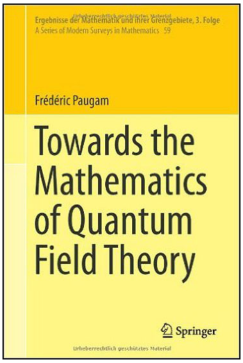
Jahresber Dtsch Math-Ver
DOI 10.1365/s13291-015-0120-3

BOOK REVIEW

Frederic Paugam: "Towards the Mathematics of Quantum Field Theory"

Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, Vol. 59, Springer-Verlag, 2014, 487 pp.

Walter D. van Suijlekom[1]



Quantum field theory can be considered as one of the cornerstones of modern physics. Historically, it originates from the problem of quantizing the electromagnetic field, dating back to shortly after the advent of quantum mechanics. This amounts to understanding the quantization of a system with infinitely many degrees of freedom, namely, the values of the electromagnetic field at all spacetime points. It turned out that precisely this property would form the main obstacle in the search for a mathematical formalism for quantum field theories.

In the 1950s this led to a separation of the field. On the one hand perturbation theory led to a computationally yet undefeated approach, where in particular Feynman's graphical method streamlined the analysis of physical probability amplitudes. On the other hand, an elegant axiomatic approach to quantum fields emerged through the work of Haag, Kastler, Wightman and others, leading to the subject known as *algebraic quantum field theory*. For a historical account we refer to [7]. Even though the two approaches ought to be complementary, it turned out that the separation only became larger and larger towards the end of the last century. This was mainly due to the fact that physically relevant interacting quantum fields were found difficult to be included in the strict algebraic formulation, while at the same time the computational approach of Feynman amplitudes mistified the mathematical structure behind this successful branch of physics.

✉ W.D. van Suijlekom
waltervs@math.ru.nl

[1] Nijmegen, The Netherlands

Published online: 18 August 2015

The last fifteen years, however, have seen a lot of progress that gives reason to be more optimistic. This is partly based on the work that is done in the analysis of the mathematical structure of Feynman amplitudes and, at the same time, on the formulation of *perturbative* algebraic quantum field theory. The book under review is a valuable witness to a wide range of recent developments that aim at unraveling the mathematics of quantum field theory. As such, it touches on several newly developed fields in mathematics that are either inspired by quantum field theory or play an important role towards understanding it.

In contrast to textbooks such as [4–6, 9] that present the physicists' rudiments of quantum field theory to a mathematical audience, the approach taken by Paugam can be described as the opposite, let us say top-down, approach. It rather describes the moving forefront of the field, where new mathematics is explored and developed to build a rigorous formalism for quantum field theory. This makes the textbook an exciting one, presenting an original view on quantum field theory, but also of a more advanced level. The graduate students to which the author addresses his book are therefore warned to come well-prepared. For one thing, the central pillar of the book is formed by categorial logic and category theory on which the author admits in his preface that "[the] distilled axioms are pretty indigestible", suggesting to read the relevant chapters simultaneously with the later, more explicit chapters. I would add to that the suggestion that to understand the motivation for setting up this mathematical formalism, some background on quantum field theory might be useful, referring the reader to one of the above more "bottom-up" textbooks.

The book is divided into three parts. The first part presents the toolbox of higher category theory and its use in an original and concise description of differential geometry, partial differential equations, affine group theory and Hopf algebras, symplectic and Poisson geometry and, finally, homotopical algebra.

The second part deals with classical physics, focussing mainly on variational problems. An extensive list of physical models are nicely described in the mathematical language of the first part. The discussion of each of the models is rather brief—the Standard Model of particle physics for instance only takes two pages—but it is always accompanied by a reference for further reading.

In the third part we come to the main topic of this book: quantum fields. In an attempt to complement the top-down viewpoint of the book, let me describe this part in somewhat more detail, also including the references back to the relevant chapters in Part I and II.

The first approach to quantum field theory addressed in Paugam's book is the conventional one, using functional integrals and perturbative methods. The central ingredient is Feynman's path integral that is supposed to compute physical probability amplitudes. A priori, this path integral is ill-defined and the main challenge is to give at least some computational significance to it. With the desired *constructive* formulation of quantum fields still lacking (see [8] for an overview) this is most successfully accomplished in a perturbative approach: one perturbs a free quantum field by interactions, organized in a nice graph theoretical way by Feynman diagrams. Each of these diagrams corresponds to a so-called Feynman integral (not to be confused with the path integral) that contributes to the physical probability amplitudes.

In Paugam's book we find the free scalar quantum field in Chap. 17 using the Klein–Gordon equation (cf. Chap. 15.1), following a discussion on quantization of classical hamiltonian systems (cf. Chap. 8). Functional integrals and Feynman’s perturbative approach are the subject of Chap. 18, also including the physically relevant class of gauge field theories. Note that for the expected link to functorial analysis developed in Chap. 4, the reader has to wait until the beginning of Chap. 20.

There is at least one caveat in the above perturbative approach to quantum field theory: the Feynman integrals corresponding to Feynman diagrams are divergent integrals. This is the beginning of the process of renormalization, which, in short, is similar to Cauchy’s principal value that assigns a finite value to a divergent integral. For Feynman integrals one finds the further complication of subdivergences corresponding to subgraphs, but whose renormalization can be organized in a recursive manner. Such renormalization schemes have been vital in the formulation of Yang–Mills gauge theories and eventually of the extremely successful Standard Model of particle physics.

At the beginning of the new millennium an algebraic structure underlying renormalization was discovered by Connes and Kreimer. They build a Hopf algebra on Feynman diagrams and realized renormalization as a Birkhoff decomposition for the affine group scheme dual to the Hopf algebra. This was later worked out by Kreimer and the reviewer for gauge theories, where Hopf ideals corresponding to quantum gauge symmetries were identified.

The renormalization Hopf algebra for Feynman diagrams in gauge theories is presented in Chap. 19 where also the dual affine group schemes are identified. Here the general theory on affine group schemes and Hopf algebras is used from Chaps. 5 and 6.

An alternative approach to quantum field theory is Wilson’s effective field theory. The idea is to enhance the classical Lagrangian or action functional to take also quantum effects into account. One then proceeds to study the flow of these effective action functionals as energy varies by analyzing the renormalization group equations.

These methods are well-established by physicists and have recently been put into a mathematical formalism by Costello in [3]. This includes gauge theories where the BV-approach plays a prominent role, even though Yang–Mills theory can only be cast into this formalism in its first-order formulation.

Paugam presents Costello’s approach in Chap. 21, including the BV-formalism that reflects some of the structure of BV-algebras found in Chap. 13.

One of the recent exciting developments in algebraic quantum field theory is the opening to a perturbative description through the work of Brunetti, Dütsch and Fredenhagen [1] and further developed by others. The idea is to formulate perturbation theory using a $*$-product deformation of the free quantum field theory, thereby using Epstein–Glaser renormalization.

Chapter 22 gives an overview of the results in this direction, including the case of gauge theories and also linking to Hopf algebraic renormalization methods in the final section.

The $*$-product of the previous paragraph is a concept coming from deformation quantization. A central result by Kontsevich is the construction of a $*$-product for any Poisson manifold, deforming the pointwise product of smooth functions thereon. It is tempting to apply this directly to the field theory setting and try to arrive at a non-perturbative algebraic quantum field theory, but for now Kontsevich’s construction is restricted to the finite-dimensional case. Nevertheless, a link between deformation quantization and quantum fields has been established by Cattaneo and Felder [2] by writing the $*$-product as a path integral expectation value for the Poisson σ-model.

Paugam gives an overview of deformation quantization including this link to quantum field theory in Chap. 23, with the mathematical preliminaries found in Chap. 10, and the Poisson σ-model briefly explained in Chap. 16.3. The author's generalization to non-topological quantum field theories using factorization spaces can be found in Chap. 24, whose mathematical basis is laid in Chap. 12.

In summary, "Towards the Mathematics of Quantum Field Theory" is precisely what the title tells you: it gives an overview of some of the mathematical research that is done on unravelling the structure of rigorous quantum field theory. Though at some points quite advanced and concise—requiring the reader to consult additional literature—the book is well written with an original organization of the material. Its reductionist viewpoint makes it a valuable textbook, especially for the more experienced practitioners in this domain of mathematical physics.

References

1. Brunetti, R., Dütsch, M., Fredenhagen, K.: Perturbative algebraic quantum field theory and the renormalization groups. Adv. Theor. Math. Phys. **13**, 1541–1599 (2009)
2. Cattaneo, A.S., Felder, G.: A path integral approach to the Kontsevich quantization formula. Commun. Math. Phys. **212**, 591–611 (2000)
3. Costello, K.: Renormalization and Effective Field Theory. Mathematical Surveys and Monographs, vol. 170. Am. Math. Soc., Providence (2011)
4. de Faria, E., de Melo, W.: Mathematical Aspects of Quantum Field Theory. Cambridge Studies in Advanced Mathematics, vol. 127. Cambridge University Press, Cambridge (2010)
5. Deligne, P., Etingof, P., Freed, D.S., Jeffrey, L.C., Kazhdan, D., Morgan, J.W., Morrison, D.R., Witten, E. (eds.): Quantum Fields and Strings: A Course for Mathematicians. Vol. 1, 2. Am. Math. Soc., Institute for Advanced Study, Providence, Princeton (1999)
6. Folland, G.B.: Quantum Field Theory. A Tourist Guide for Mathematicians. Mathematical Surveys and Monographs, vol. 149. Am. Math. Soc., Providence (2008)
7. Kuhlmann, M.: Quantum field theory. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy (2015)
8. Summers, S.J.: A perspective on constructive quantum field theory. arXiv:1203.3991
9. Zeidler, E.: Quantum Field Theory. A Bridge Between Mathematicians and Physicists (Volume I–VI). Springer, Berlin (2006, 2009, 2011) and forthcoming