Semigroup of inner perturbations in noncommutative geometry

Walter van Suijlekom

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Radboud University Nijmegen

Overview

- Spectral geometry
- Matrix algebra and noncommutative geometry
- Semigroup of perturbations
- Examples of perturbation semigroup

References

A. Chamseddine, A. Connes, WvS. Inner Fluctuations in Noncommutative Geometry without the first order condition. *J. Geom. Phys.* 73 (2013) 222-234 [arXiv:1304.7583]

WvS. The spectral model of particle physics, *Nieuw Archief voor Wiskunde*, December 2015.

WvS. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies, Springer, 2015.

and also: `http://www.noncommutativegeometry.nl`

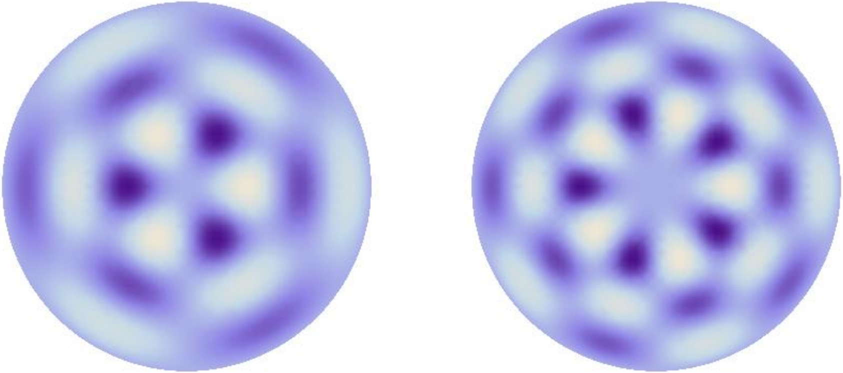
Spectral geometry

"Can one hear the shape of a drum?" (Kac, 1966)

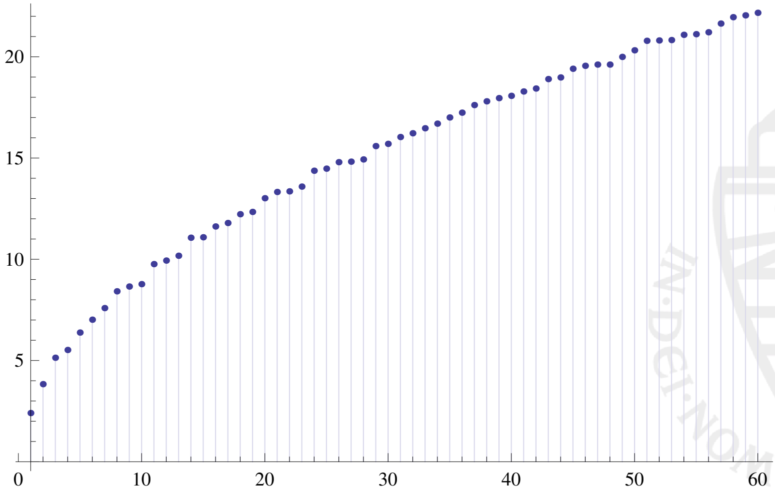
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

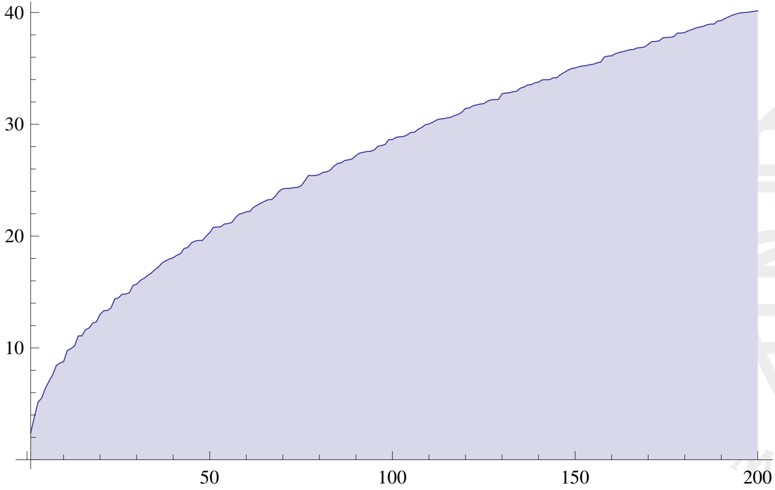
determine the geometry of M?



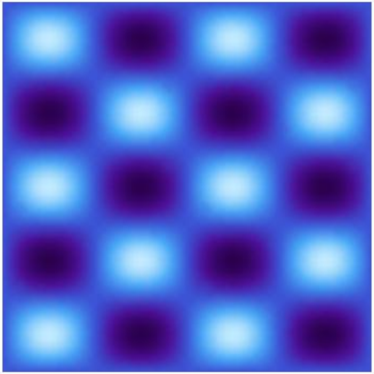
Wave numbers on the disc

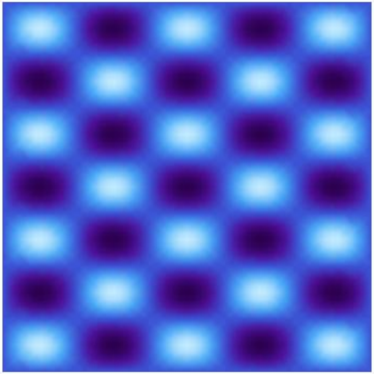


Wave numbers on the disc: high frequencies

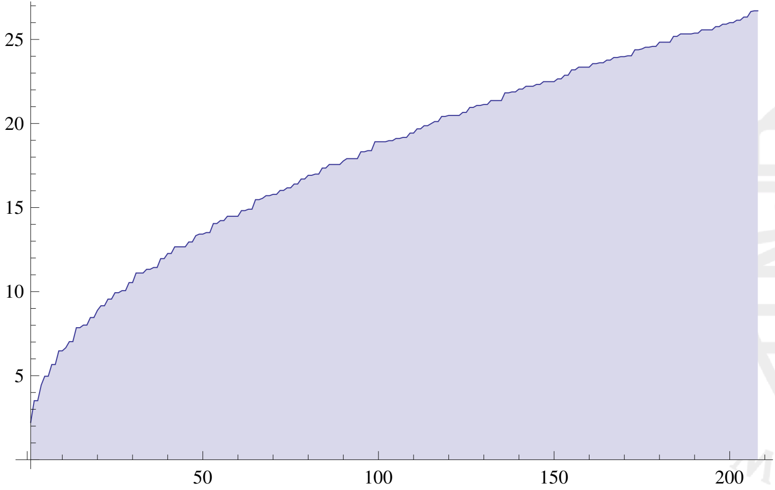


The square



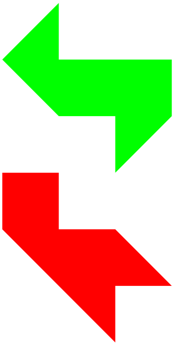


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

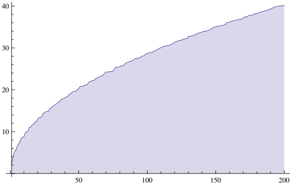
so the answer to Kac's question is no.

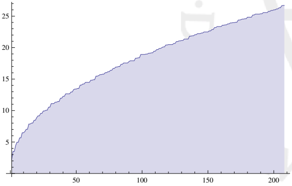
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension n of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda \sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n}\Lambda^n$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.

The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.
- The eigenfunctions of $D_{\mathbb{S}^1}$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt$$

  with eigenvalue $n \in \mathbb{Z}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2} \right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

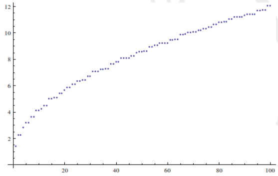
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

- The Dirac operator on $\mathbb{T}^4$ is

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$

- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

Spectral action functional
Chamseddine–Connes, 1996

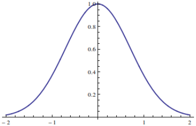
- Reconsider Weyl's estimate, in a smooth version:

$$\mathrm{Tr}\, f\left(\frac{D_M}{\Lambda}\right) = \sum_{\lambda} f\left(\frac{\lambda}{\Lambda}\right)$$

for a smooth cutoff function $f : \mathbb{R} \to \mathbb{R}$.

- For simplicity, restrict to a Gaussian function

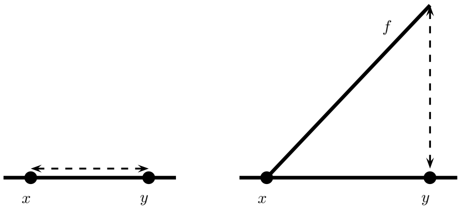
$$f(x) = e^{-x^2}$$



so that we can use heat asymptotics: $\mathrm{Tr}\, e^{-D_M^2/\Lambda^2} \sim \dfrac{\mathrm{Vol}(M)\Lambda^n}{(4\pi)^{n/2}}$

- As said, the geometry of M is not fully determined by spectrum of D_M.
- This can be improved by considering besides D_M also the algebra $C^\infty(M)$ of smooth functions on M, with pointwise product and addition
- In fact, the distance function on M is equal to

$$d(x,y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$. For example, on the circle we have $[D_{\mathbb{S}^1}, f] = -i\frac{df}{dt}$

Finite spaces

- Finite space F, discrete topology

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Smooth functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \sup_{f \in C^\infty(F)} \left\{ |f(p) - f(q)| : \|[D_F, f]\| \leq 1 \right\}$$

Example: two-point space

$$F = \ _1\bullet \qquad _2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(p,q) = \begin{cases} |c|^{-1} & p \neq q \\ 0 & p = q \end{cases}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \ldots, a_N$ are square matrices of size $n_1, n_2, \ldots, n_N$.

- Hence we will consider the matrix algebra

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A finite Dirac operator is still given by a hermitian matrix.

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the algebra A_F of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Perturbation semigroup

We make the above more dynamical by *perturbing* D_F by matrices in A_F.

Definition (Chamseddine–Connes-vS, 2013)

Let A_F be the above algebra of block diagonal matrices (fixed size). The perturbation semigroup of A_F is defined as

$$\mathrm{Pert}(A_F) := \left\{ \sum_j A_j \otimes B_j \in A_F \otimes A_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\},$$

where ^{t} denotes matrix transpose, $\mathbb{I}$ is the identity matrix in A_F, and $\bar{}$ denotes complex conjugation of the matrix entries.

The semigroup law in $\mathrm{Pert}(A_F)$ is given by the matrix product in $A_F \otimes A_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

  are called normalization and self-adjointness condition, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

  for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $A_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

  leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.
- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $A_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C})$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{matrix} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{matrix} \right\}$$

and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

Perturbation semigroup for all matrix algebras
with Niels Neumann (B.Sc.)

- More generally, consider

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- For direct sums we have

$$\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) \cong \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ)^{\mathrm{s.a.}}$$

and we compute that

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong \left\{ \begin{pmatrix} 1 & v \\ 0 & B \end{pmatrix} : \overline{v} = v\Omega, \Omega\overline{B} = B\Omega \right\} \cong V \rtimes S.$$

where

$$\Omega = \begin{pmatrix} I_{(N+2)(N-1)/2} & 0 \\ 0 & -I_{N(N-1)/2} \end{pmatrix}.$$

- This is compatible with the decomposition $\mathbb{C}^N \otimes \overline{\mathbb{C}^N} \cong \mathbb{C} \oplus \mathbb{C}^{N^2-1}$ into irreps of $U(N)$.
- Similar decompositions can be shown to hold for $\mathrm{Pert}(M_N(\mathbb{R}))$ and irreps of $O(N)$, and $\mathrm{Pert}(M_N(\mathbb{H}))$ and irreps of $Sp(N)$.

Example: perturbation semigroup of a manifold

- The perturbation semigroup can be defined for any involutive unital associative algebra A, in particular for $C^\infty(M)$.
- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two-variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \,\middle|\, \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\},$$

Proposition

Let $\mathcal{U}(A_F)$ be the unitary block diagonal matrices in A_F. This space forms a group which is a subgroup of the semigroup $\mathrm{Pert}(A_F)$ via $U \mapsto U \otimes \overline{U}$.

This is in agreement with the results for matrix algebras, for which

$$\mathcal{U}(M_N(\mathbb{R})) = O(N); \quad \mathcal{U}(M_N(\mathbb{C})) = U(N); \quad \mathcal{U}(M_N(\mathbb{H})) = Sp(N).$$

- Action of $\mathrm{Pert}(A_F)$ on hermitian matrices D_F:

$$D_F \mapsto \sum_j A_j D_F B_j^t$$

- This action is compatible with the semigroup law, since

$$\sum_{j,k} (A_j A'_k) D_F (B_j B'_k)^t = \sum_j A_j \left(\sum_k A'_k D_F (B'_k)^t \right) (B_j)^t$$

- The restriction of this action to the unitary group $\mathcal{U}(A_F)$ gives

$$D \mapsto UDU^*.$$

Perturbations on noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$:

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Physicists call ϕ_1 and ϕ_2 the Higgs field.
- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Perturbations on a Riemannian spin manifold

- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x}; \qquad (\mu = 1\ldots,n),$$

  where $f \in C^\infty(M \times M)$ is such that $f(x,x) = 1$ and $\overline{f(x,y)} = f(y,x)$.
- In physics, one writes

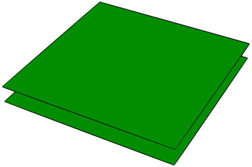
$$A_\mu := \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x}$$

  which turns out to be the electromagnetic potential

- Combine (4d) Riemannian spin manifold M with finite noncommutative space F:

$$M \times F$$

- F is internal space at each point of M



- Described by matrix-valued functions on M: algebra $C^\infty(M, A_F)$

Dirac operator on $M \times F$

- Recall the form of D_M:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

- Dirac operator on $M \times F$ is the combination

$$D_{M\times F} = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

- The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F:

$$D_{M\times F}^2 = D_M^2 + D_F^2$$

- Using this, we can expand:

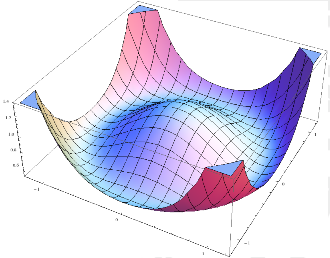
$$\mathrm{Tr}\ e^{-D_{M\times F}^2/\Lambda^2} = \frac{\mathrm{Vol}(M)\Lambda^4}{(4\pi)^2} \mathrm{Tr}\ \left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

The Higgs mechanism

We apply this to the noncommutative two-point space described before

- Algebra $A_F = \mathbb{C} \oplus M_2(\mathbb{C})$
- Perturbation of Dirac operator D_F parametrized by ϕ_1, ϕ_2.
- Potential for the perturbed Dirac operator is

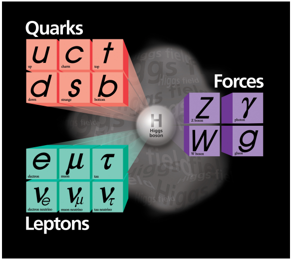
$$V(\phi) = -2\Lambda^2(|\phi_1|^2 + |\phi_2|^2) + (|\phi_1|^2 + |\phi_2|^2)^2$$



- Minimum breaks symmetry spontaneously, giving mass to Higgs boson (125.5 GeV, corresponding to $10^{-18} m$).

The spectral Standard Model and beyond

- The full Standard Model is based on the algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
- The finite Dirac operator is given by a 96×96-dimensional hermitian matrix, containing masses for the leptons and quarks.
- This allows for a derivation of the particle content of the Standard Model from pure geometry (Chamseddine–Connes–Marcolli, 2007)



- The spectral action functional describes their dynamics and interactions
- Possibility to go beyond with Pati–Salam (Chamseddine–Connes–vS):

$$A_F = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$$