The Renormalization Group and Self-avoiding Walk

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1 Introduction

I am grateful and honoured to be given this opportunity to give an introduction to the renormalisation group. It is based on recent work with Gordon Slade, and Roland Bauerschmidt. These notes are intended to be interesting for mathematicians. No knowledge of physics is assumed, but our topic began life in physics and so I begin with a review of this background. This review contains references that cannot be understood without a background in physics. I include them for historical reasons, not because they are required reading. However it would make me very happy if the mathematical developments in these notes help someone in my audience understand physics better.

The renormalisation group (RG) first appeared in quantum electrodynamics with work by Stueckelberg and Petermann [45] and Gell-Mann and Low [30]. Quantum electrodynamics is the quantum theory that extends the classical theory of electromagnetism. Classical electromagnetism is the combination of Maxwell's partial differential equations for the electromagnetic field and the Lorentz equation that describes the force experienced by a charged particle moving in an electromagnetic field. The force is proportional to the charge and charge is an example of a *coupling constant*. Maxwell's equations are invariant under a common rescaling of space and time and the charge does not change under such rescaling. However scale invariance of charge as a coupling constant does not survive in quantum electrodynamics. These founding papers were statements about how the charge coupling constant transforms under rescaling. Later the insight of these papers was put in a more useful form, called the Callan-Symanzik equations, [23, 48, 49]. These equations

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are a set of coupled differential equations for the flow of the coupling constants under scaling. In these lectures we will see how it can be that coupling constants depend on scale.

A quite different line of thought started with Symanzik [46] who began the study of quantum field theory on $\mathbb{R}^d$ with the euclidean metric instead of the Minkowski metric prescribed by Nature. He realised that these, despite the name "quantum fields", are random fields in the standard sense of probability. By formulating this idea also for fields on a lattice such as $\mathbb{Z}^d$ these theories were connected with the scaling limits of lattice spin systems such as the Ising model. This was exploited in a deep way in [31, 32, 42, 43] where euclidean quantum field theory is studied as the limit of classical spin systems and, in particular, correlation inequalities are used. In the other direction, from euclidean field theory to lattice spin systems, Wilson and Fisher [53] and Wilson [51] showed that the renormalisation group and the Feynman expansion of quantum field theory can systematically calculate critical exponents in statistical mechanics. In these lectures we will be explaining some parts of this method.

Other authors [40, 57] expressed the same ideas for calculating critical exponents in terms of the Callan-Symanzik equations. In particular, using an idea of de Gennes [24], Brezin et al. [13] and Duplantier [26] used the Callan-Symanzik equations to calculate exponents for self-avoiding walk in three and four dimensions. These authors are assuming that the scaling limit of a lattice spin system in four dimensions exists as a continuum random field with the scaling properties prescribed by the Callan-Symanzik equations and that the coefficients in these equations are given by perturbation theory. They give very efficient calculations of critical exponents based on these assumptions.

In our work, [10] and the five papers that it references, we avoid such assumptions by following the work of Ken Wilson, who invented a more detailed form of the renormalisation group that applies to statistical mechanical models on lattices. It is in fact a program to prove the existence of scaling limits as quantum field theories. We only make part of that program mathematically complete, just enough to prove that the susceptibility of a walk with weak self-repulsion has a $\log^{\frac{1}{4}}$ correction. If you are seeing this log correction for the first time then it will not seem very interesting but think of it as a signal: Wilson's program is systematic, in the same sense that calculus is systematic, and it is a calculus for critical statistical mechanics and scaling limits. It is more complicated than you will like, but most profound ideas have not been easy at first. I regard the influential lecture course [54] as still one of the best places to appreciate the ideas and scope of this program. Perhaps even better and also amusing in places, is his Nobel prize lecture [52].

Wilson and Kogut start with a random field ϕ defined on a lattice $\mathbb{Z}^d$. They "integrate out" fluctuations in the random field up to some chosen length scale L. One intuitively attractive way to do this is called the *block spin* method and it is defined as follows. The lattice is partitioned into disjoint cubes of side L and the random field is conditioned on its empirical averages over the cubes. The centres of the cubes are a new coarser lattice of spacing L and the empirical averages are a new

random field on this coarser lattice. The coarse lattice is scaled so that it becomes the unit lattice $\mathbb{Z}^d$. Thus the combination of conditioning followed by rescaling is a map RG taking the space $\mathscr{M}$ of all probability laws for a lattice random field into itself. Thus RG can be iterated and can be analysed as a dynamical system.

Suppose RG has a fixed point. Then one can define the stable set S to be the set of all points in $\mathscr{M}$ that have orbits under RG that converge to the fixed point. This S is invariant under RG. The Wilson picture is that points on S are probability distributions for random fields that are critical, for example an Ising model at its critical temperature. The set S is the universality class of all random fields whose scaling limit is the probability distribution represented by the fixed point. This special probability distribution is expected to have more symmetries than the distributions represented by points on S. For example it will be scale invariant. In fact in some cases it is conjectured to satisfy the axioms of euclidean quantum field theory. To be clearer, the fixed point is a probability distribution for a lattice random field, but in fact there will be a continuum generalised random field whose empirical averages over cubes centred on the lattice points are the random variables for the fixed point model. It is this continuum field that will be scale invariant and may satisfy the axioms of euclidean quantum field theory. One of the most important such fixed point distributions is called the massless free field. Wilson argued that in more than two dimensions the stable set for this fixed point has finite codimension: you have to carefully select values of finitely many parameters to be a critical model on the stable set for this fixed point. In four and more dimensions, and for even measures, Wilson asserts that the codimension is 2. One of the parameters is a parameter of the massless free field analogous to the variance of the normal distribution. As in the central limit theorem you have to normalise the empirical averages correctly in order that the scaling limit be a standard massless free field. Selecting the other parameter corresponds, for the Ising model, to choosing the temperature to be critical. In these lectures we will see this codimensionality of 2 for self-avoiding walk in four dimensions.

We implement the Wilson RG a little differently. Our definition of RG relies on the fact that the gaussian free field ϕ on the lattice $\mathbb{Z}^d$ is equal in distribution to a sum $\phi = \sum_j \zeta_j$ over scales j of independent gaussian fields ζ_j with a strong independence property: $\zeta_{j,x}$ and $\zeta_{j,y}$ are independent when the spatial points x and y are separated by at least $L^j/2$. This suggests an interesting open problem: characterise the gaussian random fields ϕ which in distribution can be written as such a sum. There is a nice proof that the lattice gaussian free field has such a decomposition in [7] along with references to the original constructions of such representations.

The first rigorous control of RG was achieved by Gawedzki and Kupiainen [28, 29]. Hara and Tasaki [33] used their methods to prove the existence of log corrections in the ϕ^4 lattice field theory. There is a different program with similar outcomes called the phase cell expansion described in [2]. Using the phase cell expansion Iagolnitzer and Magnen [34] considered a model that is roughly speaking an Edwards model in the continuum. They determine the asymptotics of the Greens function as one spatial argument tends to infinity. A nice feature of their approach is that it deals quite directly with walks whereas our method transforms self-avoiding

walk into massless free field language. Balaban has made the most far reaching accomplishments in the rigorous renormalisation group, for example in his series of papers on the classical lattice Heisenberg models that starts with [6].

At this time the scaling limit of self-avoiding walk in three dimensions is, for mathematicians, a complete mystery. It is believed that the scaling limit is a non-gaussian fixed point for RG and we hardly understand these at all. However in [57] you can read the remarkable non rigorous progress that started with ideas of Wilson and Fisher. With $\epsilon := 4-d$, critical exponents are obtained as expansions in powers of ϵ. The first few terms of these expansions can be calculated and results in accurate agreement with simulations of three dimensional self-avoiding walk are obtained by setting $\epsilon = 1$. A step in this direction of varying a dimension-like parameter has been made in [41]. Related to this is [1] which, for a ϕ^4 theory in three dimensions, constructs a *complete* renormalization group trajectory that at one end converges to a gaussian and at the other end to a non-gaussian fixed point.

2 The Lattice Edwards Model

We start with a continuous time random walk $\{X_t : t \geq 0\}$ on the euclidean lattice $\mathbb{Z}^d$ or on the torus $\Lambda \equiv (\mathbb{Z}/R\mathbb{Z})^d$ where R is a positive integer. The side or period R of Λ is chosen from the geometric sequence $L^N, N = 1, 2, \ldots$, where $L \geq 2$ is an integer. The (negative) generator of the walk is the lattice Laplacian $-\Delta$, acting on bounded functions f defined on $\mathbb{Z}^d$ or on Λ. It is given by

$$(-\Delta f)_x := \sum_{y \sim x} (f_x - f_y), \tag{1}$$

where $y \sim x$ means that y is a nearest neighbour to x. On the torus Λ every lattice point has $2d$ neighbours because there is no boundary. We use $\mathbb{P}_a$ to denote the law of the random walk starting at the vertex $a \in \Lambda$ and E_a is the corresponding expectation. In addition, we consider the random walk that gets killed with a rate $m^2 \geq 0$, i.e. the process on either $\mathbb{Z}^d \cup \star$ or $\Lambda \cup \star$ whose (negative) generator is given by the following block matrix

$$\begin{array}{c} \\ \Lambda \\ \star \end{array} \begin{array}{c} \begin{array}{cc} \Lambda & \star \end{array} \\ \begin{pmatrix} m^2 - \Delta, & -m^2 \\ 0, & 0 \end{pmatrix} \end{array}$$

The state $*$ is called the *cemetery*. The dynamics is easy to describe: the walk waits for a random time with distribution $\mathrm{Exp}(m^2 + 2d)$ and then jumps to a nearest neighbour with probability $1/(m^2 + 2d)$ or to the state $\star$ with probability $m^2/(m^2 + 2d)$. Further, let us denote by $\zeta := \inf\{t \geq 0 : X_t = \star\}$ the killing time, i.e. the first

hitting time of $\star$. For any $x \in \Lambda$ we define the local time spent at x by

$$L_x := \int \mathbb{1}_{\{X_s = x\}} \, \mathrm{d}s, \tag{2}$$

where the integral is over $[0, \infty)$. The time spent in self-intersection is defined to be

$$\iint \mathbb{1}_{\{X_s = X_t \neq *\}} \, \mathrm{d}s \, \mathrm{d}t \tag{3}$$

and by (2),

$$\iint \mathbb{1}_{\{X_s = X_t \neq *\}} \, \mathrm{d}s \, \mathrm{d}t \; = \; \sum_{x \in \Lambda} L_x^2. \tag{4}$$

We use $\mathbb{P}_a^{(m)}$ for the law of random walk starting from a and with killing rate m^2 and $\mathrm{E}_a^{(m)}$ for the corresponding expectation. Notice that for any function F of the local times we have

$$\mathrm{E}_a^{(m)}\big[F(L)\big] \; = \; m^2 \int \mathrm{E}_a^{(0)}\Big[F(L^T)\Big] \, \mathrm{e}^{-m^2 T} \, \mathrm{d}T \tag{5}$$

with

$$L_x^T := \int_{[0,T]} \mathbb{1}_{\{X_s = x\}} \, \mathrm{d}s. \tag{6}$$

Definition 2.1 (Susceptibility) For $g \geq 0$ and $\nu \in \mathbb{R}$ and random walk on $\mathbb{Z}^d$ starting in state a, define

$$\chi(g, \nu) := \int \mathrm{E}_a^{(0)}\Big[\mathrm{e}^{-g \sum_x (L_x^T)^2}\Big] \, \mathrm{e}^{-\nu T} \, \mathrm{d}T \tag{7}$$

with values in $(0, \infty]$. This is called the infinite volume susceptibility of the lattice Edwards model. We define the finite volume susceptibility $\chi_\Lambda(g, \nu) \in (0, \infty)$ by replacing $\mathbb{Z}^d$ by Λ so that the random walk is on Λ. In the infinite volume case we define the critical value of ν by

$$\nu_{\mathrm{c}} := -\inf\big\{\nu \in \mathbb{R} \colon \chi(g, \nu) < \infty\big\}. \tag{8}$$

In [10, Lemma A.1] we use standard subadditivity arguments to prove that $\chi(g, \nu)$ is finite if and only if ν is strictly greater than ν^{c}. The exponential term suppresses self-intersection. As g is taken larger the suppression becomes stronger, but at the same time the speed of the walk increases because the time it spends between jumps is also being suppressed. The various large g limits are discussed in

[22]. For $g = 0$ we have simple random walk for which $\nu_c = 0$. We will find that ν^c is negative for $g > 0$.

The finite volume susceptibility χ_Λ increases to the infinite volume susceptibility χ as Λ increases through tori of side-length $R = L^N$ with $N \to \infty$. This is proved in [10] by an argument based on wrapping an infinite volume walk onto a torus and noticing that this increases the number of self-intersections. Furthermore, all derivatives of χ_Λ with respect to ν converge to the corresponding derivatives of χ because this is a property of the one-sided Laplace transform and the susceptibility is a one-sided Laplace transform, with ν dual to time. This good property of the Laplace transform comes about because the one-sided Laplace transform is an analytic function of ν for ν in the half-plane $\Re\nu < \nu_c$. By the Vitali theorem pointwise convergence for real values of ν implies uniform convergence on compact subsets of the half-plane and from this it follows that derivatives converge.

The next theorem on the Edwards model is the focus for these notes. To state it we define $a(\epsilon) \sim b(\epsilon)$ to mean that $\lim_{\epsilon\downarrow 0} \frac{a(\epsilon)}{b(\epsilon)} = 1$. Let

$$\gamma = \frac{1}{4}. \tag{9}$$

We make this definition in order to be able to track the origin of the exponent in the logarithmic correction through these notes.

Theorem 2.2 (Part of Theorem 1.1 in [10]) *Let $d := 4$. For $g > 0$ sufficiently small, there exists $A_g > 0$ such that*

$$\chi\big(g, \nu_c(g) + \epsilon\big) \sim \frac{A_g}{\epsilon} \log^\gamma(1/\epsilon), \qquad \epsilon \downarrow 0. \tag{10}$$

For simple random walk, $\chi(0, \nu_c + \epsilon) = 1/\epsilon$ for each $\epsilon > 0$ (in all dimensions) so the log is an effect of $g > 0$. The conclusion of this theorem can be rewritten as

$$\int e^{-\nu_c T} \underbrace{E_a^{(0)}\Big[e^{-g\sum_x (L_x^T)^2}\Big]}_{c_T} \epsilon\, e^{-\epsilon T} dT \underset{\epsilon\downarrow 0}{\sim} A_g \log^{1/4}(1/\epsilon). \tag{11}$$

Thanks to the fact that $\epsilon\, e^{-\epsilon T} dT$ is a probability on $(0, \infty)$, this may be interpreted as a weak version of the (conjectural) statement that the quantity c_T marked above obeys

$$c_T \underset{T\to\infty}{\sim} A_g e^{\nu_c T} (\log T)^{1/4}. \tag{12}$$

Remark 2.3 For dimensions $d > 4$ there will not be a log correction. I do not know if exactly this result has been proved for the Edwards model but such results are proved for the standard discrete time self-avoiding walk by the lace expansion [8, 38]. I am not sure if anyone has considered lace expansions for continuous time

models like the Edwards model. It should be possible and they might even be neater than the standard discrete case.

3 The Free Field and Local Time

The free field is usually introduced as a real gaussian field. However we will consider a complex valued free field because the next section requires it and because the connections we are about to describe between gaussian fields and continuous time random walk are more general when complex fields are used. We only define gaussian fields on the torus Λ. Therefore we consider $\phi \in \mathbb{C}^{\Lambda}$, i.e. $\phi = \{\phi_x : x \in \Lambda\}$. We can also write $\phi_x = u_x + \mathrm{i}v_x$, where u and v are gaussian fields. A gaussian *complex valued* measure has the form

$$\mathrm{e}^{-(\phi, A\bar{\phi})} \det(A) \prod_{x \in \Lambda} \frac{\mathrm{d}u_x \mathrm{d}v_x}{\pi}, \tag{13}$$

where we are using the notation

$$(\phi, A\bar{\phi}) \ := \ \sum_{x,y \in \Lambda} \phi_x A_{xy} \bar{\phi}_y. \tag{14}$$

We say that A is *dissipative* if $\Re\mathfrak{e}\,(\phi, A\bar{\phi}) \geq m^2 \sum \phi_x \bar{\phi}_x$ for some $m^2 > 0$. The above gaussian complex measure exists for any dissipative matrix A. If $A_{xy} = A_{yx}$ and A is real, then the gaussian measure in a genuine probability distribution, in other words it is not complex valued. Furthermore, the gaussian fields u, v mentioned above are independent. We will mostly be working with a real symmetric A, but for the moment let us not make this assumption. Even though there is no probability measure we will still denote integration with respect to the complex measure by the symbol $\mathbb{E}$. The following formulas may surprise you a little. The first is an exercise. The second follows from the invariance of the gaussian density under the change of variables $\phi_x \mapsto e^{i\alpha}\phi_x$ and $\bar{\phi}_x \mapsto e^{-i\alpha}\bar{\phi}_x$ for all $x \in \Lambda$ with any real α. Recall that A is real. This symmetry is called gauge invariance.

$$\mathbb{E}\big[\bar{\phi}_x \phi_y\big] \ = \ (A^{-1})_{xy} \tag{15}$$

while

$$\mathbb{E}\big[\bar{\phi}_x \bar{\phi}_y\big] \ = \ 0 \ = \ \mathbb{E}\big[\phi_x \phi_y\big]. \tag{16}$$

Definition 3.1 If $A := m^2\,\mathrm{id} - \Delta$, then ϕ is called the (complex) free field with mass m.

The real field version of the following theorem has been called the *Dynkin Isomorphism* since [27], but Dynkin references two earlier papers. One of them is [19], where this isomorphism appears in Theorem 2.2, but stated in terms of the skeleton walk defined by a continuous time walk, and the second is [47], which is the first time a connection between local time of random walk and the square of a gaussian field was found. Since we and Symanzik are dropping out of sight in the probability literature on this theorem I am going to come back and haunt everyone who does not remember us! See [50] where the recent history of this theorem is discussed. It includes the relation between $\phi\bar{\phi}$ and the local time of loop soup discovered first in [47], but admittedly not stated very precisely.

A version [14, Proposition 3.1] of the following theorem is valid when A is a real matrix such that (1) all row sums are strictly positive, (2) A is dissipative. This is interesting because this allows A to be the generator of a nonsymmetric walk, but I do not want to pursue this here and will from now on assume that

$$A = m^2 \,\mathrm{id} - \Delta \tag{17}$$

for some $m^2 > 0$.

Theorem 3.2 *For bounded continuous* $F\colon \mathbb{R}_+^{\Lambda} \to \mathbb{R}$, $m^2 > 0$ *and* $a, b \in \Lambda$,

$$\mathbb{E}\big[F(\phi\bar{\phi})\,\bar{\phi}_a\phi_b\big] \;=\; m^{-2}\;\mathbb{E}\otimes \mathrm{E}_a\Big[F(\phi\bar{\phi}+L)\mathbb{1}_{\{X_\zeta-=b\}}\Big]. \tag{18}$$

Proof By the monotone class theorem and the linearity of both sides in the function F it suffices to check that both sides are equal for functions of the form $F(t) := \exp\{-\sum_{x\in\Lambda} w_x t_x\}$ where $w_x \geq 0$. Let W be the $\Lambda \times \Lambda$ diagonal matrix whose diagonal entries are $w_x, x \in \Lambda$. Then

$$F(\phi\bar{\phi}) = \exp\{-\sum_{x\in\Lambda} w_x\phi_x\bar{\phi}_x\} = \exp\{-(\phi, W\bar{\phi})\}. \tag{19}$$

Since this is gaussian the left hand side of (18), up to a normalisation, is the covariance of a gaussian measure whose density is the exponential of $-\big(\phi, (A + W)\bar{\phi}\big)$, where $A = m^2\,\mathrm{id} - \Delta$. Therefore the left hand side of (18) equals

$$(A + W)^{-1}_{ab}\,\mathbb{E}\big[F(\phi\bar{\phi})\big]. \tag{20}$$

Now consider the right hand side of (18): our special choice of F is such that $F(\phi\bar{\phi}+L) = F(\phi\bar{\phi})F(L)$ so the right hand side is

$$m^{-2}\;\mathbb{E}\Big[F(\phi\bar{\phi})\Big]\,\mathrm{E}_a\Big[F(L)\,\mathbb{1}_{\{X_\zeta-=b\}}\Big] \tag{21}$$

and we are reduced to checking that

$$(A+W)^{-1}_{ab} = m^{-2}\ \mathrm{E}_a\Big[F(L)\mathbb{1}_{\{X_\zeta - = b\}}\Big]. \tag{22}$$

The continuous time walk X defines the sequence $Y_0, \ldots, Y_\eta$ of lattice sites visited by X before it arrives at $*$. Let $\mathscr{W}_{n;ab}$ be the set of all nearest neighbour walks $y = (y_0, y_1, \ldots, y_n)$ such that $y_0 = a$ and $y_n = b$. For $y \in \mathscr{W}_{n;ab}$ the probability of the event $Y = y$ is the probability that the continuous time walk X will make the independent transitions $y_0 \to y_1$, $y_1 \to y_2$ and so on ending with y_n to $*$. Therefore

$$\mathrm{E}_a\Big[\mathbb{1}_{Y=y}\Big] = \left(\frac{1}{m^2+2d}\right)^n \frac{m^2}{m^2+2d}. \tag{23}$$

On the event $Y = y$, for $i = 0, \ldots, n$ the time that X waits at y_i is exponential with parameter $m^2 + 2d$ and all these exponential times are independent. Therefore, recalling that $F(L) = e^{-\sum w_x L_x}$,

$$\begin{aligned}\mathrm{E}_a\Big[F(L)\mathbb{1}_{Y=y}\Big] &= \frac{m^2+2d}{m^2+2d+w_{y_0}} \cdots \frac{m^2+2d}{m^2+2d+w_{y_n}}\ \mathrm{E}_a\Big[\mathbb{1}_{Y=y}\Big] \\ &= \frac{1}{m^2+2d+w_{y_0}} \cdots \frac{1}{m^2+2d+w_{y_n}} m^2. \end{aligned} \tag{24}$$

By summing this last formula over $y \in \mathscr{W}_{n;ab}$ and over n,

$$m^{-2}\ \mathrm{E}_a\Big[F(L)\mathbb{1}_{\{X_\zeta - = b\}}\Big] = \sum_{n\geq 0} \sum_{y\in\mathscr{W}_{n;ab}} \frac{1}{m^2+2d+w_{y_0}} \cdots \frac{1}{m^2+2d+w_{y_n}}. \tag{25}$$

To complete the proof we need the right hand side of this equation to be equal to $(A+W)^{-1}_{ab}$. To check this write $A+W$ as a diagonal matrix D minus an off-diagonal matrix J. By the expansion $(D-J)^{-1} = D^{-1} + D^{-1}JD^{-1} + D^{-1}JD^{-1}JD^{-1} + \ldots$ with matrix products written out in terms of sums over indices,

$$(A+W)^{-1}_{ab} = \sum_{n\geq 0} \sum_{y\in\mathscr{W}_{n;ab}} \frac{1}{m^2+2d+w_{y_0}} \cdots \frac{1}{m^2+2d+w_{y_n}}, \tag{26}$$

as desired. This expansion is a sum of positive terms because J has nonnegative entries and D has positive diagonal entries. The expansion is convergent because the number of terms in $\mathscr{W}_{n;ab}$ is $(2d)^n$ whereas the summand is smaller than $(2d + m^2)^{n+1}$.

Example 3.3 In Theorem 3.2 set

$$F(t) \ := \ \mathrm{e}^{-g\sum_x t_x^2}. \tag{27}$$

Then from Theorem 3.2,

$$\mathbb{E}\Big[\mathrm{e}^{-\sum_x g\,(\bar\phi_x\phi_x)^2}\,\bar\phi_a\,\phi_b\Big] \;=\; m^{-2}\int_0^\infty \mathbb{E}\otimes \mathrm{E}_a\Big[\mathrm{e}^{-\sum_x g\,(\bar\phi_x\phi_x+L_x)^2}\Big]\,\mathrm{e}^{-m^2T}\,\mathrm{d}T. \tag{28}$$

The left hand side is known as the *lattice* $|\phi|^4$ *quantum field theory*. According to, for example, Theorem 2 and the last item on page 22 of [35], in

$$\big(\bar\phi_x\phi_x + L_x\big)^2 \;=\; \big(\bar\phi_x\phi_x\big)^2 + L_x^2 + 2\,\bar\phi_x\phi_x L_x \tag{29}$$

the random variables $(\phi_x\bar\phi_x, x\in\Lambda)$ have the same distribution as the local times of loop soup plus a field of independent Γ variables that represent the local times of trivial loops that stay at one point. Thus the first term in (29) represents an interaction between the loops of loop soup and also a self-interaction for each loop. This interaction suppresses all mutual and self-intersections in the loop soup, much like the factor in the Edwards model suppresses self-intersections of a continuous time random walk. In fact, in the second term of (29) we see that as well as the loops there is a random walk with its own Edwards interaction. The third term suppresses all intersections between the loops and the walk. Thus the lattice $|\phi|^4$ quantum field theory is a model of many polymers which mutually repel. This is what Symanzik discovered in [47]. He was advocating this as a way to understand euclidean quantum field theory, but it might also be a good way to study polymers because the correlation inequalities of euclidean quantum field theory are interesting statements for the polymer model.

4 The Free Field, Local Time and Differential Forms

The main result in this section is Proposition 4.3 which is a variant of Theorem 3.2 that will be used to express the susceptibility of the Edwards model in terms of the massless free field. However it requires a conceptual extension of the massless free field and we first prepare the way with a review of differential forms.

4.1 Review of Differential Forms

A good reference for (differential) forms is [5]. As a motivational example let f be a smooth real function on $\mathbb{R}^2$. Given a point (u,v) in $\mathbb{R}^2$ let $(\dot u,\dot v)$ be also a point in $\mathbb{R}^2$, but think of it as a direction one can travel in, starting at (u,v), and call it a *tangent vector* at (u,v). Then define $df = df_{(u,v)}$, as a linear function on the vector

space of tangent vectors at (u, v), by

$$df : (\dot{u}, \dot{v}) \mapsto f_u \dot{u} + f_v \dot{v}, \tag{30}$$

where $f_u = f_u(u, v)$ and $f_v = f_v(u, v)$ are the partial derivatives of f at (u, v). With this definition with f replaced first by the function $\hat{u} : (u, v) \mapsto u$ and second by the function $\hat{v} : (u, v) \mapsto v$ the reader can verify that $df = f_u d\hat{u} + f_v d\hat{v}$. It is usual to leave off the hats as soon as we have understood that u and v are being used in two senses. Thus, by defining the symbols df, du, dv as linear functions on tangent spaces, we gain a precise meaning for

$$df = f_u du + f_v dv. \tag{31}$$

The space of forms at (u, v) *of degree one* is the vector space dual to the vector space of tangent vectors at (u, v). When we omit the phrase "at (u, v)" and say simply that "ω is a form of degree one" then we mean that for each point (u, v) in $\mathbb{R}^2$ $\omega_{(u,v)}$ is a form of degree one at (u, v) and that $(u, v) \mapsto \omega_{(u,v)}$ is smooth. This statement is clearly true for du and dv and it follows that it is true for df since $f_u(u, v)$ and $f_v(u, v)$ are smooth functions of (u, v). It is obvious how to generalise this discussion to $\mathbb{R}^n$.

Given two forms ω and ω' of degree one at a point in $\mathbb{R}^n$, we create a bilinear form $\omega \otimes \omega'$ by setting, for any two tangent vectors e and e',

$$(\omega \otimes \omega')(e, e') = \omega(e)\omega'(e') \tag{32}$$

but the bilinear form with geometrical significance is the antisymmetric tensor product defined by

$$(\omega \wedge \omega')(e, e') \ := \ \omega(e)\omega'(e') - \omega'(e)\omega(e). \tag{33}$$

For example

$$(\mathrm{d}u \wedge \mathrm{d}v)(e, e') \ := \ \mathrm{d}u(e)\mathrm{d}v(e') - \mathrm{d}u(e')\mathrm{d}v(e) \tag{34}$$

is the (signed) area of the parallelogram generated by e and e'. A form of degree two on $\mathbb{R}^n$ is, by definition, an antisymmetric bilinear function of directions assigned to points smoothly. All such objects can be written as $\sum a_{ij} \mathrm{d}u_i \wedge \mathrm{d}u_j$ where a_{ij} are smooth functions on $\mathbb{R}^n$. Similarly forms of degree p on $\mathbb{R}^n$ are antisymmetric p-linear functions of directions assigned smoothly to points. Notice that the degree is at most $p = n$ because there are no antisymmetric functions of higher degree. By definition forms of degree 0 are functions on $\mathbb{R}^n$.

Now the textbooks do something that looks strange but works out well: given two forms ω and ω' of degree $p \neq p'$ we define the direct sum $\omega \oplus \omega'$, which we can do because the set of forms of degree p is a vector space $\Omega^{(p)}$ and so $\omega \oplus \omega'$ is an element of the vector space $\Omega^{(p)} \oplus \Omega^{(p')}$. This amounts to saying that when $\omega \oplus \omega'$ is evaluated on p directions $\dot{u}_1, \ldots, \dot{u}_p$ it equals the evaluation of ω on these

p directions and likewise when $\omega \oplus \omega'$ is evaluated on p' directions it equals the evaluation of ω' on these p' directions, and evaluation on q directions is zero for q not p or p'. Let $\Omega^* = \bigoplus \Omega^{(p)}$. This is the vector space of all forms and we write the addition in this space using $+$ in place of $\oplus$. After defining $\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p}$ by antisymmetrising the tensor product the general form $\omega \in \Omega^*$ on $\mathbb{R}^n$ can be written as

$$\omega = \sum_p \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} \mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p} \tag{35}$$

where the coefficients are smooth functions on $\mathbb{R}^n$. The important fact is that the $\wedge$ product is *associative*, as well as distributive, over $+$. When a is a form of degree zero, that is, a smooth function on $\mathbb{R}^n$, and ω is a general form, then by definition

$$a\omega = a \sum_p \sum_{i_1,\dots,i_p} a_{i_1,\dots,i_p} \mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p} = \sum_p \sum_{i_1,\dots,i_p} a a_{i_1,\dots,i_p} \mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p} \tag{36}$$

which is the pointwise scalar product of the vector space Ω^*. In this case we omit the wedge. Note that

$$a(\omega \wedge \omega') = (a\omega) \wedge \omega' = \omega \wedge (a\omega'). \tag{37}$$

From now on we rarely need to know that a form is a linear combination of antisymmetric multilinear functions of tangent vectors. Most of the time we only use the fact that Ω^* is an algebra with n generators du_i, $i = 1, \dots, n$, that satisfy the relations $\mathrm{d}u_i \wedge \mathrm{d}u_i = 0$ for $i = 1, \dots, n$. An algebra whose generators satisfy such relations is called a *Grassmann algebra*. These relations imply that $\mathrm{d}u_i \wedge \mathrm{d}u_j = -\mathrm{d}u_j \wedge \mathrm{d}u_i$ for all $i, j = 1, \dots, n$. This means that multiplication is commutative for forms of even degree and anticommutative for forms of odd degree. In our case we have a Grassmann algebra *over* $\mathscr{C}^\infty(\mathbb{R}^n)$ which means that for f in $\mathscr{C}^\infty(\mathbb{R}^n)$ and forms ω and $\omega', f(\omega \wedge \omega') = (f\omega) \wedge \omega' = \omega \wedge (f\omega')$. The Grassmann algebra generated by linear functions on a vector space V is called the *exterior algebra* of V.

Given a form ω as in (35) with integrable coefficients we define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} a_{1,2,\dots,n} \mathrm{d}u_1 \dots \mathrm{d}u_n. \tag{38}$$

At first it seems strange that this definition of the integral of a form ignores all coefficients except the one in the top degree form, but it is consistent with our earlier remarks about the meaning of addition in the exterior algebra and the idea that $\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n$ is a multilinear function that assigns volume to parallelopipeds of dimension n. The important point about integration of forms is that the value of the integral is independent of the choice of coordinates for $\mathbb{R}^n$. To understand this try as an exercise case $n = 2$. Let f be a smooth orientation preserving bijection from $\mathbb{R}^2$ to itself. Rewrite $\omega = \sum a_{ij} \mathrm{d}u_i \wedge \mathrm{d}u_j$ in terms of (u_1', u_2') where $u_i = f_i(u_1', u_2')$ by

substituting $du = \sum f_i du'_i$ into ω. Check that $\int \omega$ is the same regardless of whether it is expressed in terms of u or u'. This invariance property of forms is what motivated Cartan to introduce this formalism.

4.2 Gaussian Integrals in Terms of Forms

We make the natural extension of the previous constructions for $\mathbb{R}^n$ to the complex space $\mathbb{C}^\Lambda$. A point in this space is given by $\phi = u + \mathrm{i}v$ where $u = (u_x)_{x\in\Lambda}$ and $v = (v_x)_{x\in\Lambda}$ are in $\mathbb{R}^\Lambda$. For each $x \in \Lambda$ we have a degree one form $\mathrm{d}\phi_x := \mathrm{d}u_x + \mathrm{i}\mathrm{d}v_x$ and

$$\mathrm{d}\phi_x \wedge \mathrm{d}\bar{\phi}_x \;=\; (\mathrm{d}u_x + \mathrm{i}\mathrm{d}v_x) \wedge (\mathrm{d}u_x - \mathrm{i}\mathrm{d}v_x) \;=\; -2\,\mathrm{i}\,(\mathrm{d}u_x \wedge \mathrm{d}v_x). \tag{39}$$

Let $\mathscr{N}(\Lambda)$ denote the exterior algebra over $C^\infty\big(\mathbb{R}^{2\Lambda}\big)$ generated by $\{\mathrm{d}u_x, \mathrm{d}v_x : x \in \Lambda\}$ or, alternatively, $\{\mathrm{d}\phi_x, \mathrm{d}\bar{\phi}_x : x \in \Lambda\}$.

Now let's go back to the definition of the gaussian (complex) measure (13). For F a random variable (form of degree zero)

$$\mathbb{E}\big[F\big] \;=\; \int_{R^{2\Lambda}} \mathrm{e}^{-(\phi, A\bar{\phi})} F \det(A) \prod_{x\in\Lambda} \frac{\mathrm{d}u_x \mathrm{d}v_x}{\pi}. \tag{40}$$

We claim that for any dissipative real matrix A,

$$\mathbb{E}\big[F\big] \;=\; \int_{R^{2\Lambda}} \mathrm{e}^{-(\phi, A\bar{\phi}) - \frac{1}{2\pi\mathrm{i}}(\mathrm{d}\phi\wedge, A\mathrm{d}\bar{\phi})}\; F \tag{41}$$

where

$$\big(\mathrm{d}\phi\wedge, A\,\mathrm{d}\bar{\phi}\big) \;:=\; \sum_{x,y\in\Lambda} \mathrm{d}\phi_x \wedge A_{xy} \mathrm{d}\bar{\phi}_y \tag{42}$$

and we really are claiming that there is no constant of normalisation in (41). For α a function and β a form the exponential $e^{\alpha+\beta}$ is the element of the algebra $\mathscr{N}(\Lambda)$ defined by

$$\mathrm{e}^{\alpha+\beta} \;:=\; \mathrm{e}^{\alpha} \sum_{p\geq 0} \frac{1}{p!}\, \beta^{\wedge p}. \tag{43}$$

The sum is finite because all terms of degree more than $2|\Lambda|$ are zero. For our case

$$\beta = -\frac{1}{2\pi\mathrm{i}}(\mathrm{d}\phi\wedge, A\mathrm{d}\bar{\phi}). \tag{44}$$

So an expansion of the second term in the exponent results in a linear combination of forms of various degrees. Recall that the rule of integration (38) of forms is that the integral of all but the highest-degree form vanishes. The highest degree form is the term $p = |\Lambda|$ in the sum in (43). As an exercise show that this term equals

$$\beta^{2p} = (2\pi i)^{-p} \det A \prod_{x\in\Lambda} (-\mathrm{d}\phi_x \wedge \mathrm{d}\bar\phi_x) \tag{45}$$

and complete the proof of (41) using (39).

Let $c = \frac{1}{2\pi i}$ and define

$$\begin{aligned}(\tau_\Delta)_x := \frac{1}{2}\Big(&\phi_x\,(-\Delta\bar\phi)_x + c\,\mathrm{d}\phi_x \wedge (-\Delta \mathrm{d}\bar\phi)_x \\ &+(-\Delta\phi)_x\,\bar\phi_x + c\,(-\Delta \mathrm{d}\phi)_x \wedge \mathrm{d}\bar\phi_x\Big)\end{aligned} \tag{46}$$

and

$$\tau_x \;:=\; \phi_x\bar\phi_x + c\,\mathrm{d}\phi_x \wedge \mathrm{d}\bar\phi_x. \tag{47}$$

When $(\tau_\Delta)_x$ appears under a sum over x in Λ the second two terms in $(\tau_\Delta)_x$ make the same contribution as the first two terms, which cancels the $1/2$. Let

$$\tau_\Delta(\Lambda) = \sum_{x\in\Lambda} (\tau_\Delta)_x, \qquad \tau(\Lambda) = \sum_{x\in\Lambda} \tau_x. \tag{48}$$

Then, for $A = m^2 - \Delta$, the exponent in (41) is the same as $m^2\tau(\Lambda) + \tau_\Delta(\Lambda)$, as claimed in the following definition.

Definition 4.1 (Super-Expectation) The super-expectation for massive free field is defined for $m^2 > 0$ and for bounded forms F in $\mathscr{N}(\Lambda)$ by

$$\mathbb{E}^{(m)}[F] \;:=\; \int_{\mathbb{R}^{2\Lambda}} \mathrm{e}^{-(\phi, A\bar\phi) - c(\mathrm{d}\phi, A\mathrm{d}\bar\phi)} F \;=\; \int_{\mathbb{R}^{2\Lambda}} \mathrm{e}^{-(m^2\tau + \tau_\Delta)(\Lambda)}\, F \tag{49}$$

where

$$A := m^2 - \Delta, \qquad c = \frac{1}{2\pi \mathrm{i}}. \tag{50}$$

A bounded form is a form whose coefficients are bounded. Here and from now on we are omitting the $\wedge$ between the exponential and F.

The point of this definition is that the right hand side makes sense if F is a form in $\mathscr{N}(\Lambda)$ but when we evaluate the super-expectation of a form of degree 0, in other words a random variable, by (41) the super-expectation is the same as the expectation, so we are defining an extension of the standard expectation to the

algebra of integrable forms. Of course not all properties we are used to remain valid when we are taking the super-expectation of a form. For example we do not have a Jensen inequality unless the form under $\mathbb{E}^{(m)}$ is of degree zero.

4.3 The Local Time Isomorphism and Forms

Recall the definition of τ_x from (47) and let τ denote the sequence of forms (τ_x), where x ranges over Λ. Likewise let $\phi\bar{\phi}$ denote the sequence $(\phi_x\bar{\phi}_x)$ and $c\,\mathrm{d}\phi \wedge \mathrm{d}\bar{\phi}$ denote the sequence $(c\,\mathrm{d}\phi_x \wedge \mathrm{d}\bar{\phi}_x)$. For a smooth function F defined on $\mathbb{R}^\Lambda$, there is a multivariable Taylor expansion

$$F(t+\dot{t}) \sim \sum_p \frac{1}{\alpha!} F^{(\alpha)}(t)\dot{t}^\alpha \tag{51}$$

about the point t in $\mathbb{R}^\Lambda$ in powers $\dot{t}^\alpha = \prod_{x\in\Lambda} \dot{t}_x^{\alpha_x}$ of the components (t_x) of t. Let $t = \phi\bar{\phi}$ and $\dot{t} = c\,\mathrm{d}\phi \wedge \mathrm{d}\bar{\phi}$ in this Taylor series, replacing $\prod_{x\in\Lambda} \dot{t}_x^{\alpha_x}$ by the wedge product $\wedge_{x\in\Lambda}\dot{t}_x^{\alpha_x}$. This product is well defined regardless of the order with which x ranges over Λ because the forms $c\,\mathrm{d}\phi_x \wedge \mathrm{d}\bar{\phi}_x$ are even. Also, the series terminates after finitely many terms because each term is a form of degree $2|\alpha|_1$ and forms of degree larger than $2|\Lambda|$ vanish. Therefore the Taylor expansion with these substitutions defines a form. We denote this form by $F(\tau)$. It is a good notation because $F \mapsto F(\tau)$ is an algebra homomorphism from the algebra of smooth functions into $\mathscr{N}(\Lambda)$. Also the map respects composition $f(F(\tau)) = (f \circ F)(\tau)$. We will not prove these claims, but they are consequences of the uniqueness of the Taylor expansion.

Lemma 4.2 *For any bounded smooth function $F\colon \mathbb{R}^\Lambda \to \mathbb{R}$ with bounded derivatives and for $m^2 > 0$,*

$$\mathbb{E}^{(m)}\big[F(\tau)\big] \;=\; F(0). \tag{52}$$

This also holds for any dissipative matrix A in place of A as defined in (50)*.*

Proof A complete proof is given in [21] but it is instructive to check the claim for the special case $F(\tau) := \exp\{-\sum_x w_x\tau_x\}$ with $w_x \geq 0$. We exhibit dependence on the matrix A by writing $\mathbb{E}_A$. Let W be the diagonal matrix with w_x on the diagonal. Then with this special F we have from Definition 4.1 that $\mathbb{E}_A[F] = \mathbb{E}_{A+W}[1]$ and $\mathbb{E}_{A+W}[1] = 1 = F(0)$ as desired because the super-expectation equals the expectation on forms of degree 0.

Recall that when we proved Theorem 3.2 we also checked the special case of an exponential F and this sufficed to prove the general case because both sides are linear in F. In this algebra of forms context we no longer have monotone class theorems to extend from linear combinations of exponentials to the general case. The idea in [21] is instead to write a general F as the fourier transform of its Fourier

transform to see that it is a limit of linear combinations of exponentials. The same idea is at work in the next isomorphism theorem, which I will call the "$\tau = L$" theorem.

Proposition 4.3 *For a bounded smooth function $F:\mathbb{R}^{\Lambda} \to \mathbb{R}$ with bounded derivatives,*

$$\mathbb{E}^{(m)}\big[F(\tau)\,\bar\phi_a\phi_b\big] \;=\; m^{-2}\,\mathrm{E}_a\big[F(L)\,\mathbb{1}_{\{X_\zeta - = b\}}\big]. \tag{53}$$

Proof For a complete proof see [21, Proposition 2.4]. It is again sufficient to prove it is true for the special case $F(\tau) := \exp\{-\sum_x w_x\tau_x\}$ with $w_x \ge 0$. As in the last proof,

$$\mathbb{E}^{(m)}\big[F(\phi\bar\phi)\,\bar\phi_a\phi_b\big] = \mathbb{E}_A\big[F(\phi\bar\phi)\,\bar\phi_a\phi_b\big] = \mathbb{E}_{A+W}\big[\bar\phi_a\phi_b\big]. \tag{54}$$

The super-expectation on the right coincides with the expectation so the right hand side equals $(A+W)^{-1}_{ab}$ and in the proof of Theorem 3.2 we proved that this equals the right hand side of Proposition 4.3, as desired.

5 Susceptibility as a Gaussian Integral

For the parameters g, ν that appeared in the Edwards model and a new one called z and $X \subset \Lambda$ define

$$\begin{aligned} V_{g,\nu,z;x} &:= g\,\tau_x^2 + \nu\,\tau_x + z(\tau_\Delta)_x, \\ V_{g,\nu,z}(X) &= \sum_{x\in X} V_{g,\nu,z;x}, \\ \phi(X) &= \sum_{x\in X}\phi_x. \end{aligned} \tag{55}$$

Notice that we are starting to omit $\wedge$, for example τ_x^2 is really $\tau_x \wedge \tau_x$. By choosing $F(\tau) = \exp\big[-g\,\tau_x^2(\Lambda) - (\nu - m^2)\,\tau(\Lambda)\big]$ in Proposition 4.3 and using the definition of the super-expectation we find that

$$\chi_\Lambda(g,\nu) \;=\; \int_{\mathbb{R}^{2\Lambda}} \mathrm{e}^{-V_{g,\nu,1}(\Lambda)}\;\bar\phi_a\,\phi(\Lambda). \tag{56}$$

Thus the susceptibility is represented as a $2|\Lambda|$ dimensional integral of forms. Perhaps this does not seem like a very pleasant reward for so much work, but let us see.

These differential form representations came from [39, 44] and particularly [37] in the physics literature, where they are instances of supersymmetry. The results in these papers are expressed in terms of *anticommuting numbers* which

are also known as *ghosts*. Anticommuting numbers are another name for elements of a Grassmann algebra as defined above. The definition of a *Grassmann* integral as the coefficient of the highest degree monomial in a Grassmann algebra is called the Berezin integral after the standard reference [12]. I first encountered the identification of anticommuting numbers with differential forms in [36, 55]. I discussed the isomomorphism between local time and the gaussian field for the complex case with and without Berezin integration in [14].

Lemma 4.2 for the exponential is a special case of the remarkable Duistermaat-Heckman theorem [25, Theorem 4.1 on p. 267]. We shall not need this theorem but to see why it is a more general statement note that it applies to even dimensional spheres. Since we are considering integrals over $\mathbb{R}^{2|\Lambda|}$ of functions that decay at infinity we can add a point at infinity and replace $\mathbb{R}^{2|\Lambda|}$ by an even dimensional sphere. The mathematical literature on the Duistermaat-Heckman theorem makes unfounded assumptions about my education in topology and I found the more informal Sect. 2.2.2 and the standard example in Appendix A of [56] helpful.

5.1 The Most General Split into Gaussian Plus Perturbation

At the end of the last section we found that the susceptibility has the representation

$$\chi_\Lambda(g,\nu) \;=\; \int_{\mathbb{R}^{2\Lambda}} \mathrm{e}^{-V_{g,\nu,1}(\Lambda)}\, \bar{\phi}_a\, \phi(\Lambda). \tag{57}$$

where

$$\begin{aligned} V_{g,\nu,1}(\Lambda) &= \sum_{x\in\Lambda} V_{g,\nu,1;x}, \\ V_{g,\nu,1;x} &= \left(g\tau^2 + \nu\tau + \tau_\Delta\right)_x. \end{aligned} \tag{58}$$

We are now going to try to regard this as an almost gaussian integral because in Theorem 2.2 the hypothesis was that g is positive, but small. Since $V_{g,\nu,1}$ has two quadratic terms a naive attempt is to use them to define the gaussian measure. However, recall that Theorem 2.2 concerns the case where ν is just a little larger than the critical value ν^{c} given by Definition 2.1. Furthermore ν^{c} will turn out to be negative. Therefore we cannot make this naive choice of gaussian measure because $A = \nu^{\mathrm{c}}\mathrm{id} - \Delta$ is not dissipative, which means that the gaussian is not integrable.

Actually we want to choose the gaussian part to be whatever best approximates the long distance behaviour of the model. As an analogy recall that when we want to approximate a sum of n identically distributed centred independent random variables by a gaussian as in the central limit theorem, we have to know (1) to normalise the sum by $\sqrt{n}$, (2) what the variance of the gaussian will be. (1) and (2) are both determined by computing the variance of the sum and this can be done

because the variables are independent. In the present case we cannot guess how to scale ϕ or what the best gaussian will be because there is no obvious independence. Instead, we consider a general split that is parameterised by two parameters called m^2 and z_0 and wait patiently for the renormalisation group to tell us what the values of the parameters should be.

The most general way to split

$$V_{g,\nu,1}(\Lambda) = \big(g\tau^2 + \nu\tau + \tau_\Delta\big)(\Lambda) \tag{59}$$

into a quadratic (gaussian) part and a perturbation can be parameterised by two parameters as follows. First, we introduce a parameter $z_0 > -1$ and split the coefficient 1 implicit in front of τ_Δ as

$$1 = \frac{1}{1+z_0} + \frac{z_0}{1+z_0}. \tag{60}$$

Then we introduce another parameter $m^2 > 0$ to split

$$\nu = \frac{m^2}{1+z_0} + \left(\nu - \frac{m^2}{1+z_0}\right). \tag{61}$$

The reason that $1 + z_0$ is written in the denominators is so that we can get rid of it by rescaling. Introducing the rescaled field $\hat\phi = (1+z_0)^{-1/2}\phi$, we have

$$V_{g,\nu,1}(\phi) = V_{0,m^2,1}(\hat\phi) + V_{g_0,\nu_0,z_0}(\hat\phi), \tag{62}$$

where

$$g_0 = g(1+z_0)^2, \quad \nu_0 = \nu(1+z_0) - m^2. \tag{63}$$

I will call the parameters z_0, m^2 *splitting parameters*.

From the representation (57) of χ_Λ and noticing that the exponent in the super-expectation in Definition 4.1 is $V_{0,m^2,1}(\Lambda)$, we have

$$\chi_\Lambda(g,\nu) \;=\; (1+z_0)\,\mathbb{E}^{(m)}\Big[\mathrm{e}^{-V_{g_0,\nu_0,z_0}(\Lambda)}\,\bar\phi_a\,\phi(\Lambda)\Big]. \tag{64}$$

We obtained this by rewriting the integral in terms of the scaled variable $\hat\phi$ and then renaming $\hat\phi$ back to ϕ. The change of variable does not give a Jacobian factor because it is a form integral. We define

$$\hat\chi_\Lambda(m^2, g_0, \nu_0, z_0) = \mathbb{E}^{(m)}\Big[\mathrm{e}^{-V_{g_0,\nu_0,z_0}(\Lambda)}\,\bar\phi_a\,\phi(\Lambda)\Big], \tag{65}$$

$$\hat\chi(m^2, g_0, \nu_0, z_0) = \lim_{\Lambda\to\mathbb{Z}^d} \hat\chi_\Lambda(m^2, g_0, \nu_0, z_0) \tag{66}$$

As explained in Sect. 2 the limit as Λ increases to $\mathbb{Z}^d$ exists for χ_Λ and partial derivatives with respect to coupling constants can be taken under this limit. Therefore we have proved the following splitting lemma

Lemma 5.1 *Given $\nu \in \mathbb{R}$ and $g > 0$, and given splitting parameters $z_0 > -1$ and $m^2 > 0$, let*

$$g_0 \;=\; g\,(1+z_0)^2 \quad \textit{and} \quad \nu_0 \;=\; \nu\,(1+z_0) - m^2 \tag{67}$$

then

$$\chi_\Lambda(g,\nu) \;=\; (1+z_0)\;\mathbb{E}^{(m)}\Big[\mathrm{e}^{-V_{g_0,\nu_0,z_0}(\Lambda)}\;\bar{\phi}_a\,\phi(\Lambda)\Big]. \tag{68}$$

The infinite volume limit exists and is given by

$$\chi(g,\nu) \;=\; (1+z_0)\,\hat{\chi}(m^2,g_0,\nu_0,z_0). \tag{69}$$

Moreover, for $\nu_0 > \nu^{\mathrm{c}}$,

$$\frac{\partial}{\partial\nu}\chi(g,\nu) \;=\; (1+z_0)^2\,\frac{\partial}{\partial\nu_0}\hat{\chi}(m^2,g_0,\nu_0,z_0). \tag{70}$$

5.2 The Proof of Theorem 2.2

The Edwards model that we are studying contains two parameters ν and g. Our general problem is: given ν, g calculate the susceptibility for ν slightly larger than the critical value $\nu_{\mathrm{c}}(g)$. In the last section we introduced a strategy: show that this model has a gaussian approximation and calculate the susceptibility of this approximation. This strategy was started by splitting the model into (a scaling of) a free field with mass m^2 and a perturbation described by parameters g_0, ν_0, z_0. The four new parameters m^2, ν_0, g_0, z_0, are linked by two relations (67) so we expect to need two more relations to completely specify all of them in terms of the given ν, g. Theorem 5.2 in this section provides these two relations in the form $\nu_0 = \nu_0^{\mathrm{c}}(m^2, g_0)$ and $z_0 = z_0^{\mathrm{c}}(m^2, g_0)$ and it gives enough information to prove our main Theorem 2.2.

In this Theorem 5.2 appears the expected time

$$\mathsf{B}_{m^2} = 8\iint P\big(X(t)=Y(s)\big)e^{-m^2t}e^{-m^2s}\,dt\,ds \tag{71}$$

that two independent simple random walks with killing spend intersecting each other. In $d = 4$ dimensions it can be shown that as m^2 tends to zero,

$$\mathsf{B}_{m^2} \sim 8\,\log m^{-2} \tag{72}$$

with $\mathsf{b} = 1/(2\pi^2)$. In more than four dimensions there is no divergence as $m^2 \to 0$.

Theorem 5.2 (Theorem 4.1 in [10]) *Let $d = 4$, and let $\delta > 0$ be sufficiently small. There are continuous real-valued functions $\nu_0^{\mathrm{c}}, z_0^{\mathrm{c}}$, defined for $(m^2, g_0) \in [0, \delta)^2$ and continuously differentiable in g_0, and there is a continuous function $c(g_0) = 1 + O(g_0)$, such that for all $m^2, g_0, \hat{g}_0 \in (0, \delta)$,*

$$\hat{\chi}\left(m^2, g_0, \nu_0^{\mathrm{c}}(m^2, g_0), z_0^{\mathrm{c}}(m^2, g_0)\right) = \frac{1}{m^2}, \tag{73}$$

$$\frac{\partial \hat{\chi}}{\partial \nu_0}\left(m^2, g_0, \nu_0^{\mathrm{c}}(m^2, g_0), z_0^{\mathrm{c}}(m^2, g_0)\right) \sim -\frac{1}{m^4}\frac{c(\hat{g}_0)}{(\hat{g}_0 B_{m^2})^{\gamma}} \text{ as } (m^2, g_0) \to (0, \hat{g}_0). \tag{74}$$

The functions $\nu_0^{\mathrm{c}}, z_0^{\mathrm{c}}$ obey

$$\begin{aligned} \nu_0^{\mathrm{c}}(m^2, 0) =& z_0^{\mathrm{c}}(m^2, 0) = 0, \\ \frac{\partial \nu_0^{\mathrm{c}}}{\partial g_0}(m^2, g_0) =& O(1), \\ \frac{\partial z_0^{\mathrm{c}}}{\partial g_0}(m^2, g_0) =& O(1), \end{aligned} \tag{75}$$

where $O(1)$ means that these derivatives are bounded on their whole domain by constants uniform in (m^2, g_0).

Remark 5.3 In the standard theory of renormalisation (73) is not a theorem, but merely the definition of m^2. In our work m^2 has been defined as the mass in a free field and we will instead use the renormalisation group to prove that $\nu_0^{\mathrm{c}}, z_0^{\mathrm{c}}$ exist such that at large scales this system becomes this free field.

Proof (of Theorem 2.2) Define the map

$$A : (m^2, g_0) \mapsto \left(m^2, g_0, \nu_0^{\mathrm{c}}(m^2, g_0), z_0^{\mathrm{c}}(m^2, g_0)\right) \tag{76}$$

with the domain $(m^2, g_0) \in [0, \delta)^2$ specified in the theorem for ν_0^{c} and z_0^{c}. We eliminate m^{-2} in (74) using (73). The elimination includes the m^2 in B_{m^2} using (72). We obtain

$$\left[\frac{\partial \hat{\chi}}{\partial \nu_0}\right] \circ A \sim -(\hat{\chi} \circ A)^2 \frac{c(\hat{g}_0)}{\left(\hat{g}_0 8\,\mathsf{b} \log(\hat{\chi} \circ A)\right)^{\gamma}}, \tag{77}$$

We define another map $B : (m^2, g_0, \nu_0, z_0) \mapsto (g, \nu)$ by solving (67) explicitly. Let $C = B \circ A$. By Lemma 5.1, we obtain the following equation for $\chi = \chi(g, \nu)$.

$$\begin{aligned} \left[\frac{\partial \chi}{\partial \nu}\right] \circ C &\sim -(\chi \circ C)^2 \frac{c(\hat{g}_0)}{\left(8\mathsf{b} g_0 \log(\chi \circ C) - 8\mathsf{b} g_0 \log(1 + z_0^{\mathrm{c}})\right)^{\gamma}} \\ &\sim -(\chi \circ C)^2 \frac{c(\hat{g}_0)}{\left(8\mathsf{b} g_0 \log(\chi \circ C)\right)^{\gamma}}. \end{aligned} \tag{78}$$

The $\sim$ allows us to omit the term involving $\log(1 + z_0^{\mathrm{c}})$ term because (75) implies it is bounded as $(m^2, g_0) \to (0, \hat{g}_0)$, whereas $\chi \circ C$ diverges. The divergence of $\chi \circ C$ follows from Lemma 5.1 and (73), which together assert that

$$\chi \circ C = \frac{1 + z_0^{\mathrm{c}}}{m^2}, \tag{79}$$

noting that (75), and decreasing δ if necessary, implies that $1 + z_0^{\mathrm{c}}$ does not vanish as $m^2 \downarrow 0$.

Proposition 4.2 (ii) in [10] states that the inverse $C^{-1} : (g, \nu) \mapsto (m^2, g_0)$ exists, is right-continuous in ν for g fixed and is defined on a domain $0 < g < \delta_1$, $\nu_{\mathrm{c}}(g) \leq \nu < \nu_{\mathrm{c}}(g) + \delta_1$. In the notation of Bauerschmidt et al. [10], the two components of C^{-1} are called $\tilde{m}^2, \tilde{g}$ and they are written as functions of (g, ϵ) where $\epsilon = \nu - \nu_{\mathrm{c}}(g)$. By the definition of ν_{c} in Definition 2.1 and the comment below this definition, $\chi(g, \nu)$ is finite for $\nu \in (\nu_{\mathrm{c}}(g), \infty)$ and diverges as $\nu \downarrow \nu_{\mathrm{c}}(g)$. Therefore, by (79), $m^2 = 0$ when $\nu = \nu_{\mathrm{c}}(g)$. By the right continuity of C^{-1}, $\nu \downarrow \nu_0^{\mathrm{c}}$ with g fixed implies $(m^2, g_0) \to (0, \hat{g}_0)$. Therefore (78) simplifies to

$$\frac{\partial \chi}{\partial \nu} \sim -\chi^2 \frac{c(\hat{g}_0)}{\big(8\mathsf{b} g_0 \log \chi\big)^{\gamma}}, \quad \nu \downarrow \nu_0^{\mathrm{c}} \text{ with } g \text{ fixed}, \tag{80}$$

where $\hat{g}_0$ is the g component of $C^{-1}(g, 0)$.

We fix $g < \delta_1$ and define $F(\epsilon) = \frac{1}{\chi(g,\nu)}$ for $\nu = \nu_{\mathrm{c}}(g) + \epsilon$ with $\epsilon \in (0, \delta_1)$ and we set $F(0) = 0$. By dividing (80) by χ^2 (80) becomes, for $\epsilon \downarrow 0$,

$$\frac{dF}{d\epsilon} \sim \frac{1}{A_g \big(\log F^{-1}\big)^{\gamma}}, \quad A_g = \frac{(\tilde{g}_0(g, 0)\mathsf{b})^{\gamma}}{c_0(g)} \tag{81}$$

and this differential relation can be easily integrated [10, Lemma 4.3] to show that

$$F(\nu_c + \varepsilon) \sim A_g^{-1} \varepsilon (-\log \varepsilon)^{-\gamma}. \tag{82}$$

Recalling that $F = \chi^{-1}$ this is the claim in Theorem 2.2.

5.3 *The Susceptibility in Terms of Super-Convolution*

Our main result Theorem 2.2 has been reduced to Theorem 5.2. To prepare for the proof of Theorem 5.2, we put the susceptibility into a form that suits the renormalisation group which will be introduced in the next sections. The conclusion of this section is Proposition 5.5. The proof of this Proposition contains the important idea of using a translation to approximately evaluate a generating function. This is part of the "evaluation as if gaussian" strategy that started in

Sect. 5.1. The use of "tilting" in the theory of large deviations is also an instance of this strategy.

This section and the renormalisation group use super-convolution. In order to motivate the definition of super-convolution, recall that the convolution of a function f with a probability measure P is the function $x \mapsto \int f(x+z)\,P(dz)$. In order to think of this as an operation that transforms f into a new function g we define a homomorphism θ from the algebra of functions of one variable to the algebra of functions of two variables by: for $x \mapsto f(x)$ let $\theta f : (x,z) \mapsto f(x+z)$. Then the convolution can be written as $f \mapsto \mathbb{E}^{(z)}\theta f$.

Definition 5.4 Define the algebra homomorphism $\theta\colon \mathscr{N}(\Lambda) \to \mathscr{N}(\Lambda \sqcup \Lambda)$ to be the map that replaces ϕ by $\phi + \zeta$ and $\mathrm{d}\phi$ by $\mathrm{d}\phi + \mathrm{d}\zeta$. Then the *super-convolution* of a form $F \in \mathscr{N}(\Lambda)$ by the super-expectation $\mathbb{E}^{(m)}$ is given by

$$F \mapsto \mathbb{E}^{(m,\zeta)}\big[\theta F\big], \tag{83}$$

where $\mathbb{E}^{(m,\zeta)}$ acts only on ζ; (Therefore, in Definition 4.1 rename ϕ to ζ and then replace F by θF.)

Let F be an element of $\mathscr{N}(\Lambda)$. Recall that this means that F is a form whose coefficients are functions of ϕ and $\phi \in \mathbb{C}^\Lambda$. The directional derivative of F with respect to ϕ in the direction $f \in \mathbb{C}^\Lambda$ is defined by replacing ϕ by $\phi + zf$, $\bar\phi$ by $\bar\phi + \bar z\bar f$ and evaluating $(\partial/\partial z)F$ at $z = 0$ by the rules $\partial\bar z/\partial z = 0$ and $\partial z/\partial z = 1$. These rules follow from the definition $\partial/\partial z = \frac{1}{2}(\partial/\partial x - \mathrm{i}\partial/\partial y)$ when $z = x + \mathrm{i}y$. The directional derivative of a differential is zero because the replacement of ϕ by $\phi + zf$ is a change of variable and f does not depend on ϕ so $d(\phi + zf) = d\phi$. Similarly there is also the directional derivative of F with respect to $\bar\phi$ in the direction $\bar f$.

Let 1 denote the function in $\mathbb{C}^\Lambda$ which is the constant function $1_x = 1$ for all $x \in \Lambda$. In the next Proposition $D^2F(0;1,1)$ denotes the result of taking two directional derivative of F with respect to ϕ in the direction 1 and then setting $\phi = 0$ including $d\phi = 0$. (In [10] we used the notation $D^2F(0,0;1,1)$ since ϕ and $d\phi$ are both set to zero.)

Recall that the finite volume susceptibility $\hat\chi_\Lambda = \hat\chi_\Lambda(m^2, g_0, \nu_0, z_0)$ was defined in (65). We repeat the definition here in order to introduce some new notation,

$$\hat\chi_\Lambda \;=\; \mathbb{E}^{(m)}\Big[Z_0\,\bar\phi_a\,\phi(\Lambda)\Big], \quad Z_0 = \mathrm{e}^{-V_0(\Lambda)}, \quad V_0(\Lambda) = V_{g_0,\nu_0,z_0}(\Lambda). \tag{84}$$

Proposition 5.5 *Let m^2, g_0, ν_0, z_0 be real numbers with g_0, m^2 positive and $z_0 > -1$. Then*

$$\hat\chi_\Lambda(m^2, g_0, \nu_0, z_0) \;=\; \frac{1}{m^2} + \frac{1}{m^4\,|\Lambda|}\,D^2F(0;1,1), \tag{85}$$

where $F = \mathbb{E}^{(m,\zeta)}\big[\theta Z_0\big]$.

To prepare for the proof of this result we first discuss generating functions in this context. Given an *external field* $J : \Lambda \to \mathbb{C}$, we write

$$(J, \bar{\phi}) = \sum_{x \in \Lambda} J_x \bar{\phi}_x, \quad (\bar{J}, \phi) = \sum_{x \in \Lambda} \bar{J}_x \phi_x. \tag{86}$$

Recall that 1 denotes the function in $\mathbb{C}^\Lambda$ that is identically one. By translation invariance,

$$\hat{\chi}_\Lambda = |\Lambda|^{-1} \mathbb{E}^{(m)} \Big[(1, \bar{\phi})(1, \phi) Z_0 \Big]. \tag{87}$$

We define the *generating function* $\Sigma : \mathbb{C}^\Lambda \to \mathbb{C}$ by

$$\Sigma(J, \bar{J}) = \mathbb{E}^{(m)} \Big[e^{(J,\bar{\phi}) + (\phi, \bar{J})} Z_0 \Big]. \tag{88}$$

By taking two directional derivatives, one with respect J in the direction 1, the second with respect to $\bar{J}$ in the direction 1 and setting $J = 0$ we generate a factor $(1, \bar{\phi})(1, \phi)$ and so we have

$$\hat{\chi}_\Lambda = |\Lambda|^{-1} D^2_{\bar{J},J} \Sigma(0; 1, 1), \tag{89}$$

where $D^2_{\bar{J},J}$ indicates two directional derivatives with respect to J and $\bar{J}$, the argument 0 means the derivative is at $J = \bar{J} = 0$ and the two arguments 1 indicate the directions. The evaluation of $\hat{\chi}_\Lambda$ now becomes reduced to the evaluation of $D^2_{\bar{J},J} \Sigma$ on the right-hand side of (89). Here is where the above mentioned strategy of evaluation as if gaussian ($V_0 = 0$) comes into play. Recall that gaussian integrals are evaluated by change of variables by an optimal translation that centres the gaussian. By using such a translation $\phi = \zeta + H$, where ζ is the new integration variable, we obtain

$$\Sigma(J, \bar{J}) = e^{(J, C\bar{J})} \Big[\mathbb{E}^{(m,\zeta)} \big[\theta Z_0 \big] \Big]_{|_{\phi = CJ, d\phi = 0}}, \tag{90}$$

where $C = (-\Delta + m^2)^{-1}$. In more detail, with $A = -\Delta + m^2 = C^{-1}$ and referring to Definition 4.1,

$$\begin{aligned} &\sum_{x \in \Lambda} \big(\tau_{\Delta,x} + m^2 \tau_x \big) - (J, \bar{\phi}) - (\phi, \bar{J}) \\ &\quad = (\zeta, A\bar{\zeta}) + c(d\zeta, \wedge A d\bar{\zeta}) \\ &\qquad - (J, \bar{\zeta}) - (\bar{J}, \zeta) + (H, A\bar{\zeta}) + (\zeta, A\bar{H}) \\ &\qquad\quad + (H, A\bar{H}) - (J, \bar{H}) - (\bar{J}, H) \\ &\quad = (\zeta, A\bar{\zeta}) + c(d\zeta, \wedge A d\bar{\zeta}) - (J, C\bar{J}), \end{aligned} \tag{91}$$

where the last line is obtained by choosing H to make the terms in the second line sum to zero. This happens when $H = CJ$, $\bar{H} = C\bar{J}$. Since H does not depend on ϕ, $d\phi = d(\zeta + H) = d\zeta$ and so the form part is as written. The formula (90) follows immediately. Notice that the translation also changes ϕ in Z_0 to $\zeta + CJ$ and $d\phi$ to $d\zeta$. This is implemented in (90) by θ which changes ϕ to $\phi + \zeta$ followed by evaluation at $\phi = CJ$ and $d\phi = 0$ after taking the super-expectation over ζ. Since our strategy was based on the hope that our splitting into gaussian and interaction is such that $V_0(\Lambda)$ can be neglected, $Z_0 = e^{-V_0(\Lambda)}$ should not be very dependent on this translation and so this calculation should be a good way to "almost" evaluate the generating function $\Sigma(J, \bar{J})$.

Proof (of Proposition 5.5) We use (89) followed by (90) to obtain

$$\hat{\chi}_\Lambda(m^2, g_0, \nu_0, z_0) = |\Lambda|^{-1} D^2_{\bar{J},J} \left(e^{(J,C\bar{J})} \Big[\mathbb{E}^{(m,\zeta)} \big[\theta Z_0 \big] \Big]_{|_{\phi = CJ, d\phi = 0}} \right) (0; 1, 1). \tag{92}$$

The desired result is obtained by evaluating the directional derivatives in the direction 1 noting that $C1 = (m^2 - \Delta)^{-1} 1 = m^{-2}$.

6 The Renormalisation Group

The renormalisation group is a method to evaluate $D^2 F(0; 1, 1)$ in the right hand side of Proposition 5.5. From this point on these notes become a selection of topics from the six papers that collectively comprise the proof of Theorem 2.2. There are many references to these papers, but I suggest that anyone who wishes to continue reading ignore these references until some of the general ideas emphasised by these notes start to come into focus. Many of the important ideas are also discussed in great detail in [15] for much simpler problems.

As discussed in Sect. 1 the renormalisation group can be defined in different ways, which are different interpretations of what the phrase "integrating out fluctuations" should mean. In our case we are going to write the gaussian field ϕ as a sum $\zeta_1 + \cdots + \zeta_N$ of independent gaussian fields ζ_j and then integrate over ζ_1, followed by ζ_2, and so on. First we will discuss the representation in distribution of ϕ as $\zeta_1 + \cdots + \zeta_N$. This depends on the following theorem [11, Sect. 6.1] about the inverse $(m^2 - \Delta)^{-1}$ where $\Delta = \Delta_\Lambda$ is the finite difference Laplacian for the torus Λ defined in (1). Recall that Λ has period L^N.

Theorem 6.1 (Finite Range Decomposition) *For $m^2 > 0$ let $C = (m^2 - \Delta_\Lambda)^{-1}$, regarded as a $\Lambda \times \Lambda$ matrix. There exist positive-definite $\Lambda \times \Lambda$ matrices $C_j = C_j(m^2)$ defined for $j = 1, 2, \ldots, N-1$ and $m^2 \geq 0$, and there exists $C_{N,N} =$*

$C_{N,N}(m^2)$ defined for $m^2 > 0$, such that

1. $(m^2 - \Delta_\Lambda)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}$,
2. *For* $j = 1, \dots, N-1$, $C_{j;x,y} = 0$, *if* $|x-y| \geq \frac{1}{2}L^j$,
3. $|C_{j;x,y}| \leq c\big(1 + m^2 L^{2j}\big)^{-k} L^{-2(j-1)}$, $j = 1, \dots, N-1$.

Finite difference derivatives up to any fixed order p are also bounded, according to

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \leq c(1 + m^2 L^{2(j-1)})^{-k} L^{-(j-1)(2+(|\alpha|_1+|\beta|_1))},$$

where $c = c(p, k, \bar{m}^2)$ is independent of m^2, j, L.

The matrix entries $C_{j;x,y}$ are Λ independent functions of $z \in \mathbb{Z}^d$ where $(x, y) \mapsto z$ is defined for x, y in Λ by setting z equal to the minimal $\mathbb{Z}^d$ representative of $x - y$ in Λ. Part (3) also holds for $j = N$ provided $m^2 L^{2N}$ is bounded away from zero.

Since $C_{j;x,y} = 0$ when the representative z has $|z| \geq L^j/2$ it does not matter how the minimal representative is chosen when there is more than one. If a covariance G can be written as a sum of positive-definite matrices as in (1, 2) then we say that G has a *finite range decomposition*. Quite a large class of covariances are shown to have finite range decompositions in [4, 7, 18, 20], where the proof in [7] is particularly economical. Exactly what class of covariances have finite range decompositions is an open question of great interest to us.

Property (3) is special to the covariance $(m^2 - \Delta)^{-1}$. It expresses in a crude way that the covariances C_j are approximately scalings of each other for j such that $m^2 L^j \ll 1$. When j is too large for this to hold the covariances become small because these covariances have to be consistent with the exponential decay that C_{xy} is known to have for $|x - y| \gg m^{-1}$. In fact in the finite range decomposition discovered in [20] they are double exponentially small in $j \leq N$ such that $m^2 L^j \gg 1$.

Theorem 6.2 (ϕ as Sum of Increments) *For each covariance C_j of Theorem 6.1, let $\mathbb{E}_j$ be the super-expectation given by Definition 4.1, but with $A = C_j^{-1}$, and denote by ζ_j the associated gaussian field. Let $Z : \phi \mapsto Z(\phi)$ be a bounded function defined on $\mathbb{C}^\Lambda$. Then*

$$\mathbb{E}^{(m)}\big[Z(\phi)\big] = \mathbb{E}_N \mathbb{E}_{N-1} \dots \mathbb{E}_1\big[Z(\zeta_1 + \dots + \zeta_N)\big]. \tag{93}$$

Furthermore, this also holds for smooth bounded forms $Z : (\phi, d\phi) \mapsto Z(\phi, d\phi)$ in $\mathcal{N}(\Lambda)$, with the understanding that $d\phi$ becomes $d\zeta_1 + \dots + d\zeta_N$ on the right hand side.

For the case where Z is a function on $\mathbb{C}^\Lambda$, we have a form of degree zero which is just another name for a random variable. We have seen that in this case the super-expectation is the usual expectation of probability; the theorem is just a restatement of the standard fact that the distribution of a sum of independent gaussian random variables $\zeta_1, \dots, \zeta_N$ is also gaussian with covariance equal to the

sum of the covariances of the ζ's. Thus the new content in this theorem is in the case where Z is a form. For a proof of an equivalent result see [16, Proposition 2.6].

6.1 Progressive Integration

Now we re-organise Theorem 6.2 into an iterative procedure where the fields $\zeta_1, \dots, \zeta_N$ are successively integrated out. The process is: for each $j = 1, \dots, N$, replace ϕ by $\phi + \zeta_j$, fix ϕ and integrate over ζ_j with respect to the gaussian probability distribution with covariance C_j. The accumulation of all these replacements is the same as replacing ϕ by $\phi + \zeta_1 + \cdots + \zeta_N$. Setting $\phi = 0$ and taking the $\mathbb{E}_j$ expectations reproduces the right hand side of Theorem 6.2. Since in this procedure we only deal with one fluctuation field ζ_j at a time we often write ζ in place of ζ_j, using the subscript j on $\mathbb{E}_j$ to show which field is being integrated out.

Recall the discussion around Definition 5.4 where we have defined the algebra homomorphism $\theta : \mathscr{N}(\Lambda) \to \mathscr{N}(\Lambda \sqcup \Lambda)$ to be the map that replaces ϕ by $\phi + \zeta$ and $\mathrm{d}\phi$ by $\mathrm{d}\phi + \mathrm{d}\zeta$. Returning to the context of Proposition 5.5, we define a sequence of forms in $\mathscr{N}(\Lambda)$ by the recursion

$$\begin{aligned} Z_0 &= \mathrm{e}^{-V_0(\Lambda)}, \\ Z_{j+1} &= \mathbb{E}_{j+1}\big[\theta Z_j\big], \quad j = 0, 1, \dots, N-1. \end{aligned} \tag{94}$$

We will keep referring back to this important sequence because our goal is to calculate Z_N which is the function F in Proposition 5.5 that determines the susceptibility $\hat{\chi}_\Lambda$ that appears in Theorem 5.2.

6.2 First Order Perturbation Theory

The recursion (94) defines forms Z_j. We will now be interested in how the functional dependence of Z_j on arguments $\phi, d\phi$ changes under the map $Z_j \mapsto Z_{j+1}$. In this section we examine this in a preliminary way using perturbation theory. Perturbation theory is purely algebraic: we fix an order p and regard Z_j as a power series in V_j where

$$V_j = g_j \tau^2 + L^{-2j} \mu_j \tau + z_j \tau_\Delta. \tag{95}$$

Then we calculate modulo V_j^{p+1}, or equivalently, modulo monomials of degree $p+1$ in g_j, z_j, μ_j. This is called *pth order perturbation theory*. Notice that we have started to write V_j in terms of μ_j instead of ν_j, where

$$\nu_j = L^{2j} \mu_j. \tag{96}$$

Also we are writing the parameters in the order g_j, z_j, μ_j instead of the former order g_j, μ_j, z_j because of a triangular property of some forthcoming equations for these coupling constants.

The proof of Theorem 5.2 requires second order perturbation theory, that is $p = 3$, but first order is easier and shows why coupling constants are scale dependent, so we do the first order calculation in detail and then summarise the conclusions of second order calculations. For first order perturbation theory, let g_j, μ_j, z_j be given real numbers, subject to the usual integrability constraints $g_j > 0, z_j > -1$. If $Z_j = \mathrm{e}^{-V_j(\Lambda)}$ modulo V_j^2, then I claim that, modulo V_j^2 (or V_{j+1}^2),

$$Z_{j+1} = \mathrm{e}^{-V_{j+1}(\Lambda)}, \tag{97}$$

where V_{j+1} is defined with coupling constants

$$\begin{aligned} g_{j+1} &= g_j, \\ \mu_{j+1} &= L^2\mu_j + 2L^{2j+2}C_{j+1,0,0}\, g_j, \\ z_{j+1} &= z_j. \end{aligned} \tag{98}$$

Here we see a very important idea: there is a scale dependent coupling constant $j \mapsto \mu_j$ determined by the above recursion. According to part (3) of Theorem 6.1, for $m^2 = 0$, we have $L^{2j+2}C_{j+1,0,0} = O(1)$. In fact in Proposition 6.1 of [11] we prove that $L^{2j+2}C_{j+1,0,0}$ has a positive limit as $j \to \infty$ so this recursion of μ_j becomes independent of j for j large.

Proof (of Claim) Modulo V^2 we have

$$\begin{aligned} \mathbb{E}_{j+1}\big[\theta Z_j\big] &= \mathbb{E}_{j+1}\Big[\big(1 - V_j(\Lambda)\big)\Big] \\ &= 1 - \mathbb{E}_{j+1}\big[V_j(\Lambda)\big] = \mathrm{e}^{-\mathbb{E}_{j+1}[V_j(\Lambda)]}. \end{aligned} \tag{99}$$

Therefore it is sufficient to prove that

$$\mathbb{E}_{j+1}\big[\theta V_j(\Lambda)\big] \;=\; V_{j+1}(\Lambda). \tag{100}$$

Let $Q := \mathbb{E}_{j+1}\theta\tau_x^2$. We start by calculating Q. By the definition (47) of τ,

$$\tau_x^2 = (\phi_x\bar{\phi}_x)^2 + 2\,\phi_x\bar{\phi}_x\, c\, d\phi_x \wedge d\bar{\phi}_x. \tag{101}$$

Applying θ replaces ϕ by $\phi + \zeta$ in this expression. Therefore the terms $Q(0)$ in Q that do not depend on ϕ are

$$Q(0) = \mathbb{E}_{j+1}\big((\zeta_x\bar{\zeta}_x)^2 + 2\,\zeta_x\bar{\zeta}_x\, c\, d\zeta_x \wedge d\bar{\zeta}_x\big) \tag{102}$$

which is the ζ expectation of τ_x^2 with ϕ replaced by ζ. By the last line in Lemma 4.2 applied with $F(\tau) = \tau_x^2$ this equals zero. Next we calculate the terms $Q^{(2)}$ in Q that are of degree 2 in $\phi, d\phi$. From θ applied to (101) and using gauge invariance as in (16) to escape writing some terms,

$$\begin{aligned} Q^{(2)} &= \mathbb{E}_{j+1}\big(4\phi_x\bar\phi_x\zeta_x\bar\zeta_x + 2\,\phi_x\bar\phi_x\, c\, d\zeta_x \wedge d\bar\zeta_x + 2\,\zeta_x\bar\zeta_x\, c\, d\phi_x \wedge d\bar\phi_x\big) \\ &= \mathbb{E}_{j+1}\big(2\phi_x\bar\phi_x\zeta_x\bar\zeta_x + 2\phi_x\bar\phi_x[\zeta_x\bar\zeta_x + \, c\, d\zeta_x \wedge d\bar\zeta_x] + 2\,\zeta_x\bar\zeta_x\, c\, d\phi_x \wedge d\bar\phi_x\big) \\ &= \mathbb{E}_{j+1}\big(2\phi_x\bar\phi_x\zeta_x\bar\zeta_x + 2\,\zeta_x\bar\zeta_x\, c\, d\phi_x \wedge d\bar\phi_x\big) = 2\tau_x\,\mathbb{E}_{j+1}\big(\zeta_x\bar\zeta_x\big)\,. \end{aligned} \tag{103}$$

In the third equality we used the fact that the expectation of the terms in square brackets is zero by the last line in Lemma 4.2 applied with $F(\tau) = \tau_x$. The last equality holds by the definition (47) of τ. Since the terms $Q^{(4)}$ of degree 4 in $\phi, d\phi$ in Q are τ_x^2 and since $Q = Q(0) + Q^{(2)} + Q^{(4)}$ these formulas for $Q(0)$, $Q^{(2)}$ and $Q^{(4)}$ imply that

$$\mathbb{E}_{j+1}\big[\theta\tau_x^2\big] = \tau_x^2 + 2\,\tau_x\,\mathbb{E}_{j+1}\zeta_x\bar\zeta_x. \tag{104}$$

By a much shorter version of the same calculation $\mathbb{E}_{j+1}\theta\tau_x = \tau_x$. By summing these formulas over x in Λ and using the definition $C_{j+1,x,x} = \mathbb{E}_{j+1}\zeta_x\bar\zeta_x$ of the covariance and translation invariance of the covariance we have proved (100).

6.3 Second Order Perturbation Theory

A much more efficient method for calculating in perturbation theory is used in [11]. It is based on [16, Proposition 2.6] and [11, Lemma 5.2]. In second order perturbation theory, where one calculates modulo V^3, the function $V \mapsto e^{-V(\Lambda)}$ does not retain its form under the map $\mathbb{E}_{j+1}\theta$, and we have modify it so that it is form invariant. In [11, (3.21)] we construct a sequence of explicit functions $V \mapsto W_j(V, \Lambda)$ that are quadratic in V and take values in $\mathscr{N}(\Lambda)$. There is a map [11, (4.11)]

$$\varphi^{(0)}_{\mathrm{pt},j} : \mathbb{R}^3 \to \mathbb{R}^3 \tag{105}$$

such that the function

$$I_j \,:\, (V, \Lambda) \mapsto e^{-V(\Lambda)}(1 + W_j(V, \Lambda)) \tag{106}$$

satisfies, modulo V^3,

$$\mathbb{E}_{j+1}\big[\theta I_j(V, \Lambda)\big] = I_{j+1}\big(V \circ \varphi^{(0)}_{\mathrm{pt},j}, \Lambda\big), \tag{107}$$

where we regard V as a function of its coupling constants so that the composition $V \circ \varphi^{(0)}_{\mathrm{pt},j}$ makes sense. Equation (107) says that integrating out the fluctuation field ζ_{j+1} in $\theta I_j(V, \Lambda)$ is equivalent to changing the three coupling constants g, μ, z in V according to the map $\varphi^{(0)}_{\mathrm{pt},j}$. This is what we mean when we say that I_j retains its form at second order under the map $\mathbb{E}_{j+1}\theta$.

We will not need the explicit formula [11, (4.11)] for the map $\varphi^{(0)}_{\mathrm{pt},j}$ because it is conjugate to a simpler map. According to [11, Proposition 4.3], for each scale $j = 0, \dots, N-1$ there is an explicit quadratic map $T_j : \mathbb{R}^3 \to \mathbb{R}^3$, which is almost the identity map,

$$T_0(V) = V, \quad T_j(V) = V + O(\|V\|^2). \tag{108}$$

Therefore it is invertible near the origin. Furthermore, modulo terms that only contribute to the discarded $O(V^3)$,

$$\varphi^{(0)}_{\mathrm{pt},j} = T_{j+1}^{-1} \circ \bar{\varphi}_j \circ T_j, \tag{109}$$

where $\bar{\varphi}_j : \mathbb{R}^3 \to \mathbb{R}^3$ defines, by $(\bar{g}_{j+1}, \bar{z}_{j+1}, \bar{\mu}_{j+1}) = \bar{\varphi}_j(\bar{g}_j, \bar{z}_j, \bar{\mu}_j)$, the recursion

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \tag{110}$$

$$\bar{z}_{j+1} = \bar{z}_j - \theta_j \bar{g}_j^2, \tag{111}$$

$$\bar{\mu}_{j+1} = L^2 \bar{\mu}_j (1 - \gamma \beta_j \bar{g}_j) + \eta_j \bar{g}_j - \xi_j \bar{g}_j^2 - \pi_j \bar{g}_j \bar{z}_j, \tag{112}$$

for scale dependent coupling constants.

The coefficients $\beta_j, \theta_j, \eta_j, \xi_j, \pi_j$ are real coefficients defined precisely in [11, (3.24), (3.27), (3.28)]. These coefficients, and also those of the transformations T_j, are independent of the side L^N of the torus. This means that all the recursions obtained as N varies are consistent and it is not necessary to know what N is so long as N is larger than whatever scale j one is considering. Thus there is a formal infinite volume recursion where j runs over all integers $j \geq 0$, not just $j < N$. To put it another way, we can use this infinite volume recursion for $j < N$ since it coincides with the N recursion until $j = N$.

The sequence $(\beta_j)_{0 \leq j < \infty}$ plays a key role in the analysis. These are positive numbers that have a non zero limit when $m^2 = 0$ and the dimension is 4. For $m^2 > 0$ they decay faster than exponentially to zero, but the decay does not set in until the scale j is large enough that $L^{2j} m^2$ is roughly 1. We make the following definitions [9, (1.7)]. Given $\Omega > 1$, we define a scale j_Ω as the first scale where the exponential decay sets in,

$$j_\Omega = \inf\{k \geq 0 : |\beta_j| \leq \Omega^{-(j-k)_+} \beta_{\max} \text{ for all } j \geq 0\}, \tag{113}$$

where $\beta_{\max} = \max_j \beta_j$. The infimum of the empty set is defined to equal ∞, e.g., if $\beta_j = b$ for all j then $j_\Omega = \infty$. The choice of Ω is arbitrary; let $\Omega = 10$. We also

define

$$\chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{114}$$

Thus the sequence χ_j is a normalised version of β_j which equals one for $j \le \Omega$ and decays exponentially for $j > \Omega$.

Unlike the recursion defined by $\varphi^{(0)}_{\mathrm{pt},j}$ the $\bar\varphi$ recursion (110)–(112) is triangular: the $\bar g$-equation does not depend on $\bar z$ or $\bar\mu$, the $\bar z$-equation depends only on $\bar g$, and the $\bar\mu$-equation depends both on $\bar g$ and $\bar z$. Therefore we can solve the recursion one equation at a time and thereby prove the following Proposition, which follows from [9, Proposition 1.2]. In its statement, z_∞ denotes the limit $z_\infty = \lim_{j\to\infty} z_j$, and similarly for μ_∞.

Proposition 6.3 *If $\bar g_0 > 0$ is sufficiently small, then there exists a unique global solution $(\bar g_j, \bar z_j, \bar\mu_j)_{j\in\mathbb{N}_0}$ to the recursion (110)–(112) with initial condition $\bar g_0$ and final condition $(\bar z_\infty, \bar\mu_\infty) = (0,0)$. This flow satisfies, for any real $p \in [1,\infty)$,*

$$\chi_j \bar g_j^p = O\left(\frac{\bar g_0}{1+\bar g_0 j}\right)^p, \quad \bar z_j = O(\chi_j \bar g_j), \quad \bar\mu_j = O(\chi_j \bar g_j), \tag{115}$$

with constants independent of j_Ω and $\bar g_0$, and dependent on p in the first bound. Furthermore, $(\bar g_j, \bar z_j, \bar\mu_j)$ is continuously differentiable in the initial condition $\bar g_0$ and continuous in the mass parameter $m^2 \ge 0$, for every $j \in \mathbb{N}_0$.

The proposition shows that when $m^2 = 0$ the coupling constants $\bar g_j$ tends to zero like j^{-1}. This is called *asymptotic freedom*. It suggests that perturbation theory becomes more accurate at larger scales and the theory becomes more gaussian, as needed for the "evaluation as if gaussian" strategy that started in Sect. 5.1.

6.4 The Error Coordinate

The perturbation theory of the preceding sections suggests that we approximate Z_j defined in (94) by $I_j(V_j, \Lambda)$ defined in (106) and then, by (107), Z_{j+1} should be approximately equal to $I_{j+1}(V_{j+1}, \Lambda)$ where the coupling constants in V_{j+1} are obtained by applying the map (109) to the coupling constants in V_j. However our calculations have given us formulas which hold in a very weak algebraic sense since we have worked modulo $O(V^3)$. For example there is no uniformity in Λ. Now we need a way to include the $O(V^3)$ error terms so that all our formulas hold in the usual sense of equality. For scale j in the range $0, \dots, N$ we will write Z_j as a function of (V_j, K_j) where V_j specifies the I_j approximation by perturbation theory and the *error coordinate* K_j specifies the difference between Z_j and perturbation theory. Our main result for this section is Lemma 6.6 where we show that Z_{j+1} is the scale equivalent function of V_{j+1} and the error coordinate K_{j+1} on the next scale.

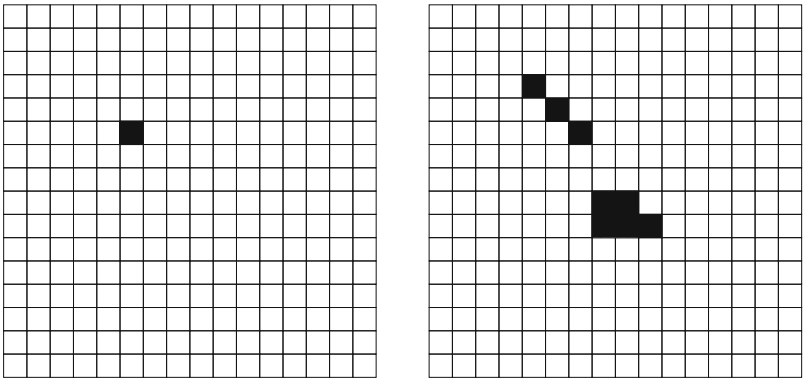

Fig. 1 Illustration of both blocks B with side length L^j and a polymer at scale j

Lemma 6.6 is an important step in our definition of the renormalisation group map, but it is not a complete description of our formalism. Firstly, we only give the version for first order perturbation theory, and secondly it is but one of a family of ways to define K_{j+1} such that Z_{j+1} is represented by (V_{j+1}, K_{j+1}). This many-to-one aspect is important for overcoming a problem that we will discuss in Sect. 7.2.

Geometry Let B be a block as in Fig. 1 with side length L^j and denote by $\mathscr{B}_j(\Lambda)$ the set of all such j-blocks or *blocks at scale j*. By definition a *j-polymer* or *polymer at scale j* is a union of j-blocks. A polymer can be the empty set. If X is a j-polymer, then $\mathscr{B}_j(X)$ denotes the set of j-blocks contained in X. Notice that polymers at scale j are also polymers at scales smaller than j because we have chosen the sides of blocks to be powers of L. Λ is a polymer at all scales and is a single block at scale N. We denote by $\mathscr{P}_j(\Lambda)$ the set of j-polymers in Λ, and for a j-polymer X let $\mathscr{P}_j(X)$ denote the set of all j-polymers contained in X. Two polymers X, Y are said to *touch* if there is a point x in X and a point y in Y such that $|x - y|_\infty = 1$. A polymer X is said to be *connected* if it is not empty and if whenever Y and Y' are nonempty polymers such that $X = Y \cup Y'$ then Y, Y' touch. For each scale j and j-polymer X let

$$X \mapsto X^{\square} \tag{116}$$

be an assignment of a polymer $X^{\square}$ that contains X. We think of this as an assignment of a neighbourhood of X to X. The assignment must be translation invariant and satisfy $(X \cup Y)^{\square} = X^{\square} \cup Y^{\square}$. Our specific choice is

$$X^{\square} = \bigcup \{Y | Y \supset X, Y \in \mathscr{S}\}, \tag{117}$$

where $\mathscr{S}$ is an important class of sets specified in

Definition 6.4 For any scale j we say that a polymer X in $\mathscr{P}_j(\Lambda)$ is a small polymer if X is connected and is a union of at most 2^d blocks. Let $\mathscr{S}_j$ be the set of small scale j polymers.

Thus the neighbourhood assigned to a block on scale j has diameter $O(L^j)$ and, for L large enough, when j-polymers X and X' are separated by a distance L^{j+1},

$$\operatorname{dist}(X^\square, X'^\square) > \frac{1}{2}L^{j+1} \tag{118}$$

holds for all scales j. We will always assume that L is at least this large.

For $X \subset \Lambda$ let $\mathscr{N}(X)$ be the set of forms in $\mathscr{N}(\Lambda)$ that only depend on $\phi_x, d\phi_x$ (and their conjugates for $x \in X$). These sets $\mathscr{N}(X)$ are the form analogues of σ-algebras of random variables. For example, when j-polymers X and X' are separated by a distance L^{j+1}, $A \in \mathscr{N}(X^\square)$ and $B \in \mathscr{N}(X'^\square)$, then θA and θB are ζ independent,

$$\mathbb{E}_{j+1}\theta\Big[AB\Big] = \mathbb{E}_{j+1}\Big[(\theta A)(\theta B)\Big] = \mathbb{E}_{j+1}\big[\theta A\big]\,\mathbb{E}_{j+1}\big[\theta B\big], \tag{119}$$

by item 2 of Theorems 6.1 and (118).

Let $B \mapsto I_j(B)$ assign to each block in $\mathscr{B}_j(\Lambda)$ a form $I_j(B) \in \mathscr{N}(B^\square)$. For example our discussion of first order perturbation theory suggests

$$I_j(B) \;=\; \mathrm{e}^{-V_j(B)}, \tag{120}$$

and in this case the condition $I_j(B) \in \mathscr{N}(B^\square)$ holds since the term $z_j\tau_\Delta(B)$ in $V_j(B)$ depends only on fields ϕ_x for x in B or x a nearest neighbour to some site in B. The other terms in $V_j(B)$ only depend on fields in B. For second order perturbation theory as in (106) the formula [11, (3.21)] for W_j also satisfies $I_j(B) \in \mathscr{N}(B^\square)$. For a polymer$X \in \mathscr{P}_j$ let

$$I^X = \prod_{B\in\mathscr{B}(X)} I(B). \tag{121}$$

Definition 6.5 Let $\mathscr{K}_j$ be the set of maps $K : \mathscr{P}_j(\Lambda) \to \mathscr{N}(\Lambda)$ such that

1. $K(\varnothing) = 1$,
2. For $X \in \mathscr{P}_j$, $K(X) \in \mathscr{N}(X^\square)$,
3. If X and Y are polymers that do not touch then $K(X \cup Y) \;=\; K(X)K(Y)$,
4. Symmetry properties.

The symmetry properties express invariance under lattice automorphism and supersymmetry. See [17, Definition 1.7] for details, but they are not needed here.

At scale $j = 0$, let

$$I_0(B) \;=\; \mathrm{e}^{-V_0(B)} \quad \text{and} \quad K_0(X) = \begin{cases} 1, & X = \varnothing \\ 0, & \text{else} \end{cases} \tag{122}$$

then

$$Z_0 = \mathrm{e}^{-V_0(\Lambda)} = \sum_{X \in \mathscr{P}_0(\Lambda)} I_0^{\Lambda \setminus X} K_0(X). \tag{123}$$

More generally, for any elements A, B of $\mathscr{K}_j$ we define a new element $A \circ B$ of $\mathscr{K}_j$ by

$$A \circ B(Y) = \sum_{X \in \mathscr{P}_j(Y)} A(X)\, B(Y \setminus X). \tag{124}$$

This product is easily verified to be commutative and associative with identity

$$\mathbb{1}_{\varnothing}(X) = \begin{cases} 1, & X = \varnothing \\ 0, & \text{else} \end{cases} \tag{125}$$

(which is the same function as K_0.) Then, after extending the domain of I_j from $\mathscr{B}_j(\Lambda)$ to $\mathscr{P}_j(\Lambda)$ by setting

$$I_0(X) = I_0^X, \quad X \in \mathscr{P}_j(\Lambda) \tag{126}$$

we can now write (123) in the shorter form

$$Z_0 = I_0 \circ K_0(\Lambda). \tag{127}$$

Notice also that the $\circ$ product depends on scale. For example at scale N, $\mathscr{B}_N(\Lambda_N)$ consists only of $\varnothing, \Lambda$, therefore for A, B that equal 1 when evaluated on the empty set $\varnothing$,

$$A \circ B(\Lambda) = A(\Lambda) + B(\Lambda). \tag{128}$$

The next result, which is the main result of this subsection, is stated for first order perturbation theory. We use a more complicated second order version in the proof of Theorem 2.2. We implicitly assume integrability. When we discuss norms there will be a property called the integration property that takes care of this issue.

Lemma 6.6 *Let j be in $\{0, 1, \ldots, N-1\}$ and for $k = j, j+1$ let $I_k = e^{-V_k}$. Given $K_j \in \mathscr{K}_j$ there exists $K_{j+1} \in \mathscr{K}_{j+1}$ such that*

$$\mathbb{E}_{j+1}\theta\Big[I_j \circ K_j(\Lambda)\Big] = I_{j+1} \circ K_{j+1}(\Lambda). \tag{129}$$

Thus if (V_j, K_j) *are such that* $Z_j = I_j \circ K_j(\Lambda)$*, then* $Z_{j+1} = I_{j+1} \circ K_{j+1}(\Lambda)$*. A formula for* K_{j+1} *is given in* (137)*.*

In particular, we can make any choice of $V_1, \dots, V_N$ and then by induction based on this Lemma and by (128) there exists K_N such that

$$Z_N = I_N \circ K_N(\Lambda) = I_N(\Lambda) + K_N(\Lambda). \tag{130}$$

By combining this result with Proposition 5.5 we have

$$\hat{\chi}_\Lambda(m^2, g_0, \nu_0, z_0) = \frac{1}{m^2} + \frac{1}{m^4\,|\Lambda|}\left(D^2 I_N(\Lambda; 0; 1, 1) + D^2 K_N(\Lambda; 0; 1, 1)\right), \tag{131}$$

Notation When we are concerned with a fixed scale and the transition to the next scale we clean up the equations by suppressing the subscript j, writing I and K in place of I_j and K_j, etc., and shorten the $j+1$ subscript to $+$ so that, for example, K_{j+1}, $\mathbb{E}_{j+1}$, $\mathscr{P}_{j+1}$ become K_+, $\mathbb{E}_+$, $\mathscr{P}_+$.

I have the following pictorial view of the proof of Lemma 6.6, but you lose nothing in the logical sense by skipping this. I give the actual proof below. The representation $I \circ K(\Lambda)$ is a sum over ways to partition Λ into a polymer $\Lambda \setminus X$ weighted by a product of I factors, one per block in $\Lambda \setminus X$, and a polymer X weighted by $K(X)$. The left hand side in Fig. 2 represents one term in this sum over partitions: the white region is the polymer $\Lambda \setminus X$ and each square in the white region represents a block B with a factor $I(B)$. The blue region is the polymer X for which there is a factor $K(X)$. If we apply the algebra homomorphism θ to $I \circ K(\Lambda)$ then the white region represents a product over blocks B of $\theta I(B)$ and the blue represents $\theta K(X)$. In the first step (133) of the proof below each $\theta I(B)$ is expanded into $I_+ + \delta$ (which defines an error term δ). Thus there arises a sum of ways to colour a subset of the white blocks red, each red block B denoting a $\delta(B)$. The remaining white blocks are

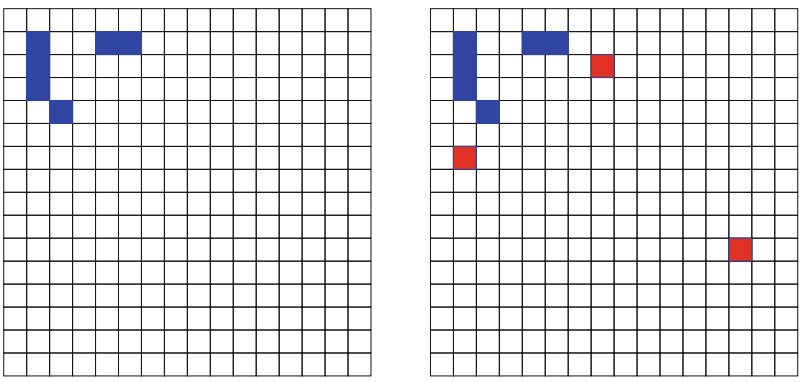

Fig. 2 Illustration of Eq. (135)

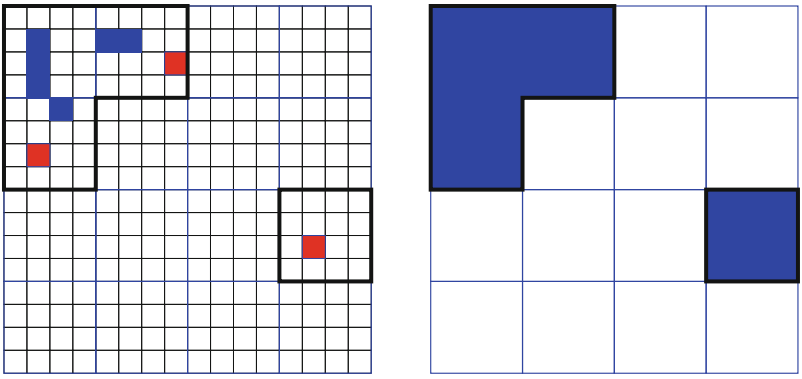

Fig. 3 Illustration of Eq. (138)

now I_+ factors as in the right hand side of Fig. 2, which stands for just one term in a sum over partitions into red, white and blue. The next step (138) in the proof is a passage to blocks on the next scale, accomplished by considering the smallest scale $j+1$ polymer Y that covers the red and blue region, as indicated in the left hand side of Fig. 3. The right hand side of Fig. 3 represents the sum $\Sigma(Y)$ over all red, white and blue partitions as in the left hand side that generate the same polymer Y. Since this right hand picture is a partition of Λ into a white polymer $\Lambda \setminus Y$ that represent products of I_+ blocks and a blue polymer Y, and since these are polymers on the next scale, this picture is a term in $I_+ \circ \Sigma$ and it tells us to define $K_+(Y) = \mathbb{E}_{j+1}\Sigma(Y)$ because then we match the desired right hand side of the Lemma.

Proof (of Lemma 6.6.) Define $B \mapsto \delta(B)$ on $\mathscr{B}(\Lambda)$ by

$$\delta = \theta I - I_+. \tag{132}$$

Notice that θI is evaluated on scale j blocks so we have to extend the domain of I_+ which is defined on scale $j+1$ block to include scale j blocks, but we can do this easily for first order perturbation theory since V_{j+1} can be summed over points in blocks of any scale: we set $I_{j+1}(X) = e^{-V_{j+1}(X)}$ for X any scale block. For later in the proof note that δ depends on ζ because of the θ, but I_+ is not dependent on ζ. For the next equation we also extend the domain of δ from blocks to polymers by writing $\delta(X) = \delta^X$ for a $X \in \mathscr{P}$. Then, for $X \in \mathscr{P}$,

$$\begin{aligned} \theta\, I^X = \big(I_+ + \delta\big)^X &= \sum_{X_\delta \in \mathscr{P}(X)} I_+^{X\setminus X_\delta}\, \delta^{X_\delta} \\ &= \sum_{X_\delta \in \mathscr{P}(X)} I_+^{X\setminus X_\delta}\, \delta(X) = I_+ \circ \delta(X). \end{aligned} \tag{133}$$

Let

$$H = \delta \circ \theta K. \tag{134}$$

Then

$$\begin{aligned}\theta(I \circ K)(\Lambda) &= \theta I \circ \theta K(\Lambda) \\ &= \big(I_+ \circ \delta\big) \circ \theta K(\Lambda) = I_+ \circ \big(\delta \circ \theta K\big)(\Lambda) \\ &= I_+ \circ H(\Lambda). \end{aligned} \tag{135}$$

For $X \in \mathscr{P}(\Lambda)$, define the closure $\overline{X}$ of X on the next scale to be smallest polymer in $\mathscr{P}_+(\Lambda)$ containing X. For Y in $\mathscr{P}_+$ let $\overline{\mathscr{P}}(Y)$ be the set of all polymers in $\mathscr{P}$ whose closure is Y, then

$$\begin{aligned} I_+ \circ H(\Lambda) &= \sum_{X \in \mathscr{P}(\Lambda)} I_+^{\Lambda\setminus X} H(X) \\ &= \sum_{Y \in \mathscr{P}_+(\Lambda)} \sum_{X \in \overline{\mathscr{P}}(Y)} I_+^{\Lambda\setminus X} H(X) \\ &= \sum_{Y \in \mathscr{P}_+(\Lambda)} I_+^{\Lambda\setminus Y} \sum_{X \in \overline{\mathscr{P}}(Y)} I_+^{Y\setminus X} H(X). \end{aligned} \tag{136}$$

For Y in $\mathscr{P}_+(\Lambda)$ let

$$\begin{aligned} K_+(Y) &= \mathbb{E}_+\Big[\sum_{X \in \overline{\mathscr{P}}(Y)} I_+^{Y\setminus X} H(X) \Big] \\ &= \sum_{X \in \overline{\mathscr{P}}(Y)} I_+^{Y\setminus X}\, \mathbb{E}_+\big[H(X)\big]. \end{aligned} \tag{137}$$

Then by (135)–(137)

$$\begin{aligned} \mathbb{E}_+\theta\big[I_j \circ K_j(\Lambda)\big] &= \sum_{Y \in \mathscr{P}_+(\Lambda)} I_+^{\Lambda\setminus Y}\, \mathbb{E}_+\Big[\sum_{X \in \overline{\mathscr{P}}(Y)} I_+^{Y\setminus X} H(X) \Big] \\ &= \sum_{Y \in \mathscr{P}_+(\Lambda)} I_+^{\Lambda\setminus Y} K_+(Y) \;=\; I_+ \circ K_+(\Lambda), \end{aligned} \tag{138}$$

which verifies the desired property with respect to $\mathbb{E}_+$ of K_+.

We now outline the proof that K_+ defined by (137) is in $\mathscr{K}_+$. This is where the finite range property of Theorem 6.1 plays its crucial role and part (2) of Definition 6.5 and the analogous property of I are used. If K_+ is evaluated on $X \cup X'$

where X and X' are polymers in $\mathscr{P}_+$ that do not touch, then the distance between X and X' is at least L^{j+1} because they are each unions of $j+1$ scale blocks which cannot touch. Therefore factors of δ evaluated on blocks in X are independent of factors of δ evaluated on blocks in X'. Furthermore, by part (3) of Definition 6.5 for K, when θK is evaluated on a subset of $X \cup X'$, it is a product of a factor of K evaluated on a subset of X and a factor of K evaluated on a subset of X'. These are also independent. Finally it is straightforward to check that the sums defining H and K_+ factor into sums of ways to partition X and a separate sum over ways to partition X'. Taken all together this implies that K_+ has property (3) of the definition of $\mathscr{K}_+$. The other properties of $\mathscr{K}_+$ are much easier to check.

7 The Norm of the Error Coordinate

So far all our analysis of the action of $\mathbb{E}_{j+1}\theta$ in terms of (V, K) has been algebraic with no hint on how the errors could be controlled. In this section we explain part of our formalism in [17] for the control of errors. In particular we will see that the error coordinate at scale j can be regarded as an element of a Banach space $\mathscr{F}_j$. The spaces $\mathscr{K}_j, j = 0, 1, \ldots, N$, for the error coordinate introduced in Definition 6.5 are not vector spaces because part (3) of Definition 6.5 is not linear, but the restrictions of elements of $\mathscr{K}_j$ to connected polymers form a vector space as in the following definition.

Definition 7.1 For $j = 0, 1, \ldots, N$, let $\mathscr{C}_j(\Lambda)$ be the set of connected polymers in $\mathscr{P}_j(\Lambda)$. Let $\mathscr{C}\mathscr{K}_j$ be the complex vector space under pointwise addition and scalar multiplication of maps $K : \mathscr{C}_j(\Lambda) \to \mathscr{N}(\Lambda)$ such that

1. For $X \in \mathscr{C}_j(\Lambda)$, $K(X) \in \mathscr{N}(X^{\square})$,
2. Symmetry properties,

where the symmetry properties are a repetition of part (4) of Definition 6.5.

Not only does an element of $\mathscr{K}_j$ determine an element of $\mathscr{C}\mathscr{K}_j$ by restriction, but, conversely, given an element K of $\mathscr{C}\mathscr{K}_j$ we can extend its domain from $\mathscr{C}_j(\Lambda)$ to $\mathscr{P}_j(\Lambda)$ by imposing property (3) of Definition 6.5: a polymer X in $\mathscr{P}_j(\Lambda)$ can be decomposed into its connected components $X_1, \ldots, X_n$ and then we define $K(X) = \prod_{i=1}^{n} K(X_i)$. If X is the empty set we define $K(X) = 1$.

7.1 The $\mathscr{F}$ Norm

7.1.1 Norm on $\mathscr{C}\mathscr{K}_j$

For each scale j the norm on $\mathscr{C}\mathscr{K}_j$ is constructed from a family of complete norms $\|F\|_j = \|F\|_{j,\mathscr{N}(X^{\square})}$, one for each space $\mathscr{N}(X^{\square})$ where X ranges over connected

j-polymers. Given such norms we define the norm $\|K\|_{\mathscr{F}_j}$ for $K \in \mathscr{C}\mathscr{K}$ by

$$\|K\|_{\mathscr{F}_j} = \sup_{X \in \mathscr{C}_j(\Lambda)} W(X) \|K(X)\|_{j,\mathscr{N}(X^\square)} \tag{139}$$

with a weight $X \mapsto W(X)$. In these notes we set

$$W(X) = \epsilon^{-|X|_j}, \tag{140}$$

where $|X|_j = |\mathscr{B}_j(X)|$ is the number of j-blocks in X and ϵ is a positive parameter smaller than one. This is not our choice in [17] but could have been and it is instructive.

7.1.2 The Norm on $\mathscr{N}(X^\square)$

The definitions are given in [16] and they take time to assimilate, so instead of repeating them, I list some desired properties and illustrate the role of these properties by using them axiomatically to prove a bound in the following section.

- *Product Property:* For all scales $j = 0, \dots, N-1$, for disjoint j-polymers X, Y, for forms A in $\mathscr{N}(X^\square)$, B in $\mathscr{N}(Y^\square)$ and AB in $\mathscr{N}((X \cup Y)^\square)$,

$$\|AB\|_j \ \le \ \|A\|_j \|B\|_j. \tag{141}$$

  See [16, Proposition 3.16]. All the spaces $\mathscr{N}(X)$, where X ranges over subsets of Λ, are subalgebras of $\mathscr{N}(\Lambda)$ so it makes sense to multiply A, B.
- *I_+ Bound:* There is a constant α_I and a coupling constant domain $\mathscr{D}_{j+1}$ for the coupling constants in V_{j+1} such that

$$\|e^{-V_{j+1}(B)}\|_{j+1} \ \le \ \alpha_I \tag{142}$$

  for $B \in \mathscr{B}_j(\Lambda)$.
- *Integration Property:* Recall the definition of δ from (132) and let $k = j$ or $j+1$. There is a constant $\alpha_{\mathbb{E}}$ such that for disjoint $X, Y \in \mathscr{P}_j$, $F(Y) \in \mathscr{N}(Y^\square)$ and V_k with coupling constants in $\mathscr{D}_k$,

$$\| \mathbb{E}_{j+1} \delta^X \theta F(Y)\|_{j+1} \ \le \ \alpha_{\mathbb{E}}^{|X|_j + |Y|_j} \epsilon_\delta^{|X|_j} \|F(Y)\|_j, \tag{143}$$

  where ϵ_δ tends to zero as the coupling constants in V_k tend to zero.

7.1.3 Comment on Norms

If we were not working with forms, but just functions of fields then the L^∞ norm would have these properties. In [16] we show how to construct norms that record information on derivatives by being equivalent to $\mathscr{C}^p$ norms, but which also have the product property. The construction is based on the idea that the Taylor expansion of a product AB is the product of the Taylor expansions for A and for B. As a corollary we can extend the norms to forms in $\mathscr{N}(X^\square)$ using the fact that coefficients of monomials in $d\phi$ and $d\bar{\phi}$ are analogous to coefficients in a Taylor expansion.

Another consideration is that at scale j an element F of $\mathscr{N}(X^\square)$ is not going to be evaluated on arbitrary fields $\phi \in \mathbb{C}^\Lambda$, but on fields that are typical for the distribution of $\phi = \sum_{k>j} \zeta_k$. Therefore, for each polymer X, we consider $K_j(X)$ as a smooth ($\mathscr{C}^p$) function defined on the vector space $\mathbb{C}^\Lambda$ with norm chosen so that the supremum of a unit norm field is $O(L^{-j})$. This factor is because part (3) of Theorem 6.1 says that the standard deviation of $\sum_{k>j} \zeta_k$ is $O(L^{-j})$, provided $m^2 \le O(L^{-2j})$. The norm on $K_j(X)$ measures directional derivatives of $K_j(X)$ as a function of fields in $\mathbb{C}^\Lambda$. For example,

$$DK_j(X;\phi;\dot{\phi}) = \frac{d}{dt}_{|_0} K_j(X;\phi + t\dot{\phi}) \tag{144}$$

is bounded in norm by the norm of $K_j(X)$ times a sup norm of $\dot{\phi}$ divided by L^{-j}. As an important example for us, let $\dot{\phi}$ be the constant test function 1, then the second directional derivative $D^2K_N(0;1,1)$ at $\phi = 0$ and $d\phi = 0$ satisfies

$$|D^2K_N(\Lambda;0;1,1)| \le \|K\|_{\mathscr{F}_N} O(L^{2N}). \tag{145}$$

7.2 The Irrelevant Parts of K_+

We will now illustrate the use of the properties postulated in Sect. 7.1.2 for the norm on $\mathscr{N}(X^\square)$. Lemma 6.6 has provided us with a formula (137) for a map $K_+ : (V_j, V_{j+1}, K_j) \mapsto K_{j+1}$ that exactly represents $Z_j \mapsto Z_{j+1}$ as in (94). The main result of this section is Proposition 7.2 which shows a very good property of K_+. It shows that K_+ is contractive in K_j provided K_j, which is a map from connected polymers to $\mathscr{N}$, restricts to be zero on the connected polymers in the class $\mathscr{S}$ of Definition 6.4.

For the rest of this section we omit j subscripts and abbreviate $j+1$ to $+$ as in the notation explained below (131). We also suppress the dependence of K_+ on V and V_+ because our estimates are pointwise and uniform for V, V_+ in domains $\mathscr{D}, \mathscr{D}_+$. Referring to the formula (137) for K_+ let $\mathscr{P}^{(2)}(Y)$ denote the set of pairs (X, X_K) such that $X \in \bar{\mathscr{P}}_j(Y)$ and $X_K \in \mathscr{P}_j(X)$. From (137) and (134),

$$K_+(Y) = \sum_{(X,X_K)\in\mathscr{P}^{(2)}(Y)} I_+^{Y\setminus X}\, \mathbb{E}_+\big[\delta^{X\setminus X_K} K(X_K)\big]. \tag{146}$$

(i) We subtract from $K_+(Y)$ the value of $K_+(Y)$ when $K = 0$ by omitting terms in the sum which do not depend on K, that is terms where $X_K = \varnothing$. (ii) We omit terms where $X = X_K \in \mathscr{S}$, where $\mathscr{S}$ is the class of small sets defined in Definition 6.4. To study the remaining part of K_+, let

$$K_+^{\mathscr{I}}(Y) = \sum_{(X,X_K)\in\mathscr{I}(Y)} I_+^{Y\setminus X}\,\mathbb{E}_+\big[\delta^{X\setminus X_K}K(X_K)\big]. \tag{147}$$

where

$$\mathscr{I}(Y) = \mathscr{P}^{(2)}(Y) \setminus \Big(\{X_K = \varnothing\} \cup \{X = X_K \in \mathscr{S}\}\Big). \tag{148}$$

Wilson called terms that contract *irrelevant* so we have used the letter $\mathscr{I}$ to label the set of terms that we can prove are collectively contracted. Wilson called terms that expand *relevant* and terms that stay the same size *marginal*. Thus we use $\mathscr{R}$ to label the complement of the set $\mathscr{I}$, but neither class is in precise correspondence with Wilson's classification because although we are capturing his intuition we are working outside his original context.

For the next section we define $K_+^{\mathscr{R}}$ be the part of K_+ that we subtracted out under item (ii). We could write it in exactly the same form as (147) with the set $\mathscr{I}(Y)$ replaced by

$$\mathscr{R}(Y) = \mathscr{P}^{(2)}(Y) \setminus \Big(\{X_K = \varnothing\} \cup \{X = X_K \notin \mathscr{S}\}\Big), \tag{149}$$

but it is easy to verify that this is the same as

$$K_+^{\mathscr{R}}(Y) = \sum_{X_K\in\mathscr{S}:\,\overline{X}_K = Y} I_+^{Y\setminus X_K}\,\mathbb{E}_+\big[K(X_K)\big]. \tag{150}$$

If the restriction of K to $\mathscr{S}$ is zero then $K_+^{\mathscr{R}} = 0$, but in general we have

$$K_+ = K_{+|_{K=0}} + K_+^{\mathscr{R}} + K_+^{\mathscr{I}}. \tag{151}$$

For the main result of this section let $\sigma = \frac{1}{2}(1+\eta)$ where $\eta > 1$ is defined by geometry in Lemma 7.3. It follows that $\eta > \sigma > 1$ and this is the only fact about σ that is important for us. Recall the definitions of the parameters $\alpha_I, \alpha_{\mathbb{E}}$ from Sect. 7.1.2 and let α be the maximum of α_I, $\alpha_{\mathbb{E}}$ and 1. We choose ϵ in (140) smaller than 1 and such that

$$\kappa < \frac{2}{3}, \quad \text{where} \quad \kappa = (3\alpha)^{L^d}\epsilon^{\sigma-1}. \tag{152}$$

We decrease, if necessary, the domain $\mathscr{D}_k$ of the coupling constants g_k, z_k, μ_k with $k = j, j+1$ so that, by the remark below (143),

$$\epsilon_\delta \leq \epsilon^\sigma \epsilon^{\sigma 2^{d+1}}. \tag{153}$$

Let $B_{\mathscr{F}}$ be the ball in $\mathscr{F}$ given by

$$\|K\|_{\mathscr{F}} \leq \left(\epsilon^{\sigma 2^{d+1}}\right)^2. \tag{154}$$

Proposition 7.2 *With the choices (148)–(154), the function $K \mapsto K_+^{\mathscr{J}}$ with domain $B_{\mathscr{F}}$ satisfies*

$$\|K_+^{\mathscr{J}}\|_{\mathscr{F}_+} \leq \kappa \|K\|_{\mathscr{F}}, \tag{155}$$

and, for K and K' in the smaller domain $\frac{1}{4}B_{\mathscr{F}}$,

$$\left\| K_+^{\mathscr{J}}(K') - K_+^{\mathscr{J}}(K) \right\|_{\mathscr{F}_+} \leq \frac{3}{2}\kappa \|K' - K\|_{\mathscr{F}}. \tag{156}$$

Thus $K_+^{\mathscr{J}}$ is a contractive map from $\frac{1}{4}B_{\mathscr{F}}$ to itself. If the restriction of K to $\mathscr{S}$ is zero then K_+ is contractive.

To prove this Proposition we need the following two geometrical estimates which show why small sets have an exceptional role. We measure the size of a polymer at scale j by counting the number of scale j blocks it contains. Similarly the size of a polymer at scale $j+1$ is measured by counting the number of scale $j+1$ blocks it contains. Does the closure map $X \mapsto \overline{X}$ make a polymer smaller? For example let X be a polymer that consists of a single block. Then $\overline{X}$ is also a polymer which consists of a single block so it is the same size as X. Lemma 7.3 says (1) that connected polymers that are not small always decrease in size under closure and (2) that no polymer increases in size under closure.

Lemma 7.3 ([15, Lemma 6.15]) *There is an $\eta = \eta(d) > 1$ such that for all $L \geq L_0(d) = 2^d + 1$ and for all connected scale j polymers that are not in $\mathscr{S}_j$,*

$$|X|_j \geq \eta |\overline{X}|_{j+1}. \tag{157}$$

In addition, (157) holds with η replaced by 1 for all $X \in \mathscr{P}_j$ (not necessarily connected, and possibly small).

Corollary 7.4 ([15, Lemma 6.16]) *Let $X \in \mathscr{P}_j$ and let n be the number of components of X. Then*

$$|X|_j \geq \frac{1}{2}(1+\eta)|\overline{X}|_{j+1} - \frac{1}{2}(1+\eta)2^{d+1}n. \tag{158}$$

Proof (of Proposition 7.2) The estimate (156) follows from (155) and [3, Lemma 1] with $\nu = \frac{1}{4}$. The final claim in Proposition 7.2 follows from (151) and the remark above this equation. Therefore we are reduced to proving (155).

By applying to (147), the product property (141), the bound (142) on I_+ and the integration property (143), for $Y \in \mathscr{C}(\Lambda)$,

$$\|K_+^{\mathscr{I}}(Y)\|_{j+1} \le \sum_{(X,X_K)\in\mathscr{I}(Y)} \alpha^{|Y|_j} \epsilon_\delta^{|X\setminus X_K|_j} \|K(X_K)\|_j. \tag{159}$$

The polymer X_K may not be connected. Suppose it has connected components $X_1, \dots, X_n$. We use property (3) of Definition 6.5 together with the product property (141) and the definition (139) of $\|K\|_{\mathscr{F}} = \|K\|_{\mathscr{F}_j}$ to obtain

$$\begin{aligned} \|K(X_K)\|_j &= \prod_{i=1,\dots,n} \|K(X_i)\|_j \\ &= \epsilon^{|X_K|_j} \prod_{i=1,\dots,n} \epsilon^{-|X_i|_j} \|K(X_i)\|_j \le \epsilon^{|X_K|_j} \|K\|_{\mathscr{F}}^n, \end{aligned} \tag{160}$$

where $n = n(X_K)$ is the number of components in X_K. We substitute (160) into (159) and estimate the sum by noting that it extends over less than $3^{|Y|_j}$ terms because this is the number of ways to partition Y into three disjoint subsets $X_K, X \setminus X_K, Y \setminus X$ that are each polymers at scale j. Also we can replace Y by $\overline{X}$ because one of the defining conditions for $\mathscr{I}(Y)$ is that they are equal. Therefore we have

$$\|K_+^{\mathscr{I}}(Y)\|_{j+1} \le \sup_{(X,X_K)\in\mathscr{I}(Y)} (3\alpha)^{|\overline{X}|_j} \epsilon_\delta^{|X\setminus X_K|_j} \epsilon^{|X_K|_j} \|K\|_{\mathscr{F}}^{n(X_K)}. \tag{161}$$

We multiply both sides by $\epsilon^{-|Y|_{j+1}}$ which equals $\epsilon^{-|\overline{X}|_{j+1}}$ and take the supremum over connected $j+1$ scale polymers in order to form the $\mathscr{F}_{j+1} = \mathscr{F}_+$ norm on the left hand side and we obtain

$$\begin{aligned} \|K_+^{\mathscr{I}}\|_{\mathscr{F}_+} &\le \sup_{(X,X_K)\in\mathscr{I}} F(X,X_K), \quad \text{where} \\ F(X,X_K) &= \epsilon^{-|\overline{X}|_{j+1}} (3\alpha)^{|\overline{X}|_j} \epsilon_\delta^{|X\setminus X_K|_j} \epsilon^{|X_K|_j} \|K\|_{\mathscr{F}}^{n(X_K)}, \end{aligned} \tag{162}$$

and $\mathscr{I} = \cup_{Y\in\mathscr{C}_+(\Lambda)} \mathscr{I}(Y)$. Recall that $Y \in \mathscr{C}_+(\Lambda)$ implies that Y is not empty. Now we cover $\mathscr{I}$ by three subsets and it suffices to prove that for (X, X_K) in each covering set we have $F(X, X_K) \le \kappa \|K\|_{\mathscr{F}}$. We give the proof in most detail for subset 1 because the proofs are similar for the other sets.

Subset 1 For $(X, X_K) \in \mathscr{I} \cap \{n(X_K) = 1, |X_K| > 2^d\}$, by using Lemma 7.3 twice, $\eta > \sigma$ and the hypothesis (153) on ϵ_δ and $\epsilon_\delta \le 1$,

$$\begin{aligned}\epsilon_\delta^{|X\setminus X_K|_j}\epsilon^{|X_K|_j} &\le \epsilon_\delta^{|\overline{X\setminus X_K}|_{j+1}}\epsilon^{\eta|\overline{X_K}|_{j+1}} \\ &\le \epsilon^{\sigma|\overline{X\setminus X_K}|_{j+1}}\epsilon^{\sigma|\overline{X_K}|_{j+1}} \le \epsilon^{\sigma|\overline{X}|_{j+1}}. \end{aligned} \tag{163}$$

The final inequality holds because $\overline{A} \cup \overline{B} \supset \overline{A \cup B}$ since the closure is the smallest next scale polymer cover. Putting this estimate into (162) and using the definition (152) of κ we have

$$\begin{aligned}F(X, X_K) &\le (3\alpha)^{|\overline{X}|_j}\epsilon^{(\sigma-1)|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}} \\ &= \left((3\alpha)^{L^d}\epsilon^{\sigma-1}\right)^{|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}} \le \kappa\|K\|_{\mathscr{F}}. \end{aligned} \tag{164}$$

Subset 2 For $(X, X_K) \in \mathscr{I} \cap \{n(X_K) = 1, |X \setminus X_K|_j \ge 1\}$, by Lemma 7.3 and Corollary 7.4 and the hypothesis (153) on ϵ_δ,

$$\epsilon_\delta^{|X\setminus X_K|_j}\epsilon^{|X_K|_j} \le \epsilon_\delta^{|\overline{X\setminus X_K}|_{j+1}}\epsilon^{\sigma|\overline{X_K}|_{j+1}}\epsilon^{-\sigma 2^{d+1}} \le \epsilon^{\sigma|\overline{X}|_{j+1}}. \tag{165}$$

Putting this estimate into (162) we have

$$\begin{aligned}F(X, X_K) &\le (3\alpha)^{|\overline{X}|_j}\epsilon^{(\sigma-1)|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}} \\ &\le (3\alpha)^{L^d}\epsilon^{\sigma-1}\|K\|_{\mathscr{F}} \le \kappa\|K\|_{\mathscr{F}}. \end{aligned} \tag{166}$$

Subset 3 For $(X, X_K) \in \mathscr{I} \cap \{n(X_K) \ge 2\}$, by Lemma 7.3 and Corollary 7.4, the hypothesis (153) on ϵ_δ and the domain (154) for K,

$$\begin{aligned}\epsilon_\delta^{|X\setminus X_K|_j}\epsilon^{|X_K|_j}\|K\|_{\mathscr{F}}^{n(X_K)} &\le \epsilon_\delta^{|\overline{X\setminus X_K}|_{j+1}}\epsilon^{\sigma|\overline{X_K}|_{j+1}}\left(\epsilon^{-\sigma 2^{d+1}}\|K\|_{\mathscr{F}}\right)^{n(X_K)} \\ &\le \epsilon^{\sigma|\overline{X}|_{j+1}}\left(\epsilon^{-\sigma 2^{d+1}}\|K\|_{\mathscr{F}}\right)^2 \\ &\le \epsilon^{\sigma|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}}. \end{aligned} \tag{167}$$

Putting this estimate into (162) we have

$$\begin{aligned}F(X, X_K) &\le (3\alpha)^{|\overline{X}|_j}\epsilon^{(\sigma-1)|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}} \\ &\le \left((3\alpha)^{L^d}\epsilon^{\sigma-1}\right)^{|\overline{X}|_{j+1}}\|K\|_{\mathscr{F}} \le \kappa\|K\|_{\mathscr{F}}. \end{aligned} \tag{168}$$

7.3 The Complete Recursion

We have seen in the previous section that the map $(V_j, V_{j+1}, K_j) \mapsto K_+$ provided by Proposition 7.2 is contractive provided K_j vanishes when evaluated on polymers in the class $\mathscr{S}$ of small sets. In this section we will discuss a better choice for K_+ which is contractive. Part of the improvement comes from specifying V_{j+1} carefully, recalling that Proposition 7.2 put essentially no constraint on it. However we need also another idea which is a change of variable formula for K. The conclusions which play a role in the sequel are the recursion (170) and its conjugation (175).

Recall from (142) and (143) that $\mathscr{D}_j$ is a domain in $\mathbb{R}^3$ for the coupling constants g_j, z_j, μ_j. Let $B_{\mathscr{F}_j}$ denote a ball in a Banach space $\mathscr{F}_j$. In [17] we define, for all scales $= 0, 1, \dots, N-1$, domains $\mathscr{D}_j \times B_{\mathscr{F}_j}$ and functions

$$R_+ : \mathscr{D}_j \times B_{\mathscr{F}_j} \to \mathbb{R}^3, \quad K_+ : \mathscr{D}_j \times B_{\mathscr{F}_j} \to B_{\mathscr{F}_{j+1}}. \tag{169}$$

These functions, together with the second order perturbative map $\varphi^{(0)}_{\mathrm{pt},j}$ appearing in (105), build a recursion,

$$\begin{aligned}(g_{j+1}, z_{j+1}, \mu_{j+1}) &= \varphi^{(0)}_{\mathrm{pt},j}(g_j, z_j, \mu_j) + R_+(g_j, z_j, \mu_j; K), \\ K_{j+1} &= K_+(g_j, z_j, \mu_j; K_j),\end{aligned} \tag{170}$$

This recursion, with initial condition $(g_0, z_0, \mu_0; K_0)$, where $K_0 = \mathbb{1}_\varnothing$, has the following properties:

1. K_+ is contractive in K_j.
2. The recursion generates a sequence $(g_j, z_j, \mu_j; K_j)_{j=0,\dots,j_{\mathrm{exit},N}}$ which terminates at scale $j_{\mathrm{exit},N}$, which is the first scale $j \wedge N$ such that (g_j, z_j, μ_j) is not in $\mathscr{D}_j$.
3. For Z_j given by (94) and $I_j = I_j(V_j)$ given by (106) and (95), we have

$$Z_j = I_j \circ K_j(\Lambda), \quad j \le j_{\mathrm{exit},N}. \tag{171}$$

4. Volume Λ compatibility.

The existence of R and K_+ with the first and third properties are parts of Theorem 2.2 of [17]. The second property follows immediately from the domains of the functions R_+ and K_+. The fourth property is a statement analogous to the statements below (112). The essential idea is that the recursions for two tori of different sizes N, N' generate the "same" $g_j, z_j, \mu_j; K_j$ for j up to the scale before the smaller of N, N'. However, it takes time to formulate the meaning of "same" so we refer to [17, Sect. 1.8.3]. This compatibility enables the definition of the infinite volume limit of (170) which is a recursion that generates a sequence $(g_j, z_j, \mu_j; K_j)_{j=0,\dots,j_{\mathrm{exit},\infty}}$ where K_j is defined on polymers in $\mathbb{Z}^d$ and

$$j_{\mathrm{exit},\infty} = \limsup_N j_{\mathrm{exit},N}. \tag{172}$$

To put this into the context of Proposition 7.2 refer to Eq. (151). The term $K_+^{\mathscr{J}}$ is contractive. Indeed, the preamble for this Proposition shows that the contractivity constant κ can be made arbitrarily small by choosing ϵ in the norm small and the ball $B_{\mathscr{F}_j}$ small. But the term $K_+^{\mathscr{R}}$ in (151) cannot be contractive. To understand why, consider (150) when Y is a single block on the scale $j+1$. Then the sum over polymers X_K has a range that includes the L^d possible choices of a single block $X_K = B$ in $\mathscr{B}_j(Y)$. This factor of L^d prevents $K_+^{\mathscr{R}}$ from being contractive.

The function R_+ in (170) is part of the solution to this problem. By choosing $g_{j+1}, z_{j+1}, \mu_{j+1}$ to be different from $\varphi^{(0)}_{\mathrm{pt},j}(g_j, z_j, \mu_j)$ we generate terms in K_{j+1} that I will call *counterterms*. These counterterms cancel most of $K_j^{\mathscr{R}}$ in (151) which enables K_+ to be contractive. However, to achieve this cancellation, one must have some way of changing the allocation of the counterterms to the different small sets $X_K \in \mathscr{S}$, because Proposition 7.2 does not match the counterterms correctly with the small set parts of K_j. We solve this allocation problem by taking advantage of the fact that there are re-allocation changes of variable $K_j \mapsto K_j'$ such that

$$I_j \circ K_j'(\Lambda) = I_j \circ K_j(\Lambda). \tag{173}$$

These changes of variable are supplied by Brydges and Slade [17, Proposition 4.1]. Roughly speaking these changes of variable cancel some part $J(X)$ of $K_j(X)$ for small sets $X \in \mathscr{S}_j$ which are not single blocks and compensate by adding $\sum_{X \supset B} J(X)$ to $K_j(B)$ for each block B. Using this re-allocation we can prove that when R_+ is chosen correctly the small set part of the re-allocated K_j' is almost zero. Therefore when K_+ given by (146) is evaluated on the re-allocated K' the term $K_+^{\mathscr{J}}$ is almost zero and Proposition 7.2 proves that the composition is contractive. Thus the composition of (146) on the right with a re-allocation gives a contractive formula. In passing let me remark that there is another re-allocation problem that is solved by a further composition with a re-allocation on the left. This problem is that one cannot prove that the first term in (151) is $O(V_j^3)$ as expected from our choice of the second order formula for I_j unless the contributions from perturbation theory are re-allocated. This is happening in Lemmas 4.2 and 5.8 of [17].

Recall from (108) that T_j is a map that conjugates $\varphi^{(0)}_{\mathrm{pt},j}$ to the triangular map $\bar{\varphi}_j$. We rewrite the recursion (170) in terms of new variables defined by

$$(\check{g}_j, \check{z}_j, \check{\mu}_j; \check{K}_j) = \big(T_j(g_j, z_j, \mu_j); K_j\big). \tag{174}$$

By (109) the new recursion is

$$\begin{aligned}(\check{g}_{j+1}, \check{\mu}_{j+1}, \check{z}_{j+1}) &= \bar{\varphi}_j(\check{g}_j, \check{\mu}_j, \check{z}_j) + \check{R}_+(\check{g}_j, \check{\mu}_j, \check{z}_j; \check{K}),\\ \check{K}_{j+1} &= \check{K}_+(\check{g}_j, \check{\mu}_j, \check{z}_j; \check{K}_j),\end{aligned} \tag{175}$$

where $\check{R}_{j+1} = T_{j+1} \circ R_{j+1} \circ T_j^{-1}$ plus the $O(V^3)$ error in (109) and on the right hand side of (175) $\check{K}_{j+1} = K_{j+1} \circ T_j^{-1}$ where T_j^{-1} acts only on the $\mathbb{R}^3$ part.

As explained in Sect. 6.6 of Bauerschmidt et al. [10] these new functions have almost the same domains and satisfy essentially the same bounds as the old ones because the conjugations T_j are close (108) to the identity. In particular $\check{K}_{j+1}$ is still contractive.

8 Outline of Proof of Theorem 5.2

This is a survey of the complete argument which is in [10, Sect. 8.3]. I omit the proofs of (75). One point that I find remarkable is that there are exact formulas for χ and its derivative in terms of the recursion of the coupling constant part of (170). The error coordinate does not appear in these formulas; the only role it has is to slightly change, via R_+, the recursion of second order perturbation theory. I should mention that this feature has been strongly emphasised in physics, but it is nice to be able to verify it here.

8.1 Construction of z_0^{c}, ν_0^{c}

Recall from (172) the definition of $j_{\mathrm{exit},\infty}$. The first step is to prove that there exist functions z_0^{c} and ν_0^{c} of (m^2, g_0) such that $j_{\mathrm{exit},\infty} = \infty$ for the (infinite volume limit of the) recursion (170) with the initial condition

$$(g_0, z_0, \mu_0; K_0) \text{ such that } z_0 = z_0^{\mathrm{c}}(m^2, g_0), \quad \mu_0 = \nu_0^{\mathrm{c}}(m^2, g_0), \quad K_0 = \mathbb{1}_{\varnothing}. \tag{176}$$

Proposition 7.1 of Bauerschmidt et al. [10] shows that z_0^{c} and ν_0^{c} exist. This proposition is proved by showing that the conjugated recursion (175) generates an infinite sequence $(\check{g}_j, \check{z}_j, \check{\mu}_j; \check{K}_j)_{j\in\mathbb{N}_0}$ with the same initial condition (176). The initial condition is the same because, by (108), T_0 is the identity. The main ideas in the proof of Proposition 7.1 of Bauerschmidt et al. [10] are that (1) the existence of infinite sequences for the $\bar{\varphi}$ recursion (110)–(112) is given by Proposition 6.3. (2) Norm estimates on the functions $\check{R}_+$ and $\check{K}_+$ in the recursion (175) show that they are always small compared with $\bar{\varphi}$ on the infinite sequence supplied by Proposition 6.3. By the main result of Bauerschmidt et al. [9] there exists a unique infinite sequence generated by the recursion (175) that stays close to the infinite sequence supplied by Proposition 6.3.

8.2 Coupling Constants at Large Scales

The following results are needed for the next steps in the proof. Recall from Sect. 8.1 that the infinite sequence generated by the recursion (175) stays close to the infinite sequence supplied by Proposition 6.3. In particular, by Proposition 7.1 of Bauerschmidt et al. [10],

$$\|K_j\|_{\mathscr{W}_j(s_j,\Lambda_N)} \le O(\chi_j \bar{g}_j^3), \quad \check{z}_j = O(\chi_j \check{g}_j), \quad \check{\mu}_j = O(\chi_j \check{g}_j), \tag{177}$$

where $\mathscr{W}_j(s_j, \Lambda_N)$ is a weighted maximum of two different choices of $\mathscr{F}_j$ norms. These bounds are the same as the coupling constant bounds (115) and K_j is third order which manifests the idea that the recursion is staying close to the second order perturbative recursion.

The coupling constant $\check{g}_j$ has the same asymptotic behaviour as the solution to (110), but does not tend to zero unless $m^2 = 0$. By Lemma 8.5 of Bauerschmidt et al. [10] it tends to a limit $\check{g}_\infty$ and, for $\hat{g}_0$ small and positive, as $m^2 \downarrow 0$ and $g_0 \to \hat{g}_0$,

$$\check{g}_\infty \sim \frac{1}{\mathsf{B}_{m^2}}. \tag{178}$$

By the formulas for the conjugation T_j which are given in [11, (6.93)], there exist constants $a_j = O(1)$ such that $\check{\mu}_j = \mu_j + a_j \mu_j^2$. By (177) and the a-priori limitation of coupling constants to be in a small domain $\mathscr{D}_j$, this implies that

$$\mu_j = O(\chi_j \check{g}_j), \quad \mu_j' = \check{\mu}_j'(1 + O(\mu_j)) \sim \check{\mu}_j', \quad \text{as} \quad j \to \infty, \tag{179}$$

where the prime denotes the derivative of with respect to v_0. Note that $\mu_0 = \check{\mu}_0 = v_0$ by (108) and (96). Therefore

$$\check{\mu}_0' = \mu_0' = 1. \tag{180}$$

We will be applying these bounds with $j = N$ which is the scale where the finite volume recursion parts company with the infinite volume recursion; at the scale where the torus becomes a single block we are integrating out a field ζ_N with the Λ dependent covariance $C_{N,N}$ of Theorem 6.1. But, provided N is large such that $m^2 L^{2N} \ge 1$, the recursion (170) obeys the same bounds as the infinite volume recursion and we can set $j = N$ in (177). Since we studying the limits in the order $N \to \infty$ followed by $m^2 \downarrow 0$ we can assume that $m^2 L^{2N} \ge 1$.

8.3 *Proof of* (73)

By (131) and (106),

$$\begin{aligned}\hat{\chi}_\Lambda(m^2, g_0, \nu_0, z_0) =& \frac{1}{m^2} + \frac{1}{m^4\,|\Lambda|} D^2 I_N(\Lambda; 0; 1, 1) + \frac{1}{m^4\,|\Lambda|} D^2 K_N(\Lambda; 0; 1, 1) \\ =& \frac{1}{m^2} + \frac{1}{m^4|\Lambda|} D^2 e^{-V_N}(\Lambda; 0; 1, 1) + \frac{1}{m^4|\Lambda|} D^2 W_N(\Lambda; 0; 1, 1) \\ &+ \frac{1}{m^4\,|\Lambda|} D^2 K_N(\Lambda; 0; 1, 1). \end{aligned} \tag{181}$$

where cross-terms in $D^2 I_N$ are zero when $\phi = 0$ and $d\phi = 0$ because W_N defined in [11, Sect. 3.5] has no monomials of odd degree, in particular of degree one. The first term on the right-hand side of (181) can be evaluated by direct calculation, using (95), (46) and (47), to give

$$\begin{aligned} D^2 e^{-V_N}(\Lambda; 0; 1, 1) &= D^2(-V_N)(\Lambda; 0; 1, 1) \\ &= -\sum_{x,y} \mu_N L^{-2N} \delta_{xy} 1_x 1_y - \sum_{x,y} z_N(-\Delta_{xy}) 1_x 1_y \\ &= -\mu_N L^{-2N} |\Lambda|, \end{aligned} \tag{182}$$

since the quartic term τ^2 does not contribute, and $\Delta 1 = 0$. Therefore

$$\begin{aligned} \hat{\chi}_N = \frac{1}{m^2} - \frac{\mu_N L^{-2N}}{m^4} + \frac{1}{m^4} \frac{1}{|\Lambda|} D^2 W_N^0(0, 0; 1, 1) \\ + \frac{1}{m^4} \frac{1}{|\Lambda|} D^2 K_N^0(0, 0; 1, 1). \end{aligned} \tag{183}$$

For m^2 fixed the final term tends to zero as $N \to \infty$ like $O(\chi_N g_N^3) L^{-2N}$. This follows from (145) and

1. $\frac{1}{|\Lambda|} = L^{-4N}$.
2. $\|K_N\|_{\mathscr{F}_N} \le O(\chi_N g_N^3)$ by (177).

By (177) μ_N is bounded and therefore the second term tends to zero like L^{-2N}. The third term is estimated from the explicit formula for W_N and tends to zero as $O(\chi_N \bar{g}_N^2 L^{-2N})$ by Bauerschmidt et al. [10, (8.56)]. Thus we have proved Part (1) of Theorem 5.2.

8.4 *Proof of* (74)

By the remark above Theorem 2.2 we can interchange a derivative with respect to ν_0 with the infinite volume limit. Thus we take the derivative of both sides of (183) with respect to ν_0 and obtain

$$\frac{\partial \hat{\chi}}{\partial \nu_0} = -\frac{1}{m^4} \lim_{N\to\infty} L^{-2N} \frac{\partial \mu_N}{\partial \nu_0} = -\frac{1}{m^4} \lim_{N\to\infty} L^{-2N} \check{\mu}'_N, \tag{184}$$

where we used (179) to obtain the second equality. As in Sect. 8.3, there are no contributions to this derivative from the third and fourth terms in (183) because they decay to zero as $N \to \infty$. To calculate the derivative $\check{\mu}'_N$ we differentiate the recursion (175) with respect to ν_0 and obtain a recursion for the derivatives $(\check{g}'_j, \check{z}'_j, \check{\mu}'_j; \check{K}'_j)$. Since there are many terms and the details are given in Sect. 8.3 of Bauerschmidt et al. [10] we write only the terms that will turn out to be dominant. From (175) and (112) the $\check{\mu}$ equation is

$$\check{\mu}_{j+1} = L^2 \check{\mu}_j (1 - \gamma \beta_j \check{g}_j) + r_j, \tag{185}$$

where r_j is the sum of all the other terms. It changes from line to line in the next equations. Therefore

$$\begin{aligned} \check{\mu}'_{j+1} &= L^2 \check{\mu}'_j (1 - \gamma \beta_j \check{g}_j) + r'_j \\ &= L^2 \check{\mu}'_j (1 - \beta_j \check{g}_j)^\gamma + r'_j. \end{aligned} \tag{186}$$

From (175) and (110) and an estimate on the $\check{g}$ component of $\check{R}_+$,

$$\check{g}_{j+1} = \check{g}_j - \beta_j \check{g}_j^2 + O(\chi_j \check{g}_j^3) = \check{g}_j (1 - \beta_j \check{g}_j) + O(\chi_j \check{g}_j^3). \tag{187}$$

Using this to eliminate $(1 - \beta_j \check{g}_j)$ in (186) and dropping r'_j we obtain

$$\check{\mu}'_{j+1} = L^2 \check{\mu}'_j \left(\frac{\check{g}_{j+1}}{\check{g}_j} \right)^\gamma \left(1 + O(\chi_j \check{g}_j^2) \right)^\gamma. \tag{188}$$

By iterating this equality and recalling (180) we have

$$\lim_{N\to\infty} L^{-2N} \check{\mu}'_N = c(m^2, g_0) \check{\mu}'_0 \left(\frac{\check{g}_\infty}{\check{g}_0} \right)^\gamma = c(m^2, g_0) \left(\frac{\check{g}_\infty}{\check{g}_0} \right)^\gamma, \tag{189}$$

where $\check{g}_\infty = \lim \check{g}_N$ and $c(m^2, g_0)$ arises from the r'_j terms and the factors $1 + O(\chi_j \check{g}_j^2)$. The coefficient $c(m^2, g_0)$ has a limit as $m^2 \downarrow 0$. We insert (189) into (184)

and obtain

$$\frac{\partial \hat{\chi}}{\partial \nu_0} = -\frac{1}{m^4} c(m^2, g_0) \left(\frac{\check{g}_\infty}{\check{g}_0} \right)^\gamma . \tag{190}$$

By (178), we obtain the desired (74).

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