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Lectures on the Renormalisation Group

David C. Brydges

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Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: db5d@math.ubc.ca
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LECTURE 1

Scaling Limits and Gaussian Measures

1.1. Introduction

These lectures are about a class of probability measures which are common to diverse problems in statistical mechanics. In theoretical physics the corresponding integrals are called lattice *functional integrals*. We start with a finite subset Λ of the lattice $\mathbb{Z}^d$. The probability space is $\mathbb{R}^\Lambda$ which is the set of all functions $\varphi : \Lambda \to \mathbb{R}$. An element φ of this space is called a *field*. We are given a function $S : \mathbb{R}^\Lambda \to \mathbb{R}$ which is called the *action*. Letting $d\varphi$ denote Lebesgue measure on $\mathbb{R}^\Lambda$ the action determines the probability measure on $\mathbb{R}^\Lambda$ by

$$d\mu_\Lambda(\varphi) = \Xi^{-1}\, e^{-S}\, d\varphi. \tag{1.1}$$

The constant $\Xi = \Xi(\Lambda)$ normalises this to a probability measure on $\mathbb{R}^\Lambda$. It is called the *partition function*. This probability measure is also known as the finite volume *Gibbs measure*.

For example, we may visualise $\Lambda \subset \mathbb{Z}^d$ as a perfect crystal, in which the atoms are located at sites x in Λ. However, in the presence of thermal fluctuations small sound waves deform the crystal and the atom at x is slightly displaced by an amount $\vec{\varphi}(x)$ to position $x + \vec{\varphi}(x)$. The collection of displacements $\vec{\varphi} = (\vec{\varphi}(x), x \in \Lambda)$ is a vector-valued field. The function $S(\vec{\varphi})$ is the energy of the field. Thinking of the restoring forces as little springs, it is natural to choose

$$S(\vec{\varphi}) = (\beta/2) \sum \big(\vec{\varphi}(x) - \vec{\varphi}(y)\big) \cdot \big(\vec{\varphi}(x) - \vec{\varphi}(y)\big), \tag{1.2}$$

where the sum is over all pairs $\{x, y\}$ of nearest neighbours in Λ. The parameter $\beta \geq 0$ is called the *inverse temperature*. This choice of S defines an associated Gibbs probability measure as in (1.1) which is called the finite volume d-component lattice *massless Gaussian*. It is Gaussian because S is a quadratic form in $\vec{\varphi}$. The "d-components" specifies the vector-valued range of $\vec{\varphi}$. Notice that a large relative displacement $\vec{\varphi}(x) - \vec{\varphi}(y)$ contributes an exponentially small factor $\exp\big(-\beta|\vec{\varphi}(x) - \vec{\varphi}(y)|^2/2\big)$ to the Gibbs measure so a large β means that thermal fluctuations are highly suppressed, which fits in with thinking of low temperature as a land of icy perfection. In these lectures we shall study massless Gaussian measures and their anharmonic perturbations, but since this is only an introduction to a large subject we confine our attention to scalar-valued fields.

Let us set $\beta = 1$ and think of the quadratic form (1.2) as the lattice analogue of $\int (\nabla f)^2\, dx$. Recalling that integration by parts leads to $\int f(-\Delta f)$, we expect the matrix $\Delta_{x,y}$ associated to the quadratic form by $S(\vec{\varphi}) = \sum \varphi(x)(-\Delta_{x,y})\varphi(y)/2$ to be a lattice analogue of the continuum Laplacian. Perhaps a suitably defined inverse $(-\Delta)^{-1}_{x,y}$ will then resemble the continuum Green's function $|x - y|^{-d+2}$

when $x - y$ is very large and Λ is much larger still. The inverse $(-\Delta)^{-1}_{x,y}$ is in fact the covariance of the massless Gaussian.

Crystals contain an enormous number of atoms and this is recognised by taking a limit as Λ becomes large. We choose a sequence Λ_n of increasing sets and attempt to define a probability measure $d\mu$ on $\mathbb{R}^{\mathbb{Z}^d}$ by the condition,

$$\int d\mu \, e^{i(f,\varphi)} = \lim_{n\to\infty} \int d\mu_{\Lambda_n} \, e^{i(f,\varphi)} \tag{1.3}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is zero at all but finitely many points $x \in \mathbb{Z}^d$ and

$$(f,\varphi) = \sum_{x\in\mathbb{Z}^d} f(x)\varphi(x). \tag{1.4}$$

A probability measure $d\mu$ defined in this way is called an *infinite volume limit*. We want to understand macroscopic features of statistical mechanical systems so we focus on the long distance behaviour of correlations. Thus, the first fundamental question is to determine how the covariance

$$\operatorname{cov}\big(\varphi(x), \varphi(y)\big) = \int \varphi(x)\varphi(y) d\mu - \int \varphi(x) d\mu \int \varphi(y) d\mu \tag{1.5}$$

decays as $x - y \to \infty$. If the covariance decays with a power law then we write the power law in the form

$$O\big(|x-y|^{-2[\varphi]}\big) \tag{1.6}$$

where the real number $[\varphi]$ is called the *dimension of the field*. Power law exponents are called *critical exponents*. In particular, for the massless Gaussian $[\varphi]$ will turn out to be $(d-2)/2$.

The next fundamental question concerns the scaling limit. The scaling limit embodies the idea of looking at the lattice from very far away in order to lose the details of the lattice and pass to a continuum process. To formalise it, let ℓ be a positive real number. We define a scaled field, which is a random function[1] on $\mathbb{R}^d$, by

$$\varphi_\ell(x) = \ell^{[\varphi]} \varphi(\lfloor \ell x \rfloor). \tag{1.7}$$

The scaling is defined this way because from (1.6) the covariance of $\varphi_\ell(x)$ becomes const. $|x-y|^{-2[\varphi]}$ in the limit as $\ell \to \infty$. Next, anticipating that as $\ell \to \infty$ the function $\varphi_\ell(x)$ may oscillate itself out of existence, we try to save some part of it from this horrible fate by defining, for f a smooth function of compact support, the random variable

$$\int \varphi_\ell(x) f(x) \, dx. \tag{1.8}$$

The scaling limit, if it exists, is the joint distribution μ_{scale} for random variables $(\varphi_{\text{scale}}(f), f \in \mathcal{C}_0^\infty(\mathbb{R}^d))$, such that

$$\int d\mu_{\text{scale}} \, e^{i\varphi_{\text{scale}}(f)} = \lim_{\ell\to\infty} \int d\mu \, e^{i\int \varphi_\ell(x) f(x)\, dx}. \tag{1.9}$$

[1]In the usual notation capital letters are random variables and lower case letters are values of random variables but I use the same lower case letter for both concepts.

Thus, in the limit, we have a theory of random distributions in the sense of Schwartz, because although the map $f \mapsto \varphi_{\text{scale}}(f)$ is linear, we are allowing for the possibility that there is no function $\varphi_{\text{scale}}(x)$ such that

$$f \mapsto \int \varphi_{\text{scale}}(x) f(x)\, dx. \tag{1.10}$$

We shall shortly discuss the Donsker Invariance Principle as a special example of a scaling limit for which $\varphi_{\text{scale}}(x)$ does exist, but in other important cases there turns out to be no way to define a random variable for each continuum point, but, intuitively speaking, only for each neighbourhood of a point, formalised as above by the choice of a smooth function supported in such a neighbourhood.

1.2. Theoretical Physics

The renormalisation group (RG) is a program based on ideas of theoretical physicists to determine scaling limits. Theoretical physicists are usually making assumptions about the existence and properties of the scaling limit, in which case RG becomes a systematic procedure that tells them which perturbation theories around Gaussian are correct and whether their assumptions are consistent. At the most basic level it is a theory for how the scaling limit reacts to dilation (scaling) of lengths by L. Don't worry too much what this means yet, but think of it as looking at the system from L times further away, so ℓ would become ℓL in our previous formulas. For example, suppose

$$S(\varphi) = \sum_{x \in \Lambda} \Big(\frac{1}{2} (\nabla \varphi)^2(x) + g \varphi^4(x) \Big), \tag{1.11}$$

where $\nabla \varphi$ is a finite difference gradient defined more carefully later in (1.33), and you wish to test if the φ^4 term can alter the scaling limit from what it would have been if $g = 0$. (1) We find the dimension for φ from the known covariance for the Gaussian case $g = 0$. The covariance of the Gaussian is the Green's function discussed in Appendix 1.9 so this dimension is $[\varphi] = \frac{d-2}{2}$. (2) We assign a dimension to the term we are testing according to the rule

$$\underbrace{\sum_{x \in \Lambda}}_{[L]^d} \underbrace{(\varphi)^4(x)}_{[L]^{-4[\varphi]}} \tag{1.12}$$

giving $[L]^{d-4(d-2)/2}$. For dimensions $d > 4$ this is a negative power, suggesting that this term would disappear under the scaling that respects the $g = 0$ measure. Notice that the Gaussian part of $S(\varphi)$ is dimensionless when $[\varphi] = \frac{d-2}{2}$ because

$$\underbrace{\sum_{x \in \Lambda}}_{[L]^d} \big(\underbrace{\nabla}_{[L]^{-1}} \underbrace{\varphi}_{[L]^{-[\varphi]}} \big)^2(x) \tag{1.13}$$

combines to give $[L]^0$. Besides these scaling considerations, the renormalisation group contains another ingredient (integrating out small scale fluctuations) which is much deeper than a scaling argument because it allows one to know that the φ^4 term can influence the scaling limit indirectly by having a little baby before it is killed by scaling. The baby is

$$\sum_{x\in\Lambda} \varphi^2(x) \tag{1.14}$$

and if you calculate the dimension of this term in the same way you find it is $[L]^2$ which means that this baby is a Great Dane! Under scaling it will grow and invalidate the assumption that $(\nabla\varphi)^2$ determines the scaling limit. Therefore, one should now test if instead φ^2 is dominant in the limit – dominance of φ^2 will mean that the scaling limit is the same as the scaling limit for independent Gaussians at each site, whose scaling limit is called *white noise*. Terms that grow under scaling are, in Wilson's terminology, called "relevant operators". This particular term is called the *mass* term.

Here is a glimpse at the nativity of the Great Dane. We should check if the assignment of dimension in (2) is consistent with the definition (1.6) by regarding φ^4 as a new field and see if the Gaussian measure thinks that $[\varphi^4] = [\varphi]^4$ by calculating the covariance of φ^4 for the $g = 0$ measure to see if it decays like $|x-y|^{-8[\varphi]}$. Using a formula, given later, for moments of Gaussian measures one finds that $[\varphi^4] = [\varphi]^4$ is false, but instead,

$$\operatorname{cov}\bigl(\varphi^4(x) - 6\sigma^2\varphi^2(x),\ \varphi^4(y) - 6\sigma^2\varphi^2(y)\bigr) = O\bigl(|x-y|^{-8[\varphi]}\bigr), \tag{1.15}$$

where σ^2 is the variance of $\varphi(x)$. The variance is independent of x. Thus the combination $\varphi^4 - 6\sigma^2\varphi^2$ does have dimension $8[\varphi]$ suggesting that we view φ^4 as the baby $6\sigma^2\varphi^2$ and a remainder $\varphi^4 - 6\sigma^2\varphi^2$. This also suggests another idea: if we replace $g\varphi^4$ in (1.11) by

$$g\varphi^4 - a_0\varphi^2 \tag{1.16}$$

with a very cleverly chosen a_0 perhaps the mass term would not be born and we would escape from white noise into more interesting scaling limits. This special $a_0 = a_0(g)$ is called *critical* or we say that a_0 lies on the *critical manifold*. In general a_0 cannot be computed explicitly. The calculation (1.15) is only an approximation.

On the basis of the picture sketched above this more interesting scaling limit would, for dimension $d > 4$, be the same as the scaling limit for the action

$$S(\varphi) = \sum_{x\in\Lambda} \Bigl(\frac{1}{2}(\nabla\varphi)^2(x)\Bigr), \tag{1.17}$$

which is called the "massless Gaussian". For dimension d less than four the massless Gaussian would be invalidated by the $g\varphi^4$ term in (1.11) and would not be Gaussian. Non-Gaussian scaling limits are at the research frontier of our subject.

The renormalisation group gives the big picture behind these consistency checks. Much of this picture originated in the work of Ken Wilson who was awarded the Nobel prize in theoretical physics in 1982. The lectures by Wilson and Kogut [**57**] sparked my lifelong interest in this subject, partly because they looked to me like the beginnings of a complete calculus instead of a collection of special tricks. The Wilson-Kogut lectures make clear that the RG program has the capacity to *prove* the existence of scaling limits as well. In this sense RG promises a complete analytic theory for the functional integrals of theoretical physics. There is a very long way to go to realise this, but in the coming lectures, I hope to convey that it is feasible by using an example called the dipole gas as a model for a general program.

The first three lectures are about statistical mechanical models on hierarchical lattices. The notation and organisation of these lectures prepare the way for models on the Euclidean lattice $\mathbb{Z}^d$ which is the subject of the last three lectures. The gradient increases in the last three lectures not because I got tired but because the dipole model is hard! The ideas that are most critical to the proof are in lectures 4 and 5. Lecture 6 is supposed to be a rather complete collection of answers to all the questions a really determined reader might ask in order of increasing technicality.

1.3. Some Results

A complete list of rigorous RG results would be very imposing and very unhelpful, because the subject began in the unfamiliar territory of quantum field theory. Here are a few results chosen on the basis of having a close relation to these lectures or being situated in widely understood contexts. They are all long and hard proofs which is a signal that the good organisation for this subject is not yet known. Perhaps the convexity methods being developed in [**48**] can be part of a simpler program. The methods in these lectures are also an attempt to find simplicity, but it is a relative term.

- (Fröhlich and Spencer 1981) Existence of the Kosterlitz-Thouless phase transition in the two dimensional Coulomb gas: charged particles on a lattice bind into neutral clusters [**28**].
- (Bricmont and Kupiainen 1988) The ground state at low temperature of the $3d$ Ising model in a small random external field is ordered [**6**].
- (Pinson and Spencer 2000) Universality of critical exponents for the two dimensional Ising model: if the nearest neighbour Ising model is perturbed by the addition of more complicated interactions, it still has the same scaling limit. Preprint, and reported at [**54**]. See also (Giuliani and Mastropietro, 2005, [**34**]).
- (Brydges and Imbrie 2003) Self-avoiding walk in four dimensions: the end-to-end distance of a self-repelling walk on a four dimensional hierarchical lattice grows as $\sqrt{N}\log^{1/8} N$ [**16, 17**]. The same law is conjectured for the Euclidean lattice.
- (Bricmont-Kupiainen 1991, Sznitman-Zeitouni 2006) For three or more dimensions, if X_n is a possibly asymmetric random walk in a weakly random environment with reflection invariant law, then EX_n^2 is $O(n)$ [**7, 55**].
- (Brydges, Dimock and Hurd, 1998) Existence of non-Gaussian fixed points in $4-\epsilon$ dimensions [**9**].
- (Abdesselam 2007) Complete renormalisation group trajectory [**1**].
- (Mitter and Scoppola 2007) The beginning of an analysis of the scaling limit of a critical self-avoiding walk [**45**].

The results I have not mentioned here are in areas such as non-Abelian gauge theory and condensed matter and constructive quantum field theory. Some of these results go far beyond these lectures and will be briefly surveyed in Section 3.4.

1.4. Gaussian Measures on $\mathbb{R}^\Lambda$

This is the case where the action S is a quadratic form. A thorough understanding of this case is needed because in these lectures it is the basic approximation for understanding more general actions. The content of this section is standard and given in books that construct Brownian motion.

Definition 1.1. A probability measure on $\mathbb{R}^\Lambda$ is *Gaussian* (with mean zero) if

$$d\mu(\varphi) = \text{const.}\, d\varphi\, e^{-\frac{1}{2}Q(\varphi,\varphi)} \tag{1.18}$$

where $Q : \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ is a quadratic form which is strictly positive, that is $Q(\varphi, \varphi) > 0$ for $\varphi \neq 0$.

The form Q arises from a unique symmetric matrix $A = (A(x, y))_{x,y\in\Lambda}$ via

$$Q(\varphi, \varphi) = \big(\varphi, A\varphi\big) = \big(A\varphi, \varphi\big) \tag{1.19}$$

where $(f, g) = \sum_{x\in\Lambda} f(x)g(x)$. A symmetric matrix A is said to be *positive-definite* when the associated form $(\varphi, A\varphi)$ is positive-definite. In this case the eigenvalues of A are strictly positive so A is invertible. Therefore A^{-1} is also symmetric with strictly positive eigenvalues and so A^{-1} is positive-definite. Conversely, given any positive-definite symmetric matrix $C = (C(x, y))_{x,y\in\Lambda}$, a Gaussian measure on $\mathbb{R}^\Lambda$ is defined by taking $A = C^{-1}$. Therefore we can parameterise Gaussian measures by positive-definite matrices and we write

$$\mu \in N(C) \text{ or } \varphi \sim N(C) \tag{1.20}$$

when φ is distributed according to (1.18) with $A = C^{-1}$. The Laplace and Fourier transforms for $\mu \in N(C)$ are,

$$\int d\mu\, e^{(f,\varphi)} = e^{\frac{1}{2}(f,Cf)}, \quad \int d\mu\, e^{i(f,\varphi)} = e^{-\frac{1}{2}(f,Cf)}, \tag{1.21}$$

for any $f \in \mathbb{R}^\Lambda$. By taking partial derivatives with respect to components of f at $f = 0$ we have

$$\int d\mu\, \varphi(x)\varphi(y) = C(x, y). \tag{1.22}$$

Thus C is the covariance of $d\mu$.

A measure on $\mathbb{R}^\Lambda$ is said to be *degenerate Gaussian* if it is supported on a linear subspace of $\mathbb{R}^\Lambda$ and is Gaussian on the subspace. An example is the measure on $\mathbb{R}^2$ given by

$$\text{const.}\, d\varphi_1 \delta(d\varphi_2)e^{-\varphi_1^2}. \tag{1.23}$$

Degenerate Gaussian measures are on the boundary of the family of Gaussian measures; for example $\delta(d\varphi_2)$ is the weak limit of $\sqrt{2\pi N}\exp(-N\varphi_2^2/2)\, d\varphi_2$ as $N \to \infty$. Degenerate Gaussian measures have semi-definite covariances; in the example, the variance of φ_2 is zero. These measures are so similar to Gaussian measures that we will be sloppy and refer to them also as Gaussian measures.

Definition 1.2. Let μ_Y be a measure on $\mathbb{R}^Y$ and let μ_X be a measure on $\mathbb{R}^X$ where X is a finite subset of Y. We say μ_X is *compatible* with μ_Y if $\mu_X(E) = \mu_Y(E \times \mathbb{R}^{Y\setminus X})$ whenever E is $\mathbb{R}^X$ Borel measurable.

Now comes a very important property of Gaussian measures: if Y is a finite set and μ_Y is a Gaussian measure on $\mathbb{R}^Y$ and μ_X is compatible with μ_Y then μ_X is Gaussian. This is an easily checked consequence of (1.21) because the Fourier transform characterises the measure. Furthermore, if μ_Y and μ_X are Gaussian then μ_X is compatible with μ_Y iff the covariance of μ_X is a submatrix of the covariance of μ_Y, because (1.21) shows that the covariances characterise the measures.

Definition 1.3. A measure μ on $\mathbb{R}^{\mathbb{Z}^d}$ is said to be *Gaussian* if the compatible measure on $\mathbb{R}^\Lambda$ is Gaussian for all finite $\Lambda \subset \mathbb{Z}^d$.

Definition 1.4. A matrix $(C(x,y))_{x,y\in\mathbb{Z}^d}$ is said to be positive-definite if the submatrix $C_\Lambda = (C(x,y))_{x,y\in\Lambda}$ is positive-definite for all finite $\Lambda \subset \mathbb{Z}^d$.

Given a positive-definite matrix $(C(x,y))_{x,y\in\mathbb{Z}^d}$, for each finite Λ there exists a unique Gaussian measure μ_Λ on $\mathbb{R}^\Lambda$ with covariance C_Λ. Whenever $\Lambda \subset \Lambda'$, μ_Λ is compatible with $\mu_{\Lambda'}$. By Kolmogorov's theorem there exists a measure μ on $\mathbb{R}^{\mathbb{Z}^d}$ such that μ_Λ is compatible with μ for every finite $\Lambda \subset \mathbb{Z}^d$. By the last definition μ is a Gaussian measure on $\mathbb{R}^{\mathbb{Z}^d}$ and it has covariance C. It is unique if we choose the sigma algebra of measurable sets to be the smallest sigma algebra containing $E \times \mathbb{R}^{\mathbb{Z}^d\setminus X}$ for every Borel measurable $E \subset \mathbb{R}^X$ where X is finite.

1.4.1. Sums of Gaussian Fields.

Suppose that we have a sequence C_j, $j \in \{1,2,\dots,n\}$, of positive-definite matrices. Then

$$\zeta_j \sim N(C_j) \text{ and independent } \Rightarrow \sum \zeta_j \sim N(\sum C_j). \tag{1.24}$$

This also follows from (1.21).

1.4.2. Formula for Moments of Gaussian Measures

The following formula can be obtained from (1.21) by differentiating several times with respect to f and setting $f = 0$.

$$\int \prod_{i=1}^{2n} \varphi(x_i)\, d\mu = \sum_{\text{pairings}} \prod_{\{x,y\}\in\text{pairings}} C(x,y), \tag{1.25}$$

where pairings is the set of all partitions of $\{1,2,\dots,2n\}$ into subsets which each have two elements. This formula is the gateway to the *Feynman perturbation series* because it is natural to classify the pairings by graphs as in Figure 1. Figure 2 shows the graphs that result when $x_1 = x_3$ and $x_2 = x_4$ in Figure 1.

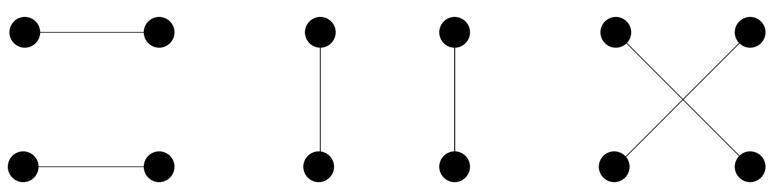

Figure 1. The three possible pairings for $\int \prod_{i=1}^4 \varphi(x_i)\, d\mu$.

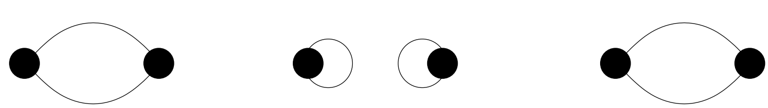

Figure 2. Pairs of points are identified in the pairings in Figure 1 to evaluate $\int \prod_{i=1}^2 \varphi^2(x_i)\, d\mu$.

An equivalent way to write this formula, is

$$\int P\, d\mu = \exp\Big(\frac{1}{2} \sum_{x,y\in\Lambda} C(x,y) \frac{\partial}{\partial\varphi(x)} \frac{\partial}{\partial\varphi(y)}\Big)_{\varphi=0} P, \tag{1.26}$$

where P is a polynomial in $\varphi(x), x \in \Lambda$ and the exponential is defined by expanding it as a power series. A good introduction to these formulas and Feynman graphs is given in Salmhofer [**51**].

Exercise 1.5. Prove (1.24).

Exercise 1.6. Prove "the very important property of Gaussian measures": if Y is a finite set and μ_Y is a Gaussian measure on $\mathbb{R}^Y$ and μ_X is compatible with μ_Y then μ_X is Gaussian.

Exercise 1.7. Prove (1.15) for $d > 2$ using (1.26) and (1.75), in the appendix. This problem is a good context to appreciate the use of Feynman diagrams [**51**].

Exercise 1.8. If the combinatorics of the previous exercise seemed ugly, deduce from (1.26) and $\partial_t f(t)g(t) = (\partial_u + \partial_v) f(u)g(v)_{u=v=t}$ a lemma of the form,

$$\int PQ\, d\mu = e^{\frac{1}{2}\Delta_P + \Delta_{PQ} + \frac{1}{2}\Delta_Q} PQ\big|_0 = e^{\Delta_{PQ}}\Big(\big(e^{\frac{1}{2}\Delta_P}P\big)\big(e^{\frac{1}{2}\Delta_Q}Q\big)\Big)\Big|_0. \tag{1.27}$$

Exercise 1.9. In the case where Λ is a single point, explain why (1.26) is a special case of the well known fact that the fundamental solution of the heat equation is the Gaussian.

1.5. Example: One Dimension

In the one dimensional case, we can think of $\varphi(x)$ as the position of a random walker at time x. In higher dimensional cases it is sometimes helpful to think of $\varphi(x)$ as the height of a random surface $z = \varphi(x)$ over $x \in \mathbb{Z}^d$.

Consider the one dimensional case, $\Lambda = \Lambda_n = \{1, 2, \ldots, n\}$, and let μ_{Λ_n} be the Gaussian measure for which

$$Q(\varphi, \varphi) = \sum_{x\in\Lambda} \big(\underbrace{\varphi(x) - \varphi(x-1)}_{\nabla\varphi(x)}\big)^2 \tag{1.28}$$

with $\varphi(0) = 0$. This choice of quadratic form Q is called the Dirichlet form and it is associated, via (1.19), with the finite difference Laplacian; indeed, the corresponding matrix A is

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -1 & 1 \end{bmatrix} \tag{1.29}$$

which is the finite difference Laplacian with the lattice analogue of the PDE notion of a Dirichlet boundary condition, because we set $\varphi_0 = 0$ in Q. The measure μ_{Λ_n} is easily checked to be the joint distribution for the first n positions, $X_k = \varphi(k),\ k \in \Lambda = \{1, 2, \ldots, n\}$, of a random walk that starts at the origin at time $k = 0$, in fact $(\varphi(k),\ k \in \Lambda)$ are, in distribution, the same as the positions of Brownian motion B_t sampled at times $t \in \mathbb{N}$. The infinite volume limit $n \to \infty$ is the law for the whole walk. To define the scaling limit, we let

$$\varphi_\ell(t) = \ell^{-\frac{1}{2}}\varphi(\lfloor \ell t \rfloor) \tag{1.30}$$

The covariance of this field does not decay, as required by (1.6), but we still continue to say that the dimension of the field is $[\varphi] = -\frac{1}{2} = (d-2)/2$ with $d = 1$ because there is a scaling limit with this choice of scaling. The cases $d = 1, 2$ are exceptional in this and other ways. The scaling limit is Brownian motion in the sense of (1.10), that is

$$\varphi_{\text{scale}}(f) = \int B_t f(t)\, dt, \tag{1.31}$$

so all the random variables $(\varphi_{\text{scale}}(f), f \in \mathcal{C}_0^\infty(\mathbb{R}))$ are determined by B_t which lives in the space of continuous functions and there is no need to consider the larger space of Schwartz distributions. According to the Donsker invariance principle [**21**] much more is true; even if we start with a more general action $S(\varphi) = \sum f(\nabla\varphi(x))$, where f is not quadratic, the scaling limit is *still* Brownian motion provided f is such that the increments have mean zero and second moments.

From the theoretical physics point of view, the Donsker Invariance principle is saying that we do not have to know the details of the microscopic world in order to make predictions on large length scales[2] because the scaling limit will bundle all our ignorance into a few parameters; in this case there is one, namely the speed of the Brownian motion. These parameters are the "renormalised coupling constants" of theoretical physics. In this view the central limit theorem is just the first example of a conjectural family of Invariance Principles associated to quantum field theories, with Brownian motion being a quantum field theory for a world with one spacetime dimension, namely time. It used to be believed that quantum field theory is a fundamental description of Nature at small scales, but this example suggests that it is more likely to be a phenomenological description of Nature at large scales. This viewpoint was nicely expressed in a very clear paper on renormalisation by Polchinski [**49**].

Exercise 1.10. Verify (1.31) in the Gaussian case.

1.6. Local and Global Functions

In this section we use an example to informally introduce the idea of a "translation invariant local function". Roughly speaking a local function assigns a function of the field, and finitely many finite difference derivatives of the field, to each point $x \in \mathbb{Z}^d$, but since the renormalisation group is a multi-scale analysis, when the same idea recurs at larger length scales, points become regions called blocks, which at each scale are regions that are pointlike in the sense that fields at that scale do not separate points inside a block because their typical fluctuations are on larger length scales.

As we have seen, infinite volume Gaussian measures can be constructed by Kolmogorov's theorem. The infinite volume limit is hidden inside Kolmogorov's theorem and compatibility. For statistical mechanics we have to work with measures whose actions are not Gaussian. A common example, known as *lattice φ^4 Euclidean quantum field theory*, is

$$S(\varphi) = \sum_{x\in\Lambda} \Big(\frac{1}{2}(\nabla\varphi)^2(x) + \frac{1}{2}m^2\varphi^2(x) + g\big(\varphi^2(x) - a_0\big)^2\Big), \tag{1.32}$$

[2]The diameter of a proton is an example of a large length scale

where $m^2 > 0$ is a parameter and $g > 0$ is called the *bare coupling constant*, $a_0 \in \mathbb{R}$ and

$$(\nabla\varphi)^2(x) = \sum_{\hat{e}\in\mathcal{E}} (\nabla_{\hat{e}}\varphi(x))^2, \quad \nabla_{\hat{e}}\varphi(x) = \varphi(x+\hat{e}) - \varphi(x) \tag{1.33}$$

with $\mathcal{E}$ being the standard basis vectors for $\mathbb{Z}^d$. A boundary condition such as $\varphi(x) = 0$ outside Λ is needed so that $\nabla\varphi$ is defined at every point $x \in \Lambda$. The Dirichlet form

$$(\nabla\varphi, \nabla\varphi) = \sum_{x\in\Lambda} (\nabla\varphi)^2(x) \tag{1.34}$$

defines the lattice Laplacian Δ as the unique symmetric matrix such that

$$(\nabla\varphi, \nabla\varphi) = (\varphi, -\Delta\varphi). \tag{1.35}$$

This matrix depends on the boundary condition and consequently, so does the Gaussian measure in the next definition, but we will not make this apparent in the notation because we will shortly restrict our attention to periodic boundary conditions.

Definition 1.11. The Gaussian measure on $\mathbb{R}^\Lambda$ with quadratic form

$$Q(\varphi, \varphi) = \sum_{x\in\Lambda} \left((\nabla\varphi)^2(x) + m^2\varphi^2(x)\right), \tag{1.36}$$

where $m^2 > 0$, is called the *massive Gaussian (free field)*. m^2 is called the *mass squared*.

Let $d\mu$ be the massive Gaussian measure and introduce the peculiar notation

$$F(\{x\}) = e^{-g\left(\varphi^2(x) - a_0\right)^2}, \quad F^\Lambda = \prod_{x\in\Lambda} F(\{x\}) \tag{1.37}$$

so that we can write $\Xi^{-1} e^{-S}\, d\varphi$ as

$$\Xi^{-1} d\mu(\varphi)\, F^\Lambda, \tag{1.38}$$

where a Gaussian normalisation constant has been absorbed into Ξ.

We call F a "local function" to contrast it with F^Λ which is called a "global function" because it depends on all fields in Λ. A *local function* $F : x \mapsto F(\{x\})$ assigns to each site $x \in \mathbb{Z}^d$ an *interaction* $F(\{x\})$. $F(\{x\})$ depends on $\varphi(x)$ and the lattice derivatives, up to some fixed order, of φ evaluated at x. Since we allow dependence on derivatives $F(\{x\})$ is a function of the fields $(\varphi(y), y \in \{x\}^*)$ in a neighbourhood $\{x\}^*$ of $\{x\}$. The example (1.37) is translation invariant; there is one function $f(t) = e^{-g(t^2 - a_0)^2}$ such that $F(\{x\}) = f(\varphi(x))$ for every $x \in \Lambda$. Translation T_y by $y \in \mathbb{Z}^d$ acts on a local function F by $(T_yF)(\{x\}, \varphi) = F(\{x + y\}, T_y\varphi)$ where $T_y\varphi(x) = \varphi(x - y)$. We say a local function F is translation invariant if $T_yF = F$ for all $y \in \mathbb{Z}^d$.

The scope of these lectures is the analysis of probability measures that have the form $\Xi^{-1} d\mu(\varphi)\, F^\Lambda$ with $F(\{x\})$ being translation invariant. The objective is to understand infinite volume limits of these measures as perturbations of μ, so there will be a $F \approx 1$ hypothesis.

In contrast to the Gaussian case, the infinite volume limit is now a difficult problem because as Λ grows the global function F^Λ makes the normalisation constant behave like $\exp(O(\text{volume}))$ so that a characteristic function such as

$$\int e^{i\varphi(a)} e^{i\varphi(b)} F^\Lambda d\mu / \int F^\Lambda d\mu \tag{1.39}$$

is the ratio of two $\exp(O(\text{volume}))$ factors which must be canceled perfectly in order to study the infinite volume limit. I will call this the *exp(volume) problem.*

We write the ratio as

$$\int F^\Lambda_{a,b}\, d\mu \Big/ \int F^\Lambda d\mu \tag{1.40}$$

with

$$F_{a,b}(\{x\}) = F(\{x\}) e^{i\varphi(x)(1_{x=a}+1_{x=b})}, \tag{1.41}$$

to emphasise we are studying ratios of integrals with local functions F and $F_{a,b}$ which are the same except at the sites $\{a, b\}$. More generally we suppose that the local function F in the denominator is translation invariant and the local function in the numerator is the same except at finitely many points, which are $\{a, b\}$ in this case.

1.7. Example: Particles on a Lattice

This example shows one of the ways that models in statistical mechanics are of the form (1.38). Theoretical physicists have succeeded in writing almost every model they encounter in this form. In some cases they end up with S imaginary which leads into the largely unknown land of oscillatory infinite dimensional integrals. In other cases, S is "Fermionic". This is also known as "Grassmann" integration. Renormalisation group methods for these cases are discussed in [**51**]. These are avenues to such problems as self-avoiding walk [**16, 17**] and delocalisation [**53**].

Suppose each point in Λ can either be empty, or occupied by a positively charged particle, or occupied by a negatively charged particle. In order to later study fractional charges we will take the charge to be 2. Then the space of particle configurations is $\{-2, 0, 2\}^\Lambda$. For $\eta \in \{-2, 0, 2\}^\Lambda$ define the *interaction energy* by

$$U(\eta) = \frac{1}{2} \sum_{x,y\in\Lambda} \eta(x) C(x, y) \eta(y), \tag{1.42}$$

where C is some positive-definite matrix called a *two body potential.* Let

$$n = \sum_{x\in\Lambda} |\eta(x)|/2 \tag{1.43}$$

be the total number of particles. The probability distribution on $\{-2, 0, 2\}^\Lambda$ is given by

$$\mathbb{P}(\{\eta\}) = \Xi^{-1} z^n e^{-\beta U(\eta)}, \tag{1.44}$$

where $z \geq 0$ and $\beta \geq 0$ are parameters. z is called the *activity* and β is called the *inverse temperature.* The constant Ξ is called the *partition function* and since it normalises the measure it equals

$$\Xi = \sum_{\eta\in\{-2,0,2\}^\Lambda} z^n e^{-\beta U(\eta)}. \tag{1.45}$$

Theorem 1.12. *(Kac-Siergert transformation). Let* $F(\{x\}) = 1 + 2z\cos(2\varphi(x))$. *Then*

$$\Xi = \int d\mu(\varphi) F^{\Lambda} \tag{1.46}$$

where $\mu \in N(\beta C)$.

The message in the following proof is that a charge q at x in the φ language is a factor $e^{iq\varphi(x)}$. For example, $1 + 2z\cos(2\varphi(x))$ arises as $\sum_{q\in\{-2,0,2\}} z^{|q|/2} e^{iq\varphi(x)}$ which is a sum over what charges can occupy the single point x.

Proof. If the interaction energy U is set to zero we can calculate the characteristic function,

$$\sum_{\eta\in\{-2,0,2\}^{\Lambda}} z^n e^{i(\varphi,\eta)} = \prod_{x\in\Lambda} \underbrace{\sum_{\eta(x)\in\{-2,0,2\}} z^{|\eta(x)|/2} e^{i\eta(x)\varphi(x)}}_{F(\{x\})} = F^{\Lambda}. \tag{1.47}$$

Independently, Kac and Siegert, [**41**], [**52**], noticed that one can return to the case $U \neq 0$ by integrating over φ. Let $\mu \in N(\beta C)$, then

$$\begin{aligned} \int d\mu(\varphi) F^{\Lambda} &= \int d\mu(\varphi) \sum_{\eta\in\{-2,0,2\}^{\Lambda}} z^n e^{i(\varphi,\eta)} \\ &= \sum_{\eta\in\{-2,0,2\}^{\Lambda}} z^n \int d\mu(\varphi) e^{i(\varphi,\eta)} \\ &= \sum_{\eta\in\{-2,0,2\}^{\Lambda}} z^n e^{-\beta U} = \Xi. \end{aligned} \tag{1.48}$$

□

1.8. The Importance of the Partition Function

Throughout these notes we emphasise the partition function. This is standard in theoretical physics because the partition function turns out to know everything about $\mathbb{P}$ if you put questions to it in a polite way. By politeness we mean that the reaction of the partition function to modifications at finitely many sites can be probed. We already see this idea at work in (1.41).

Consider, for example, the partition function (1.45). It contains the factor

$$z^n = \prod_{x\in\Lambda} z^{|\eta(x)|/2}. \tag{1.49}$$

We generalise the notation and allow z to depend on the site x in the lattice as in

$$z^n \mapsto \prod_{x\in\Lambda} z_x^{|\eta(x)|/2}. \tag{1.50}$$

The effect of the derivative $z_a\partial/\partial z_a$ is to replace $z_a^{|\eta(a)|/2}$ by $z_a^{|\eta(a)|/2}|\eta(a)|/2$. Therefore, by differentiating with respect to z_a and then setting all $z_x = z$ we obtain

$$z\frac{\partial}{\partial z_a}\log\Xi = \int d\mathbb{P}\,|\eta_a|/2. \tag{1.51}$$

By differentiating twice we obtain the covariance as in

$$z^2 \frac{\partial^2}{\partial z_a z_b} \log \Xi = \operatorname{cov}\big(|\eta_a|/2; |\eta_b|/2\big). \tag{1.52}$$

More generally, derivatives of the log of the partition function with respect to local parameters give linear combinations of moments of $\mathbb{P}$ which, in statistics, are called *cumulants*.

Exercise 1.13. Referring to (1.41), why is E_{ab}, which is defined by

$$e^{-\beta E_{a,b}} = \int d\mu\, F^{\Lambda}_{a,b} \Big/ \int d\mu\, F^{\Lambda}, \tag{1.53}$$

interpreted as the energy $E_{a,b}$ of two immersed fractional charges?

Exercise 1.14. *Dipole gas* and functions of $\nabla\varphi$. A dipole is a pair of opposite charges on adjacent sites in Λ so a dipole is specified by giving the site x where the positive charge is located and a unit vector $\hat{e} \in \mathcal{E}(\pm)$,

$$\mathcal{E}(\pm) = \{\hat{e} \in \mathbb{Z}^d : \|\hat{e}\| = 1\}, \tag{1.54}$$

such that the negative charge is at site $x + \hat{e}$. Let E be the set of all oriented edges $(x, x+\hat{e})$ such that x and $x+\hat{e}$ are in Λ. If each oriented edge $(x, \hat{e}) \in E$ is permitted to be empty or occupied by only one dipole then the space of dipole configurations is $\{0,1\}^E$, a configuration in $\{0,1\}^E$ is denoted by η and $n = \sum_{(x,\hat{e})\in E} \eta(x,\hat{e})$ is the number of dipoles. The probability is given by (1.44), but where $U(\eta)$ is the potential energy of the charges specified by the dipole configuration η. Show that the partition function $\Xi = \int d\mu(\varphi) F^E$ where

$$F^E = \prod_{(x,\hat{e})\in E} \big(1 + z e^{-i\nabla_{\hat{e}}\varphi(x)}\big) = \prod_{\{x,y\}} \big(1 + z^2 + 2z\cos(\varphi(x) - \varphi(y))\big), \tag{1.55}$$

where the second product is over unoriented pairs $\{x, y\}$ of nearest neighbours in Λ.

Exercise 1.15. Can the partition function for an Ising model be expressed in the form $\int d\mu\, F^{\Lambda}$? Let Λ be periodic as in (2.1). Consider the model with interaction

$$\Xi = 2^{-\Lambda} \sum_{\sigma \in \{-1,1\}^{\Lambda}} e^{\beta \sum_{x,y\in\mathbb{Z}^d} w_{x,y}\sigma_x\sigma_y} \tag{1.56}$$

with the matrix inverse $w_{x,y} = (1-\Delta)^{-1}_{x,y}$. Prove that $\Xi = \int d\mu\, F^{\Lambda}$ where $d\mu$ is the massive Gaussian measure of Defn 1.11 with $m^2 = 1$ and

$$F^{\Lambda} = e^{\sum_{x\in\Lambda} \log\cosh(\sqrt{\beta}\varphi(x))} = e^{-\sum_{x\in\Lambda}\left(\frac{1}{12}\beta^2\varphi^4(x) - \frac{1}{2}\beta\varphi^2(x) + O(\beta^3\varphi^6(x))\right)}. \tag{1.57}$$

Details may be found in [**8**]. The result of this problem in combination with the scaling arguments in Section 1.2 indicate that the Ising model will have the same scaling limit as the massless free field in dimensions greater than four. This is an open problem which could be addressed by the methods of these lectures. It was partially settled by completely different ideas in [**2, 26**].

1.9. Appendix. Green's Functions

This appendix shows some of the standard methods for proving existence and estimates for inverses of lattice partial differential operators and the meaning of the phrase "mass" is tied to whether or not the inverses decay exponentially.

1.9.1. Continuum Greens Functions

First let us recall basic facts about the continuum Green's functions. $G(x,y)$ is said to be a Green's function for $m^2 - \Delta$ if $G(x,y)$ is an inverse to the differential operator $m^2 - \Delta$ and "inverse" is defined by demanding that G is a weak solution to

$$(m^2 - \Delta) G(x,y) = \delta(x-y). \tag{1.58}$$

We shall not need it, but if you want to verify this equation, the definition of weak solution means that G satisfies

$$\int G(x,y)(m^2 - \Delta) f(x)\, dx = f(y) \tag{1.59}$$

for all $y \in \mathbb{R}^d$ and all $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. The solutions are not unique. In dimensions $d > 2$ or if $m^2 > 0$ the condition that they tend to zero at infinity fixes them. For the massless case, that is $m^2 = 0$, the standard choices are

$$G(x,y) = \begin{cases} \frac{c_d}{|x-y|^{d-2}} & d \neq 2 \\ \frac{1}{2\pi} \ln |x-y|^{-1} & d = 2 \end{cases}, \tag{1.60}$$

where c_d^{-1} is the volume of the unit $d-1$ dimensional sphere. For the massive case

$$G(x,y) = \begin{cases} \frac{1}{2m} e^{-m|x-y|} & d = 1 \\ \frac{c_d}{|x-y|^{d-2}} e^{-m|x-y|} & d \text{ odd} \end{cases}. \tag{1.61}$$

There are explicit formulas in terms of Bessel functions in even dimensions. The massive cases have exponential decay and the massless cases decay with power-laws.

1.9.2. Lattice Green's Functions

Now we want to solve (1.58) but for the lattice Laplacian and with $\delta(x-y)$ being the identity matrix, also known as the Kronecker delta. Explicit formulas can be found in terms of the Fourier transform.

Let $B = [-\pi, \pi]^d$. The Fourier transform of a summable function $f : \mathbb{Z}^d \to \mathbb{C}$ is given by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{-ik\cdot x} \tag{1.62}$$

where $k \in B$ and, if $\hat{f}$ is integrable, then there is the inversion theorem,

$$f(x) = (2\pi)^{-d} \int_B \hat{f}(k) e^{ik\cdot x}\, dk. \tag{1.63}$$

The lattice Δ of a function f at lattice site x is $\sum (f(x+\hat{e}) - f(x))$. Taking the case $f(x) = e^{ik\cdot x}$ and we obtain

$$\big(m^2 - \Delta\big) e^{ik\cdot x} = \big(m^2 + \underbrace{\sum_{\hat{e} \in \mathcal{E}(\pm)} (1 - e^{ik\cdot\hat{e}})}_{\widehat{\Delta}(k)}\big) e^{ik\cdot x}. \tag{1.64}$$

Before continuing, notice the low momentum approximation $\widehat{\Delta}(k) = k\cdot k + O(k^4)$ is the same as (minus) the continuum Laplacian. This is the reason why the lattice Green's function resembles the continuum Green's function at large distances when there is no mass. For $m^2 > 0$ we now come to a nice formula for the inverse G of $m^2 - \Delta$

$$G(x,y) = (2\pi)^{-d} \int_B \frac{1}{m^2 + \widehat{\Delta}(k)} e^{ik\cdot(x-y)}\, dk. \tag{1.65}$$

The sense in which this is an inverse is that $m^2 - \Delta$ is a symmetric bounded operator on the Hilbert space $l^2(\mathbb{Z}^d)$ and this is the inverse operator. To verify this it is useful to know that the Fourier transform has properties that are analogous to the continuum case. For example, after reapportioning the $(2\pi)^{-d}$, it is unitary as a map from $l^2(\mathbb{Z}^d)$ to $L^2(B)$.

One can use this formula to prove that G decays exponentially. For simplicity, suppose $d = 1$. Then

$$G(x,y)e^{\kappa(x-y)} \propto \int_{[-\pi,\pi]} \frac{1}{m^2 + \widehat{\Delta}(k)} e^{i(k-i\kappa)(x-y)}\, dk. \tag{1.66}$$

Consider the closed path in the complex plane consisting of the straight line segments

$$[-\pi,\pi], \quad [\pi, \pi - i\kappa], \quad [\pi - i\kappa, -\pi - i\kappa], \quad [-\pi - i\kappa, -\pi]. \tag{1.67}$$

By the periodicity of $m^2 + \widehat{\Delta}$ contributions from the vertical segments to the contour integral along this closed path cancel. By Cauchy's theorem,

$$G(x,y)e^{\kappa(x-y)} \propto \int_{[-\pi,\pi]} \frac{1}{m^2 + \widehat{\Delta}(t + i\kappa)} e^{it(x-y)}\, dt. \tag{1.68}$$

The closed path must not contain any singularities, which is the same as saying it must not contain zeros of $m^2 + \widehat{\Delta}$ and this is so if $|\kappa| < m$. If this condition holds the integrand is a continuous function on the compact set $[-\pi,\pi]$ so it is integrable and therefore $G(x,y)e^{\kappa(x-y)}$ is bounded uniformly in x, y. Therefore $G(x,y)$ decays exponentially

$$|G(x,y)| \leq \text{const.}\, e^{-\kappa|x-y|} \tag{1.69}$$

for any $\kappa < m$. For $d > 1$ one can get exponential decay by the deformation of the contour of integration carried out in each component of k.

This proof of exponential decay seems to rest completely on Fourier transform which in turn depends on the fact that $m^2 - \Delta$ commutes with translations. It is often useful to know that Combes and Thomas [**18**] discovered how to carry out this proof in a way that does not need translation invariance. Suppose that $G = (m^2 - L)^{-1}$ where L is a lattice differential operator. Let M be the linear operator that multiplies a function $f \in l^2(\mathbb{Z}^d)$ by $e^{\kappa\cdot x}$,

$$M : f(x) \mapsto e^{\kappa\cdot x} f(x). \tag{1.70}$$

If κ is pure imaginary then M and M^{-1} are bounded linear operators and

$$M^{-1}GM = (m^2 - M^{-1}LM)^{-1}. \tag{1.71}$$

It is reasonable to guess that this relation will analytically continue to κ real, and it does, provided $m^2 - M^{-1}LM$ remains invertible during the analytic continuation. The conjugation of L by M is equal to the replacement of ∇ by $M\nabla M^{-1}$ in L and

$$M\nabla_{\hat{e}}M^{-1} : f(x) \mapsto e^{-\kappa\cdot\hat{e}} f(x + \hat{e}) - f(x). \tag{1.72}$$

For operators L with bounded coefficients it follows that for κ/m small

$$MLM^{-1} - L \tag{1.73}$$

is a bounded operator whose bound tends to zero as $\kappa \to 0$. Therefore $(m^2 - L)$ has a bounded inverse for κ/m small and exponential decay of G follows immediately by

$$\begin{aligned} |e^{\kappa\cdot x}G(x,y)e^{-\kappa\cdot y}| &= |\langle \delta_x, MGM^{-1}\delta_y\rangle| \\ &\leq \|\delta_x\|\|MGM^{-1}\|\|\delta_y\| \leq \|MGM^{-1}\|. \end{aligned} \tag{1.74}$$

In dimension $d > 2$ the Fourier formula continues to define an inverse to $m^2 - \Delta$ even for $m = 0$, but the inverse is unbounded, as an operator on $l^2(\mathbb{Z}^d)$. The formula can be used to prove that as $x - y \to \infty$ the Green's function is asymptotic to the continuum Green's function,

$$G(x,y) \sim \frac{c_d}{|x-y|^{d-2}}. \tag{1.75}$$

For $m > 0$ and $\mathbb{Z}^d$ periodic, Green's functions are obtained by summing over periods as in (2.14).

LECTURE 2

Renormalisation group in hierarchical models

2.1. Massless Gaussian Measure

The most important case of (1.38) is when m^2 is very small or zero. This case where it is zero is called the *massless Gaussian*. When it is zero, the quadratic form Q becomes the Dirichlet form which annihilates constant fields, leading to a failure to be positive-definite, which must be cured either by a boundary condition such as $\varphi = 0$ outside Λ or by changing the measure space from $\mathbb{R}^\Lambda$ to equivalence classes $\mathbb{R}^\Lambda/\{\text{constant fields}\}$. Regardless of how the form is made positive-definite, the key problem emerges as the infinite volume limit is taken. As Λ grows, the eigenvalues of Δ become close to zero and Δ is losing invertibility. The covariance of the Gaussian measure μ is becoming an unbounded operator on $l^2(\mathbb{Z}^d)$ and this shows up in long range correlations expressed as power law decay of the covariance. We want to regard F^Λ as a perturbation, but it is not obvious that assuming $F \approx 1$ in (1.38) helps because a long range correlation in μ allows all the factors $F(\{x\})$ in a huge subset of Λ to collude in forming a big perturbation. Think, for example of the behaviour of $\prod_x \exp(-g\varphi^4(x))$ when φ is constant in a large region. In these lectures we are aiming for a systematic theory for perturbations of the massless Gaussian and the prelude is to organise the fluctuations of φ according to scales. In this section we explain a particularly convenient way to do this.

2.2. Finite Range Decompositions

From now on we assume that $\Lambda = \Lambda_N$ is a cube of side L^N, where $L \geq 3$ is an odd integer and N is a positive integer. Thus,

$$\Lambda = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq \frac{1}{2}(L^N - 1)\} \tag{2.1}$$

where, for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, $\|x\|_\infty = \max_i |x_i|$. Except for Sections 2.4 to 3.1 on hierarchical models, we assume periodic boundary conditions; this means that opposite sides of the cube are identified so that x is identified with $x + L^N \hat{e}$ for $\hat{e} \in \mathcal{E}$, where $\mathcal{E}$ is the set of standard basis vectors that generate $\mathbb{Z}^d$. For $x, y \in \Lambda$, $x \pm y$ are defined by addition/subtraction of coordinates modulo L^N. A matrix $C = (C(x,y))_{x,y\in\Lambda}$ is said to be translation invariant if it is a function of $x - y$.

Definition 2.1. Let $C = (C(x,y))_{x,y\in\Lambda}$ be positive-definite and translation invariant. We say that C admits a *finite range decomposition* if there are translation invariant positive-definite matrices $C_j = (C_j(x,y))_{x,y\in\Lambda}$ such that

$$\begin{aligned} &C = \sum_{j=1}^{N} C_j \\ &C_j(x,y) = 0 \quad \text{if } |x-y| \geq \frac{1}{2}L^j. \end{aligned} \tag{2.2}$$

The important consequence comes from (1.24); if $\varphi \sim N(C)$ and C has a finite range decomposition, then there exist independent $\zeta_j \sim N(C_j)$ such that

$$\varphi = \sum_{j=1}^{N} \zeta_j \tag{2.3}$$

where the equality is in distribution and where ζ_j is called a *fluctuation field* on scale j. Notice that $\zeta_j(x)$ and $\zeta_j(y)$ are independent if $|x-y| \geq \frac{1}{2}L^j$ because they are Gaussian and their covariance is zero.

For much of our development nothing prevents the stupid choice $C_j = 0,\ j = 1, \dots, N-1$, but the only decompositions that lead to interesting conclusions *saturate* bounds which are uniform in N as in the following definition.

Definition 2.2. We say a finite range decomposition satisfies a *dimension* $[\varphi]$ *estimate* if,

$$|\nabla^{2\alpha} C_j(0,0)| \leq c_{[\varphi]}(\alpha, L) L^{-2(j-1)([\varphi]+|\alpha|_1)}, \quad \forall j \in \{1, \dots, N-1\}, \tag{2.4}$$

where $\alpha \in \mathbb{N}_0^{\mathcal{E}}$ and $\nabla^\alpha = \prod_{\hat{e} \in \mathcal{E}} \nabla_{\hat{e}}^{\alpha(\hat{e})}$ and $|\alpha|_1 = \sum \alpha_{\hat{e}}$. The constants $c_{[\varphi]}(\alpha, L)$ are independent of L when $[\varphi] + |\alpha|_1 > 0$. The finite difference derivative $\nabla_{\hat{e}}$ was defined in (1.33).

Remark 2.3. If there is a finite range decomposition for some L, for example for $L = 3$, then there exists a finite range decomposition $\sum C_j'$ where $L = 3$ is replaced by $L = 3^p$, with p any positive integer, because we can set $C_1' = C_1 + C_2 + \dots + C_p$ and $C_2' = C_{p+1} + \dots + C_{2p}$ etc. Thus we can assume L is large in proofs. This observation also determines the L dependence of the constants in a dimension $[\varphi]$ estimate: by inserting (2.4) with the $L = 3$ into $C_j' = \sum_{k=1}^p C_{(j-1)p+k}$ we obtain (2.4) with

$$c_{[\varphi]}(\alpha, L) = c_{[\varphi]}(\alpha, 3) \sum_{k=0}^{p-1} L^{-2k([\varphi]+|\alpha|_1)}, \qquad L = 3^p \tag{2.5}$$

Therefore, $c_{[\varphi]}(\alpha, L)$ is bounded in L when $[\varphi] + |\alpha|_1 > 0$ because the right hand side is convergent as $p \to \infty$, whereas when, for example $[\varphi] + |\alpha|_1 = 0$, then

$$c_{[\varphi]}(\alpha, L) = O(p) = O(\ln L). \tag{2.6}$$

Existence. The existence of such decompositions is not trivial. In [**10**] we prove, for dimension $d > 2$ and

$$[\varphi] = \frac{d-2}{2}, \tag{2.7}$$

that the covariance of the massless Gaussian admits a finite range decomposition that saturates (2.4). Since the massless Gaussian field is an equivalence class it is more correct to express the decomposition in the form

$$\varphi(x) - \varphi(x_*) = \sum (\zeta_j(x) - \zeta_j(x_*)), \tag{2.8}$$

where x_* is a basepoint. In three or more dimensions the base point can be chosen on the boundary, but in two dimensions this cannot be done, if the boundary is subsequently to be removed to infinity by taking an infinite volume limit, because the variance of $\varphi(x) - \varphi(x_*)$ will become infinite. Since functionals of the massless Gaussian field have to be well defined on equivalence classes modulo constants the subtraction of the random constants $\varphi(x_*), \zeta_j(x_*)$ have no effect and are often omitted.

Two dimensional massless Gaussian. No finite range decomposition for this case was given in [**10**], but I think that paper can be extended to give a finite range decomposition for $\nabla^{(x)}\nabla^{(y)}C(x,y)$, which is the covariance of $\nabla\varphi$. From this it will follow that there is a decomposition $\varphi(x) - \varphi(0) = \sum(\zeta_j(x) - \zeta_j(0))$ where ζ_j are Gaussian and their covariances are finite range and satisfy (2.4) with $[\varphi] = 0$. In particular, from (2.6), $c_{[\varphi]}(0, L) = O(\ln L)$. From (2.4) we see that this constant controls the variance of the fluctuations ζ_j so that

$$\operatorname{var}(\zeta_j(x)) = O(\ln L). \tag{2.9}$$

2.3. Motivation

Def. 2.1 and 2.2 axiomatise the properties of Gaussian measures that will be used in the coming analysis. In this section we motivate why the massless Gaussian saturates (2.4) with $[\varphi] = (d-2)/2$ and explain how the massive Gaussian fits in. Most of this section is not needed for the rest of our logical development.

Def. 2.1 and 2.2 are suggested by the following Lemma. Recall that a continuous function $f(x)$ defined on $\mathbb{R}^d$ is said to be *positive-definite* if the $n \times n$ matrix $f(x_i - x_j)$ is positive-definite for all finite sets $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$.

Lemma 2.4. *Let $L > 1$, $[\varphi] > 0$. There exists a $\mathcal{C}^\infty$ positive-definite function $u(x)$ supported in the ball of radius $\frac{1}{2}$ such that*

$$|x - y|^{-2[\varphi]} = \sum_{j \in \mathbb{Z}} L^{-2j[\varphi]} u(L^{-j}(x - y)) \tag{2.10}$$

for $x \neq y$.

Proof. Let $v(x)$ be a smooth positive-definite function supported in the ball of radius $\frac{1}{2}$ which is a function of $|x|$. Then

$$f(x) = \int_0^\infty \frac{d\ell}{\ell} \ell^{-2[\varphi]} v(x/\ell) \tag{2.11}$$

is an absolutely convergent integral for $x \neq 0$. Furthermore, by change of variables, $f(ax) = a^{-2[\varphi]} f(x)$ for any $a > 0$. Therefore $f(x) = \text{const.}\, |x|^{-2[\varphi]}$. Absorb the constant into a redefinition of v. Then

$$|x|^{-2[\varphi]} = \sum_{j \in \mathbb{Z}} \int_{L^{j-1}}^{L^j} \frac{d\ell}{\ell} \ell^{-2[\varphi]} v(x/\ell). \tag{2.12}$$

Let

$$u(x) = \int_{L^{-1}}^{1} \frac{d\ell}{\ell} \ell^{-2[\varphi]} v(x/\ell). \tag{2.13}$$

Then the Lemma is valid with this choice of u.

$\square$

The relevance of this Lemma is that the decaying Green's function for the continuum Laplacian on $\mathbb{R}^d$ is proportional to $|x - y|^{-2[\varphi]}$ with $[\varphi] > 0$ given by (2.7) in dimensions $d > 2$.

One can get a one sided decomposition labeled by $j \in \mathbb{N}$ by conglomerating all $j \leq 1$ terms into a single new $j = 1$ term but this term would be singular at $x = y$.

This singularity is because, unlike the lattice case, the continuum Greens function is singular for $x = y$. This is also the reason why the decomposition is two sided. In three dimensions the decaying Green's function for $m^2 - \Delta$ is proportional to $G(x-y) = |x-y|^{-1}e^{-m|x-y|}$. Finite range decompositions analogous to Lemma 2.4 for this G were found in [**38**]. The main effect of the mass is to accelerate the convergence of the sum over j when j exceeds the threshold defined by $mL^j > 1$. For $m^2 > 0$ a periodic Green's function on a cube of side $R = L^N$ exists and is given by

$$G_{\text{periodic}}(x-y) = \sum_{p \in \text{periods}} G(x-y-p), \quad \text{periods} = L^N \mathbb{Z}^d. \tag{2.14}$$

Through this formula a finite range decomposition for G_{periodic} is immediately obtained from the decomposition for G. On the periodic cube there are no distances greater than L^N so we can conglomerate the terms with $j \geq N$ into one term and rename it as the $j = N$ term. This is why a sum over $j \in \{1, \ldots, N\}$ is natural in Def. 2.1. However, as $m^2 \to 0$ this redefined $j = N$ term diverges with $m^2 \to 0$ so we do not impose an estimate on this last term in Def. 2.1.

Coming now to the lattice, the finite difference Laplacian Δ associated with $\mathbb{Z}^d$ is a bounded nonnegative symmetric operator acting on $\ell^2(\mathbb{Z}^d)$. Therefore it has a resolvent $G = (m^2 - \Delta)^{-1}$ for $m^2 > 0$. The associated matrix $G(x,y) = (\delta_x, G\delta_y)$ is positive-definite and so there is an infinite volume Gaussian measure μ_G defined in $\mathbb{R}^{\mathbb{Z}^d}$ whose covariance is $G(x,y)$. This is called the lattice *massive free field*. $G(x,y)$ is finite on the diagonal $x = y$. In [**10**], we prove that G has a finite range decomposition except that the sum is infinite, $j \in \mathbb{N}$. A finite range decomposition then holds for the periodic cube Λ, by the same idea (2.14) of summing over periods. As in the continuum case, all terms in the decomposition have limits as $m^2 \to 0$ except the $j = N$ term which diverges as $m^2 \to 0$. This reflects the fact that the massless Gaussian free field is defined on equivalence classes $\mathbb{R}^\Lambda/\{\text{constant fields}\}$ as we discussed earlier.

Exercise 2.5. Prove that a smooth (isotropic) positive-definite function of compact support exists. Cf. first line in proof of Lemma 2.4.

Exercise 2.6. In Section 1.7 choose C to be the inverse of $m^2 - \Delta$ and Λ as in (2.1). Let $m^2 \to 0$ and thereby find a version of Theorem 1.12 which is valid in the limit $m^2 \to 0$. Hint: neutrality.

2.4. The Renormalisation Group and Hierarchical Models

Let μ be a Gaussian measure on $\mathbb{R}^\Lambda$ whose covariance admits a finite range decomposition $C = \sum C_j$ and let μ_j be the Gaussian measure with covariance C_j. Then,

$$\int d\mu\, F^\Lambda = \int d\mu_N \int d\mu_{N-1} \ldots \int d\mu_1\, F^\Lambda \tag{2.15}$$

where on the left hand side $F = F(\varphi)$ and on the right hand side we insert $\sum \zeta_j$ in place of φ. Define the expectation $\mathbb{E}_j = \int d\mu_j$ and rewrite this equation as

$$Z_0 = F^\Lambda, \quad Z_j = \mathbb{E}_j Z_{j-1}, \text{ for } j = 1, \ldots, N, \quad Z_N = \int d\mu\, Z_0. \tag{2.16}$$

The expectation $\mathbb{E}_j$, viewed as a map on global functions is called a *renormalisation group transformation or* $\tilde{\mathrm{RG}}$ *map* and the *renormalisation group* is the set of $\tilde{\mathrm{RG}}$ maps $\{\tilde{\mathrm{RG}}_j = \mathbb{E}_j : j = 1, \dots, N\}$. You will be disappointed to learn that it is not a group in this context. There was a group in an original formulation of Gell-Mann and Low [**33**], but Wilson had to trade the group in as the price for a clear foundation for his theory in the statistical mechanics of lattice models. For Gell-Mann and Low it was a statement about a scaling limit whose existence was not proved. We write

$$\varphi_j = \sum_{k>j}^{N} \zeta_k, \quad \varphi_j = \varphi_{j+1} + \zeta_{j+1}, \text{ for } j \in \{0, 1, 2, \dots, N-1\}, \quad \begin{cases} \varphi_0 = \varphi \\ \varphi_N = 0 \end{cases} . \tag{2.17}$$

All this will remind probabilists of backward martingales and in fact the global function F^Λ is the first random variable of a backward martingale where the time is the scale j. But, martingale language is not at the heart of the challenge of the infinite volume, whereas all the coming discussion on the "global to local problem" is.

The purpose behind the coming blizzard of definitions is to show that the action of $\tilde{\mathrm{RG}}$ on the global function F^Λ is equivalent to an action (RG) on the local function F. We will call this the *global to local* program. To begin with this will be achieved by making an unreasonable change in the structure of the covariance C. When making this "hierarchical" assumption it is customary to make apologies for not enduring the full resentment of the Euclidean lattice, but it is a useful prelude and the message of lectures IV – VI will be that a natural extension of the hierarchical machinery suffices for Euclidean models.

Definition 2.7. (a) *Blocks.* For each $j = 0, 1, \dots, N$, the torus Λ is paved in a natural way by L^{N-j} disjoint cubes of side L^j. Recalling that $L \geq 3$ is an odd integer, the cube that contains the origin has the form,

$$\{x \in \Lambda : |x| \leq \frac{1}{2}(L^j - 1)\} \tag{2.18}$$

and all the other cubes are translates of this one by vectors in $L^j\mathbb{Z}^d$. We call these cubes *j-blocks*, or *blocks* for short, and denote the set of j-blocks by $\mathcal{B}_j = \mathcal{B}_j(\Lambda)$. The $j = 0$ blocks are single points, $B = \{x\},\ x \in \Lambda$.
(b) *Polymers.* A union of j-blocks is called a *polymer* or *j-polymer*, and the set of j-polymers is denoted $\mathcal{P}_j = \mathcal{P}_j(\Lambda)$. The empty set is included in $\mathcal{P}_j$. For $X \in \mathcal{P}_j$, the set of j-blocks in X is denoted $\mathcal{B}_j(X)$ and $|X|_j = |\mathcal{B}_j(X)|$ is the number of j-blocks in X. For $X, Y \in \mathcal{P}_j$ we define $X \setminus Y \in \mathcal{P}_j$ by $X \setminus Y = \cup_{B \in X, B \notin Y} B$.

The name "polymer" was introduced in an important paper by Gruber and Kunz [**37**], but do not let the terminology make you hope for a close connection to the theory of long chain molecules right here.

2.5. Hierarchical Models

The pictures of hierarchical lattices in these notes are for $L = 2$ which violates the assumptions on L but are easier to draw.

Definition 2.8. The hierarchical distance between two points $x, y \in \Lambda$ is the side L^j of the smallest cube in $\cup_j \mathcal{B}_j$ that contains x and y. We say a covariance is

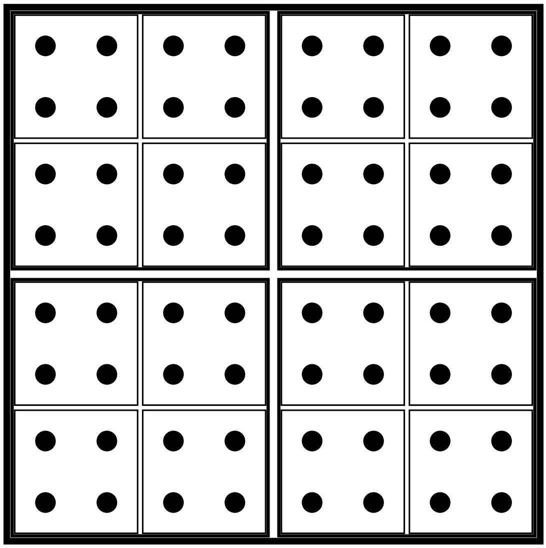

FIGURE 1. Part of the $d = 2, L = 2$ hierarchical lattice: the squares are blocks of diameter L, L^2, L^3. The hierarchical distances between any two of the four central points is L^3 because the L^3 block is the smallest that contains them.

hierarchical if $C = \sum_{j=1}^{N} C_j$ where C_j is positive-semi-definite and $C_j(x,y) = 0$ when $\mathrm{dist}_h(x,y) > L^j$.

The hierarchical distance is a metric, in fact it satisfies the ultrametric triangle inequality $\mathrm{dist}_h(x,y) \le \mathrm{dist}_h(x,z) \vee \mathrm{dist}_h(z,y)$. Another way to say this is that in this topology, two open balls are either disjoint or one is contained in the other. Some people will have encountered this kind of metric already as the natural metric on the leaves of a rooted tree.

Translation Invariance. The hierarchical lattice is homogeneous under an Abelian group of "translations" so that with respect to this group structure the hierarchical metric has the form $\mathrm{dist}_h(x,y) = \mathrm{dist}_h(x-z,0)$, but the group structure is not the same as on the Euclidean lattice [**23**]. We will not need to know more about this beautiful property. The words F is "translation invariant" just mean the consequence that $F(B)$ is the same function of the fields near B for every $j-$block B in the hierarchical lattice.

The important consequence of having a hierarchical covariance is that $\zeta_j(x)$ and $\zeta_j(y)$ are independent if x, y are not in the same j-block. This independence will allow $\tilde{\mathrm{RG}}$ to preserve products of local functions. We are now going to formalise this in a slightly heavy way in order to prepare the way for Euclidean lattices at the same time. Given $X \subset \Lambda$, let $\mathcal{N}_j(X)$ be the algebra of functions measurable with respect to the σ-algebra generated by $\{\varphi_j(x) : x \in X\}$. In more down to earth terms, an element of $\mathcal{N}_j(X)$ is a function only of the fields at points $x \in X$. By (2.17), $\mathcal{N}_j(X)$ are functions of $\varphi_{j+1}, \zeta_{j+1}$, but only through the combination $\varphi_{j+1} + \zeta_{j+1}$. We also need the larger sigma algebra $\tilde{\mathcal{N}}_j(X)$ generated by $\{\zeta_{j+1}(x), \varphi_{j+1}(x) : x \in X\}$. We write $\mathcal{N}_j = \mathcal{N}_j(\Lambda)$ and similarly for $\tilde{\mathcal{N}}_j$. We let $\mathcal{N}_j^{\mathcal{B}_j}$ be the set of maps

$$F : \mathcal{B}_j \to \mathcal{N}_j, \text{ such that } F(B) \in \mathcal{N}_j(B^*), \quad B \in \mathcal{B}_j. \tag{2.19}$$

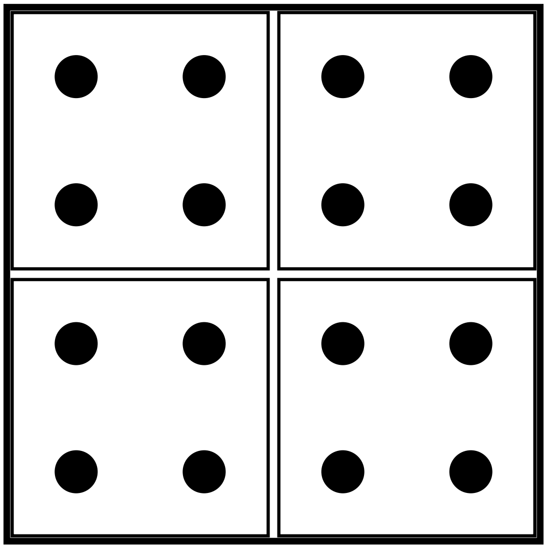

FIGURE 2. Hierarchical lattice, $d = 2, L = 2$. Visualise Lemma 2.9 and Def. 2.10 as having collapsed the balls of diameter L^1 in Figure 1 to points.

where $B^* = B$ for hierarchical case, but for Euclidean models B^* will be a neighbourhood of B. For $X \in \mathcal{P}_j$ and $F \in \mathcal{N}_j^{\mathcal{B}_j}$ we define

$$F^X = \prod_{B \in \mathcal{B}_j(X)} F(B). \tag{2.20}$$

We always adopt the convention that a product taken over a null index set is 1, in particular, $F^\varnothing = 1$, and that a sum over a null index set is zero. When $X = \Lambda$ this builds the global function F^Λ from the local function F. We make the same definitions with $\mathcal{N}_j$ replaced by $\tilde{\mathcal{N}}_j$.

Lemma 2.9. *If the covariance of μ is hierarchical and $F \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ is integrable, then*

$$\mathbb{E}_{j+1} F^\Lambda = \left(F'\right)^\Lambda \tag{2.21}$$

where $F' \in \mathcal{N}_{j+1}^{\mathcal{B}_{j+1}}$ is defined for $B' \in \mathcal{B}_{j+1}$ by

$$F'(B') = \mathbb{E}_{j+1} F^{B'} \tag{2.22}$$

and $j = 0, \ldots, N-1$.

Definition 2.10. Define $(\mathrm{RG}) : \tilde{\mathcal{N}}_j^{\mathcal{B}_j} \to \mathcal{N}_{j+1}^{\mathcal{B}_{j+1}} \subset \tilde{\mathcal{N}}_{j+1}^{\mathcal{B}_{j+1}}$ by $(\mathrm{RG})F = F'$.

Proof. Since $\zeta_{j+1}(x)$ and $\zeta_{j+1}(y)$ are independent if x, y are not in the same $j+1$-block,

$$\begin{aligned}
\mathbb{E}_{j+1} F^{\Lambda} &= \mathbb{E}_{j+1} \prod_{B \in \mathcal{B}_j} F(B) \\
&= \mathbb{E}_{j+1} \prod_{B' \in \mathcal{B}_{j+1}} \prod_{B \in \mathcal{B}_j(B')} F(B) \\
&= \prod_{B' \in \mathcal{B}_{j+1}} \mathbb{E}_{j+1} \prod_{B \in \mathcal{B}_j(B')} F(B) = \prod_{B' \in \mathcal{B}_{j+1}} \underbrace{\mathbb{E}_{j+1} F^{B'}}_{=(\mathrm{RG})(F)(B')}
\end{aligned} \tag{2.23}$$

□

This lemma accomplishes the global to local program for the hierarchical case.

2.6. The Formal Infinite Volume Limit and Trivial Fixed Point

With the aid of (RG) we can return to (1.39) and (1.40) and rewrite them as

$$\int e^{i\varphi(a)} e^{i\varphi(b)} F^{\Lambda} d\mu \Big/ \int F^{\Lambda} d\mu = (\mathrm{RG})^N F_{a,b} / (\mathrm{RG})^N F. \tag{2.24}$$

The notation conceals the fact that each map (RG) in $(\mathrm{RG})^N$ is acting on a different space, but for the hierarchical example in the next section they are isomorphic in a natural way, except for the final (RG) which is where the model finally realises it is in a finite lattice Λ and "feels" the Λ because the final covariance C_N is different. It is best to ignore this issue until I elaborate on it in Section 3.2. Referring to (2.22), the great advantage in passing from $\widetilde{\mathrm{RG}}$ acting on global functions to (RG) acting on local functions is that (RG) does not know about Λ, until the final (RG). Indeed, if we have a hierarchical decomposition with infinitely many scales, $C = \sum_{j \in \mathbb{N}} C_j$, for a covariance C defined on an infinite lattice, (RG) can still be defined by the same formula. Then infinite iteration of (RG) is definable as a limit of $(\mathrm{RG})^{N-1} F$. We will consider cases where infinite iteration brings F to the fixed point $F(B) = 1$ for all B. In these cases iteration of (RG) on $F_{a,b}$ may also converge to the same fixed point up to a constant. If so, we call this constant the "formal (infinite volume) limit", cf. Def. 2.15. To identify it with the infinite volume limit

$$\lim_{N \to \infty} \int e^{i\varphi(a)} e^{i\varphi(b)} F^{\Lambda} d\mu \Big/ \int F^{\Lambda_N} d\mu \tag{2.25}$$

one has to prove that the final (RG) is continuous near 1. This, by the way, will not be feasible if one makes a stupid choice of finite range decomposition as in the comment above Def. 2.2. In other words bad choices of finite range decomposition lead to formal infinite volume limits which are not the same as the infinite volume limit. Until we reach Section 3.2 I ignore the last step and concentrate on the formal limit. Also from now on I concentrate on analysing the denominator $\int d\mu(\varphi)\, F^{\Lambda}$, where F is translation invariant because this contains the important ideas and the corresponding analysis of the numerator is similar, but clumsy and tedious at the present stage of development. When we come to Section 3.3 you will see a calculation of the numerator. The important point is that nothing in the coming analysis forces the local function to be translation invariant.

2.7. Analysis

Now we set up the domains on which we can study (RG). Suppose for each scale j there exists a norm on $\tilde{\mathcal{N}}_j$ such that

$$\|F^X\| \leq \|F\|^X \quad \text{where} \quad \|F\|^X = \prod_{B \in \mathcal{B}_j(X)} \|F(B)\|, \quad \text{for } F \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j} \tag{2.26}$$

and

$$\mathbb{E}_{j+1} : \tilde{\mathcal{N}}_j \to \mathcal{N}_{j+1} \subset \tilde{\mathcal{N}}_{j+1}, \quad \|\mathbb{E}_{j+1} Z\| \leq \|Z\|, \tag{2.27}$$

and the norm is complete so that the finite norm elements of $\tilde{\mathcal{N}}_j$ are a Banach space. We denote this Banach space also by $\tilde{\mathcal{N}}_j$ and the norm must be such that $\mathcal{N}_j$ is a closed subspace. Other spaces acquire their norms as subspaces of cartesian products. In particular, the norm of $F \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ is $\max\{\|F(B)\| : B \in \mathcal{B}_j\}$.

Definition 2.11. For X a Banach space, B_X denotes an open ball in X centred on the origin. We say that a function defined on a ball centred on the origin in a Banach space with values in another Banach space is *smooth (near the origin)* if it is $\mathcal{C}^2$ in the sense of having two Frechet derivatives which are defined and continuous on the ball.

Lemma 2.12. *The map* (RG) $: \tilde{\mathcal{N}}_j^{\mathcal{B}_j} \to \mathcal{N}_{j+1}^{\mathcal{B}_{j+1}}$ *is a smooth map of Banach spaces and the derivative* $D(\mathrm{RG})_F$ *of* (RG) *at* F *in the direction* $\dot{F}$ *is*

$$D(\mathrm{RG})_F \dot{F}(B') = \sum_{B \in \mathcal{B}_j(B')} \mathbb{E}_{j+1} F^{B' \setminus B} \dot{F}(B). \tag{2.28}$$

Proof. The formula for $D(\mathrm{RG})_F \dot{F}$ is an easy calculation. We have to prove that it is bounded as a linear map of Banach spaces,

$$\|D(\mathrm{RG})_F \dot{F}(B')\| \leq \sum_{B \in \mathcal{B}_j(B')} \|F\|^{B' \setminus B} \|\dot{F}(B)\| \leq L^d \big(\|F\|\big)^{L^d - 1} \|\dot{F}\|. \tag{2.29}$$

Take the maximum over $B' \in \mathcal{B}_{j+1}$,

$$\|D(\mathrm{RG})_F \dot{F}\| \leq L^d \big(\|F\|\big)^{L^d - 1} \|\dot{F}\| \tag{2.30}$$

so $D(\mathrm{RG})$ is bounded. The rest of the proof (second derivatives and continuity) is omitted.

□

Remark 2.13. In fact, all derivatives exist.

2.8. Expanding Directions, Relevant Operators

The L^d in the bound is a symptom of expanding directions in (RG). The most obvious of these is manifested by

$$(\mathrm{RG})\big(e^{\lambda} F\big) = e^{L^d \lambda} (\mathrm{RG})\big(F\big).$$

In Wilson's terminology, 1 is a *relevant operator*. For Wilson, the "1" is a function of the field which happens not to depend on the field and λ is a coefficient in front of this function. Disregard the word "operator" which came to us from quantum field theory. The point is that a constant factor in the local function expands. This is where the exp(volume) problem is concealed. Our strategy for studying the action of (RG) on $F \in \mathcal{N}_j^{\mathcal{B}_j}$ is to decompose

$$F = I + K, \tag{2.31}$$

into a part $I \in \mathcal{N}_j^{\mathcal{B}_j}$, for which we can calculate the action of $\mathbb{E}_{j+1}$ explicitly and which carries the part of F that expands, and an error $K \in \mathcal{N}_j^{\mathcal{B}_j}$ that contains the contracting part and remains small compared with I. Since the directions that expand and contract are changing along the orbit of (RG) we have to make a change of the coordinates I, K each time (RG) acts, which is the role of $\tilde{I}$ in the next calculation:

Lemma 2.14. *For any integrable* $\tilde{I} \in \mathcal{N}_{j+1}^{\mathcal{B}_j}$,

$$(\mathrm{RG})(I + K) = I' + K' \tag{2.32}$$

where, for $B' \in \mathcal{B}_{j+1}$,

$$\begin{aligned} I'(B') &= \tilde{I}^{B'}, \\ K'(B') &= \sum_{B \in \mathcal{B}_j(B')} \tilde{I}^{B' \setminus B} \mathbb{E}_{j+1}\big(K + I - \tilde{I}\big)(B) + O(\|K + I - \tilde{I}\|^2) \end{aligned} \tag{2.33}$$

where $O(\|K+I-\tilde{I}\|^2)$ *is a smooth function of* $(K, I, \tilde{I}) \in \mathcal{N}_j^{\mathcal{B}_j} \times \mathcal{N}_j^{\mathcal{B}_j} \times \mathcal{N}_{j+1}^{\mathcal{B}_j}$ *whose norm is bounded as indicated.*

PROOF.

$$\begin{aligned} (\mathrm{RG})(I + K) &= (\mathrm{RG})\big(\tilde{I} + (K + I - \tilde{I})\big) \\ &= (\mathrm{RG})(\tilde{I}) + D(\mathrm{RG})_{\tilde{I}}(K + I - \tilde{I}) + O(\|K + I - \tilde{I}\|^2). \end{aligned} \tag{2.34}$$

Since $\tilde{I}$ has values in $\mathcal{N}_{j+1}$ we can move it outside the expectation $\mathbb{E}_{j+1}$ and rewrite the first term in the right hand side of (2.34) with

$$(\mathrm{RG})(\tilde{I})(B') = \tilde{I}^{B'} = I'(B'). \tag{2.35}$$

Consider now the second term in the right hand side of (2.34). We complete the proof by applying (2.28) with $F = \tilde{I}$ and $\dot{F} = K + I - \tilde{I}$.

□

2.9. Trivial Fixed Point

For each scale j define an element $F = 1$ of $\mathcal{N}_j^{\mathcal{B}_j}$ by

$$F(B) = 1 \ \forall \ B \in \mathcal{B}_j. \tag{2.36}$$

The map (RG) takes 1 in $\mathcal{N}_j^{\mathcal{B}_j}$ to 1 in $\mathcal{N}_{j+1}^{\mathcal{B}_{j+1}}$. Even though the Banach spaces are different we call this the trivial fixed point. From (2.28) we find that

$$D(\mathrm{RG})_1 \dot{F}(B') = \sum_{B \in \mathcal{B}_j(B')} \mathbb{E}_{j+1} \dot{F}(B). \tag{2.37}$$

Referring to the discussion before (2.25),

Definition 2.15. If $\lim_{N \to \infty} \|(\mathrm{RG})^N F - 1\| = 0$ and there exists a constant A such that $\lim_{N \to \infty} \|(\mathrm{RG})^N F_{a,b} - A1\| = 0$ then we call A the formal (infinite volume) limit.

The plan now is to regard (RG) as a map on pairs (I, K). K will be an element of a Banach space that may depend on the scale j, and I will be explicitly determined by parameters $\lambda \in \mathbb{R}^{\text{some } n}$ called *coupling constants* and such that the trivial fixed point is $(\lambda, K) = (0, 0)$. The coupling constants are coordinates for the expanding directions or more accurately the non-contracting directions, relevant and *marginal* operators in Wilson's terminology. We have to prove that (RG) satisfies the hypotheses of the stable manifold Theorem 2.16 and then we conclude from this theorem that (λ, K) is driven to the trivial fixed point if λ is correctly chosen. This is called *tuning*. In the coming examples it is possible to tune by exploiting the non-uniqueness of the representation (1.38). In other words, by putting a well chosen part of F into $d\mu$ at the beginning, we arrange that the remaining part of F is driven to the trivial fixed point 1 and the scaling limit is the $d\mu$ in this modified representation (1.38). In contrast, referring back to the discussion in Section 1.2 the choice of a in (1.16) is not achieved by a different factorisation, but reflects the fact that this model would generically scale to white noise and there is only one choice of a such that the model is on the stable manifold for the massless Gaussian fixed point in dimensions $d > 4$.

2.10. Appendix. Stable Manifold Theorem

Notation. For any Banach space X, $B_{X,r}$ denotes the open ball of radius r centred on the origin. For $j \in \mathbb{N}_0$ let E_j, F_j be Banach spaces. Let $B_{E_j,r} \subset E_j$ and $B_{F_j,r} \subset F_j$ be balls of radius r centred on the origin. Suppose for each $j \in \mathbb{N}$ we have a map from $B_{E_{j-1},r} \times B_{F_{j-1},r}$ to $E_j \times F_j$, given by

$$\begin{aligned} x_j &= A_j x_{j-1} + B_j y_{j-1} + f_j(x_{j-1}, y_{j-1}), \\ y_j &= C_j y_{j-1} + g_j(x_{j-1}, y_{j-1}), \\ x_j &\in E_j,\ y_j \in F_j, \end{aligned} \tag{2.38}$$

where A_j, B_j, C_j are linear and f_j, g_j are smooth functions satisfying $f_j(0,0) = g_j(0,0) = 0$, $Df_j(0,0) = Dg_j(0,0) = 0$.

Theorem 2.16. [1] *For $j \in \mathbb{N}$ let f_j, g_j be smooth uniformly in j, let A_j be invertible, $\sup_{j,k} \|A_j^{-1}\|\|C_k\| < 1$, $\sup \|C_j\| < 1$ and $\sup \|B_j\| < \infty$. Then there exists a ball $B_{F_0,\rho}$ and a smooth function $h : B_{F_0,\rho} \to B_{E_0}$ such that if (x_0, y_0) lies in the graph $\{(h(y), y) : y \in B_{F_0,\rho}\}$, then $(x_j, y_j) \to 0$.*

The idea of the proof, which is based on a standard proof [**40**], is to define a map T acting on a space whose points are sequences

$$\big((x_0, y_0), \underbrace{(x_1, y_1), (x_2, y_2), \ldots}_{\mathcal{Z}}\big) \text{ such that } \|x_j\| \vee \|y_j\| \to 0 \text{ as } j \to \infty \tag{2.39}$$

where y_0 is fixed and such that a fixed point of T is a sequence that satisfies (2.38). Since y_0 is fixed, we regard it as a parameter and define our map on $E_0 \times \mathcal{Z}$ where $\mathcal{Z}$ is a space whose points are sequences $z = (x_j, y_j)_{j \in \mathbb{N}}$. After $E_0 \times \mathcal{Z}$ is made into a Banach space we can use the implicit function theorem to determine h. $\mathcal{Z}$ is a Banach space with the norm

$$\|z\| = \sup_{j \in \mathbb{N}} \mu^{-j} \max\{\|x_j\|, \alpha\|y_j\|\} \tag{2.40}$$

[1] The author thanks Jon Dimock for communicating a correction to the hypotheses of this theorem.

where $\mu \in (0,1)$ so that $(x_j, y_j) \to 0$ as $j \to \infty$ and $\alpha \geq 1$. $E_0 \times \mathcal{Z}$ is a Banach space with the norm $\|(x_0, z)\| = \|x_0\| \vee \|z\|$. With $(A, B, C, f, g) = (A_j, B_j, C_j, f_j, g_j)$, let $T : (x_0, z) \mapsto (Tx_0, Tz)$ be defined (on a ball) by

$$\begin{aligned} Tx_j &= A^{-1}\big(x_{j+1} - By_j - f(x_j, y_j)\big), \quad j \geq 0, \\ Ty_j &= Cy_{j-1} + g(x_{j-1}, y_{j-1}), \quad j \geq 1, \end{aligned} \tag{2.41}$$

Note that these equations are the same as the recursion (2.38) if $Tx_j = x_j$ and $Ty_j = y_j$. The rewriting of the first equation as a backwards recursion is a very clever idea that allows us to choose μ, α so that the derivative $DT_{(x_0,z)}$ at $(x_0, z) = 0$ and $y_0 = 0$ is contractive. Therefore the fixed point principle, which is hidden inside the implicit function theorem in the proof given below, shows that there is a sequence (x_0, z) such that $T(x_0, z) = (x_0, z)$.

Proof of Theorem 2.16. For $z \in B_{\mathcal{Z},r}$ and $j \in \mathbb{N}$,

$$\|x_j\| \leq \mu^j \|z\| \leq \|z\| \leq r, \qquad \|y_j\| \leq \alpha^{-1}\mu^j \|z\| \leq \|z\| \leq r. \tag{2.42}$$

Therefore $z \in B_{\mathcal{Z},r}$ implies (x_j, y_j) is in the domain of (f_j, g_j) for all $j \in \mathbb{N}$. By hypothesis $y_0 \in B_{F_0,\rho}$ where we can take $\rho < r$. Then T is defined on $B_{E_0 \times \mathcal{Z},r}$ and takes values in $E_0 \times \mathcal{Z}$.

Let $DT = DT_{(x_0,z)}$ be the derivative of the map at the point (x_0, z). The derivative is the linear map whose action on $(\dot{x}_0, \dot{z}) \in E_0 \times \mathcal{Z}$ is given by

$$\begin{aligned} DT\dot{x}_j &= A^{-1}\big(\dot{x}_{j+1} - B\dot{y}_j - Df(\dot{x}_j, \dot{y}_j)\big), \quad j \geq 0 \\ DT\dot{y}_j &= C\dot{y}_{j-1} + Dg(\dot{x}_{j-1}, \dot{y}_{j-1}), \quad j \geq 1, \end{aligned} \tag{2.43}$$

where $\dot{y}_0 = 0$. From these equations, $\|\dot{x}_j\| \leq \|(\dot{x}_0, \dot{z})\|\mu^j$ and $\|\dot{y}_j\| \leq \alpha^{-1}\|(\dot{x}_0, \dot{z})\|\mu^j$,

$$\begin{aligned} \mu^{-j}\|DT\dot{x}_j\| &\leq \|A^{-1}\|\big(\mu + \|B\|\alpha^{-1} + \|Df\|\big)\|(\dot{x}_0, \dot{z})\|, \\ \alpha\mu^{-j}\|DT\dot{y}_j\| &\leq \big(\|C\|\mu^{-1} + \alpha\|Dg\|\mu^{-1}\big)\|(\dot{x}_0, \dot{z})\|, \end{aligned} \tag{2.44}$$

which implies

$$\begin{aligned} &\|DT(\dot{x}_0, \dot{z})\| \leq \|DT\|\|(\dot{x}_0, \dot{z})\|, \qquad \|DT\| \leq \\ &\sup_{j\in\mathbb{N}} \Big(\|A_j^{-1}\|\big(\mu + \|B_j\|\alpha^{-1} + \|Df_j\|\big) \vee \big(\|C_j\|\mu^{-1} + \alpha\|Dg_j\|\mu^{-1}\big)\Big). \end{aligned} \tag{2.45}$$

Recall that $DT = DT_{(x_0,z)}$ is the derivative at (x_0, z). By the hypotheses on f_j, $\|Df_{j,(x_j,y_j)}\|$ is uniformly bounded in j for $(x_0, z) \in B_{E_0\times\mathcal{Z},r}$ and the same is true for g_j. Therefore, referring to (2.45), $\|DT_{(x_0,z)}\|$ is bounded uniformly for $(x_0, z) \in B_{E_0\times\mathcal{Z},r}$. We find that the second derivative $D^2T_{(x_0,z)} : E_0 \times \mathcal{Z} \times E_0 \times \mathcal{Z} \to E_0 \times \mathcal{Z}$ is also bounded uniformly and continuous in $(x_0, z) \in B_{E_0\times\mathcal{Z},r}$ because $D^2 f_{j,(x_j,y_j)}$ is bounded and continuous uniformly in j and likewise for $D^2 g_j$. We conclude that $T \in \mathcal{C}^2(B_{E_0\times\mathcal{Z},r})$.

Let DT_0 be $DT_{(x_0,z)}$ evaluated at $(x_0, z) = (0, 0)$ and with the parameter $y_0 = 0$. We now prove that μ, α can be chosen so that

$$\|DT_0\| < 1. \tag{2.46}$$

The intersection over j of the intervals $\big(\|C_j\|, \|A_j^{-1}\|^{-1}\big) \cap (0, 1)$ is non-empty by the hypotheses. Choose $\mu < 1$ in this intersection. Noting that the choice of μ makes $\sup \|A_j^{-1}\|\mu < 1$ choose α large so that $\sup \|A_j^{-1}\|\big(\mu + \|B_j\|\alpha^{-1}\big) < 1$. The choice of μ also makes $\|C_j\|\mu^{-1} < 1$. Referring to (2.45) we conclude that $\|DT_0\| < 1$.

Recall that T depends on y_0 and write $T = T_{y_0}$. Define a map $S : B_{F_0} \times B_{E_0 \times \mathcal{Z}} \to E_0 \times \mathcal{Z}$ by $S : (y_0, (x_0, z)) \mapsto (x_0, z) - T_{y_0}(x_0, z)$. Then S is smooth. The derivative of S at $(0, (0, 0))$ in the $E_0 \times \mathcal{Z}$ arguments with y_0 fixed equals $I - DT_0$, which is invertible with inverse given by the convergent series $\sum DT_0^n$ because $\|DT_0\| < 1$. By the implicit function theorem for Banach spaces there exists $B_{F_0,\rho}$ and a smooth function $H : B_{F_0,\rho} \to E_0 \times \mathcal{Z}$ such that $S(y_0, H(y_0)) = S(0, (0, 0)) = (0, 0)$. By construction of S, $H(y_0)$ is a sequence in $E_0 \times \mathcal{Z}$ which is a fixed point for T_{y_0} so it solves (2.38) and tends to zero. Define $h(y_0)$ to be the x_0 component of $H(y_0)$ and the theorem is proved.

□

Background. In the case where the Banach spaces and maps are the same for all j, (2.38) defines a smooth dynamical system $M : E \times F \to E \times F$. If the maps C, B are surjective then the conclusion of Theorem 2.16 follows from the stable manifold theorem [**50**, Theorem 6.1]. To see this we have to check that M is a ρ-pseudohyperbolic system as defined in [**50**, page 26] with $\rho \leq 1$. Letting DM_0 denote the linearisation of M at the fixed point $(0, 0)$, this means that DM_0 has spectrum disjoint from the circle of radius ρ. DM_0 is obtained by setting $f = g = 0$ in (2.38). Therefore $DM_0(x, 0) = (Ax, 0)$ and the subspace $\{(x, 0) : x \in E\}$ is invariant. Let $\alpha = \|C\|$. The spectrum associated with this invariant subspace is in $\{\lambda : |\lambda| > \alpha\}$ because $(A - \lambda) = A(I - \lambda A^{-1})$ is invertible when $|\lambda|\|A^{-1}\| < 1$ which by hypothesis holds for $|\lambda| \leq \alpha$. Therefore there exists $\rho > \alpha$ such that the spectrum is disjoint from $\{\lambda : |\lambda| \geq \rho\}$. There is another invariant subspace $\{(Ry, y) : y \in F\}$ where $R : F \to E$ is given by

$$R = \sum_{j \in \mathbb{N}_0} A^{-j-1} B C^j. \tag{2.47}$$

The sum is norm convergent by hypotheses and it is easy to check that $(Ry, y) \mapsto (RCy, Cy)$, so $\{(Ry, y) : y \in F\}$ is invariant. Surjectivity of C, B implies that R is surjective and this implies the two subspaces span $E \times F$. Finally we have to prove that the spectrum associated with this subspace is in $|\lambda| < \rho$. Let $E \times F$ have the norm $\|x\| \vee \|R\|\|y\|$. Then

$$\|DM_0(Ry, y)\| = \|(RCy, Cy)\| = \|RCy\| \vee \|R\|\|Cy\| \leq \alpha\|R\|\|y\| = \alpha\|(Ry, y)\| \tag{2.48}$$

so DM_0 is bounded by α in this norm. Therefore, $(\lambda - DM_0)$ is invertible on this subspace for $|\lambda| > \alpha$ which means that the spectrum is contained in $|\lambda| \leq \alpha$ and so is disjoint from $|\lambda| \geq \rho$.

LECTURE 3

Example: the hierarchical Coulomb gas

3.1. Example: Hierarchical Coulomb gas

For an illustration we turn to the Coulomb gas on the hierarchical lattice. We are following Marchetti and Perez in [**43**], but it looks more elaborate here because we are preparing the way for models on the Euclidean lattice as well. The goal is to evaluate the asymptotics as $a - b \to \infty$ for the energy of a pair of fractional charges at positions a, b in order to see if they are "confined". I should also advertise that Marchetti and Perez emphasise the role of bifurcation theory and study the critical point.

Background. The Coulomb gas on $\mathbb{Z}^d$ in $d = 2$ consists of positively and negatively charged particles experiencing the Coulomb potential which is a Green's function for the lattice Laplacian. In [**42**] Kosterlitz and Thouless discussed a phase transition in which charged particles bind into pairs of oppositely charged particles, called dipoles. The first rigorous proof of some of the assertions in their paper was given by Fröhlich and Spencer [**28**] and the technique of that paper is a precursor to the finite range decompositions in these notes. The effect of the phase transition on the correlations is drastic; in the plasma phase, characterised by unbound charges, correlations decay exponentially [**60**], whereas in the dipole phase there is power law decay. As in Ex. 1.13 one can define the energy $E_{a,b}$ of a pair of fractionally charged particles immersed in the medium at positions $a, b \in \mathbb{Z}^d$. This energy includes the distortion in the system caused by having this pair present. In the plasma phase $E_{a,b}$ tends to a constant exponentially fast as $a - b \to \infty$. In contrast, in the dipole phase, $E_{a,b}$ grows as $\varepsilon^{-1} \ln |a - b|$, where ε is called the dielectric constant. Growth is interpreted as confinement: the pair of charges have a huge energy unless they are close together. Thus the dipole phase *confines* fractional charges. If the charges are integral, then $E_{a,b}$ does not grow with separation. This is reminiscent of quark confinement. If you want to win a million dollars by solving the Yang Mills Clay problem you have to prove there is a mechanism in four dimensions which confines quarks [**59**].

I am only going to discuss the dipole phase of the hierarchical model. It is an artifact of this model that the transition from confinement to screening phase is much less noticeable than in the Euclidean case; according to [**43**], all that happens is that whereas the dielectric constant $\varepsilon = 1$ in the dipole phase, it is less than one in the screening phase. Furthermore, the hierarchical model has a simpler dipole phase than the Euclidean model, as manifested by $\varepsilon = 1$, meaning that $E_{a,b}$ is behaving as if all the other particles were not there; they are bound into neutral pairs in a stronger sense than in the Euclidean model, for which $\varepsilon \neq 1$ in the dipole phase.

We consider the model in Section 1.7 with a hierarchical potential C. The particles are in a box Λ as in (2.1), but without periodic boundary conditions, which are not compatible with the hierarchical metric. In d dimensions the simplest hierarchical C with the same behaviour at infinity as the Coulomb potential is $C = \sum_{j \in \mathbb{N}} C_j$ with

$$C_j(x, y) = \text{const.}\,(L) \begin{cases} L^{-2(j-1)[\varphi]} & \text{if } \operatorname{dist}_h(x, y) \leq L^j \\ 0 & \text{if } \operatorname{dist}_h(x, y) > L^j \end{cases}. \tag{3.1}$$

with $[\varphi] = (d-2)/2$. Cf. Def. 2.1. We now specialise to case $d = 2$ so $[\varphi] = 0$ and, following (2.9),

$$\text{const.}\,(L) = \ln L. \tag{3.2}$$

Since C_j is positive-semi-definite the associated fluctuation field satisfies the relation $\zeta_j(x) = \zeta_j(y)$ for all x, y with $\operatorname{dist}_h(x, y) \leq L^j$, cf. comments near (1.23). In other words ζ_j is constant on blocks in $\mathcal{B}_j$ and in each $B \in \mathcal{B}_j$ there is really only one fluctuation $\zeta_j(B) = \zeta_j(x)$ for $x \in B$. Referring to Theorem 1.12 note that there is a factor of β in front of the covariance so $\zeta_j \sim N(\beta C)$.

The basic fact that makes everything work is

$$\begin{aligned} \mathbb{E}_{j+1} e^{iq(\varphi_{j+1}(x) + \zeta_{j+1}(x))} &= e^{iq\varphi_{j+1}(x)} \mathbb{E}_{j+1} e^{iq\zeta_{j+1}(x)} \\ &= e^{iq\varphi_{j+1}(x)} e^{-\frac{1}{2}\beta q^2 \operatorname{var}(\zeta_{j+1}(x))} \\ &= L^{-\frac{1}{2}\beta q^2} e^{iq\varphi_{j+1}(x)} \end{aligned} \tag{3.3}$$

which follows from the formula for C_j and (1.21). After looking at the following example you will see that (RG) maps the parameter z to $L^2 L^{-\frac{1}{2}\beta q^2} z$ which is strictly smaller than z when $\beta > 1$. In the particle model z is a rate at which particles occur so if $\beta > 1$ the charges become exponentially rare as j increases. They are rare because they are grouping into neutral clusters.

Example 3.1. Let us calculate $F' = (\text{RG})F$ when

$$F(\{x\}) = 1 + ze^{i\varphi(x)} + ze^{-i\varphi(x)}. \tag{3.4}$$

when the lattice is one dimensional. Figure 1 shows the blocks of the one dimensional hierarchical lattice for the case $L = 2$. The $0-$blocks are the points. The

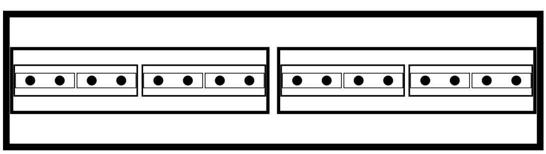

FIGURE 1. One dimensional hierarchical lattice

$1-$blocks are the pairs of points and the $2-$blocks have four points. The dimension d is *one*, in the sense that the ball of radius 2^j contains 2^j points. F' is defined on

1−blocks. Let $B' = \{x, y\}$ be a 1−block, where $y = x + 1$.

$$\begin{aligned} F'(B') &= \mathbb{E}_1 F^{B'} \\ &= \mathbb{E}_1 \Big(\big(1 + z e^{i\varphi(x)} + z e^{-i\varphi(x)}\big)\big(1 + z e^{i\varphi(y)} + z e^{-i\varphi(y)}\big) \Big) \\ &= \mathbb{E}_1 \Big(1 + z e^{i\varphi(x)} + z e^{-i\varphi(x)} + z e^{i\varphi(y)} + z e^{-i\varphi(y)} + O(z^2) \Big). \end{aligned} \tag{3.5}$$

Let $\sigma^2 = \operatorname{var}(\zeta(x))$, which is independent of x. Then

$$\mathbb{E}_1 e^{i\varphi(x)} = \mathbb{E}_1 e^{i\varphi'(B') + i\zeta(x)} = e^{i\varphi'(B')} \mathbb{E}_1 e^{i\zeta(x)} = e^{i\varphi'(B')} e^{-\frac{1}{2}\sigma^2}, \tag{3.6}$$

where we wrote $\varphi'(B')$ because φ' is constant on the block B'. Then

$$\begin{aligned} F'(B') &= 1 + z e^{-\frac{1}{2}\sigma^2} \big(e^{i\varphi'(B')} + e^{-i\varphi'(B')} + e^{i\varphi'(B')} + e^{-i\varphi'(B')} \big) + \mathbb{E}_1 O(z^2). \\ &= 1 + 2z e^{-\frac{1}{2}\sigma^2} \big(e^{i\varphi'(B')} + e^{-i\varphi'(B')} \big) + \mathbb{E}_1 O(z^2). \end{aligned} \tag{3.7}$$

The main point is that we have returned to the same F as we started with except that the coupling constant z has been replaced (renormalised) by $2ze^{-\frac{1}{2}\sigma^2}$ so it is being increased by the factor 2 which is the volume of B' and decreased by a factor $e^{-\frac{1}{2}\sigma^2}$. Thus we can see explicitly the local action of (RG) as an evolution of coupling constants. However there are also the $O(z^2)$ terms which were not in the original F which points at the difficulties we would face if we were to try to pursue the action of (RG) in terms of explicit formulas – there is always a tendency for new terms to appear. We introduced K for this reason. We can see another effect by calculating the $O(z^2)$ terms,

$$O(z^2) = z^2 \big(e^{i\varphi(x) + i\varphi(y)} + e^{-i\varphi(x) - i\varphi(y)} + e^{i\varphi(x) - i\varphi(y)} + e^{-i\varphi(x) + i\varphi(y)} \big). \tag{3.8}$$

In particular, in the "dipole" term $e^{-i\varphi(x)+i\varphi(y)}$ the φ' cancels out when we insert $\varphi = \varphi' + \zeta$ and then applying the expectation $\mathbb{E}_1$ gives const. z^2 which causes the 1 in F also to be renormalised (vacuum energy). This concludes the example.

We need a norm with properties (2.26) and (2.27). The L^∞ norm is the obvious choice but instead we use the l^1 norm of the Fourier transform because it has better contractive properties with respect to $\mathbb{E}_{j+1}$. Referring to the definition of F in Theorem 1.12, we see that the initial local functions are π periodic and therefore 2π periodic in φ. We consider the space of 2π periodic functions $Z \in \tilde{\mathcal{N}}_j$ of $\varphi_{j+1}, \zeta_{j+1} \in \mathbb{R}^{\mathcal{B}_{j+1}}$ with absolutely convergent Fourier series

$$Z(\varphi_{j+1}, \zeta_{j+1}) = \sum_{q, r \in \mathbb{Z}^{\mathcal{B}_{j+1}}} \widehat{Z}(q, r) e^{i(q, \varphi_{j+1}) + i(r, \zeta_{j+1})} \tag{3.9}$$

and give $\tilde{\mathcal{N}}_j$ the norm

$$\|Z\| = \sum_{q,r} |\widehat{Z}(q, r)|. \tag{3.10}$$

If $Z \in \mathcal{N}_j$ then there is another set of Fourier coefficients $\widehat{Z}(r)$ such that

$$Z(\varphi_j) = \sum_{r \in \mathbb{Z}^{\mathcal{B}_{j+1}}} \widehat{Z}(r) e^{i(r, \varphi_{j+1} + \zeta_{j+1})}. \tag{3.11}$$

The norm (3.10) is also equal to the l^1 norm of these coefficients because Fourier expansions are unique,

$$\|Z\| = \sum_r |\widehat{Z}(r)|. \tag{3.12}$$

This norm is larger than the L^∞ norm. The norm satisfies (2.26) and also (2.27). There is also a strict contractive property. If $Z \in \mathcal{N}_j$ is such that $\widehat{Z}(r) = 0$ for all $r \in \mathbb{Z}^{\mathcal{B}_j}$ with $\|r\|_\infty < p$ for some $p \in \mathbb{N}$ then

$$\|\mathbb{E}_{j+1} Z\| \leq L^{-\frac{\beta}{2}p^2} \|Z\|. \tag{3.13}$$

For the rest of this section $\mathcal{N}_j$ is the Banach space of 2π periodic functions of finite norm.

Exercise 3.2. Check (3.12) and prove that the norm satisfies (2.26), (2.27), (3.13).

Recall that Theorem 1.12 expresses the partition function Ξ for the Coulomb gas with parameters z, β as the integral with respect to μ of F_0^Λ where $F_0 \in \mathcal{N}_0^{\mathcal{B}_0}$ is, apart from the inserted e^{λ_0},

$$F_0(B) = e^{\lambda_0}\big(1 + 2z\cos(2\varphi)\big), \quad B \in \mathcal{B}_0. \tag{3.14}$$

The next theorem says that this iterates towards the trivial fixed point.

Proposition 3.3. *For $\beta > 1$ there exists a smooth $\lambda_0(z)$ defined on a neighbourhood $B_\mathbb{R}$ of the origin such that, for $z \in B_\mathbb{R}$, $\lim_{n\to\infty} \|(\mathrm{RG})^n F_0 - 1\| = 0$.*

In [**43**], the factor e^{λ_0} hardly appears because it cancels out in correlations, and it is usual to take advantage of this fact to get rid of it, but I prefer to let it remain as it is the very simplest example of a general phenomenon in RG. λ_0 is a coordinate for the expanding direction under the action of (RG) in the spaces $\mathcal{N}_j^{\mathcal{B}_j}$. If it were the only expanding direction we would ever encounter then the following idea would be overkill, but in other models there are other expanding directions. The general procedure for all expanding directions is exemplified by *tuning* the initial λ_0 to a special value so that the dynamical system evolves on the stable manifold of the trivial fixed point. We are going to have to use a version of the stable manifold theorem, Theorem 2.16, to do this tuning. This expanding direction is sometimes called *vacuum energy*.

Remark 3.4. This tendency for the vacuum energy to go out of control is a major issue in cosmology because it is a source for gravity. How did the vacuum energy get chosen so precisely at the microscopic scale so that the Universe does not get rolled up into a tight little ball by gravity?

Referring to Section 1.2, the mass term was an example of an expanding direction, coordinatised by a, that has to be tuned in order to get on the stable manifold of the massless free field, at least for dimensions $d > 4$.

Proof of Proposition 3.3. Recall the strategy explained with (2.31) and under Def. 2.15. (I_0, K_0) are given by

$$F_0(B) = \underbrace{e^{\lambda_0}}_{I_0(B)} + \underbrace{2e^{\lambda_0} z\cos(2\varphi_0(x))}_{K_0(B)}, \quad B = \{x\} \in \mathcal{B}_0. \tag{3.15}$$

This initial interaction is translation invariant; every site has the same interaction. The renormalisation group preserves this property. For $j \geq 0$, given $(I, K) =$

(I_j, K_j) with $I(B) = e^{\lambda}$ we determine $(I', K') = (I_{j+1}, K_{j+1})$ by Lemma 2.14 together with a choice of $\tilde{I}$ at each scale j. Given (I, K) we choose $\tilde{I}$ constant in (φ_{j+1}, B) so that the zero Fourier mode, $\widehat{K}(B, 0)$, is canceled,

$$\widehat{K}(B,0) + I(B) - \tilde{I}(B) = 0. \tag{3.16}$$

For (λ, K) near the origin $\tilde{I}$ is positive and we can write it as $\tilde{I}(B) = e^{L^{-d}\lambda'}$ so that by Lemma 2.14 $I'(B') = e^{\lambda'}$. Therefore, near the trivial fixed point (3.16) can be rewritten as

$$e^{\lambda} - e^{L^{-d}\lambda'(\lambda,K)} + \widehat{K}(B,0) = 0 \tag{3.17}$$

and this defines a map $(\lambda, K) \mapsto \lambda'$. By translation invariance this equation for λ' has the same solution in every block B. The map $(\lambda, K) \mapsto \lambda'$ is smooth near the origin because $\widehat{K}(B, 0)$ is a bounded linear functional of K, so the left hand side of (3.17) is a smooth function on $\mathbb{R} \times \mathcal{N}_j$ and we can use the Banach space implicit function theorem.

Now we turn to the action of (RG) on K. Referring to (2.33) and putting in our choices for $I, \tilde{I}$,

$$\begin{aligned} K'(\lambda, K)(B') = &\sum_{B \in \mathcal{B}_j(B')} e^{\lambda' L^{-d}|B' \setminus B|_j} \mathbb{E}_{j+1}\big(K(B) - \widehat{K}(B,0)\big) \\ &+ O(\|K + I - \tilde{I}\|^2). \end{aligned} \tag{3.18}$$

This formula and (3.17) define the action of (RG) as a map $(\lambda, K) \to (\lambda', K')$ which is smooth on a ball $B_{\mathbb{R}\times\mathcal{N}_j}$. The derivative of this map at the trivial fixed point, $(\lambda, K) = (0, 0)$ is

$$(\dot{\lambda}, \dot{K}) \mapsto \Big(\underbrace{L^d\dot{\lambda}}_{A\dot{\lambda}} + \underbrace{L^d\widehat{\dot{K}}(0)}_{B\dot{K}},\quad \underbrace{\sum_{B\in\mathcal{B}_j(\cdot)} \mathbb{E}_{j+1}\big(\dot{K}(B) - \widehat{\dot{K}}(B,0)\big)}_{C\dot{K}(\cdot)}\Big). \tag{3.19}$$

$\widehat{\dot{K}}(0) = \widehat{\dot{K}}(B, 0)$ does not depend on the $j-$block B. Thus, (RG) is a smooth map which has the form described in Theorem 2.16,

$$\begin{aligned} \lambda' &= A\lambda + BK + \tilde{f}(\lambda, K), \\ K' &= CK + \tilde{g}(\lambda, K) \end{aligned} \tag{3.20}$$

where $\tilde{f}(0, 0) = \tilde{g}(0, 0) = 0$ and $D\tilde{f}(0, 0) = D\tilde{g}(0, 0) = 0$. The inverse of A is obviously a contractive map. It is now sufficient to prove that C is contractive because then the existence of $\lambda = \lambda_0(K)$ follows from Theorem 2.16. From (3.19),

$$\|C\dot{K}\| \le L^d \|\mathbb{E}_{j+1}\big(\dot{K} - \widehat{\dot{K}}(0)\big)\| \overset{(3.13)}{\le} L^{-\frac{1}{2}2^2\beta} L^d \|\dot{K}\|. \tag{3.21}$$

Therefore, since $d = 2$, with the hypothesis $\beta > 1$, C is contractive. □

Exercise 3.5. Why is there 2^2 as opposed to 1^2 in (3.21).

3.2. Finite Volume

What does the infinite volume Proposition 3.3 tell us about finite volume? The infinite volume hierarchical Gaussian field has the decomposition $\varphi = \sum_{j\geq 1}\zeta_j$ into infinitely many scales. One obtains a decomposition into N scales, appropriate for Λ with side L^N, by combining and integrating out all the scales $\sum_{j\geq N}\zeta_j$ in one final step. This is natural because, by (3.1), the fields ζ_j with $j > N$ are all constant on Λ. However one finds from (3.1) with $d = 2$ that the covariance $\sum_{j\geq N} C_j$ is divergent! This rather drastically illustrates the comment in Section 2.6 about the last (RG) being different from the previous ones. The problem has arisen because Theorem 1.12 was proved under the assumption that the interaction U of the particle system is a positive-definite form on $\mathbb{R}^\Lambda$ but actually it is only defined on the subspace consisting of $q \in \mathbb{R}^\Lambda$ such that $\sum_{x\in\Lambda} q(x) = 0$. These are called *neutral* charge configurations. Intuitively the Coulomb energy of a non-neutral charge configuration is plus infinity and so two dimensional Coulomb systems forbid non-neutrality. If one examines the proof carefully one finds that Theorem 1.12 extends to the hierarchical Coulomb when written as a limit

$$\Xi = \lim_{n\to\infty} \int d\mu_n \int d\mu_{n-1} \ldots \int d\mu_1 \, F^\Lambda. \tag{3.22}$$

so that the final (RG) map is really

$$F \mapsto \lim_{n\to\infty} \int d\mu_n \int d\mu_{n-1} \ldots \int d\mu_N \, F^\Lambda. \tag{3.23}$$

It is an easy exercise to prove

$$\lim_{n\to\infty} \int d\mu_n \int d\mu_{n-1} \ldots \int d\mu_{N+1} e^{iq\sum_{j>N}\zeta_j(\Lambda)} = 0 \tag{3.24}$$

for integral $q \neq 0$ so the final (RG) forces F^Λ to be well defined on equivalence classes by projecting out the neutral charge $q = 0$ part. For the Euclidean model Ex. 2.6 describes an analogous limit as the mass tends to zero. Notice that the neutral part of F^Λ is a well defined function on the equivalence classes $\mathbb{R}^\Lambda/\{\text{constants}\}$.

By Proposition 3.3, for any $\epsilon > 0$, let $N(\epsilon)$ be so large that

$$\| \underbrace{(\mathrm{RG})^N F_0}_{e^{\lambda_0|\Lambda|}\Xi(\Lambda)} - 1\| < \epsilon \tag{3.25}$$

for $N > N(\epsilon)$. Since the norm bounds the $q = 0$ component,

$$|e^{\lambda_0|\Lambda|}\Xi(\Lambda) - 1| < \epsilon. \tag{3.26}$$

This implies existence of (β times the pressure) which by definition is

$$\lim |\Lambda|_0^{-1} \log \Xi(\Lambda) = -\lambda_0, \tag{3.27}$$

but it is a *far stronger statement*; existence of the pressure is a large deviations result, in which any Λ dependence which is subexponential is lost, but (3.26) rules out prefactors [1] such as

$$\Xi(\Lambda) \sim |\Lambda| e^{\beta|\Lambda|\text{Pressure}}. \tag{3.28}$$

[1] I thank Guowei Zhao for many corrections to these equations.

Remark 3.6. I did not check, but since iteration of (RG) drives F to 1 it may be easy to prove that the scaling limit for the model is the same as the scaling limit of the Gaussian measure $d\mu$ by making a translation $\varphi \to \varphi + g$ in $\int e^{\varphi_\ell(f)} F^\Lambda d\mu$ to eliminate $\varphi(f)$ and then applying (RG) to the translated F. (RG) commutes with translations and if $(\mathrm{RG})^{N-1}F$ tends to 1, the translation drops out. This calculates the Laplace transform instead of the Fourier transform in (1.9).

3.3. Fractional Charge Observable and Confinement

In this section we sketch how to calculate the fractional charge correlation via (2.24). The new idea is that observables are also to be understood in terms of evolution of coupling constants, cf. α_j in equations below. I regard this calculation as a pointer towards a more complete theory of renormalisation. At present coupling constants appear as coordinates for expanding directions in the space of translation invariant local functions. The action of (RG) on this space defines the expanding directions. It would be desirable to construct a clean extension of this picture in which coupling constants are also associated to local deviations from translation invariance. These local deviations can live at points, which is the case for the fractional charges in the present discussion, or submanifolds of higher dimension. The submanifold case would allow us to analyse boundary conditions and interfaces.

Referring to Def. 2.15 we need to calculate $(\mathrm{RG})^N F_{a,b}$. Referring to Ex. 1.13, this means that the initial local function is given by (1.41). First we consider the case where there is only one fractional charge at a so that the initial $F_{a,b}$ is replaced by

$$F_a(\{x\}) = F(\{x\}) e^{i\varphi(x) 1_{x=a}}. \tag{3.29}$$

For each scale j, let $B_a \in \mathcal{B}_j$ be the block that contains a. We will proceed as in the proof of Proposition 3.3 but with $I_j = e^{\lambda_j}$ replaced by

$$I_j(B) = e^{\lambda_j} \begin{cases} \sum_{q\in\{-1,1\}} \alpha_j(q) e^{iq\varphi_j(B)} & \text{if } B = B_a \\ 1 & \text{otherwise} \end{cases}. \tag{3.30}$$

We say $\alpha_j \sim L^{-\frac{1}{2}\beta j}$ if $\lim_{j\to\infty} \frac{1}{j}\log_L \alpha_j = -\frac{\beta}{2}$. Let $\mathbb{I}_{B_a} = \mathbb{I}_{B_a}(B)$ be the indicator for $\{B = B_a\}$.

Proposition 3.7. *Under the hypotheses of Proposition 3.3,* $(\mathrm{RG})^j F_a = I_j + K_j$ *where* $\alpha_j \sim L^{-\frac{1}{2}\beta j}$ *and* $\|L^{\frac{1}{2}\beta j \mathbb{I}_{B_a}} K_j\| \to 0$ *as* $j \to \infty$.

Proof. (Sketch) We define (I_0, K_0) by

$$(3.29) = \underbrace{e^{\lambda_0} e^{i\varphi_0(x)1_{x=a}}}_{I_0(B)} + \underbrace{e^{\lambda_0} e^{i\varphi_0(x)1_{x=a}} 2z\cos(2\varphi_0(x))}_{K_0(B)}, \quad B = \{x\} \tag{3.31}$$

which is of the form (3.30) with $\alpha_0(q) = 1, 0$ for $q = 1, -1$. Let $\tilde{I}_{j+1}$ have the form

$$\tilde{I}_{j+1}(B) = e^{L^{-d}\lambda_{j+1}} \begin{cases} \sum_{q\in\{-1,1\}} \alpha_{j+1}(q) e^{iq\varphi_{j+1}(B)} & \text{if } B = B_a \\ 1 & \text{otherwise} \end{cases}. \tag{3.32}$$

With this choice, by Lemma 2.14, I_j keeps the form (3.30) when j advances to $j+1$. By induction using Lemma 2.14 the Fourier coefficients of K_j satisfy

$$\widehat{K_j}(B, q) = \begin{cases} 0 & B = B_a \text{ and } q \text{ even} \\ 0 & B \neq B_a \text{ and } q \text{ odd} \end{cases}. \tag{3.33}$$

Define $\tilde{K}_j$ by

$$\tilde{K}_j = \mathbb{E}_{j+1} K_j + \mathbb{E}_{j+1} I_j - \tilde{I}_{j+1}. \tag{3.34}$$

Choose λ_j in $\tilde{I}_{j+1}$ to be the same sequence as was determined in the proof of Proposition 3.3. Then, for $B \neq B_a$ the evolution of K_j is the same as in Proposition 3.3. We now have to determined the evolution of α_j in this "background".

Choose $\alpha_{j+1} \in \mathbb{R}^{\{-1,1\}}$ so that $\tilde{K}_j(B_a)$ has vanishing Fourier coefficients for $q \in \{-1, 1\}$. Let D be the derivative with respect to the initial α_0. By differentiating the last equation,

$$D\tilde{I}_{j+1} = P\mathbb{E}_{j+1} DK_j + \mathbb{E}_{j+1} DI_j \tag{3.35}$$

where P projects onto the $q \in \{-1, 1\}$ components,

$$P\Big(\sum_q c(q) e^{iq\varphi}\Big) = \sum_{q \in \{-1,1\}} c(q) e^{iq\varphi}. \tag{3.36}$$

Evaluating the derivatives in (3.35), we have, as functions on $\{-1, 1\}$,

$$e^{L^{-d}\lambda_{j+1}} D\alpha_{j+1} = e^{\lambda_j} L^{-\frac{1}{2}\beta} D\alpha_j + \mathbb{E}_{j+1} D\widehat{K_j}(B_a). \tag{3.37}$$

Our objective is to solve this recursion for $D\alpha_j$ and recover α_j by integrating the initial α_0 from zero to one. Actually, everything is linear in α_0 so these derivatives are constant in α_0 and the integral does nothing. If we could set DK_j and λ_j to zero we would easily obtain Proposition 3.7 from the contractive factor $L^{-\frac{1}{2}\beta}$. From Proposition 3.3 we know $\lambda_j \to 0$ and in fact the proof of Proposition 3.3 shows that $\lambda_j \to 0$ exponentially fast and this is good enough. To prove that we can neglect DK_j we differentiate (2.33) and take the norm,

$$\begin{aligned}
\|DK_{j+1}\| &\leq \|\mathbb{E}_{j+1} D\tilde{K}_j\| + O(\|\tilde{K}_j\| \, \|D\tilde{I}_{j+1}\|) + O(\|\tilde{K}_j\| \, \|D\tilde{K}_j\|) \\
&\leq L^{-\frac{9}{2}\beta} \|DK_j\| + O(\|K_j\| \, \|D\tilde{I}_{j+1}\|) + O(\|K_j\| \, \|DK_j\|) \\
&= \big(L^{-\frac{9}{2}\beta} + O(\|K_j\|)\big) \|DK_j\| + O(\|K_j\|) \, \|D\tilde{I}_{j+1}\|.
\end{aligned} \tag{3.38}$$

The factor $L^{-\frac{9}{2}\beta}$ is obtained from (3.13) because $\tilde{I}_{j+1}$ was chosen to cancel the $q \in \{-1, 1\}$ Fourier coefficients and the next smallest is $|q| = 3$. By inspecting the formulas in (2.33) the $O(\|K_j\|)$ are the norms on blocks $B \neq B_a$ and we know from Proposition 3.3 that these tend to zero so the bound has the form $d_{j+1} \leq a d_j + b_j$, where $a \approx L^{-\frac{9}{2}\beta}$. If $b_j \to 0$ more slowly than a^j this recursion is comparable with $(1 - a) d_{j+1} \leq b_j$ which says $\|DK_{j+1}\| \leq O(\|K_j\| \, \|D\tilde{I}_{j+1}\|) \leq O(\|K_j\|) \, |D\alpha_{j+1}|$. Otherwise it is even smaller. Either way it is negligible in (3.37). □

Referring to Def. 2.15 we conclude from Proposition 3.7 that the formal infinite volume limit for a single immersed fractional charge is zero because $\alpha_j e^{\pm i\varphi_j(B)} \to A1$ with $A = 0$. The exponential decay $\alpha_j \sim L^{-\frac{1}{2}\beta j}$ expresses the fact that the system hates to have an isolated single fractional charge and the energy of such an immersed charge grows with scale of isolation. What happens if there are two fractional charges? Under (RG) the two charges evolve independently as in Proposition 3.7 until the scale j such that $\mathrm{dist}_h(a, b) = L^j$ is reached. At this scale $B_a = B_b$ and φ_j does not separate the points a, b so $\exp(\pm i\varphi_j(B_a) \pm i\varphi_j(B_b))$ combine into $\exp(iq\varphi_j(B))$ with $q \in \{-2, 0, 2\}$. Further analysis based on the same ideas as in the proof of Proposition 3.3 shows there is no further exponential decay as the scale advances after $B_a = B_b$. This is because the combined immersed charge is no longer fractional and under (RG) evolution a nonzero $q = 0$ term appears in I_j, which reflects system charges combining with the immersed charge to make a neutral configuration. Thus

$$e^{-\beta E_{a,b}} = \lim_{N\to\infty} (\mathrm{RG})^N F_{a,b} / (\mathrm{RG})^N F \approx \alpha_j^2, \quad \mathrm{dist}_h(a,b) = L^j. \tag{3.39}$$

By putting in α_j from Proposition 3.7 we have the skeleton of a proof of

Proposition 3.8. *Under the hypotheses of Proposition 3.3, the energy E_{ab} of two immersed fractional charges, as defined in Ex. 1.13, grows logarithmically with separation,*

$$\underbrace{\lim_{\mathrm{dist}_h(a,b)\to\infty} \frac{E_{ab}}{\ln \mathrm{dist}_h(a,b)}}_{\varepsilon^{-1}} = 1. \tag{3.40}$$

If $\beta < 1$, it is proved in [**43**] that $\varepsilon \neq 1$.

Remark 3.9. Here is a physical explanation for why the observable need not be neutral:

$$F_{a,b}(\{x\}) = F(\{x\}) e^{i\varphi(x)(1_{x=a}+1_{x=b})}, \tag{3.41}$$

is not neutral but it is neutral mod 2 so the particle system can neutralise the fractional charges.

Exercise 3.10. Fill in details in the proof of Proposition 3.7.

Exercise 3.11. Referring to the remark above Proposition 3.8, why is there no more exponential decay in j after $B_a = B_b$.

3.4. Appendix. Notes on the Rigorous Renormalisation Group

The "group" that gave rise to the name "renormalisation group" appeared at the very beginning of this subject in the Gellman-Low equations [**33**]. If one considers the lattice to be embedded in a continuum $\mathbb{R}^d$ then dilation acts on $\mathbb{R}^d$ but changes the embedding, so that no useful consequences come immediately. However, if the continuum limit exists then it is a theory on $\mathbb{R}^d$ and dilations will act on it. Under the assumption that the scaling limit exists and is described by a renormalised perturbation theory, Gell-Mann and Low noticed that dilation combined with a simultaneous change of renormalised coupling constants leaves the correlation functions of the scaling limit invariant. Thus scale and coupling constants are a redundant description of the scaling limit. The renormalisation *group* is this group action on the redundant description.

K. Wilson [**57, 58**] introduced the "renormalisation semigroup". In the context of lattice field theory such as (1.32) he defined a conditional expectation by conditioning on the averages $\{Q\varphi(B) : B \in \mathcal{B}_{j+1}\}$ defined by

$$Q\varphi(B) = |B|^{-1} \sum_{x\in B} \varphi(x). \tag{3.42}$$

This creates a new lattice theory by regarding $\mathcal{B}_{j+1}$ as a new lattice, for example by identifying $B \in \mathcal{B}_{j+1}$ with the point at the centre of B. Using perturbation theory he argued that the dominant effect of this operation is to return to (1.32), but with a rescaled field and new coupling constants m^2, g. In his program there are many other terms in the new action because this conditioning destroys the simple form (1.32), but he was able to show that the additional terms are "irrelevant" in the sense that the rescaling of the fields contracts them so that they do not accumulate. He was only able to do this term by term and not for the sum of all the terms, which is not expected to be convergent. Thus Wilson's semigroup acts on an infinite dimensional space of actions, which he was not able to completely define. The difficulty in defining this space of actions is part of the global to local problem; defining the renormalisation group on a space of actions is, by definition, the same as requiring the renormalisation group to act on a space of Gibbsian measures. The review [**56**] discusses the obstacles to such a program, but does not completely rule it out. In these lectures we define the renormalisation group without making the assumption that the measures are Gibbsian.

A related set of ideas called the phase-cell expansion was introduced by Glimm and Jaffe in [**35**]. This was a landmark paper in the constructive field theory program which came close to Wilson's great contribution, but did not have the dynamical system insight into fixed points and the role of four dimensions as a bifurcation for the renormalisation group. The phase-cell expansion was shown to have some of the same scope as the renormalisation group, [**24**], and is at work in the very complete program of Feldman et al. [**25**] in condensed matter.

The use of block spin averages as advocated by Wilson was taken up by Gawedzki and Kupiainen [**29, 30, 32**] who achieved the first good solution to the global to local problem, that is, the definition of the space on which Wilson's semigroup acts. Balaban found a more flexible solution in his work on Yang Mills and classical spin models which starts with [**3**]. Because these papers are difficult everyone who has followed in their footsteps, including me, has invested in alternative formulations, which has led to fragmentation, but since no formulation has yet achieved both simplicity and power it is important to keep searching. Recently, Spencer et al. [**47, 53, 48**] have introduced new ideas based on convexity [**39**], which may in time lead to better formulations. I believe that the method of Balaban will, in time, triumph. It has a beautiful structure based on the Laplace method in infinite dimensions: this is to integrate subject to constraints by minimising the action subject to the constraints. The order in which the constraints are released by integrating over them can be very general. Someone with patience and conceptual ability is needed to reveal this well kept secret.

The solution to the global to local problem I am presenting in the next three lectures is based on papers with Yau, Dimock and Hurd, Mitter, and Scoppola, which are summarised in the encyclopedia article by Pronob Mitter [**44**], and papers in preparation with Slade [**13**]. There is an error (the norms are not complete) undermining parts of [**9, 12**] which was noticed and corrected by Malek Abdessalam [**1**].

The study of Hierarchical models began with the paper of Dyson [**22**]. In order to understand the validity and defects of simple approximations to the renormalisation group flow for Euclidean models, new models were designed such that the approximate flows become exact and these models have features like hierarchical models. See for example [**36**]. This line of investigation escapes the straightjacket of perturbation theory because the recursions can be solved numerically. The review by Fisher [**27**] makes the case for not losing the intuition that these approximations provide by submerging the renormalisation group within Gaussian approximation and perturbation theory, but in the absence of theorems that relate the hierarchical model approximations to Euclidean models, we continue to rely on perturbation theory. The very helpful hierarchical decomposition of covariance used in these lectures was first proposed in [**4**]. This paper also originated the attractive idea of decomposing the field into a sum of independent fluctuations as is being done in these notes and began the renormalisation group program reviewed in [**5**]. Finite range decompositions are a natural continuation of this idea and were first used in [**46**].

LECTURE 4

Renormalisation group for Euclidean models

4.1. Euclidean Lattice and the Dipole Model

We extend the ideas we have introduced to the Euclidean lattice. We consider formal infinite volume limits of $d\mu\, F^{\Lambda}$ where F is a function of $\nabla\varphi$, and μ is a Gaussian measure whose covariance has a finite range decomposition as in Def. 2.1 which satisfies a dimension $[\varphi] = (d-2)/2$ estimate as in Def. 2.2. In particular, μ can be the massless Gaussian measure on $\mathbb{R}^{\Lambda}$ for $d > 2$ and for $d \leq 2$ if some details about finite range decompositions are checked.

These measures include the dipole system (Ex. 1.14) and also sound waves in crystals. These are interesting applications, but what makes this class of measures appropriate in these lectures is that they are the first rung on a ladder of increasingly difficult Euclidean lattice scaling limit problems. We will refer to these measures collectively as the *dipole gas*. The word *gas* is used because the $F \approx 1$ hypothesis translates into the requirement that the density of dipoles be low.

The $F \approx 1$ hypothesis: the perturbation $F(\{x\})$ is a three times continuously differentiable function of $\nabla\varphi(x)$ satisfying, for all $x \in \Lambda$ and $p = 0, 1, 2, 3$,

$$|D^p(F(\{x\}) - 1)| \leq A^{*-1} h^{-p} e^{h^{-2}(\nabla\varphi)^2(x)/2} \tag{4.1}$$

where D^p is p partial derivatives with respect to $\nabla\varphi$. In this hypothesis, A^{*-1}, h^{-1} are small constants depending on dimension $d \geq 1$. The role of the important parameter h is to fix a large scale for φ: the hypothesis says that the interaction is more or less constant with respect to $\pm h$ fluctuations in $\nabla\varphi$. On the other hand the massless Gaussian measure only allows the typical fluctuation in $\nabla\varphi$ to be $O(h^0)$ so in this sense $F \approx 1$.

We assume that $F(\{x\})$ is even in φ and invariant under lattice symmetries that fix x. For example

$$\exp(\nabla_{\hat{e}_1}\varphi(x)\nabla_{\hat{e}_2}\varphi(x)) \tag{4.2}$$

is not invariant under these symmetries because a reflection or rotation which fixes x can move $\hat{e}_1$.

Theorem 4.1. *Under these assumptions the scaling limit of the dipole gas is the same as the scaling limit of the massless free field with covariance multiplied by some constant* $\frac{1}{\varepsilon}$.

The large class of allowed initial local functions means that many systems share the same scaling limit. In the physics literature this phenomenon is called *Universality*. It is an important attribute of the RG that it naturally proves Universality results and in this sense is complementary to the exact solution programs.

Theorems of this type as well as decay of correlations were first proved by Gawedzki and Kupiainen [**31**]. A much easier proof for actions which are convex in $\nabla\varphi$ was given by Naddaf and Spencer [**47**], but the Gawedzki and Kupiainen argument is able to do more; for example they also proved that the pressure is analytic in the activity z for z near the origin. The method I will use began with [**15**].

4.1.1. Steps in proof of Theorem 4.1

The main steps in proving Theorem 4.1 parallel the hierarchical case.

(1) solve the global to local problem. This is accomplished by Proposition 5.1, which replaces Lemma 2.14.
(2) prove that the map (RG) : $(I, K) \mapsto (I', K')$ defined in Proposition 5.1 is smooth. This is in Section 6.1.
(3) identify the operator that corresponds to C in Theorem 2.16. In Section 5.2 we show that this is essentially the operator $\mathcal{L}$ of Def. 5.4.
(4) prove that $\mathcal{L}$ is contractive. This is in Section 6.2.3 and 6.2.4.
(5) Apply Theorem 2.16 to obtain a global (RG) trajectory that converges to the trivial fixed point provided the initial interaction is tuned. Preparation for the tuning of the initial interaction is discussed in Section 4.2.

These steps establish existence of a global trajectory for (RG) in the formal infinite volume limit which is the key to proving Theorem 4.1. We will not have time and space to do more, but with this accomplished the way is open, by arguments similar to the hierarchical case in Sections 3.2 and 3.3, to obtain hard estimates on the decay of correlation functions and the scaling limit. This program has been started in [**19, 11**].

4.2. The Initial I_0, K_0

In this section we outline step 5 of Section 4.1.1. In the hierarchical Coulomb gas we had just one coupling constant, which is the vacuum energy. In this example we will have two,

$$\begin{aligned} &\text{Vacuum energy} \quad \lambda_0^{\text{vac}}, \\ &\text{Dielectric constant} \quad \varepsilon_0. \end{aligned} \tag{4.3}$$

Here, the tradition in mathematics that ε_0 is a small number gives way to the tradition in electromagnetism that ε_0 denotes the dielectric constant which in this model is almost equal to one. Recall that vacuum energy is a multiplicative constant in the definition of the partition function which is an ambiguity in how to define the partition function because it cancels out in correlation functions. In Proposition 3.3 we chose the the vacuum energy so that the interaction tends to the trivial fixed point under the action of (RG). In the present case there is also the ambiguity in how to define the factors $d\mu$ and F^Λ in $d\mu\, F^\Lambda$ because we can multiply and divide by a Gaussian factor as in

$$d\mu\, F^\Lambda \propto \underbrace{\text{const.}\, d\mu\, e^{\frac{1}{2}(1-\varepsilon_0)(\nabla\varphi_0, \nabla\varphi_0)}}_{d\mu_{\varepsilon_0}} \left(e^{-\frac{1}{2}(1-\varepsilon_0)(\nabla\varphi_0)^2} F\right)^\Lambda. \tag{4.4}$$

The $d\mu_{\varepsilon_0}$ factor is a normalised massless Gaussian measure whose covariance is inverse to $\varepsilon_0(-\Delta)$ and the phrase "tuning the dielectric constant" means to prove that ε_0 can be chosen so that (RG) drives $e^{-\frac{1}{2}(1-\varepsilon_0)(\nabla\varphi_0)^2} F$ to the trivial fixed point. The dimensional analysis of Section 1.2 predicts that $\sum(\nabla\varphi_0)^2$ is marginal.

It is easier not to have the unknown ε_0 in the Gaussian measure as well as the perturbation so we scale the field, $\varphi_0 \mapsto \varphi_0/\sqrt{\varepsilon_0}$ and obtain

$$\text{const.}\, d\mu\, \big(e^{-\frac{1}{2}(\varepsilon_0^{-1}-1)(\nabla\varphi_0)^2} F_{\varphi_0 \mapsto \varphi_0/\sqrt{\varepsilon_0}}\big)^{\Lambda}. \tag{4.5}$$

Set

$$\lambda_0^{\text{diel}} = \varepsilon_0^{-1} - 1.$$

For $j \in \mathbb{N}_0$ and $B \in \mathcal{B}_j$ let

$$I_j(B) = \prod_{x\in B} e^{-\lambda_j^{\text{vac}} - \frac{1}{2}\lambda_j^{\text{diel}}(\nabla\varphi_j)^2(x)} \tag{4.6}$$

and

$$\lambda_j = (\lambda_j^{\text{vac}}, \lambda_j^{\text{diel}}). \tag{4.7}$$

Then, K_0 is determined by

$$e^{-\lambda_0^{\text{vac}} - \frac{1}{2}\lambda_0^{\text{diel}}(\nabla\varphi_0)^2(x)} F_{\varphi_0 \mapsto \varphi_0/\sqrt{\varepsilon_0}} = I_0(\{x\}) + K_0(\{x\}). \tag{4.8}$$

This is a little different from our previous analysis in that the unknown λ_0^{diel} occurs in K_0 as well as I_0. Recall that Theorem 2.16 will provide us with the function h (not to be confused with the parameter h in (4.1)) such that if $\lambda_0 = h(K_0)$ then (K_0, λ_0) maps to the trivial fixed point. Therefore, we have to solve $\lambda_0 = h(K_0(\lambda_0))$. After we have defined norms and Banach spaces this is done by an application of the implicit function theorem to prove that the solution exists. We will not say any more about this but it should be plausible that it works because the hypothesis (4.1) makes the dependence of K_0 on λ_0 very weak.

4.3. The Basic Scaling Mechanism

For the Coulomb gas the basic mechanism was in (3.3). For our dipole model this is replaced by the estimate of Def. 2.2 with $[\varphi] = (d-2)/2$ which follows from the finite range decomposition of the Euclidean massless free field. By Def. 2.2

$$\text{var}(\nabla\varphi_j) = \sum_{k>j} \text{var}(\nabla\zeta_k) = O(L^{-2j([\varphi]+1)}) = O(L^{-dj}). \tag{4.9}$$

In other words, the standard deviation of $\nabla\varphi_{j+1}$ scales down by a factor $L^{-d/2}$ relative to the standard deviation of $\nabla\varphi_j$. Referring to (2.33), we have to prove that the linear part is contractive. If we can choose $\tilde{I}$ to cancel (most of) the first two terms in the Taylor expansion of $\mathbb{E}_{j+1}I_j + \mathbb{E}_{j+1}K_j$ in powers of $\nabla\varphi$ about the origin, then the remainder will contain $(\nabla\varphi_{j+1})^3$ which scales down by $L^{-3d/2}$ and this is enough to beat the L^d coming from the sum over $B \in \mathcal{B}_j(B')$ in (2.33). This is an explanation using the hierarchical equations but we will find essentially the same equations for the Euclidean model.

4.4. Coordinates (I_j, K_j)

In this section we begin step 1 of Section 4.1.1. We define $\tilde{\text{RG}} = \mathbb{E}_{j+1}$ as before, but now the finite range decomposition is in terms of the Euclidean norm as in Def. 2.1. Lemma 2.9 fails, but a small change in the definition of the (I_j, K_j) "coordinates" enables us to recover a global to local program substituting for Lemma 2.9. All we have to do is to extend the domain of K_j so it becomes a function defined on polymers instead of just blocks and to impose a factorisation property on K_j.

The remainder of this section collects in one place most of our notation for later reference, so instead of reading it carefully we recommend just looking to see what is in it and coming back to it when necessary.

Definition 4.2. (a) *Connectivity.* A subset $X \subset \Lambda$ is said to be connected if for any two points $x_a, x_b \in X$ there exists a path $(x_i, i = 0, 1, \dots n) \in X$ with $\|x_{i+1} - x_i\|_\infty = 1$, $x_0 = x_a$ and $x_n = x_b$.[1] Connected sets are not empty. According to this definition, a nonempty polymer X can be decomposed into connected components. We let $\mathcal{C}(X)$ be the set of connected components of X. Two sets $X, Y \subset \Lambda$ are said to be *strictly disjoint* if there is no path from x to y when $x \in X$ and $y \in Y$.
(b) We let $\mathcal{P}_{j,c}$ be the subset of $\mathcal{P}_j$ consisting of connected polymers.
(c) *Closure.* Let $X \in \mathcal{P}_j$. The *closure* $\overline{X}$ is the smallest $Y \in \mathcal{P}_{j+1}$ such that $X \subset Y$.
(d) A connected polymer $X \in \mathcal{P}_{j,c}$ is said to be a *small set* if $|X|_j \leq 2^d$. It is said to be *large* if it is not small. Let $\mathcal{S}_j$ be the set of all small sets in $\mathcal{P}_j$ and, for any fixed $B \in \mathcal{B}_j$ let $S = |\{X \in \mathcal{S}_j : X \supset B\}|$ be the number of small sets containing B. By translation invariance S is a dimension dependent constant independent of B. For $B \in \mathcal{B}_j$ define the small set neighbourhood B^* to be the smallest cube of lattice points in Λ that contains $\bigcup\{Y \in \mathcal{S}_j : Y \supset B\}$.
(e) $\mathcal{N}_j^{\mathcal{P}_j}$ is the set of maps $K : \mathcal{P}_j \to \mathcal{N}_j$ such that $K(X) \in \mathcal{N}_j(X^*)$ where $X^* = \bigcup\{B^* : B \in \mathcal{B}_j(X)\}$. Other spaces such as $\tilde{\mathcal{N}}_j^{\mathcal{P}_j}$ are defined in the same way. $\mathcal{N}_j$ and the bigger space $\tilde{\mathcal{N}}_j$ were defined in the paragraph below Def. 2.8.
(f) $K \in \tilde{\mathcal{N}}_j^{\mathcal{P}_j}$ is said to *factor* if

$$K(X) = \prod_{Y \in \mathcal{C}(X)} K(Y). \tag{4.10}$$

When X is empty, $K(X) = 1$ by the convention that an empty product equals one.
(g) If $I_j \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ we define $I_j(X) = I_j^X$ for $X \in \mathcal{P}_j$.

The finite range property (2.2) enables $\mathbb{E}_{j+1}$ to preserve factorisation on scale $j+1$: Suppose $K \in \tilde{\mathcal{N}}_j^{\mathcal{P}_j}$ factors and let $Y_1, Y_2 \in \mathcal{P}_j$, be such that the closures $\overline{Y_i} \in \mathcal{P}_{j+1}$ are strictly disjoint. Then

$$\mathbb{E}_{j+1} K(Y_1 \cup Y_2) = \prod_{i=1,2} \mathbb{E}_{j+1} K(Y_i). \tag{4.11}$$

This holds provided $L > 2^{d+2}$ because this implies that the distance between strictly disjoint sets in $\mathcal{P}_{j+1}$, which is at least L^{j+1}, is larger than $\frac{1}{2}L^{j+1} + 2 \cdot 2^d L^j$. In this expression $\frac{1}{2}L^{j+1}$ is the range of the covariance of ζ_{j+1} and $2^d L^j$ is there because $K(Y) \in \mathcal{N}_j(Y^*)$. This is the first of several choices of the form $L > L(d)$.

Definition 4.3. *Circle product.* Given $I, K \in \tilde{\mathcal{N}}_j^{\mathcal{P}_j}$, we define a commutative and associative product by

$$(K \circ I)(X) = \sum_{Y \in \mathcal{P}_j(X)} K(Y) I(X \setminus Y) \tag{4.12}$$

(including the important terms $Y = \varnothing, X$). The product depends on j but we do not make this explicit in the notation.

[1]For example, $\{(0,0),(1,1)\}$ is connected.

Lemma 4.4. *If F_1 and $F_2 \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ and $X \in \mathcal{P}_j$ then*

$$(F_1 + F_2)^X = (F_1 \circ F_2)(X). \tag{4.13}$$

Proof. By (2.20),

$$(F_1 + F_2)^X = \prod_{B \in \mathcal{B}_j(X)} (F_1 + F_2)(B) = \sum_{Y \in \mathcal{P}_j(X)} F_1^Y F_2^{X \setminus Y}, \tag{4.14}$$

and the right-hand side is $(F_1 \circ F_2)(X)$ by definition. □

4.5. Euclidean Replacement for Lemma 2.14

In this section we continue step 1 of Section 4.1.1. To shorten notation we suppress the index j on all scale j objects and signal scale $j + 1$ objects by primes. For hierarchical models we wrote a global function $Z(\Lambda) \in \mathcal{N}_j$ in the form $Z(\Lambda) = F^\Lambda$. Recalling that $F = I + K$ and using Lemma 4.4, this gives the representation

$$Z(\Lambda) = \sum_{X \in \mathcal{P}_j(\Lambda)} K^X I^{\Lambda \setminus X}. \tag{4.15}$$

For the Euclidean lattice we use the very similar representation by a pair $(I, K) \in \mathcal{N}_j^{\mathcal{B}_j} \times \mathcal{N}_j^{\mathcal{P}_j}$,

$$Z(\Lambda) = \underbrace{\sum_{X \in \mathcal{P}_j(\Lambda)} K(X) I^{\Lambda \setminus X}}_{(I \circ K)(\Lambda)}, \quad K(\emptyset) = 1. \tag{4.16}$$

The difference is that K no longer has a complete factorisation into contributions labeled by blocks, but has the weaker factorisation property (4.10). We look for an action on the local pair (I, K) which represents the action of $\tilde{\mathrm{RG}}$ on $Z(\Lambda)$.

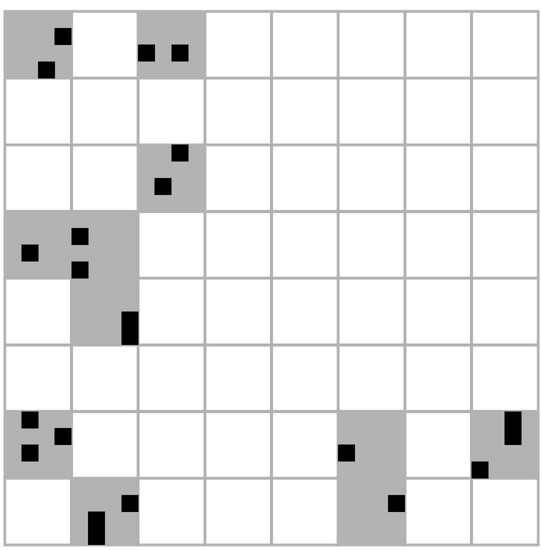

Figure 1. Polymer on next scale is smallest union of next scale blocks that covers the current scale polymer.

The main idea in finding such representations is quite parallel to the proof of Lemma 2.9. Suppose $\tilde{I} \in \mathcal{N}_{j+1}^{\mathcal{B}_j}$, which means that $\tilde{I}$ does not depend on the fluctuation field ζ_{j+1}, and consider

$$\mathbb{E}_{j+1}\Big(\sum_{X\in\mathcal{P}_j} K(X)\tilde{I}^{\Lambda\setminus X}\Big) = \sum_{X\in\mathcal{P}_j} \mathbb{E}_{j+1}\big(K(X)\tilde{I}^{\Lambda\setminus X}\big). \tag{4.17}$$

The union of the dark regions in Figure 1 is the set X for one of the terms $X \in \mathcal{P}_j$ under the sum. The individual dark regions are the connected components of X and each dark region represents a factor $K(Y)$ as in (4.10). Similarly, the complement of the dark regions, $\Lambda \setminus X$, represents the factor $\tilde{I}^{\Lambda\setminus X}$. The gray regions also represent factors, but these are factors in the coarser product over connected components of the closure $\overline{X}$. By (4.11), these gray regions are independent, conditionally on φ_{j+1}, so we can bring $\mathbb{E}_{j+1}$ inside the products over the gray regions. This means that $\mathbb{E}_{j+1}\big(\text{gray region}\big)$ has the factorisation property (4.10) on the next scale and is a candidate for K'.

Proposition 4.5. *For any $\tilde{I} \in \mathcal{N}_{j+1}^{\mathcal{B}_j}$, define $I' \in \mathcal{N}_{j+1}^{\mathcal{B}_{j+1}}$ by*

$$I'(B') = \tilde{I}^{B'}, \quad B' \in \mathcal{B}_{j+1} \tag{4.18}$$

and $\delta I \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ by

$$\delta I = I - \tilde{I}. \tag{4.19}$$

Then

$$\mathbb{E}_{j+1}(K \circ I)(\Lambda) = (K' \circ I')(\Lambda), \tag{4.20}$$

where, for $U \in \mathcal{P}_{j+1}$,

$$K'(U) = \sum_{X\in\mathcal{P}_j(U)} \mathbb{I}_{\overline{X}=U}\, \tilde{I}^{U\setminus X}\mathbb{E}_{j+1}(K \circ \delta I)(X). \tag{4.21}$$

$K' \in \mathcal{N}_{j+1}^{\mathcal{P}_{j+1}}$ inherits the factorisation property (4.10) at the next scale.

Proof. By Lemma 4.4, $I^X = (\delta I \circ \tilde{I})(X)$, so

$$K \circ I = K \circ (\delta I \circ \tilde{I}) = \underbrace{(K \circ \delta I)}_{\tilde{K}} \circ \tilde{I}. \tag{4.22}$$

Therefore

$$\begin{aligned}
\mathbb{E}_{j+1}(K \circ I)(\Lambda) &= \mathbb{E}_{j+1}(\tilde{K} \circ \tilde{I})(\Lambda) \\
&= \mathbb{E}_{j+1} \sum_{X\in\mathcal{P}_j(\Lambda)} \tilde{K}(X)\tilde{I}^{\Lambda\setminus X} \\
&= \sum_{X\in\mathcal{P}_j(\Lambda)} \mathbb{E}_{j+1}\big(\tilde{K}(X)\big)\tilde{I}^{\overline{X}\setminus X} I'^{\Lambda\setminus\overline{X}} \\
&= \sum_{U\in\mathcal{P}_{j+1}(\Lambda)} \Big(\sum_{X\in\mathcal{P}_j(U)} \mathbb{I}_{\overline{X}=U}\mathbb{E}_{j+1}\big(\tilde{K}(X)\big)\tilde{I}^{U\setminus X}\Big) I'^{\Lambda\setminus U} \\
&= K' \circ I'(\Lambda),
\end{aligned} \tag{4.23}$$

where we switched the order of the sums and used $\sum_{U\in\mathcal{P}_{j+1}(\Lambda)} \mathbb{I}_{\overline{X}=U} = 1$. Factorisation follows from (4.11). $\square$

Exercise 4.6. Check that $\tilde{K}$ factors as in (4.10) and K' factors on scale $j+1$.

As Einstein said, "everything should be as simple as possible, but not simpler". This Proposition illustrates the main idea, which is that the finite range property together with the separation of connected components of next scale polymers preserves factorisation, but it is not sufficient for our purposes. According to Ex. 4.7 the part of the map $(I, K) \mapsto (I', K')$ that corresponds to C in Theorem 2.16 is

$$K(\cdot) \mapsto \sum_{X \in \mathcal{P}_{j,c}} \mathbb{I}_{\overline{X}=(\cdot)} \, \mathbb{E}_{j+1}\big(K(X) + \delta I(X)\mathbb{I}_{X \in \mathcal{B}_j}\big). \tag{4.24}$$

Consider the case where the right hand side is evaluated on a block $B' \in \mathcal{B}_{j+1}$. The $O(L^d)$ terms in the sum makes the map want to expand the size of K by L^d. By choice of $\tilde{I}$ the contribution to the sum from the part of K that lives on single blocks can be canceled, but we will need to cancel part of K that lives on $X \notin \mathcal{B}_j$ to overcome the expansion by L^d and make this contractive. Thus we have to elaborate on the basic idea in Proposition 4.5 and find a more flexible way to define (I', K').

Exercise 4.7. Under the assumption that $\tilde{I}$ is chosen so that δI is linear in K check that the linear operator in (4.24) is the contractive part C of Theorem 2.16 for the map defined by Proposition 4.5.

LECTURE 5

Coordinates and action of renormalisation group

5.1. Euclidean Replacement for Lemma 2.14 continued

We continue with step 1 of Section 4.1.1. Referring to the previous section it is necessary to cancel parts of $K(X)$. We now will describe a way to cancel parts of K that live on small sets, $X \in \mathcal{S}_j$. This section contains an idea that is critical to control of Euclidean (RG).

Our method for canceling parts of $K(X)$ avoids the infamous pitfall known in the literature as "the large field problem", which was first successfully solved in [**30**] by different methods. The large field problem is that K must be permitted to grow as a function of the field when the field is large because the discussion below (4.9) shows that K is a Taylor remainder which will grow as $O((\nabla\varphi)^2)$. Most formulas that do more than cancel single block parts of K also lose control over this growth of K. The formula for K' in the coming Proposition 5.1 does not contain products of $K(X)$ such that the sets X in the range of the product overlap. Overlap causes the growth of K as a function of φ to be compounded which is the prelude to the large field problem of losing control of the large field growth.

5.1.1. A Preliminary Calculation

To explain the main idea, we make some unreasonable assumptions to simplify formulas. Thus suppose that $K(X)$ vanishes unless every connected component of X is a small set and suppose $I = 1$ so that

$$(I \circ K)(\Lambda) = \sum_{X \in \mathcal{P}_j} K(X). \tag{5.1}$$

Furthermore, suppose that K has somehow been normalised so that

$$\sum_{X \in \mathcal{S}_j} \mathbb{I}_{X \supset B} \frac{1}{|X|_j} K(X) = 0, \quad B \in \mathcal{B}_j. \tag{5.2}$$

We will exhibit a way to define the next scale (I', K') so that the linearisation of this map about $(1,0)$ vanishes!

For each summand labeled by $X \in \mathcal{P}_j$ let $\{X_1, \dots, X_n\}$ be the connected components of X. In each X_i we pick a $j-$block $B_i \in \mathcal{B}_j(X_i)$ and sum over the possible choices of these $j-$blocks, introducing factors $|X_i|_j^{-1}$ to normalise these sums to unity. We can write the resulting identity as

$$(I \circ K)(\Lambda) = \sum_{X \in \mathcal{P}_j} K(X) = \sum_{\mathcal{X}} \prod_{(B,X) \in \mathcal{X}} \frac{1}{|X|_j} K(X). \tag{5.3}$$

where $\mathcal{X} = \{(B_i, X_i) : i = 1, \dots, n\}$ is summed over finite subsets of

$$\hat{\mathcal{S}}_j = \big\{(B, X) : B \in \mathcal{B}_j(X),\ X \in \mathcal{S}_j\big\}. \tag{5.4}$$

such that $X_1, \dots X_n$ are strictly disjoint. We use the $j-$blocks $\{B_1, \dots, B_n\}$ in $\mathcal{X}$ to construct a polymer $U \in \mathcal{P}_{j+1}$ on the next scale,

$$U = \overline{B_1^* \cup \cdots \cup B_n^*}, \tag{5.5}$$

where B^* is the small set neighbourhood of B defined in Def. 4.2. Since $K(X)$ vanishes unless every connected component of X is a small set, U contains all the components X_i determined by $\mathcal{X}$ and

$$\begin{aligned}(I \circ K)(\Lambda) &= \sum_{\mathcal{X}} \prod_{(B,X)\in\mathcal{X}} \frac{1}{|X|_j} K(X)\\ &= \sum_{U\in\mathcal{P}_{j+1}} \underbrace{\sum_{\mathcal{X}} \mathbb{I}_{(5.5)} \prod_{(B,X)\in\mathcal{X}} \frac{1}{|X|_j} K(X)}_{K'(U)} = (I' \circ K')(\Lambda)\end{aligned} \tag{5.6}$$

where $I' = 1$. The linearisation of K' about $(1, 0)$ vanishes because

$$\sum_{(B,X)} \mathbb{I}_{\overline{B^*}=U} \frac{1}{|X|_j} K(X) = \sum_{B} \mathbb{I}_{\overline{B^*}=U} \sum_{X\supset B} \frac{1}{|X|_j} K(X) = 0. \tag{5.7}$$

In the coming arguments we lift the unnatural assumptions on I and K. We impose the condition (5.2) not on all of K, but on a part called J and (5.2) turns up in (5.23) and (5.30) below.

5.1.2. The Main Proposition

For $X \in \mathcal{S}_j$, we will split $\tilde{K}(X)$ into separate terms associated with blocks $B \in \mathcal{B}_j(X)$ and cancel these terms by a clever choice of $\tilde{I}(B)$. Let $\mathcal{N}_{j+1}^{\hat{\mathcal{S}}_j}$ denote the set of all functions

$$J : \hat{\mathcal{S}}_j \to \mathcal{N}_{j+1} \quad \text{such that } J(B, X) \in \mathcal{N}_{j+1}(B^*) \tag{5.8}$$

and extend J to be a function of all pairs (B, X) by setting $J(B, X) = 0$ if $(B, X) \notin \hat{\mathcal{S}}_j$. $J(B, X)$ is the part of $\tilde{K}(X)$ that we will cancel by a choice of $\tilde{I}(B)$. The uncanceled part of $\tilde{K}$ is $\check{K}$ defined by

$$\tilde{K}(X) = \sum_{B\in\mathcal{B}_j(X)} J(B, X) + \check{K}(X), \quad X \in \mathcal{P}_{j,c} \tag{5.9}$$

and

$$\check{K}(X) = \prod_{Y\in\mathcal{C}(X)} \check{K}(Y), \quad X \in \mathcal{P}_j. \tag{5.10}$$

Given a possibly empty subset $\mathcal{X} = \{(B_i, X_i) : i = 1, \dots, n = n(\mathcal{X})\}$ of $\hat{\mathcal{S}}_j$ we define

$$J^{\mathcal{X}} = \prod_{(B,X)\in\mathcal{X}} J(B, X) \quad \text{and} \quad X_{\mathcal{X}} = \cup\{X : (B, X) \in \mathcal{X}\}. \tag{5.11}$$

For $U \in \mathcal{P}_{j+1}$ we say $(\mathcal{X}, \check{X}) \in \mathcal{G}(U)$ if

$$\begin{aligned}&\check{X} \in \mathcal{P}_j, \quad \mathcal{X} \subset \hat{\mathcal{S}}_j,\\ &X_1, \dots, X_n, \check{X} \text{ are strictly disjoint,}\end{aligned} \tag{5.12}$$

$$\overline{B_1^* \cup \cdots \cup B_n^* \cup \check{X}} = U. \tag{5.13}$$

Proposition 5.1. *For any* $\tilde{I} \in \mathcal{N}_{j+1}^{\mathcal{B}_j}$ *and any* $J \in \mathcal{N}_{j+1}^{\hat{\mathcal{S}}_j}$*,*

$$\mathbb{E}_{j+1}\big(I \circ K\big)(\Lambda) = \big(I' \circ K'\big)(\Lambda), \tag{5.14}$$

where $I' \in \mathcal{N}_{j+1}^{\mathcal{B}_{j+1}}$ *is defined by*

$$I'(B') = \tilde{I}^{B'} \tag{5.15}$$

for $B' \in \mathcal{B}_{j+1}$ *and* $K' \in \mathcal{N}_{j+1}^{\mathcal{P}_{j+1}}$ *is defined by* $K'(\emptyset) = 1$ *and*

$$K'(U) = \sum_{(\mathcal{X},\check{X}) \in \mathcal{G}(U)} J^{\mathcal{X}} \tilde{I}^{X_I} \mathbb{E}_{j+1} \check{K}(\check{X}), \tag{5.16}$$

for $U \in \mathcal{P}_{j+1}$ *with* $X_I = U \setminus (\check{X} \cup X_{\mathcal{X}})$*.* K' *factors.*

Proof.

$$\tilde{K}(X) \overset{\text{Ex. 4.6}}{=} \prod_{Y \in \mathcal{C}(X)} \tilde{K}(Y) \overset{(5.9)}{=} \prod_{Y \in \mathcal{C}(X)} \Big(\sum_{B \in \mathcal{B}_j(Y)} J(B,Y) + \check{K}(Y) \Big). \tag{5.17}$$

When we expand the product over $Y \in \mathcal{C}(X)$, the result is a sum over all ways to assign to each $Y \in \mathcal{C}(X)$ a factor $\sum_B J(B,Y)$ or a factor $\check{K}(Y)$. We let $\check{X}$ be the union of the sets Y which are assigned $\check{K}(Y)$,

$$\tilde{K}(X) = \sum_{\check{X} \in \mathrm{U}\mathcal{C}(X)} \prod_{Y \in \mathcal{C}(X \setminus \check{X})} \Big(\sum_{B \in \mathcal{B}_j(Y)} J(B,Y) \Big) \underbrace{\check{K}(\check{X})}_{\text{using (5.10)}}, \tag{5.18}$$

where $\mathrm{U}\mathcal{C}(X)$ is the set of all unions of connected components of X, or the empty set. By substituting this into $Z = \tilde{K} \circ \tilde{I}$, we obtain

$$Z(\Lambda) = \sum_{X \in \mathcal{P}_j} \sum_{\check{X} \in \mathrm{U}\mathcal{C}(X)} \prod_{Y \in \mathcal{C}(X \setminus \check{X})} \Big(\sum_{B \in \mathcal{B}_j(Y)} J(B,Y) \Big) \check{K}(\check{X}) \tilde{I}^{\Lambda \setminus X}, \tag{5.19}$$

Inside the product over Y we have a sum over $B \in \mathcal{B}_j(Y)$ which is the same as summing over the set of choices, one B for each Y. This is the same as summing over a set $\mathcal{X}$ of pairs (B,Y). Therefore we can rewrite the last equation in the form

$$Z(\Lambda) = \sum_{\mathcal{X},\check{X} \in \{(5.12)\}} J^{\mathcal{X}} \check{K}(\check{X}) \tilde{I}^{\Lambda \setminus (X_{\mathcal{X}} \cup \check{X})}. \tag{5.20}$$

Using $\sum_U \mathbb{I}_{\{(5.13)\}} = 1$, we write this as

$$\mathbb{E}_{j+1} Z(\Lambda) = \sum_{U \in \mathcal{P}_{j+1}(\Lambda)} \underbrace{\sum_{(\mathcal{X},\check{X}) \in \mathcal{G}(U)} J^{\mathcal{X}} \mathbb{E}_{j+1} \check{K}(\check{X}) \tilde{I}^{X_I}}_{=K'(U)} I'^{\Lambda \setminus U}. \tag{5.21}$$

The proof that K' factors is Ex. 5.2.

∎

Exercise 5.2. Check that K' factors on scale $j+1$.

5.2. Formulas for $\tilde{I}, J$.

In this section we carry out step 3 of Section 4.1.1, which is to find the linear part of the renormalisation group map that corresponds to C in Theorem 2.16. The conclusion will be that C is essentially given by $\mathcal{L}$ as defined in Def. 5.4. The "essentially" is put in because we are going to drop some terms. The other important point in this section is that the formula (5.32) determines $I'(B') = \tilde{I}^{B'}$ and therefore the coupling constants $\vec{\lambda}'$ in I'. Therefore the renormalisation group map $(I,K) \mapsto (I',K')$ is equivalent to a map $(\vec{\lambda},K) \mapsto (\vec{\lambda}',K')$.

The formula (5.16) defines K' as a function.

$$\begin{aligned} &\big(I,\tilde{I},J,K\big) \mapsto K' \\ &K' : \mathcal{N}_j^{\mathcal{B}_j} \times \mathcal{N}_{j+1}^{\mathcal{B}_j} \times \mathcal{N}_{j+1}^{\hat{\mathcal{S}}_j} \times \mathcal{N}_j^{\mathcal{P}_{j,c}} \to \mathcal{N}_{j+1}^{\mathcal{P}_{j+1,c}}. \end{aligned} \tag{5.22}$$

In Section 6.1 we will define Banach spaces so that it makes sense to say that K is $\mathcal{C}^3$ so in the next Proposition where we calculate the linearisation we are in fact calculating the derivative and accordingly use the standard notation for a derivative. Note the almost invisible subscript c on $\mathcal{P}_{j,c}$ which signals the restriction of $K(X)$ to connected sets X.

Proposition 5.3. *Under the condition*

$$\sum_{X\in\mathcal{S}_j} \dot{J}(B,X) = 0 \tag{5.23}$$

the linearisation of K' at the fixed point

$$(I,\tilde{I},J,K) = (1,1,0,0) \tag{5.24}$$

is

$$\begin{aligned} DK'_{(1,1,0,0)}\big(\dot{I},\dot{\tilde{I}},\dot{J},\dot{K}\big)(U) = &\sum_{X\in\mathcal{P}_{j,c}(U):\overline{X}=U} \Big(\mathbb{E}_{j+1}\dot{K}(X) + \mathbb{E}_{j+1}\dot{I}(X)1_{X\in\mathcal{B}_j} \\ &- \dot{\tilde{I}}(X)1_{X\in\mathcal{B}_j} - \sum_{B\in\mathcal{B}_j(X)} \dot{J}(B,X)\Big). \end{aligned} \tag{5.25}$$

PROOF. Ex. 5.6. ☐

We replace the "tangent vector" $(\dot{I},\dot{\tilde{I}},\dot{J},\dot{K})$ by $(I-1,\tilde{I}-1,J,K)$ in order to obtain the linear terms in the forthcoming Taylor expansion of K' about the trivial fixed point. In the next section in Lemma 6.2 we show that all the terms in (5.25) for which X is a large set are negligible. The remaining terms are

$$\begin{aligned} R(U) = &\sum_{X\in\mathcal{S}_j:\overline{X}=U} \Big(\mathbb{E}_{j+1}K(X) + \\ &\mathbb{E}_{j+1}I(X)1_{X\in\mathcal{B}_j} - \tilde{I}(X)1_{X\in\mathcal{B}_j} - \sum_{B\in\mathcal{B}_j(X)} J(B,X)\Big). \end{aligned} \tag{5.26}$$

We have to jump ahead of the logical development in order to explain how to choose $\tilde{I}$ and J to cancel expanding parts of K. $\mathbb{E}_{j+1}K(X)$ is an element of $\mathcal{N}_{j+1}$, in other words, a function of $\varphi' = \varphi_{j+1}$. Later we will introduce a Banach space structure on $\mathcal{N}_{j+1}$ so that it makes sense to say that $\mathbb{E}_{j+1}K(X) \in \mathcal{C}^3(\mathcal{N}_{j+1})$, in other words a $\mathcal{C}^3$ function of φ'. Then $\mathbb{E}_{j+1}K(X)$ admits a second order Taylor expansion in powers of φ'. This Taylor expansion is denoted by $\operatorname{Tay}\mathbb{E}_{j+1}K(X)$ and it is a linear combination of

$$1, \quad \nabla_{\hat{e}_1}\varphi'(x)\nabla_{\hat{e}_2}\varphi'(y), \quad \hat{e}_1,\hat{e}_2 \in \mathcal{E}(\pm), \quad x,y \in X. \tag{5.27}$$

$\mathcal{E}(\pm)$ was defined in (1.54). There are no odd powers of $\nabla\varphi'$ because we assumed the initial interaction is even and it is easy to check that this property is inherited by $\mathbb{E}_{j+1}K$ because Proposition 5.1 propagates it to successive scales, provided J and $\tilde{I}$ are even. The remainder after the Taylor expansion is the contribution to $R(U)$ defined by

Definition 5.4.

$$\mathcal{L}(U) = \sum_{X\in\mathcal{S}_j:\overline{X}=U} (1-\operatorname{Tay})\mathbb{E}_{j+1}K(X) \tag{5.28}$$

Proposition 5.5. *For $X \in \mathcal{S}_j \setminus \mathcal{B}_j$ let*

$$J(B,X) = \operatorname{Tay}\frac{1}{|X|_j}\mathbb{E}_{j+1}K(X) \tag{5.29}$$

and let $J(B,B)$ solve

$$\sum_{X\in\mathcal{S}_j} J(B,X) = 0. \tag{5.30}$$

Then

$$R(U) = \mathcal{L}(U) + R_2(U) + R_3(U) \tag{5.31}$$

where

$$R_2(U) = \sum_{B\in\mathcal{B}_j} \mathbb{I}_{\overline{B}=U}\operatorname{Tay}\Big(-\tilde{I}(B)+\mathbb{E}_{j+1}I(B)+\sum_{X\in\mathcal{S}_j}\mathbb{I}_{X\supset B}\frac{1}{|X|_j}\mathbb{E}_{j+1}K(X)\Big) \tag{5.32}$$

and

$$R_3(U) = \sum_{B\in\mathcal{B}_j} \mathbb{I}_{\overline{B}=U}(1-\operatorname{Tay})\Big(\mathbb{E}_{j+1}I(B) - \tilde{I}(B)\Big). \tag{5.33}$$

Proof. It is easy to check that the choice (5.29) is such that for $X \notin \mathcal{B}_j$ the last term in (5.26) cancels the Taylor expansion of $\mathbb{E}_{j+1}K(X)$. By subtracting and adding $\operatorname{Tay}\mathbb{E}_{j+1}K(X)$ for $X = B \in \mathcal{B}_j$,

$$\begin{aligned} R(U) &\overset{(5.26)}{=} \sum_{X\in\mathcal{S}_j}\mathbb{I}_{\overline{X}=U}(1-\operatorname{Tay})\mathbb{E}_{j+1}K(X) \\ &\quad + \sum_{B\in\mathcal{B}_j}\mathbb{I}_{\overline{B}=U}\Big(\operatorname{Tay}\mathbb{E}_{j+1}K(B) + \mathbb{E}_{j+1}I(B) - \tilde{I}(B) - J(B,B)\Big) \end{aligned} \tag{5.34}$$

The first line is $\mathcal{L}(U)$. In the second line we split the third and fourth terms according to $1 = \operatorname{Tay} + (1-\operatorname{Tay})$. The $1-\operatorname{Tay}$ parts become $R_3(U)$,

$$\begin{aligned} R(U) &= \mathcal{L}(U) + R_3(U) \\ &\quad + \sum_{B\in\mathcal{B}_j}\mathbb{I}_{\overline{B}=U}\Big(\operatorname{Tay}\Big(\mathbb{E}_{j+1}K(B) + \mathbb{E}_{j+1}I(B) - \tilde{I}(B)\Big) - J(B,B)\Big). \end{aligned} \tag{5.35}$$

We substitute in $J(B,B)$ given by (5.30) and (5.29) and obtain $R_2(U)$. $\square$

From now on we drop the term $R_3(U)$. A complete argument would prove that R_3 is smooth and nonlinear in coupling constants and is part of g_j in Theorem 2.16. It requires calculations based on the explicit forms of I and $\tilde{I}$ given in (4.6).

We are also going to drop R_2. The idea is that $\tilde{I}$ is chosen so that $R_2(U)$ vanishes. However, this cannot be completely achieved because $\tilde{I}$ must have the form

$$\tilde{I}(B) = \prod_{x \in B} e^{-\lambda^{\mathrm{vac}\prime} - \frac{1}{2}\lambda^{\mathrm{diel}\prime}(\nabla\varphi')^2(x)} \tag{5.36}$$

because it determines I' by (5.15) and I' is given (4.6) with j replaced by $j+1$. Therefore the Taylor expansion of $\tilde{I}(B)$ has the form

$$-\sum_{x \in B}\Big(\lambda^{\mathrm{vac}\prime} + \frac{1}{2}\lambda^{\mathrm{diel}\prime}(\nabla\varphi')^2(x)\Big). \tag{5.37}$$

But the Taylor expansion of the other terms in R_2 contains nonlocal terms like $\nabla\varphi'(x)\cdot\nabla\varphi'(y)$. Therefore the new coupling constants $\vec{\lambda}'$ are determined so that $R_2(U)$ is zero modulo

$$\nabla\varphi'(x)\cdot\nabla\varphi'(y) - |B|^{-1}\sum_{z \in B}\nabla\varphi'(z)\cdot\nabla\varphi'(z). \tag{5.38}$$

These terms would be zero if $\nabla\varphi'$ were constant so they can be rewritten in terms of higher spatial derivatives like $\nabla^2\varphi'\nabla\varphi'$ which scale down faster than L^d. The symmetry hypotheses below (4.1) are respected by the renormalisation group and are needed to forbid such as (4.2).

Exercise 5.6. Prove Proposition 5.3. Note that there is no term in (5.16) with $X_I = U$ so the linearisation of $\tilde{I}^{X_I}$ cannot contribute.

LECTURE 6

Smoothness of (RG)

6.1. Choice of Spaces and Smoothness of (RG)

In this section we carry out step 2 (smoothness) of Section 4.1.1. We introduce Banach spaces and prove that our map (RG) : $(I, K) \to (I', K')$ is smooth along with estimates on the linearisations. In each proof, as soon as the estimates reach combinatoric/geometric principles, we drop the reader into Appendix 6.4. We adopted this organisation not because the remaining details are too steamy for polite society, but because there is a simple set of combinatoric principles at work and it is good to collect them in one place.

Referring to (4.6), $I, \tilde{I}$ are explicitly known functions of coupling constants and we expect to know more about them than K, so we allow a strong norm for $I, \tilde{I}$ and a weaker norm for K. Thus, for each $X \in \mathcal{P}_j$[1], there are two norms $\|\cdot\| \le \|\cdot\|_{\mathrm{str}}$ on $\mathcal{N}_j(X) \subset \tilde{\mathcal{N}}_j(X)$. For $K \in \tilde{\mathcal{N}}_j^{\mathcal{P}_j}$ that factor and $F \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$, the two norms are required to satisfy

$$\|K(X)\| \le \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|, \tag{6.1}$$

$$\|F^X K(Y)\| \le \|F\|_{\mathrm{str}}^X \, \|K(Y)\|, \quad X, Y \text{ disjoint} \tag{6.2}$$

$$\|1(B)\|_{\mathrm{str}} = 1 \tag{6.3}$$

$$\|\mathbb{E}_{j+1} K(X)\| \le 2^{|X|_j} \|K(X)\|, \quad X \in \mathcal{P}_j. \tag{6.4}$$

The sets X, Y in (6.2) are disjoint, but they could "touch" in the sense that there could be $x \in X$ and $y \in Y$ such that x, y are nearest neighbours in $\mathbb{Z}^d$. This is not allowed in (6.1); components, by Def. 4.2, are strictly disjoint.

The argument X in $\|K(X)\|$ signals that the norm is taken as an element of the space $\tilde{\mathcal{N}}_j(X^*)$, so $\|K(X)\| = \|K(X)\|_{\tilde{\mathcal{N}}_j(X^*)}$. This dependence of space on X could make difficulties with the triangle inequality but, as the following Lemma shows, the triangle inequality holds in the form we need.

Lemma 6.1. *If $X, X_i \in \mathcal{P}_j$, $\cup_i X_i \subset X$ and $K(X) = \sum_i K_i(X_i)$, then*

$$\|K(X)\| \le \sum_i \|K_i(X_i)\|. \tag{6.5}$$

Proof. For $Y \subset X$ and $X, Y \in \mathcal{P}_j$ we have $Y^* \subset X^*$ so $\tilde{\mathcal{N}}_j(Y^*) \subset \tilde{\mathcal{N}}_j(X^*)$.

$$\|K(Y)\|_{\tilde{\mathcal{N}}_j(X^*)} = \|K(Y) 1^{X \setminus Y}\|_{\tilde{\mathcal{N}}_j(X^*)} \overset{(6.2),(6.3)}{\le} \|K(Y)\|_{\tilde{\mathcal{N}}_j(Y^*)}. \tag{6.6}$$

[1] which includes X^* as in Def. 4.2 (e)

Then

$$\begin{aligned}\|K(X)\| &= \|K(X)\|_{\tilde{\mathcal{N}}_j(X^*)} \leq \sum_i \|K_i(X_i)\|_{\tilde{\mathcal{N}}_j(X^*)} \\ &\leq \sum_i \|K_i(X_i)\|_{\tilde{\mathcal{N}}_j(X_i^*)} = \sum_i \|K_i(X_i)\|.\end{aligned} \tag{6.7}$$

□

The above properties are all estimates at scale j. In particular, in (6.4), $\mathbb{E}_{j+1}K(X)$ is normed as an element of $\mathcal{N}_{j+1}$ regarded as a subspace of $\tilde{\mathcal{N}}_j$; in other words it is a function of (φ', ζ) which is constant in ζ. We pass to the next scale when we regard $\mathcal{N}_{j+1}$ as a subspace of $\tilde{\mathcal{N}}_{j+1}$ by writing $\varphi' = \varphi_{j+1} = \varphi_{j+2} + \zeta_{j+2}$. We relate the norms in this map from $\mathcal{N}_{j+1} \subset \tilde{\mathcal{N}}_j$ to $\mathcal{N}_{j+1} \subset \tilde{\mathcal{N}}_{j+1}$ by requiring

$$\|K'(U)\|' \leq \|K'(U)\|, \quad U \in \mathcal{P}_{j+1} \subset \mathcal{P}_j, \quad K'(U) \in \mathcal{N}_{j+1}(U^*), \tag{6.8}$$

where primed norm is the norm on $\tilde{\mathcal{N}}_{j+1}$. U^* is the small set neighbourhood of U as a set in $\mathcal{P}_j$, which is consistent with the $j+1$-scale norm because it is smaller than the small set neighbourhood of U as an element of $\mathcal{P}_{j+1}$.

$\mathcal{N}_j^{\mathcal{B}}, \mathcal{N}_{j+1}^{\hat{\mathcal{S}}_j}$ are Banach spaces with the norms

$$\begin{aligned}\|I\|_{\text{str}} &= \sup_{B \in \mathcal{B}_j} \|I(B)\|_{\text{str}}, \\ \|J\| &= \sup_{(B,X) \in \hat{\mathcal{S}}_j} \|J(B,X)\|\end{aligned} \tag{6.9}$$

and $\mathcal{N}_j^{\mathcal{P}_{j,c}}$ is a Banach space with

$$\|K\| = \sup_{X \in \mathcal{P}_{j,c}} \|K(X)\| A^{|X|_j}, \tag{6.10}$$

where $A \geq 1$. Notice that the norm only measures K on connected sets and connected sets are non-empty. Disconnected sets are handled by the algebra of factorisation.

Our first result explains why we only need to concentrate on small sets in defining $\mathcal{L}$. This important Lemma says that if we define the norms with A sufficiently large depending on the prior choices of d, L then the part of K that lives on large sets is a negligible proportion and this is why we dropped that contribution in the definition of R in (5.26); they make a negligible contribution to C in Theorem 2.16.

Lemma 6.2. *In the formula* (5.25) *the contribution from* $\dot{K}$ *on large sets is negligible in the sense*

$$\lim_{A \to \infty} \left(\|DK'_{(1,1,0,0)}(0,0,0,\dot{K}\mathbb{I}_{\notin \mathcal{S}_j})\|' / \|\dot{K}\| \right) = 0. \tag{6.11}$$

where $(\dot{K}\mathbb{I}_{\notin \mathcal{S}_j})(X) = \dot{K}(X)\mathbb{I}_{X \notin \mathcal{S}_j}$.

Proof.

$$DK'_{(1,1,0,0)}(0,0,0,\dot{K}\mathbb{I}_{\notin \mathcal{S}_j}) \overset{(5.25)}{=} \sum_{X \in \mathcal{P}_{j,c}(U) : \overline{X} = U} \mathbb{E}_{j+1} \dot{K}(X) \mathbb{I}_{X \notin \mathcal{S}_j}. \tag{6.12}$$

Therefore,

$$
\begin{aligned}
&A^{|U|_{j+1}} \|DK'_{(1,1,0,0)}(0,0,0,\dot{K}\mathbb{I}_{\notin \mathcal{S}_j})(U)\|' \\
&\overset{(6.8)}{\leq} A^{|U|_{j+1}} \|DK'_{(1,1,0,0)}(0,0,0,\dot{K}\mathbb{I}_{\notin \mathcal{S}_j})(U)\| \\
&\overset{(6.5),(6.4)}{\leq} A^{|U|_{j+1}} \sum_{X\in\mathcal{P}_{j,c}(U)} \mathbb{I}_{\overline{X}=U, X\notin\mathcal{S}_j} 2^{|X|_j} \|\dot{K}(X)\| \\
&\overset{(6.10)}{\leq} \|\dot{K}\| A^{|U|_{j+1}} \sum_{X\in\mathcal{P}_{j,c}(U)} \mathbb{I}_{\overline{X}=U, X\notin\mathcal{S}_j} 2^{|X|_j} A^{-|X|_j} \\
&\leq \|\dot{K}\| \sup_{U\in\mathcal{P}_{j+1,c}} A^{|U|_{j+1}} \underbrace{k(A/2,U)}_{\text{cf. Lemma 6.18}} .
\end{aligned}
\tag{6.13}
$$

The supremum tends to zero as $A \to \infty$ by Lemma 6.18 in the combinatoric appendix (which is where the condition $X \notin \mathcal{S}_j$ comes into play). □

Next we prepare to prove smoothness for the full nonlinear (RG) map. From now on the domain of the function K' defined by (5.16) is a ball

$$
B_{I,\tilde{I},J,K} \subset \mathcal{N}_j^{\mathcal{B}_j} \times \mathcal{N}_{j+1}^{\mathcal{B}_j} \times \mathcal{N}_{j+1}^{\hat{\mathcal{S}}_j} \times \mathcal{N}_j^{\mathcal{P}_{j,c}}
\tag{6.14}
$$

defined by

$$
\|I-1\|_{\mathrm{str}} < A^{*-1}, \quad \|\tilde{I}-1\|_{\mathrm{str}} < A^{*-1},
\tag{6.15}
$$

$$
\|J\| < A^{*-1},
\tag{6.16}
$$

$$
\|K\| < A^{*-1}.
\tag{6.17}
$$

Lemma 6.3. *On the domain $B_{I,\tilde{I},J,K}$, for $X \in \mathcal{P}_{j,c}$, $A^* \geq 2^{d+1}A^{2^d}$, $A \geq 1$,*

$$
\|\check{K}(X)\| \leq \left(\frac{A^*}{2^{d+1}A^{2^d}}\right)^{-|\mathcal{C}(X)|} (A/2)^{-|X|_j}.
\tag{6.18}
$$

Proof. (a) Estimate on K,

$$
\begin{aligned}
\|K(X)\| &\overset{(6.1)}{\leq} \prod_{X_c\in\mathcal{C}(X)} \|K(X_c)\| \leq \prod_{X_c\in\mathcal{C}(X)} \left(\|K\| A^{-|X_c|_j}\right) \\
&\overset{(6.17)}{\leq} A^{*-|\mathcal{C}(X)|} A^{-|X|_j}.
\end{aligned}
\tag{6.19}
$$

(b) Estimate on δI,

$$
\|\delta I(B)\|_{\mathrm{str}} = \|I(B) - \tilde{I}(B)\|_{\mathrm{str}} \overset{(6.15)}{\leq} 2A^{*-1}
\tag{6.20}
$$

which implies

$$
\|\delta I^X\|_{\mathrm{str}} \overset{(6.2)}{\leq} \left(\frac{A^*}{2A}\right)^{-|X|_j} A^{-|X|_j}
\tag{6.21}
$$

(c) Estimate on $\tilde{K}$. For $X \in \mathcal{P}_{j,c}$,

$$\|\tilde{K}(X)\| = \|(K \circ \delta I)(X)\| \overset{(6.2)}{\leq} \sum_{Y \in \mathcal{P}_j(X)} \|K(Y)\| \, \|\delta I\|_{\mathrm{str}}^{X \setminus Y}$$

$$\overset{(6.19),(6.21)}{\leq} \sum_{Y \in \mathcal{P}_j(X)} A^{*-|\mathcal{C}(Y)|} A^{-|Y|_j} \left(\frac{A^*}{2A}\right)^{-|X \setminus Y|_j} A^{-|X \setminus Y|_j} \tag{6.22}$$

$$\leq \left(\frac{A^*}{2A}\right)^{-1} \sum_{Y \in \mathcal{P}_j(X)} A^{-|Y|_j} A^{-|X \setminus Y|_j} = \left(\frac{A^*}{2A}\right)^{-1} (A/2)^{-|X|_j}.$$

(d) Estimate on $J(B,X)$. There are at most 2^d blocks B in $\mathcal{B}_j(X)$ because $X \in \mathcal{S}_j$, so

$$\|\sum_{B \in \mathcal{B}_j(X)} J(B,X)\| \overset{(6.9),(6.16)}{\leq} 2^d A^{*-1} \leq 2^d \left(\frac{A^*}{A^{2^d}}\right)^{-1} A^{-|X|_j}. \tag{6.23}$$

(e) Conclusion. For $X \in \mathcal{P}_{j,c}$,

$$\|\check{K}(X)\| \overset{(5.9)}{\leq} \|\tilde{K}(X)\| + \|\sum_{B \in \mathcal{B}_j(X)} J(B,X)\|$$

$$\overset{(6.22),(6.23)}{\leq} (\frac{A^*}{2A})^{-1} (A/2)^{-|X|_j} + 2^d \left(\frac{A^*}{A^{2^d}}\right)^{-1} A^{-|X|_j}. \tag{6.24}$$

$$\leq 2 \cdot 2^d \left(\frac{A^*}{A^{2^d}}\right)^{-1} (A/2)^{-|X|_j}.$$

□

Proposition 6.4. *Let $B'_{K'}$ be a ball centred on the origin in $\tilde{\mathcal{N}}_{j+1}^{\mathcal{P}_{j+1}}$. There exist $A(d, L, B'_{K'})$ and $A^*(d, A)$ such that for $A > A(d, L, B'_{K'})$ and $A^* > A^*(d, A)$, the function K' defined by* (5.16) *is a smooth map from $B_{I,\tilde{I},J,K}$ to the ball $B'_{K'}$.*

Proof. The main part is to prove that the range of K' is contained in $B'_{K'}$ and we only do that part. Derivatives are easy to bound by the same ideas because, for each U, $K'(U)$ is a polynomial in its arguments $(I, \tilde{I}, J, K)$. Let $U \in \mathcal{P}_{j+1,c}$. We take the norm of (5.16) using the same ideas shown in detail in the proof of Lemma 6.2. Thus, by the next scale bound (6.8) followed by the triangle inequality Lemma 6.1, the $\mathbb{E}_{j+1}$ bound, (6.4), norm factorisation (6.1),(6.2) together with strict disjointness of components of $\mathcal{X}, \check{X}$,

$$\|K'(U)\|' \leq \sum_{(\mathcal{X}, \check{X}) \in \mathcal{G}(U)} \|J\|^{\mathcal{X}} \|\tilde{I}\|_{\mathrm{str}}^{X_I} 2^{|\check{X}|_j} \prod_{\check{X}_c \in \mathcal{C}(\check{X})} \|\check{K}(\check{X}_c)\|. \tag{6.25}$$

Let $\alpha = A/2$ and $\alpha^* = A^*/(2^{d+1} A^{2^d})$. By Lemma 6.3,

$$\|\check{K}(\check{X}_c)\| \leq \alpha^{*-1} \alpha^{-|\check{X}_c|_j}. \tag{6.26}$$

By definition of $B_{I,\tilde{I},J,K}$, $\|J(B,X)\| \leq A^{*-1} \leq \alpha^{*-1}$ and $\|\tilde{I}\|_{\text{str}} \leq 2$. By definition of $\mathcal{G}(U)$, $X_I \cup \check{X} \subset U$ so the factors of 2 are bounded by $2^{|U|_j}$. Therefore,

$$\|K'(U)\|' \leq 2^{|U|_j} \underbrace{\sum_{(\mathcal{X},\check{X})\in\mathcal{G}(U)} \alpha^{*-|\mathcal{X}|-|\mathcal{C}(\check{X})|}\alpha^{-|\check{X}|_j}}_{=k(0,\alpha,\alpha^*,U),\text{ cf. Lemma 6.17}}. \tag{6.27}$$

Therefore,

$$\begin{aligned}
\|K'\|' &\overset{(6.10)}{=} \sup_{U\in\mathcal{P}_{j+1,c}} \|K'(U)\|' A^{|U|_{j+1}} \\
&\overset{(6.27)}{\leq} \sup_{U\in\mathcal{P}_{j+1,c}} 2^{|U|_j}(2\alpha)^{|U|_{j+1}} k(0,\alpha,\alpha^*,U) \\
&= \sup_{U\in\mathcal{P}_{j+1,c}} 2^{(L^d+1)|U|_{j+1}}\alpha^{|U|_{j+1}} k(0,\alpha,\alpha^*,U).
\end{aligned} \tag{6.28}$$

For any $c > 1$, $\lim_{\alpha\to\infty} 2^{(L^d+1)|U|_{j+1}}\alpha^{(1-c)|U|_{j+1}} = 0$, so by Lemma 6.17,

$$\lim_{A\to\infty}\lim_{A^*\to\infty} \|K'\|' = \lim_{\alpha\to\infty}\lim_{\alpha^*\to\infty} \|K'\|' \overset{(6.28)}{=} 0. \tag{6.29}$$

Therefore $K' \subset B'_{K'}$ if A and A^* are large as specified in the Proposition. □

The final result, which is a Corollary of Proposition 6.4 is out of logical order because we have to completely specify the norms in order to be able to check that the choices of J and $\tilde{I}$ made in Section 5.2 are smooth in I, K and that $\tilde{I} \to 1$ and $J \to 0$ as $I \to 1$ and $K \to 0$. These norms will be specified in the remaining sections, but we have omitted the verifications.

Corollary 6.5. *With J and $\tilde{I}$ chosen as in Section 5.2 and norms as given below K' is a smooth function of I, K in a ball whose radius is independent of scale j.*

Since I is given as a (smooth) function of the coupling constants by (4.6) this shows that (RG) is a smooth map on $(\vec{\lambda}, K)$ in a ball centred on the origin of radius independent of j. To accomplish this we have had to choose A large depending on d, L. To complete the program set out in Section 4.1.1 we have to prove that $\mathcal{L}$ is contractive and that will require choosing L large, depending on d. Thus the order of choice is L large depending on d, A large depending on d, L, the radius of the ball for $(\vec{\lambda}, K)$ small depending on d, L, A.

6.2. Norms

We have left until last step 4 of Section 4.1.1, which is to prove that $\mathcal{L}$ is contractive. Recall that we have axiomatised the properties of the norms in the criteria (6.1), (6.2), (6.4), (6.8). These criteria express compatibility with the product structure of the renormalisation group map together with integrability on all scales. The difficulty, which is once again the large field problem, is to satisfy these criteria with norms that permit growth in $\nabla\varphi$, recalling from the introduction to Section 5 that there is no hope to prove that $\mathcal{L}$ is contractive unless the norm permits growth in the field.

6.2.1. The $\|\cdot\|_{T_\varphi}$ Seminorm

We introduce a seminorm which (a) controls the Taylor remainder and (b) has a good product property. The product property originates in the fact that the Taylor expansion of a product is the product of the Taylor expansions.

Suppose $F = F(\varphi)$ is a $\mathcal{C}^3$ function defined on a Banach space Φ. The Taylor expansion to third order about φ in direction $\dot\varphi$ is

$$\sum_{p=0}^{3} \frac{1}{p!} D^p F_\varphi(\underbrace{\dot\varphi, \dots, \dot\varphi}_{p \text{ factors}}). \tag{6.30}$$

We measure the size of the third order Taylor expansion at φ by seminorms,

$$\|F\|_{T^p_\varphi(\Phi)} = \sup_{\dot\varphi \in B_{\Phi^p}} |D^p F_\varphi(\dot\varphi)|, \qquad \|F\|_{T_\varphi(\Phi)} = \sum_{p=0}^{3} \frac{1}{p!} \|F\|_{T^p_\varphi(\Phi)}, \tag{6.31}$$

where B_{Φ^p} is the unit ball in

$$\Phi^p = \underbrace{\Phi \times \cdots \times \Phi}_{p \text{ factors}}, \qquad \|(\dot\varphi_1, \dots, \dot\varphi_p)\|_{\Phi^p} = \max \|\dot\varphi_i\|_\Phi. \tag{6.32}$$

An easy consequence of the product rule of differentiation is

$$\|F_1 F_2\|_{T_\varphi(\Phi)} \le \|F_1\|_{T_\varphi(\Phi)} \|F_2\|_{T_\varphi(\Phi)}. \tag{6.33}$$

The seminorm of 1 is 1.

Lemma 6.6 (Monotonicity). *If $\Phi_2 \hookrightarrow \Phi_1$ is a bounded injection with norm γ then, for $p \in \{0,1,2,3\}$,*

$$\|F\|_{T^p_\varphi(\Phi_2)} \le \gamma^p \|F\|_{T^p_\varphi(\Phi_1)} \le p!\gamma^p \|F\|_{T_\varphi(\Phi_1)}. \tag{6.34}$$

Proof. The first inequality follows immediately from the definition and $\dot\varphi \in B_{\Phi_2} \Rightarrow \|\dot\varphi\|_{\Phi_1} \le \gamma$. The second follows from the definition of the seminorms. □

We assume that fields defined on $X \subset \Lambda$ belong to a Banach space $\Phi(X)$ and that the family of Banach spaces $(\Phi(X),\ X \subset \Lambda)$ satisfies

$$X \subset Y \Rightarrow \Phi(Y) \hookrightarrow \Phi(X) \text{ is a contraction.} \tag{6.35}$$

where the inclusion $\hookrightarrow$ is induced by restriction: a field defined on a region Y defines a field on $X \subset Y$ by restriction to the smaller domain X.

Lemma 6.7. *If* (6.35) *holds and $F \in \mathcal{N}_j(X)$ then*

$$\|F\|_{T_\varphi(\Phi(Y))} \le \|F\|_{T_\varphi(\Phi(X))}, \quad Y \supset X \tag{6.36}$$

and, for any, not necessarily disjoint, $X, Y \in \mathcal{P}_j$, if $F \in \mathcal{N}_j^{\mathcal{P}_j}$,

$$\|F(X)F(Y)\|_{T_\varphi(\Phi(X^* \cup Y^*))} \le \|F(X)\|_{T_\varphi(\Phi(X^*))} \|F(Y)\|_{T_\varphi(\Phi(Y^*))}. \tag{6.37}$$

Proof. The first part is a corollary of Lemma 6.6. For the second part, in (6.33), replace Φ by $\Phi(X^* \cup Y^*)$ and use the first part with the set inclusions $X, Y \hookrightarrow X \cup Y$. □

We shorten the notation by writing

$$\|F\|_{T_\varphi} = \|F\|_{T_\varphi(\Phi(X))}, \quad F \in \mathcal{N}_j(X). \tag{6.38}$$

Lemma 6.8. *For $F \in \mathcal{N}_j(X)$ let* $\operatorname{Tay} F = \operatorname{Tay} F(\varphi)$ *be the second order Taylor expansion about* 0*,*

$$\operatorname{Tay} F(\varphi) = \sum_{p=0}^{2} \frac{1}{p!} D^p F_0(\varphi, \dots, \varphi). \tag{6.39}$$

Then

$$\|F - \operatorname{Tay} F\|_{T_\varphi} \leq \left(1 + \|\varphi\|_{\Phi(X)}\right)^3 \sup_{t \in (0,1)} \|F\|_{T^3_{t\varphi}}. \tag{6.40}$$

Proof. We separately estimate $\|F - \operatorname{Tay} F\|_{T^p_\varphi}$ for $p = 0, 1, 2, 3$. For example, for $p = 1$, we use the Cauchy form of the Taylor remainder theorem to write

$$D\big(F - \operatorname{Tay} F\big)_\varphi(\dot\varphi) = \int \frac{(1-t)^1}{1!} \frac{d^2}{dt^2} DF_{t\varphi}(\dot\varphi)\, dt \tag{6.41}$$

where the integral is over the unit interval and where $\dot\varphi$ is in the unit ball. The third derivative of $\operatorname{Tay} F$ is not in the right hand side because it is zero. This equals

$$\int \frac{(1-t)^1}{1!} D^3 F_{t\varphi}(\dot\varphi, \varphi, \varphi)\, dt. \tag{6.42}$$

This is bounded in absolute value by

$$\sup_{t \in (0,1)} \|F\|_{T^3_{t\varphi}} \|\varphi\|^2_{\Phi(X)} \int \frac{(1-t)^1}{1!}\, dt. \tag{6.43}$$

In the same way we can bound the other derivatives and the result follows from the binomial theorem,

$$\sum_p \frac{1}{p!} \frac{(1-t)^{3-p}}{(3-p)!} \|\varphi\|^{(3-p)}_{\Phi(X)} \leq \left(1 + \|\varphi\|_{\Phi(X)}\right)^3. \tag{6.44}$$

□

6.2.2. The Weak and Strong Norms

Recall that objects in $\mathcal{N}_j$ are functions of $\varphi = \varphi_j$. We write $\varphi' = \varphi_{j+1}$ and $\zeta = \zeta_{j+1}$. Since $\varphi = \varphi' + \zeta$, objects in $\mathcal{N}_j$ are a special case of objects in $\tilde{\mathcal{N}}_j$ which are functions of two fields ζ, φ'. If $F \in \tilde{\mathcal{N}}_j$, then we define

$$\|F\|_{T_{\zeta,\varphi'}} = \|F_\zeta\|_{T_{\varphi'}} \tag{6.45}$$

where F_ζ is the function of φ' obtained by holding ζ fixed in F. Thus F is a function of (ζ, φ'), but the seminorm is applied to F as a function only of φ', pointwise in ζ. For future reference notice that when $F \in \mathcal{N}_j$, in other words, is a function of $\varphi = \varphi' + \zeta$, then

$$\|F\|_{T_{\zeta,\varphi'}} = \|F\|_{T_\varphi}. \tag{6.46}$$

When we want to shorten the notation we suppress the (ζ, φ') by writing

$$\|F\|_T = \|F\|_{T_{\zeta,\varphi'}}. \tag{6.47}$$

Lemma 6.9.

$$\|\mathbb{E}_{j+1} F\|_{T_{\varphi'}} \leq \mathbb{E}_{j+1} \|F\|_{T_{\zeta,\varphi'}}. \tag{6.48}$$

Proof. This is easily checked. The derivatives with respect to φ' commute with $\mathbb{E}_{j+1}$, which is an integration over ζ. □

Growth in φ is "regulated" by two local functions $G_{\text{strong}} \in \tilde{\mathcal{N}}_j^{\mathcal{B}_j}$ and $G \in \tilde{\mathcal{N}}_j^{\mathcal{P}_j}$. We assume that $G = G_{\varphi'} G_\zeta$ is a product of a function $G_{\varphi'} \in \mathcal{N}_{j+1}^{\mathcal{P}_j}$ of φ' and a function G_ζ of ζ. All these functions are called *regulators*. Regulators take values in $[1, \infty)$, are normalised to equal one at $\varphi' = \zeta = 0$ and are increasing functions of $|t|$ when fields φ', ζ are replaced by $t\varphi', t\zeta$. Given such regulators, norms on $\tilde{\mathcal{N}}_j(X^*)$ and $\tilde{\mathcal{N}}_j(B^*)$ are defined by

$$\begin{aligned} \|K(X)\| &= \sup_{\zeta, \varphi' \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda} \|K(X)\|_{T_{\zeta,\varphi'}} G(X, \zeta, \varphi')^{-1} \\ \|I(B)\|_{\text{str}} &= \sup_{\zeta, \varphi' \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda} \|I(B)\|_{T_{\zeta,\varphi'}} G_{\text{strong}}(B, \zeta, \varphi')^{-1}, \end{aligned} \tag{6.49}$$

It is easy to check that (6.3) holds. Our proofs and intuition are based on the equivalent statement that the norms are best constants in the estimates

$$\begin{aligned} \|K(X)\|_{T_{\zeta,\varphi'}} &\le \underbrace{\text{const.}}_{\|K(X)\|} G(X, \zeta, \varphi') \\ \|I(B)\|_{T_{\zeta,\varphi'}} &\le \underbrace{\text{const.}}_{\|I(B)\|_{\text{str}}} G_{\text{strong}}(B, \zeta, \varphi'). \end{aligned} \tag{6.50}$$

At every scale there are norms and regulators and we use a prime to denote the next scale norms, spaces and regulators.

Lemma 6.10. *The product properties* (6.1), (6.2), *the integration property* (6.4) *and the next scale property* (6.8) *are respectively implied by*

$$\prod_{Y \in \mathcal{C}(X)} G(Y) \le G(X) \tag{6.51}$$

$$G^X_{\text{strong}} G(Y) \le G(X \cup Y), \quad X, Y \text{ disjoint} \tag{6.52}$$

$$\mathbb{E}_{j+1} G_\zeta(X) \le 2^{|X|_j}. \tag{6.53}$$

$$\forall\, U \in \mathcal{P}_{j+1} \subset \mathcal{P}_j, \quad \begin{cases} G_{\varphi'}(U) \le G'(U) \\ \Phi'(U) \hookrightarrow \Phi(U) \text{ contractive.} \end{cases} \tag{6.54}$$

Proof. For each claim it suffices to prove a bound of the form (6.50), which we write with (ζ, φ') suppressed.

Proof that (6.51) $\Rightarrow$ (6.1). By the multiplicative property in Lemma 6.7,

$$\begin{aligned} \|K(X)\|_T &\le \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|_T \\ &\overset{(6.50)}{\le} \prod_{Y \in \mathcal{C}(X)} \|K(Y)\| G(Y) \overset{(6.51)}{\le} G(X) \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|. \end{aligned} \tag{6.55}$$

(6.52) $\Rightarrow$ (6.2) is proved in the same way.

Proof that (6.53) $\Rightarrow$ (6.4).

$$\begin{aligned} \|\mathbb{E}_{j+1}F(X)\|_T &\overset{(6.48)}{\leq} \mathbb{E}_{j+1}\|F(X)\|_T \\ &\overset{(6.50)}{\leq} \|F(X)\|\ \mathbb{E}_{j+1}G(X) \\ &\overset{(6.53)}{\leq} \|F(X)\|2^{|X|_j}G_{\varphi'}(X) \\ &\overset{G_\zeta\geq 1}{\leq} \|F(X)\|2^{|X|_j}G(X). \end{aligned} \tag{6.56}$$

Proof that (6.54) $\Rightarrow$ (6.8). We write $\zeta' = \zeta_{j+2}$, $\varphi'' = \varphi_{j+2}$ and $\varphi' = \zeta' + \varphi''$.

$$\begin{aligned} \|K'(U)\|_{T'_{\varphi'',\zeta'}} &\overset{(6.46)}{=} \|K'(U)\|_{T'_{\varphi'}} \overset{(6.34),(6.54)}{\leq} \|K'(U)\|_{T_{\varphi'}} \\ &\overset{(6.50)}{\leq} \|K'(U)\| \inf_\zeta G(U,\varphi',\zeta) \\ &= \|K'(U)\|G_{\varphi'}(U,\varphi') \overset{(6.54)}{\leq} \|K'(U)\|G'(U,\varphi'',\zeta'). \end{aligned} \tag{6.57}$$

We can insert the infimum because $K'(U)$ does not depend on ζ. $\square$

6.2.3. Proof that C is contractive

In this section we reduce the proof that C is contractive to criteria on G, Φ. This is an important part of step 4 of Section 4.1.1. Recall from Section 5.2 that this reduces to proving that $\mathcal{L}$ is contractive. Choices of G, Φ that satisfy these criteria are given after this section.

Proposition 6.11. *Let $\bar\gamma(d,L) = 2^{-2^d-d-1}L^{-d}S^{-1}$ where S was defined in Definition 4.2. If, for $X \in \mathcal{S}_j$,*

$$\left(1 + \|\varphi'\|_{\Phi'(X^*)}\right)^3 G_{\varphi'}(X) \leq qG'(\overline{X}), \tag{6.58}$$

where q is a constant, and if $\Phi' \hookrightarrow \Phi$ is a contraction with norm γ such that

$$3!2^{2^d}\gamma^3 q < \bar\gamma(d,L), \tag{6.59}$$

then $\mathcal{L}$ is contractive,

$$\|\mathcal{L}\| \leq \frac{1}{2}\|K\|. \tag{6.60}$$

Lemma 6.12. *Let $X \in \mathcal{S}_j$ and let $U = \overline{X}$. Define $R_X \in \tilde{\mathcal{N}}_{j+1}(U^*)$ by*

$$R_X = (1 - \mathrm{Tay}\,)\mathbb{E}_{j+1}K(X). \tag{6.61}$$

If

$$\|R_X\|' \leq \bar\gamma(d,L)\|K(X)\|, \tag{6.62}$$

then (6.60) *holds*

PROOF.

$$
\begin{aligned}
\|\mathcal{L}(U)\|' &\le \sum_{X\in\mathcal{S}_j:\overline{X}=U} \|R_X\|' \\
&\overset{(6.62)}{\le} \sum_{X\in\mathcal{S}_j:\overline{X}=U} \bar{\gamma}(d,L)\|K(X)\| \\
&\overset{(6.10)}{\le} \bar{\gamma}(d,L)\|K\| \underbrace{\sum_{X\in\mathcal{S}_j:\overline{X}=U} A^{-|X|_j}}_{k(A,U),\ \text{cf. Lemma 6.19}}.
\end{aligned}
\tag{6.63}
$$

The proof is completed by Lemma 6.19 and the definition of $\bar{\gamma}(d,L)$. □

PROOF OF PROPOSITION 6.11. In the next equations the $\sup_{t\in(0,1)}$ inherited from (6.40) is (incorrectly) omitted because monotonicity of G in t ultimately makes it drop out at the end anyway. We use the notations $\zeta' = \zeta_{j+2}$, $\varphi'' = \varphi_{j+2}$ and $\varphi' = \zeta' + \varphi''$, $W(\varphi') = \big(1 + \|\varphi'\|_{\Phi'(X^*)}\big)^3$,

$$
\begin{aligned}
\|R_X(U)\|_{T'_{\zeta',\varphi''}} &\overset{(6.46)}{=} \|R_X(U)\|_{T'_{\varphi'}}. \\
&\overset{(6.40)}{\le} W(\varphi')\|\mathbb{E}_{j+1}K(X)\|_{T'^{3}_{\varphi'}} \\
&\overset{(6.34)}{\le} 3!\gamma^3 W(\varphi')\|\mathbb{E}_{j+1}K(X)\|_{T_{\varphi'}} \\
&\overset{(6.48),(6.50)}{\le} 3!\gamma^3 W(\varphi')\|K(X)\|\mathbb{E}_{j+1}G(X,\zeta,\varphi') \\
&= 3!\gamma^3 W(\varphi')\|K(X)\|G_{\varphi'}(X,\varphi')\mathbb{E}_{j+1}G_{\zeta}(X,\zeta) \\
&\overset{(6.53),(6.58)}{\le} 3!\gamma^3 q 2^{|X|_j}\|K(X)\|G'(\overline{X},\zeta',\varphi'').
\end{aligned}
\tag{6.64}
$$

which proves (6.62) because $X \in \mathcal{S}_j$ so $|X|_j \le 2^d$ and we use the hypothesis (6.59). □

6.2.4. Choice of Φ

In this section we prove that $\mathcal{L}$ is contractive by verifying the hypothesis (6.59) when L is sufficiently large depending on d. The underlying reason why this works is given in the discussion in Section 4.3; namely, the standard deviation of $\nabla\varphi$ scales down as $L^{-dj/2}$ as the scale j increases because the finite range decomposition has decreasing covariances. This consideration is built into the choice of norm on Φ which is such that typical fields have norm of order h^{-1}. The change in the variance of the typical field then becomes the statement that $\Phi' \hookrightarrow \Phi$ is contractive with norm $\gamma \le L^{-d/2}$ and so we obtain (6.59) for L large. Notice that this implies also that the second half of (6.54) is satisfied.

Definition 6.13. $\Phi(X)$ is the vector space $\mathbb{R}^X/\{\|\varphi\|_{\Phi(X)} = 0\}$ with norm

$$
\begin{aligned}
&\|\varphi\|_{\Phi(X)} = \max_{p=1,2} \|\nabla_j^p \varphi\|_{L_j^\infty(X)}, \\
&\|\nabla_j^p\varphi\|_{L_j^\infty(X)} = h_j^{-1} \max_{x\in X, \hat{e}\in\mathcal{E}(\pm)^p} |\nabla^p_{j,\hat{e}}\varphi(x)|, \\
&\nabla_j = L^j\nabla, \quad h_j = L^{-[\varphi]j}h.
\end{aligned}
\tag{6.65}
$$

$\mathcal{E}(\pm)$ was defined in (1.54).

It is clear that (6.35) holds. The inclusion $\Phi' \hookrightarrow \Phi$ is a contraction with norm $\gamma \leq L^{-d/2}$ because the factors of L^{-1} are gained when ∇_j is compared with ∇_{j+1} and h_j with h_{j+1}. Therefore, for L large enough depending on dimension d, we have (6.59).

6.2.5. Choice of Regulator

In this section we choose the regulators and in the next section we prove that our choices have the properties displayed in Lemma 6.10 and Proposition 6.11. The choice of regulators is based on work in [**15, 19**], but changes are necessary for the present lattice context.

The strong regulator is defined on blocks $B \in \mathcal{B}_j$ by

$$G_{\text{strong}}(B) = e^{\|\varphi'\|^2_{\Phi(B^*)} + \|\zeta\|^2_{\Phi(B^*)}} \tag{6.66}$$

and it extends to $X \in \mathcal{P}_j$ by $G_{\text{strong}}(X) = G^X_{\text{strong}}$.

For $X \subset \Lambda$ let

$$\begin{aligned} \|\nabla_j \varphi\|^2_{L^2_j(X)} &= h_j^{-2} L^{-dj} \sum_{x \in X} (\nabla_j \varphi)^2(x), \\ \|\nabla_j \varphi\|^2_{L^2_j(\partial X)} &= h_j^{-2} L^{-(d-1)j} \sum_{(x,\hat{e}) \in \partial X} |\nabla_{j,\hat{e}} \varphi(x)|^2 \end{aligned} \tag{6.67}$$

where

$$\partial X = \{(x, \hat{e}) \in X \times \mathcal{E}(\pm) : x + \hat{e} \notin X\}. \tag{6.68}$$

The weak regulator is

$$\begin{aligned} &G(X) = G_{\varphi'}(X) G_\zeta(X), \quad X \in \mathcal{P}_j, \\ &G_{\varphi'}(X) = \exp \Big[\\ &\qquad \sum_{B \in \mathcal{B}_j(X)} \left(c_1 \|\nabla_j \varphi'\|^2_{L^2_j(B)} + c_2 \|\nabla_j^2 \varphi'\|^2_{L^\infty_j(B^*)} \right) \\ &\qquad + c_3 \|\nabla_j \varphi'\|^2_{L^2_j(\partial X)} \Big] \\ &G_\zeta(X) = G_\zeta^X, \quad G_\zeta(B) = e^{c_4 \max_{p=0,1,2} \|\nabla_j^p \zeta\|^2_{L^\infty_j(B^*)}}. \end{aligned} \tag{6.69}$$

Theorem 6.14. *There exist $c_1 = c_2, c_3, c_4, L(d), h(d, L)$ such that for $L > L(d)$ and $h > h(d, L)$ the regulators defined in* (6.69) *have the properties, (6.51, 6.52, 6.53,6.54,6.58).*

PROOF. See Appendix 6.5. □

This choice of regulators fixes the norm (6.49), (6.9) in the smoothness hypothesis (6.17). The hypothesis (4.1) on our initial interaction F implies the estimate $\|(F-1)\|_{\text{str}} < A^{*-1}$ for the scale $j = 0$ norm. For K_0 given by (4.8) we verify (I_0, K_0) is in the domain of (RG) for λ_0 in a small ball centred on the origin in $\mathbb{R}^2$ of radius $O(A^* h^2)^{-1}$ so that we can get the renormalisation group started!

6.3. Open Problems

(1) Reduce the complexity of renormalisation group proofs by exploiting convexity. Cf. [**48**] for a start on this program.
(2) Scaling limits for more general dipole systems where we no longer make all the symmetry assumptions below (4.1).
(3) Which covariances admit finite range decompositions? Can any positive-definite function be decomposed into a non-zero finite range positive-definite function and a positive-definite remainder? Cf. [**14**].
(4) Observables and boundary conditions. See the introduction to Section 3.3. For previous attempts see [**19**] and [**11**].
(5) RG analysis of the critical point for the Kosterlitz Thouless phase transition and decay of correlations at the critical point. See [**20**] before reading the earlier papers by Dimock and Hurd for the correction of an error.
(6) Improve the current best result, [**60**], on screening in the 2D Coulomb gas which is valid only for small β and special boundary conditions. Screening is conjectured to hold up to the critical point and ought not to depend on boundary conditions.
(7) In two dimensional models there is recent progress by Fermionic RG in [**54, 34**]. Is there also a "Bosonic" RG approach to these models? In vague terms the idea is that in some sense two dimensional models can also be written in terms of the fields $e^{iq\varphi}$ where $field$ is the massless Gaussian.

6.4. Appendix. Geometry and Counting Lemmas

Despite being in an appendix these results are built around a general principle that can have wide application in renormalisation group proofs and they are quite essential for this one. The main idea is in Lemma 6.15, which says that when you measure the size of a set by counting the number of blocks in it, then the set becomes smaller when you close it, unless it is a *small set.* As a consequence,

$$A^{-|X|_j} \ll A^{-|\overline{X}|_{j+1}}, \tag{6.70}$$

so bounds by $A^{-|X|_j}$ become stronger on passing to the next scale. The natural reaction on first encountering polymer representations is to be nervous about the part of K that lives on large complicated connected sets, but the remark we just made shows that only the part that lives on small sets is dangerous. For those readers who look at Wilson's papers, this is where our proof encodes Wilson's observation that only local terms in the effective action play a major role.

Another key principle is to reduce estimates on sums over configurations of sets to counting the number of subsets of X by $2^{|X|}$. To do this we "tilt" the sums by exponential weights as in Lemma 6.17.

The first lemma is stated above [**19**, Lemma 2]. A consequence of this lemma is that $\overline{X} \in \mathcal{P}_{j+1}$ is small whenever $X \in \mathcal{P}_j$ is small.

Lemma 6.15. *There is an $\eta = \eta(d) > 1$ such that for all $L \geq L_0(d) = 2^d + 1$ and for all large connected sets $X \in \mathcal{P}_j$,*

$$|X|_j \geq \eta|\overline{X}|_{j+1}. \tag{6.71}$$

In addition, (6.71) holds with $\eta = 1$ for all $X \in \mathcal{P}_j$ (not necessarily connected, and possibly small).

Proof. Fix $L \geq L_0(d) = 2^d + 1$ (this restriction enters only in the third paragraph of the proof). It is clear that for any $m \geq 1$ the closure of any set of m j-blocks contains at most m $(j+1)$-blocks, and hence (6.71) always holds with $\eta = 1$.

Assume that X is a large connected set. Let $\Delta = \Delta(d)$ denote the maximum possible number of blocks that touch a connected set of $2^d + 1$ blocks. Then $\Delta \leq (2^d+1)3^d$. We will prove (6.71) by induction on $|\overline{X}|_{j+1}$, with $\eta = 1+1/(2^d+1+2^d\Delta)$.

To begin the induction, we claim that if $|\overline{X}|_{j+1} = 2^d + 1$ then $|X|_j \geq 2^d + 2$, and hence

$$\frac{|X|_j}{|\overline{X}|_{j+1}} \geq \frac{2^d+2}{2^d+1} = 1 + \frac{1}{2^d+1} \geq \eta. \tag{6.72}$$

To prove the claim, we proceed as follows. The maximum possible value of $|\overline{X}|_{j+1}$ is $|X|_j$, so we only need to rule out the case $|X|_j = |\overline{X}|_{j+1} = 2^d + 1$, which we now assume. Let $D(X)$ be the integer part of $L^{-j} \max_{x,y\in X} \|x-y\|_\infty$; this is a measure of the diameter of X counted in number of j-blocks. Then $D(X) \leq 2^d + 1 \leq L$. Also, every j-block in X lies in a different $(j+1)$-block in $\overline{X}$. However, any set of $2^d + 1$ $(j+1)$-blocks contains a pair of blocks B_1, B_2 that do not touch. Therefore $D(b_1 \cup b_2) > L$ for every pair of j-blocks $b_1 \in B_1$ and $b_2 \in B_2$, so that $b_1 \cup b_2 \subset X$ is not possible. This contradiction proves the claim.

To advance the induction, suppose that (6.71) holds when $2^d+1 \leq |\overline{X}|_{j+1} \leq n$, and suppose now that $|\overline{X}|_{j+1} = n+1$. We remove from $\overline{X}$ a connected subset of $2^d + 1$ blocks. The complement of this connected subset consists of no more than Δ connected components.[2] We list these components as $\overline{X}_1, \ldots, \overline{X}_\Delta$, and choose $k \in \{0, 1, \ldots, \Delta\}$ such that $|\overline{X}_i|_{j+1} \geq 2^d + 1$ for $i \leq k$ and $|\overline{X}_i|_{j+1} \leq 2^d$ for $i > k$ (some of the latter components may be empty). Let $M = \sum_{i=1}^k |\overline{X}_i|_{j+1}$ and $m = \sum_{i=k+1}^\Delta |\overline{X}_i|_{j+1}$. By the induction hypothesis applied to $\overline{X}_i$ for $i \leq k$, and by (6.71) with $\eta = 1$ for $i > k$,

$$\begin{aligned}
\frac{|X|_j}{|\overline{X}|_{j+1}} &\geq \frac{2^d+2+\eta M + m}{2^d+1+M+m} \\
&= 1 + \frac{1+(\eta-1)M}{2^d+1+M+m} \\
&\geq 1 + \frac{1+(\eta-1)M}{2^d+1+M+\Delta 2^d} \\
&= 1 + \frac{1+(\eta-1)M}{\frac{1}{\eta-1}+M} \\
&= \eta,
\end{aligned} \tag{6.73}$$

where we used our specific choice for the value of η in the penultimate step (note that the last equality is true no matter what the value of M). This advances the induction and completes the proof. □

---

[2] If there were more, then one of these components is not adjacent to the removed subset nor to any of the at most Δ components adjacent to the removed subset, and hence X would be disconnected.

Lemma 6.16. *Let $X \in \mathcal{P}_j$ and let n be the number of components of X. Then*

$$|X|_j \geq \frac{1}{2}(1+\eta)|\overline{X}|_{j+1} - \frac{1}{2}(1+\eta)2^{d+1}n. \tag{6.74}$$

PROOF. We write

$$X = W \cup Y \tag{6.75}$$

where W is the union of the small components of X and Y is the union of the large components of X. Since either (i) $|\overline{Y}|_{j+1} \geq \frac{1}{2}|\overline{X}|_{j+1}$, or (ii) $|\overline{W}|_{j+1} \geq \frac{1}{2}|\overline{X}|_{j+1}$, it suffices to prove (6.74) for each of these two cases.

Case (i). $|\overline{Y}|_{j+1} \geq \frac{1}{2}|\overline{X}|_{j+1}$. By Lemma 6.15, in this case (6.74) follows from

$$\begin{aligned} |X|_j &= |W|_j + |Y|_j \\ &\geq |\overline{W}|_{j+1} + \eta|\overline{Y}|_{j+1} \\ &\geq |\overline{X}|_{j+1} + (\eta - 1)|\overline{Y}|_{j+1}, \\ &\geq \frac{1}{2}(1+\eta)|\overline{X}|_{j+1}. \end{aligned} \tag{6.76}$$

Case (ii). $|\overline{W}|_{j+1} \geq \frac{1}{2}|\overline{X}|_{j+1}$. In this case (6.74) follows from

$$\frac{1}{2}|\overline{X}|_{j+1} \leq |\overline{W}|_{j+1} \leq |W|_j \leq 2^d n. \tag{6.77}$$

$\square$

Lemma 6.17. *For $U \in \mathcal{P}_{j+1}$ and $p \in \mathbb{N}_0$ and $\alpha \geq 1$ define*

$$k = k(p, \alpha, \alpha^*, U) = \sum_{(\mathcal{X}, X) \in \mathcal{G}(U)} \alpha^{*-|\mathcal{X}| - |\mathcal{C}(X)|} \alpha^{-|X|_j} \big(|\mathcal{X}| + |\mathcal{C}(X)|\big)^p. \tag{6.78}$$

There exists $c = c(d) > 1$ such that,

$$\lim_{\alpha \to \infty} \lim_{\alpha^* \to \infty} \sup_{U \in \mathcal{P}_{j+1}} \alpha^{c|U|_{j+1}} k = 0. \tag{6.79}$$

PROOF. Let $a = \frac{1}{2}(1+\eta)$, where η is the geometric constant in Lemma 6.15. Recalling that $\mathcal{X}$ is a set of pairs (B_i, X_i),

$$|U|_{j+1} \overset{(5.13)}{\leq} 2^d|\mathcal{X}| + |\overline{X}|_{j+1}. \tag{6.80}$$

Therefore,

$$\begin{aligned} a|U|_{j+1} &\leq a2^d|\mathcal{X}| + a|\overline{X}|_{j+1} \\ &\overset{(6.74)}{\leq} a2^d|\mathcal{X}| + |X|_j + a2^{d+1}|\mathcal{C}(X)|. \end{aligned} \tag{6.81}$$

Therefore, with $n = |\mathcal{X}| + |\mathcal{C}(X)|$,

$$\alpha^{a|U|_{j+1}} k \leq \sum_{(\mathcal{X}, X) \in \mathcal{G}(U)} \alpha^{*-n} \alpha^{a2^{d+1}n} n^p. \tag{6.82}$$

We now evaluate part of the sum over $\mathcal{G}(U)$. Referring to the definition of $\mathcal{G}(U)$ near (5.13) let $X_B = \bigcup\{B : (B, X) \in \mathcal{X}\}$. For each block $B \in X_B$ there are S pairs (B, X). Cf. Def. 4.2(d). Therefore the number of sets $\mathcal{X}$ with X_B is fixed is $S^{|\mathcal{X}|} \leq S^n$. Therefore,

$$\alpha^{a|U|_{j+1}} k \leq \sum_{(X_B, X)} S^n \alpha^{*-n} \alpha^{a2^{d+1}n} n^p \tag{6.83}$$

where (X_B, X) is summed over the set of disjoint polymers $X_B \in \mathcal{P}_j(U)$ and $X \in \mathcal{P}_j(U)$. There are $3^{|U|_j}$ ways to split U into three disjoint subsets X_B, X and their complement in U so

$$\alpha^{a|U|_{j+1}} k \leq 3^{|U|_j} \big(\sup_{n \in \mathbb{N}} S^n \alpha^{*-n} \alpha^{a2^{d+1}n} n^p \big). \tag{6.84}$$

Let $c \in (1, a)$. Then

$$\sup_{U \in \mathcal{P}_{j+1}} \alpha^{c|U|_{j+1}} k \leq \sup_{U \in \mathcal{P}_{j+1}} \big(\alpha^{-(a-c)|U|_{j+1}} 3^{L^d |U|_{j+1}} \big) \sup_{n \in \mathbb{N}} \big(S^n \alpha^{*-n} \alpha^{a2^{d+1}n} n^p \big). \tag{6.85}$$

For α sufficiently large (depending on L) the first supremum is finite and then the limit as $\alpha^* \to \infty$ drives the expression to zero.

□

Lemma 6.18. *If $U \in \mathcal{P}_{j+1}$ and*

$$k(\alpha, U) = \sum_{X \in \mathcal{P}_{j,c}} \mathbb{I}_{\overline{X}=U, X \notin \mathcal{S}_j} \alpha^{-|X|_j} \tag{6.86}$$

then, for any $\lambda \in (0, 1]$, $\alpha \geq 1$,

$$\lim_{\alpha \to \infty} \sup_{U \in \mathcal{P}_{j+1}} k(\lambda\alpha, U) \alpha^{|U|_{j+1}} = 0. \tag{6.87}$$

Proof. By Lemma 6.15 and estimating the number of sets $X \in \mathcal{P}_{j,c}$ by $2^{|U|_j}$,

$$\begin{aligned} &\alpha^{|U|_{j+1}} \sum_{X \in \mathcal{P}_{j,c}} \mathbb{I}_{\overline{X}=U, X \notin \mathcal{S}_j} (\lambda\alpha)^{-|X|_j} \\ &\quad \leq \sup_{U \in \mathcal{P}_{j+1}, U \neq \emptyset} 2^{L^d |U|_{j+1}} \lambda^{-\eta |U|_{j+1}} \alpha^{-(\eta-1)|U|_{j+1}} \to 0. \end{aligned} \tag{6.88}$$

□

Lemma 6.19. *If $U \in \mathcal{P}_{j+1}$ and*

$$k(\alpha, U) = \sum_{X \in \mathcal{S}_j} \mathbb{I}_{\overline{X}=U} \alpha^{-|X|_j} \tag{6.89}$$

then, for any $\lambda \in (0, 1]$, $\alpha \geq 1$,

$$\sup_{U \in \mathcal{P}_{j+1}} k(\lambda\alpha, U) \alpha^{|U|_{j+1}} \leq \lambda^{-2^d} (2L)^d S \tag{6.90}$$

where S was defined in Definition 4.2, part (d).

Proof. By Lemma 6.15, $|U|_{j+1} \leq |X|_j$ and by definition of $\mathcal{S}_j$ $\lambda^{-|X|_j} \leq \lambda^{-2^d}$ so

$$\begin{aligned} k(\lambda\alpha, U) \alpha^{|U|_{j+1}} &\leq \lambda^{-2^d} \sum_{X \in \mathcal{S}_j} \mathbb{I}_{\overline{X}=U} \\ &\leq \lambda^{-2^d} \sum_{X \in \mathcal{S}_j} \sum_{B \in \mathcal{B}_j} \mathbb{I}_{B \in \mathcal{B}(X)} \frac{1}{|X|_j} \mathbb{I}_{\overline{X}=U} \\ &\leq \lambda^{-2^d} \sum_{B \in \mathcal{B}_j(U)} \sum_{X \in \mathcal{S}_j} \mathbb{I}_{B \in \mathcal{B}(X)} \\ &\leq \lambda^{-2^d} |U|_j S \leq \lambda^{-2^d} (2L)^d S. \end{aligned} \tag{6.91}$$

□

6.5. Appendix. Proof of Theorem 6.14

In this section we prove Theorem 6.14. This completes the program set out in Section 4.1.1. The purpose of the regulators is to put a bound on the allowed growth of local functions as $\nabla\varphi$ becomes large so that we will have integrability at every scale. Dipole systems are extremely delicate in this respect. Roughly speaking, we want $G_j = G_j(X, \varphi_{j+1}, \zeta_{j+1})$ with the supermartingale property $\mathbb{E}_{j+1} G_j \le G_{j+1}$ and we also want $G_j(X)$ to have the factorised form G_j^X. A good starting point is a Gaussian candidate such as

$$G_j(X, \varphi_{j+1}, \zeta_{j+1}) = \exp\left(c\|\varphi_{j+1}\|^2_{L^2_j(X)} + c\|\zeta_{j+1}\|^2_{L^2_j(X)}\right). \tag{6.92}$$

Then $\mathbb{E}_{j+1} G_j$ gives something like $\exp\left(\|\varphi_{j+1}\|^2_{L^2_j(X)}\right)$. Now we write $\varphi_{j+1} = \varphi_{j+2} + \zeta_{j+2}$ and use $(\varphi_{j+2} + \zeta_{j+2})^2 \le 2\varphi^2_{j+2} + 2\zeta^2_{j+2}$, to get a bound by G_{j+1}, but unfortunately c has become $2c$. This increase in c will happen at every scale, which is a disaster. The solution is the clever integration by parts in Lemma 6.26, which has been exploited in previous works [**30, 15, 19**]. Because this integration by parts generates boundary terms we are forced to complicate G_j by adding boundary terms and this gets in the way of a G_j^X structure. Welcome to the Jungle!

Proof of (6.51). Referring to Def. 4.2, ∂X is the disjoint union over $Y \in \mathcal{C}(X)$ of ∂Y so every term in the exponent of G splits into a sum of terms corresponding to the decomposition of X into connected components $Y \in \mathcal{C}(X)$ and (6.51) is actually an equality. □

The proof of (6.52) will be harder because $G_{\text{strong}}(X)G(Y)$ contains a factor

$$G_{\varphi'}(\partial Y) = \exp\left[c_3\|\nabla_j\varphi'\|^2_{L^2_j(\partial Y)}\right] \tag{6.93}$$

whereas $G(X\cup Y)$ contains $G_{\varphi'}(\partial(X\cup Y))$ and it is not obvious how to compare them since ∂Y need not be a subset of $\partial(X\cup Y)$. The next three Lemmas prepare a solution for this problem.

Let $X \in \mathcal{P}_j$. Define the distance $\text{dist}_X(x,y)$ between two points $x, y \in X$ to be the length of a shortest path of nearest neighbour points in X that joins x to y. If there is no such path then $\text{dist}_X(x,y) = \infty$. Define

$$\text{diam}_j(X) = L^{-j}\max_{x,y\in X}\text{dist}_X(x,y). \tag{6.94}$$

Lemma 6.20. *Let $Y \subset X$ be sets in $\mathcal{P}_j$ and let $Z \in \mathcal{P}_j(X\setminus Y)$. Then,*

$$\begin{aligned} |Z|_j^{-1}\|f\|^2_{L^2_j(Z)} &\le \|f\|^2_{L^\infty_j(X)} \\ &\le \left(2/|Y|_j\right)\|f\|^2_{L^2_j(Y)} + 2\,\text{diam}^2_j(X)\,\|\nabla_j f\|^2_{L^\infty_j(X)}, \end{aligned} \tag{6.95}$$

$$\|\nabla_j f\|^2_{L^2_j(\partial B)} \le c\|\nabla_j f\|^2_{L^2_j(B)} + c\|\nabla^2_j f\|^2_{L^\infty_j(B)}, \quad c = c(d), \quad B \in \mathcal{B}_j.$$

Proof. For $x \in X$ and $y \in Y \subset X$, summing the finite difference derivatives along a shortest path joining x to y easily leads to the estimate,

$$|f(x) - f(y)| \le \text{diam}_j(X)\max_{z\in X}|\nabla_j f(z)|. \tag{6.96}$$

By the triangle inequality this implies

$$|f(x)| \leq |f(y)| + \operatorname{diam}_j(X) \max_{z \in X} |\nabla_j f(z)|. \tag{6.97}$$

By $(a+b)^2 \leq 2a^2 + 2b^2$, followed by averaging both sides over $y \in Y \subset X$,

$$|f(x)|^2 \leq 2|Y|^{-1} \sum_{y \in Y} |f(y)|^2 + 2 \operatorname{diam}_j^2(X) \max_{z \in X} |\nabla_j f(z)|^2. \tag{6.98}$$

We replace f by $h_j^{-1} f$. To obtain the L_j^∞ estimate we take the maximum over $x \in X$, noting that $|Y| = L^{dj}|Y|_j$. The $L_j^2(Z)$ estimate is immediate. To obtain the $L_j(\partial B)$ estimate we replace X by B and f by $\nabla_{j,\hat{e}} f$ and average $(x, \hat{e})$ over ∂B. The constant $c = c(d)$ arises because the number of edges in the boundary of B is a geometric constant times $L^{j(d-1)}$. $\square$

Lemma 6.21. *Let $c_4 \geq 2$ and*

$$\begin{aligned} &G_{\mathrm{strong},\varphi'}(B) = e^{\|\varphi'\|^2_{\Phi(B^*)}}, \qquad G_{\mathrm{strong},\zeta}(B) = e^{\|\zeta\|^2_{\Phi(B^*)}}, \\ &E(B) = \exp\big(c_1 \|\nabla_j \varphi'\|^2_{L_j^2(B)} + c_2 \|\nabla_j^2 \varphi'\|^2_{L_j^\infty(B^*)}\big). \end{aligned} \tag{6.99}$$

There exists a constant c_1 such that for $B \in \mathcal{B}_j$ and $X \in \mathcal{P}_j$,

$$G^2_{\mathrm{strong},\varphi'}(B) \leq E(B), \qquad G_{\mathrm{strong}}(X)^2 \leq G(X). \tag{6.100}$$

Proof. It suffices to prove the first inequality. Noting that $E(X) \leq G_{\varphi'}(X)$, the second inequality is obtained by multiplying the first one on the left by $G^2_{\mathrm{strong},\zeta}(B)$ and on the right by $G_\zeta(B)$, which is larger by the hypothesis on c_4. Then take the product over $b \in \mathcal{B}_j(X)$.

In Lemma 6.20 we choose $Y = B$ and $X = B^*$ and $f = \nabla_j^p \varphi'$ and we obtain, with $p = 1$,

$$\|\nabla_j^p \varphi'\|^2_{L_j^\infty(B^*)} \leq 2\|\nabla_j \varphi'\|^2_{L_j^2(B)} + 2 \operatorname{diam}_j^2(B^*) \|\nabla_j^2 \varphi'\|^2_{L_j^\infty(B^*)}. \tag{6.101}$$

The same inequality trivially holds when $p = 2$ and since $2 \leq 2 \operatorname{diam}_j^2(B^*)$,

$$\|\varphi'\|^2_{\Phi(B^*)} \leq 2 \operatorname{diam}_j^2(B^*) \Big(\|\nabla_j \varphi'\|^2_{L_j^2(B)} + \|\nabla_j^2 \varphi'\|^2_{L_j^\infty(B^*)} \Big). \tag{6.102}$$

Therefore, the Lemma holds with $c_1 = 4 \operatorname{diam}_j^2(B^*)$. $\square$

We assume, from now on, that c_1 is fixed according to Lemma 6.21 and that $c_4 \geq 2$.

Lemma 6.22. *There exists $\bar{c}_3$ such that for all $c_3 \leq \bar{c}_3$, $c_2 \geq c_1$ and all disjoint $X, Y \in \mathcal{P}_j$,*

$$G_{\mathrm{strong},\varphi'}(X) G_{\varphi'}(Y) \leq G_{\varphi'}(X \cup Y). \tag{6.103}$$

Proof. The part of ∂Y not in $\partial(X \cup Y)$ is the boundary of Y which is also boundary of X and, in particular, is boundary to blocks $B \in \mathcal{B}_j(X)$. Therefore

$$\|\nabla_j \varphi'\|^2_{L_j^2(\partial Y)} \leq \|\nabla_j \varphi'\|^2_{L_j^2(\partial(X \cup Y))} + \sum_{B \in \mathcal{B}_j(X)} \|\varphi'\|^2_{L_j^2(\partial B)}. \tag{6.104}$$

By estimating the last term by Lemma 6.20 we find that there exists $\bar{c}_3$ such that for $c_3 \leq \bar{c}_3$,

$$\begin{aligned} c_3\|\nabla_j\varphi'\|^2_{L^2_j(\partial Y)} &\leq c_3\|\nabla_j\varphi'\|^2_{L^2_j(\partial(X\cup Y))} \\ &\quad + (1/2)\sum_{B\in\mathcal{B}_j(X)}\Big(c_1\|\nabla_j\varphi'\|^2_{L^2_j(B)} + c_2\|\nabla^2_j\varphi'\|^2_{L^\infty_j(B^*)}\Big). \end{aligned} \tag{6.105}$$

Then, with $E(X)$ as defined in Lemma 6.21, we have

$$G_{\varphi'}(\partial Y) \leq G_{\varphi'}(\partial(X\cup Y))E^{1/2}(X). \tag{6.106}$$

By Lemma 6.21,

$$G_{\mathrm{strong},\varphi'}(X) \leq E^{(1/2)X}. \tag{6.107}$$

Then

$$\begin{aligned} G_{\mathrm{strong},\varphi'}(X)G_{\varphi'}(Y) &\overset{(6.93),(6.99)}{=} G_{\mathrm{strong},\varphi'}(X)G_{\varphi'}(\partial Y)E^Y \\ &\overset{(6.106)}{\leq} G_{\mathrm{strong},\varphi'}(X)G_{\varphi'}(\partial(X\cup Y))E^{1/2}(X)E^Y \\ &\overset{(6.107)}{\leq} G_{\varphi'}(\partial(X\cup Y))E^XE^Y \\ &= G_{\varphi'}(\partial(X\cup Y))E^{X\cup Y} = G_{\varphi'}(X\cup Y). \end{aligned} \tag{6.108}$$

□

From now on we fix $c_2 = c_1$, $c_3 = 2\,\bar{c}_3/3$ and continue to assume $c_4 \geq 2$.

Proof of (6.52). In the conclusion of Lemma 6.22 multiply the left hand side by $G_{\mathrm{strong},\zeta}(X)G_\zeta(Y)$ and multiply the right hand side by the larger $G_\zeta(X\cup Y)$ to obtain (6.52). □

The next Lemmas prepare for the proof of (6.54). We use the notations

$$\begin{aligned} &\varphi'' = \varphi_{j+2}, \qquad \zeta' = \zeta_{j+2}, \\ &W_2(X,f) = \sum_{B\in\mathcal{B}_j(X)}\|\nabla^2_jf\|^2_{L^\infty_j(B^*)}, \\ &W(X,f) = \sum_{B\in\mathcal{B}_j(X)}\max_{p=0,1,2}\|\nabla^p_jf\|^2_{L^\infty_j(B^*)}, \end{aligned} \tag{6.109}$$

where $X\in\mathcal{P}_j$ and we use a prime to denote the same construction on the next scale; for example,

$$W'_2(X,f) = \sum_{B\in\mathcal{B}_{j+1}(X)}\|\nabla^2_{j+1}f\|^2_{L^\infty_{j+1}(B^*)}.$$

Lemma 6.23. *For $X\in\mathcal{P}_j$,*

$$\begin{aligned} &\|\nabla_jf\|^2_{L^2_j(\partial X)} \leq 2d\,W(X,f), \\ &\|\nabla^p_jf\|^2_{L^2_j(X)} \leq (2d)^pW(X,f), \qquad p=0,1,2. \end{aligned} \tag{6.110}$$

PROOF. Since ∂X is contained in $\cup\{\partial B : B \in \mathcal{B}_{j+1}(X)\}$ and since there are $2d\, L^{j(d-1)}$ pairs $(x, \hat{e})$ in ∂B,

$$\begin{aligned} \|\nabla_j f\|^2_{L^2_j(\partial X)} &\overset{(6.67)}{\leq} \sum_{B \in \mathcal{B}_j(X)} L^{-j(d-1)} \sum_{(x,\hat{e}) \in \partial B} \|\nabla_j f\|^2_{L^\infty_j(B^*)} \\ &\leq \sum_{B \in \mathcal{B}_j(X)} 2d\, \|\nabla_j f\|^2_{L^\infty_j(B^*)} \leq 2d\, W(X, f). \end{aligned} \tag{6.111}$$

A similar proof gives the other inequality. There is $(2d)^p$ because we included a sum over directions of differentiation in $\nabla^p_j f$. □

Lemma 6.24. *For $X \in \mathcal{P}_{j+1}$,*

$$\|\nabla_j \varphi'\|^2_{L^2_j(\partial X)} \leq 2L^{-1}\big(\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial X)} + 2d\, W'(X, \zeta')\big). \tag{6.112}$$

PROOF. By the triangle inequality followed by $(a+b)^2 \leq 2a^2 + 2b^2$ followed by

$$\begin{aligned} \|\nabla_j f\|^2_{L^2_j(\partial X)} &\overset{(6.67)}{=} L^{d-1}(h_{j+1}/h_j)^2 L^{-2} \|\nabla_{j+1} f\|^2_{L^2_{j+1}(\partial X)} \\ &= L^{-1}\|\nabla^p_{j+1} f\|^2_{L^2_{j+1}(X)}, \end{aligned} \tag{6.113}$$

we have

$$\|\nabla_j \varphi'\|^2_{L^2_j(\partial X)} \leq 2L^{-1}\big(\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial X)} + \|\nabla_{j+1}\zeta'\|^2_{L^2_{j+1}(\partial X)}\big). \tag{6.114}$$

We estimate the second term using Lemma 6.23. □

Lemma 6.25. *For $X \in \mathcal{P}_{j+1}$,*

$$W_2(X, \varphi') \leq 2L^{-2}\Big(W_2'(X, \varphi'') + W'(X, \zeta')\Big). \tag{6.115}$$

PROOF. It suffices to prove the case $X = B' \in \mathcal{B}_{j+1}$. We use the triangle inequality followed by $(a+b)^2 \leq 2a^2 + 2b^2$ followed by

$$\begin{aligned} W_2(B', f) &= \sum_{B \in \mathcal{B}_j(B')} \|\nabla^2_j f\|^2_{L^\infty_j(B^*)}, \\ &\leq \sum_{B \in \mathcal{B}_{j+1}(B')} L^d L^{-4}(h_{j+1}/h_j)^2 \|\nabla^2_{j+1} f\|^2_{L^\infty_{j+1}(B'^*)}, \\ &= L^d L^{-4}(h_{j+1}/h_j)^2 \|\nabla^2_{j+1} f\|^2_{L^\infty_{j+1}(B'^*)} \leq L^{-2} W'(B', f). \end{aligned} \tag{6.116}$$

□

The contractive factors $2L^{-1}, 2L^{-2}$ displayed by Lemmas 6.24 and 6.25 will make $W_2(B, \varphi')$ easy to handle in the coming proof of (6.54). The marginal term appearing in the next Lemma is the dangerous part of G because it has no such contraction.

Lemma 6.26. *For $X \in \mathcal{P}_{j+1}$, $\alpha > 0$,*

$$\begin{aligned} \|\nabla_j \varphi'\|^2_{L^2_j(X)} \leq{} & \|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(X)} + \alpha \|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial X)} \\ & + \alpha W_2'(X, \varphi'') + O(\alpha^{-1}) W'(X, \zeta'), \end{aligned} \tag{6.117}$$

where $O(\alpha^{-1})$ denotes a d-dependent constant times α^{-1}.

Proof. By multiplying out the squared L^2 norm,

$$\begin{aligned}
\|\nabla_j\varphi'\|^2_{L^2_j(X)} &\overset{(6.67)}{=} \|\nabla_{j+1}\varphi'\|^2_{L^2_{j+1}(X)} \\
&= \|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(X)} + \|\nabla_{j+1}\zeta'\|^2_{L^2_{j+1}(X)} \\
&\quad + 2h_{j+1}^{-2}L^{-d(j+1)}\sum_{x\in X}\nabla_{j+1}\varphi''(x)\nabla_{j+1}\zeta'(x).
\end{aligned} \tag{6.118}$$

The key idea is to use summation by parts to rewrite the crossterm as

$$2h_{j+1}^{-2}L^{-d(j+1)}\sum_{x\in X}\nabla^2_{j+1}\varphi''\zeta' + \text{ bt}, \tag{6.119}$$

where bt is the boundary term and ∇^2_{j+1} contains a sum over directions of differentiation. Using $2ab \le a^2 + b^2$ this is less than

$$\alpha\|\nabla^2_{j+1}\varphi''\|^2_{L^2_{j+1}(X)} + \alpha^{-1}\|\zeta'\|^2_{L^2_{j+1}(X)} + \text{ bt}. \tag{6.120}$$

Recalling that we sum over forward and backward derivatives the boundary term is

$$\begin{aligned}
&2h_{j+1}^{-2}L^{-d(j+1)}L^{j+1}\sum_{(x,\hat{e})\in\partial X}\nabla_{j+1,\hat{e}}\varphi''(x)\big(\zeta'(x)+\zeta'(x+\hat{e})\big) \\
&\le \alpha\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial X)} + \alpha^{-1}\|\zeta'\|^2_{L^2_{j+1}(\partial X)}.
\end{aligned} \tag{6.121}$$

$\|\zeta'\|^2_{L^2_{j+1}(\partial X)}$ is defined in terms of $\zeta'(x)^2 + \zeta'(x+\hat{e})^2$. Therefore $\|\nabla_j\varphi'\|^2_{L^2_j(X)} - \|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(X)}$ is bounded by

$$\begin{aligned}
&\|\nabla_{j+1}\zeta'\|^2_{L^2_{j+1}(X)} + \alpha\|\nabla^2_{j+1}\varphi''\|^2_{L^2_{j+1}(X)} + \alpha^{-1}\|\zeta'\|^2_{L^2_{j+1}(X)} \\
&\quad + \alpha\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial X)} + \alpha^{-1}\|\zeta'\|^2_{L^2_{j+1}(\partial X)}
\end{aligned} \tag{6.122}$$

and we obtain (6.117) by Lemma 6.23. □

Lemma 6.27. *Let*

$$E(\kappa,U) = c_1\|\nabla_j\varphi'\|^2_{L^2_j(U)} + \kappa_1 c_2 W_2(U,\varphi') + \kappa_2 c_3\|\nabla_j\varphi'\|^2_{L^2_j(\partial U)} \tag{6.123}$$

There exists $\bar{c}_4$ and $L(d,\kappa)$ such that for $c_4 \ge \bar{c}_4$, $\kappa = (\kappa_1,\kappa_2)$ and $L > L(d,\kappa)$,

$$e^{E(\kappa,U)} \le G'(U). \tag{6.124}$$

Proof. By Lemmas 6.26, 6.24 and 6.25,

$$\begin{aligned}
E(\kappa,U) \le\; & c_1\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(U)} + c_1\alpha\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial U)} \\
&+ c_1\alpha W_2'(U,\varphi'') + c_1 O(\alpha^{-1})W'(U,\zeta') \\
&+ \kappa_1 c_2 2L^{-2}\Big(W_2'(U,\varphi'') + W'(U,\zeta')\Big) \\
&+ \kappa_2 c_3 2L^{-1}\big(\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(\partial U)} + 2d\,W'(U,\zeta')\big).
\end{aligned} \tag{6.125}$$

We fix α such that $c_1\alpha = c_3/2 \wedge c_2$ and we choose $\bar{c}_4 > c_1 O(\alpha^{-1})$. Then the limit of this expression as $L \to \infty$ is less than the exponent of $G'(U)$, which is

$$c_1\|\nabla_{j+1}\varphi''\|^2_{L^2_{j+1}(U)} + c_2 W_2'(U,\varphi'') + c_3\|\nabla_{j+1}\varphi'\|^2_{L^2_{j+1}(\partial U)} + c_4 W'(U,\zeta'). \tag{6.126}$$

Then (6.124) holds if L sufficiently large, depending on κ. □

From now on we fix $c_4 = \max(2,\bar{c}_4)$.

Proof of (6.54). From (6.69) $G_{\varphi'} = \exp(E(1,U))$ so it is a special case of Lemma 6.27. $\square$

Proof of (6.58). For $\alpha > 0$ there exists $q(\alpha)$ such that for $s \geq 0$, $(1+s)^3 \leq q(\alpha)e^{\alpha s^2}$. Therefore, there exists q such that

$$\begin{aligned}(1 + \|\varphi'\|_{\Phi'(X^*)})^3 &\leq q e^{(2/3)\|\varphi''\|^2_{\Phi'(X^*)} + (2/3)\|\zeta'\|^2_{\Phi'(X^*)}} \\ &\leq q {G'_{\text{strong}}}^{2/3}(\overline{X}) \overset{(6.100)}{\leq} q {G'}^{1/3}(\overline{X}).\end{aligned} \tag{6.127}$$

Therefore (6.58) is implied by

$$G_{\varphi'}(X) \leq {G'}^{2/3}(\overline{X}). \tag{6.128}$$

To prove this, recall that X is a small set. For all L sufficiently large ($L \geq 2^{d+1}+1$), there exists a translate TX of X such that TX and X do not touch, but $(TX)^*$ and X^* intersect, and TX is a subset of $\overline{X}$. By Lemma 6.20, there exists a constant $c = c(d)$ such that

$$\begin{aligned}(3/2)\|\nabla_j \varphi'\|^2_{L^2_j(X)} &= \|\nabla_j \varphi'\|^2_{L^2_j(X)} + (1/2)\|\nabla_j \varphi'\|^2_{L^2_j(X)} \\ &\leq \|\nabla_j \varphi'\|^2_{L^2_j(X)} + \|\nabla_j \varphi'\|^2_{L^2_j(TX)} + c\, W_2(X \cup TX, \varphi').\end{aligned} \tag{6.129}$$

Therefore, with $E(\kappa, X)$ as defined in (6.123),

$$\begin{aligned}G^{3/2}_{\varphi'}(X) &= e^{\left((3/2)c_1\|\nabla_j\varphi'\|^2_{L^2_j(X)} + (3/2)c_2 W_2(X,\varphi') + (3/2)c_3\|\nabla_j\varphi'\|^2_{L^2_j(\partial X)}\right)} \\ &\overset{(6.129)}{\leq} e^{E(\kappa, X\cup TX)} \overset{(6.103)}{\leq} e^{E(\kappa,\overline{X})} \overset{(6.124)}{\leq} G'(\overline{X}),\end{aligned} \tag{6.130}$$

where $\kappa_2 = 3/2$ and $\kappa_1 c_2$ is $[(3/2)c_2 + (1/2)c_1 c]$. We are permitted to use (6.103) with $G_{\varphi'}$ replaced by $E(\kappa, X)$ because we chose c_3 so that $\kappa_2 c_3 \leq \bar{c}_3$, and $\kappa_1 c_2 \geq c_1$. $\square$

The following Lemmas 6.28, 6.29, 6.30 prepare for the proof of the integration property (6.53). This integration property is where the bound (2.4) comes into play and proves that our choice of Φ is such that the typical fluctuation $\zeta = \zeta_{j+1}$ has a norm $O(h^{-1})$.

Lemma 6.28. *Let $(\xi_\alpha,\ \alpha \in \mathcal{A})$ be a finite set of Gaussian random variables with covariance $C(\alpha,\beta)$. Assume that the largest eigenvalue of C is less than $1/2$. Then*

$$\mathbb{E} e^{\frac{1}{2}\sum \xi_\alpha^2} \leq e^{\sum C(\alpha,\alpha)}. \tag{6.131}$$

Proof. Let $(\xi,\xi) = \sum \xi_\alpha^2$, let $\kappa \in (0,1)$, let A be the inverse of the matrix C. Then the eigenvalues of A are at least 2 by the hypothesis on C so $A - \kappa$ is invertible. Let $C_\kappa = (A-\kappa)^{-1}$. Then

$$\begin{aligned}\frac{d}{d\kappa}\ln \mathbb{E} e^{\frac{\kappa}{2}(\xi,\xi)} &= \frac{1}{2}\mathbb{E}\Big(e^{\frac{\kappa}{2}(\xi,\xi)}(\xi,\xi)\Big)\Big/\Big(\mathbb{E}e^{\frac{\kappa}{2}(\xi,\xi)}\Big) \\ &= \frac{1}{2}\sum C_\kappa(\alpha,\alpha).\end{aligned} \tag{6.132}$$

We obtained the second equality by rewriting the ratio of expectations in terms of the second moment for a Gaussian integral with $Q(\xi,\xi) = \big(\xi, (A-\kappa)\xi\big)$. The last line is the trace of C_κ which is the sum over the eigenvalues of C_κ and each eigenvalue of

C_κ equals $(\lambda^{-1}-\kappa)^{-1}$ where λ is an eigenvalue of C. Also $(\lambda^{-1}-\kappa)^{-1} = \lambda(1-\kappa\lambda)^{-1}$ and $\kappa\lambda \leq 1/2$ so that

$$\begin{aligned} \frac{d}{d\kappa} \ln \mathbb{E} e^{\frac{\kappa}{2}(\xi,\xi)} &= \frac{1}{2} \sum_\lambda \frac{\lambda}{1-\kappa\lambda} \\ &\leq \sum_\lambda \lambda = \operatorname{Trace} C = \sum C(\alpha,\alpha). \end{aligned} \tag{6.133}$$

The conclusion of the Lemma is obtained by integrating this inequality over $\kappa \in (0,1)$. □

Lemma 6.29. *If $f \in \mathbb{R}_+^{\mathcal{P}_j}$ has the additive property $f(X) = \sum_{B \in \mathcal{B}_j(X)} f(B)$ then*

$$\sum_{B \in \mathcal{B}_j(X)} f(B^*) = \sum_{C \in \mathcal{B}_j} |X \cap C^*|_j f(C) \leq k f(X^*), \tag{6.134}$$

where $k = |B^*|_j$ is the volume of the small set neighbourhood of a $B \in \mathcal{B}_j$. In particular, k is a constant that depends on dimension d.

PROOF.

$$\begin{aligned} \sum_{B \in \mathcal{B}_j(X)} f(B^*) &= \sum_{B \in \mathcal{B}_j(X)} \sum_{C \in \mathcal{B}_j(B^*)} f(C) \\ &= \sum_{B,C \in \mathcal{B}_j} \mathbb{I}_{B \in \mathcal{B}_j(X)} \mathbb{I}_{C \in \mathcal{B}_j(B^*)} f(C) \\ &= \sum_{B,C \in \mathcal{B}_j} \mathbb{I}_{B \in \mathcal{B}_j(X)} \mathbb{I}_{B \in \mathcal{B}_j(C^*)} f(C) \\ &= \sum_{C \in \mathcal{B}_j} \underbrace{|X \cap C^*|_j}_{\leq k \mathbb{I}_{C \in \mathcal{B}_j(X^*)}} f(C) \leq k f(X^*). \end{aligned} \tag{6.135}$$

□

Lemma 6.30. *[Lattice Sobolev estimate] Let $x \in C$ where $C \subset \mathbb{Z}^d$ is a cube of side ℓ. Then*

$$|f(x)|^2 \leq \text{const.} \sum_{p=0}^{d} \ell^{-d+2p} \sum_{y \in C} |\nabla^p f(y)|^2. \tag{6.136}$$

PROOF. [**10**, Appendix B]. □

Lemma 6.31. *Let $G_\zeta(X)$ be defined as in* (6.69).

$$\mathbb{E}_{j+1} G_\zeta(X) \leq e^{O(h^{-2})|X|_j}, \tag{6.137}$$

when $h \to \infty$ with L fixed.

PROOF. By Lemma 6.30 with $C = B^*$ it is enough to prove, for $a > 0$, $h \to \infty$ with a, L fixed,

$$\mathbb{E}_{j+1} e^{a \sum_{B \in \mathcal{B}_j(X)} \sum_{p=0}^{d+2} \|\nabla_j^p \zeta\|^2_{L^2_j(B^*)}} \leq e^{O(ah^{-2})|X|_j}. \tag{6.138}$$

By the Holder inequality this is implied by

$$\mathbb{E}_{j+1} e^{a \sum_{B \in \mathcal{B}_j(X)} \|\eta\|^2_{L^2_j(B^*)}} \leq e^{O(ah^{-2})|X|_j}, \tag{6.139}$$

with $\eta(x) = \nabla_j^p \zeta(x)$, where ∇_j^p is any fixed sequence of p forward or backward derivatives and $p \in \{0, 1, \ldots, d+2\}$. By Lemma 6.29 this is implied by

$$\mathbb{E}_{j+1} e^{a\|\eta\|^2_{L^2_j(X^*)}} \leq e^{O(ah^{-2})|X|_j}. \tag{6.140}$$

By Lemma 6.28; with $\xi(x) = \sqrt{2a} h_j^{-1} L^{-dj/2} \eta(x)$,

$$\begin{aligned} \mathbb{E}_{j+1} e^{a\|\eta\|^2_{L^2_j(X^*)}} &= \mathbb{E}_{j+1} e^{\frac{1}{2}\sum_{x \in X^*} \xi^2(x)} \\ &\leq e^{2aL^{-dj} h_j^{-2} \sum_{x \in X^*} (-\nabla_j^{2p} C_{j+1})(x-x)} \\ &= e^{2a|X^*|_j h_j^{-2} (-\nabla_j^{2p} C_{j+1})(0)} \end{aligned} \tag{6.141}$$

where $C_{j+1}(x-y)$ is the covariance of η. By (2.4)

$$h_j^{-2} |\nabla_j^{2p} C_{j+1}(0)| \leq O(h^{-2}) \tag{6.142}$$

and $|X^*|_j \leq k|X|_j$ so the proof is complete, except for checking the hypothesis in Lemma 6.28 on the largest eigenvalue of the covariance of ξ. The largest eigenvalue is smaller than the norm of the covariance as a convolution operator and by Young's inequality this is less than the l^1 norm which is less than the maximum times the volume of the range. The maximum is assumed when $x = y$ because a positive definite function assumes its maximum at the origin. Therefore the maximum is

$$\underbrace{O(L^{d(j+1)})}_{\text{vol. range}} \underbrace{a h_j^{-2} L^{-dj} (-\nabla_j^{2p} C_{j+1})(0)}_{\text{maximum}} \overset{(2.4)}{\leq} c(L) O(ah^{-2}) \tag{6.143}$$

which is less than $1/2$ if h is sufficiently large. $\square$

Proof of (6.53). This follows immediately from Lemma 6.31 by choosing h large depending on L. $\square$

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