally trivial almost-commutative manifold describes a gauge theory, this no longer holds for arbitrary globally non-trivial almost-commutative manifolds. In Chapter 7 we therefore focus our attention on those internal spaces that will allow us to obtain a gauge theory. After briefly recalling the classification of finite spectral triples, we define the notion of a *principal module*, which is an internal space built from a finite spectral triple $F$ and a principal fibre bundle $\mathtt{P}$ over $M$. We show that the algebraic definition of the gauge group of a principal module (defined similarly to (5.2)) coincides precisely with the usual definition of the gauge group of $\mathtt{P}$ (i.e. the vertical automorphisms of $\mathtt{P}$), provided that the underlying manifold $M$ is simply connected.

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. Whereas principal modules are constructed from geometric objects (namely principal fibre bundles), we devote Section 7.3 to the purely *algebraic* notion of what we call a *gauge module*. We prove that these gauge modules form a proper subclass of the principal modules, which are characterised by a lift of $\mathtt{P}$ to a principal $\mathcal{U}(A_F)$-bundle (where $A_F$ is the algebra of the finite spectral triple $F$).

By equipping a principal module with a connection and a 'mass matrix', we construct the corresponding *principal* almost-commutative manifold in Chapter 8. The remainder of this chapter is used to establish the main goal of this part of the thesis; namely, we describe in detail how this principal almost-commutative manifold describes a gauge theory on $M$.

[1] The isomorphism $\mathcal{G}(M \times F) \cong C^\infty(M, G_F)$, stated in [11, Proposition 4.3] and [89, §2.4.3], is only valid under some additional conditions, and simply-connectedness of $M$ is always sufficient. We shall prove this in general for the globally non-trivial case in Theorem 7.2.7.

In Section 8.5 we provide two basic but illustrative examples of such gauge theories, namely Yang-Mills theory and electrodynamics. The Yang-Mills example in particular shows that not every principal module is a gauge module. However, we also show that the Yang-Mills example *is* a gauge module when the underlying manifold is simply-connected and 4-dimensional. Hence on such manifolds we have no example of a principal module which is not a gauge module.

We finish with an outlook on possible future work.

### Notation

All $C^*$-algebras and Hilbert modules will be denoted with capital letters (e.g. $A, B, E \ldots$), their smooth sub-algebras or pre-$C^*$-algebras (i.e. densely defined $*$-sub-algebras that are closed under the holomorphic functional calculus) and Hilbert pre-modules will be denoted with curly letters (e.g. $\mathcal{A}, \mathcal{B}, \mathcal{E}, \ldots$). The main exception to these conventions is the notation $\mathcal{H}$, which always denotes a complex Hilbert space. By $M$ we denote a smooth connected compact Riemannian spin manifold. From now on, bundles over $M$ are denoted with 'typewriter font', where we use $\mathtt{B}$ for algebra bundles, $\mathtt{E}$ for vector bundles, $\mathtt{P}$ for principal fibre bundles, $\mathtt{G}$ for group bundles, and $\mathtt{S}$ for the spinor bundle. Continuous (resp. smooth) sections of a bundle $\mathtt{E} \to M$ are denoted by $\Gamma(\mathtt{E})$ (resp. $\Gamma^\infty(\mathtt{E})$). In this part we omit the underlying manifold in the notation for the space of sections, because the underlying manifold will always be $M$. Also, in this part, unbounded operators are always assumed to be closed.

# 5.2 Preliminaries

## 5.2.1 Fibre bundles

Different kinds of fibre bundles occur frequently in gauge theories. The definitions concerning fibre bundles may differ from the definitions in some other literature, including [11], so that we find it necessary to include a list of the definitions we use. All manifolds are assumed to be *smooth* and all maps between them are also assumed to be smooth.

A ***fibre bundle*** (*cf.* [57]) with fibre $F$ over a smooth manifold $M$, is a smooth manifold $\mathtt{E}$ together with a surjective smooth map $\pi : \mathtt{E} \to M$, such that $\pi^{-1}(x)$ is diffeomorphic to $F$ for each $x \in M$, and such that for each $x \in M$ there exist an open neighbourhood $U$ of $x$ and a diffeomorphism $h_U : \pi^{-1}(U) \to U \times F$ such that $\pi = \mathrm{proj}_1 \circ h_U$ on $\pi^{-1}(U)$. The pair $(U, h_U)$ is called a ***local trivialisation*** for $\mathtt{E}$. For two local trivialisations $(U, h_U)$ and $(V, h_V)$ for which $U \cap V \neq \emptyset$, we define the ***transition function*** $g_{VU} := h_V \circ h_U^{-1} \in C^\infty(U \cap V, \mathrm{Diff}(F))$, that is, for each $x \in U \cap V$ we have a diffeomorphism $g_{VU}(x) \colon F \to F$ which depends smoothly on $x$. We denote a fibre bundle by $\pi : \mathtt{E} \to M$ or by $\pi_{\mathtt{E}} : \mathtt{E} \to M$ if we want to distinguish the projection map of $\mathtt{E}$ from projections maps of other bundles.

In many cases the fibres of the bundle $\mathtt{E}$ will be assumed to have additional structure, compatible with the local triviality. The following definition captures all relevant possibilities in an abstract manner.

**Definition 5.2.1.** Let $\mathcal{C}$ be some category with objects $\mathrm{Ob}_{\mathcal{C}}$ and morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ for all objects $A, B \in \mathrm{Ob}_{\mathcal{C}}$. Let $M$ be a smooth manifold. A fibre bundle $\pi\colon \mathtt{E} \to M$ with fibre $F$ is called a ***$\mathcal{C}$-bundle*** if $F \in \mathrm{Ob}_{\mathcal{C}}$ and if on each local trivialisation $(U, h_U)$ the map $h_U|_{\pi^{-1}(x)}\colon \pi^{-1}(x) \to x \times F$ is an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(\pi^{-1}(x), F)$.

If $\mathcal{C}$ is the category of *finite-dimensional* vector spaces, *finite-dimensional* ($*$-)algebras, or Lie groups, then $\mathcal{C}$-bundles are referred to as vector bundles, ($*$-)algebra bundles, or group bundles[2] (respectively).

**Remark 5.2.2.** Note that according to Definition 5.2.1 a ($*$-)algebra bundle is always locally trivial, in contrast with the definition of ($*$-)algebra bundle in [11]. The weaker notion given in [11] will here be referred to as algebra fibration, and is defined as follows.

**Definition 5.2.3** (see also [11, Definition 3.1] for more details)**.** An ***algebra fibration*** $\mathtt{B}$ over $M$ is a vector bundle $\mathtt{B}$ over $M$ together with a smooth vector bundle morphism $\mu\colon \mathtt{B}\otimes\mathtt{B} \to \mathtt{B}$ that satisfies $\mu(b_1 \otimes \mu(b_2 \otimes b_3)) = \mu(\mu(b_1 \otimes b_2) \otimes b_3)$ for all $b_1, b_2, b_3$ that are in the same fibre. Thus, the map $(b_1, b_2) \mapsto \mu(b_1 \otimes b_2)$, $(b_1, b_2 \in \mathtt{B}_x)$ induces an associative product on the fibres.

The algebra fibration $\mathtt{B}$ is called a ***$*$-algebra fibration*** if in addition there is a smooth anti-linear fibre bundle morphism $*\colon \mathtt{B} \to \mathtt{B}$ such that $*_x$ turns $\mathtt{B}_x$ into a $*$-algebra.

The space of sections, $\Gamma^\infty(\mathtt{B})$, of a ($*$-)algebra fibration $\mathtt{B}$ forms a ($*$-)algebra with fibre-wise addition, multiplication and involution. The fibration $\mathtt{B}$ is called ***unital*** if its algebra of sections $\Gamma^\infty(\mathtt{B})$ is unital.

Note that $C^\infty(M) \subset \Gamma^\infty(\mathtt{B})$ if and only if $\mathtt{B}$ is unital.

**Example 5.2.4.** An algebra fibration is not necessarily locally trivial as an algebra bundle. An example is given by the following non-unital algebra fibration. Choose $M = (-1, 1)$ and consider the trivial vector bundle $\mathtt{B} := M \times \mathbb{C} \to M$, where on each fibre $\mathbb{C}_t$ ($t \in (-1, 1)$) the product is given by $(a_t, b_t)_t = t a_t b_t$. This is a non-unital algebra fibration (in the sense of Definition 5.2.3), but *not* an algebra bundle (in the sense of Definition 5.2.1). For each $s, t \neq 0$ the fibres are isomorphic by the isomorphism $\phi_{s,t}$ sending $\mathbb{C}_s \ni a \to \frac{s}{t} a$ (since $\phi_{s,t}(a \cdot_s b) = \phi_{s,t}(sab) = \frac{s^2 ab}{t}$ and $\phi_{s,t}(a) \cdot_t \phi_{s,t}(b) = \frac{sa}{t} \cdot_t \frac{sb}{t} = t\frac{s^2 ab}{t^2} = \frac{s^2 ab}{t}$). However, the product in $\mathbb{C}_0$ is the trivial zero product. Hence the fibres are not all isomorphic as algebras.

[2]Note that group bundles are not the same as principal fibre bundles, which we will define in Definition 5.2.13.

**Definition 5.2.5.** Let $\pi_1\colon \mathtt{E}_1 \to M$ and $\pi_2\colon \mathtt{E}_2 \to M$ be fibre bundles. A ***bundle morphism*** $\phi\colon \mathtt{E}_1 \to \mathtt{E}_2$ is a smooth map such that $\pi_2 \circ \phi = \pi_1$. If $\mathtt{E}_1$ and $\mathtt{E}_2$ are $\mathcal{C}$-bundles, then $\phi$ is called a ***$\mathcal{C}$-bundle morphism*** if $\phi|_{\pi_1^{-1}(x)}\colon \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is an element of $\mathrm{Mor}_{\mathcal{C}}(\pi_1^{-1}(x), \pi_2^{-1}(x))$ for each $x \in M$.

The space of smooth sections of a vector bundle $\mathtt{E}$ is denoted by $\Gamma^\infty(\mathtt{E})$. This is a finitely generated projective $C^\infty(M)$-module, where the functions in $C^\infty(M)$ act on $\Gamma^\infty(\mathtt{E})$ by fibrewise addition and scalar multiplication, i.e. $(s+t)(x) = s(x) + t(x)$, $(fs)(x) = f(x)s(x)$ for all $f \in C^\infty(M)$, $s, t \in \Gamma^\infty(\mathtt{E})$, $x \in M$. If $\phi : \mathtt{E}_1 \to \mathtt{E}_2$ is a vector bundle morphism, then

$$\phi_* : \Gamma^\infty(\mathtt{E}_1) \to \Gamma^\infty(E_2), \quad (\phi_* s)(x) = \phi(s(x))$$

is a $C^\infty(M)$-module morphism. In fact, the assignment $\mathtt{E} \mapsto \Gamma^\infty(\mathtt{E})$ on objects and the assignment $s \mapsto \phi_* s$ on morphisms determines an equivalence of the categories of vector bundles over $M$ and the category of finitely generated projective modules over $C^\infty(M)$. This is the Serre-Swan theorem [86].

Note that, similarly to the case of vector bundles, the set of sections $\Gamma^\infty(\mathtt{G})$, where $\mathtt{G}$ is a group bundle, forms a group with fibre-wise multiplication and inverse.

**Example 5.2.6** (Unitary group bundle)**.** If $\mathtt{B}$ is a unital $*$-algebra bundle (i.e. if $\Gamma^\infty(\mathtt{B})$ is unital), we can define the ***unitary group bundle*** of $\mathtt{B}$ as

$$\mathcal{U}(\mathtt{B}) := \{b \in \mathtt{B} \mid bb^* = b^*b = 1\}.$$

Then $\mathcal{U}(\mathtt{B})$ is a fibre subbundle of $\mathtt{B}$, which forms a group bundle with group multiplication of $\mathcal{U}(\mathtt{B})_x = \mathcal{U}(\mathtt{B}_x)$ inherited from the algebra multiplication of $\mathtt{B}_x$, and group inverse given by the involution $*$. We also find that the sections of the unitary group bundle are equal to the unitary sections of the algebra bundle:

$$\Gamma^\infty(\mathcal{U}(\mathtt{B})) = \mathcal{U}(\Gamma^\infty(\mathtt{B})).$$

**Remark 5.2.7.** Using a partition of unity argument one can show that for each point $e \in \mathtt{E}_x$ there exists a section $s \in \Gamma^\infty(\mathtt{E})$ such that $s(x) = e$. The same is true for $\mathcal{U}(\mathtt{B})$, for $\mathtt{B}$ a $*$-algebra bundle with the fibre isomorphic to a finite-dimensional $C^*$-algebra, as the following argument shows. Let $u \in \mathcal{U}(\mathtt{B})_x$ be given. Then $u$ can be written as $u = \exp(ia)$ for some hermitian element $a \in \mathtt{B}_x$. Take a section $s = s^* \in \Gamma^\infty(\mathtt{B})$ such that $a = s(x)$. Then $\tilde{s} = \exp(is(x))$ is a smooth section of $\mathcal{U}(\mathtt{B})$ such that $\tilde{s}(x) = u$.

**Definition 5.2.8.** Let $\pi\colon \mathtt{E} \to M$ be a $\mathcal{C}$-bundle with fibre $F$. A fibre subbundle $\pi'\colon \mathtt{E}' \to M$ with fibre $F'$ is a ***$\mathcal{C}$-subbundle*** if $F' \in \mathrm{Ob}_{\mathcal{C}}$ and there exist local trivialisations $\{(U, h_U)\}$ for $\mathtt{E}$ such that $h_U(\mathtt{E}'|_U) \cong U \times \iota(F')$, where $\iota$ is an injective morphism in $\mathrm{Mor}_{\mathcal{C}}(F', F)$.

**Example 5.2.9** (Endomorphism bundle)**.** Let $\pi_{\mathtt{E}}\colon \mathtt{E} \to M$ be a (hermitian) vector bundle with fibre $V$ and local trivialisations $(U, h_U^{\mathtt{E}})$. Then the bundle of endomorphisms $\mathrm{End}(\mathtt{E})$ is a unital $(*\text{-})$algebra bundle over $M$ with fibre $\mathrm{End}(V)$ and local trivialisations $(U, h_U^{\mathrm{End}(\mathtt{E})})$ induced from $(U, h_U^{\mathtt{E}})$. Every $(*\text{-})$algebra subbundle of $\mathrm{End}(\mathtt{E})$ is also a $(*\text{-})$algebra bundle over $M$.

We conclude this subsection by recalling a Serre-Swan-like result for $(*\text{-})$algebra fibrations. We first need a definition.

**Definition 5.2.10.** Let $R$ be a commutative (involutive) ring. An ***R-module algebra*** is an $R$-module $A$ with an associative multiplication $A \times A \to A\colon (a, b) \mapsto ab$ which is $R$-bilinear:

$$r(ab) = (ra)b = a(rb), \qquad \forall a, b \in A, r \in R.$$

An $R$-module algebra is called ***involutive*** if there exists a map $*\colon A \to A$ such that

$$(ab)^* = b^*a^*; \quad (a+b)^* = a^* + b^*; \quad (ra)^* = r^*a^*; \quad (r, s \in R, \quad a, b \in A).$$

An ***R-module algebra homomorphism*** is an $R$-linear map that preserves multiplication.

If $\mathtt{B}$ is a $(*\text{-})$algebra fibration, then the finitely generated projective module $\Gamma^\infty(\mathtt{B})$ has a natural multiplication and $*$-structure defined by the fibrewise operations, e.g. if $s, t \in \Gamma^\infty(\mathtt{B})$ are sections, then the product section $st$ is defined by $st(x) = s(x)t(x)$ for all $x \in M$. This turns $\Gamma^\infty(\mathtt{B})$ into an (involutive) $C^\infty(M)$-module algebra that is finitely generated projective as a $C^\infty(M)$-module.

**Theorem 5.2.11** ([11, Theorem 3.8])**.** *Let $M$ be a compact manifold. There is an equivalence between the category of (unital) $(*\text{-})$algebra fibrations over $M$ and the category of (unital) (involutive) $C^\infty(M)$-module algebras that are finitely generated projective as $C^\infty(M)$-modules.*

**Remark 5.2.12.** We again emphasise the difference between algebra bundles and algebra fibrations as mentioned in Remark 5.2.2. It would be interesting to generalise the above theorem to algebra bundles. However, it is unclear what algebraic conditions one needs to impose on a $C^\infty(M)$-module algebra $\mathcal{B} = \Gamma^\infty(\mathtt{B})$ to ensure that the algebra fibration $\mathtt{B}$ is in fact an algebra bundle.

### 5.2.2 Principal fibre bundles and (classical) gauge theories

In this subsection, we briefly recall the definition of a principal fibre bundle, and some basic results. We refer to [57, Chapter I] and [9] for more details.

**Definition 5.2.13.** A ***principal fibre bundle*** $\mathtt{P}$ over $M$ with ***structure group*** $G$ (or a principal $G$-bundle for short) consists of a fibre bundle $\mathtt{P} \xrightarrow{\pi} M$ equipped with a smooth right action of $G$ that acts freely and transitively on the fibres, such that for a local trivialisation $(U, h_U)$ of $\mathtt{P}$, the map $h_U$ intertwines the right action of $G$ on $\mathtt{P}|_U$ with the natural right action of $G$ on $U \times G$.

If $(U_i, h_i)$ and $(U_j, h_j)$ are two local trivialisations for the principal fibre bundle $\mathtt{P}$, then, for fixed $x \in U_i \cap U_j$, there is an element $g_x \in G$ such that the composition $h_i \circ h_j^{-1}$ maps $(x, h)$ to $(x, g_x h)$ for all $h \in G$. The function $g_{ij} : x \mapsto g_x$ is known as the ***transition function*** between the local trivialisations $(U_i, h_i)$ and $(U_j, h_j)$. If $(U_i, h_i)$, $(U_j, h_j)$, $(U_k, h_k)$ are local trivialisations, then

$$g_{ij}(x)g_{jk}(x)g_{ki}(x) = \mathrm{id} \in G \tag{5.3}$$

on $U_i \cap U_j \cap U_k$. Equation (5.3) is also known as the cocycle condition. If $\{(U_i, h_i)\}$ is a set of local trivialisations such that $\cup U_i = M$, then we say that $\mathtt{P}$ has transition functions $\{g_{ij}\}$. One can construct the bundle $\mathtt{P}$ as soon as one knows its ($G$-valued) transition functions for some set of local trivialisations $\{(U_i, h_i)\}$ such that $\cup_i U_i = M$:

**Theorem 5.2.14** (Reconstruction theorem [57, Chapter I, Proposition 5.2.])**.** *Let $M$ be a compact manifold, $G$ a Lie-group, and $\{U_i\}_{i \in I}$ an open covering of $M$. Suppose that for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, there is a smooth map $g_{ij} : U_i \cap U_j \to G$ such that $g_{ij}(x)g_{jk}(x)g_{ki}(x) = e$ for all $x \in U_i \cap U_j \cap U_k$. Then there exists a unique principal $G$-bundle $\mathtt{P}$ over $M$ with the $\{U_i\}$ as trivialising neighbourhoods and the $\{g_{ij}\}$ as transition functions.*

Principal bundles can be endowed with connections.

**Definition 5.2.15.** Let $\{(U_i, h_i)\}$ be a set of local trivialisations of $P$ such that $\cup_i U_i = M$. A ***connection*** $\omega$ on $\mathtt{P}$ is a set of local $\mathfrak{g}$-valued 1-forms $\omega_i \in \Omega^1(U_i, \mathfrak{g})$ such that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij} \tag{5.4}$$

for $i, j$ such that $U_i \cap U_j \neq \emptyset$.

Connections always exist (see e.g. [57]). Equation (5.4) is also known as the transformation rule for gauge potentials in physics. We are now ready to define what we mean by a classical gauge theory over a manifold.

**Definition 5.2.16.** Let $M$ be a manifold and $G$ a Lie group. A ***classical $G$-gauge theory*** over $M$ is a principal fibre bundle $\mathtt{P}$ with structure group $G$. Connections $\omega$ on $\mathtt{P}$ are also called ***gauge potentials***.

More precisely, the bundle $\mathtt{P}$ forms the *setting* for a classical gauge theory. The particle fields can be added by introducing associated bundles of $\mathtt{P}$. Also, associated to a gauge theory is a gauge group. This gauge group should be seen as a group of symmetries of the physical system, in the sense that the corresponding dynamics, given by an action principle, is invariant under this gauge group (the presence of these symmetries is what makes the theory a gauge theory). We now recall the precise definition of an associated bundle and of the gauge group associated to the principal bundle $\mathtt{P}$. The discussion regarding the action principle is postponed until Section 8.3.

**Definition 5.2.17.** Suppose we are given an action of $G$ on a smooth manifold $F$ (e.g. a vector space or a Lie group), that is, we have a group homomorphism $\rho\colon G \to \mathrm{Aut}(F)$. Consider the product manifold $\mathtt{P}\times F$ with the equivalence relation given by $(pg, f) \sim (p, \rho(g)f)$. We then define the ***associated bundle*** $\mathtt{P} \times_\rho F$ as the quotient of $\mathtt{P} \times F$ with respect to this equivalence relation, and we write $[p, g]$ for the image of $(p, g)$ under this quotient. The projection map $\mathtt{P} \times_\rho F \to M$ is defined as $[p, f] \mapsto \pi_{\mathtt{P}}(p)$.

By properness and freeness of the action of $G$ on $\mathtt{P}$, the action of $G$ on the manifold $\mathtt{P} \times F$ is free and proper as well, which implies that the quotient $\mathtt{P} \times_\rho F$ is naturally a manifold, and that the projection map $\mathtt{P} \times_\rho F \to M$ is smooth. In general, if the manifold $F$ has some more structure and the action $\rho$ of $G$ on $F$ preserves this structure, this structure can be carried over to the fibres of the associated bundle $\mathtt{P} \times_\rho F$. For instance, if $F$ is a (finite-dimensional) vector space and $G$ acts by linear transformations, then $\mathtt{P} \times_\rho F$ is a vector bundle (e.g. fibre-wise vector addition in $(\mathtt{P} \times_\rho F)_x$ is given by $[p, v_1] + [p, v_2] := [p, v_1 + v_2]$, which is independent of the choice of $p \in \mathtt{P}_x$). If $F$ in addition carries an inner product and $G$ acts by unitary transformations, then this induces a hermitian structure on $\mathtt{P} \times_\rho F$ (as defined in Section 5.2.3 below).

**Example 5.2.18.** Consider the case where $F = G$, and the action $\rho : G \to \mathrm{Aut}(G)$ is given by the adjoint action $\mathrm{Ad}(g)f := gfg^{-1}$. The corresponding associated bundle $\mathtt{P} \times_{\mathrm{Ad}} G$ is called the ***adjoint bundle*** $\mathrm{Ad}\,\mathtt{P}$. Since $\mathrm{Ad}\,G \subset \mathrm{Aut}(G)$, the bundle $\mathrm{Ad}\,\mathtt{P}$ is a group bundle with fibres isomorphic to the group $G$. The sections $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$ then form a group with fibre-wise multiplication.

We now state the definition of the gauge group.

**Definition 5.2.19.** A ***gauge transformation*** of a principal $G$-bundle $\mathtt{P}$ is a principal bundle automorphism of $\mathtt{P}$ over $\mathrm{id} : M \to M$, that is, a smooth invertible map $\phi\colon \mathtt{P} \to \mathtt{P}$ such that $\pi(\phi(p)) = \pi(p)$ and $\phi(pg) = \phi(p)g$ for all $p \in \mathtt{P}$ and $g \in G$. The set of all such $\phi$ is called the ***gauge group*** $\mathcal{G}(\mathtt{P})$ of $\mathtt{P}$, where the group multiplication is given by composition.

**Remark 5.2.20.** To avoid confusion let us make the following remark on our terminology. If P is a principal $G$-bundle the term *gauge group* refers to $\mathcal{G}(\mathtt{P})$. The group $G$ will be referred to as the *structure group* of P.

The following result is well known (see for instance [9]).

**Theorem 5.2.21.** *The gauge group* $\mathcal{G}(\mathtt{P})$ *is isomorphic to the group* $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$.

Definition 5.2.19 is the usual definition for a gauge group. However, for our purposes it is easier to work with the group $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$.

### Structure group

Let E be a vector bundle with fibre $V$. A set of local trivialisations $\{(U_i, h_i)\}$ satisfying $\cup_i U_i = M$, and with transition functions $g_{ij} = h_i \circ h_j^{-1} \in C^\infty(U_i \cap U_j, \mathrm{Aut}(V))$ for $U_i \cap U_j \neq \emptyset$ is called an ***atlas*** on E, and is often simply denoted as $\{(U_i, g_{ij})\}$. An atlas $\{(U_i, g_{ij})\}$ is called a ***G-atlas*** if the transition functions $\{g_{ij}\}$ in fact take values in a subgroup $G \subset \mathrm{Aut}(V)$. If E admits a $G$-atlas, then we say that E has ***structure group*** $G$. Given two $G$-atlases $\{(U_i, g_{ij})\}$ and $\{(U_i, g'_{ij})\}$ (where, after taking a common refinement, we may assume (without loss of generality) that both atlases are given on the same open covering by $U_i$), we say that they are equivalent if their union is a $G$-atlas, that is, if there are functions $g_i \in C^\infty(U_i, G)$ such that

$$g'_{ij}(x) = g_i(x)^{-1} g_{ij}(x) g_j(x), \qquad \text{for all } x \in U_i \cap U_j.$$

Given a $G$-atlas $\{(U_i, g_{ij})\}$ on E, Theorem 5.2.14 constructs a unique principal $G$-bundle P. In fact, P only depends (up to isomorphism) on the equivalence class of the $G$-atlas. Hence any (equivalence class of a) $G$-atlas on E uniquely defines a principal $G$-bundle P. Conversely, given a principal $G$-bundle P, we can construct the associated vector bundle $\mathtt{P} \times_G V$. A given set of transition functions of P induces a $G$-atlas on $\mathtt{P} \times_G V$, and any other set of transition functions of P would give an equivalent $G$-atlas. In this way, P uniquely determines the equivalence class of $G$-atlases on $\mathtt{P} \times_G V$.

In some cases, the equivalence class of $G$-atlases on E is determined by some additional structure on the vector bundle E. Of particular interest to us is:

**Example 5.2.22.** Let E be a complex vector bundle of rank $N$, with a given hermitian structure. Then the equivalence class of $U(N)$-atlases is uniquely determined by the isometry class of the given hermitian structure, and vice versa. (This can be proved similarly to the case of $O(N)$-atlases on tangent bundles, for which we refer to e.g. [57, Ch. 1, §5]. See also [85, Part I, 12.13].)

**Definition 5.2.23** (Lifting of structure group)**.** Let $\pi : H \to G$ be a surjective group homomorphism. A principal $G$-bundle P is said to ***lift*** to a principal $H$-bundle Q along $\pi$ if there is a bundle morphism $\tau\colon \mathtt{Q} \to \mathtt{P}$ (over $M$) such that

$\tau(qh) = \tau(q)\pi(h)$ for all $q \in \mathtt{Q}$, $h \in H$. Equivalently, $\mathtt{Q}$ is a lift of $\mathtt{P}$ if

$$\mathtt{Q} \times_H G \cong \mathtt{P}$$

as principal $G$-bundles. If $\tau : \mathtt{Q} \to \mathtt{P}$ is such a lift and $\rho : G \to GL(V)$ is a finite-dimensional representation, then

$$\mathtt{Q} \times_{H,\rho\circ\pi} V \cong \mathtt{P} \times_{G,\rho} V.$$

We stress that a lift need not always exist, and if it exists, it need not be unique.

### 5.2.3 Conjugate modules and vector bundles

In the construction of gauge modules in Section 7.3 we will make explicit use of the notion of a conjugate module. For completeness, we recall the definition of conjugate modules and vector bundles here. Since most of the modules are endowed with a hermitian structure, we recall the definition of a hermitian module first (see e.g. [62]).

If $\mathcal{A}$ is a $*$-algebra and $a \in \mathcal{A}$, then we write $a \geq 0$ if $a = b^*b$ for some $b \in \mathcal{A}$.

**Definition 5.2.24.** Let $\mathcal{A}$ be a $*$-algebra and let $\mathcal{E}$ be a right $\mathcal{A}$-module. A ***hermitian structure*** $(\cdot,\cdot)_\mathcal{A} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ on $\mathcal{E}$ is a sesquilinear map (anti-linear in the first variable) satisfying

$$\begin{aligned}(e_1, e_2a)_\mathcal{A} &= (e_1, e_2)_\mathcal{A} a;\\ (e_2, e_1)_\mathcal{A} &= (e_1, e_2)^*_\mathcal{A};\\ (e, e)_\mathcal{A} &\geq 0;\\ (e, e)_\mathcal{A} = 0 &\iff e = 0,\end{aligned}$$

for all $a \in \mathcal{A}$, $e_1, e_2, e \in \mathcal{E}$. We also write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_\mathcal{A}$ when no confusion can arise. A module endowed with a hermitian structure is also called a ***hermitian module***.

A hermitian structure is called ***non-degenerate*** if the map

$$\mathcal{E} \to \mathcal{E}^* := \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}), \qquad e_0 \mapsto (e \mapsto (e_0, e))$$

is an anti-linear isomorphism.

Note that the assumption that the hermitian structure is positive-definite implies that the map $\mathcal{E} \to \mathcal{E}^*$ is injective. Non-degeneracy implies surjectivity of this map.

A finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ is of the form $p\mathcal{A}^N$, for some $N \in \mathbb{N}$ and some projection $p \in M_N(\mathcal{A})$. The restriction of the standard hermitian structure on $\mathcal{A}^N$ then gives a non-degenerate hermitian structure on $\mathcal{E}$. If $\mathcal{A} = C^\infty(M)$ (so that $\mathcal{E} = \Gamma^\infty(\mathtt{E})$ for some vector bundle $\mathtt{E} \to M$ by the Serre-Swan theorem [86]), then the hermitian structure is non-degenerate if and only if the corresponding sesquilinear forms on the fibres of $\mathtt{E}$ are all inner products.

**Definition 5.2.25.** Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule. Its ***conjugate module*** $\overline{\mathcal{E}}$ is equal to $\mathcal{E}$ itself as an additive group. It can naturally be endowed with a $\mathcal{B}$-$\mathcal{A}$-bimodule structure by setting

$$b\overline{e} = \overline{eb^*}, \quad \overline{e}a = \overline{a^*e},$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $e \in \mathcal{E}$. Moreover, if $\mathcal{E}$ carries a (right) $\mathcal{B}$-valued hermitian structure $(\cdot,\cdot)_{\mathcal{B}}$, then $\overline{\mathcal{E}}$ carries a (left) $\mathcal{B}$-valued hermitian structure ${}_{\mathcal{B}}(\cdot,\cdot)$ given by

$${}_{\mathcal{B}}(\overline{e_1}, \overline{e_2}) := (e_1, e_2)_{\mathcal{B}}, \quad e_1, e_2 \in \mathcal{E},$$

and similarly $\overline{\mathcal{E}}$ carries a (right) $\mathcal{A}$-valued hermitian structure if $\mathcal{E}$ carries a (left) $\mathcal{A}$-valued hermitian structure.

If $\mathcal{E} = \Gamma^\infty(\mathtt{E})$ is the $C^\infty(M)$-module of sections of some (hermitian) vector bundle $\mathtt{E}$, then the conjugate module $\overline{\mathcal{E}}$ is equal to the $C^\infty(M)$-module of sections of the conjugate vector bundle $\overline{\mathtt{E}}$ which is defined as:

**Definition 5.2.26.** Let $\mathtt{E} \to M$ be a complex vector bundle. Take $\overline{\mathtt{E}}$ to be equal to $\mathtt{E}$ as fibre bundles over $M$, and write $\overline{e}$ for the element in $\overline{\mathtt{E}}$ that corresponds to $e \in \mathtt{E}$ under this identification. The bundle $\overline{\mathtt{E}}$ is turned into a vector bundle over $M$ by defining the vector space structure in $\mathtt{E}_x$ by

$$(\overline{e_1}, \overline{e_2}) \mapsto \overline{e_1 + e_2}, \quad \lambda \cdot \overline{e} = \overline{\overline{\lambda} e},$$

for all $\lambda \in \mathbb{C}$, $e, e_1, e_2 \in \mathtt{E}_x$. The vector bundle $\overline{\mathtt{E}} \to M$ is called the ***conjugate vector bundle*** of $\mathtt{E}$.

Note that the identification $\mathtt{E} \ni e \mapsto \overline{e} \in \overline{\mathtt{E}}$ in the above definition is an anti-linear isomorphism of vector bundles.

Let $(U, h)$ be a local trivialisation of the bundle $\mathtt{E}$, that is, there exists a finite-dimensional complex vector space $V$ and a fibre-preserving map $h : \pi_{\mathtt{E}}^{-1} \to U \times V$ that is linear on the fibres over $U$. Such a local trivialisation of $\mathtt{E}$ induces a local trivialisation of the conjugate vector bundle $\overline{\mathtt{E}}$ given by the map

$$\overline{h} : \pi_{\overline{\mathtt{E}}}^{-1}(U) \ni \overline{e} \to \overline{he} \in U \times \overline{V},$$

where $\overline{(x,v)}$ is defined to be $(x, \overline{v})$. If $\{(U_i, h_i)\}$ is a complete set of local trivialisations for the bundle $\mathtt{E}$, then $\{(U_i, \overline{h_i})\}$ is a complete set of local trivialisations for the bundle $\overline{\mathtt{E}}$. If $g_{ij}$ is a transition function between two local trivialisations $(U_i, h_i)$ and $(U_j, h_j)$ of $\mathtt{E}$, then the transition function $\overline{g_{ij}}$ between the corresponding local trivialisations $(U_i, \overline{h_i})$ and $(U_j, \overline{h_j})$ is equal to

$$\overline{h_i} \circ \overline{h_j}^{-1}(x, \overline{v}) = \overline{h_i}\left(\overline{h_j^{-1}(x,v)}\right) = \overline{h_i h_j^{-1}(x,v)} = (x, \overline{g_{ij}(x)v}) = (x, \overline{v} \cdot g_{ij}(x)^*). \tag{5.5}$$

The sections of $\overline{\mathcal{E}} = \Gamma^\infty(\overline{\mathtt{E}})$ are related to $\Gamma^\infty(\mathtt{E})$ through an anti-linear $C^\infty(M)$-module isomorphism. Under this identification an element $s : x \mapsto s(x)$ corresponds to the element $\overline{s} : x \mapsto \overline{s(x)}$. Since $\overline{\mathcal{E}}$ is the space of sections of a vector bundle, it has a natural $C^\infty(M)$-module structure. This module structure is related to the module structure on $\mathcal{E}$ by the relation $f\overline{s} = \overline{\overline{f}s}$. Hence $\overline{\mathcal{E}}$ is conjugate to $\mathcal{E}$.

From here on, we consider $\mathcal{A} := C^\infty(M)$. Suppose that $\mathcal{E}$ is a hermitian right $\mathcal{A}$-module with hermitian structure $(\cdot,\cdot)_\mathcal{A}$. Let $\nabla$ be a hermitian connection on $\mathcal{E}$, that is, a map $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega^1(M)$ satisfying the rule

$$(\nabla e_1, e_2)_{\Omega^1(M)} + (e_1, \nabla e_2)_{\Omega^1(M)} = d(e_1, e_2)_\mathcal{A}, \quad e_1, e_2 \in \mathcal{E},$$

where the map $(\cdot,\cdot)_{\Omega^1(M)}\colon \mathcal{E} \times (\mathcal{E} \otimes_\mathcal{A} \Omega^1(M)) \to \Omega^1(M)$ is defined as

$$(e_1, e_2 \otimes \alpha)_{\Omega^1(M)} := (e_1, e_2)_\mathcal{A}\alpha.$$

We then define $(\cdot,\cdot)_{\Omega^1(M)}\colon (\mathcal{E} \otimes_\mathcal{A} \Omega^1(M)) \times \mathcal{E} \to \Omega^1(M)$ as $(e_1 \otimes \alpha, e_2)_{\Omega^1(M)} := \big((e_2, e_1 \otimes \alpha)_{\Omega^1(M)}\big)^*$.

The ***conjugate connection*** $\overline{\nabla}\colon \overline{\mathcal{E}} \to \Omega^1(M) \otimes_\mathcal{A} \overline{\mathcal{E}}$ is given by

$$\overline{\nabla}\overline{e} = \overline{\nabla e}, \quad (e \in \mathcal{E}),$$

where $\overline{e \otimes \omega} = \omega^* \otimes \overline{e}$ for all $e \otimes \omega \in \mathcal{E} \otimes \Omega^1(M)$. Here $* : \Omega^1(M) \to \Omega^1(M)$ is defined as $(fdg)^* = f^*(dg^*)$. It then follows that $\overline{\nabla}$ is also hermitian for the map ${}_{\Omega^1(M)}(\cdot,\cdot)\colon \Omega^1(M) \otimes_\mathcal{A} \overline{\mathcal{E}} \times \overline{\mathcal{E}} \to \Omega^1(M)$ defined as ${}_{\Omega^1(M)}(\alpha \otimes \overline{e_1}, \overline{e_2}) := (e_1 \otimes \alpha^*, e_2)_{\Omega^1(M)} = \alpha(e_1, e_2)_\mathcal{A}$.

For a commutative algebra $\mathcal{A}$ the notion of left and right modules are equivalent. If $\mathcal{E}$ is a left $\mathcal{A}$-module with $\mathcal{A}$-valued inner product ${}_\mathcal{A}(\cdot,\cdot)$, then

$$(e_1, e_2)_\mathcal{A} := {}_\mathcal{A}(e_2, e_1)$$

defines a $\mathcal{A}$-valued inner product when $\mathcal{E}$ is considered as as right $\mathcal{A}$-module. If $\mathcal{A} = C^\infty(M)$, we will always consider $\mathcal{E}$ as a right $\mathcal{A}$-module. One can verify that $\nabla$ is a hermitian connection for ${}_\mathcal{A}(\cdot,\cdot)$ if and only if $\nabla$ is a hermitian connection for $(\cdot,\cdot)_\mathcal{A}$.

### 5.2.4 Covering maps

We observe that, for a surjective group bundle morphism $\phi\colon \mathtt{H} \to \mathtt{G}$, the induced map $\phi_*\colon \Gamma^\infty(\mathtt{H}) \to \Gamma^\infty(\mathtt{G})$ need *not* always be surjective, as the following example shows.

**Example 5.2.27.** Take $M = SO(3)$ and consider the *globally trivial* group bundles $\mathtt{H} = M \times U(2)$ and $\mathtt{G} = M \times PSU(2)$, with the obvious group bundle morphism $\phi\colon \mathtt{H} \to \mathtt{G}$ given by the quotient $\pi\colon U(2) \to PSU(2)$.

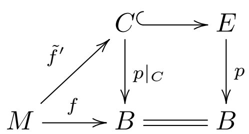
Since the bundles $\mathtt{H}$ and $\mathtt{G}$ are globally trivial, we can make the identifications $\Gamma^\infty(\mathtt{H}) \cong C^\infty(SO(3), U(2))$ and $\Gamma^\infty(\mathtt{G}) \cong C^\infty(SO(3), PSU(2))$. Consider the map $f : SO(3) \to PSU(2)$ given by the identification of $PSU(2)$ with $SO(3)$, i.e. $f = \mathrm{id}$ on $SO(3)$. We will show that there is no lift $\widetilde{f}\colon SO(3) \to U(2)$ such that $f = \pi \circ \widetilde{f}$.

To see that such a map does not exist, note that a map $\widetilde{g}\colon SO(3) \to U(2)$ such that $f = \pi \circ \widetilde{g}$ is nothing but a global section of the $U(1)$-principal bundle $\pi : U(2) \to SO(3)$. However, as this bundle is not globally trivial, such a section does not exist.[3] Hence the map $f$, seen as a section in $\Gamma^\infty(\mathtt{G})$, is not contained in the image of $\phi_*$.

In this subsection we aim to find sufficient conditions for the surjectivity of $\phi_*$. In other words, we would like to have sufficient conditions to ensure that for any section $s\colon M \to \mathtt{G}$ there exists a lift $\tilde{s}\colon M \to \mathtt{H}$ such that $\phi_*(\tilde{s}) = s$. Though the existence of lifts for covering maps has been well-studied, we will typically be dealing with more general fibrations $\phi\colon \mathtt{H} \to \mathtt{G}$, for which the problem of existence of lifts is more complicated. We avoid this problem by reducing it to the case of covering maps, as follows.

**Lemma 5.2.28.** *Let $p\colon E \to B$ be a fibration, and consider some map $f\colon M \to B$. Suppose there exists a submanifold $C \subset E$ such that $p|_C\colon C \to B$ is a covering space, satisfying $f_*(\pi_1(M, m)) \subset p_*(\pi_1(C, c))$, where $m \in M$ and $c \in C$ are such that $f(m) = p(c)$. Then there exists a lift $\tilde{f}\colon M \to E$ satisfying $p \circ \tilde{f} = f$ and $\tilde{f}(m) = c$.*

*Proof.* Consider the diagram




The assumption $f_*(\pi_1(M, m)) \subset p_*(\pi_1(C, c))$ implies (see e.g. [41, Proposition 1.33]) that there exists a lift $\tilde{f}'\colon M \to C$ satisfying $\tilde{f}'(m) = c$, and then we can simply define $\tilde{f}\colon M \to E$ as the composition $M \xrightarrow{\tilde{f}'} C \hookrightarrow E$. □

We now translate the above lemma into the setting of group bundles, where we will need it later.

**Corollary 5.2.29.** *Let $M$ be a simply connected manifold, and let $\mathtt{G}, \mathtt{H}$ be group bundles over $M$. If $\mathtt{G}$ is covered by a subbundle $\mathtt{U}$ of $\mathtt{H}$ via a group bundle morphism $\phi : \mathtt{H} \to \mathtt{G}$, then the map $\phi_* : \Gamma^\infty(\mathtt{H}) \to \Gamma^\infty(\mathtt{G})$, given by $s \mapsto \phi \circ s$, is surjective.*

[3]The fundamental group of $U(2)$ is $\mathbb{Z}$, whereas the fundamental group of $SO(3) \times U(1)$ is $\mathbb{Z}_2 \times \mathbb{Z}$.

*Proof.* By assumption, $\phi|_{\mathtt{U}} \colon \mathtt{U} \to \mathtt{G}$ is a covering space. Since $\pi_1(M, m)$ is trivial (by definition of simply-connectedness) it follows from Lemma 5.2.28 that each section $s \colon M \to \mathtt{G}$ can be lifted to a section $\widetilde{s} \colon M \to \mathtt{U} \subset \mathtt{H}$ such that $\phi_*(\widetilde{s}) = s$. □

### 5.2.5 Spectral triples and Kasparov modules

Spectral triples were introduced in [19] as a noncommutative analogue of a spin manifold.

**Definition 5.2.30.** A ***spectral triple*** $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint (generally unbounded) operator $D$ with compact resolvent (or equivalently, $(1 + D^2)^{-1/2}$ is a compact operator) such that $a \cdot \operatorname{Dom} D \subset \operatorname{Dom} D$ and the commutator $[D, a]$ is bounded for each $a \in \mathcal{A}$.

A spectral triple is called ***even*** if there exists a $\mathbb{Z}_2$-grading $\gamma$ on $\mathcal{H}$ that commutes with any $a \in \mathcal{A}$ and anti-commutes with $D$.

A spectral triple is called ***real*** if there exists an anti-unitary isomorphism $J \colon \mathcal{H} \to \mathcal{H}$ satisfying

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon''\gamma J \text{ (if } \gamma \text{ exists)},$$
$$[a, JbJ^*] = 0, \qquad [[D, a], JbJ^*] = 0, \qquad \forall a, b \in \mathcal{A}.$$

The signs $\varepsilon$, $\varepsilon'$ and $\varepsilon''$ determine the *KO-dimension* $n$ modulo 8 of the real spectral triple, according to the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |
| $\varepsilon'$ | 1 | −1 | 1 | 1 | 1 | −1 | 1 | 1 |
| $\varepsilon''$ | 1 | | −1 | | 1 | | −1 | |

We will also refer to the conditions $[a, JbJ^*] = 0$ and $[[D, a], JbJ^*]$ as the ***zeroth-*** and ***first-order*** conditions, respectively.

Given an algebra $\mathcal{A}$, we define the ***opposite algebra*** as the vector space $\mathcal{A}^{\mathrm{op}} := \{a^{\mathrm{op}} \mid a \in \mathcal{A}\}$ with the ***opposite product*** $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$. For a real spectral triple, we therefore have a linear representation of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$ given by $a^{\mathrm{op}} \mapsto Ja^*J^*$.

The notion of spectral triple can be seen as an unbounded version of a Fredholm module. The generalisation of Fredholm modules from Hilbert spaces to Hilbert modules was performed by Kasparov in [55], where for any two $C^*$-algebras $A$ and $B$ the set $KK(A, B)$ was defined as the set of equivalence classes of certain Kasparov $A$-$B$-modules. In addition, there exists a Kasparov product $KK(A, B) \times KK(B, C) \to KK(A, C)$. More details can be found in e.g. [8]. These Kasparov module were subsequently generalised to the unbounded picture by Baaj and Julg [2]. We will only focus on the unbounded picture, which we briefly recall below.

**Definition 5.2.31** ([2])**.** Given $\mathbb{Z}_2$-graded $C^*$-algebras $A$ and $B$, an ***unbounded Kasparov** A-B**-module*** $({}_{\phi(A)}E_B, D)$ is given by

- a $\mathbb{Z}_2$-graded, countably generated, right Hilbert $B$-module $E_B$;
- a $\mathbb{Z}_2$-graded $*$-homomorphism $\phi\colon A \to \mathrm{End}_B(E)$;
- a self-adjoint, regular, odd operator $D\colon \mathrm{Dom}\, D \subset E \to E$ such that, for all $a$ in a dense sub-algebra $\mathcal{A}$ of $A$, $\phi(a) \cdot \mathrm{Dom}\, D \subset \mathrm{Dom}\, D$ and $[D, \phi(a)]_\pm$ is (or extends to) a bounded endomorphism, and $\phi(a)(1 + D^2)^{-\frac{1}{2}}$ is a compact endomorphism (i.e. it lies in $\mathrm{End}_B^0(E)$).

The set of all unbounded Kasparov $A$-$B$-modules is denoted by $\Psi(A, B)$.

A right Hilbert $\mathbb{C}$-module is just a Hilbert space. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ may then be seen as an unbounded Kasparov $A$-$\mathbb{C}$-module $({}_A\mathcal{H}_\mathbb{C}, D)$, where the $C^*$-closure $A$ of $\mathcal{A}$ is trivially graded.

There is a natural map from the unbounded picture to the bounded one. This map is defined by replacing the operator $D$ in $({}_{\phi(A)}E_B, D)$ by $b(D) = D(1+D^2)^{-\frac{1}{2}}$, where $b : \mathbb{R} \to \mathbb{R}$ denotes the function $b(x) = x(1 + x^2)^{-\frac{1}{2}}$.

**Theorem 5.2.32** ([8], Theorems 17.10.7 and 17.11.4)**.** *If $({}_{\phi(A)}E_B, D) \in \Psi(A, B)$, then $({}_{\phi(A)}E_B, b(D)) \in KK(A, B)$. Moreover, if $A$ is separable and $B$ is $\sigma$-unital, then this map $\Psi(A, B) \to KK(A, B)$ is surjective.*

The Kasparov product also has an unbounded analogue. To be precise, we say that a unbounded Kasparov $A$-$C$-module $({}_{\phi(A)}E_C, D)$ represents the Kasparov product of two unbounded Kasparov modules $({}_{\phi_1(A)}E_{1B}, D_1)$ and $({}_{\phi_2(B)}E_{2C}, D_2)$ if $[(E, b(D))] \in KK(A, C)$ is the Kasparov product of $[(E_1, b(D_1))] \in KK(A, B)$ and $[(E_2, b(D_2))] \in KK(B, C)$.

We will show in Chapter 6 that the construction of an almost-commutative manifold as the product of an internal space $I$ with the underlying manifold $M$ corresponds to an unbounded Kasparov product on the level of $KK$-classes. Although this follows from the (more general) framework of Mesland [73], we will prove it directly using the following result.

**Theorem 5.2.33** (Kucerovsky [61])**.** *Let $({}_{\phi_1(A)}E^1_B, D_1)$ and $({}_{\phi_2(B)}E^2_C, D_2)$ be unbounded Kasparov modules. Write $E := E^1 \hat{\otimes}_B E^2$. Suppose that $({}_{\phi_1(A)}E_C, D)$ is an unbounded Kasparov module such that:*

*i) for all $e_1$ in a dense subspace of $\phi_1(A)E^1$, the commutators*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T^*_{e_1} & 0 \end{pmatrix}\right]$$

*are bounded on* $\mathrm{Dom}(D \oplus D_2) \subset E \oplus E^2$*;*

*ii)* $\mathrm{Dom}(D) \subset \mathrm{Dom}(D_1 \hat{\otimes} 1)$*;*

*iii)* $((D_1 \hat{\otimes} 1)e, De) + (De, (D_1 \hat{\otimes} 1)e) \geq K(e,e)$ *for some* $K \in \mathbb{R}$*, for all* $e \in \mathrm{Dom}(D)$*.*

*Then* $(E, D)$ *represents the Kasparov product. Here* $\hat{\otimes}$ *denotes the* graded *tensor product and for* $e_1 \in E_1$ *the operator* $T_{e_1} : E_2 \to E$ *is given by* $T_{e_1}(e_2) = e_1 \otimes e_2$*.*

# Chapter 6

# Almost-commutative manifolds as a KK-product

Almost-commutative manifolds $M \times F$ of the form (5.1) were first studied in [22] and [23, 24, 25, 26]. They were later used in [18, 15] to geometrically describe Yang-Mills theories and the Standard Model of elementary particles. The name almost-commutative manifolds was coined in [52], their classification starting in [60].

Let $M$ be a smooth compact even-dimensional Riemannian spin manifold. We assume (throughout this chapter) that $M$ has dimension 4. The manifold $M$ can be completely characterised [21] by the real even spectral triple

$$(C^\infty(M), L^2(\mathtt{S}), \not{D}, \gamma_5, J_M),$$

which is often referred to as the ***canonical*** spectral triple for $M$. Here $\mathtt{S}$ is a spinor bundle over $M$, $\not{D} = -ic \circ \nabla^{\mathtt{S}}$ is the corresponding Dirac operator (where $\nabla^{\mathtt{S}}$ is the lift of the Levi-Civita connection on $M$, and $c$ denotes Clifford multiplication[1]), $\gamma_5$ is the grading of the spinor bundle, and $J_M$ is the charge conjugation operator. Given a real even *finite* spectral triple $(A_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ (for which $\dim \mathcal{H}_F < \infty$), we can construct the product triple

$$M \times F := \big(C^\infty(M, A_F), L^2(\mathtt{S}) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \gamma_5 \otimes \gamma_F, J_M \otimes J_F\big) .$$

Defining the (globally trivial) algebra bundle $\mathtt{B} = M \times A_F$ and the (globally trivial) vector bundle $\mathtt{E} = M \times \mathcal{H}_F$, we can rewrite $C^\infty(M, A_F) \cong \Gamma^\infty(\mathtt{B})$ and $L^2(\mathtt{S}) \otimes \mathcal{H}_F \cong L^2(\mathtt{S} \otimes \mathtt{E})$. The purpose of this chapter is to generalise the construction of $M \times F$ to *globally non-trivial* bundles over $M$. At the same time, we will put this generalised

[1] We use the conventions $c(v)c(w) + c(w)c(v) = 2g(v, w)$ and $c(v)^* = c(v)$ for any $v, w \in \Gamma^\infty(TM)$.

construction in the context of the Kasparov product between unbounded Kasparov modules. The globally non-trivial case was first considered in [11] for the case of algebra bundles with fibre $M_N(\mathbb{C})$, and has also been studied more generally in [13].

## 6.1 The internal space

**Definition 6.1.1.** A (smooth) ***internal space*** $I^\infty$ over a compact manifold $M$ is given by the data

$$I^\infty := (\Gamma^\infty(\mathtt{B}),\ \Gamma^\infty(\mathtt{E}),\ D_I)\,,$$

where $\mathtt{E}$ is a hermitian vector bundle over $M$, $\mathtt{B}$ is a unital $*$-algebra subbundle of $\mathrm{End}(\mathtt{E})$, and $D_I$ is a hermitian element of $\Gamma^\infty(\mathrm{End}(\mathtt{E})) \cong \mathrm{End}_{C^\infty(M)}(\Gamma^\infty(\mathtt{E}))$.

An internal space is called ***even*** if there is a grading $\gamma_I$, i.e. an endomorphism $\gamma_I \in \Gamma^\infty(\mathrm{End}(\mathtt{E}))$ such that

$$\gamma_I^* = \gamma_I, \qquad \gamma_I^2 = 1, \qquad \gamma_I D_I = -D_I\gamma_I, \qquad \gamma_I a = a\gamma_I \quad \forall a \in \Gamma^\infty(\mathtt{B}).$$

An even internal space is called ***real*** if there is a real structure $J_I$, i.e. an anti-unitary endomorphism $J_I$ on $\mathtt{E}$ such that

$$\begin{aligned} &J_I^2 = \varepsilon, && J_I D_I = \varepsilon' D_I J_I, && J_I\gamma_I = \varepsilon''\gamma_I J_I, \\ &[a, Jb^*J^*] = 0, && \big[[D_I, a], Jb^*J^*\big] = 0, && \forall a, b \in \Gamma^\infty(\mathtt{B}), \end{aligned}$$

where the signs determine the $KO$-dimension of the internal space according to the same table as in Definition 5.2.30.

We shall write $\mathcal{A} = C^\infty(M)$, $\mathcal{B} = \Gamma^\infty(\mathtt{B})$, and $\mathcal{E} = \Gamma^\infty(\mathtt{E})$. Their respective $C^*$-closures are denoted by $A = C(M)$, $B = \Gamma(\mathtt{B})$, and $E = \Gamma(\mathtt{E})$.

**Proposition 6.1.2.** *An even internal space $I^\infty = (\Gamma^\infty(\mathtt{B}), \Gamma^\infty(\mathtt{E}), D_I)$ yields an unbounded Kasparov $B$-$A$-module $I = ({}_{\phi(B)}\Gamma(\mathtt{E})_A, D_I)$.*

*Proof.* The algebras $A$ and $B$ are trivially graded $C^*$-algebras, and $E = \Gamma(\mathtt{E})$ is a $\mathbb{Z}_2$-graded, finitely generated, right Hilbert $A$-module, with a left action of $B$ that commutes with the (right) action of $A$. The properties of $\gamma_I$ guarantee that all conditions with respect to the grading are satisfied. For instance, the condition $(E^{(m)}, E^{(n)}) \subset A^{(m+n)}$, where $m, n \in \mathbb{Z}_2$, is satisfied, since the condition $\gamma_I^* = \gamma_I$ implies that $\langle s, t\rangle = 0$ as soon as one of the arguments is odd and the other is even. The operator $D_I$ is a bounded, self-adjoint, odd operator by definition (and hence it is automatically regular). The boundedness of $D_I$ implies that $[D_I, b]$ is also bounded for all $b \in \mathcal{B}$.

For a compact manifold $M$ the compact endomorphisms of the Hilbert-$C(M)$-module $\Gamma(\mathtt{E})$ are exactly the sections of the endomorphism bundle $\mathrm{End}(\mathtt{E})$, i.e.

$\mathrm{End}^0_{C(M)}(\Gamma(\mathtt{E})) = \Gamma(\mathrm{End}(\mathtt{E}))$.[2] Thus, $\phi(b)(1+D_I^2)^{-\frac{1}{2}}$ is compact for all $b \in \mathcal{B}$, because both $(1+D_I^2)^{-\frac{1}{2}}$ and $\phi(b)$ are compact.

Hence $({}_{\phi(B)}\Gamma(\mathtt{E})_A, D_I)$ has all the properties mentioned in Definition 5.2.31.

□

## 6.2 The product space

**Definition 6.2.1.** Let $I^\infty := (\Gamma^\infty(\mathtt{B}), \Gamma^\infty(\mathtt{E}), D_I, \gamma_I, J_I)$ be a real even internal space over $M$, with $M$ a compact 4-dimensional manifold. Let $\nabla^I$ be a hermitian connection on $\mathtt{E}$. We define a real even ***almost-commutative manifold*** to be

$$I^\infty \times_\nabla M := \left(\Gamma^\infty(\mathtt{B}),\ L^2(\mathtt{E}\otimes\mathtt{S}),\ \not{D}_{\mathtt{E}} + D_I \otimes \gamma_5, \gamma_I \otimes \gamma_5, J_I \otimes J_M\right),$$

where $L^2(\mathtt{E}\otimes\mathtt{S}) \cong \Gamma(\mathtt{E}) \otimes_{C(M)} L^2(\mathtt{S})$ are the $L^2$-sections of the twisted spinor bundle $\mathtt{E}\otimes\mathtt{S}$, and $\not{D}_{\mathtt{E}}$ is the twisted Dirac operator

$$\not{D}_{\mathtt{E}} := \mathbb{I} \otimes_\nabla \not{D} := \mathbb{I} \otimes \not{D} - i(\mathbb{I} \otimes c) \circ (\nabla^I \otimes \mathbb{I}).$$

Note that by definition the underlying manifold of an almost-commutative manifold is always assumed to be of dimension 4.

We note that our definition of almost-commutative manifolds fits within the slightly more general definition of almost-commutative spectral triples given in [13, Definition 2.3].

The order of $I^\infty$ and $M$ in the notation $I^\infty \times_\nabla M$ is reversed in comparison with the order of $F$ and $M$ in $M \times F$. The reason is that the order $I^\infty \times_\nabla M$ is more natural from a $KK$-theoretical viewpoint, whereas the notation $M \times F$ for the globally trivial case is quite standard in the literature. Below we show in detail that an almost-commutative manifold $I^\infty \times_\nabla M$ determines an unbounded Kasparov $B$-$\mathbb{C}$-module (i.e. a spectral triple over $\mathcal{B}$) whose $KK$-class represents the Kasparov product between the $KK$-classes of the internal space $I^\infty$ and the canonical spectral triple for $M$.

**Proposition 6.2.2.** *Let $I^\infty = (\Gamma^\infty(\mathtt{B}), \Gamma^\infty(\mathtt{E}), D_I, \gamma_I, J_I)$ be a real even internal space over $M$ of even $KO$-dimension $k$. Let $\nabla^I$ be a hermitian connection on $\mathtt{E}$ that commutes with the grading $\gamma_I$ and the real structure $J_I$ in the sense that $\nabla^I_\mu J_I = J_I \nabla^I_\mu$, such that the induced connection $[\nabla^I, \cdot]$ on $\mathrm{End}\,\mathtt{E}$ restricts to a connection on $\mathtt{B}$. Then the real even almost-commutative manifold $I^\infty \times_\nabla M$ is a real even spectral triple of $KO$-dimension $4+k$ (mod 8).*

[2] Since $\Gamma(\mathrm{End}(\mathtt{E}))$ is already unital, the compact endomorphisms of $\Gamma(\mathtt{E})$ are actually all the bounded endomorphisms [31, Proposition 3.9].

*Proof.* Let us write $D := \not{D}_{\mathtt{E}} + D_I \otimes \gamma_5$. We need to show that $[D, a]$ is bounded for all $a \in \Gamma^\infty(\mathtt{B})$. Since $D_I$ is bounded itself, we need only check this for the twisted Dirac operator $\not{D}_{\mathtt{E}}$, and we find[3]

$$[\not{D}_{\mathtt{E}}, a] = -ic([\nabla^I, a]),$$

which for smooth $a$ indeed acts as a bounded operator on $L^2(\mathtt{E} \otimes \mathtt{S})$. Furthermore we need to show that $D$ has compact resolvent, and (as $M$ is compact) for this it is sufficient to show that $D^2$ is elliptic. The Lichnerowicz-Weitzenböck formula shows that the square of the twisted Dirac operator $\not{D}_{\mathtt{E}}$ is a generalised Laplacian, and hence is elliptic. The bounded (zeroth-order) perturbation $\not{D}_{\mathtt{E}} \to \not{D}_{\mathtt{E}} + D_I \otimes \gamma_5$ does not affect this ellipticity. Hence $I^\infty \times_\nabla M$ is indeed a spectral triple.

Given the grading operators $\gamma_I$ and $\gamma_5$, it is straightforward to check that $D(\gamma_I \otimes \gamma_5) = -(\gamma_I \otimes \gamma_5)D$, provided that $[\nabla^I, \gamma_I] = 0$.

Given the real structures $J_I$ and $J_M$, the operator $J_I \otimes J_M$ is anti-unitary and satisfies

$$\begin{aligned}(J_I \otimes J_M)^2 &= -\varepsilon, \\ D(J_I \otimes J_M) &= (J_I \otimes J_M)D, \\ (J_I \otimes J_M)(\gamma_I \otimes \gamma_5) &= \varepsilon''(\gamma_I \otimes \gamma_5)(J_I \otimes J_M),\end{aligned} \tag{6.1}$$

where the signs $\varepsilon, \varepsilon''$ are determined by the $KO$-dimension $k$ of $J_I$. The first equality in Equation (6.1) is immediate from $J_M^2 = -1$ and $J_I^2 = \varepsilon$. Using the relations

$$J_M \not{D} = \not{D} J_M, \qquad \gamma^\mu J_M = -J_M \gamma^\mu, \qquad \gamma_5 J_M = J_M \gamma_5,$$

and

$$J_I D_I = D_I J_I, \qquad \nabla^I_\mu J_I = J_I \nabla^I_\mu,$$

the second equality in Equation (6.1) is checked by a local calculation (writing $(\mathbb{I} \otimes c) \circ (\nabla^I \otimes \mathbb{I}) = \nabla^I_\mu \otimes \gamma^\mu$):

$$\begin{aligned}D(J_I \otimes J_M)(s \otimes \psi) &= (J_I s) \otimes (\not{D} J_M \psi) - i(\nabla^I_\mu J_I s) \otimes (\gamma^\mu J_M \psi) \\ &\quad + (D_I J_I s) \otimes (\gamma_5 J_M \psi) \\ &= (J_I s) \otimes (J_M \not{D} \psi) + i(J_I \nabla^I_\mu s) \otimes (J_M \gamma^\mu \psi) \\ &\quad + (J_I D_I s) \otimes (J_M \gamma_5 \psi) \\ &= (J_I s) \otimes (J_M \not{D} \psi) - (J_I \nabla^I_\mu s) \otimes (J_M i \gamma^\mu \psi) \\ &\quad + (J_I D_I s) \otimes (J_M \gamma_5 \psi) \\ &= (J_I \otimes J_M) D(s \otimes \psi).\end{aligned}$$

[3] With some abuse of notation, we write $c(T \otimes \alpha) = T \otimes c(\alpha)$ for $T \in \Gamma^\infty(\mathrm{End}\,\mathtt{E})$ and $\alpha \in \Omega^1(M)$.

The third equality in Equation (6.1) immediately follows from $[J_M, \gamma_5] = 0$ and $J_I \gamma_I = \varepsilon'' \gamma_I J_I$. From the values of $-\varepsilon$ and $\varepsilon''$ it is immediate that the $KO$-dimension of $I^\infty \times M$ should be $4 + k$ (mod 8) (see the table in Definition 5.2.30).

The zeroth-order condition on $I^\infty \times_\nabla M$ is immediate from the zeroth-order condition on $I^\infty$. Moreover,

$$[[\not{D}_{\mathtt{E}}, a], JbJ^*] = -i[c([\nabla^I, a]), JbJ^*] = -ic([[\nabla^I, a], JbJ^*]) = 0,$$

because, by assumption, $[\nabla^I, a] \in \Gamma^\infty(\mathtt{B}) \otimes_{C^\infty(M)} \Omega^1(M)$, which commutes with $JbJ^*$. Together with the first-order condition on $D_I$, this implies that $D$ satisfies the first-order condition. □

For a real spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J)$, the gauge group is defined as [89, Definition 2.5]

$$\mathcal{G}(T) := \left\{ uJuJ^* \mid u \in \mathcal{U}(\mathcal{A}) \right\} \cong \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A}_J), \tag{6.2}$$

where the central subalgebra $\mathcal{A}_J$ is defined as $\mathcal{A}_J := \{a \in \mathcal{A} \mid aJ = Ja^*\}$. For the above almost-commutative manifold, we thus obtain the gauge group

$$\mathcal{G}(I^\infty \times_\nabla M) = \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J),$$

for the real structure $J = J_I \otimes J_M$. However, since $\mathcal{B}_J \cong \mathcal{B}_{J_I}$, we find that the gauge group of the almost-commutative manifold is completely determined by the internal space, and we write

$$\mathcal{G}(I^\infty \times_\nabla M) \cong \mathcal{G}(I^\infty) := \{uJ_I u J_I^* \mid u \in \mathcal{U}(\mathcal{B})\}. \tag{6.3}$$

## 6.3 The Kasparov product

We now show that the product $I^\infty \times_\nabla M$ represents the unbounded Kasparov product of the $KK$-classes of $I^\infty$ and the canonical spectral triple for $M$. We first prove this for the cases where $D_I = 0$, and then show that the presence of $D_I$ is irrelevant at the level of $KK$-classes.

Let $I^\infty$ be an internal space over $M$, where $D_I = 0$, and consider the unbounded Kasparov module $I := ({}_B E_A, 0)$, where $E = \Gamma(\mathtt{E})$. We know from Proposition 6.2.2 that $I^\infty \times_\nabla M = (\mathcal{B}, L^2(\mathtt{E} \otimes \mathtt{S}), D)$ is a spectral triple, which thus yields an unbounded Kasparov module $I \times_\nabla M = ({}_B L^2(\mathtt{E} \otimes \mathtt{S})_{\mathbb{C}}, D) \in \Psi(B, \mathbb{C})$ (Definition 5.2.31).

**Proposition 6.3.1.** *The unbounded Kasparov module $I \times_\nabla M$ represents the Kasparov product of (the classes of) $I \in \Psi(B, A)$ and $({}_A L^2(\mathtt{S})_{\mathbb{C}}, \not{D}) \in \Psi(A, \mathbb{C})$.*

*Proof.* It suffices to check the conditions of Theorem 5.2.33. Since $D_I = 0$, conditions ii) and iii) are trivial, and we only need to check Condition i). For all $e$ in a dense subspace of $BE = E$, we need to check boundedness of

$$DT_e - T_e \not{D} \qquad \text{on } \mathrm{Dom}(\not{D}) \subset L^2(\mathtt{S}),$$
$$\not{D} T_e^* - T_e^* D \qquad \text{on } \mathrm{Dom}(D) \subset E \otimes_A L^2(\mathtt{S}) \cong L^2(\mathtt{E} \otimes \mathtt{S}),$$

where $D = \not{D}_{\mathtt{E}} = -i(\mathbb{I} \otimes c) \circ (\mathbb{I} \otimes \nabla^{\mathtt{S}} + \nabla^I \otimes \mathbb{I})$. For $\psi \in \mathrm{Dom}(\not{D})$ we obtain

$$(DT_e - T_e \not{D})\psi = -i(\mathbb{I} \otimes c) \circ (\mathbb{I} \otimes \nabla^{\mathtt{S}} + \nabla^I \otimes \mathbb{I}) e \otimes \psi - e \otimes \not{D}\psi = -ic(\nabla^I e) \otimes \psi,$$

which is indeed bounded for all $e$ in the dense subspace $\mathcal{E}$. Next, for $f \otimes \psi \in \mathrm{Dom}(D)$ we obtain

$$(\not{D} T_e^* - T_e^* D)(f \otimes \psi) = \not{D}(e, f)\psi - (e, f)\not{D}\psi + i\big(e, c(\nabla^I f)\big)\psi = -ic\big(\nabla^I e, f\big)\psi,$$

where we have used the compatibility of the connection $\nabla^I$ with the hermitian form $(\cdot|\cdot)_{\mathcal{A}}$, and this is again bounded for smooth $e$. □

To prove a similar result for the case where $D_I \neq 0$, we use the following two lemmas.

**Lemma 6.3.2.** *If ${}_{\phi(B)}E_A$ is finitely generated projective as a right $A$-module, then for any self-adjoint, odd endomorphism $F \in \mathrm{End}_A(E)$, the unbounded Kasparov $B$-$A$-modules $({}_{\phi(B)}E_A, F)$ and $({}_{\phi(B)}E_A, 0)$ represent the same class in $KK(B, A)$.*

*Proof.* Since $E$ is a finitely generated projective $A$-module, all bounded endomorphisms are in fact compact, i.e. $\mathrm{End}_A(E) = \mathrm{End}^0_A(E)$. The equivalence of the compact operators 0 and $b(F) = F(1 + F^2)^{-\frac{1}{2}}$ is then simply obtained via the operator homotopy $t \mapsto tb(F)$, for $t \in [0, 1]$. Hence the modules $({}_{\phi(B)}E_A, b(F))$ and $({}_{\phi(B)}E_A, 0)$ are equivalent bounded Kasparov $B$-$A$-modules. □

**Lemma 6.3.3** (see also [61, Corollary 17])**.** *Let $({}_{\phi(B)}E_A, D) \in \Psi(B, A)$ and let $T \in \mathrm{End}_A(E)$ be self-adjoint and odd. Then*

1. *$({}_{\phi(B)}E_A, D + T)$ is also an unbounded Kasparov module in $\Psi(B, A)$, and;*
2. *$({}_{\phi(B)}E_A, D + T)$ and $({}_{\phi(B)}E_A, D)$ represent the same class in $KK(B, A)$.*

*Proof.* 1. Since $T$ is bounded and self-adjoint, it follows from the Kato-Rellich theorem for Hilbert modules (see [54, Theorem 4.5]) that the sum $D + T$ remains self-adjoint *and* regular. The only non-trivial thing to prove is that $D + T$ has compact resolvent, i.e. $\phi(b)(1 + (D + T)^2)^{-1/2} \in \mathrm{End}^0_A(E)$ for all $b \in \mathcal{B} \subset B$. This is equivalent to showing that $\phi(b)(\pm i + D + T)^{-1}$ is compact. The operator $(\pm i + D + T)^{-1}$ maps $E$ into $\mathrm{Dom}(D + T) = \mathrm{Dom}\, D$, so that $(\pm i + D)(\pm i + D + T)^{-1}$ is a well-defined bounded operator on $E$. From

$$\phi(b)(\pm i + D + T)^{-1} = \phi(b)(\pm i + D)^{-1}(\pm i + D)(\pm i + D + T)^{-1}$$

we then see that $\phi(b)(\pm i + D + T)^{-1}$ is compact.

2. The idea is to prove that $({}_{\phi(B)}E_A, D + T) \in \Psi(B, A)$ represents the Kasparov product $[({}_{\phi(B)}E_A, D)] \otimes_A [({}_A A_A, 0)]$. It is enough to show that all the conditions in Theorem 5.2.33 are satisfied. First of all,

$$A \ni a \mapsto (D + T)T_e(a) = ((D + T)e)a,$$

and

$$(f \otimes a) \mapsto T_e^*(D + T)(f \otimes a) = ((D + T)e, f)_A a,$$

are both clearly bounded for all $e \in \mathrm{Dom}(D{+}T) = \mathrm{Dom}\, D$. As $({}_{\phi(B)}E_A, D{+}T)$ is an unbounded Kasparov module, there exists a dense subalgebra $\mathcal{B}$ such that $\phi(\mathcal{B})\,\mathrm{Dom}\, D \subset \mathrm{Dom}\, D$. Consequently, $\phi(\mathcal{B})\,\mathrm{Dom}\, D$ is a dense subset of $\phi(B)E$. This proves that Condition (i) in Theorem 5.2.33 is satisfied.

Since Dom $D = \mathrm{Dom}(D + T)$, Condition (ii) is also satisfied. For the final condition, a small calculation shows that

$$\begin{aligned}((D + T)e, De) + (De, (D + T)e) &= ((D + T)e, (D + T)e) - ((D + T)e, Te) \\ &\quad + (De, (D + T)e) \\ &= ((D + T)e, (D + T)e) - (Te, Te) \\ &\quad + (De, De) \\ &\geq -\|T\|^2(e, e),\end{aligned}$$

for all $e \in$ Dom $D$, since $((D{+}T)e, (D{+}T)e)$ and $(De, De)$ are positive. □

**Corollary 6.3.4.** *The unbounded Kasparov module* $I \times_\nabla M = ({}_B E \otimes_A L^2(\mathtt{S})_\mathbb{C}, \mathbb{I} \otimes_\nabla \not{D} + D_I \otimes \gamma_5)$ *represents the Kasparov product of* $I = ({}_B E_A, D_I)$ *with* $({}_A L^2(\mathtt{S})_\mathbb{C}, \not{D})$.

*Proof.* By Lemma 6.3.2 we know that $({}_B E_A, D_I)$ and $({}_B E_A, 0)$ represent the same Kasparov class. From Proposition 6.3.1 it then follows that the cycle $({}_B E \otimes_A L^2(\mathtt{S})_\mathbb{C}, \not{D}_\mathtt{E})$ also represents the Kasparov product of $({}_B E_A, D_I)$ with $({}_A L^2(\mathtt{S})_\mathbb{C}, \not{D})$. According to Lemma 6.3.3 the cycle $({}_B E \otimes_A L^2(\mathtt{S})_\mathbb{C}, \not{D}_\mathtt{E} + D_I \otimes \gamma_5)$ represents the same Kasparov class as $({}_B E \otimes_A L^2(\mathtt{S})_\mathbb{C}, \not{D}_\mathtt{E})$, so it also represents this Kasparov product. □

**Remark 6.3.5.** 1. The construction of $I \times_\nabla M$ via Kasparov products fits naturally in the framework of Mesland's category of spectral triples [73], where the internal space $I^\infty$ with the connection $\nabla$ can be seen as (a representative of) a morphism from the canonical triple for $M$ to the almost-commutative manifold $I^\infty \times_\nabla M$.

2. As is clear from the above discussion, the presence of the operator $D_I$ (or $D_I \otimes \gamma_5$) is completely irrelevant on the level of $KK$-classes. In this sense the $KK$-equivalence is too strong for our purposes, because in the models under consideration the presence of the operator $D_I$ certainly does matter. We will describe in Chapter 8 how this operator plays the role of a 'mass matrix' for the elementary fermions of the gauge theory, and gives rise to the Higgs field (see also Section 8.5.2 for a concrete example of $D_I$ as a mass matrix).

# Chapter 7

# Principal and gauge modules

We would like to describe a classical gauge theory on a manifold $M$ by considering an almost-commutative manifold $I^\infty \times_\nabla M$. For this purpose we will now restrict our attention to a special case of internal spaces.

In Section 7.1 we first recall (part of) the classification of finite-dimensional real spectral triples that has been done by Krajewski ([60]). In Section 7.2 we then define the notion of principal modules, and we show that, when the base manifold is simply connected, the gauge group of a principal module (as defined for internal spaces in Equation (6.3)) is isomorphic to the classical notion of the gauge group of a principal fibre bundle (as defined in Definition 5.2.19). Principle modules are of an entirely *geometric* nature.

In Section 7.3 we introduce so-called *gauge modules*, which are of a purely *algebraic* nature. We show that each gauge module is in fact also a principal module, but unfortunately not all principal modules can be obtained from gauge modules.

## 7.1 Real finite spectral triples

Finite-dimensional real spectral triples have been classified by Krajewski [60] for the case of $KO$-dimension 0. With similar arguments, this can be generalised to arbitrary $KO$-dimension (*cf.* for instance [91]). In the following theorem we give the result for complex algebras, while also setting the matrix $D_F = 0$. Below c. c. denotes complex conjugation of the coefficients with respect to the standard basis of $\mathbb{C}^{m_{ij}}$.

**Proposition 7.1.1.** *Let* $F := (A_F, \mathcal{H}_F, 0, J_F)$ *be a real finite spectral triple over* *a complex $*$-algebra $A_F$. Up to unitary equivalence, this triple is of the form*

$$A_F = \bigoplus_{i=1}^{l} M_{N_i}(\mathbb{C}), \qquad \mathcal{H}_F = \bigoplus_{i,j=1}^{l} \mathcal{H}_{ij}, \qquad \mathcal{H}_{ij} := \bigoplus_{i,j=1}^{l} M_{N_i,N_j}(\mathbb{C}) \otimes \mathbb{C}^{m_{ij}},$$

*such that $m_{ij} = m_{ji}$, and the inner product on each copy of $M_{N_i,N_j}(\mathbb{C})$ is given by $\langle t_1, t_2 \rangle = \mathrm{Tr}(t_1^* t_2)$. If $J_F^2 = \varepsilon$, then $J_F$ acts on $\mathcal{H}_{ij} \oplus \mathcal{H}_{ji}$, $(i < j)$, as*

$$\begin{pmatrix} 0 & \varepsilon(\cdot)^* \\ (\cdot)^* & 0 \end{pmatrix} \otimes (\mathrm{Id}_{m_{ij}} \circ \mathrm{c.\,c.}).$$

*The real structure $J_F$ acts on $\mathcal{H}_{ii} \cong M_{N_i}(\mathbb{C}) \otimes \mathbb{C}^{m_{ii}}$ as*

$$(\cdot)^* \otimes (\mathrm{Id}_{m_{ii}} \circ \mathrm{c.\,c.}),$$

*if $J_F^2 = 1$. If $J_F^2 = -1$, then $m_{ii}$ is even and $J_F$ acts on $(M_{N_i}(\mathbb{C}) \oplus M_{N_i}(\mathbb{C})) \otimes \mathbb{C}^{\frac{m_{ii}}{2}}$ as*

$$\begin{pmatrix} 0 & -(\cdot)^* \\ (\cdot)^* & 0 \end{pmatrix} \otimes (\mathrm{Id}_{\frac{m_{ii}}{2}} \circ \mathrm{c.\,c.}).$$

*The different copies of $M_{N_i,N_j}(\mathbb{C})$ (with respect to the above decomposition) in $\mathcal{H}_{ij}$ are denoted by $\mathcal{H}_{ij}^{\alpha}$, where $1 \leq \alpha \leq m_{ij}$.*

**Remark 7.1.2.** Write $V_i = \mathbb{C}^{N_i}$, endowed with the standard inner product. Consider the linear isomorphism

$$L : V \otimes \overline{W} \to \mathrm{Hom}(W, V), \quad v \otimes \overline{w} \mapsto (w' \mapsto v\langle w, w' \rangle), \quad v \in V,\ w, w' \in W,$$

where $V$, $W$ are finite-dimensional complex vector spaces, and $\overline{W}$ denotes the conjugate vector space. Then the finite-dimensional Hilbert space $\mathcal{H}_{ij}$ can also be put in the form

$$\mathcal{H}_F = \bigoplus_{(i,j) \in K} V_i \otimes \overline{V}_j,$$

endowed with its standard inner product. Here $K$ is a *multiset* consisting of pairs in $I \times I$ such that the multiplicity of $(i, j)$ is equal to $(j, i)$ and such that the projection $K \to I$ on either of the factors is surjective (this last condition is equivalent to the faithfulness of the action of $A_F$ on $\mathcal{H}_F$). The algebra $A_F \otimes A_F^{\mathrm{op}}$ acts on a summand $V_i \otimes \overline{V_j}$ as

$$(a, b^{\mathrm{op}})(v \otimes \overline{w}) = a_i v \otimes \overline{b_j^* w},$$

and the corresponding real structure on $V_i \otimes \overline{V_j} \to V_j \otimes \overline{V_i}$ is simply given by

$$J_F(v \otimes \overline{w}) = \pm w \otimes \overline{v},$$

where the signs are determined by the $KO$-dimension of $F$. We will use this form of the real finite spectral triple in Section 7.3.

From now on we assume that every real finite spectral triple (with $D_F = 0$) is of the form as mentioned in Proposition 7.1.1. Later on, the algebra $(A_F)_{J_F}$ will also be of interest, so we conclude this section by determining its precise form.

Recall that, in general, for any real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$, the complex central subalgebra $\mathcal{A}_J$ is defined as $\mathcal{A}_J = \{a \in \mathcal{A} \mid Ja = a^* J\}$.

**Proposition 7.1.3.** *With notation as above (and with $\lambda_i \in \mathbb{C}$ for each $i$), we have*

$$(A_F)_{J_F} = \Big\{ a = \bigoplus_{i \in I} \lambda_i \, \mathrm{id}_{N_i} \in A_F \;\Big|\; \lambda_i = \lambda_j \text{ if } \mathcal{H}_{ij} \neq \{0\} \Big\}.$$

*Proof.* We can assume that $J$ is in standard form. Write $A_F = \bigoplus_i M_{N_i}(\mathbb{C})$ and consider an element $a = \bigoplus_{i \in I} a_i \in A_F$. If $t \in \mathcal{H}_{ij}^\alpha$ $(1 \leq \alpha \leq m_{ij})$, then

$$a(J_F t) = \pm a_j t^* \quad \text{and} \quad J_F(a^* t) = \pm t^* a_i.$$

Choose $t^* = e_{kl}$, where $1 \leq k \leq N_j$ and $1 \leq l \leq N_i$. Then

$$(a_j e_{kl})_{\gamma\beta} = (a_j)_{\gamma k} \delta_{\beta l}, \quad \text{and} \quad (e_{kl} a_i)_{\gamma\beta} = \delta_{\gamma k} (a_i)_{l\beta}.$$

Therefore, $aJ_F = J_F a^*$ if and only if

$$(a_j)_{\gamma k} \delta_{\beta l} = (a_i)_{l\beta} \delta_{\gamma k},$$

for all $1 \leq k, \gamma \leq N_j$ and $1 \leq l, \beta \leq N_i$. It follows that $a_i$, $a_j$ are diagonal and $(a_j)_{kk} = (a_i)_{ll}$ for all $1 \leq k \leq N_j$ and $1 \leq l \leq N_i$. Hence, $a \in (A_F)_{J_F}$ if and only if each $a_i = \lambda_i \mathrm{id}_{N_i}$ and $\lambda_i = \lambda_j$ if $\mathcal{H}_{ij} \neq \{0\}$. □

The following definition is inspired by the proof of Proposition 7.1.3.

**Definition 7.1.4.** Let $A_F = \bigoplus_{i \in I} M_{N_i}(\mathbb{C})$ act on $\mathcal{H}_F = \bigoplus_{i,j \in I} \mathcal{H}_{ij}$ as above. We define an equivalence relation on $I$ as follows. For $i \neq j \in I$ we set $i \sim j$ if there exists a sequence $i = i_0, \ldots, i_k = j$ such that $\mathcal{H}_{i_l, i_{l+1}} \neq \{0\}$ for all $0 \leq l < k$. If $i \sim j$ we say that $i$ ***is connected to*** $j$.

Proposition 7.1.3 in particular shows that $\mathbb{C} \subset (A_F)_{J_F} \subset Z(A_F)$.

**Corollary 7.1.5.** *We have the isomorphism $(A_F)_{J_F} \cong \bigoplus_{[i] \in I/\sim} \mathbb{C}$. In particular, the two extreme cases are:*

- *$(A_F)_{J_F} = Z(A_F)$ if and only if $\mathcal{H}_{ij} = 0$ for all $i \neq j$ (that is, $I/\sim \; \cong I$).*
- *$(A_F)_{J_F} = \mathbb{C}$ if and only if $i$ is connected to $j$ for all $i, j \in I$ (that is, $I/\sim \; \cong \{0\}$).*

## 7.2 Principal modules

We now want to find spectral triples for gauge theories that are globally non-trivial. Recall from Definition 5.2.16 that a general gauge theory with structure group $G_F$ on a manifold $M$ is given by a principal $G_F$-bundle $\mathtt{P}$ over $M$ (along with a prescribed action functional or Lagrangian).

If $(A_F, \mathcal{H}_F, D_F, J_F)$ is a finite-dimensional real spectral triple, then the corresponding gauge group $G_F$ is given by (see also Equation (6.2))

$$G_F := \{uJ_F uJ_F^* \mid u \in \mathcal{U}(A_F)\} \cong \mathcal{U}(A_F)/\mathcal{U}((A_F)_{J_F}).$$

Such finite spectral triples can be used to describe globally trivial gauge theories over $M$ (see the Introduction). Any finite spectral triple $F$ automatically yields an internal space

$$I_F^\infty = \big(\Gamma^\infty(M \times A_F), \Gamma^\infty(M \times \mathcal{H}_F), D_F, J_F\big),$$

where now $D_F$ and $J_F$ are seen as constant bundle endomorphisms acting on the fibre $\mathcal{H}_F$. We now want to generalise this construction in order to describe globally non-trivial gauge theories. Of course, fibrewise we want to obtain the finite-dimensional situation that has been obtained by Krajewski in [60], and has been explained in Section 7.1.

The most straightforward way to obtain (examples of) globally non-trivial gauge theories over $M$ would then be as follows (see also [13, Lemma 2.5] and [11]). Take any real finite spectral triple $F := (A_F, \mathcal{H}_F, D_F, J_F)$ with gauge group $G_F$, and let $M$ be a smooth compact 4-dimensional Riemannian spin manifold. Take any principal $G_F$-bundle $\mathtt{P} \to M$. We construct the globally non-trivial triple of the form

$$\mathtt{P} \times_{G_F} F := \big(\Gamma^\infty(\mathtt{P} \times_{G_F} A_F), \Gamma^\infty(\mathtt{P} \times_{G_F} \mathcal{H}_F), D_{\mathtt{P}}, 1 \times J_F\big).$$

Here $D_{\mathtt{P}}$ could be $1 \times D_F$, but we also allow for more general endomorphisms acting on the vector bundle $\mathtt{P} \times_{G_F} \mathcal{H}_F$ satisfying certain compatibility requirements (see Definition 8.1.1).

**Remark 7.2.1.** Note that (in contrast to [13]) we do not require $D_{\mathtt{P}}$ to be of the form $1 \times D_F$, where $D_F$ is a $G_F$-invariant operator on $\mathcal{H}_F$, as such an assumption is too strong for our purposes. In particular, in specific examples (such as the noncommutative Standard Model) that requirement would prevent the appearance of a scalar (Higgs-like) field through inner fluctuations.

For the remainder of this section we ignore the endomorphism $D_{\mathtt{P}}$, since it is not relevant for the definition of the gauge group. Since $\mathtt{P} \times_{G_F} F$ is constructed from a *principal* bundle we introduce the following terminology.

**Definition 7.2.2.** Let $F := (A_F, \mathcal{H}_F, 0, J_F)$ be a real finite spectral triple of the same form as in Proposition 7.1.1. Write $G_F$ for the corresponding gauge group. Let $M$ be a smooth compact Riemannian spin manifold and let $\mathtt{P} \to M$ be any principal $G_F$-bundle. A triplet of the form

$$\mathtt{P} \times_{G_F} F := \big(\Gamma^\infty(\mathtt{P} \times_{G_F} A_F), \Gamma^\infty(\mathtt{P} \times_{G_F} \mathcal{H}_F), 1 \times J_F\big),$$

is called a ***principal*** $G_F$***-module*** over $M$ (or $C^\infty(M)$) with fibre $F$. For brevity, we introduce the notation $\mathtt{B} := \mathtt{P} \times_{G_F} A_F$, $\mathtt{E} := \mathtt{P} \times_{G_F} \mathcal{H}_F$, $\mathcal{B} := \Gamma^\infty(\mathtt{B})$, $\mathcal{E} := \Gamma^\infty(\mathtt{E})$, and $J := 1 \times J_F$.

**Remark 7.2.3.** The principal fibre bundle $\mathtt{P}$ is an explicit ingredient in the definition of a principal module. From $\mathtt{P}$ we construct the associated vector bundle $\mathtt{E} = \mathtt{P} \times_{G_F} \mathcal{H}_F$, and (as discussed in Section 5.2.2) $\mathtt{P}$ equips $\mathtt{E}$ with a unique equivalence class of $G_F$-atlases. Whenever we consider transition functions of $\mathtt{E}$, we therefore assume that they form a $G_F$-atlas in the equivalence class obtained from $\mathtt{P}$. Given a $G_F$-atlas, the vector bundle $\mathtt{E}$ inherits a hermitian structure from the inner product on $\mathcal{H}_F$, which is well-defined because the action of $G_F$ on $\mathcal{H}_F$ is unitary. For two equivalent $G_F$-atlases, the corresponding hermitian structures are isometric (see Example 5.2.22).

We stress that, given only the vector bundle $\mathtt{E}$ (with structure group $G_F$), we cannot reconstruct the principal $G_F$-bundle $\mathtt{P}$. In order to reconstruct $\mathtt{P}$, we also need to know the corresponding equivalence class of $G_F$-atlases.

**Proposition 7.2.4.** *A principal module* $\mathtt{P} \times_{G_F} F$ *is a real internal space* $(\Gamma^\infty(\mathtt{P} \times_{G_F} A_F), \Gamma^\infty(\mathtt{P} \times_{G_F} \mathcal{H}_F), 0, 1 \times J_F)$ *over* $M$*.*

*Proof.* The action of $G_F$ on $A_F$ is given by conjugation when $A_F$ is considered as a $*$-subalgebra of $\mathrm{End}(\mathcal{H}_F)$. Consequently, the fibre-wise action of the $*$-algebra bundle $\mathtt{B} = \mathtt{P} \times_{G_F} A_F$ on $\mathtt{E}$ is well defined, and hence $\mathtt{B}$ is a unital $*$-algebra subbundle of $\mathrm{End}(\mathtt{E})$. The operator $D_I = 0$ is trivially a hermitian endomorphism. Since the operator $J_F$ commutes with $G_F$, it induces a real structure $J_x$ on each fibre of $\mathtt{E}$. The operator $J = 1 \times J_F$ denotes the anti-linear operator on $\mathtt{E}$ that is induced by these real structures $J_x$ on the fibres. $\square$

**Remark 7.2.5.** Because $(uJ_FuJ_F^*)a(J_Fu^*J_F^*u^*) = uau^*$ for all $a \in A_F$, $u \in \mathcal{U}(A_F)$, we see that the given action of an element $uJ_FuJ_F^* \in G_F$ on $A_F$ coincides with the usual conjugation of the element $u \in \mathcal{U}(A_F)$. Since $(A_F)_{J_F} \subset Z(A_F)$, the map $\tau : G_F \ni uJ_FuJ_F^* \mapsto \mathrm{Ad}(uJ_FuJ_F^*) = \mathrm{Ad}\, u \in \mathrm{Inn}(A_F)$ does not depend on the choice of $u$. Thus, the surjective map $\tau : G_F \to \mathcal{U}(A_F)/\mathcal{U}(Z(A_F)) \cong \mathrm{Inn}(A_F)$ is induced by the usual map $\mathcal{U}(A_F) \to \mathrm{Inn}(A_F)$ (recall that $G_F$ is the quotient $\mathcal{U}(A_F)/\mathcal{U}((A_F)_{J_F})$).

### 7.2.1 The gauge group

Consider a principal module $\mathtt{P} \times_{G_F} F = \big(\mathcal{B}, \mathcal{E}, J\big)$ over $M$. Using the classification of $A_F$ and $\mathcal{H}_F$, as given in Section 7.1, we can decompose the bundles $\mathtt{B} = \mathtt{P} \times_{G_F} A_F$ and $\mathtt{E} = \mathtt{P} \times_{G_F} \mathcal{H}_F$ in a similar way:

$$\mathtt{B} = \bigoplus_{i \in I} \mathtt{B}_i, \qquad \mathtt{B}_i = \mathtt{P} \times_{G_F} M_{N_i}(\mathbb{C}),$$
$$\mathtt{E} = \bigoplus_{i,j \in I} \mathtt{E}_{ij}, \qquad \mathtt{E}_{ij} = \mathtt{P} \times_{G_F} \mathcal{H}_{ij}.$$

Each vector bundle $\mathtt{E}_{ij}$ carries the obvious action by $\mathtt{B} \otimes \mathtt{B}^{\mathrm{op}}$. Note, however, that even though $\mathcal{H}_{ij} = \mathbb{C}^{N_i} \otimes \mathbb{C}^{N_j} \otimes \mathbb{C}^{m_{ij}}$, $\mathtt{E}_{ij}$ is not necessarily of the form $\mathtt{E}_i \otimes \mathtt{E}_j \otimes \mathbb{C}^{m_{ij}}$ for some vector bundles $E_i$ and $E_j$ (see also Section 8.5.1).

Note that, for the case $i = j$, the bundle $\mathtt{E}_{ii}$ is necessarily isomorphic to (a number of copies of) $\mathtt{B}_i$. Indeed, the $G_F$-valued transition functions act on the fibres of $\mathtt{E}_{ii}$, which are isomorphic to (copies of) $M_{N_i}(\mathbb{C})$, by conjugation with an element $u \in U(N_i)$, and are therefore inner automorphisms of the algebra $M_{N_i}(\mathbb{C})$. By Remark 7.2.5 these transition functions are equal to those for the $*$-algebra bundle $\mathtt{B}_i$.

Denote by $[i]$ the equivalence class of all $j \in I$ that are connected to $i$ (see Definition 7.1.4). Write

$$\mathtt{B}_{[i]} = \bigoplus_{s \in [i]} \mathtt{B}_s,$$

and write $b_{[i]}$ for the projection of an element $b$ onto $\mathtt{B}_{[i]}$. As $(\mathcal{B}_{[i]})_J = C^\infty(M)$ we get (see also Corollary 7.1.5)

$$\mathcal{B}_J = \bigoplus_{[i] \in I/\sim} C^\infty(M).$$

The ***gauge group*** of the principal module $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ is defined as (see Equation (6.3))

$$\mathcal{G}(\mathtt{P} \times_{G_F} F) := \big\{ uJuJ^* \mid u \in \mathcal{U}(\mathcal{B}) \big\} \cong \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J).$$

At the same time, a principal $G_F$-bundle $\mathtt{P} \to M$ is equipped with the gauge group $\mathcal{G}(\mathtt{P}) = \Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$ (see Section 5.2.2). We now aim at showing that for a principal module $\mathtt{P} \times_{G_F} F$, the gauge groups $\mathcal{G}(\mathtt{P} \times_{G_F} F)$ and $\mathcal{G}(\mathtt{P})$ coincide, provided that $M$ is simply connected.

Consider the group bundle map

$$\phi \colon \mathcal{U}(\mathtt{B}) \cong \mathtt{P} \times_{G_F} \mathcal{U}(A_F) \to \mathtt{P} \times_{G_F} \mathcal{U}(\mathcal{H}_F), \qquad u_x \mapsto u_x J_x u_x J_x^*,$$

where $\mathcal{U}(\mathcal{H}_F)$ denotes the group of unitary operators on $\mathcal{H}_F$. The image $\phi(\mathcal{U}(\mathtt{B}))$ is a group subbundle of $\mathtt{P} \times_{G_F} \mathcal{U}(\mathcal{H}_F)$, with fibres isomorphic to $G_F$. In fact, this subbundle is isomorphic to the group bundle $\mathrm{Ad}\,\mathtt{P}$. The induced map $\phi_*$ on the sections $\mathcal{U}(\mathcal{B}) \cong \mathcal{U}(\Gamma^\infty(\mathtt{B})) \to \mathcal{U}(\Gamma^\infty(\mathrm{End}(\mathtt{E})))$ is precisely the map $u \mapsto uJuJ^*$, $u \in \mathcal{U}(\mathcal{B})$. Thus, $\phi_*$ maps $\mathcal{U}(\mathcal{B})$ into $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$. However, as discussed in Section 5.2.4, this map $\phi_*$ does not always map onto $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$. We will proceed by showing that in our case, under the assumption that $M$ is simply connected, $\phi_*$ does map $\mathcal{U}(\mathcal{B})$ onto $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$.

**Proposition 7.2.6.** *Let $\mathtt{P} \times_{G_F} F$ be a principal module over $M$. There exists a Lie group subbundle $\mathtt{U} \subset \mathcal{U}(\mathtt{B})$ such that the restriction $\phi : \mathtt{U} \to \mathrm{Ad}\,\mathtt{P}$ is a covering map.*

*Proof.* Consider the subbundle $\mathtt{E}_{[i]} := \mathtt{B}_{[i]} \cdot \mathtt{E}$ (i.e. the subbundle on which $\mathtt{B}_{[i]}$ acts non-trivially). Define the group subbundle

$$\mathtt{U} := \{u \in \mathcal{U}(\mathtt{B}) \mid \det_{[i]} u_{[i]} = 1 \text{ for all } [i]\},$$

where $\det_{[i]}$ denotes the determinant taken on the fibre of the subbundle $\mathtt{E}_{[i]}$.[1] Denote the rank of $\mathtt{E}_{[i]}$ by $N_{[i]}$. Since any element $u \in \mathcal{U}(\mathtt{B})$ can be written as $u = vw$, where $v \in \mathtt{U}$ and $w \in \mathcal{U}(\mathtt{B}_J)$ (just take $w_{[i]} = \big(\det_{[i]} u_{[i]}\big)^{\frac{1}{N_{[i]}}} \mathrm{id}_{N_{[i]}}$ and $v = uw^{-1}$), the image $\phi(\mathtt{U})$ is equal to the image $\phi(\mathcal{U}(\mathtt{B})) = \mathrm{Ad}\,\mathtt{P}$.

Let us calculate the kernel $\phi_x : \mathtt{U}_x \to (\mathrm{Ad}\,\mathtt{P})_x$. Choose $u \in \mathtt{U}_x \cap \ker \phi_x$. Since $u \in \ker \phi_x$, each $u_{[i]}$ is diagonal. Because $\det_{[i]} u_{[i]} = 1$, we obtain that $u_{[i]} = \lambda_{[i]} \mathrm{id}_{N_{[i]}}$, where $\lambda_{[i]}$ is an $N_{[i]}$-th root of unity. Since there are only finitely many equivalence classes $[i]$, the group $\mathtt{U}_x \cap \ker \phi_x$ is finite.

The condition for a map to be a covering map is of a local nature, so we can assume that all bundles are globally trivial. In that case, it follow from the fact that $\mathtt{U}_x \cap \ker \phi_x$ is finite, that $\mathtt{U} \to \mathrm{Ad}\,\mathtt{P}$ is a covering map. □

Combining Proposition 7.2.6 with Corollary 5.2.29 and Theorem 5.2.21 immediately yields the desired result:

**Theorem 7.2.7.** *Let $\mathtt{P} \times_{G_F} F$ be a principal module over $M$. If $M$ is simply connected, then*

$$\mathcal{G}(\mathtt{P} \times_{G_F} F) \cong \Gamma^\infty(\mathrm{Ad}\,\mathtt{P}) \cong \mathcal{G}(\mathtt{P}).$$

**Remark 7.2.8.** We emphasise that the isomorphism $\mathcal{G}(\mathtt{P} \times_{G_F} F) \cong \Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$ need not hold if $M$ is not simply connected (*cf.* Section 5.2.4). In general, $\mathcal{G}(\mathtt{P} \times_{G_F} F)$ is identified with a subgroup of $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$. If $\mathcal{G}(\mathtt{P} \times_{G_F} F)$ is a proper subgroup of $\Gamma^\infty(\mathrm{Ad}\,\mathtt{P})$, then it fails to be the full gauge group of $\mathtt{P}$ that was defined in Definition 5.2.19. We do not yet know how to interpret $\mathcal{G}(\mathtt{P} \times_{G_F} F)$ in that case.

[1] This definition makes sense, because all transition functions of $\mathtt{B}$ take values in the group of inner automorphisms of $A_F$.

**Remark 7.2.9.** It follows from the above that for each element $g$ of the gauge group $\mathcal{G}(\mathtt{P}\times_{G_F}F)$, there exists a unitary section $u \in \mathcal{B}$ with (fibrewise) determinant equal to 1, such that $g = uJuJ^*$. In this sense, the gauge group is *unimodular* by default. This only holds for complex algebras $\mathcal{B}$. For real algebras (including the one describing the noncommutative Standard Model [18, 15]) one needs to impose unimodularity by hand (see also [66] and references therein).

## 7.3 Gauge modules

In Section 7.2 we introduced the notion of principal modules, which have an entirely *geometric* nature: because we were unable to find an algebraic characterisation for principal modules, a principal bundle $\mathtt{P}$ still appears in the definition of a principal module. In this section we introduce *gauge modules*, which are in fact special instances of principal modules (Proposition 7.3.2 below), but they are of a purely *algebraic* nature in the sense that their definition can be formulated entirely in algebraic terms. Unfortunately not all principal modules can be obtained from gauge modules (*cf.* Section 8.5.1). As we will show below, in the case of gauge modules one *can* (re)construct the corresponding principal bundle.

Inspired by the standard form of finite spectral triples as obtained in Proposition 7.1.1 and Remark 7.1.2, we introduce the following definition. The idea is to globalise Krajewski diagrams [60].

**Definition 7.3.1.** Let $\mathcal{A} := C^\infty(M)$. Suppose we are given a finite set of non-degenerate hermitian finitely generated projective $\mathcal{A}$-modules $\mathcal{E}_i$ (for $i \in I = \{1, \dots, l\}$), and define the module algebras $\mathcal{B}_i := \mathrm{End}_{\mathcal{A}}(\mathcal{E}_i)$. Take a *multiset* $K$ consisting of pairs in $I \times I$ such that the multiplicity of $(i, j)$ is equal to the multiplicity of $(j, i)$, and such that the projection $K \to I$ on either of the factors is surjective. Denote the multiplicity of the pair $(i, j)$ by $m_{ij}$ and write $(i_\alpha, j_\alpha)$ $(1 \leq \alpha \leq m_{ij})$ to distinguish the pairs in $K$ that occur more than once (see also Proposition 7.1.1 for this notation).

A ***gauge module*** $(\mathcal{B}, \mathcal{E}, J)$ is of the form

$$\mathcal{B} := \bigoplus_{i\in I} \mathcal{B}_i, \qquad \mathcal{E} := \bigoplus_{(i,j)\in K} \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}_j}, \qquad J\colon \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}_j} \to \mathcal{E}_j \otimes_{\mathcal{A}} \overline{\mathcal{E}_i},$$

where $J$ is of the same standard form as the finite operator $J_F$ in Proposition 7.1.1 (and which depends on the value of $J^2 = \varepsilon = \pm 1$, e.g. $J_{ij}(e_{i_\alpha} \otimes \overline{e_{j_\alpha}}) = \varepsilon e_{j_\alpha} \otimes \overline{e_{i_\alpha}}$, for $e_{i_\alpha} \otimes \overline{e_{j_\alpha}} \in \mathcal{E}_{i_\alpha} \otimes \overline{\mathcal{E}_{j_\alpha}}$ if $j < i$).

The assumption that the projection $K \to I$ is surjective ensures that the action of $\mathcal{B}$ on $\mathcal{E}$ is faithful. From the Serre-Swan theorem we know that each module $\mathcal{E}_i$ is given by the smooth sections of a vector bundle $\mathtt{E}_i \to M$. Because the hermitian structure on $\mathcal{E}_i$ is non-degenerate, this yields a hermitian structure on $\mathtt{E}_i$. By

Theorem 5.2.11 the module algebra $\mathcal{B}_i$ is given by the smooth sections of a unital $*$-algebra fibration $\mathtt{B}_i \to M$. Since $\mathcal{B}_i = \mathrm{End}_{\mathcal{A}}(\mathcal{E}_i)$ we obtain $\mathtt{B}_i = \mathrm{End}(\mathtt{E}_i)$. The local triviality of $\mathtt{B}_i$ then follows from the local triviality of $\mathtt{E}_i$, which means that $\mathtt{B}_i$ is in fact a unital $*$-algebra *bundle*.

As mentioned in Remark 7.2.3, given a principal module $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ (but not $P$ itself), it is not possible to reconstruct $\mathtt{P}$, unless we are given the equivalence class of $G$-atlases on the vector bundle $\mathtt{E} = \mathtt{P} \times_{G_F} \mathcal{H}_F$. However, we will show below that for gauge modules it is possible to uniquely reconstruct the corresponding principal $G_F$-bundle. The main distinctive feature of gauge modules is that the vector bundle $\mathtt{E}$ decomposes as a direct sum of tensor products of hermitian vector bundles $\mathtt{E}_i$. For each $\mathtt{E}_i$, the hermitian structure allows us to uniquely construct a corresponding principal $U(N_i)$-bundle. From these principal $U(N_i)$-bundles we can subsequently construct the corresponding principal $G_F$-bundle $\mathtt{P}$.

**Proposition 7.3.2.** *Let $(\mathcal{B}, \mathcal{E}, J)$ be a gauge module. Then:*

1. *There exist a real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ and a principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ such that $\mathtt{B} \cong \mathtt{Q} \times_{\mathcal{U}(A_F)} A_F$, $\mathtt{E} \cong \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathcal{H}_F$, and $J = 1 \times J_F$.*
2. *There exists a principal $G_F$-bundle $\mathtt{P}$ such that $(\mathcal{B}, \mathcal{E}, J) = \mathtt{P} \times_{G_F} F$.*

*Proof.* 1. The gauge module $(\mathcal{B}, \mathcal{E}, J)$ is constructed from a given set of hermitian vector bundles $\mathtt{E}_i$ of rank $N_i$ and the index (multi)sets $I$ and $K$. By assumption $\mathtt{B}_i = \mathrm{End}(\mathtt{E}_i)$, and so $\mathtt{B}_i$ has the fibre $M_{N_i}(\mathbb{C})$. We define

$$A_F := \bigoplus_{i \in I} M_{N_i}(\mathbb{C}), \qquad \mathcal{H}_F := \bigoplus_{(i,j) \in K} \mathbb{C}^{N_i} \otimes \overline{\mathbb{C}^{N_j}}.$$

For each $\mathtt{E}_i$, (the isometry class of) the given hermitian structure uniquely determines an equivalence class of $U(N_i)$-atlases on $\mathtt{E}$ (see Example 5.2.22), from which we construct a principal $U(N_i)$-bundle $\mathtt{Q}_i$ (which is unique up to isomorphism) such that $\mathtt{E}_i \cong \mathtt{Q}_i \times_{U(N_i)} \mathbb{C}^{N_i}$. Let $(U, h^i_U)$ be local trivialisations of $\mathtt{E}_i$ corresponding to local trivialisations of $\mathtt{Q}_i$, and write the corresponding $U(N_i)$-valued transition functions as $u^i_{UV}$. Denoting the induced local trivialisations on $\overline{\mathtt{E}_i}$ as $(U, \overline{h^i_U})$, we obtain local trivialisations of $\mathtt{E}$ of the form

$$h_U = \bigoplus_{(i,j) \in K} h^i_U \otimes \overline{h^j_U}.$$

The transition functions $\overline{u^j_{UV}}$ of $\overline{\mathtt{E}_j}$ are given by the right action of $(u^j_{UV})^*$ on $\overline{\mathbb{C}^{N_j}}$ (see Equation (5.5)), which is implemented as $(v_i \otimes \overline{w_j})(u^j_{UV})^* = J u^j_{UV} J^* (v_i \otimes \overline{w_j})$. Hence we obtain transition functions for $\mathtt{E}$ of the form

$$g_{UV} = \bigoplus_{(i,j) \in K} u^i_{UV} \otimes (u^j_{UV})^{*\mathrm{op}} = \bigoplus_{(i,j) \in K} u^i_{UV} J u^j_{UV} J^*.$$

Writing $u_{UV} = \bigoplus_{i\in I} u^i_{UV} \in C^\infty(U\cap V, \mathcal{U}(A_F))$, we see that

$$g_{UV} = u_{UV} J u_{UV} J^* \in C^\infty(U\cap V, G_F).$$

Since the $u^i_{UV}$ are transition functions of $\pi_i\colon \mathtt{Q}_i \to M$, we see that the $u_{UV}$ are the transition functions of the principal $\mathcal{U}(A_F)$-bundle

$$\mathtt{Q} := \mathtt{Q}_1 \times_M \cdots \times_M \mathtt{Q}_l := \{(q_1,\ldots,q_l) \in \mathtt{Q}_1 \times \cdots \times \mathtt{Q}_l \mid \pi_1(q_1) = \cdots = \pi_l(q_l)\}.$$

Since the action of $u_{UV}$ on $\mathcal{H}_F$ is given by $g_{UV} = u_{UV} J u_{UV} J^*$, we see that $\mathtt{E} \cong \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathcal{H}_F$ as hermitian vector bundles. As conjugation by $u_{UV}$ coincides with conjugation by $g_{UV}$ on the algebra $A_F$, we also have $\mathtt{B} \cong \mathtt{Q} \times_{\mathcal{U}(A_F)} A_F$. It is straightforward to check that $J$ is invariant under conjugation by a transition function $g_{UV}$, and hence it is simply of the form $J = 1 \times J_F$, where $J_F$ is a real structure on $\mathcal{H}_F$.

2. Given the principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ from the first part of this lemma, we simply construct a principal $G_F$-bundle as

$$\mathtt{P} := \mathtt{Q} \times_{\mathcal{U}(A_F)} G_F,$$

where $u \in \mathcal{U}(A_F)$ acts on $G_F$ as left multiplication by the element $uJ_F uJ_F^*$. The transition functions of $\mathtt{P}$ are given by $g_{UV} = u_{UV} J u_{UV} J^* \in C^\infty(U \cap V, G_F)$. It then straightforwardly follows that

$$\mathtt{P} \times_{G_F} \mathcal{H}_F \cong (\mathtt{Q} \times_{\mathcal{U}(A_F)} G_F) \times_{G_F} \mathcal{H}_F \cong \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathcal{H}_F \cong \mathtt{E},$$

and similarly we obtain $\mathtt{P} \times_{G_F} A_F \cong \mathtt{B}$. □

The above proposition shows that each gauge module is in fact a principal module $\mathtt{P} \times_{G_F} F$ (where we can uniquely reconstruct $F$ and $\mathtt{P}$). Furthermore, it shows not only that $\mathtt{P}$ allows for a lifting of the structure group from $G_F$ to $\mathcal{U}(A_F)$, but also that we have a preferred lift $\mathtt{Q}$ (corresponding to the hermitian structures of the bundles $\mathtt{E}_i$). We will now show the converse, namely that a principal module $\mathtt{P} \times_{G_F} F$ with a preferred lift $\tau\colon \mathtt{Q} \to \mathtt{P}$ uniquely corresponds to a gauge module.

**Proposition 7.3.3.** *Let $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J)$ be a principal module, and suppose we have a principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ that lifts $\mathtt{P}$. Then $\mathtt{Q}$ naturally induces a gauge module structure on $(\mathcal{B}, \mathcal{E}, J)$.*

*Proof.* As we have seen in Section 7.1, the (massless) real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$ has a decomposition of the form

$$A_F = \bigoplus_{i\in I} M_{N_i}(\mathbb{C}), \qquad \mathcal{H}_F = \bigoplus_{(i,j)\in K} \mathbb{C}^{N_i} \otimes \overline{\mathbb{C}^{N_j}}.$$

Thus we have $\mathcal{U}(A_F) = \times_{i\in I} U(N_i)$, and the principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ then decomposes as $\mathtt{Q}_1 \times_M \cdots \times_M \mathtt{Q}_l$, where each $\mathtt{Q}_i$ is a principal $U(N_i)$-bundle given by $\mathtt{Q}_i := \mathtt{Q} \times_{\mathcal{U}(A_F)} U(N_i)$. We then construct

$$\mathtt{B}_i := \mathtt{Q} \times_{\mathcal{U}(A_F)} M_{N_i}(\mathbb{C}) \cong \mathtt{Q}_i \times_{U(N_i)} M_{N_i}(\mathbb{C}),$$

and

$$\mathtt{E}_i := \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathbb{C}^{N_i} \cong \mathtt{Q}_i \times_{U(N_i)} \mathbb{C}^{N_i},$$

where $\mathcal{U}(A_F) = \times_{i\in I} U(N_i)$ acts on $\mathbb{C}^{N_i}$ as left multiplication by the factor $U(N_i)$, and on $M_{N_i}(\mathbb{C})$ as conjugation by $U(N_i)$. The bundle $\mathtt{E}_i$ naturally inherits a hermitian structure from the standard inner product on $\mathbb{C}^{N_i}$. Because $\mathtt{Q}$ lifts $\mathtt{P}$, the bundles $\mathtt{B}$ and $\mathtt{E}$ corresponding to the principal module $\mathtt{P} \times_{G_F} F$ are in fact of the form

$$\mathtt{B} := \mathtt{Q} \times_{\mathcal{U}(A_F)} A_F = \bigoplus_{i\in I} \mathtt{B}_i, \qquad \mathtt{E} := \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathcal{H}_F = \bigoplus_{(i,j)\in K} \mathtt{E}_i \otimes \overline{\mathtt{E}_j}.$$

Furthermore, as the transition functions of $\mathtt{B}_i$ are given by conjugation by the transition functions of $\mathtt{E}_i$, and as its fibre equals $M_{N_i}(\mathbb{C}) = \mathrm{End}(\mathbb{C}^{N_i})$, it follows that $\mathtt{B}_i = \mathrm{End}(\mathtt{E}_i)$ and $\mathtt{B}_i$ acts as such on $\mathtt{E}$. Hence we have shown that the principal module $\mathtt{P} \times_{G_F} F$ is equal to the gauge module given by the modules $\mathcal{E}_i := \Gamma^\infty(\mathtt{E}_i)$ and the real structure $J = 1 \times J_F$. □

The previous two propositions then lead us to the main result of this section:

**Theorem 7.3.4.** *A gauge module is characterised uniquely (up to isomorphism) by a principal module* $\mathtt{P} \times_{G_F} F$ *with a given principal* $\mathcal{U}(A_F)$*-bundle* $\mathtt{Q}$ *that lifts* $\mathtt{P}$*.*

*Proof.* Given a gauge module, we have shown in Proposition 7.3.2 that we can uniquely construct a (massless) real finite spectral triple $F = (A_F, \mathcal{H}_F, 0, J_F)$, a principal $G_F$-bundle $\mathtt{P}$, and a principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ that lifts $\mathtt{P}$. Conversely, given such $F$, $\mathtt{P}$, and $\mathtt{Q}$, Proposition 7.3.3 shows that $\mathtt{P} \times_{G_F} F$ is in fact given by a gauge module. It remains to show that these two constructions are inverse to each other. This follows because the bundles $\mathtt{E}_i$ constructed in the proof of Proposition 7.3.3 inherit a hermitian structure from their fibre $\mathbb{C}^{N_i}$, and these hermitian structures suffice to reconstruct this same bundle $\mathtt{Q}$ as in the proof of Proposition 7.3.2. □

**Remark 7.3.5.**

1. Every *globally trivial* principal module, constructed from a finite spectral triple $F$ and the principal bundle $\mathtt{P} = M \times G_F$, is in fact a gauge module, characterised by the lift $\mathtt{Q} = M \times \mathcal{U}(A_F)$.
2. An example of a principal module that is (in general) *not* a gauge module (except when for instance the underlying manifold is simply connected and 4-dimensional) is described in Section 8.5.1.

# Chapter 8

# Gauge theory

In this chapter we show how principal modules describe gauge theories on 4-dimensional compact spin manifolds. First we will introduce a 'mass matrix'. Viewing the (now massive) principal module as an internal space and endowing it with a (suitable) connection, we can then use it to construct an almost-commutative manifold. Subsequently, we determine the inner fluctuations and provide an explicit formula for the spectral action of this almost-commutative manifold. This eventually leads to the main result of Part II of this thesis, namely that such an almost-commutative manifold indeed describes a gauge theory in the sense of Definition 5.2.16. We end this chapter by giving some examples.

## 8.1 Principal almost-commutative manifolds

**Definition 8.1.1.** Consider a principal module $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I)$.[1] In order to be able to describe massive gauge theories, we now introduce a ***'mass matrix'***

$$D_I \in \Gamma^\infty(\mathrm{End}(\mathtt{E})) \cong \mathrm{End}_{\mathcal{A}}(\mathcal{E}),$$

satisfying

$$D_I = D_I^*, \qquad D_I J_I = \varepsilon' J_I D_I, \qquad \big[[D_I, a], JbJ^*\big] = 0 \quad \forall a, b \in \mathcal{B},$$

where the sign $\varepsilon'$ (along with the signs $\varepsilon, \varepsilon''$ obtained through the finite spectral triple $F$) is determined by the $KO$-dimension according to the same table as in Definition 5.2.30. We then call $I_{\mathtt{P}}^\infty := (\mathcal{B}, \mathcal{E}, D_I, J_I)$ a ***massive principal module over*** $M$. If there is a grading operator $\gamma_I$ on $\mathcal{E}$, we require in addition that $D_I \gamma_I = -\gamma_I D_I$.

[1] From here on we include a subscript $I$ in order to differentiate between the different operators occurring.

It is an immediate consequence of the definition that a massive principal module over $M$ is a real internal space over $M$. If $(\mathcal{B}, \mathcal{E}, J_I)$ is in fact a gauge module, we shall call $(\mathcal{B}, \mathcal{E}, D_I, J_I)$ a ***massive gauge module***.

Let $\mathtt{P} \times_{G_F} F$ be a principal module. Denote by $\mathfrak{g}_F$ the Lie algebra of the structure group $G_F$. Take a connection on $\mathtt{P}$, i.e. for each local trivialisation $(U_i, h_i)$ of $\mathtt{P}$ we have a (local) $\mathfrak{g}_F$-valued 1-form $\omega_i \in \Omega^1(U_i, \mathfrak{g}_F)$ such that

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_i g_{ij}$$

for all $i, j$ such that $U_i \cap U_j \neq \emptyset$ (see Definition 5.2.15). These connection one-forms yield a connection $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(M)$ by defining locally, that is, on local trivialisations $(U_i, h_i)$ of $\mathtt{E}$ that are induced by those of $\mathtt{P}$, the expression

$$\nabla|_{U_i} := h_i^{-1} \circ (d + \omega_i) \circ h_i,$$

where $d$ is the exterior derivative acting on the components of the local trivialisation. The transformation property of $\omega_i$ ensures that $\nabla$ is globally well-defined. Connections on $\mathcal{E}$ of this form are also referred to as ***$G_F$-compatible connections***, or simply ***$G_F$-connections***.

Consider the associated vector bundle $\operatorname{ad}\mathtt{P} := \mathtt{P} \times_{\mathrm{ad}} \mathfrak{g}_F$, where $\mathfrak{g}_F$ is the Lie algebra of $G_F$ and ad is the adjoint action of $G_F$ on $\mathfrak{g}_F$. Since $\mathfrak{g}_F$ is (isomorphic to) the image of $\mathfrak{u}(A_F)$ in $\mathfrak{u}(\mathcal{H}_F)$ under the map $t \mapsto t + J_F t J_F^*$, the bundle $\operatorname{ad}\mathtt{P}$ is (isomorphic to) the image of $\mathfrak{u}(\mathtt{B})$ in $\mathfrak{u}(\mathtt{E})$ under the map $\tau \colon t \mapsto t + J_I t J_I^*$. The kernel of this map is equal to the set of all elements $t \in \mathfrak{u}(\mathtt{B})$ satisfying $t = -J_I t J_I^* = J_I t^* J_I^*$, or equivalently,

$$\ker \tau = \{t \in \mathfrak{u}(\mathtt{B}) \mid t J_I = J_I t^*\} = \mathfrak{u}(\mathtt{B}_J).$$

Hence, we see that $\operatorname{ad}\mathtt{P}$ is isomorphic to $\mathfrak{u}(\mathtt{B})/\mathfrak{u}(\mathtt{B}_J)$. In particular, we see that $\mathfrak{g}_F = \mathfrak{u}(A_F)/\mathfrak{u}((A_F)_{J_F})$.

**Lemma 8.1.2.** *The induced map $\tau \colon \mathfrak{u}(\mathcal{B}) \to \Gamma^\infty(\operatorname{ad}\mathtt{P})$ is surjective, and*

$$\Gamma^\infty(\operatorname{ad}\mathtt{P}) \cong \mathfrak{u}(\mathcal{B})/\mathfrak{u}(\mathcal{B}_J).$$

*Moreover, $\operatorname{ad}\mathtt{P}$ is isomorphic to the subbundle*

$$\mathfrak{u} = \{t \in \mathfrak{u}(\mathtt{B}) \mid \operatorname{Tr}_{[i]} t_{[i]} = 0 \text{ for all } [i]\}$$

*of $\mathfrak{u}(\mathcal{B})$, where $\operatorname{Tr}_{[i]}$ denotes the trace taken on the fibre of the subbundle $\mathtt{E}_{[i]}$, and $\mathfrak{u}(\mathtt{B}) = \ker\tau \oplus \mathfrak{u}$, with $\ker\tau = \mathfrak{u}(\mathtt{B}_J)$.*

*Proof.* Though the first two statements follow immediately from the exactness of the Serre-Swan equivalence functor $\Gamma^\infty$, we prove them directly by showing that $\operatorname{ad}\mathtt{P}$ is isomorphic to the subbundle $\mathfrak{u}$ (compare also Proposition 7.2.6). Indeed, every $t \in \mathfrak{u}(\mathtt{B})$ can be written as $s + q$, where $s \in \mathfrak{u}$ and $q \in \mathfrak{u}(\mathtt{B}_J)$ (just take $q_{[i]} = \frac{1}{N_{[i]}} \operatorname{Tr}_{[i]}(t_{[i]}) \cdot \mathrm{id}_{[i]}$ and $s = t - q$). Hence $\tau|_{\mathfrak{u}}$ is surjective.

Suppose now that $t \in \ker \tau|_{\mathfrak{u}}$. Because $t \in \ker \tau$, we obtain $t_{[i]} = \lambda_{[i]} \mathrm{id}_{N_{[i]}}$, where $\lambda_{[i]} \in i\mathbb{R}$ (see Proposition 7.1.3). Since $t \in \mathfrak{u}$, each of the $t_{[i]}$ is traceless. Hence each of the $\lambda_{[i]}$ is zero, and consequently, the kernel of $\tau|_{\mathfrak{u}}$ is trivial. □

**Lemma 8.1.3.** *Let* $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I, \gamma_I)$ *be an even principal module. Any* $G_F$*-compatible connection on* $\mathcal{E}$ *commutes with the real structure* $J_I$ *and the grading* $\gamma_I$*.*

*Proof.* It is sufficient to show that $J_F$ and $\gamma_F$ commute with elements in $\mathfrak{g}_F$. Any element in $\mathfrak{g}_F$ is of the form $t + J_F t J_F^*$, with $t \in \mathfrak{u}(A_F)$. In particular, $J_F$ commutes with these elements. Since $\gamma_F$ commutes with elements in $A_F$, and (anti-)commutes with $J_F$, the grading $\gamma_F$ commutes with elements in $\mathfrak{g}_F$, too. □

If the principal module is obtained from a gauge module $(\mathcal{B}, \mathcal{E}, J_I)$, we can construct such a connection explicitly as follows. Consider the decomposition $\mathcal{E} = \bigoplus_{(i,j)} \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}_j}$, and choose a hermitian connection $\nabla^i$ on each $\mathcal{E}_i$. We define

$$\nabla := \bigoplus_{(i,j)} \left( \nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j} \right),$$

where the conjugate connection $\overline{\nabla^j}$ is defined in Section 5.2.3. In order to see that $\nabla$ corresponds to a connection on the principal bundle $\mathtt{P}$, we need to check that its local connection one-forms take values in the Lie algebra $\mathfrak{g}_F$. If $(U, h^i_U)$ are local trivialisations of $\mathcal{E}_i$, we can write $\nabla^i|_U = \left(h^i_U\right)^{-1} \circ (d + \omega^i_U) \circ h^i_U$ for some local connection one-forms $\omega^i_U \in \Omega^1(U, \mathfrak{u}(N_i))$. The connection $\nabla$ then locally has the connection 1-form

$$\omega_U := \bigoplus_{(i,j)} \left( \omega^i_U \otimes \mathbb{I} + \mathbb{I} \otimes ({\omega^j_U}^*)^{\mathrm{op}} \right) \in \Omega^1(U, A_F \otimes A_F^{\mathrm{op}}).$$

This ensures that $[\nabla, \cdot]$ yields a connection on $\mathcal{B} \otimes \mathcal{B}^{\mathrm{op}}$, which preserves $\mathcal{B}$ and $\mathcal{B}^{\mathrm{op}}$. Writing $t_U = \bigoplus_{i \in I} \omega^i_U$, we can write $\omega_U = t_U + J_F t_U J_F^* \in \Omega^1(U, \mathfrak{g}_F)$. To verify that $\omega_U$ defines a connection on the principal $G_F$-bundle $\mathtt{P}$ we need to show that $\omega_U$ transforms correctly under the $G_F$-valued transition functions.

So, consider two neighbourhoods $U$ and $V$ such that $U \cap V \neq \emptyset$, and let $u = \times u_i \in C^\infty(U \cap V, \mathcal{U}(A_F))$ be a transition function for the principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$. The corresponding transition function for the principal $G_F$-bundle $\mathtt{P}$ is $g := u J_F u J_F^*$. Since the $\omega^i_U$ are connection forms on $\mathtt{E}_i$, $t_U$ transforms as

$$t_V = \sum_{i \in I} \omega^i_V = \sum_{i \in I} (u_i^* \omega^i_U u_i + u_i^* du_i) = u^* t_U u + u^* du.$$

We then see that

$$\begin{aligned}\omega_V &= t_V + J_F t_V J_F^* = u^* t_U u + u^* du + J_F(u^* t_U u + u^* du)J_F^* \\ &= u^* J_F u^* J_F^* t_U u J_F u J_F^* + J_F(u^* J_F u^* J_F^* t_U u J_F u J_F^*)J_F^* \\ &\quad + u^* J_F u^* J_F^*(du) J_F u J_F^* + u^* J_F u^* J_F^* u J_F (du) J_F^* \\ &= g^{-1}(t_U + J_F t_U J_F^*)g + g^{-1}dg = g^{-1}\omega_U g + g^{-1}dg.\end{aligned}$$

Thus, $U \mapsto \omega_U$ indeed defines a $G_F$-connection.

**Proposition 8.1.4.** *Let $(\mathcal{B}, \mathcal{E}, J)$ be a gauge module. A connection on $\mathtt{E}$ is of the form $\bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j}\right)$ if and only if it induces a connection on the principal $\mathcal{U}(A_F)$-bundle $\mathtt{Q}$ from Proposition 7.3.2.*

*Proof.* Consider a local trivialisation $(U, h_U)$ of $\mathtt{P}$, and let $\omega_U \in \Omega^1(U, \mathfrak{u}(A_F))$ be a local connection form on $\mathtt{Q}$, yielding a connection $\nabla$ on $\mathtt{E} = \mathtt{Q} \times_{\mathcal{U}(A_F)} \mathcal{H}_F$. Since the decomposition $\mathfrak{u}(A_F) = \bigoplus_{i \in I} \mathfrak{u}(N_i)$ is preserved by the action of $\mathcal{U}(A_F)$, we can write $\omega_U = \bigoplus_{i \in I} \omega_i$, where each $\omega_i \in \Omega^1(U, \mathfrak{u}(N_i))$ yields a connection $\nabla^i$ on $\mathtt{E}_i$. For $x \in U$, the connection form $\omega_U$ acts on $(\mathtt{E}_i \otimes \overline{\mathtt{E}_j})|_x \cong \mathbb{C}^{N_i} \otimes \overline{\mathbb{C}^{N_j}}$ as

$$\omega(v_i \otimes \overline{w_j}) = \omega_i v_i \otimes \overline{w_j} + v_i \otimes \overline{w_j}\omega_j^*,$$

from which it follows that $\nabla = \bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j}\right)$.

For the converse, consider a connection on $\mathtt{E}$ of the form

$$\nabla = \bigoplus_{(i,j)} \left(\nabla^i \otimes \mathbb{I} + \mathbb{I} \otimes \overline{\nabla^j}\right).$$

On a local trivialisation $(U, h_U)_i$ of $\mathtt{E}_i$, each connection $\nabla^i$ yields a local connection form $\omega_i \in \Omega^1(U, \mathfrak{u}(N_i))$. Then $\omega_U := \bigoplus_{i \in I} \omega_i \in \Omega^1(U, \mathfrak{u}(A_F))$ is a connection form on $\mathtt{Q}$ that induces $\nabla$. □

**Definition 8.1.5.** Let $I_{\mathtt{P}}^\infty = (\mathcal{B}, \mathcal{E}, D_I, J_I)$ be a massive principal module of $KO$-dimension $k$ over $M$, where $M$ now has dimension 4. Let $\nabla$ be a $G_F$-compatible connection on $\mathcal{E}$ . We construct the almost-commutative manifold $I_{\mathtt{P}}^\infty \times_\nabla M$ as in Definition 6.2.1. Since $I_{\mathtt{P}}^\infty$ is now a massive principal module (instead of a more general internal space), we will refer to $I_{\mathtt{P}}^\infty \times_\nabla M$ as a ***principal almost-commutative manifold***.

If $I_{\mathtt{P}}^\infty$ has a grading $\gamma_I$, we obtain an even almost-commutative manifold $I_{\mathtt{P}}^\infty \times_\nabla M$. Since the connection $\nabla$ is $G_F$-compatible, it automatically commutes with $J_I$ and $\gamma_I$ (see Lemma 8.1.3). Moreover, the same condition implies that the induced connection $[\nabla, \cdot]$ on $\operatorname{End}\mathtt{E}$ restricts to $\mathtt{B}$. It then follows from Proposition 6.2.2 that $I_{\mathtt{P}}^\infty \times_\nabla M$ is a real even spectral triple of $KO$-dimension $4 + k$ (mod 8).

As in the usual approach for globally trivial almost-commutative manifolds (see [18, 15] or the review [89]), we continue by generating the gauge fields and scalar fields via inner fluctuations, and subsequently calculating the spectral action.

## 8.2 Inner fluctuations

Let $(\mathcal{B}, \mathcal{H}, D)$ be a spectral triple. Consider the ***generalised one-forms*** given by

$$\Omega^1_D(\mathcal{B}) := \Big\{ \sum_j a_j[D, b_j] \;\Big|\; a_j, b_j \in \mathcal{B} \Big\}.$$

For the canonical triple $(\mathcal{A}, L^2(\mathtt{S}), \not{D})$ of a spin manifold $M$, the generalised one-forms $\Omega^1_{\not{D}}(\mathcal{A})$ are simply given by the Clifford multiplication $c$ of the usual one-forms $\Omega^1(M)$. To be precise, for smooth functions $f_1, f_2 \in \mathcal{A}$, we obtain $f_1[\not{D}, f_2] = -if_1c(df_2)$.

**Definition 8.2.1.** Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a real spectral triple. An ***inner fluctuation*** of the operator $D$ is a self-adjoint element $A = A^* \in \Omega^1_D(\mathcal{B})$. Such an inner fluctuation yields the *fluctuated operator*

$$D_A := D + A + \varepsilon' JAJ^*,$$

where the sign $\varepsilon' = \pm 1$ is determined by the $KO$-dimension of the spectral triple (see Definition 5.2.30).

In what follows we assume that the dimension of $M$ is equal to 4. We would like to show that, for a principal almost-commutative manifold, these inner fluctuations yield gauge fields and Higgs fields. The inner fluctuations of the twisted Dirac operator $\not{D}_{\mathtt{E}} := \mathbb{I} \otimes_\nabla \not{D}$ are (finite sums of) elements of the form

$$a[\not{D}_{\mathtt{E}}, b] = -i(\mathbb{I} \otimes c) \circ (a[\nabla, b] \otimes \mathbb{I}),$$

for $a, b \in \mathcal{B}$, where $c$ denotes Clifford multiplication. The fact that $\nabla$ is a $G_F$-compatible connection ensures that $a[\nabla, b] \in \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M) \cong \Omega^1(M, \mathcal{B})$. Requiring that $a[\not{D}_{\mathtt{E}}, b]$ is self-adjoint then implies that $a[\nabla, b] \in \Omega^1(M, \mathfrak{u}(\mathcal{B}))$, where $\mathfrak{u}(\mathcal{B})$ contains the anti-hermitian elements of $\mathcal{B}$. An arbitrary inner fluctuation of $\not{D}_{\mathtt{E}}$ is therefore given by

$$\alpha := \sum_j a_j[\nabla, b_j] \in \Omega^1(M, \mathfrak{u}(\mathcal{B})).$$

The inner fluctuations of the operator $D_I \otimes \gamma_5$ are of the form $\phi \otimes \gamma_5$, where

$$\phi = \phi^* := \sum_j a_j[D_I, b_j] \in \Gamma^\infty(\mathrm{End}(\mathtt{E})).$$

We also see that

$$Ja[\not{D}_{\mathtt{E}}, b]J^* = -i(\mathbb{I} \otimes c) \circ (J_I \alpha J_I^* \otimes \mathbb{I})$$

and consequently,

$$a[\not{D}_{\mathtt{E}}, b] + Ja[\not{D}_{\mathtt{E}}, b]J^* = -i(\mathbb{I} \otimes c) \circ ((\alpha + J_I \alpha J_I^*) \otimes \mathbb{I}).$$

**Proposition 8.2.2.** *The fluctuated Dirac operator* $D_A := D + A + JAJ^*$ *for an even almost-commutative manifold is of the form*

$$D_A = 1 \otimes_{\nabla'} \not{\partial} + \Phi \otimes \gamma_5,$$

*where* $\nabla' := \nabla + \beta$ *for some* $\beta \in \Omega^1(M, \mathrm{ad}\,\mathtt{P})$*, and* $\Phi = \Phi^* := D_I + \phi + J_I \phi J_I^* \in \Gamma^\infty(\mathrm{End}(\mathtt{E}))$ *for some* $\phi = \phi^* := \sum_j a_j [D_I, b_j]$*.*

*Proof.* As $\alpha + J_I \alpha J_I^*$ vanishes identically if $\alpha \in \Omega^1(M, \mathfrak{u}(\mathcal{B}_J))$, we find that $\beta = \alpha + J_I \alpha J_I^*$ is in fact uniquely determined by an element $\alpha \in \Omega^1(M, \mathfrak{u})$ (here $\mathfrak{u}$ is defined in the proof of Lemma 8.1.2, and $\alpha \leftrightarrow \beta$ via the isomorphism $\Gamma^\infty(\mathfrak{u}) \cong \Gamma^\infty(\mathrm{ad}\,\mathtt{P})$ constructed there). Noting that $\varepsilon' = 1$ by assumption, the statement follows straightforwardly. ☐

The construction of $I^\infty_{\mathtt{P}} \times_\nabla M$ explicitly uses the choice of a connection $\nabla$. However, we will now show that this choice is irrelevant once we take the inner fluctuations into account. We will need the following lemma.

**Lemma 8.2.3.** *Let* $\mathtt{B} \to M$ *be a unital* $*$*-algebra bundle, and let* $\widetilde{\nabla}$ *be a connection on* $\mathcal{B} = \Gamma^\infty(\mathtt{B})$ *such that* $\widetilde{\nabla}(1) = 0$*, where* 1 *denotes the identity section. Write* $\mathcal{A} = C^\infty(M)$*. Then*

$$\Big\{ \sum_j a_j \widetilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \Big\} = \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M) \cong \Omega^1(M, \mathcal{B}). \tag{8.1}$$

*Consequently,* $\Omega^1(M, \mathfrak{u}(\mathcal{B}))$ *is given by the subspace of anti-hermitian elements in* $\big\{ \sum_j a_j \widetilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \big\}$*.*

*Proof.* Since $\widetilde{\nabla}(b) \in \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M)$, the left hand side of Equation (8.1) is clearly contained in the right hand side of Equation (8.1). For the converse inclusion, first suppose that both $a_j$ and $b_j$ are in $\mathcal{A} \subset Z(\mathcal{B})$. In that case,

$$\Big\{ \sum_j f_j \widetilde{\nabla}(g_j \,\mathrm{Id}_{\mathcal{B}}) \mid f_j, g_j \in \mathcal{A} \Big\} \cong \Big\{ \sum_j f_j dg_j \mid f_j, g_j \in \mathcal{A} \Big\} = \Omega^1(M).$$

It follows from this that

$$\Big\{ \sum_j a_j \widetilde{\nabla}(g_j 1) \mid a_j \in \mathcal{B}, g_j \in \mathcal{A} \Big\} = \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M).$$

Of course, the left-hand side of the previous equation is contained in $\big\{ \sum_j a_j \widetilde{\nabla}(b_j) \mid a_j, b_j \in \mathcal{B} \big\}$, which proves the other inclusion. ☐

**Proposition 8.2.4.** *Let* $\mathtt{P} \times_{G_F} F = (\mathcal{B}, \mathcal{E}, J_I)$ *be a principal module over* $M$ *(for simplicity we consider here the massless case* $D_I = 0$*) with two (*$G_F$*-compatible) connections* $\nabla$ *and* $\nabla'$*. Then* $\mathbb{I} \otimes_{\nabla'} \not{\partial}$ *is obtained as an inner fluctuation of* $\mathbb{I} \otimes_\nabla \not{\partial}$*.*

*Proof.* The difference between the two connections $\beta := \nabla' - \nabla$ is an element in $\Omega^1(M, \mathrm{ad}\,\mathtt{P})$. By Lemma 8.1.2 there exists a (unique) element $\alpha \in \Omega^1(M, \mathfrak{u}) \subset \Omega^1(M, \mathfrak{u}(\mathtt{B}))$ such that $\beta = \alpha + J_I \alpha J_I^*$. The connection $\widetilde{\nabla} = [\nabla, \cdot]$ on $\mathrm{End}(\mathcal{E})$ restricts to a connection on $\mathcal{B}$, and satisfies $\widetilde{\nabla}(1) = 0$. Lemma 8.2.3 now implies that $\beta$ is obtained as an inner fluctuation. □

**Remark 8.2.5.** We have seen that considering inner fluctuations of the Dirac operator essentially replaces the connection $\nabla$ (chosen in the construction of the almost-commutative manifold $I_{\mathtt{P}}^\infty \times_\nabla M$) by a different (arbitrary) connection $\nabla'$. Therefore, after taking into account the inner fluctuations, our construction of principal almost-commutative manifolds is essentially independent of the initial choice of the connection $\nabla$.

We also note that the endomorphisms $\Phi$ obtained through inner fluctuations in general remain dependent on the initial choice of $D_I$.

## 8.3 The spectral action

As mentioned immediately below Definition 5.2.16, the dynamics of a gauge theory can be obtained from a gauge-invariant action functional. In the case of almost-commutative manifolds, such an action functional can be formulated in terms of the spectral triple.

Let us first recall the definitions of the bosonic and fermionic action functionals for an arbitrary spectral triple $T = (\mathcal{A}, \mathcal{H}, D)$. The bosonic part of the action functional is given by the ***spectral action*** [14], defined as.

$$S_b(T) := \mathrm{Tr}\left( f\left(\frac{D_A}{\Lambda}\right)\right).$$

Here Tr denotes the operator trace on $B(\mathcal{H})$, $D_A$ is the fluctuated Dirac operator, $f : \mathbb{R} \to \mathbb{R}$ is some positive even function, and $\Lambda \in \mathbb{R}$ is a (large) cut-off parameter. The function $f$ is assumed to decay sufficiently rapidly at infinity so that the trace of $f(D_A/\Lambda)$ exists. In particular, $f$ could be considered as a smooth approximation to a cut-off function (and as such it counts the number of eigenvalues of $D_A$ smaller than $\Lambda$), but this viewpoint is not necessary for the following.

If the spectral triple is even (with grading $\gamma$) and has a real structure $J$ of $KO$-dimension 2, the ***fermionic action*** [20] is defined as

$$S_f(T) := \frac{1}{2}\langle J\widetilde{\xi}, D_A\widetilde{\xi}\rangle,$$

where $\widetilde{\xi}$ is the Grassmann variable corresponding to a vector $\xi \in \mathcal{H}^+$ (i.e. $\gamma\xi = \xi$).

We quote the following well-known result:

**Proposition 8.3.1** (see e.g. [89, §2.6.1]). *For a real even spectral triple $T = (\mathcal{A}, \mathcal{H}, D, J, \gamma)$ of KO-dimension 2, the action functionals $S_b(T)$ and $S_f(T)$ are invariant under the action of the gauge group $\mathcal{G}(T)$.*

We will now provide explicit formulas for the spectral action of principal almost-commutative manifolds (formulas for the fermionic action will only be given for the example of electrodynamics in Section 8.5.2). The spectral action was calculated in [15, 20] for the product triple $M \times F$, where $F$ was chosen in order to describe the full Standard Model of elementary particle physics. In what follows we will largely follow the notation of [89], where also detailed derivations of the formulas provided here can be found.

For the canonical triple $(C^\infty(M), L^2(\mathtt{S}), D\!\!\!/)$ of a smooth compact 4-dimensional Riemannian spin manifold $M$, the spectral action yields the asymptotic formula

$$S_b(M) \sim_{\Lambda\to\infty} \int_M \mathcal{L}_M(g_{\mu\nu})\sqrt{|g|}d^4x + \mathcal{O}(\Lambda^{-1}),$$

where $g$ is the Riemannian metric on $M$. The Lagrangian $\mathcal{L}_M$ is given by

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s + \frac{f(0)}{16\pi^2}\Big(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\Big). \quad (8.2)$$

Here $s$ denotes the scalar curvature of $M$, $\Delta$ is the scalar Laplacian, $C$ is the Weyl curvature, and $R^*R^*$ is a topological term, which integrates to (a multiple of) the Euler class. The coefficients $f_k$ (for $k > 0$) are the moments of $f$, defined as

$$f_k := \int_0^\infty f(t)t^{k-1}dt.$$

We will now provide the spectral action for a principal almost-commutative manifold. As all calculations are local, the result is exactly the same as for the spectral action of a product triple $M \times F$, and we refer to [89] for the detailed calculations.

In Proposition 8.2.2 we saw that the fluctuated Dirac operator is determined by a connection $\nabla' = \nabla + \beta$ and an endomorphism $\Phi$ on $\mathtt{E}$. From here on we shall work on a local trivialisation $(U, h_U)$, where we can write $\nabla|_U = h_U^{-1}\circ(d+\omega_U)\circ h_U$, and define the local $\mathfrak{g}_F$-valued 1-form $B := \omega_U + h_U \circ \beta|_U \circ h_U^{-1} \in \Omega^1(U, \mathfrak{g}_F)$ (for ease of notation we do not make the dependence of $B$ on the local chart $U$ explicit). Thus $B$ is the local connection form for $\nabla'$. Using a local coordinate basis $\partial_\mu$, we define $B_\mu := B(\partial_\mu) \in C^\infty(U, \mathfrak{g}_F)$. We will omit the local trivialisation $h_U$ from our notation, so we write e.g. $\nabla'_\mu = \partial_\mu + B_\mu$. Furthermore, we introduce the notation

$$D_\mu\Phi := [\nabla'_\mu, \Phi] = \partial_\mu\Phi + [B_\mu, \Phi], \qquad F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu].$$

**Proposition 8.3.2.** *The spectral action for a principal almost-commutative manifold $I_{\mathtt{P}}^{\infty} \times_{\nabla} M$ is asymptotically given by the local formula*

$$S_b(I_{\mathtt{P}}^{\infty} \times_{\nabla} M) \sim_{\Lambda\to\infty} \int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{|g|}d^4x + \mathcal{O}(\Lambda^{-1}),$$

*for*

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) := N\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_B(g_{\mu\nu}, B_\mu) + \mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi).$$

*Here $\mathcal{L}_M(g_{\mu\nu})$ is given in* (8.2)*, and $N$ is the rank of* $\mathtt{E}$. *$\mathcal{L}_B$ gives the kinetic term of the gauge field and equals*

$$\mathcal{L}_B(g_{\mu\nu}, B_\mu) := \frac{f(0)}{24\pi^2} Tr\ (F_{\mu\nu}F^{\mu\nu}),$$

*where Tr denotes the fibrewise trace for endomorphisms on the bundle* $\mathtt{E}\otimes\mathtt{S}$. *$\mathcal{L}_H$ gives the scalar Lagrangian given by*

$$\mathcal{L}_H(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2} Tr\ (\Phi^2) + \frac{f(0)}{8\pi^2} Tr\ (\Phi^4) + \frac{f(0)}{24\pi^2}\Delta\big(Tr\ (\Phi^2)\big) + \frac{f(0)}{48\pi^2} s\, Tr\ (\Phi^2) + \frac{f(0)}{8\pi^2} Tr\ \big((D_\mu\Phi)(D^\mu\Phi)\big),$$

*where the first two terms form the scalar potential, the third is a boundary term, the fourth couples the scalar field to the scalar curvature, and finally we have the kinetic term including interactions with the gauge field.*

## 8.4 Gauge theory

The results can be summarised as follows, leading to the following main result:

**Theorem 8.4.1.** *Let $M$ be a smooth compact* 4*-dimensional Riemannian spin manifold. Consider a massive even principal module $I_{\mathtt{P}}^{\infty} = (\mathcal{B}, \mathcal{E}, D_I, \gamma_I, J_I)$ of KO-dimension $k$ over $M$. Let $\nabla$ be a $G_F$-compatible connection on $\mathcal{E}$. If $M$ is simply connected, then the principal almost-commutative manifold $I_{\mathtt{P}}^{\infty} \times_{\nabla} M$ of KO-dimension $4+k$ (mod 8) describes a classical gauge theory over $M$ with gauge group $\mathcal{G}(I_{\mathtt{P}}^{\infty} \times_{\nabla} M)$.*

*Proof.* The principal module $I_{\mathtt{P}}^{\infty}$ is constructed from a principal $G_F$-bundle $\mathtt{P}$ over $M$, such that $\mathcal{B}$ and $\mathcal{E}$ are given by smooth sections of bundles associated to $\mathtt{P}$. By assumption $M$ is simply connected, so it follows from Theorem 7.2.7 that we have the isomorphism $\mathcal{G}(I_{\mathtt{P}}^{\infty} \times_{\nabla} M) \cong \mathcal{G}(\mathtt{P})$. We have seen in Section 8.2 that the inner fluctuations transform a $G_F$-compatible connection on $\mathcal{E}$ to another $G_F$-compatible connection, which hence corresponds to a connection on $\mathtt{P}$ (and by

Proposition 8.2.4 any connection on $\mathtt{P}$ can be obtained in this way). Finally, the spectral action and the fermionic action provide a gauge-invariant action functional (see Proposition 8.3.1). □

## 8.5 Examples

In this section we adapt two simple examples of (globally trivial) gauge theories from the context of noncommutative geometry to the globally non-trivial case. In each example, we assume (as before) that the underlying manifold $M$ is a smooth compact 4-dimensional Riemannian spin manifold.

In Section 8.5.1 we describe the Yang-Mills case that was studied in [11], and provided the motivation for this work. In particular, we show that the Yang-Mills case provides examples of principal modules that cannot be described by gauge modules. In Section 8.5.2 we discuss the abelian gauge theory of electrodynamics, based on the (globally trivial) description in [90]. We will describe the resulting (globally non-trivial) gauge theory, and provide explicit formulas for both the spectral action and the fermionic action.

### 8.5.1 Yang-Mills

Globally trivial Yang-Mills theory was already studied in the setting of spectral triples by Chamseddine and Connes in [14]. It is described by the (real, even) finite spectral triple

$$F_{\text{YM}} := (M_N(\mathbb{C}), M_N(\mathbb{C}), D_F = 0, J_F = (\cdot)^*, \gamma_F = \text{id}),$$

where the algebra $M_N(\mathbb{C})$ acts on the Hilbert space $M_N(\mathbb{C})$ by left-multiplication. The $KO$-dimension of this spectral triple is 0 and the structure group $G_F$ is equal to $PSU(N)$.

This has been generalised to the globally non-trivial case in [11]. Let $\mathtt{B} \to M$ be an arbitrary $*$-algebra bundle with fibre $M_N(\mathbb{C})$, and let $\mathcal{B} = \Gamma^\infty(\mathtt{B})$ be its unital, involutive $C^\infty(M)$-module algebra of sections. We consider the real even internal space

$$I^\infty_{\text{YM}} := (\mathcal{B}, \mathcal{B}, D_I = 0, J_I = (\cdot)^*, \gamma_I = \text{id}).$$

For a general principal module $\mathtt{P} \times_{G_F} F$ we do not know how to reconstruct the principal bundle $\mathtt{P}$ from the module. However, in the Yang-Mills case we do.

**Lemma 8.5.1.** *There exists a principal $PSU(N)$-bundle $\mathtt{P} \to M$ (unique up to isomorphism) such that $I^\infty_{\text{YM}} \cong \mathtt{P} \times_{PSU(N)} F_{\text{YM}}$.*

*Proof.* The transition functions of the $*$-algebra bundle $\mathtt{B}$ take values in the group $\text{Aut}(M_N(\mathbb{C})) \cong PSU(N)$ (where $PSU(N)$ acts on $M_N(\mathbb{C})$ by conjugation). Hence by Theorem 5.2.14 we can reconstruct a principal $PSU(N)$-bundle $\mathtt{P}$ such that

$\mathtt{B} \cong \mathtt{P} \times_{PSU(N)} M_N(\mathbb{C})$. Since $PSU(N)$ is the full automorphism group of the fibre, this reconstruction does not depend on the choice of transition functions, and hence $\mathtt{P}$ is uniquely defined. As the action of $PSU(N)$ on $M_N(\mathbb{C})$ commutes with the real structure $J_F = (\cdot)^*$, it follows that $I^\infty_{\text{YM}} \cong \mathtt{P} \times_{PSU(N)} F_{\text{YM}}$. □

**Remark 8.5.2.** Note that $I^\infty_{\text{YM}}$ will in general *not* be a gauge module. If this were the case, the structure group $PSU(N)$ of $\mathtt{B}$ could be lifted to $U(N)$ by Proposition 7.3.2. This is only possible if the Dixmier-Douady class $\delta(\mathtt{B}) \in \check{H}^3(M,\mathbb{Z})$ is identically zero (see e.g. [78, Ch.5] or [80] for more details on Dixmier-Douady classes), which is equivalent to saying that $\mathtt{B}$ is an endomorphism bundle (note that this is consistent with the condition $\mathtt{B}_i = \text{End}(\mathtt{E}_i)$ in Definition 7.3.1). Since not every $*$-algebra bundle with fibre $M_N(\mathbb{C})$ has zero Dixmier-Douady class (see e.g. [80]), this example shows that there exist principal modules that are not gauge modules. However, in our description of gauge theories in Chapter 8 we have restricted our attention to simply connected, 4-dimensional manifolds, and it turns out that in this case the Dixmier-Douady class always vanishes (as we will prove below). It is unclear if there exist other examples of principal modules that are not gauge modules.

**Proposition 8.5.3.** *Let* $\mathtt{B}$ *be an algebra bundle with fibre* $M_N(\mathbb{C})$ *over a simply connected,* 4*-dimensional, oriented, compact manifold* $M$*. Then the Dixmier-Douady class of* $\mathtt{B}$ *is identically zero.*

*Proof.* Since $M$ is simply connected, its fundamental group is trivial, and hence (see e.g. [41, Theorem 2.A.1]) the first singular homology group $H_1(M,\mathbb{Z})$ is trivial. By Poincaré duality (see e.g. [41, Proposition 3.25 & Theorem 3.30]) it then follows that the third cohomology group $H^3(M,\mathbb{Z})$ is also trivial. The Dixmier-Douady class by definition takes values in the third Čech cohomology group $\check{H}^3(M,\mathbb{Z})$. Since for compact manifolds these cohomology groups are equal, it follows that $\check{H}^3(M,\mathbb{Z})$ is trivial and hence that the Dixmier-Douady class of $\mathtt{B}$ must vanish. □

A connection $\nabla\colon \mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \Omega^1(M)$ is $PSU(N)$-compatible (*cf.* Section 8.1) if and only if it satisfies the algebraic identities [11, §3.2]

$$\nabla(ab) = \nabla(a)b + a\nabla(b), \qquad (\nabla a)^* = \nabla(a^*), \qquad \forall a, b \in \mathcal{B}.$$

Such a connection thus corresponds to a connection form $\omega$ on $\mathtt{P}$. If we pick any such connection, we can then consider the (principal) almost-commutative manifold

$$I^\infty_{\text{YM}} \times_\nabla M := \big(\Gamma^\infty(\mathtt{B}), L^2(\mathtt{B} \otimes \mathtt{S}), \not{D}_{\mathtt{B}}, J_I \otimes J_M, \gamma_I \otimes \gamma_5\big).$$

If $M$ is simply connected, the group $\mathcal{G}(I^\infty_{\text{YM}} \times_\nabla M)$ is isomorphic to $\mathcal{G}(\mathtt{P})$, and $I^\infty_{\text{YM}} \times_\nabla M$ describes a $PSU(N)$ gauge theory $(\mathtt{P}, \omega)$ over $M$. We denote the local connection form of $\nabla$ by $B_\mu$, and its curvature tensor by $F_{\mu\nu}$. From Proposition 8.3.2 we find that the spectral action yields the Lagrangian

$$\mathcal{L}(g_{\mu\nu}, B_\mu) = N^2 \mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_{\mathrm{YM}}(g_{\mu\nu}, B_\mu),$$

where the Yang-Mills Lagrangian is given (up to a normalisation constant) by the usual expression:

$$\mathcal{L}_{\mathrm{YM}}(g_{\mu\nu}, B_\mu) := \frac{f(0)}{24\pi^2} \operatorname{Tr}(F_{\mu\nu} F^{\mu\nu}).$$

### 8.5.2 Electrodynamics

The example of (globally trivial) Electrodynamics in the context of noncommutative geometry appeared in [90]. Here we describe its generalisation to the globally non-trivial case. The finite spectral triple for electrodynamics is given in [90] by

$$F_{\mathrm{ED}} := (\mathbb{C}^2, \mathbb{C}^4, D_F, \gamma_F, J_F).$$

We shall generalise this finite triple to a massive even gauge module $I^\infty_{\mathrm{ED}}$ over $M$. First, we set the algebra $\mathcal{B}$ to be of the form

$$\mathcal{B} := \mathcal{A} \oplus \mathcal{A} = C^\infty(M) \oplus C^\infty(M).$$

Let $\mathtt{L}$ be a complex line bundle over $M$, with a given hermitian structure, so that its structure group is $U(1)$. We shall take two identical copies of this line bundle, which we denote by $\mathtt{E}_L$ and $\mathtt{E}_R$, with smooth sections $\mathcal{E}_L = \Gamma^\infty(\mathtt{E}_L)$ and $\mathcal{E}_R = \Gamma^\infty(\mathtt{E}_R)$. Then the Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ is defined as

$$\mathcal{E} := (\mathcal{E}_L \oplus \mathcal{E}_R) \oplus (\overline{\mathcal{E}_L} \oplus \overline{\mathcal{E}_R}),$$

where the first component of $\mathcal{B}$ acts on $\mathcal{E}_L \oplus \mathcal{E}_R$, and the second component acts on its conjugate. On this decomposition, the grading is defined as $\gamma_I := 1 \oplus (-1) \oplus (-1) \oplus 1$. The real structure $J_I$ is the anti-linear map $\mathcal{E}_{L,R} \mapsto \overline{\mathcal{E}_{L,R}}$ and $\overline{\mathcal{E}_{L,R}} \mapsto \mathcal{E}_{L,R}$ of $KO$-dimension 6 (see Definition 5.2.30). We then have the subalgebra $\mathcal{B}_J \cong \mathcal{A} \subset \mathcal{B}$, where the injection is given by $a \mapsto a \oplus a$. Imposing all conditions in Definition 8.1.1, the 'mass matrix' $D_I$ is restricted to be of the form

$$D_I := \begin{pmatrix} 0 & d & 0 & 0 \\ \overline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} \\ 0 & 0 & d & 0 \end{pmatrix},$$

where $d \in C^\infty(M)$ (see [90, §4.1.1]).

**Remark 8.5.4.** In order to interpret $d$ as a mass parameter, it would have to be given by a single real-valued parameter. For this reason we restrict ourselves to the case $d = -im$ (see [90, Remark 4.4]). We stress here that in general the mass $m$ is not a single parameter, but a function on $M$ (although it can be chosen to be constant). In other words, the mass of a particle is allowed to vary from point to point in $M$, so essentially the Yukawa mass parameter is replaced by a Yukawa field. This could of course have significant physical implications, which we intend to study in future work.

The module $I^\infty_{\text{ED}} = (\mathcal{B}, \mathcal{E}, D_I, \gamma_I, J_I)$ defined in this way is in fact a massive even gauge module. To be precise, if we write $\mathcal{E}_1 := \Gamma^\infty(\mathtt{L}) = \mathcal{E}_L = \mathcal{E}_R$ and $\mathcal{E}_2 := \mathcal{A}$, then we have $\mathcal{B}_1 = \operatorname{End}_{\mathcal{A}}(\mathcal{E}_1) = \Gamma^\infty(\mathtt{L} \otimes \mathtt{L}^*) \cong \mathcal{A}$ and also $\mathcal{B}_2 \cong \mathcal{A}$. Furthermore, the module $\mathcal{E}$ can be written as

$$\mathcal{E} \cong \bigoplus_{(i,j)\in K} \mathcal{E}_i \otimes_{\mathcal{A}} \overline{\mathcal{E}_j}, \qquad K := \big\{(1,2), (1,2), (2,1), (2,1)\big\}.$$

The hermitian structure on $\mathtt{L}$ determines a class of transition functions of $\mathtt{L}$ taking values in $U(1)$, so using Theorem 5.2.14 we can uniquely reconstruct a principal $U(1)$-bundle $\mathtt{P}$, and we have $I^\infty_{\text{ED}} \cong \mathtt{P} \times_{U(1)} F_{\text{ED}}$ as *massless* modules (i.e. ignoring the mass matrices $D_F$ and $D_I$). Assuming that $M$ is simply connected, it follows (see Theorem 7.2.7) that the gauge group is given by

$$\mathcal{G}(I^\infty_{\text{ED}}) \cong \mathcal{U}(\mathcal{B})/\mathcal{U}(\mathcal{B}_J) \cong \Gamma^\infty(\operatorname{Ad}\mathtt{P}) \cong C^\infty(M, U(1)),$$

where the group bundle $\operatorname{Ad}\mathtt{P} \cong M \times U(1)$ is globally trivial, because the structure group $U(1)$ is abelian. An element $\lambda \in \mathcal{G}(I^\infty_{\text{ED}})$ acts on $\mathcal{E}_L \oplus \mathcal{E}_R$ as multiplication by $\lambda$, and acts on $\overline{\mathcal{E}_L} \oplus \overline{\mathcal{E}_R}$ as multiplication by $\overline{\lambda}$.

Pick a connection $\nabla^{\mathtt{L}}$ on $\mathtt{L}$, and let the connection $\nabla$ on $\mathcal{E}$ be given by

$$\nabla := \nabla^{\mathtt{L}} \oplus \nabla^{\mathtt{L}} \oplus \overline{\nabla^{\mathtt{L}}} \oplus \overline{\nabla^{\mathtt{L}}}.$$

On a local trivialisation (say on a neighbourhood $U$), the connection $\nabla^{\mathtt{L}}$ is determined by a local connection form $\omega^{\mathtt{L}}_U \in \Omega^1(U, i\mathbb{R})$, where $i\mathbb{R}$ is the Lie algebra of $U(1)$. For the connection $\nabla$ on $\mathcal{E}$ this yields the connection form

$$\omega_U = \omega^{\mathtt{L}}_U \oplus \omega^{\mathtt{L}}_U \oplus \overline{\omega^{\mathtt{L}}_U} \oplus \overline{\omega^{\mathtt{L}}_U} = \omega^{\mathtt{L}}_U \left(1 \oplus 1 \oplus (-1) \oplus (-1)\right),$$

where the last equality follows because the action of $\overline{\omega^{\mathtt{L}}_U}$ is given by (right) multiplication with ${\omega^{\mathtt{L}}_U}^* = -\omega^{\mathtt{L}}_U$.

Now consider the almost-commutative manifold $I^\infty_{\text{ED}} \times_\nabla M$ of $KO$-dimension 2, which (by Theorem 8.4.1) describes a $U(1)$-gauge theory over $M$. Taking inner fluctuations simply amounts to choosing a different connection $\nabla^{\mathtt{L}}$ (see Proposition 8.2.4), while there will be no Higgs field (because $D_I$ commutes with $\mathcal{B}$). Hence we will ignore these inner fluctuations, and simply consider the local gauge field $A_\mu := \omega_U^{\mathrm{L}}(\partial_\mu)$, on some coordinate basis $\partial_\mu$. Its curvature is defined as $\mathcal{F}_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. From Proposition 8.3.2 (see also [90, Proposition 4.2]) we find that the spectral action for $I_{\mathrm{ED}}^\infty \times_\nabla M$ is asymptotically given by the local formula

$$S_b(I_{\mathrm{ED}}^\infty \times_\nabla M) \sim_{\Lambda\to\infty} \int_M \mathcal{L}(g_{\mu\nu}, A_\mu, m)\sqrt{|g|}d^4x + \mathcal{O}(\Lambda^{-1}),$$

for

$$\mathcal{L}(g_{\mu\nu}, A_\mu, m) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_A(g_{\mu\nu}, A_\mu) + \mathcal{L}_H(g_{\mu\nu}, m).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is the Lagrangian (8.2), and $\mathcal{L}_H(g_{\mu\nu}, m)$ yields additional terms depending on the mass $m$ and the scalar curvature $s$:

$$\mathcal{L}_H(g_{\mu\nu}, m) := -\frac{2f_2\Lambda^2 m^2}{\pi^2} + \frac{f(0)m^4}{2\pi^2} + \frac{f(0)m^2 s}{12\pi^2}.$$

The Lagrangian for the gauge field is given by

$$\mathcal{L}_A(g_{\mu\nu}, A_\mu) := \frac{f(0)}{6\pi^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}.$$

The interaction of the $U(1)$ gauge field with the fermions is described by the fermionic action, and is given by [90, Proposition 4.3 and Theorem 4.5]

$$S_f(I_{\mathrm{ED}}^\infty \times_\nabla M) = \int_M \mathcal{L}_f(g_{\mu\nu}, A_\mu, m)\sqrt{|g|}d^4x,$$

for the Lagrangian

$$\mathcal{L}_f(g_{\mu\nu}, A_\mu, m, \widetilde{\chi}, \widetilde{\psi}) := -i\left(J_M\widetilde{\chi}, \left(\gamma^\mu(\nabla_\mu^S - A_\mu) - m\right)\widetilde{\psi}\right),$$

where $\chi$ and $\psi$ are two Dirac spinors in $L^2(\mathtt{S})$ and where $\widetilde{\xi}$ is the Grassmann variable corresponding to a vector $\xi \in \mathcal{H}^+$ (i.e. $\gamma\xi = \xi$). We summarise this as follows:

**Proposition 8.5.5.** *The total Lagrangian for $I_{\mathrm{ED}}^\infty \times_\nabla M$ is given by a gravitational part*

$$\mathcal{L}_{\mathrm{grav}}(g_{\mu\nu}, m) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_H(g_{\mu\nu}, m),$$

*and a part for electrodynamics*

$$\mathcal{L}_{\mathrm{ED}}(g_{\mu\nu}, A_\mu, m, \widetilde{\chi}, \widetilde{\psi}) := -i\left(J_M\widetilde{\chi}, \left(\gamma^\mu(\nabla_\mu^S - A_\mu) - m\right)\widetilde{\psi}\right) + \frac{f(0)}{6\pi^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}.$$

## 8.6 Outlook

One of the main ideas in the development of noncommutative geometry has been the translation of geometric data into (operator-)algebraic data. In this light, it is somewhat unsatisfactory that our definition of principal modules relies entirely on the geometric notion of a principal bundle. Our discussion of gauge modules is an attempt to provide a purely algebraic approach, but as we have shown, these gauge modules only yield a proper subclass of principal modules. It is still an open question how arbitrary principal modules should be described algebraically, that is, what algebraic structure on a triplet $(\mathcal{B}, \mathcal{E}, J)$ would completely characterise the properties of a principal module. The decompositions $\mathcal{E} = \oplus_{i,j\in I}\mathcal{E}_{ij}$ and $\mathcal{B} = \oplus\mathcal{B}_i$ (as described in Section 7.2.1) are not yet enough to ensure that $(\mathcal{B}, \mathcal{E}, J)$ is a principal module. On the other hand, the condition that $\mathcal{E}_{ij} = \mathcal{E}_i \otimes_{\mathcal{A}} \mathcal{E}_j$ (modulo multiplicities) along with $\mathcal{B}_i = \mathrm{End}(\mathcal{E}_i)$, as for gauge modules, is in fact too strong.

As mentioned in Remark 7.2.3, the principal bundle $\mathtt{P}$ can only be reconstructed from the associated vector bundle $\mathtt{E} = \mathtt{P} \times_{G_F} \mathcal{H}_F$ if we also know the corresponding equivalence class of $G_F$-atlases. It is not clear if there exists a geometric structure on $\mathtt{E}$, for which this equivalence class corresponds precisely to those transition functions that preserve the geometric structure (just like the equivalence class of unitary transition functions on a complex vector bundle corresponds to (the isometry class of) a hermitian structure, see Example 5.2.22). If one has such a geometric structure on $\mathtt{E}$, this might provide the possibility of finding an algebraic equivalent structure on the module $\mathcal{E}$.

Furthermore, we also need conditions on a (unital) involutive finitely generated projective $C^\infty(M)$-module algebra to ensure that the corresponding (unital) $*$-algebra fibration is actually a (unital) $*$-algebra *bundle*. This is automatically the case for gauge modules, since in that case the $*$-algebra bundle $\mathtt{B}$ decomposes into $*$-algebra subbundles $\mathtt{B}_i$ that are isomorphic to End $\mathtt{E}_i$ (which is always locally trivial). We intend to return to these questions in the future.

In Section 8.5 we described two basic examples, namely Yang-Mills theory and electrodynamics. It would of course be more interesting to also put the description of the noncommutative Standard Model [15] into our globally non-trivial framework. This should certainly be possible, though it would require some small modifications to accommodate real algebras (as we have always assumed that our algebras are complex). In particular, for real algebras the resulting gauge group would not automatically be unimodular (see also Remark 7.2.9), and one would have to impose unimodularity by hand (as in [15, §2.5]). More importantly, as we have also mentioned in Remark 8.5.4, the mass parameters (i.e. the Yukawa couplings and the Majorana terms) of the theory are not restricted to be constant, but they are allowed to vary on spacetime. Such variation of the Majorana mass then naturally leads to a new scalar field $\sigma$, which was used in [16] to restore the consistency of the noncommutative Standard Model with the experimental value of the Higgs mass. In addition however, the variation of the Yukawa couplings will also have its effect on the physical theory. We hope to provide a more detailed study of these physical implications in a future work.

# Bibliography


[1] R. Abraham and J. E. Marsden. *Foundations of mechanics.* Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[2] S. Baaj and P. Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.* 296 (1983) 875–878.

[3] W. Ballmann. *Lectures on Kähler manifolds.* ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2006.

[4] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and $K$-theory of group $C^*$-algebras. In *$C^*$-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[5] P. F. Baum and R. J. Sánchez-García. $K$-theory for group $C^*$-algebras. In *Topics in algebraic and topological $K$-theory*, volume 2008 of *Lecture Notes in Math.*, pages 1–43. Springer, Berlin, 2011.

[6] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1992.

[7] J. Bertin, J.-P. Demailly, L. Illusie, and C. Peters. *Introduction to Hodge theory*, volume 8 of *SMF/AMS Texts and Monographs.* American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002. Translated from the 1996 French original by James Lewis and Peters.

[8] B. Blackadar. *$K$-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications.* Cambridge University Press, Cambridge, Second edition, 1998.

[9] D. Bleecker. *Gauge theories and Variational Principles.* Addison-Wesley Publishing Company, 1981.

[10] J. Boeijink and K. Van Den Dungen. Gauge theories as noncommutative manifolds. Pre-print, 2014.

[11] J. Boeijink and W. D. van Suijlekom. The noncommutative geometry of Yang-Mills fields. *J. Geom. Phys.* 61 (2011) 1124–1134.

[12] N. Bourbaki. *Lie groups and Lie algebras. Chapters 7–9.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[13] B. Ćaćić. A reconstruction theorem for almost-commutative spectral triples. *Lett. Math. Phys.* 100 (2012) 181–202.

[14] A. H. Chamseddine and A. Connes. The spectral action principle. *Comm. Math. Phys.* 186 (1997) 731–750.

[15] A. H. Chamseddine, A. Connes, and M. Marcolli. Gravity and the Standard Model with neutrino mixing. *Adv. Theor. Math. Phys* 11 (2007) 991–1089.

[16] A. H. Chamseddine and A. Connes. Resilience of the spectral standard model. *J. High Energy Phys.* (2012) 104, front matter+10.

[17] P. R. Chernoff. Essential self-adjointness of powers of generators of hyperbolic equations. *J. Functional Analysis* 12 (1973) 401–414.

[18] A. Connes. Gravity coupled with matter and the foundation of noncommutative geometry. *Comm. Math. Phys.* 182 (1996).

[19] A. Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[20] A. Connes. Noncommutative geometry and the standard model with neutrino mixing. *J. High Energy Phys.* (2006) 081, 19 pp. (electronic).

[21] A. Connes. On the spectral characterization of manifolds. *J. Noncommut. Geom.* 7 (2013) 1–82.

[22] A. Connes and J. Lott. Particle models and noncommutative geometry. *Nuclear Phys. B Proc. Suppl.* 18B (1990) 29–47 (1991). Recent advances in field theory (Annecy-le-Vieux, 1990).

[23] M. Dubois-Violette, R. Kerner, and J. Madore. Classical bosons in a noncommutative geometry. *Class. Quant. Grav* 6 (1989) 1709.

[24] M. Dubois-Violette, R. Kerner, and J. Madore. Gauge bosons in a noncommutative geometry. *Phys. Lett. B* 217 (1989) 485 – 488.

[25] M. Dubois-Violette, R. Kerner, and J. Madore. Noncommutative differential geometry and new models of gauge theory. *J. Math. Phys.* 31 (1990) 323. Preprint Orsay 1988, SLAC PPF 88-49.

[26] M. Dubois-Violette, R. Kerner, and J. Madore. Noncommutative differential geometry of matrix algebras. *J. Math. Phys.* 31 (1990) 316. Preprint Orsay 1988, SLAC PPF 88-45.

[27] J. J. Duistermaat. *The heat kernel Lefschetz fixed point formula for the spin-c Dirac operator*. Progress in Nonlinear Differential Equations and their Applications, 18. Birkhäuser Boston Inc., Boston, MA, 1996.

[28] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[29] E. Fischer, G. Rudolph, and M. Schmidt. A lattice gauge model of singular Marsden-Weinstein reduction. I. Kinematics. *J. Geom. Phys.* 57 (2007) 1193–1213.

[30] T. Friedrich. *Dirac operators in Riemannian geometry*, volume 25 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[31] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[32] G. Grubb. *Distributions and operators*, volume 252 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[33] V. Guillemin and S. Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.* 67 (1982) 515–538.

[34] V. Guillemin, V. Ginzburg, and Y. Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman.

[35] V. Guillemin and M. Stenzel. Grauert tubes and the homogeneous Monge-Ampère equation. *J. Differential Geom.* 34 (1991) 561–570.

[36] V. Guillemin and S. Sternberg. *Symplectic techniques in physics*. Cambridge University Press, Cambridge, 1984.

[37] B. C. Hall. The Segal-Bargmann "coherent state" transform for compact Lie groups. *J. Funct. Anal.* 122 (1994) 103–151.

[38] B. C. Hall. The inverse Segal-Bargmann transform for compact Lie groups. *J. Funct. Anal.* 143 (1997) 98–116.

[39] B. C. Hall. Phase space bounds for quantum mechanics on a compact Lie group. *Comm. Math. Phys.* 184 (1997) 233–250.

[40] B. C. Hall. Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type. *Comm. Math. Phys.* 226 (2002) 233–268.

[41] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[42] N. Higson and J. Roe. *Analytic K-homology.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[43] P. Hochs. *Quantisation commutes with reduction for cocompact Hamiltonian group actions.* PhD thesis, Radboud University Nijmegen, 2008.

[44] P. Hochs and N. P. Landsman. The Guillemin-Sternberg conjecture for non-compact groups and spaces. *J. of K-Theory* 1 (2008) 473–533.

[45] P. Hochs and V. Mathai. Quantizing tame action. 2013.

[46] G. Hochschild. *The structure of Lie groups.* Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[47] L. Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[48] J. Huebschmann, G. Rudolph, and M. Schmidt. A gauge model for quantum mechanics on a stratified space. *Comm. Math. Phys.* 286 (2009) 459–494.

[49] J. Huebschmann. Singular Poisson-Kähler geometry of stratified Kähler spaces and quantization. In *Geometry and quantization*, volume 19 of *Trav. Math.*, pages 27–63. Univ. Luxemb., Luxembourg, 2011.

[50] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[51] D. Huybrechts. *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[52] B. Iochum, T. Schücker, and C. Stephan. On a classification of irreducible almost commutative geometries. *J. Math. Phys.* 45 (2004) 5003–5041.

[53] J. Jost. *Riemannian Geometry and Geometric Analysis.* Universitext. Springer-Verlag, Berlin, fourth edition, 2005.

[54] J. Kaad and M. Lesch. A local global principle for regular operators in Hilbert $C^*$-modules. *J. Funct. Anal.* 262 (2012) 4540–4569.

[55] G. G. Kasparov. The operator $K$-functor and extensions of $C^*$-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.* 44 (1980) 571–636, 719.

[56] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1996.

[57] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[58] K. Kodaira. On a differential-geometric method in the theory of analytic stacks. *Proc. Nat. Acad. Sci. U. S. A.* 39 (1953) 1268–1273.

[59] B. Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[60] T. Krajewski. Classification of finite spectral triples. *J. Geom. Phys.* 28 (1998) 1–30.

[61] D. Kucerovsky. The $KK$-product of unbounded modules. *$K$-Theory* 11 (1997) 17–34.

[62] G. Landi. *An Introduction to Noncommutative Spaces and Their Geometry.* Springer, 1997.

[63] N. P. Landsman. Quantization as a functor. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 9–24. Amer. Math. Soc., Providence, RI, 2002.

[64] N. P. Landsman. Functorial quantization and the Guillemin-Sternberg conjecture. In *Twenty years of Bialowieza: a mathematical anthology*, volume 8 of *World Sci. Monogr. Ser. Math.*, pages 23–45. World Sci. Publ., Hackensack, NJ, 2005.

[65] N. P. Landsman, M. J. Pflaum, and M. Schlichenmaier, editors. *Quantization of singular symplectic quotients*, volume 198 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2001.

[66] S. Lazzarini and T. Schücker. A farewell to unimodularity. *Phys. Lett. B* 510 (2001) 277–284.

[67] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972.

[68] X. Ma and W. Zhang. Geometric quantization for proper moment maps. *C. R. Math. Acad. Sci. Paris* 347 (2009) 389–394.

[69] J. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.* 5 (1974) 121–130.

[70] V. Mathai and W. Zhang. Geometric quantization for proper actions. *Adv. Math.* 225 (2010) 1224–1247. With an appendix by Ulrich Bunke.

[71] E. Meinrenken. Symplectic surgery and the $\mathrm{Spin}^c$-Dirac operator. *Adv. Math.* 134 (1998) 240–277.

[72] E. Meinrenken and R. Sjamaar. Singular reduction and quantization. *Topology* 38 (1999) 699–762.

[73] B. Mesland. Unbounded bivariant K-theory and correspondences in noncommutative geometry. To appear in J. Reine Angew. Math.

[74] G. Mislin and A. Valette. *Proper group actions and the Baum-Connes conjecture.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[75] J.-P. Ortega and T. Ratiu. *Momentum Maps and Hamiltonian Reduction.* Birkhauser, 2004.

[76] P.-É. Paradan. Formal geometric quantization II. *Pacific J. Math.* 253 (2011) 169–211.

[77] M. J. Pflaum. *Analytic and geometric study of stratified spaces*, volume 1768 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[78] I. Raeburn and D. P. Williams. *Morita equivalence and continuous-trace $C^*$-algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[79] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012.

[80] C. Schochet. The Dixmier-Douady invariant for dummies. *Notices Amer. Math. Soc.* (2009) 809–816.

[81] R. Sjamaar. *Singular orbit spaces in Riemannian and symplectic geometry.* PhD thesis, Rijksuniversiteit te Utrecht, 1990.

[82] R. Sjamaar and E. Lerman. Stratified symplectic spaces and reduction. *Ann. of Math. (2)* 134 (1991) 375–422.

[83] R. Sjamaar. Symplectic reduction and Riemann-Roch formulas for multiplicities. *Bull. Amer. Math. Soc. (N.S.)* 33 (1996) 327–338.

[84] J.-M. Souriau. *Structure des systèmes dynamiques.* Maîtrises de mathématiques. Dunod, Paris, 1970.

[85] N. Steenrod. *The Topology of Fibre Bundles.* Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[86] R. G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.* 105 (1962) 264–277.

[87] F. Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York, 1967.

[88] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[89] K. van den Dungen and W. D. van Suijlekom. Particle physics from almost-commutative spacetimes. *Rev. Math. Phys.* 24 (2012) 1230004, 105.

[90] K. van den Dungen and W. D. van Suijlekom. Electrodynamics from noncommutative geometry. *J. Noncommut. Geom.* 7 (2013) 433–456.

[91] W. D. van Suijlekom. *Noncommutative geometry and gauge theories.* Springer, 2014. To appear.

[92] N. M. J. Woodhouse. *Geometric quantization.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.

# Index

# Index of Notation

# Samenvatting

Het wetenschappelijke deel van dit proefschrift vindt u op de voorgaande pagina's. In deze samenvatting zou ik graag op een wat luchtigere manier de inhoud van dit proefschrift uit de doeken doen in de hoop alle lezers aan te spreken.[1] Daarnaast zou ik deze samenvatting ook graag gebruiken om uit te leggen wat ik zo bijzonder vind aan de toepassing van de wiskunde in de natuurkunde. Want hoewel mijn methoden wiskundig van aard zijn, zit er altijd een natuurkundige motivatie achter.[2]

## Mathematische fysica

De mathematische fysica is een gebied binnen de wiskunde dat zich bezighoudt met wiskundige vraagstukken die een belangrijke rol spelen in de natuurkunde. Dit heeft door de eeuwen heen geleid tot een nauwe interactie tussen natuur- en wiskunde. In de geschiedenis is het al vaak voorgekomen dat hele wiskundige vakgebieden zijn ontstaan vanuit de natuurkunde. Deze wiskunde is op zichzelf staand doorgaans ook erg interessant en wordt dan ook door wiskundigen vanuit een puur wiskundig perspectief benaderd. Een voorbeeld hiervan is de bewegingsvergelijking van Newton:

$$\vec{F} = m\vec{a}, \quad (kracht) = (massa) \times (versnelling). \tag{A}$$

Deze vergelijking wordt door natuurkundigen gebruikt om te berekenen welk pad een object zal volgen onder invloed van een kracht. Met pad bedoelen we overigens niet alleen de 'route' die het object aflegt. We bedoelen er ook mee dat we op elk

[1] Mocht u de rest van dit proefschrift abstracte onzin vinden en denken dat het papier waarop het gedrukt is, beter gebruikt kan worden om u in de winter warm te houden door het te verbranden in de open haard (wiskundigen worden hopelijk al warm van de inhoud zodat verbranding niet nodig en zelfs onverstandig is), leest u dan eerst deze samenvatting of scheur deze pagina's eruit alvorens de chemische reactie tot stand te brengen die dit werk omzet in as. Velen van u kunnen de inhoud overigens als droge kost ervaren, waardoor het risico op roetvorming in uw schoorsteen hopelijk wel tot een minimum beperkt blijft.

[2] Wiskundigen zijn het liefst volledig. Deze samenvatting is daardoor wat aan de lange kant. :)

tijdstip weten waar op de route het object zich bevindt. Dit pad hangt natuurlijk wel af van de beginsnelheid en beginpositie van het object. Hoe het object zich vervolgens voortbeweegt, hangt verder alleen af van de kracht die op het object werkt. Preciezer, deze kracht $\vec{F}$ is weergegeven aan de linkerkant van vergelijking (A) en het feit dat de linkerkant gelijkgesteld wordt aan de rechterkant, betekent dat de versnelling van het object alleen wordt bepaald door de kracht.[3] En wanneer we de beginsnelheid en beginpositie van het object weten, is het genoeg op elk volgend moment de versnelling te weten om het pad te bepalen voor latere tijdstippen. Denk hierbij aan een auto die op een bepaald tijdstip met een bepaalde snelheid op een autosnelweg rijdt. U kunt zich hopelijk misschien wel voorstellen dat we precies weten waar de auto is, wanneer we weten hoeveel de auto harder of langzamer is gaan rijden.

De kracht in vergelijking (A) zou bijvoorbeeld de zwaartekracht kunnen zijn, een aandrijvende kracht, veerkracht, maar ook een elektrische of magnetische kracht. Met vergelijking (A) kunnen we bijvoorbeeld de beweging van een vallend object afleiden door voor de kracht $\vec{F}$ in vergelijking (A) de zwaartekracht te noteren. Wanneer een natuurkundige een natuurkundig proces bestudeert (dat valt binnen het kader waarin vergelijking (A) van toepassing is), dan schrijft hij voor dat proces op wat $\vec{F}$ is. Vervolgens wil hij het pad vinden dat door het object afgelegd wordt. Dit pad wordt gevonden als het pad dat vergelijking (A) oplost.

Vergelijking (A) heet in de wiskunde een *differentiaalvergelijking*. De oplossing van differentiaalvergelijkingen is een wiskundig probleem en dit is een voorbeeld waar wiskunde een rol speelt in de natuurkunde. Het is namelijk zo dat voor elke natuurkundige situatie de formule voor $\vec{F}$ anders is. Dit betekent dat voor elke nieuwe situatie vergelijking (A) weer anders is en opnieuw opgelost moet worden. Het is niet erg efficiënt om voor elke nieuwe situatie opnieuw te gaan bepalen of er een oplossing is (er hoeft er namelijk niet per se een te bestaan en zeker niet voor elk tijdstip) en wat vervolgens die oplossing is. Een wiskundige probeert daarom voor zo algemeen mogelijke $\vec{F}$, dus de precieze vorm van $\vec{F}$ is niet geven, te laten zien dat er een oplossing bestaat, hoe deze oplossing gevonden kan worden en wat voor eigenschappen deze oplossing heeft. Dit doen ze door differentiaalvergelijkingen van een zo algemeen mogelijke vorm te bestuderen. Die differentiaalvergelijkingen hoeven hierbij zelfs niet van de vorm (A) te zijn. De studie van differentiaalvergelijkingen is een erg interessant gebied in de wiskunde, waarvan de antwoorden ook nog eens de natuurkundigen helpen. Kort gezegd, wiskundigen bekijken dus niet specifieke systemen uit de natuurkunde, maar proberen de vraagstukken in een zo algemeen mogelijk kader te bestuderen.[4]

Ik heb met bovenstaand voorbeeld uit de *klassieke* of *Newtoniaanse mechanica*

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[3] En massa natuurlijk, maar die beschouwen we als constant en vergeten we daarom voor het gemak even. Wat betreft de massa zegt de formule niets anders dan dat je bij zwaardere objecten meer kracht moet uitoefenen om ze dezelfde versnelling te geven.

[4] Wiskundigen letten als het ware meer op de abstracte structuur van het probleem. Vaak geven deze structuren veel inzicht.

proberen duidelijk te maken waar wiskunde een rol speelt in de natuurkunde en hoe de vragen die een wiskundige stelt in zo'n situatie verschillen van die van een natuurkundige. Vergelijking (A) staat overigens bekend als de tweede wet van Newton, vernoemd naar de Engelse natuur- en wiskundige *Isaac Newton* (1642-1727), en is al meer dan drie eeuwen oud. Sindsdien hebben de natuurkunde en wiskunde reusachtige ontwikkeling ondergaan waardoor de huidige vraagstukken binnen de natuurkunde veel ingewikkelder zijn dan het voorbeeld dat ik hierboven heb geschetst. Mijns inziens is hierbij een goed wiskundig begrip van deze problemen van essentieel belang om de volgende stap te zetten. Het gaat hier niet alleen om het oplossen van bewegingsvergelijkingen maar ook om het creëren van een raamwerk waarbinnen deze natuurkunde beschouwd kan worden, de wiskundige structuur van het probleem. Helaas wijkt door de toegenomen complexiteit de wiskundige formulering vaak zo veel af van de natuurkundige dat het voor een rasnatuurkundige lastig is de natuurkunde in deze wiskunde te herkennen. Dat is in dit proefschrift niet anders en dit spijt me dan ook enorm voor mijn natuurkundevrienden.

## Dit proefschrift

Na dit uitstapje over de mathematische fysica zal ik wat meer vertellen over de inhoud van dit proefschrift en ik zal dit doen aan de hand van de titel van dit proefschrift: 'Dirac operators, gauge systems and quantisation'. Hierbij zal ik ook meer uitleg geven over de puzzelsstukjes die op de voorkant van dit proefschrift staan afgebeeld. In feite zijn de drie begrippen in de titel - in het Nederlands Dirac-operatoren, ijksystemen en kwantisatie geheten - *ontstaan in de natuurkunde*. Ik zal kort proberen uit te leggen wat de natuurkundige betekenis is van deze begrippen en vertellen hoe de wiskunde met ze aan de haal gegaan is. Hierbij komt ook naar voren hoe de natuurkunde en de wiskunde elkaar beïnvloed hebben.

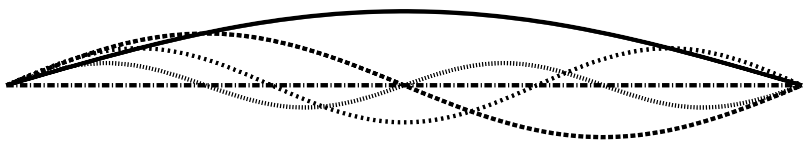
### Dirac-operatoren

De klassieke mechanica waarover ik hierboven schreef, geeft een goede beschrijving voor de natuurkunde in het alledaagse leven. Echter, op erg kleine schaal, denk daarbij aan schalen die kleiner zijn dan een miljardste meter, gelden er totaal andere wetten. Daar gelden namelijk de wetten van de *kwantummechanica*, die zijn opmars in de natuurkunde begon in de jaren 20 van de vorige eeuw.[5] Twintig jaar eerder had Albert Einstein zijn *speciale relativiteitstheorie* opgeschreven, een uitbreiding van de klassieke mechanica naar situaties waarin snelheden erg hoog zijn. Bij de beschrijving van erg kleine deeltjes die ook erg snel voortbewegen, hebben

[5] Om er achter te komen op welke schalen de kwantummechanica van toepassing is pakt u een strookje papier dat ongeveer de lengte heeft van de lange zijde van een A4'tje. Als u dit strookje nu 28 keer dubbel vouwt, komt u op de lengteschalen van de kwantummechanica. Probeert u dit maar eens!

we daarom te maken met zowel de kwantummechanica als de speciale relativiteitstheorie. Het was *Paul Dirac* die op zoek was naar een geschikte (eerste-orde) kwantummechanische vergelijking die ook voldeed aan de wetten van de speciale relativiteitstheorie. Zijn antwoord was de *Dirac-vergelijking* met als belangrijkste object daarin de Dirac-operator.[6] De Dirac-vergelijking geeft de voortbeweging van (een veld van) deeltjes, net zoals de wet van Newton (A) dat doet voor objecten in de klassieke mechanica. Deze Dirac-operator was geformuleerd voor een vier-dimensionale wereld (drie ruimterichtingen en een tijdrichting), maar kan ook geformuleerd worden voor ruimtes van willekeurige dimensie en vorm. Met deze laatste generalisatie zijn we weer in de wiskunde beland. We bekijken nu immers niet uitsluitend Dirac-operatoren op ruimtes die natuurkundig van belang zijn. Wiskundig gezien heeft de Dirac-operator veel bijzondere eigenschappen. Eén ervan is dat het kwadraat van de Dirac-operator een *Laplaciaan* is. Laplacianen spelen een belangrijke rol binnen de meetkunde, waarmee de Dirac-operator dus een meetkundige status heeft.

De Laplaciaan is een bijzonder meetkundig object en het belang ervan voor ons verhaal valt misschien het beste uit te leggen aan de hand van een *gitaarsnaar*. Wanneer een gitaarsnaar trilt, dan kunnen we deze trilling opgebouwd zien uit (oneindig veel) "eenvoudigere" trillingen waarvan ik er vier geschetst heb in de figuur hieronder.



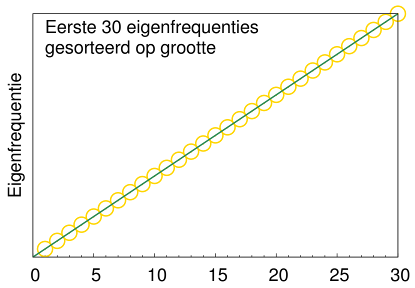
Deze trillingen worden ook wel *eigentrillingen* genoemd. Het bijzondere of "eenvoudigere" aan deze trillingen is dat, op de mate van uitwijking na, de vorm niet verandert bij het trillen. Dit is anders bij trillingen van een andere vorm. Elk van deze eigentrillingen beweegt met een vaste snelheid op en neer (dat wil zeggen, er is alleen beweging in de verticale richting), maar niet elk van de eigentrillingen trilt even snel. Deze snelheid wordt ook wel frequentie genoemd en er geldt hoe meer "toppen" (buiken genoemd) de golf heeft, hoe sneller deze golf trilt in de tijd. De frequentie bepaalt de toonhoogte die we horen, wanneer het geproduceerde gitaargeluid opgevangen wordt door onze oren.[7]

[6] In de kwantummechanica noteren natuurkundigen operatoren door boven het symbool ˆ toe te voegen. Op de omslag ziet u een foto van Paul Dirac met dit symbool boven zijn hoofd.

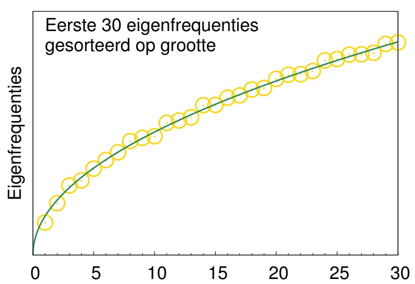
[7] Wanneer een gitaarsnaar trilt, hoor je dus niet één maar vele toonhoogtes, namelijk de toonhoogtes behorende bij elk van de voorkomende eigentrillingen. Deze tonen verschillen echter altijd een veelvoud van een octaaf van elkaar. Overigens is de trilling met een enkele buik meestal het duidelijkst aanwezig (tenzij men flageoletten speelt) en dit is de daadwerkelijke noot die je speelt. De overige eigentrillingen, waarvan we de frequenties ook wel boventonen noemen, bepalen de klankkleur.

De eigentrillingen worden zo genoemd omdat ze de *eigenfuncties* van de Laplaciaan zijn. Wat dit allemaal betekent, is niet zo van belang. Voor deze samenvatting is het voldoende om te weten dat er een manier is om met de Laplaciaan deze eigentrillingen inclusief de bijbehorende frequentie te bepalen.

We zouden de snaar ook kunnen zien als een meetkundige vorm of object, in feite is de snaar niets anders dan een eindig lijnstuk. Voor andere vormen, zoals driehoeken, cirkelschijven of tetraëders, kunnen we ook met behulp van de Laplaciaan bepalen welke eigentrillingen er voor kunnen komen (waarbij we altijd aannemen dat de rand niet trilt, net zoals bij de gitaarsnaar of bij een trommel). Wat we vervolgens doen is een lijst maken van de eigen*frequenties*. Voor de snaar vinden we dan ongeveer de volgende grafiek:




Doen we hetzelfde voor een cirkelschijf (pannenkoek) dan vinden we iets van de vorm:




We zien nu een duidelijk verschil in de vorm van de grafiek en dit heef te maken met het verschil in dimensie. Zo is een lijnstuk een-dimensionaal en een cirkelschijf twee-dimensionaal. Het is precies dit verschil in dimensie dat ervoor zorgt dat de vorm van de grafiek anders is. Met andere woorden, als ik een wiskundige alleen een lijst geef van *alle* eigenfrequenties van de eigentrillingen op het object, dan kan deze zien wat de dimensie is van het object. En zo zijn er nog veel meer eigenschappen die men uit de lijst van eigenfrequenties kan afleiden (maar niet in alle gevallen genoeg om twee willekeurige vormen te kunnen onderscheiden!).

Kortgezegd kunnen we stellen dat de lijst van eigenfrequenties van de Laplaciaan informatie geeft over de eigenschappen van het meetkundige object. Een wiskundige kan zich nu de vraag stellen of we de eigenschappen van het object ook zouden kunnen afleiden door alleen te kijken naar de (oneindige) lijst van eigenfrequenties. Als we ons beperken tot twee-dimensionale vormen, dan is deze vraag ook wel bekend als 'kun je horen wat de vorm van een trommel is?'. Hierbij zien we het twee-dimensionale object als een trommel. Net zoals bij de snaar wordt de klank van de trommel bepaald door de eigenfrequenties die je hoort wanneer je op de trommel slaat. Als de eigenfrequenties de vorm van de trommel zouden bepalen, dan zouden we dus in feite de vorm van de trommel kunnen afleiden uit het geluid dat deze produceert.[8]

Het idee om naar de eigenfrequenties te kijken, is ook het leidende idee binnen de niet-commutatieve meetkunde, behalve dat er daar niet gekeken wordt naar de eigenfrequenties van de Laplaciaan, maar naar die van de Dirac-operator.[9] Hiermee zijn we weer terug bij de Dirac-operator, die dus voor het eerst in de natuurkunde onstond, maar nu ook binnen de meetkunde een belangrijke rol speelt. Een cruciaal resultaat uit de niet-commutatieve meetkunde is dat men door het optellen van de eigenfrequenties van de Dirac-operator een formule vindt die de zwaartekracht beschrijft! Echter, in de natuurkunde interageren deeltjes niet alleen met elkaar door zwaartekracht. Er spelen ook nog andere krachten een rol, zoals de *zwakke* en *sterke kernkracht* en de *elektromagnetische kracht*. Het zijn deze drie krachten die worden beschreven door ijktheorieën.[10] Door het optellen van de eigenfrequenties van de Dirac-operator vonden we eerder alleen de zwaartekracht, niet de andere krachten. Dit kan opgelost worden, zonder bovenstaand principe van het optellen van eigenfrequenties van de Dirac-operator los te laten, door Dirac-operatoren te bekijken op *niet-commutatieve ruimten*. Hiermee zijn we in de niet-commutatieve meetkunde beland.

### Niet-commutatieve meetkunde en deel 2 van dit proefschrift

Waar bij meetkunde nog gesproken kan worden over meetkundige objecten zoals bollen, cirkels of tetraëders, is de niet-commutatieve meetkunde abstracter van aard. Van bollen, cirkels of tetraëders kunt u zich een voorstelling maken. Al deze objecten bestaan uit punten (zeg maar, een punt is een plek op het object). Niet-

[8] Het antwoord op de vraag of je 'kunt horen wat de vorm van de trommel is' is in het algemeen 'nee'. Maar het is opmerkelijk hoeveel eigenschappen we uit de lijst van eigenfrequenties kunnen halen.

[9] Dit staat niet los van hetzelfde probleem met de Laplaciaan aangezien het de Laplaciaan het kwadraat van de Dirac-operator is.

[10] Over ijktheorieën vertel ik later meer. Zwaartekracht zoals beschreven door de algemene relativiteitstheorie is *geen* ijktheorie. Er zit dus een fundamenteel verschil tussen de zwaartekracht en de overige krachten.

commutatieve ruimten hoeven niet uit punten te bestaan, wat een voorstelling van deze ruimten onmogelijk maakt. Ik zal niet uitleggen wat niet-commutatieve meetkunde precies is, maar ik zal uitleggen wat we met *niet-commutativiteit* bedoelen. Dit begrip is later in deze samenvatting ook nog van belang.

Het zal jullie niet verbazen dat

$$3 \times 4 = 4 \times 3.$$

Met andere woorden, we kunnen 3 met 4 vermenigvuldigen of 4 met 3, maar de volgorde doet er niet toe. In beide gevallen is het antwoord 12. Dit geldt natuurlijk niet alleen voor de getallen 3 en 4. We kunnen 3 en 4 door twee willekeurig andere getallen vervangen en dan geldt nog steeds dat de volgorde niet uitmaakt. In formuletaal zeggen we dat

$$n \times m = m \times n, \quad \text{voor alle getallen } m, n.$$

Deze eigenschap van getallen wordt *commutativiteit* genoemd. In gewoon Nederlands: de volgorde doet er niet toe. Nu is het zo dat dit niet voor alle wiskundige operaties geldt. Binnen de natuur- en wiskunde geldt het volgende standaardvoorbeeld dat ik, ondanks het hoge clichégehalte, toch wil noemen: stel dat $m$ betekent dat je in bad stapt en $n$ dat je je kleren aan doet, en laten we zeggen dat $m \times n$ betekent dat we eerst $m$ uitvoeren en dan $n$. Met deze afspraken zien we dat

$$m \times n \neq n \times m,$$

want in het laatste geval heb je natte kleren.[11]

Maar hoe heeft niet-commutativiteit nu met meetkunde te maken? Deze vraag zal ik nu helaas niet beantwoorden. Het enige wat ik er hier over wil zeggen is dat we bij een meetkundig object altijd een zogenoemde *algebra* kunnen construeren die altijd *commutatief* is. Een niet-commutatieve ruimte ontstaat wanneer we met een *niet-commutatieve* algebra werken. In het niet-commutatieve geval hebben we dan alleen een algebra en niet een bijbehorend meetkundig object. Immers, voor een meetkundig object is de algebra altijd *commutatief*. Deze generalisatie van meetkunde naar niet-commutatieve meetkunde is dus erg abstract, en u vraagt zich zeker af of dit ons wat oplevert. En dat doet het zeker!

De Dirac-operator is namelijk ook te generaliseren naar niet-commutatieve ruimten en nu ontstaat de magie.[12] Misschien herinnert u zich nog dat we door het tellen van de eigenfrequenties van de Dirac-operator op een meetkundig object de zwaartekrachtstheorie kunnen verkrijgen, maar dat de overige krachten niet

[11] Op de achterkant van dit proefschrift en op de legger zie je Pythagoras, een wiskundige uit de Griekse oudheid, met z'n kleren aan in bad stappen.

[12] Een niet-commutatieve ruimte bestaat dus in het algemeen uit (1.) een niet-commutatieve algebra, (2.) Dirac operator en (3.) (combinaties van) eigentrillingen. Gezamenlijk noemen we dit een *spectraal tripel* en dit vormt de kern van de niet-commutatieve meetkunde.

hiermee verkregen worden. We combineren nu dit meetkundige object met een (goed gekozen) niet-commutatief deel. Door het tellen van eigenfrequenties van de Dirac-operator op deze gecombineerde ruimte krijgt men een formule die niet alleen de zwaartekracht beschrijft op het meetkundig object maar ook de ijktheorieën. Het opmerkelijke is dat de ijktheorieën hierbij uit het *niet-commutatieve deel* volgen! In deze zin correspondeert de zwaartekracht met de commutatieve component van de niet-commutatieve ruimte en de ijktheorie met de niet-commutatieve component.

Dit is een van de belangrijkste toepassingen van de niet-commutatieve meetkunde in de natuurkunde. Het bijzondere is dat de zwaartekracht en de ijktheorieën worden verkregen uit hetzelfde formalisme, namelijk het tellen van eigenfrequenties van de Dirac-operator. Dit is opmerkelijk omdat, zoals ik eerder vermeld had, zwaartekracht en ijktheorieën vanuit natuurkundig opzicht fundamenteel anders zijn. In het tweede deel van dit proefschrift heb ik, in samenwerking met Koen van den Dungen van de Australian National University en de University of Wollongong, deze beschrijving uitgebreid naar wat men noemt *globaal niet-triviale ijktheorieën*.

Het kan natuurlijk zijn dat bovenstaande u toch wat deed duizelen. Er kwamen waarschijnlijk nogal een hoop voor u onbekende termen voor. De boodschap die ik u wil geven laat zich als volgt samenvatten en hopelijk valt u ook op hoe de natuur- en wiskunde elkaar beïnvloed hebben. De Dirac-operator is ontstaan in de *natuurkunde* als bewegingsvergelijking voor erg kleine deeltjes die ontzettend hard voortbewegen, soms wel met een snelheid die de lichtsnelheid nadert. De Dirac-operator bleek ook belangrijk te zijn in de *wiskunde*, en vooral in de meetkunde, mede doordat zijn kwadraat een Laplaciaan is. In de wiskunde kon men veel informatie uit meetkundige objecten halen door te kijken naar de lijst van eigenfrequenties van de Laplaciaan. Door nu de Laplaciaan door de Dirac-operator te vervangen, kan men door het optellen van de eigenfrequenties van de Dirac-operator, een *wiskundige operatie*, de zwaartekrachtstheorie uit de *natuurkunde* verkrijgen. Echter, om door het optellen van eigenfrequenties van een Dirac-operator de overige drie fundamentele krachten te verkrijgen, die door ijktheorieën beschreven worden, dient men een Dirac-operator op een *niet-commutatieve ruimte* te gebruiken!

Maar wat zijn ijktheorieën nu eigenlijk en wat bedoelen we met kwantisatie, de andere twee begrippen in de titel?

### Kwantisatie

Hierboven heb ik al gesproken over klassieke mechanica en kwantummechanica. Daarbij heb ik opgemerkt dat de wetten van de kwantummechanica gelden voor erg kleine objecten en dat de wetten van de klassieke mechanica gelden voor onze alledaagse waarnemingen. Voor dit verschil is een theoretische verklaring nodig. Om geen tegenstrijdige theorieën te hebben, moet er bovendien een geleidelijke overgang zijn van de wetten van de kwantummechanica naar de wetten van de klassieke mechanica als de we lengteschalen opschroeven. Een belangrijke vraag die natuur- en wiskundigen zich stellen is hoe, gegeven een klassieke theorie, de bijbehorende kwantummechanische theorie er uit ziet. Het construeren van een kwantummechanische theorie bij een klassieke theorie wordt ook wel *kwantisatie* genoemd. Voor veel natuurkundige systemen is deze stap uit te voeren en de natuurkunde is op dit gebied erg succesvol gebleken. Echter, een vast recept voor het construeren van een kwantisatie ontbreekt nog. Vooral de kwantisatie van ijksystemen is wiskundig gezien verre van begrepen.[13]

### IJktheorie en ijksystemen

IJktheorieën komen veelvuldig voor in de natuurkunde, met name in de theoretische hoge-energiefysica. Voor het Standaard Model van de elementaire deeltjes, dat de interacties tussen de tot nu toe bekende kleinste deeltjes beschrijft, vormen ze zelfs de belangrijkste onderdelen. Ik zal proberen een idee te geven van het principe van een ijktheorie aan de hand van een simpel voorbeeld dat gerelateerd is aan een ijktheorie bekend onder de naam *kwantumchromodynamica*. Dit voorbeeld lijkt misschien wat kinderachtig maar het is (in een wat ingewikkeldere vorm) zeker relevant voor de natuurkunde.

Stelt u zich een kinderfeestje voor waar twaalf kinderen ingedeeld worden in vier groepjes van drie op de volgende manier. Ieder kind krijgt een petje op zijn hoofd. Het petje is rood, groen of blauw gekleurd en van elke kleur zijn er vier petjes. De opdracht voor de kinderen is dat ze nu vier groepjes mogen vormen van drie personen, maar dat in elk groepje elk kind een ander kleur petje moet dragen. Na verloop van tijd, de duur hangt af van de mate van medewerking van de kinderen, zullen er waarschijnlijk vier groepjes ontstaan elk bestaande uit drie kinderen waarbij elk kind in zo'n groepje een ander kleur hoofddeksel heeft. Wie er bij wie in het groepje komt, hebben de kinderen nu niet helemaal voor het uitkiezen. Immers, als jij en jouw beste vriendje een hoofddeksel dragen met de kleur rood, dan hebben jullie beide vette pech, want jullie zullen in verschillende groepjes komen. Nu zouden we er natuurlijk voor kunnen kiezen om iedereen met een rood petje zijn petje te laten wisselen met iemand met een blauw petje. Weliswaar is de situatie op het oog nu anders, maar deze verandering beïnvloedt de uitkomst niet. Welke groepjes gevormd kunnen worden, is in beide situaties gelijk. Als nu de natuurwet zou zijn dat elk kind in een groepje een ander kleur petje moet hebben, dan hangt dit alleen af van het feit hoeveel verschillende kleuren er

[13] De kwantisatie hangt af van een parameter $\hbar$ (je zou voor $\hbar$ kunnen denken aan de grootte van de lengteschalen, al is dit niet juist). Het plaatje met de verschillende groottes van $\hbar$ op de voorkant, representeert de kwantisatieprocedure.

zijn, maar de kleuren zelf zijn irrelevant.[14]

Het feit dat we de rode en blauwe petjes kunnen verwisselen (of de rode en groene, of de groene en blauwe) zonder dat de structuur van de situatie wordt aangepast, noemen we ook wel een *symmetrie* of een *symmetrietransformatie*. Deze symmetrietransformaties komen in de kwantumchromodynamica[18] ook voor en het zijn deze symmetrieën die we ijksymmetrieën noemen. Voor de bijbehorende theorie wordt daarom de term *ijktheorie* gebruikt. Belangrijk bij de kwantisatie van dit soort theorieën is dat deze symmetrieën overgevoerd worden. Dus als de klassieke theorie deze symmetrie heeft, dan moet ook de bijbehorende kwantisatie deze symmetrie hebben. Het zijn juist deze symmetrieën die de kwantisatie van ijktheorieën lastig maken.

We spreken van een symmetrie in de natuurkunde wanneer de natuurkunde niet verandert onder zo'n symmetrietransformatie. Met andere woorden, we kunnen twee systemen die in de theorie verbonden zijn door een symmetrietransformatie in de praktijk niet onderscheiden. We zouden dus ook in de theorie al die verschillende toestanden die dezelfde natuurkundige situatie geven, als dezelfde toestand willen zien. Daarmee *reduceren* we als het ware het totaal aantal toestanden. Deze procedure staat ook wel bekend als *reductie*. Wanneer we ook kwantisatie in het verhaal betrekken, rijst natuurlijk de volgende vraag: moeten we vóór of ná kwantisatie reduceren? Het antwoord is dat een wiskundige vindt dat een goede

[14]Dit voorbeeld heeft als doel om aan te geven dat de situatie symmetrisch is onder verwisseling van kleuren, maar de 'natuurwet' die ik noemde geeft daadwerkelijk het idee weer achter een interessant kwantummechanisch verschijnsel, namelijk het Pauli-uitsluitingsprincipe. Het Pauli-uitsluitingsprincipe stelt dat twee (fermionische) ononderscheidbare[15] deeltjes niet alle eigenschappen gelijk hebben. Laten we voor het gemak daarom aannemen dat ons kinderfeestje bestaat uit een twaalfling of, beter nog, twaalf *identieke* klonen in identieke kleren (behalve dan de kleur van het petje).[16] Dan zijn we alleen in staat de kinderen te onderscheiden op basis van hun petje. Kinderen met dezelfde kleur pet hebben nu volledig gelijke eigenschappen en zouden volgens het Pauli-uitslutingsprincipe niet naast elkaar mogen staan. Er kunnen daardoor alleen groepjes van drie gevormd worden wanneer elk kind in het groepje een ander kleur petje draagt.

Men zou overigens kunnen aanvoeren dat we ook groepjes van twee kunnen maken, maar volgens een ander principe in kwantumchromodynamica komen alleen groepjes voor die *kleurloos* zijn (rood+groen+blauw wordt gezien als wit).[17] De naam *kleur* wordt overigens echt gebruikt als eigenschap voor een *quark*, een van de elementaire deeltjes in het Standaard Model. Kleur als eigenschap voor quarks werd geïntroduceerd toen er deeltjes gevonden werden die opgebouwd waren uit drie quarks die alle precies dezelfde eigenschappen hadden. Op grond van het Pauli-uitsluitingsprincipe zou dit echter niet mogen. Met de introductie van *kleurlading* werd dit probleem opgelost door elk van deze drie quarks een andere kleur toe te kennen, precies zoals we hierboven op het kinderfeestje gedaan hebben.

[15]verwarrend woord

[16]Dit is wat we in de fysica noemen een gedachte-experiment. Gelukkig is het meestal voldoende om het in dit soort voorbeelden bij gedachte-experimenten te laten zodat problemen met kinderbeschermingen, gebrek aan technische mogelijkheden of ethische commissies doorgaans voorkomen kunnen worden.

[17]Dit zou een verklaring kunnen zijn waarom Kwik, Kwek en Kwak onafscheidelijk zijn. Dit overigens geheel terzijde.

[18]het Griekse woord voor kleur is '$\chi\rho\acute{\omega}\mu\alpha$'.

definitie van kwantisatie (en reductie) zo moet zijn dat de volgorde van kwantiseren en reduceren niet uit maakt. Wanneer dit zo is, dan zeggen we dat *kwantisatie commuteert met reductie*.[19] Wanneer kwantisatie commuteert met reductie, dan maakt het in het volgende diagram

$$\begin{array}{ccc} \text{klassiek systeem + symmetrieën} & \xrightarrow{\text{kwantisatie}} & \text{kwantumsysteem + symmetrieën} \\ \Big\downarrow{\scriptstyle\text{reductie}} & & \Big\downarrow{\scriptstyle\text{reductie}} \\ \text{gereduceerd klassiek systeem} & \xrightarrow{\text{kwantisatie}} & \text{gereduceerd kwantumsysteem} \end{array}$$

niet uit of we linksom of rechtsom van linksboven naar rechtsonder gaan. Zo'n diagram als hierboven heet in de wiskunde ook wel een *commutatief diagram*.

IJktheorieën laten zich lastig kwantiseren en wiskundigen weten nog niet hoe dit in het algemeen moet. De situaties in de natuurkunde zijn wiskundig erg lastig. Daarom bekijken we eerst eenvoudigere situaties, zoals ijktheorieën op een ruimte die slechts uit (eindig veel) losse punten bestaat. Dit is ook wat we in dit proefschrift doen. In het eerste deel van dit proefschrift geven we een kwantisatie voor een speciale klasse van wiskundige ruimten, welke in de natuurkunde overeenkomen met ijktheorieën op een ruimte die bestaat uit vier losse punten. Ook laten we zien dat in deze voorbeelden kwantisatie commuteert met reductie. Voor de wiskundigen onder de lezers: we bekijken de Hilbertruimte-kwantisatie van coraaksbundels van compacte, samenhangende Lie-groepen met daarop de actie die geïnduceerd wordt door de actie van de Lie-groep op zichzelf door conjugatie.

[19]Het woord 'commuteren' is verwant aan het woord 'commutativiteit'. Hier spreken we van commutativiteit omdat de volgorde van kwantiseren en reduceren er niet toe doet.

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# Curriculum Vitae

Jord Boeijink is geboren op 7 januari 1986 te Enschede. Na in de zomer van 2003 zijn atheneum afgerond te hebben aan Scholengemeenschap 'Het Assink' in Haaksbergen, is hij in september 2003 begonnen aan de studie 'Natuur- en Sterrenkunde' aan de Radboud Universiteit Nijmegen, toen nog de Katholieke Universiteit Nijmegen geheten. Tijdens zijn studie heeft hij met liefde voor het onderwijs vele werkcolleges verzorgd in zowel de natuur- als de wiskunde. In augustus 2009 studeerde hij cum laude af met de scriptie getiteld 'Noncommutative geometry of Yang-Mills fields', die hij geschreven heeft onder begeleiding van dr. W.D. van Suijlekom, werkzaam in de vakgroep 'Mathematische Fysica', en prof.dr. R.H.P. Kleiss, werkzaam in de vakgroep 'Theoretische Hoge-Energiefysica'. Na zijn studie begon Jord te werken als promovendus in de Mathematische-Fysicagroep aan de Radboud Universiteit Nijmegen onder begeleiding van dr. W.D. van Suijlekom en prof.dr. N.P. Landsman. Het proefschrift dat nu voor u ligt, is hiervan het resultaat.