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III-9: The renormalization group

1 Introduction

The renormalization group is one of those brilliant ideas that lets you get something for nothing through clever reorganization of things you already know. It is hard to underestimate the importance of the renormalization group in shaping the way we think about quantum field theory. The phrase *renormalization group* refers to an invariance of observables under changes in the way things are calculated. There are two versions of the renormalization group used in quantum field theory:

1. **Wilsonian renormalization group**. In a finite theory with a UV cutoff Λ, physics at energies $E \ll \Lambda$ is independent of the precise value of Λ. Changing Λ changes the couplings in the theory so that observables remain the same.

2. **Continuum renormalization group**: Observables are independent of the renormalization conditions, in particular, of the scales $\{p_0\}$ at which we choose to define our renormalized quantities. This invariance holds after the theory is renormalized and the cutoff is removed ($\Lambda = \infty$, $d = 4$). In dimensional regularization with $\overline{\text{MS}}$, the scales $\{p_0\}$ are replaced by μ, and the continuum renormalization group comes from μ-independence.

The two versions are closely related, but technically different. Much confusion arises from conflating them, for example trying to take Λ all the way down to physical low energy scales in the Wilsonian case or taking $\mu \to \infty$ in the continuum case. Although the renormalization group equations have essentially the same form in the two versions, the two methods really are conceptually different and we will try to keep them separate as much as possible, concentrating on the continuum method, which is more practical for actual QFT calculations.

In both cases, the fact that the theory is independent of something means one can set up a differential equations like $\frac{d}{d\Lambda} X = 0$, $\frac{d}{dp_0} X = 0$ or $\frac{d}{d\mu} X = 0$ where X is some observable. Solving these differential equations leads to a trajectory in the space of theories. The term **renormalization group** (RG) or renormalization group evolution refers to the flow along these trajectories. In practice, there are 3 types of things whose renormalization group evolution we often look at: coupling constants (like the electric charge) operators (like the current $J_\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x)$) and Green's functions.

The Wilsonian renormalization group has its origins in condensed matter physics. Suppose you have a solid with atoms at evenly-spaced lattice sites. Many physical quantities, such as resistivity, are independent of the precise inter-atomic spacing. In other words, the lattice spacing Λ^{-1} is a UV cutoff which should drop out of calculations of properties of the long-distance physics. It is therefore reasonable to think about coarse-graining the lattice. This means that, instead of taking as input to your calculation the spin degrees of freedom for an atom on a site, one should be able to use the average spin over a group of nearby sites and get the same answer, with an appropriately rescaled value of the spin-spin interaction strength. Thus there should be a transformation by which nearby degrees of freedom are replaced by a single effective degree of freedom and parameters of the theory are changed accordingly. This is known as a block-spin renormalization group, and was first introduced by Leo Kadanoff in 1966. In the continuum limit, this replacement becomes a differential equation known as the RG equation, which was first understood by Ken Wilson in the early 1970s.

The Wilsonian RG can be implemented through the path integral, an approach clarified by Joe Polchinski in the mid 1980s. There, one can literally integrate out all the short-distance degrees of freedom of a field, say at energies $E > \Lambda$, making the path integral a function of the cutoff Λ. Changing Λ to Λ' and demanding physics be the same (since Λ is arbitrary) means that the couplings in the theory, such as the gauge coupling g, must depend on Λ. Taking Λ' close to Λ induces a differential equation on the couplings, also known as the RG equation. This induces a flow in the coupling constants of the theory as a function of the effective cutoff. Note, the renormalization group is not a group in the traditional mathematical sense, only in the sense that it maps $\mathcal{G} \to \mathcal{G}$ where $\mathcal{G}$ is the set of couplings in a theory.

Implementing the Wilsonian RG picture in field theory, either through a lattice or through the path integral, is a mess from a practical point of view. For actual calculations, at least in high energy contexts, the continuum renormalization group is exclusively used. Then the RG is an invariance to the arbitrary scale one chooses to define the renormalized couplings. In dimensional regularization, this scale is implicitly set by the dimensionful parameter μ. This approach to renormalization was envisioned by Stueckelberg and Petermann in 1953 and made precise the year after by Gell-Mann and Low. It found widespread application to particle physics in the early 1970s through the work of Callan and Symanzik who applied the renormalization group to correlation functions in renormalizable theories. Applications of the enormous power of the continuum renormalization group to precision calculations in non-renormalizable theories, such as the chiral Lagrangian, the four-Fermi theory, heavy-quark effective theory, etc., have been developing since the 1970s, and continue to develop today. We will cover these examples in detail in Part IV.

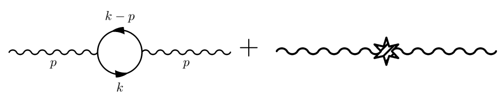
The continuum renormalization group is an extremely practical tool for getting partial results for high-order loops from low-order loops. Recall from Lecture III-2 that the difference between the momentum-space Coulomb potential $\tilde{V}(t)$ at two scales t_1 and t_2 was proportional to $\alpha^2 \ln\frac{t_1}{t_2}$ for $t_1 \ll t_2$. The renormalization group is able to reproduce this logarithm, and similar logarithms of physical quantities. Moreover, the solution to the RG equation is equivalent to summing series of logarithms to all orders in perturbation theory. With these all-orders results, qualitatively important aspects of field theory can be understood quantitatively. Two of the most important examples are the asymptotic behavior of gauge theories, and critical exponents near second-order phase transitions. Many other examples will be discussed in later lectures. We begin our discussion with the continuum renormalization group, since it leads directly to important physical results. The Wilsonian picture is discussed in Section 7.

2 Running couplings

Let's begin by reviewing what we have already shown about scale-dependent coupling constants. The scale dependent electric charge $e_{\text{eff}}(\mu)$ showed up as a natural object in Lecture III-2, where we calculated the vacuum polarization effect, and also in Lecture III-6, where it played a role in the total cross section for $e^+e^- \to \mu^+\mu^-(+\gamma)$. In this section, we review the effective coupling and point out some important features exploited by the renormalization group.

2.1 Large logarithms

In Lectures III–2 and III-5 we calculated the vacuum polarization diagrams at 1-loop and found



$$= -i\,(p^2 g^{\mu\nu} - p^\mu p^\nu)(e_R^2 \Pi_2(p^2) + \delta_3)$$

where δ_3 is the 1-loop counterterm and

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m_R^2 - p^2 x(1-x)}\right)\right] \tag{1}$$

in dimensional regularization, with $d=4-\varepsilon$. Then by embedding this off-shell amplitude into a scattering diagram, we extracted an effective Coulomb potential whose Fourier transform was

$$\tilde{V}(p^2)=e_R^2\frac{1-e_R^2\Pi_2(p^2)}{p^2} \tag{2}$$

Defining the gauge coupling e_R so that $\tilde{V}(p_0^2)=\frac{e_R^2}{p_0^2}$ exactly at the scale p_0 fixes the counterterm δ_3 and lets us write the renormalized potential as

$$\tilde{V}(p^2)=\frac{e_R^2}{p^2}\left\{1+\frac{e_R^2}{2\pi^2}\int_0^1 dx\,x(1-x)\ln\left(\frac{p^2x(1-x)-m^2}{p_0^2x(1-x)-m^2}\right)\right\} \tag{3}$$

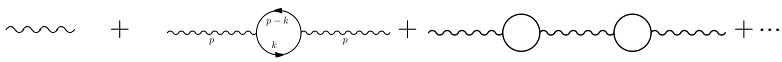
which is finite and ε- and μ-independent.

The entire functional form of this potential is phenomenologically important, especially at low energies, where we saw it gives the Uehling potential and contributes to the Lamb shift. However, when $p\gg m$, the mass drops out and the potential simplifies to

$$\tilde{V}(p^2)=\frac{e_R^2}{p^2}\left(1+\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}\right) \tag{4}$$

In this limit, we can see clearly the problem of **large logarithms**, which the renormalization group will solve. Normally, one would expect that since the correction is proportional to $\frac{e_R^2}{12\pi^2}\sim 10^{-3}$ higher order terms would be proportional to the square, cube, etc. of this term and therefore would be negligible. However, there exist scales $p^2\gg p_0^2$ where $\ln\frac{p^2}{p_0^2}>10^3$ so that this correction is of order 1. When these logarithms are this large, then terms of the form $\left(\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}\right)^2$, which would appear at the next order in perturbation theory, will also be order 1 and so perturbation theory breaks down.

The running coupling was also introduced in Lecture III-2, where we saw that we could sum additional 1PI insertions into the photon propagator



to get

$$\tilde{V}(p^2)=\frac{e_R^2}{p^2}\left[1+\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}+\left(\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}\right)^2+\cdots\right]=\frac{1}{p^2}\left[\frac{e_R^2}{1-\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}}\right] \tag{5}$$

We then defined the effective coupling through the potential by $e_{\text{eff}}^2(p^2)\equiv p^2\tilde{V}(p^2)$. So that

$$e_{\text{eff}}^2(p^2)=\frac{e_R^2}{1-\frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}} \tag{6}$$

This is the effective coupling including the 1-loop 1PI graphs, This is called **leading-logarithmic resummation**.

Once all of these 1PI 1-loop contributions are included, the next terms we are missing should be subleading in some expansion. The terms included in the effective charge are of the form $e_R^2\left(e_R^2\ln\frac{p^2}{p_0^2}\right)^n$ for $n\geqslant 0$. For the 2-loop 1PI contributions to be subleading, they should be of the form $e_R^4\left(e_R^2\ln\frac{p^2}{p_0^2}\right)^n$. However, it is not obvious at this point that there cannot be a contribution of the form $e_R^6\ln^2\frac{p_0^2}{p^2}$ from a 2-loop 1PI graph. To check, we would need to perform the full $\mathcal{O}(e_R^4)$ calculation, including graphs with loops and counterterms. As you might imagine, trying to resum large logarithms beyond the leading-logarithmic level diagrammatically is extremely impractical. The renormalization group provides a shortcut to systematic resummation beyond the leading-logarithmic level.

The key to systematizing the above QED calculation is to first observe that the problem we are trying to solve is one of large logarithms. If there were no large logarithms, we would not need the renormalization group – fixed-order perturbation theory would be fine. For the Coulomb potential, the large logarithms related the physical scale p^2 where the potential was to be measured to an arbitrary scale p_0^2 where the coupling was defined. The **renormalization group equation** (RGE) then comes from requiring that the potential is independent of p_0^2

$$p_0^2 \frac{d}{dp_0^2} \tilde{V}(p^2) = 0 \tag{7}$$

$\tilde{V}(p^2)$ has both explicit p_0^2 dependence, as in Eq. (4), and implicit p_0^2 dependence, through the scale where e_R is defined. In fact, recalling that e_R was defined so that $p_0^2 \tilde{V}(p_0^2) = e_R^2$ exactly, and that the effective charge is defined by $e_{\text{eff}}^2(p^2) \equiv p^2 \tilde{V}(p^2)$, we can make the p_0^2 dependence of $\tilde{V}(p^2)$ explicit by replacing e_R by $e_{\text{eff}}(p_0^2)$.

So, Eq. (4) becomes

$$\tilde{V}(p^2) = \frac{e_{\text{eff}}^2(p_0^2)}{p^2}\left(1 - \frac{e_{\text{eff}}^2(p_0^2)}{12\pi^2}\ln\frac{p_0^2}{p^2}\right) + \cdots \tag{8}$$

Then at 1-loop the RGE is

$$0 = p_0^2 \frac{d}{dp_0^2}\tilde{V}(p^2) = \frac{1}{p^2}\left(p_0^2 \frac{de_{\text{eff}}}{dp_0^2} 2e_{\text{eff}} - \frac{e_{\text{eff}}^4}{12\pi^2} - p_0^2 \frac{de_{\text{eff}}}{dp_0^2}\frac{e_{\text{eff}}^3}{3\pi^2}\ln\frac{p_0^2}{p^2} + \cdots\right) \tag{9}$$

To solve this equation perturbatively, we note that $\frac{de_{\text{eff}}}{dp_0^2}$ must scale like e_{eff}^3 and so the third term inside the brackets is subleading. Thus the 1-loop RGE is

$$p_0^2 \frac{de_{\text{eff}}}{dp_0^2} = \frac{e_{\text{eff}}^3}{24\pi^2} \tag{10}$$

Solving this differential equation with boundary condition $e_{\text{eff}}(p_0) = e_R$ gives

$$e_{\text{eff}}^2(p^2) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\ln\frac{p^2}{p_0^2}} \tag{11}$$

which is the same effective charge that we got above by summing 1PI diagrams.

Note however, that we did not need to talk about the geometric series or 1PI diagrams at all to arrive at equation Eq. (11): we only used the 1-loop graph. In this way, the renormalization group efficiently encodes information about some higher-order Feynman diagrams without having to be explicit about which diagrams are included. This improvement in efficiency is extremely helpful, especially in problems with multiple couplings, or beyond 1-loop.

2.2 Universality of large logarithms

Before getting to the systematics of the renormalization group, let us think about the large logarithms in a little more detail. Large logarithms arise when one scale is much bigger or much smaller than every other relevant scale. In the vacuum polarization calculation, we considered the limit where the off-shellness p^2 of the photon was much larger than the electron mass, m^2. In the limit where one scale is much larger than all the other scales, we can set all the other physical scales to zero to first approximation. If we do this in the vacuum polarization diagram we find from Eq. (1) that full vacuum polarization function $\Pi(p^2) = e_R^2 \Pi_2(p^2) + \delta_3$ at order e_R^2 is

$$\Pi(p^2) = \frac{e_R^2}{12\pi^2}\left[\frac{2}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right) + \text{const}\right] + \delta_3 \quad \text{(DR)} \tag{12}$$

The equivalent result using a regulator with a dimensional UV cutoff, such as Pauli-Villars, is

$$\Pi(p^2) = \frac{e_R^2}{12\pi^2}\left[\ln\left(\frac{\Lambda^2}{-p^2}\right) + \text{const}\right] + \delta_3 \quad (\text{PV}) \tag{13}$$

As was discussed in Lectures III-7 and III-8, the logarithmic, non-analytic dependence on momentum is characteristic of a loop effect and a true quantum prediction. The renormalization group focuses in on these logarithmic terms, which give the dominant quantum effects in certain limits.

If the only physical scale is p^2, the logarithm of p^2 must be compensated by a logarithm of some other *unphysical* scale, in this case, the cutoff Λ^2 (or μ^2 in dimensional regularization). If we renormalize the theory at some scale p_0 by defining $\delta_3 = -\frac{e_R^2}{12\pi^2}\ln\frac{\Lambda^2}{-p_0^2}$ then this becomes

$$\Pi(p^2) = \frac{e_R^2}{12\pi^2}\left[\ln\left(\frac{p_0^2}{p^2}\right) + \text{const}\right] \qquad (\text{PV}) \tag{14}$$

In dimensional regularization, the $\overline{\text{MS}}$ prescription is that $\delta_3 = \frac{e_R^2}{12\pi^2}\left(-\frac{2}{\varepsilon}\right)$ so that

$$\Pi(p^2) = \frac{e_R^2}{12\pi^2}\left[\ln\left(\frac{\mu^2}{p^2}\right) + \text{const}\right] \qquad (\text{DR}) \tag{15}$$

In Eqs. (12)-(15), the logarithmic dependence on the unphysical scales, Λ^2, p_0^2, or μ^2 uniquely determines the logarithmic dependence of the amplitude on the physical scale p^2. The Wilsonian RG extracts physics from the $\ln \Lambda^2$ dependence (see Section 7), while the continuum renormalization group uses p_0^2 or μ^2.

In practical applications of the renormalization group, dimensional regularization is almost exclusively used. It is therefore important to understand the roles of $\varepsilon = 4 - d$, the arbitrary scale μ^2 and scales like p_0^2 where couplings are defined. UV divergences show up as poles of the form $\frac{1}{\varepsilon}$. Don't confuse the scale μ, which was added to make quantities dimensionally correct, with a UV cutoff! Removing the cutoff is taking $\varepsilon \to 0$, *not* $\mu \to \infty$. In minimal subtraction, renormalized amplitudes depend on μ. In observables, such as the difference $p_1^2\tilde{V}(p_1^2) - p_2^2\tilde{V}(p_2^2)$, μ necessarily drops out. However, one can imagine choosing

$$\delta_3 = \frac{e_R^2}{12\pi^2}\left[-\frac{2}{\varepsilon} - \ln\frac{\mu^2}{p_0^2}\right] \tag{16}$$

in dimensional regularization so that Eq. (12) turns into Eq. (14). This is equivalent to *choosing* $\mu^2 = p_0^2$ in Eq. (12) and minimally subtracting the $\frac{1}{\varepsilon}$ term. Thus, people usually think of μ as a physical scale where amplitudes are renormalized and μ is often called the **renormalization scale**.

Although we choose μ to be a physical scale, observables should be independent of μ. At fixed-order in perturbation theory, verifying μ-independence can be a strong theoretical cross check on calculations in dimensional regularization. As we will see by generalizing the vacuum polarization discussion above, the μ-independence of physical amplitudes comes from a cancellation between μ-dependence of loops and μ-dependence of couplings. Since μ is the renormalization point, the effective coupling becomes $e_{\text{eff}}(\mu)$ and the renormalization group equation in Eq. (10) becomes

$$\mu\frac{d e_{\text{eff}}(\mu)}{d\mu} = \frac{e_{\text{eff}}^3(\mu)}{12\pi^2} \tag{17}$$

and we never have to talk about the physical scale p_0 explicitly.

Although μ is a physical, low-energy scale, not taken to ∞, the dependence of amplitudes on μ is closely connected with the dependences on $\frac{1}{\varepsilon}$. For example, in the vacuum polarization calculation, the $\ln \mu^2$ dependence came from the expansion

$$\mu^{\varepsilon}\left(\frac{2}{\varepsilon} - \ln p^2 + \cdots\right) = \frac{2}{\varepsilon} + \ln \frac{\mu^2}{p^2} + \cdots \tag{18}$$

The $\frac{1}{\varepsilon}$ pole and the $\ln \mu^2$ in unrenormalized amplitudes are inseparable – in 4 dimensions, there is no ε and no μ. In particular, the numerical coefficient of $\frac{2}{\varepsilon}$ is the same as the coefficient of $\ln \frac{\mu^2}{p^2}$. Thus, even in dimensional regularization, the large logarithms of the physical scale p^2 are connected to UV divergences as they would be in a theory with a UV regulator Λ. Since the large logarithms correspond to UV divergences, it is possible to resum them entirely from the ε dependence of the counterterms. This leads to the more efficient, but more abstract, derivation of the continuum RGE, as we now show.

3 Renormalization group from counterterms

We have seen how large logarithms of the form $\ln \frac{p_1^2}{p_2^2}$ can be resummed though a differential equation which establishes that physical quantities are independent of the scale p_0^2 where the renormalized coupling is defined. Dealing directly with physical renormalization conditions for general amplitudes is extremely tedious. In this section, we will develop the continuum renormalization group with dimensional regularization exploiting the observations made in the previous section: the large logarithms are associated with UV divergences, which determine the μ dependence of amplitudes; μ^2 can be used as a proxy for the (arbitrary) physical renormalization scale p_0^2; the renormalization group equation will then come from μ-independence of physical quantities.

Let us first recall where the factors of μ come from. Recall our bare Lagrangian for QED:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}^0(i\partial\!\!\!/ - e^0\,\gamma^\mu A_\mu^0 - m^0)\psi^0 \tag{19}$$

The quantities appearing here are infinite, or if we are in $d = 4 - \varepsilon$ dimensions, they are finite but scale like inverse powers of ε. The dimensions of the fields can be read off from the Lagrangian:

$$[A_\mu^0] = \frac{d-2}{2}, \quad [\psi^0] = \frac{d-1}{2}, \quad [m^0] = 1, \quad [e^0] = \frac{4-d}{2} \tag{20}$$

In particular, notice that the bare charge is only dimensionless if $d = 4$. In renormalized perturbation theory, the Lagrangian is expressed instead in terms of physical renormalized fields and renormalized charges. In particular, we would like the charge e_R we expand in to be a number, and the fields to have canonical normalization. We therefore rescale by

$$A_\mu = \frac{1}{\sqrt{Z_3}}A_\mu^0, \quad \psi = \frac{1}{\sqrt{Z_2}}\psi^0, \quad m_R = \frac{1}{Z_m}m^0, \quad e_R = \frac{1}{Z_e}\mu^{\frac{d-4}{2}}e^0 \tag{21}$$

which leads to

$$\mathcal{L} = -\frac{1}{4}Z_3 F_{\mu\nu}^2 + i\,Z_2\bar{\psi}\,\partial\!\!\!/\psi - m_R Z_2 Z_m \bar{\psi}\psi - \mu^{\frac{4-d}{2}} e_R Z_e Z_2\sqrt{Z_3}\,\bar{\psi}\,A\!\!\!/\psi \tag{22}$$

with e_R and the Z_X dimensionless, even in $d = 4 - \varepsilon$ dimensions. (Note that we will be using the charge renormalization Z_e instead of Z_1, which defined in Lecture III-5 as $Z_1 = Z_e Z_2\sqrt{Z_3}$.) Recall also from Lecture III-5 the 1-loop MS counterterms

$$\delta_2 = \frac{e_R^2}{16\pi^2}\left[-\frac{2}{\varepsilon}\right], \quad \delta_3 = \frac{e_R^2}{16\pi^2}\left[-\frac{8}{3\epsilon}\right], \quad \delta_m = \frac{e_R^2}{16\pi^2}\left[-\frac{6}{\varepsilon}\right], \quad \delta_e = \frac{e_R^2}{16\pi^2}\left[\frac{4}{3\varepsilon}\right] \tag{23}$$

where each of these counterterms is defined by $Z_X = 1 + \delta_X$.

Now notice that, since there is μ-dependence in the renormalized Lagrangian but not in the bare Lagrangian, we must have

$$0 = \mu\frac{d}{d\mu}e^0 = \mu\frac{d}{d\mu}\left[\mu^{\frac{\varepsilon}{2}}e_R Z_e\right] = \mu^{\frac{\varepsilon}{2}}e_R Z_e\left[\frac{\varepsilon}{2} + \frac{\mu}{e_R}\frac{d}{d\mu}e_R + \frac{\mu}{Z_e}\frac{dZ_e}{d\mu}\right] \tag{24}$$

At leading order in e_R, $Z_e = 1$ and so

$$\mu\frac{d}{d\mu}e_R = -\frac{\varepsilon}{2}e_R \tag{25}$$

At next order, we have

$$\mu\frac{d}{d\mu}Z_e = \mu\frac{d}{d\mu}\left(1 + \frac{e_R^2}{16\pi^2}\frac{4}{3\varepsilon}\right) = \frac{1}{\varepsilon}\frac{e_R}{6\pi^2}\mu\frac{d}{d\mu}e_R = -\frac{e_R^2}{12\pi^2} \tag{26}$$

where Eq. (25) has been used in the last step. So,

$$\boxed{\beta(e_R) \equiv \mu\frac{d}{d\mu}e_R = -\frac{\varepsilon}{2}e_R + \frac{e_R^3}{12\pi^2}} \tag{27}$$

This is the leading order QED $\boldsymbol{\beta}$**-function**. Taking $\varepsilon \to 0$, this agrees with Eq. (17) when we identify $e_R(\mu) = e_{\text{eff}}(\mu)$, but here we calculated the RGE using only counterterms with no mention of logarithms.

It is worth tracing back to which diagrams contributed to the β-function. The β-function depended on $Z_e = \frac{Z_1}{Z_2\sqrt{Z_3}}$. In Lecture III-5 we found Z_1 from the $\bar{\psi}A_\mu\psi$ vertex, Z_3 came from the vacuum polarization diagrams, and Z_2 from the electron self energy. In QED, since $Z_1 = Z_2$, the β-function can be calculated from Z_3 alone, which is why Eq. (27) agrees with Eq. (17). In other theories, such as quantum chromodynamics, $Z_1 \neq Z_2$ and all three diagrams will contribute. As we will see in Lecture IV-2, we will need to use the full relation $\delta_e = \delta_1 - \delta_2 - \frac{1}{2}\delta_3$. There, and in other examples in this Lecture, it will be clearer why having an abstract way to calculate the running coupling, through the μ-independence of the bare Lagrangian, is better than having to deal with explicit observables like $\tilde{V}(p^2)$.

The β-function is sometimes written as a function of $\alpha = \frac{e_R^2}{4\pi}$ defined by

$$\beta(\alpha) \equiv \mu\frac{d\alpha}{d\mu} \tag{28}$$

The expansion is conventionally written as

$$\beta(\alpha) = -2\alpha\left[\frac{\varepsilon}{2} + \left(\frac{\alpha}{4\pi}\right)\beta_0 + \left(\frac{\alpha}{4\pi}\right)^2\beta_1 + \left(\frac{\alpha}{4\pi}\right)^2\beta_2 + \cdots\right] \tag{29}$$

Matching to Eq. (27) in $d = 4$ then gives $\beta_0 = -\frac{4}{3}$. At leading order (at $\varepsilon = 0$), the solution is

$$\alpha(\mu) = \frac{2\pi}{\beta_0}\frac{1}{\ln\frac{\mu}{\Lambda_{\text{QED}}}} \tag{30}$$

which increases with μ. Here, Λ_{QED} is an integration constant fixed by the boundary condition of the RGE. Using $\alpha(m_e = 511\,\text{keV}) = \frac{1}{137}$ we find $\Lambda_{\text{QED}} = 10^{286}$ eV. Since α blows up when $\mu = \Lambda_{\text{QED}}$, Λ_{QED} is the location of the **Landau pole**.

In writing the solution to the RGE in Eq. (30) we have swapped a dimensionless number, $\frac{1}{137}$, for a dimensionful scale Λ_{QED}. This is known as **dimensional transmutation**. When we introduced the effective charge, we had specified a scale and the value of α measured at that scale. But now we see that only a scale is needed. This uncovers a very profound misconception about nature: electrodynamics is fundamentally not defined by the electric charge, as you learned classically, but by a dimensionful scale Λ_{QED}. Moreover, this scale only has meaning if there is another scale in the theory, such as the electron mass, so really it is the ratio m_e/Λ that specifies QED completely.

While we have the counterterms handy, let us work out the RGE for the electron mass. The bare mass m^0 must be independent of μ, so

$$0 = \mu\frac{d}{d\mu}m^0 = \mu\frac{d}{d\mu}\Big(Z_m m_R\Big) = Z_m m_R\left[\frac{\mu}{m_R}\frac{dm_R}{d\mu} + \frac{\mu}{Z_m}\frac{dZ_m}{d\mu}\right] \tag{31}$$

We conventionally define

$$\gamma_m \equiv \frac{\mu}{m_R}\frac{dm_R}{d\mu} \tag{32}$$

γ_m is called an **anomalous dimension**. (This terminology will be explained in Section 5.4.) Since Z_m only depends on μ through e_R, we have

$$\gamma_m = -\frac{\mu}{Z_m}\frac{dZ_m}{d\mu} = -\frac{1}{Z_m}\frac{dZ_m}{de_R}\mu\frac{de_R}{d\mu} \tag{33}$$

At 1-loop, $Z_m = 1 - \frac{3e_R^2}{8\pi^2\varepsilon}$ and to leading non-vanishing order $\mu\frac{de_R}{d\mu} = \beta(e_R) = -\frac{\varepsilon}{2}e_R$, so

$$\gamma_m = -\frac{1}{1+\delta_m}\left(\frac{2}{e_R}\delta_m\right)\left(-\frac{\varepsilon}{2}e_R\right) = \delta_m\varepsilon = -\frac{3e_R^2}{8\pi^2} \tag{34}$$

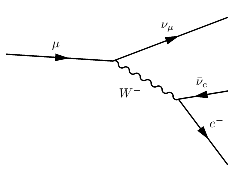
We will give a physical interpretation of a running mass in Section 6.

4 RGE for the 4-Fermi theory

We have seen that the renormalization group equation for the electric charge allows us to resum large logarithms of kinematic scales, for example, in Coulomb scattering. In that case, the logarithms were resummed through the running electric charge. Large logarithms can also appear in pretty much any scattering process, with any Lagrangian, whether renormalizable or not. In fact, non-renormalizable theories, with their infinite number of operators, provide a great arena for understanding the variety of possible renormalization group equations. We will begin with a concrete example: large logarithmic corrections to the muon decay rate from QED. Then we discuss the generalization for renormalizing operators in the Lagrangian and external operators inserted into Green's functions.

The muon decays into an electron and two neutrinos through an intermediate off-shell W-boson. In the standard model, the decay rate comes from the following tree-level diagram, which leads to

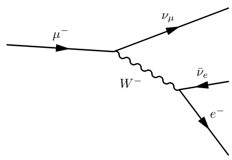
$$\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{1}{2m_\mu}\int d\Pi_{\mathrm{LIPS}}\left|\phantom{\frac{a}{b}}\right.$$



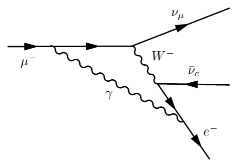
$$\left.\phantom{\frac{a}{b}}\right|^2 = \left(\frac{\sqrt{2}g^2}{8m_W^2}\right)^2\frac{m_\mu^5}{192\pi^3} \tag{35}$$

plus corrections suppressed by additional factors of $\frac{m_\mu}{m_W}$ or $\frac{m_e}{m_\mu}$, with $g = 0.65$ the weak coupling constant and $m_W = 80.4$ GeV (see Lecture IV-4 for more details). A photon loop gives a correction to this decay rate of the form

$$\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = \frac{1}{2m_\mu}\int d\Pi_{\text{LIPS}} \Bigg|$$



$$+$$

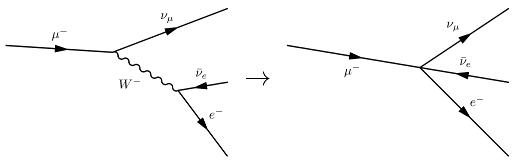


$$\Bigg|^2 \qquad (36)$$

$$= \left(\frac{\sqrt{2}g^2}{8m_W^2}\right)^2 \frac{m_\mu^5}{192\pi^3}\left(1 + A\frac{\alpha}{4\pi}\ln\frac{m_W}{m_\mu} + \cdots\right) \qquad (37)$$

We have only shown the term in this correction which dominates for $m_\mu \ll m_W$, which is a large logarithm. To extract the coefficient A of this logarithm we would need to evaluate the diagram, which is both difficult and unnecessary. At higher order in perturbation theory, there will be additional large logarithms, proportional to $\left(A\frac{\alpha}{4\pi}\ln\frac{m_W}{m_\mu}\right)^n$. While we could attempt to isolate the series of diagrams which contributes these logarithms (as we isolated the geometric series of 1PI corrections to the Coulomb potential in the Section 2) such an approach is not nearly as straightforward in this case – there are many relevant diagrams with no obvious relation between them. Instead, we will resum the logarithms using the renormalization group.

In order to use the renormalization group to resum logarithms besides those in the effective charge, we need another parameter to renormalize besides e_R. To see what we can renormalize, we first expand in the limit that the W is very heavy, so that we can replace $\frac{i}{p^2 - m_W^2} \to -\frac{i}{m_W^2}$ for $p^2 \ll m_W^2$. Graphically, this means



(38)

This approximation leads to the 4-Fermi theory, discussed briefly in Lecture III-8, and to be discussed in more detail here and extensively in Part IV. The 4-Fermi theory replaces the W boson with a set of effective interactions involving 4 fermions. The relevant Lagrangian interaction in this case is

$$\mathcal{L}_{4F} = \frac{G_F}{\sqrt{2}}\bar{\psi}_\mu \gamma^\mu P_L \psi_{\nu_\mu} \bar{\psi}_e \gamma^\mu P_L \psi_{\nu_e} + h.c \qquad (39)$$

where $P_L = \frac{1-\gamma_5}{2}$ projects onto left-handed fermions and $G_F = \frac{\sqrt{2}g^2}{8m_W^2} = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$ (see Lecture IV-5 for the origin of P_L). This leads to a decay rate of $\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = G_F^2 \frac{m_\mu^5}{192\pi^3}$ which agrees with Eq. (35). The point of doing this is twofold: first, the four-Fermi theory is simpler than the theory with the full propagating W boson; second, we can use the renormalization group to compute the RG evolution of G_F which will reproduce the large logarithms in Eq. (37) and let us resum them to all orders in α.

It is not hard to go from the RGE for the electric charge to the RGE for a general operator. Indeed, the electric charge can be thought of as the coefficient of the operator $\mathcal{O}_e = \bar{\psi} A\!\!\!/ \psi$ in the QED Lagrangian. The RGE was determined by the renormalization factor $Z_e = \frac{Z_1}{Z_2\sqrt{Z_3}}$, which was calculated from the radiative correction to the $\bar{\psi} A\!\!\!/ \psi$ vertex (this gave Z_1), and then subtracting off the field strength renormalizations which came from the electron self-energy graph and vacuum polarization graphs (giving Z_2 and Z_3, respectively).

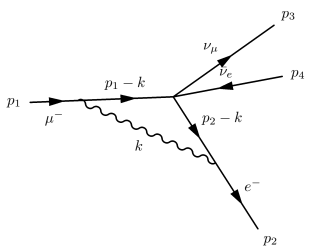
Unfortunately for the pedagogical purposes of this example, in the actual weak theory, the coefficient A of the large logarithm in Eq. (37) is 0 (see Problem 1). This fact is closely related to the non-renormalization of the QED current (see Section 5.1 below) and is somewhat of an accident. For example, a similar process for the weak decays of quarks does have a nonzero coefficient of the large logarithm, proportional to the strong coupling constant α_s (see Lecture IV-6). To get something non-zero, let us pretend that the weak interaction is generated by the exchange of a neutral scalar instead, so that the four-Fermi interaction is

$$\mathcal{L}_4 = \frac{G}{\sqrt{2}}(\bar{\psi}_\mu \psi_e)(\bar{\psi}_{\nu_e}\psi_{\nu_\mu}) + h.c$$

In this case, we will get a non-zero coefficient of the large logarithm.

To calculate the renormalization factor for G, we must compute the one-loop correction to this four-Fermi interaction. There is only one diagram,

$$i\mathcal{M} = \tag{40}$$



$$= \frac{G}{\sqrt{2}} e_R^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}(p_2)\gamma^\mu(\not{p}_2 - \not{k} + m_e)(\not{p}_1 - \not{k} + m_\mu)\gamma^\mu u(p_1)\bar{u}(p_3)v(p_4)}{[(p_1-k)^2 - m_\mu^2][(p_2-k)^2 - m_e^2]k^2} \tag{41}$$

To get at the renormalization group equation, we just need the counterterm, which comes from the coefficient of the the UV divergence of this amplitude. To that end, we can set all the external momenta and masses to zero. Thus,

$$\mathcal{M} = \mathcal{M}_0 \left(-i e_R^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{d}{k^4} \right) + \text{finite} \tag{42}$$

with the d coming from $\gamma^\mu \not{k}\not{k}\gamma^\mu = d k^2$ and

$$\mathcal{M}_0 = \frac{G}{\sqrt{2}} \bar{u}(p_2)u(p_1)\bar{u}(p_3)v(p_4) \tag{43}$$

is the tree-level matrix element from $\mathcal{L}_4$. Extracting the pole gives

$$\mathcal{M} = \mathcal{M}_0 \left(\frac{e_R^2}{2\pi^2} \mu^\varepsilon \frac{1}{\varepsilon} \right) + \text{finite} \tag{44}$$

which is all we will need for the renormalization group analysis.

To remove this divergence, we have to renormalize G. We do so by writing $G = G_R Z_G$ giving

$$\mathcal{L} = \frac{G_R}{\sqrt{2}} Z_G (\bar{\psi}_\mu \psi_e)(\bar{\psi}_{\nu_e}\psi_{\nu_\mu}) + h.c \tag{45}$$

To extract the counterterm, we expand $Z_G = 1 + \delta_G$. The counterterm then contributes $\mathcal{M}_0 \delta_G$. To remove the divergence we therefore need to take

$$\delta_G = -\frac{e_R^2}{16\pi^2} \frac{8}{\varepsilon} \tag{46}$$

Now that we know the counterterm, we can calculate the RGE, just like for the electric charge. Expressing the four-Fermi term in terms of bare fields, we find

$$\frac{G_R}{\sqrt{2}} Z_G (\bar{\psi}_\mu \psi_e)(\bar{\psi}_{\nu_\mu}\psi_{\nu_e}) = \frac{G_R}{\sqrt{2}} \frac{Z_G}{\sqrt{Z_{2\mu} Z_{2e} Z_{2\nu_\mu} Z_{2\nu e}}} \left(\bar{\psi}_\mu^{(0)} \psi_e^{(0)}\right)\left(\bar{\psi}_{\nu_e}^{(0)} \psi_{\nu_\mu}^{(0)}\right) \tag{47}$$

The coefficient of the bare operator must be independent of μ, since there is no μ in the bare Lagrangian. So, setting $Z_{2\nu}=1$ since the neutrino is neutral and therefore not renormalized until higher order in e_R, and using $Z_{2\mu}=Z_{2e}=Z_2$ since the muon and electron have identical QED interactions, we find

$$0=\mu\frac{d}{d\mu}\left(\frac{G_R Z_G}{Z_2}\right)=\frac{G_R Z_G}{Z_2}\left[\frac{\mu}{G_R}\frac{dG_R}{d\mu}+\frac{1}{Z_G}\frac{\partial Z_G}{\partial e_R}\mu\frac{de_R}{d\mu}-\frac{1}{Z_2}\frac{\partial Z_2}{\partial e_R}\mu\frac{de_R}{d\mu}\right] \tag{48}$$

where we have used that Z_G and Z_2 only depend on μ through e_R in the last step. Using the 1-loop results $Z_G=1-\frac{e_R^2}{16\pi^2}\frac{8}{\varepsilon}$ and $Z_2=1-\frac{e_R^2}{16\pi^2}\frac{2}{\varepsilon}$ and keeping only the leading terms, we have

$$\gamma_G\equiv\frac{\mu}{G_R}\frac{dG_R}{d\mu}=\left(-\frac{\partial Z_G}{\partial e_R}+\frac{\partial Z_2}{\partial e_R}\right)\beta(e_R)=\frac{3e_R}{4\varepsilon\pi^2}\left(-\frac{\varepsilon}{2}e_R\right)=-\frac{3e_R^2}{8\pi^2}=-\frac{3\alpha}{2\pi} \tag{49}$$

where γ_G is the anomalous dimension for $\mathcal{O}_G=Z_G(\bar{\psi}_\mu\psi_e)(\bar{\psi}_{\nu_\mu}\psi_{\nu_e})$.

Using $\mu\frac{d\alpha}{d\mu}=\beta(\alpha)$, the solution to this differential equation is

$$G_R(\mu)=G_R(\mu_0)\exp\left[\int_{\alpha(\mu_0)}^{\alpha(\mu)}\frac{\gamma_G(\alpha)}{\beta(\alpha)}d\alpha\right] \tag{50}$$

In particular, with $\beta(\alpha)=-\frac{\alpha^2}{2\pi}\beta_0=\frac{2\alpha^2}{3\pi}$ at leading order we find

$$G_R(\mu)=G_R(\mu_0)\exp\left[-\frac{9}{4}\int_{\alpha(\mu_0)}^{\alpha(\mu)}\frac{d\alpha}{\alpha}\right]=G_R(\mu_0)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\frac{9}{4}} \tag{51}$$

Now, we are assuming that we know the value for G at the scale $\mu_0=m_W$ where the W boson (or its equivalent in our toy model) is integrated out and we would like to know the value of G at the scale relevant for muon decay, $\mu=m_\mu$. Using Eq. (30), we find $\alpha(m_\mu)=0.00736$ and $\alpha(m_W)=0.00743$ so that

$$G_R(m_\mu)=1.024\times G(m_W) \tag{52}$$

which would have given a 4.8% correction to the muon decay rate, if the muon decay were mediated by a neutral scalar. In the actual weak theory, where muon decay is mediated by a vector boson coupled to left-handed spinors, the anomalous dimension for the operator in Eq. (39) is zero and so G_F does not run in QED.

5 RGE for general interactions

In the muon decay example, we calculated the running of G, defined as the the coefficient of the local operator $\mathcal{O}_G=Z_G(\bar{\psi}_\mu\psi_e)(\bar{\psi}_{\nu_e}\psi_{\nu_\mu})$ in a four-Fermi Lagrangian. More generally, we can consider adding additional operators to QED, with an effective Lagrangian of the form

$$\mathcal{L}=-\frac{1}{4}Z_3F_{\mu\nu}^2+Z_2^i\bar{\psi}_i i\partial\!\!\!/\psi_i-Z_2^iZ_m^i m_i^R\bar{\psi}_i\psi_i+Z_eZ_2{}^i\sqrt{Z_3}Q_ie_R\bar{\psi}_i A\!\!\!/\psi_i+\sum_j C_j\mathcal{O}_j(x) \tag{53}$$

These operators $\mathcal{O}_j=Z_j\partial^n\gamma^m A_\mu(x)\cdots A_\nu(x)\bar{\psi}_i(x)\cdots\psi_j(x)$ are **composite local operators**, with all fields evaluated at the same spacetime point. They can have any number of photons, fermions, γ-matrices, factors of the metric, etc. and analytic (power law) dependence on derivatives. Keep in mind that the fields A_μ and ψ_j are the renormalized fields. The C_j are known as **Wilson coefficients**. Note that in this convention, each Z_j is grouped with its corresponding operator, which is composed of renormalized fields; the Z_j is not included in the Wilson coefficient so that Wilson coefficient will be a finite number at any given scale. Since the Lagrangian is independent of μ, if we assume no mixing, the renormalization group equations take the form

$$\mu \frac{d}{d\mu}(C_j \mathcal{O}_j) = 0 \quad (\text{no sum on } j) \tag{54}$$

These equations (one for each j) let us extract the RG evolution of Wilson coefficients from the μ-dependence of matrix elements of operators. In general, there can be mixing among the operators (see Section 6.2 and Lecture IV-6), in which case this equation must be generalized to $\mu \frac{d}{d\mu}\left(\sum_j C_j \mathcal{O}_j\right) = 0$. One can also have mixing between the operators and the other terms in the Lagrangian in Eq. (53), in which case the RGE is just $\mu \frac{d}{d\mu}\mathcal{L} = 0$.

The way these effective Lagrangians are used is that the C_j are first either calculated or measured at some scale μ_0. We can calculate them if we have a (full) theory which is equivalent to this (effective) one at a particular scale. For example, we found G_F by designing the four-Fermi theory to reproduce the muon decay rate from the full electroweak theory, to leading order in $\frac{1}{m_W^2}$ at the scale $\mu_0 = m_W$. This is known as **matching**. Alternatively, if a full theory to which our effective Lagrangian can be matched is not known (or is not perturbative), one can simply measure the C_j at some scale μ_0. For example, in the chiral Lagrangian (describing the low-energy theory of pions) one could in principle match to the theory of strong interactions (QCD), but in practice it's easier just to measure the Wilson coefficients. In either case, once the values of the C_j are set at some scale, we can solve the RGE to resum large logarithms. In the toy muon decay example, we evolved G_R to the scale $\mu = m_\mu$ in order to incorporate large logarithmic corrections of the form $\alpha \ln \frac{m_\mu}{m_W}$ into the rate calculation.

5.1 External operators

Eq. (54) implies that the RG evolution of Wilson coefficients is exactly compensated for by the RG evolution of the operators. Operator running provides a useful language in which to consider physical implications of the renormalization group. An important example is the running of the current $J_\mu(x) = Z_J \bar{\psi}(x)\gamma^\mu \psi(x)$, which we will now explore. Rather than thinking of J^μ as the coefficient of A_μ in the QED interaction, we will treat $J_\mu(x)$ as an **external operator**: an operator which is not part of the Lagrangian, but which can be inserted into Green's functions.

The running of J_μ is determined by the μ-dependence of Z_J and of the renormalized fields $\bar{\psi}(x)$ and $\psi(x)$ appearing in the operator. To find Z_J, we can calculate any Green's function involving J^μ. The simplest non-vanishing one is the 3-point function with the current and two fields, whose Fourier transform we already discussed in the context of the Ward-Takahashi identity in Lectures II-7 and the proof of $Z_1 = Z_2$ in Lecture III-5. We define

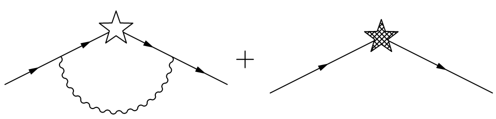
$$\langle \Omega | T\{J^\mu(x)\psi(x_1)\bar{\psi}(x_2)\}|\Omega\rangle = \int \frac{d^4 p}{(2\pi)^4}\frac{d^4 q_1}{(2\pi)^4}\frac{d^4 q_2}{(2\pi)^4} e^{-ipx} e^{-iq_1 x_1} e^{iq_2 x_2}$$

$$\times i\mathcal{M}^\mu(p, q_1, q_2)(2\pi)^4 \delta^4(p + q_1 - q_2) \tag{55}$$

so that $\mathcal{M}^\mu$ is given by Feynman diagrams without truncating the external lines or adding external spinors. At tree-level

$$i\,\mathcal{M}^\mu_{\text{tree}}(p, q_1, q_2) = \frac{i}{\not{q}_1 - m}\gamma^\mu \frac{i}{\not{q}_2 - m} \tag{56}$$

At next-to-leading order, there is a 1PI loop contribution and a counterterm

 (57)

Here the hollow star indicates an insertion of the current and the solid star indicates the counterterm for the current, both with incoming momentum p^μ. The counterterm contribution to the Green's function comes from expanding $Z_J = 1 + \delta_J$ directly in the Green's function (we have not added J^μ to the Lagrangian). These two graphs give, in Feynman gauge,

$$i\mathcal{M}^\mu_{1-\text{loop}} = \frac{i}{\not{q}_1 - m}\left[(-ie_R)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i\gamma^\nu(\not{q}_1 - \not{k} + m)}{(q_1-k)^2 - m^2}\gamma^\mu\frac{i(\not{q}_2 - \not{k} + m)\gamma_\nu}{(q_2-k)^2 - m^2}\frac{-i}{k^2} + \gamma^\mu\delta_J\right]\frac{i}{\not{q}_2 - m}$$

Since we are just interested in counterterm we take $k \gg q_1, q_2$. Then this reduces to

$$i\mathcal{M}^\mu_{1-\text{loop}} = \frac{i}{\not{q}_1 - m}\gamma^\mu\frac{i}{\not{q}_2 - m}\left[-ie_R^2\mu^{4-d}\frac{(2-d)^2}{d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^4} + \delta_J\right] \tag{58}$$

$$= i\mathcal{M}^\mu_{\text{tree}}\left\{\frac{e_R^2}{16\pi^2}\left[\frac{2}{\varepsilon}\right] + \delta_J\right\} \tag{59}$$

Thus $\delta_J = \frac{e_R^2}{16\pi^2}\left[-\frac{2}{\varepsilon}\right]$, which also happens to equal δ_2 and δ_1. Thus $Z_2 = Z_J$ at 1-loop.

Now that we know Z_J we can calculate the renormalization of the current. The bare current is independent of μ. This is $J^\mu_{\text{bare}}(x) = \bar{\psi}_0\gamma^\mu\psi_0 = \frac{1}{Z_J}Z_2 J^\mu(x)$.

$$0 = \mu\frac{d}{d\mu}J^\mu_{\text{bare}} = \mu\frac{d}{d\mu}\left(\frac{Z_2}{Z_J}J^\mu(x)\right) = \mu\frac{d}{d\mu}J^\mu(x) \tag{60}$$

Thus, the current does not run. In other words, whatever scale we measure the current at, it will have the same value. To be clear, the current *is* renormalized, but it does not *run*. Defining the anomalous dimension γ_J for the current by

$$\mu\frac{d}{d\mu}J^\mu = \gamma_J J^\mu \tag{61}$$

we have found that

$$\gamma_J = 0 \tag{62}$$

That is, the anomalous dimension for the current vanishes.

As you might have figured out by now, the Ward-Takahashi identity implies $\gamma_J = 0$ to all orders. In fact, $\gamma_J = 0$ is just the renormalization group version of the Ward-Takahashi identity, which actually has a nice physical interpretation. The vanishing anomalous dimension of the current is equivalent to the statement that the total number of particles minus the number of antiparticles does not depend on the scale at which we count them. To see this, observe that 0 component of the renormalized current when integrated over all space gives a conserved total charge:

$$Q = \int d^3x\, J_0 = \int d^3x \psi^\dagger\psi = \text{total charge} \tag{63}$$

This does not change with time, since the current vanishes at infinity and

$$\partial_t Q = \int d^3x\, \partial_0 J_0 = \int d^3x \vec{\nabla}\cdot\vec{J} = \vec{J}(\infty) = 0 \tag{64}$$

To see what Q does, note that since the (renormalized) fields at the same time anticommute $\{\psi^\dagger(x), \psi(y)\} = \delta^3(x-y)$, we have

$$\psi(x)Q = \int d^3y \psi(x)\psi^\dagger(y)\psi(y) = Q\psi(x) + \int d^3y\delta^3(x-y)\psi(y) = Q\psi(x) + \psi(x) \tag{65}$$

$$Q\psi^\dagger(x) = \int d^3y \psi^\dagger(y)\psi(y)\psi^\dagger(x) = \psi^\dagger(x)Q + \int d^3y\delta^3(x-y)\psi^\dagger(y) = \psi^\dagger(x)Q + \psi^\dagger(x) \tag{66}$$

So,

$$[Q,\psi]=-\psi, \qquad [Q,\psi^\dagger]=\psi^\dagger \tag{67}$$

That is, Q counts the number of particles minus antiparticles. The fields ψ can be (and are) scale dependent. Thus the only way for these equations to be satisfied is if Q does not have scale dependence itself. Thus the current cannot run.

5.2 Lagrangian operators versus external operators

There is of course not much difference between the calculation of the RGE for the coefficients of operators in a Lagrangian or for external operators. In fact, the relation

$$\mu\frac{d}{d\mu}(C_j\mathcal{O}_j)=0 \tag{68}$$

implies that the RGE for the Wilson coefficient and the operator it multiplies carry the same information.

Some distinctions between external operators and operators in the Lagrangian include:

1. External operators do not have to be Lorentz invariant, while operators in the Lagrangian do.

2. External operators can insert momentum into a Feynman diagram, while operators in the Lagrangian just give Feynman rules which are momentum conserving.

3. For external operators, it is the operators itself which run, whereas for operators in a Lagrangian we usually talk about their Wilson coefficients as having the scale dependence.

In this sense, an operator in the Lagrangian is a special case of an external operator, which is Lorentz invariant and evaluated at $p=0$. For example, we can treat the 4-Fermi operator $\mathcal{O}_F=\bar{\psi}\gamma^\mu P_L\psi\bar{\psi}\gamma^\mu P_L\psi$ as external. Then we can determine its anomalous dimension by evaluating

$$\langle\Omega|T\{\mathcal{O}_F(x)\bar{\psi}(x_1)\gamma^\mu P_L\psi(x_2)\bar{\psi}(x_3)\gamma^\mu P_L\psi(x_4)\}|\Omega\rangle \tag{69}$$

This will amount to the same Feynman diagram as in Section 4, but now we can have momentum p^μ coming in at the vertex. As far as the RGE is concerned, we only need the UV divergences, which are independent of external momentum. So in this case we would find that the operator runs with the same RGE that its Wilson coefficient had before. That is, it runs with exactly what is required by $\frac{d}{d\mu}(G_F\mathcal{O}_F)=0$.

5.3 RGE for Green's functions

We have now discussed the renormalization group equation for operators, coupling constants, and scalar masses. We can also consider directly the running of Green's functions. Consider, for example, the bare correlation function of n photons and m fermions in QED

$$G_{n,m}^{(0)}=\langle\Omega|T\{A_{\mu_1}^0\cdots A_{\mu_n}^0\psi_1^0\cdots\psi_m^0\}|\Omega\rangle \tag{70}$$

This is constructed out of bare fields, and since there is no μ in the bare Lagrangian, this is μ-independent. The bare Green's function is infinite, but it is related to the renormalized Green's function by

$$G_{n,m}^{(0)}=Z_3^{\frac{n}{2}}Z_2^{\frac{m}{2}}G_{n,m} \tag{71}$$

where

$$G_{n,m}(p,e_R,m_R,\mu)=\langle\Omega|T\{A_{\mu_1}\cdots A_{\mu_n}\psi_1\cdots\psi_m\}|\Omega\rangle \tag{72}$$

The renormalized Green's function is finite. It can depend on μ explicitly as well as on momenta, collectively called p, and the parameters of the renormalized Lagrangian, namely the renormalized coupling e_R and the mass m_R, which themselves depend on μ. Then,

$$0 = \mu \frac{d}{d\mu} G_{n,m}^{(0)} \tag{73}$$

$$= Z_3^{\frac{n}{2}} Z_2^{\frac{m}{2}} \left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \frac{\mu}{Z_3} \frac{dZ_3}{d\mu} + \frac{m}{2} \frac{\mu}{Z_2} \frac{dZ_2}{d\mu} + \mu \frac{\partial e_R}{\partial \mu} \frac{\partial}{\partial e_R} + \mu \frac{\partial m_R}{\partial \mu} \frac{\partial}{\partial m_R} \right) G_{n,m} \tag{74}$$

Defining

$$\gamma_3 = \frac{\mu}{Z_3} \frac{d\, Z_3}{d\mu}, \qquad \gamma_2 = \frac{\mu}{Z_2} \frac{d\, Z_2}{d\mu} \tag{75}$$

this reduces to

$$\left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \gamma_3 + \frac{m}{2} \gamma_2 + \beta \frac{\partial}{\partial e_R} + \gamma_m m_R \frac{\partial}{\partial m_R} \right) G_{n,m} = 0 \tag{76}$$

This equation is known variously as the **Callan-Symanzik equation**, the **Gell-Mann-Low equation**, the **'t Hooft-Weinberg equation** and the **Georgi-Politzer equation**. (The differences refer to different schemes, such as $\overline{\text{MS}}$ or the on-shell physical renormalization scheme.) We will restrict our discussion to the $\overline{\text{MS}}$ form given above, and just call it the renormalization group equation for Green's functions.

One can also calculate Green's functions with external operators inserted, such as $\langle \Omega | T\{ J^\mu(x) \psi_1(x_1) \bar{\psi}_2(x_2) \} | \Omega \rangle$ considered in Section 5.1. For a general operator, we define

$$\mu \frac{d}{d\mu} \mathcal{O} = \gamma_{\mathcal{O}} \mathcal{O} \tag{77}$$

Then a Green's function with an operator $\mathcal{O}$ in it satisfies

$$\left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \gamma_3 + \frac{m}{2} \gamma_2 + \beta \frac{\partial}{\partial e_R} + \gamma_m m_R \frac{\partial}{\partial m_R} + \gamma_{\mathcal{O}} \right) G = 0 \tag{78}$$

If there are more operators, there will be more $\gamma_{\mathcal{O}}$ terms.

5.4 Anomalous dimensions

Now let us discuss the term anomalous dimension. We have talked about the mass dimension of a field many times. For example, in four dimensions, $[\phi] = M^1$, $[m] = M^1$, $[\psi] = M^{3/2}$ and so on. These numbers just tell us what happens if we change units. To be more precise, consider the action for ϕ^4:

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2} \phi (\Box + m^2) \phi + g \phi^4 \right] \tag{79}$$

This has a symmetry under $x^\mu \to \frac{1}{\lambda} x^\mu$, $\partial_\mu \to \lambda \partial_\mu$, $m \to \lambda m$, $g \to g$ and $\phi \to \lambda \phi$. This operation is called **dilatation** and denoted by $\mathcal{D}$. Thus,

$$\mathcal{D}\colon \phi \to \lambda^{d_0} \phi \tag{80}$$

The d_0 are called the **classical** or **canonical scaling dimensions** of the various fields and couplings in the theory.

Now consider a correlation function

$$G_n = \langle \Omega | T\{ \phi_1(x_1) \cdots \phi_n(x_n) \} | \Omega \rangle \tag{81}$$

In a classical theory, this Green's function can only depend on the various quantities in the Lagrangian raised to various powers:

$$G_n(x, g, m) = m^a g^b x_1^{c_1} \cdots x_n^{c_n} \tag{82}$$

By dimensional analysis, we must have $a - c_1 - \cdots - c_n = n$. Thus we expect that $\mathcal{D}\colon G_n \to \lambda^n G_n$.

In the quantum theory, G_n can also depend on the scale where the theory is renormalized, μ. So we could have

$$G_n(x, g, m, \mu) = m^a g^b x_1^{c_1} \cdots x_n^{c_n} \mu^\gamma \tag{83}$$

where now $a - c_1 - \cdots - c_n = n - \gamma$. Note that μ does not transform under $\mathcal{D}$ since it does not appear in the Lagrangian – it is the subtraction point used to connect to experiment. So when we act with $\mathcal{D}$, only the x and m terms change, thus we find $\mathcal{D}$: $G_n \to \lambda^{n-\gamma} G_n$. Thus G_n does not have the canonical scaling dimension. In particular,

$$\mu \frac{d}{d\mu} G_n = \gamma G_n \tag{84}$$

which is how we have been defining anomalous dimensions. Thus the anomalous dimensions tell us about deviations from the classical scaling behavior.

6 Scalar masses and RG flows

In this section we will examine the RG evolution of a super-renormalizable operator, namely a scalar mass term $m^2\phi^2$. To extract physics from running masses, we have to think of masses more generally than just the location of the renormalized physical pole in an S-matrix, since by definition the pole mass is independent of scale. Rather, we should think of them as a term in a potential, like a ϕ^4 interaction would be. This language is very natural in condensed matter physics. As we will now see, in an off-shell scheme (like $\overline{\text{MS}}$) masses can have scale dependence. This scale dependence can induce phase-transitions and signal spontaneous symmetry breaking (cf Lectures IV-4 and IV-9).

6.1 Yukawa potential correction

Recall that the exchange of a massive particle generates a Yukawa potential, with the mass giving the characteristic scale of the interactions. Just as the Coulomb potential let us understand the physics of a running coupling, the Yukawa potential will help us understand running scalar masses. For example, consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{1}{4!}\lambda\phi^4 + g\phi J \tag{85}$$

which has the scalar field interacting with some external current J. The current-current interaction at leading order comes from an exchange of ϕ, which generates the Yukawa potential. For the static potential, we can drop time derivatives and then Fourier transform the propagator, giving

$$V(r) = \langle\Omega|\phi(\vec{x})\phi(0)|\Omega\rangle = -\int \frac{d^3k}{(2\pi)^3}\frac{g^2}{\vec{k}^2 + m^2} e^{i\vec{k}\cdot\vec{x}} = -\frac{g^2}{4\pi r} e^{-mr} \tag{86}$$

In the language of condensed matter physics, this correlation function has a correlation length ξ given by the inverse mass, $\xi = \frac{1}{m}$. In this language, we can easily give a physical interpretation to a running mass: the Yukawa potential will be modified by $m \to m(r)$ with calculable logarithmic dependence on r.

To calculate $m(r)$ we'll solve the renormalization group evolution induced by the $\lambda\phi^4$ interaction. The first step to studying the RGE for this theory is to renormalize it at 1-loop, for which we need to introduce the various Z-factors into the Lagrangian. In terms of renormalized fields

$$\mathcal{L} = -\frac{1}{2} Z_\phi \phi\Box\phi - \frac{1}{2} Z_m Z_\phi m_R^2 \phi^2 - \mu^{4-d}\frac{\lambda_R}{4!} Z_\lambda Z_\phi^2 \phi^4 \tag{87}$$

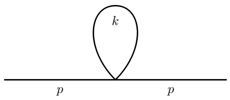
Since ϕ has mass dimension $\frac{d-2}{2}$ an extra factor of μ^{4-d} has been added to keep λ_R dimensionless, as was done for the electric charge in QED. The RGE for the mass comes from the μ-independence of the bare mass $m^2 = m_R^2 Z_m$:

$$0 = \mu\frac{d}{d\mu}(m^2) = \mu\frac{d}{d\mu}(m_R^2 Z_m) = m_R^2 Z_m\left(\frac{1}{m_R^2}\mu\frac{d}{d\mu}m_R^2 + \frac{1}{Z_m}\mu\frac{d}{d\mu}\delta_m\right) \tag{88}$$

Since the only μ-dependence in the Lagrangian comes from the ϕ^4 interaction, we need to compute the dependence of δ_m on λ_R and the dependence of λ_R on μ.

We can extract Z_m (and Z_ϕ) from corrections to the scalar propagator. The leading graph is a seagull graph

$$i\Sigma_2(p^2) =$$



$$= \frac{-i\lambda_R}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m_R^2} = \frac{-i\lambda_R\mu^{4-d}}{2(4\pi)^{d/2}}\left(\frac{1}{m_R^2}\right)^{1-\frac{d}{2}}\Gamma\left(1-\frac{d}{2}\right)$$

The quadratic divergence in this integral shows up in dimensional regularization as a pole at $d = 2$ but is hidden if one expands near $d = 4$. Nevertheless, since quadratic divergences are just absorbed into the counterterms, we can safely ignore them and focus on the logarithmic divergences. After all, it is the non-analytic logarithmic momentum dependence which we will resum using the renormalization group.

Expanding in $d = 4 - \varepsilon$ dimensions, $\Sigma_2(p^2) = \frac{\lambda_R m_R^2}{16\pi^2}\frac{1}{\varepsilon}$. The counterterms from $Z_\phi = 1 + \delta_\phi$ and $Z_m = 1 + \delta_m$ give a contribution

$$i\Sigma_{\rm ct}(p^2) =$$

$$= i\delta_\phi(p^2 - m_R^2) - i\delta_m m_R^2 \tag{89}$$

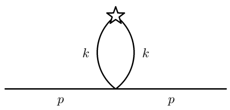
So to order λ_R, $\delta_\phi = 0$ and $\delta_m = \frac{\lambda_R}{16\pi^2}\frac{1}{\varepsilon}$.

An alternative way to extract these counterterms is to use the propagator of the massless theory and to treat $m_R^2\phi^2$ as a perturbation. This does not change the physics, since the massive propagator is reproduced by summing the usual geometric series of 1PI insertions of the mass

$$\frac{i}{p^2} + \frac{i}{p^2}(-im_R^2)\frac{i}{p^2} + \frac{i}{p^2}(-im_R^2)\frac{i}{p^2}(-im_R^2)\frac{i}{p^2} + \cdots = \frac{i}{p^2 - m_R^2} \tag{90}$$

However, one can look at just the first mass insertion to calculate the counterterms. The leading graph with a insertion of the mass and the coupling λ_R is

$$i\Sigma_2(p^2) =$$



$$= (-im_R^2)\frac{-i\lambda_R}{2}\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{k^2} \tag{91}$$

This is now only logarithmically divergent. Extracting the UV divergence with the usual trick gives $\Sigma_2(p^2) = \frac{\lambda_R m_R^2}{16\pi^2}\frac{1}{\varepsilon}$ and so $\delta_m = \frac{\lambda_R}{16\pi^2}\frac{1}{\varepsilon}$, which is the same result we got from the quadratically divergent integral.

Next, we need the dependence of λ_R on μ. The RGE for λ_R is derived by using that the bare coupling $\lambda = \mu^{4-d}\lambda_R Z_\lambda$ is μ independent, so

$$0 = \mu\frac{d}{d\mu}(\lambda) = \mu\frac{d}{d\mu}(\mu^{4-d}\lambda_R Z_\lambda) = \mu^\varepsilon\lambda_R Z_\lambda\left(\varepsilon + \frac{\mu}{\lambda_R}\frac{d}{d\mu}\lambda_R + \frac{\mu}{Z_\lambda}\frac{d}{d\mu}\delta_\lambda\right) \tag{92}$$

Then, since δ_λ starts at order λ_R we have $\mu\frac{d}{d\mu}\lambda_R = -\varepsilon\lambda_R + \mathcal{O}(\lambda_R^2)$. Although not necessary for the running of m_R, it's not hard to calculate δ_λ at 1-loop. We can extract it from the radiative correction to the 4-point function. With zero external momenta, the loop gives

$$(-i\lambda_R)^2\frac{3}{2}\mu^{2(4-d)}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{k^2} = \mu^{2(4-d)}\frac{3\lambda_R^2}{16\pi^2}\frac{i}{\varepsilon} \tag{93}$$

So that $\delta_\lambda = \frac{3\lambda_R}{16\pi^2}\frac{1}{\varepsilon}$ and then the β-function to order λ_R^2 is

$$\beta(\lambda_R) \equiv \mu\frac{d}{d\mu}\lambda_R(\mu) = -\varepsilon\lambda_R - \frac{3\lambda_R^2}{16\pi^2}\frac{1}{\varepsilon}(-\varepsilon) = -\varepsilon\lambda_R + \frac{3\lambda_R^2}{16\pi^2} \tag{94}$$

We will use this result below.

Using the RGE for the mass, Eq. (88), $\mu\frac{d}{d\mu}\lambda_R = -\varepsilon\lambda_R$ and $\delta_m = \frac{\lambda_R}{16\pi^2}\frac{1}{\varepsilon}$ we find

$$\gamma_m \equiv \frac{\mu}{m_R^2}\frac{d}{d\mu}m_R^2 = -\frac{1}{Z_m}\frac{\partial\delta_m}{\partial\lambda_R}\mu\frac{d\lambda_R}{d\mu} = \frac{\lambda_R^2}{16\pi^2} + \mathcal{O}(\lambda_R^3) \tag{95}$$

The solution, treating γ_m as constant, is

$$m_R^2(\mu) = m_R^2(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m} \tag{96}$$

You can check in Problem 2 that the more general solution (including the μ-dependence of λ_R following Eq. (50)) reduces to Eq. (96) for small λ_R.

Now let us return to the Yukawa potential. Since μ just represents an arbitrary scale with dimensions of mass, we can equally well write the solution to the RGE in position space as

$$m^2(r) = m_0^2\left(\frac{r}{r_0}\right)^{-\gamma_m} \tag{97}$$

where $m_0 = m(r = r_0)$. This leads to a corrected Yukawa potential

$$V(r) = -\frac{g^2}{4\pi r}\exp[-r\, m(r)] = -\frac{g^2}{4\pi r}\exp\left[-r^{1-\frac{\gamma_m}{2}} r_0^{\frac{\gamma_m}{2}} m_0\right] \tag{98}$$

which is in principle measurable. The final form has been written in a suggestive way to connect to what we will discuss below. Indeed, extracting a correlation length by dimensional analysis, we find

$$V(r) = -\frac{g^2}{4\pi r}\exp\left[-(r/\xi)^{1-\frac{\gamma_m}{2}}\right], \quad \xi = r_0^{1-2\nu} m_0^{-2\nu}, \quad \nu = \frac{1}{2-\gamma_m} \tag{99}$$

In the free theory, ξ scales like m_0^{-1}, by dimensional analysis. With interactions we see it scales like m_0 to a different power of the mass, determined by ν. This quantity ν is known as a **critical exponent**. Dimensional transmutation has given us another scale with dimensions of mass, r_0^{-1}, which has changed the scaling relation predicted by dimensional analysis. These critical exponents have been measured in a number of situations. In fact, we are very close to being able to compare the result of our RG calculation to experimental results.

6.2 Wilson-Fisher fixed point

The classic example of a scalar field with a mass is magnetization $M(x)$. This is a 3-vector field $\vec{M}(x)$ indicating the local orientation of the magnetic dipole moment in some system. Its norm $M(x) = |\vec{M}(x)|$ is a scalar field. If you take some magnetic material and heat it up past its critical temperature, T_C, its magnetization disappears. Indeed, the magnetization dies with distance with a characteristic length scale ξ, the **correlation length**, which has been found to scale with temperature across many materials like $\xi \sim (T - T_C)^{-0.63}$. This 0.63 is known as a **critical exponent** and conventionally defined by $\xi \sim (T - T_C)^{-\nu}$. In addition, many other apparently unrelated second order phase transitions are characterized by nearly identical scaling.

The universality of this critical exponent suggests that it should be describable without detailed knowledge of the microscopic system. This leads to a Ginzburg-Landau model, in which one describes the system with an effective Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{kin}} - \frac{1}{2}(T - T_C)M^2 - \frac{1}{4!}\lambda M^4 + \cdots \tag{100}$$

Here the $T - T_C$ factor just comes from assuming a Taylor exists near T_C and nothing special forces the linear term to vanish. This Lagrangian has the property that, below T_C, the effective mass-squared $m^2 = T - T_C$ becomes negative, signaling spontaneous symmetry breaking into the magnetic phase. Moreover, the transition is smooth across T_C, as required for a second-order phase transition. We will discuss spontaneous symmetry breaking more in Lecture IV-4.

As a quick check, we already know that the 2-point function in a scalar theory like this should behave like a Yukawa potential

$$\langle\Omega|M(r)M(0)|\Omega\rangle \sim \frac{1}{r}e^{-rm} = \frac{1}{r}\exp\left(-r(T-T_C)^{1/2}\right) \tag{101}$$

Thus the classical theory predicts $\nu = \frac{1}{2}$, which is not far from the observed universal value. But now we know how to correct this value using quantum field theory – we just did the calculation. We found

$$\nu = \frac{1}{2-\gamma_m} \tag{102}$$

So an anomalous dimension allows for this universal scaling behavior, that is, the existence of universal critical exponents. This suggests that the Ginzburg-Landau effective Lagrangian may be relevant even if we have no idea how to calculate γ_m.

There actually is a way to calculate ν using what we've already done. Recall that we found that λ_R and the effective mass parameter $m_R^2 = (T-T_C)$ run according to

$$\begin{aligned} \mu\frac{d}{d\mu}m_R^2 &= \frac{\lambda_R^2}{16\pi^2}m_R^2 + \mathcal{O}(\lambda_R^2) \\ \mu\frac{d}{d\mu}\lambda_R &= -\varepsilon\lambda_R + \frac{3\lambda_R^2}{16\pi^2} + \mathcal{O}(\lambda_R^3) \end{aligned} \tag{103}$$

For any ε, these equations have a solution with $\lambda_R = m_R = 0$. This is known as the **Gaussian fixed point**, since at this point the Lagrangian is a free theory of a massless scalar field and the path integral is an exact Gaussian.

In condensed matter physics we are interested in the macroscopic, long-distance behavior of a system. In particle physics, we are interested usually in the low-energy limit of a system, which is most accessible experimentally. So in either case we would like to know what happens as we *lower* μ. The behavior of a system as μ is lowered gives the renormalization group trajectory or **RG flow** of the couplings in a system. For example, suppose we start near (but not on) the Gaussian fixed point. Then the RG equation for λ_R at leading order is $\frac{d\ln\lambda_R}{d\ln\mu} = -\varepsilon$ which implies that if $d>4$ ($\varepsilon<0$), the system will flow back towards the fixed point as μ decreases, while for $d<4$ ($\varepsilon>0$), the system will flow away from the fixed point. Since many interesting systems take place in $d=3$ where the flow is away from the fixed point, the natural question is, where do they flow to? As $\mu\to 0$, they can either blow up, go to zero, or go to some non-zero fixed point.

Instead of going all the way to $d=3$, let us explore what happens in $d=4-\varepsilon$ dimensions. For $0<\varepsilon\ll 1$, there exists a value of λ_R for which $\frac{d}{d\mu}\lambda_R = 0$, namely

$$\lambda_\star = \frac{16\pi^2\varepsilon}{3} \tag{104}$$

This is known as the **Wilson-Fisher** fixed point. At this value of the coupling $\gamma_m = \frac{\varepsilon}{3}$ from Eq. (95) and so, from Eq. (102)

$$\nu = \frac{3}{6-\varepsilon} \tag{105}$$

For $\varepsilon = 1$ corresponding to 3 dimensions, $\nu = 0.6$ at this point, which is quite close to the observed value of 0.63. This questionable practice of expanding around $d=4$ to get results in $d=3$ is known as the **epsilon expansion**. You can compute the two-loop value of ν in Problem ?.

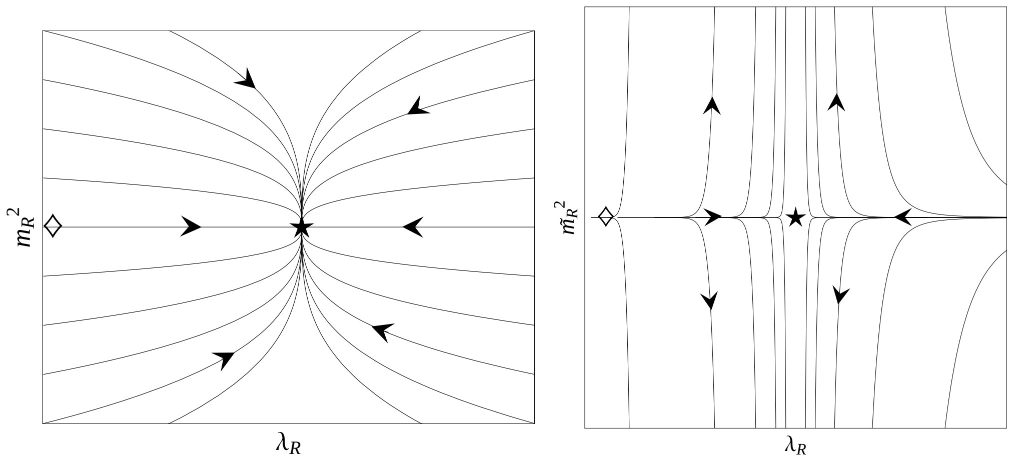
Regardless of the validity of $\varepsilon=1$, we can at least trust the qualitative observation of Wilson and Fisher, that there is a **nontrivial fixed point** (couplings do not all vanish) in this effective theory for $d<4$. As ε increases, the fixed point will move away from the $\lambda_\star$, due to large ε^2 corrections. This justifies the universality of the critical exponents in 3-dimensional systems – even if we cannot calculate the anomalous dimension, we expect that for $d<3$ it should still exist and should be separate from the Gaussian fixed point.

Fixed points are interesting places. Exactly on the fixed point, the theory is scale invariant, since $\mu\frac{d}{d\mu}m_R^2 = \mu\frac{d}{d\mu}\lambda_R = 0$. While there are many classical theories which are scale invariant (such as QED with massless fermions) theories which are scale invariant at the quantum level are much rarer. Such theories are known as **conformal field theories**. In a conformal theory, the Poincare group is enhanced to a larger group called the **conformal group**. Recall that the Poincare group acting on functions of spacetime is generated by translations, $P_\mu = -i\partial_\mu$, and Lorentz transformations, $\Lambda_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. In the conformal group, these are supplemented with a generator for scale transformations $D = -ix_\mu\partial_\mu$ and four generators for special-conformal transformations: $K_\mu = i(x^2\partial_\mu - 2x_\mu x_\nu\partial_\nu)$. Invariance under the conformal group is so restrictive that correlation functions in conformal field theories are strongly constrained. On the other hand, conformal field theories do not have massive particles. In fact they do not have particles at all. That is, there is no sensible way to define asymptotic single-particle states in such a theory. Thus they do not have an S-matrix.

One way to find conformal field theories is by looking for fixed points of RG flows in non-conformal field theories, as in the Wilson-Fisher example. Since conformal field theories have no inherent scales, dimensional parameters such as m_R in the Wilson-Fisher theory become dimensionless. To see how the fixed point is approached, it is natural to rescale away any classical scaling dimension of the various couplings. In the Wilson-Fisher case, we do this by defining $\tilde{m}_R(\mu) \equiv \frac{1}{\mu}m_R(\mu)$ so that $\tilde{m}_R$ is dimensionless. Then the RG equations become

$$\begin{aligned} \mu\frac{d}{d\mu}\tilde{m}_R^2 &= \left(-2 + \frac{\lambda_R^2}{16\pi^2}\right)\tilde{m}_R^2 \\ \mu\frac{d}{d\mu}\lambda_R &= -\varepsilon\lambda_R + \frac{3\lambda_R^2}{16\pi^2} \end{aligned} \tag{106}$$

The fixed point is at the same place, $\lambda_\star = \frac{16\pi^2\varepsilon}{3}$ and $m_R^2 = 0$. The RG flow for m_R^2 and $\tilde{m}_R^2$ are shown in Figure 1.



RG flow for λ_R
and a dimensionful scale m_R

Conventional RG flow diagram
in which all parameters are dimensionless

Figure 1. Renormalization group flow in the Wilson-Fisher theory for $0 < \varepsilon \ll 1$. The Wilson-Fisher fixed point indicated by a $\star$ at $m = 0$ and $\lambda = \lambda_\star$ is attractive. The Gaussian fixed point at $m = \lambda = 0$ is indicated with a $\diamond$. The arrows denote flow as the length scale is increased, or equivalently, as μ is decreased. On the left is the RG flow of a dimension one mass parameter, in which it appears that the fixed-point is attractive. On the right is the flow in terms of a dimensionless $\tilde{m}_R$ which shows that the fixed point is only attractive for $\tilde{m}_R = 0$.

The different trajectories in an RG flow diagram represent different values of m_R^2 and λ_R which might correspond to different microscopic systems. For example, changing the temperature of a system moves it from one trajectory to another. The temperature for which $m_R = 0$ is the critical temperature where the theory intersects the nontrivial fixed point. In the diagram on the right, in which all inherent scales have been removed, we see that to get close to the nontrivial fixed point, one would have to be very close to the $m_R = 0$ trajectory.

6.3 Varieties of Asymptotic behavior

One can easily imagine more complicated RG flows than those described by the Wilson-Fisher theory. With just one coupling, such as in QED or in QCD, the RG flow is determined by the β function $\beta(\alpha) = \mu \frac{d}{d\mu}\alpha$. When the coupling is small, the theory is perturbative, and then the coupling must either increase or decrease with scale. If the coupling increases with μ, as in QED, it goes to zero at long distances. In this case it is said to be **infrared free.** If it decreases with μ (as in QCD, see Lecture IV-2), it goes to zero at short distances and the theory is said to be **asymptotically free**. The third possibility in a perturbative theory is that $\beta(e) = 0$ exactly, in which case the theory is scale invariant. If the coupling is nonperturbative one can still define a coupling through the value of a Green's function. Then, as long as $\beta(\alpha) > 0$ at one α and $\beta(\alpha) < 0$ at a larger α, there is guaranteed to be an intermediate value where $\beta(\alpha_\star) = 0$. With multiple couplings there are other possibilities for solutions to the RGEs. For example, one could imagine a situation in which couplings circle around each other. It's certainly easy to write down coupled differential equations with bizarre solutions; whether such equations correspond to anything in nature or in a laboratory is another question.

There are not many known examples of perturbative conformal field theories in 4 dimensions. One is called $\mathcal{N} = 4$ super Yang-Mills theory. Another possibility is if the leading β-function coefficient is small, for example if $\beta(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \cdots$ where β_0 happens to be of order α. Then there could be a cancellation between β_0 and β_1 and a nontrivial fixed point at some finite value of α. That this might happen in a non-Abelian gauge theory with a large enough number of matter fields was conjectured by Banks and Zaks and is known as the **Banks-Zaks theory**.

7 Wilsonian RGE

So far we have been discussing the RGE as an invariance of physical quantities to the scale μ where the renormalization conditions are imposed. This is the continuum renormalization group, where all comparisons are made after the UV regulator has been completely removed. The Wilsonian picture instead supposes that there is an actual physical cutoff Λ, as there would be in a metal (the atomic spacing) or string theory (the string scale). Then all loops are finite and the theory is well-defined. In this case, one can (in principle) integrate over a shell of momentum in the path integral $\Lambda' < p < \Lambda$ and change the couplings of the theory so that low energy physics is the same. The Wilsonian RGE describes the resulting flow of coupling constants under infinitesimal changes in Λ. The reason we focused on the continuum renormalization group first is that it is easier to connect to observables, which coupling constants are not. However, the Wilsonian RGE helps explain why renormalizable theories play such an important role in physics.

You have perhaps heard people say mysterious phrases like "a dimension 6 operator, such as $\bar{\psi}\psi\bar{\psi}\psi$ is *irrelevant* since it *should* have a coefficient $\frac{1}{\Lambda^2}$, where Λ is an arbitrarily large cutoff." You may also have wondered how the word "should" earned a place in scientific discourse. There is indeed something very odd about this language, since if $\Lambda = 10^{19}$ GeV the operator $\frac{1}{\Lambda^2}\bar{\psi}\psi\bar{\psi}\psi$ can be safely be ignored at low energy, but if Λ is lowered to 1 GeV, this operator becomes extremely important. This language, although imprecise, actually is logical. It originates from the Wilsonian renormalization group, as we will now explain.

To begin, imagine that you have a theory with a physical short distance cutoff Λ_H which is described by a Lagrangian with a finite or infinite set of operators $\mathcal{O}_r$ of various mass dimensions r. For example, in a metal with atomic spacing ξ the physical cutoff would be $\Lambda_H \sim \xi^{-1}$ and some operators might look like $\frac{1}{\Lambda_H^2}\bar{\psi}\psi\bar{\psi}\psi$ where ψ correspond to atoms. Let us write a general Lagrangian with cutoff Λ_H as $\mathcal{L}(\Lambda_H) = \sum C_r(\Lambda_H)\Lambda_H^{4-r}\mathcal{O}_r$ with $C_r(\Lambda_H)$ some dimensionless numbers. These numbers can be large and are probably impossible to compute. In principle they could all be measured, and we would need an infinite number of renormalization conditions for all the $C_r(\Lambda_H)$ to completely specify the theory. The key point, however, as we will show, is that not all the $C_r(\Lambda_H)$ are important for long-distance physics.

At low energies, we don't need to take Λ to be as large as ξ^{-1}. As long as Λ is much larger than any energy scale of interest we can perform loops as if $\Lambda = \infty$ and cutoff-dependent effects will be suppressed by powers of $\frac{E}{\Lambda}$. (For example, for observables with $E \sim 100$ GeV, you don't need $\Lambda = 10^{19}$ GeV; $\Lambda \sim 10^{10}$ GeV works just as well.) So let us compute a different Lagrangian $\mathcal{L}(\Lambda) = \sum C_r(\Lambda)\Lambda^{4-r}\mathcal{O}_r$ with a cutoff $\Lambda < \Lambda_H$ by demanding that physical quantities computed with the two Lagrangians be the same. With $\Lambda = \Lambda_L \ll \Lambda_H$, the coefficients $C_r(\Lambda_L)$ will be some other dimensionless numbers, which may be big or small, and which are (in principle) computable in terms of $C_r(\Lambda_H)$.

Now, if we're making large distance measurements only, we should be able to work with $\mathcal{L}(\Lambda_L)$ just as well as $\mathcal{L}(\Lambda_H)$. So we might as well measure $C_r(\Lambda_L)$ to connect our theory to experiment. The important point, which follows from the Wilsonian RG is that $C_r(\Lambda_L)$ is *independent* of $C_r(\Lambda_H)$ if $r > 4$. Since there will only be a finite number of operators in a given theory with mass dimension $r \leqslant 4$, if we measure $C_{r\leqslant 4}(\Lambda_L)$ for these operators (as renormalization conditions), we can then calculate $C_{r>4}(\Lambda_L)$ for all the other operators as functions of the $C_{r\leqslant 4}(\Lambda_L)$. An explicit example is given below.

This result motivates the definition of **relevant** operators as those with $r < 4$ and **irrelevant** operators as those with $r > 4$. Operators with $r = 4$ are called **marginal**. We only need to specify renormalization conditions for the relevant and marginal operators, of which there are always a finite number. The Wilson coefficients for the irrelevant operators can be computed with very weak dependence any boundary condition related to short-distance physics, that is, on the values of $C_r(\Lambda_H)$.

Thus, it is true that with $\Lambda = \Lambda_H$ or $\Lambda = \Lambda_L$ the Lagrangian *should* have operators with coefficients determined by Λ to some power. Therefore, irrelevant operators *do* get more important as the cutoff is lowered. However, the important point is not the size of these operators but that their Wilson coefficients are computable. In other words

- Values of couplings when the cutoff is low are insensitive to the boundary conditions associated with *irrelevant* operators when the cutoff is high.

If we take the high cutoff to infinity then the irrelevant operators are precisely those for which there is zero effect on the low-cutoff Lagrangian. Only relevant operators remain when the cutoff is removed. So,

- The space of renormalizable field theories is the space for which the limit $\Lambda_H \to \infty$ exists, holding the couplings fixed when the cutoff is Λ_L.

Another important point is that in the Wilsonian picture one does not want to take Λ_L down to physical scales of interest. One wants to lower Λ enough so that the irrelevant operators become insensitive to boundary conditions, but then to leave it high enough so one can perform loop integrals as if $\Lambda = \infty$. That is

- The Wilsonian cutoff Λ should always be much larger than all relevant physical scales. This is in contrast the μ in the continuum picture which should be taken equal to a relevant physical scale.

For example, in the electroweak theory, one can imagine take $\Lambda = 100$ TeV, not $\Lambda = 10^{19}$ GeV and not $\Lambda = 100\,\text{GeV}$.

7.1 Wilson-Polchinski RGE

To prove the above statements, we need to sort out what is being held fixed and what is changing. Since the theory is supposed to be finite with UV cutoff Λ, the path integral is finite (at least to a physicist), and all the physics is contained in the generating functional $Z[J]$. The RGE is then simply $\Lambda\frac{d}{d\Lambda}Z[J]=0$. If we change the cutoff Λ, then the coupling constants in the Lagrangian must change to hold $Z[J]$ constant. For example, in a scalar theory, we might have

$$Z[J]=\int^{\Lambda_H}\mathcal{D}\phi\exp\left\{i\int d^4x\left(-\frac{1}{2}\phi(\Box+m^2)\phi+\frac{g_3}{3!}\phi^3+\frac{g_4}{4!}\phi^4+\frac{g_6}{6!}\phi^6\cdots+\phi J\right)\right\} \tag{107}$$

for some cutoff Λ_H on the momenta of the fields in the path integral. All the couplings, m, g_3, g_4, etc. are finite. If we change the cutoff to Λ then the couplings change to m', g_3', g_4' etc., so that $Z[J]$ is the same.

Unfortunately, actually performing the path integral over a Λ-shell is extremely difficult to do in practice. A more efficient way to phrase the Wilsonian RGE in field theory was developed by Polchinski. Polchinski's idea was first to cut off the path integral more smoothly by writing

$$Z[J]=\int\mathcal{D}\phi e^{iS+\phi J}=\int\mathcal{D}\phi\exp\left\{i\int d^4x\left(-\frac{1}{2}\phi(\Box+m^2)e^{\frac{\Box}{\Lambda^2}}\phi+\frac{g_3}{3!}\phi^3+\frac{g_4}{4!}\phi^4+\cdots+\phi J\right)\right\} \tag{108}$$

The $e^{\Box/\Lambda^2}$ factor makes the propagator go like $e^{-p^2/\Lambda^2}\to 0$ at high energy. You can get away with this only in a scalar theory in Euclidean space, but we will not let such technical details prevent us from making very general conclusions. It's easiest to proceed in momentum space, where $\phi(x)^2\to\phi(p)\phi(-p)$. Then

$$Z[J]=\int\mathcal{D}\phi e^{iS+\phi J}=\int\mathcal{D}\phi\exp\left\{i\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{2}\phi(p)(p^2-m^2)e^{-\frac{p^2}{\Lambda^2}}\phi(-p)+\mathcal{L}_{\mathrm{int}}(\phi)+\phi J\right)\right\}$$

Taking $\frac{d}{d\Lambda}$ on both sides gives

$$\Lambda\frac{d}{d\Lambda}Z[J]=i\int\mathcal{D}\phi\int\frac{d^4p}{(2\pi)^4}\left(\phi(p)(p^2-m^2)\phi(-p)\frac{p^2}{\Lambda^2}e^{-\frac{p^2}{\Lambda^2}}+\Lambda\frac{d}{d\Lambda}\mathcal{L}_{\mathrm{int}}(\phi)\right)e^{iS+\phi J} \tag{109}$$

Since $\frac{p^2}{\Lambda^2}e^{-\frac{p^2}{\Lambda^2}}$ only has support near $p^2\sim\Lambda^2$, this says that the change in $\mathcal{L}_{\mathrm{int}}$ comes from that momentum region. Therefore, the RGE will be local in Λ. This is a general result, independent of the precise way the cutoff is imposed. It can also be used to define a functional differential equation known as the **exact renormalization group** (see Problem 4), which we will not make use of here.

As a concrete example, consider a theory with a dimension 4 operator (with dimensionless coupling g_4) and a dimension 6 operator (with coupling g_6 with mass dimension -2). Then the RGE $\Lambda\frac{d}{d\Lambda}Z[J]=0$ would imply some equations we can write as

$$\Lambda\frac{d}{d\Lambda}g_4=\beta_4(g_4,\Lambda^2g_6) \tag{110}$$

$$\Lambda\frac{d}{d\Lambda}g_6=\frac{1}{\Lambda^2}\beta_6(g_4,\Lambda^2g_6) \tag{111}$$

where β_4 and β_6 are some general, complicated functions. The factors of Λ have all been inserted by dimensional analysis, since as we just showed, no other scale can appear in $\Lambda\frac{d}{d\Lambda}Z[J]$. To make these equations more homogeneous, let us define dimensionless couplings $\lambda_4=g_4$ and $\lambda_6=\Lambda^2g_6$. Then,

$$\Lambda\frac{d}{d\Lambda}\lambda_4=\beta_4(\lambda_4,\lambda_6) \tag{112}$$

$$\Lambda\frac{d}{d\Lambda}\lambda_6-2\lambda_6=\beta_6(\lambda_4,\lambda_6) \tag{113}$$

The $-2\lambda_6$ term implies that if β_6 is small, then $\lambda_6(\Lambda) = \lambda_6(\Lambda_H)\left(\frac{\Lambda}{\Lambda_H}\right)^2$ is a solution. We would like this to mean that as the coupling Λ is taken small, $\Lambda \ll \Lambda_H$, the higher dimension operators die away. However, the actual coupling of the operator for this solution is just $g_6(\Lambda) = \frac{1}{\Lambda_H^2}\lambda_6(\Lambda_H) = g_6(\Lambda_H)$ which does not die off (it doesn't run since we have set $\beta = 0$), so things are not quite that simple. We clearly need to work beyond 0^{th} order.

It is not hard to solve the RGEs explicitly in the case when β_4 and β_6 are small. Actually, one does not need the β_i to be small, rather one can start with an exact solution to the full RGEs and then expand perturbatively around the solution. For simplicity, we will just assume that the β_i can be expanded in their arguments. To linear order, we can write

$$\Lambda\frac{d}{d\Lambda}\lambda_4 = a\lambda_4 + b\lambda_6 \tag{114}$$

$$\Lambda\frac{d}{d\Lambda}\lambda_6 = c\lambda_4 + (2+d)\lambda_6 \tag{115}$$

and we assume a, b, c, d are small real numbers, so that the anomalous dimension does not overwhelm the classical dimension (otherwise perturbation theory would not be valid). It is now easy to solve this vector of homogeneous linear differential equations by changing to a diagonal basis

$$\tilde{\lambda_4} = -\frac{c}{\Delta}\lambda_4 - \frac{2+d-a-\Delta}{2\Delta}\lambda_6, \qquad \tilde{\lambda_6} = \frac{c}{\Delta}\lambda_4 + \frac{2+d-a+\Delta}{2\Delta}\lambda_6 \tag{116}$$

where $\Delta = \sqrt{4bc + (d-a+2)^2}$. The RGEs are easy to solve now:

$$\tilde{\lambda_4}(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a-\Delta}{2}}\tilde{\lambda_4}(\Lambda_0), \quad \tilde{\lambda_6}(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a+\Delta}{2}}\tilde{\lambda_6}(\Lambda_0) \tag{117}$$

Back in terms of the original basis, we then have

$$\begin{aligned}\lambda_4(\Lambda) = {}& \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a-\Delta}{2}}\left[\left(\frac{2+d-a+\Delta}{2\Delta}\right)\lambda_4(\Lambda_0) - \frac{b}{\Delta}\lambda_6(\Lambda_0)\right] \\ & + \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a+\Delta}{2}}\left[-\left(\frac{2+d-a-\Delta}{2\Delta}\right)\lambda_4(\Lambda_0) + \frac{b}{\Delta}\lambda_6(\Lambda_0)\right]\Bigg\}\end{aligned} \tag{118}$$

$$\begin{aligned}\lambda_6(\Lambda) = {}& \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a-\Delta}{2}}\left[-\frac{c}{\Delta}\lambda_4(\Lambda_0) - \left(\frac{2+d-a-\Delta}{2\Delta}\right)\lambda_6(\Lambda_0)\right] \\ & + \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a+\Delta}{2}}\left[\frac{c}{\Delta}\lambda_4(\Lambda_0) + \left(\frac{2+d-a+\Delta}{2\Delta}\right)\lambda_6(\Lambda_0)\right]\Bigg\}\end{aligned} \tag{119}$$

which is an exact solution to Eqs (114) and (115). In these solutions, $\lambda_4(\Lambda_0)$ and $\lambda_6(\Lambda_0)$ are free parameters to be set by boundary conditions.

What we would like to know is the sensitivity of λ_6 at some low scale Λ_L to its initial condition at some high scale Λ_H for fixed, renormalized, value of $\lambda_4(\Lambda_L)$. For simplicity, let us take $\lambda_6(\Lambda_H) = 0$ (any other boundary value would do just as well, but the solution is messier). Then, Eqs. (118) and (119) can be combined into

$$\lambda_6(\Lambda) = \frac{2c\left[\left(\frac{\Lambda}{\Lambda_H}\right)^{\Delta} - 1\right]}{(2+d-a+\Delta) - (2+d-a-\Delta)\left(\frac{\Lambda}{\Lambda_H}\right)^{\Delta}}\lambda_4(\Lambda) \tag{120}$$

Setting $\Lambda = \Lambda_L \ll \Lambda_H$ and assuming $a, b, c, d \ll 2$, so that $\Delta \approx 2$, we find

$$\lambda_6(\Lambda_L) = -\frac{c}{2}\left(1 - \frac{\Lambda_L^2}{\Lambda_H^2}\right)\lambda_4^L(\Lambda_L) \tag{121}$$

In particular, the limit $\Lambda_H \to \infty$ exists. Back in terms of g_4 and g_6 we have fixed $g_4(\Lambda_L)$ and set $g_6(\Lambda_H) = 0$. Thus as $\Lambda_H \to \infty$ we have $g_6(\Lambda_L) = -\frac{c}{2}\frac{1}{\Lambda_L^2} g_4(\Lambda_L)$. That is, the boundary condition at large Λ_H is totally irrelevant to the value of g_6 at the low scale. That is why operators with dimension greater than 4 are called **irrelevant**. This result is shown in Figure 2.

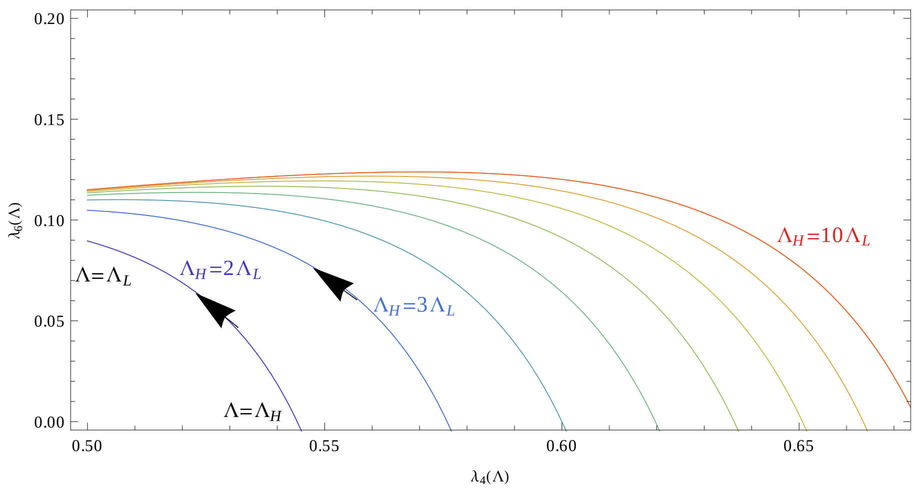


Figure 2. Solutions of the Wilsonian RGEs with $a = 0.1, b = 0.2, c = -0.5$ and $d = 0.3$. We fix $\lambda_4(\Lambda_L) = 0.5$ and look at how the value of $\lambda_6(\Lambda_L)$ depends on $\lambda_6(\Lambda_H)$ for some higher Λ_H. As $\Lambda_H \to \infty$ the value of $\lambda_6(\Lambda)$ goes to a constant value, entirely set by $\lambda_4(\Lambda)$ and the anomalous dimensions. Arrows denote RG flow to decreasing Λ. Note the convergence is extremely quick.

To relate all this rather abstract manipulation to physics, recall the calculation of the electron magnetic moment from Lecture III-3. We found that the moment was $g = 2$ at tree-level and $g = 2 + \frac{\alpha}{\pi}$ at 1-loop. If we had added to the QED Lagrangian an operator of the form $\mathcal{O}_\sigma = \frac{e}{4}\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ with some coefficient C_σ this would have given $g = 2 + \frac{\alpha}{\pi} + C_\sigma$. Since the measured value of g is in excellent agreement with the calculation ignoring C_σ, we need an explanation of why $\mathcal{O}_\sigma$ should be absent or have a small coefficient. The answer is given by the above calculation, with g_4 representing α and g_6 representing the coefficient of $\mathcal{O}_\sigma$. Say we do add $\mathcal{O}_\sigma$ to the QED Lagrangian with even a very large coefficient, but with the cutoff set to some very high scale, say $\Lambda_H \sim M_{\mathrm{Pl}} \sim 10^{19}$ GeV. Then when the cutoff is lowered down, even a little bit (say to 10^{15} GeV), whatever you set your coefficient to at M_{Pl} would be totally irrelevant: the coefficient of $\mathcal{O}_\sigma$ would now be determined completely in terms of α, like g_6 is determined by g_4. Hence g becomes a calculable function of α. The operator $\mathcal{O}_\sigma$ is *irrelevant* to the $g-2$ calculation.

Note that if we lowered the cutoff down to say 1 MeV, then $\mathcal{O}_\sigma$ would indeed give a contribution to g, but a contribution calculable entirely in terms of α. With such a low cutoff, there would be cutoff dependence in the 1-loop calculation of $g - 2$ as well (which is tremendously difficult to actually calculate). Indeed, these two contributions must precisely cancel, since the theory is independent of cutoff. That is why one does not want to take the cutoff Λ_L down to scales near physics of interest in the Wilsonian picture. To repeat, in the continuum picture, μ is of order of physical scales, but in the Wilsonian picture, Λ is always much higher than all of the relevant physical scales.

Returning to our toy RGEs, suppose we set $\lambda_4(\Lambda_H) = 0$. Then we would have found

$$\lambda_4(\Lambda) = \frac{2b\left[1 - \left(\frac{\Lambda}{\Lambda_H}\right)^{\Delta}\right]}{2 + d - a - \Delta - (2 + d - a + \Delta)\left(\frac{\Lambda}{\Lambda_H}\right)^{\Delta}} \lambda_6(\Lambda) \tag{122}$$

Expanding this for $a, b, c, d \ll 2$ gives

$$\lambda_4(\Lambda_L) = \frac{b}{2}\left(1 - \frac{\Lambda_H^2}{\Lambda_L^2}\right)\lambda_6(\Lambda_L) \tag{123}$$

which diverges as $\Lambda_H \to \infty$! Thus, we cannot self-consistently hold the irrelevant couplings fixed at low energy and take the high energy cutoff to infinity.

The same would be true if we had a dimension 4 coupling (like a gauge coupling) and a dimension 2 parameter, like m^2 for a scalar. Then we would have found an extraordinary sensitivity of $m^2(\Lambda_L)$ to the boundary condition $m^2(\Lambda_H)$ if $g(\Lambda_L)$ is held fixed. Of course, like any renormalizable coupling, one should fix $m^2(\Lambda_L)$ through a low-energy experiment, for example, measuring the Higgs mass. The Wilsonian RG simply implies that if there is a short-distance theory with cutoff Λ_H in which m_h is calculable then $m_h(\Lambda_H)$ should have a very peculiar looking value. For example, suppose $m(\Lambda_L) = 10$ GeV when $\Lambda_L = 10^5$ GeV. Then, there is some value for $m^2(\Lambda_H)$ with $\Lambda_H = 10^{19}$ GeV. If there were a different short-distance theory for which $m^2(\Lambda_H)$ were different by factor of order $\frac{\Lambda_L^2}{\Lambda_H^2} = 10^{-38}$, then $m^2(\Lambda_L)$ would differ by a factor of order 1 (see Problem 5). This is the **fine-tuning problem**. It is a sensitivity of long-distance measurements to small deformations of a theory defined at some short distance scale. The general result is that relevant operators, like scalar masses are **UV-sensitive**.

7.2 Generalization and discussion

The generalization of the above 2-operator example is a theory with an arbitrary set of operators $\mathcal{O}_n$. To match onto the Wilson operator language (this is, after all, the Wilsonian RGE), let us write

$$Z[J] = \int^{\Lambda} \mathcal{D}\phi \exp\left\{ i \int d^4x \sum_n C_n \mathcal{O}_n(\phi) \right\} \tag{124}$$

Since there is a cutoff, all couplings (Wilson coefficients C_n) in the theory are finite. The RGE in the Wilsonian picture is $\Lambda \frac{d}{d\Lambda} Z[J] = 0$ which forces

$$\Lambda \frac{d}{d\Lambda} C_n = \beta_n(\{C_m\}, \Lambda) \tag{125}$$

for some β_n. In the continuum picture, the RGE we used was

$$\mu \frac{d}{d\mu} C_n = \gamma_{nm} C_m \tag{126}$$

which looks a lot like the linear approximation to the Wilsonian RGE. In fact, we can linearize the Wilsonian RGE not necessarily by requiring that all the couplings be small, but simply by expanding around a fixed point, which is a solution of Eq.(125) for which $\beta_n = 0$.

In the continuum language, although the cutoff is removed the anomalous dimensions γ_{mn} are still determined by the UV divergences. So these two equations are very closely related. However, there is one very important difference: in the continuum picture quadratic and higher-order power-law divergences are exactly removed by counterterms. In dimensional regularization, this is trivial, since power-law divergent integrals just give 0 (they give poles at $d = 2$, $d = 0$, etc, but 0 when $d \to 4$). But for any regulator, the power divergences can be absorbed into the renormalization of operators in the theory. In the continuum picture of renormalization, the only UV divergences corresponding to physically observable effects are logarithmic ones (cf. Lecture III-8). With a finite cutoff, one simply has Λ^2 terms in the RGE. This Λ^2-dependence was critical for the analysis of g_4 and g_6 in the previous subsection.

For a theory with general, possibly non-perturbative β_n, consider a given subset $\mathcal{S}$ of the operators and its complements $\overline{\mathcal{S}}$. Choose coefficients for the operators in $\mathcal{S}$ to be fixed at a scale Λ_L and set the coefficients for the operators in $\overline{\mathcal{S}}$ to 0 at a scale Λ_H. If it is possible to take the limit $\Lambda_H \to \infty$ so that *all* operators have finite coefficients at Λ_L, the theory restricted to the set $\mathcal{S}$ is called a **renormalizable** theory. Actually, one does not have to set all the operators in $\overline{\mathcal{S}}$ to 0 at Λ_H; if there is *any way* to choose their coefficients as a function of Λ_H so that the theory at Λ_L is finite, then the theory is still considered renormalizable.

It is not hard to see that this definition coincides with the one we have been using all along. As you might imagine, generalizing the g_4/g_6 example above, any operator with dimension greater than 4 will be **non-renormalizable** and **irrelevant**. Operators with dimension less than 4 are **super-renormalizable** and **relevant**. **Marginal** operators have dimension equal to 4; however, if the operator has any anomalous dimension at all it will become marginally relevant or marginally irrelevant. From the Wilsonian point of view, marginally irrelevant operators are the same as irrelevant ones – one cannot keep their couplings fixed at low energy and remove the cutoff.

Technically, the terms relevant and irrelevant should be applied only to operators corresponding to eigenvectors of the renormalization group. Otherwise there is operator mixing. So let us diagonalize the matrix γ_{mn} and consider its eigenvalues. Any eigenvalue λ_n of γ_{mn} with $\lambda_n > 0$ will will cause the couplings C_n to decrease as μ is lowered. Thus these operators decrease in importance at long distances. They are the irrelevant operators. Relevant operators have $\lambda_n < 0$. These operators increase in importance as μ is lowered. If we try to take the long-distance limit, the relevant operators blow up. It is sometimes helpful to think of all possible couplings in the theory as a large multidimensional surface. An RG fixed point therefore lies on the subsurface of irrelevant operators. Any point on this surface will be attracted to the fixed point, while any point off the surface will be repelled away from it.

In practice, we do not normally work in a basis of operators which are eigenststates of the renormalization group. In a perturbative theory (near a gaussian fixed point), operators are usually classified by their classical scaling dimension d_n. The coefficient of such an operator (in 4 dimensions) has classical dimension $[C_n] = 4 - d_n$. If we rescale $C_n \to C_n \mu^{d_n - 4}$ to make the coeffieicnt dimensionless, then the γ_{nn} component in the matrix Eq. (126) becomes $\gamma_{nn} = d_n - 4$. Thus, at leading order, irrelevant operators are those with $d_n > 4$. In the quantum theory, loops induce non-diagonal components in γ_{mn}. If a relevant or relevant operator mixes into an irrelevant one, this mixing completely dominates the RG evolution of C_n at low energy. In this way, an operator which is classified as irrelevant based on its scaling dimension can become more important at large distances. However, the value of its coefficent quickly becomes a caclulable function of coupling constants corresping to more relevant operators. We saw this through direct calculation.

8 Problems

1. Show that $A = 0$ in Eq. (37) by evaluating the anomalous dimension of G_F from Eq (39) in QED. At an intermediate stage, you may want to use the Fierz identity

$$(\bar{\psi}_1 P_L \gamma^\mu \gamma^\alpha \gamma^\beta \psi_2)(\bar{\psi}_3 P_L \gamma^\mu \gamma^\alpha \gamma^\beta \psi_4) = 16(\bar{\psi}_1 P_L \gamma^\mu \psi_2)(\bar{\psi}_3 P_L \gamma^\mu \psi_4) \tag{127}$$

which you derived in problem ??.

2. Show that Eq. (96) follows from the small λ_R limit of the general solution to $m_R(\mu)$.
3. Compute the two-loop value of the critical exponent ν in the Wilson-Fisher theory.
4. Derive

$$\Lambda \frac{d}{d\Lambda} \mathcal{L}_{\text{int}}(\phi) = \int d^4p \frac{(2\pi)^4}{p^2 + m^2} \frac{p^2}{\Lambda^2} e^{\frac{p^2}{\Lambda^2}} \left[\frac{\delta \mathcal{L}_{\text{int}}}{\delta\phi(p)} \frac{\delta \mathcal{L}_{\text{int}}}{\delta\phi(-p)} - \frac{\delta^2 \mathcal{L}_{\text{int}}}{\delta\phi(p)\delta\phi(-p)} \right] \tag{128}$$

using the Wilson-Polchinski RGE. Show that the first term corresponds to integrating out tree-level diagram and the second from loops.

5. Consider a theory with a dimension 2 mass parameter m^2 and a dimensionless coupling g.
   a) Write down and solve generic RGEs for this theory, as in Eqs. (114) and (115).
   b) Fix $g(\Lambda_L) = 0.1$ for concreteness with $\Lambda_L = 10^5$ GeV. What value of $m^2(\Lambda_H)$ would lead to $m^2(\Lambda_L) = 100$ GeV?
   c) What would $m^2(\Lambda_L)$ be if you changed $m^2(\Lambda_H)$ by 1 part in 10^{20}?
   d) Sketch the RG flows for this theory.