

Spectral action and particle physics

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- Fermions in spacetime and emerging bosons
- Noncommutative fine structure of spacetime
- Examples: electroweak model, Standard Model
- Beyond the Standard Model: Pati–Salam unification



A fermion in a spacetime background

Minimal ingredients to describe a free fermion:

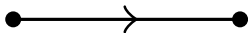
- **coordinates** on spacetime M :

$$x_\mu \cdot x_\nu(p) = x_\mu(p)x_\nu(p), \text{ etc.},$$

with $\mu, \nu = 1, \dots, 4$.

- **propagation**, described by **Dirac operator** $\not{\partial}_M = i\gamma^\mu \partial_\mu$, acting on wavefunctions ψ :

$$S[\psi] = \int \bar{\psi} \not{\partial}_M \psi \quad \rightsquigarrow \text{EOM: } \not{\partial}_M \psi = 0.$$



- This combination of **coordinate algebra** and **operators** is central to the **spectral**, or **noncommutative** approach [C 1994].

$$(\mathcal{A}, \mathcal{H}, D)$$

Our fermionic starting point induces a bosonic theory:

- “Inner perturbations” by the coordinates [C 1996]:

$$\partial_M \rightsquigarrow \partial_M + \sum_j a_j [\partial_M, a'_j]$$

for functions a_j, a'_j depending on the coordinates x_μ .

- Then, by the chain rule:

$$\sum_j a_j [\partial_M, a'_j] = A^\nu \gamma^\mu (\partial_\mu x^\nu) = A^\mu \gamma_\mu$$

where A^μ is the **electromagnetic 4-potential**.

Moreover, it is possible to derive a bosonic action from the (Euclidean) Dirac operator via the **spectral action** [CC 1996]:

$$\text{Trace } f(\not{D}_M/\Lambda)$$

which is usually understood as an effective field theory using the asymptotic heat expansion:

$$\text{Trace } e^{-\not{D}_M^2/\Lambda^2} \sim c_4 \Lambda^4 \text{Vol}(M) + c_2 \Lambda^2 \int R \sqrt{g} + c_0 \int (\partial_{[\mu} A_{\nu]})^2 + \dots$$

for some coefficients c_4, c_2, \dots

We recognize

- The Einstein-Hilbert action $\int R \sqrt{g}$ for (Euclidean) gravity
- The Lagrangian $\int (\partial_{[\mu} A_{\nu]})^2$ for the electromagnetic field



Replace spacetime by
spacetime \times **noncommutative space**: $M \times F$

- F is a **finite internal space** (Kaluza–Klein like)
- F is described by **noncommutative matrices**, that play the role of coordinates, just as spacetime is described by $x_\mu(p)$.
- 'Propagation' of particles in F is described by a '**Dirac operator**' \not{D}_F which is actually simply a hermitian matrix.

Note that the **spectral approach** is now the only way to describe the **geometry of F** .

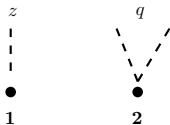
Example: electroweak theory

Coordinates on F are elements in $\mathbb{C} \oplus \mathbb{H}$

- A complex number z
- A quaternion $q = q_0 + iq_k \sigma^k$; in terms of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It describes a two-point space, with internal structure:



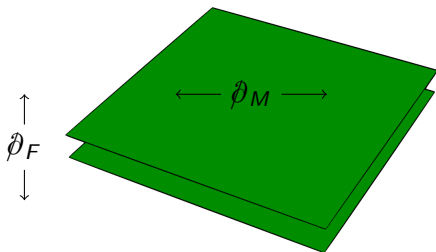
Gauge group is given by unitaries: $U(1) \times SU(2)$.

'Dirac operator'

$$\not{\partial}_F = \begin{pmatrix} 0 & \bar{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- “Inner perturbations” can be defined as before but now yield:

$$\sum_j \begin{pmatrix} z_j & 0 \\ 0 & q_j \end{pmatrix} \left[\not{\partial}_F, \begin{pmatrix} z'_j & 0 \\ 0 & q'_j \end{pmatrix} \right] =: \begin{pmatrix} 0 & \bar{c}\bar{\phi}_1 & \bar{c}\bar{\phi}_2 \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$



We combine this mild (matrix) noncommutativity with spacetime:

- **coordinates** of the **almost-commutative spacetime** $M \times F$:

$$\hat{x}^\mu(p) = (z^\mu(p), q^\mu(p))$$

as elements in $\mathbb{C} \oplus \mathbb{H}$ (for each μ and each point p of M)

- The **combined Dirac operator** becomes

$$\hat{\partial}_{M \times F} = \hat{\partial}_M + \gamma_5 \hat{\partial}_F$$

Note that $\hat{\partial}_{M \times F}^2 = \hat{\partial}_M^2 + \hat{\partial}_F^2$, which will be useful later on.

Inner perturbations on $M \times F$

So, we describe $M \times F$ by:

$$\hat{x}^\mu = (z^\mu, q^\mu); \quad \not{\partial}_{M \times F} = \not{\partial}_M + \gamma_5 \not{\partial}_F$$

As before, we consider inner perturbations of $\not{\partial}_{M \times F}$ by $\hat{x}^\mu(p)$:

- The inner perturbations of $\not{\partial}_F$ become **scalar fields** ϕ_1, ϕ_2 .
- The inner perturbations of $\not{\partial}_M$ become matrix-valued:

$$\sum_j a_j [\not{\partial}_M, a'_j] = a_\nu \gamma^\mu (\partial_\mu \hat{x}^\nu) =: \not{\partial}_M + A_\mu \gamma^\mu$$

with A_μ taking values in $\mathbb{C} \oplus \mathbb{H}$:

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ \\ 0 & W_\mu^- & -W_\mu^3 \end{pmatrix}$$

corresponding to **hypercharge and the W-bosons**.

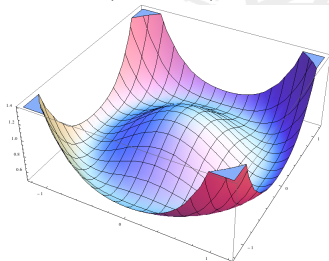
Action functional: electroweak theory

Use $\partial_{M \times F}^2 = \partial_M^2 + \partial_F^2$ to compute the spectral action

$$\begin{aligned} \text{Trace } e^{-\partial_{M \times F}^2 / \Lambda^2} &= \text{Trace } e^{-\partial_M^2 / \Lambda^2} \left(1 - \frac{\partial_F^2}{\Lambda^2} + \frac{1}{2} \frac{\partial_F^4}{\Lambda^4} - \dots \right) \\ &\sim \left(c_4 \Lambda^4 \text{Vol}(M) + c_2 \Lambda^2 \int R \sqrt{g} + c_0 \int F_{\mu\nu} F^{\mu\nu} \right) \left(1 - \frac{|\phi|^2}{\Lambda^2} + \frac{|\phi|^4}{2\Lambda^4} \right) + \dots \end{aligned}$$

We now recognize in terms of the field-strength $F_{\mu\nu}$ for A_μ :

- The Yang–Mills term $F_{\mu\nu} F^{\mu\nu}$ for hypercharge and W -boson
- The Higgs potential $-c_4 \Lambda^2 |\phi|^2 + \frac{1}{2} c_4 |\phi|^4$



Standard Model as an almost-commutative spacetime

Describe $M \times F_{SM}$ by [CCM 2007]

- **Coordinates:** $\hat{x}^\mu(p) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).
- **Dirac operator** $\hat{D}_{M \times F} = \hat{D}_M + \gamma_5 \hat{D}_F$ where

$$\hat{D}_F = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$$

is a 96×96 -dimensional hermitian matrix where 96 is:

$$3 \times 2 \times (\underline{2} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{2} \otimes \underline{3} + \underline{1} \otimes \underline{3} + \underline{1} \otimes \underline{3})$$

families anti-particles

(ν_L, e_L) ν_R e_R (u_L, d_L) u_R d_R

$$\not{D}_F = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

where Y_ν , Y_e , Y_u and Y_d are 3×3 mass matrices acting on the three generations.

- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\bar{\nu}_R$ for a 3×3 symmetric Majorana mass matrix Y_R , and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Just as before, we find

- Inner perturbations of $\not{\partial}_M$ give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

corresponding to **hypercharge, weak and strong interaction**.

- Inner perturbations of $\not{\partial}_F$ give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \bar{\phi}_2 \\ Y_\nu \phi_2 & Y_e \bar{\phi}_1 \end{pmatrix}$$

corresponding to **SM-Higgs field**. Similarly for Y_u, Y_d .

If we reconsider the spectral action:

$$\text{Trace } e^{-\not{D}_{M \times F}^2 / \Lambda^2} \sim \left(c_4 \Lambda^4 \text{Vol}(M) + c_0 \int F_{\mu\nu} F^{\mu\nu} \right) \left(1 - \frac{|\phi|^2}{\Lambda^2} + \frac{|\phi|^4}{2\Lambda^4} \right) + \dots$$

we observe [CCM 2007]:

- The coupling constants of hypercharge, weak and strong interaction are expressed in terms of the **single constant** c_0 which implies

$$g_3^2 = g_2^2 = \frac{5}{3} g_1^2$$

In other words, there should be **unification** of the couplings.

- Moreover, the quartic Higgs coupling λ is related via

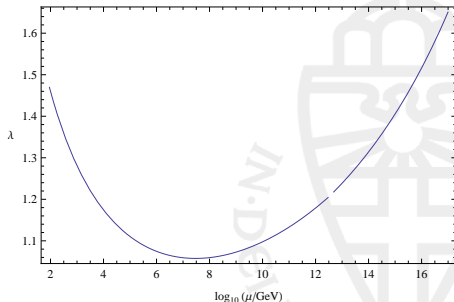
$$\lambda \approx 24 \frac{3 + \rho^4}{(3 + \rho^2)^2} g_2^2; \quad \rho = \frac{m_\nu}{m_{\text{top}}}$$

Phenomenology of the noncommutative Standard Model

This can be used to derive predictions as follows:

- Interpret the spectral action as an **effective field theory** at $\Lambda_{\text{GUT}} \approx 10^{13} - 10^{16}$ GeV.
- Run the quartic coupling constant λ to SM-energies to predict

$$m_h^2 = \frac{4\lambda M_W^2}{3g_2^2}$$

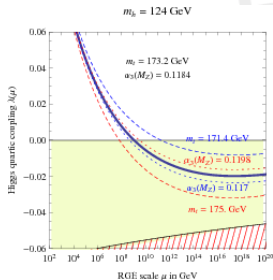
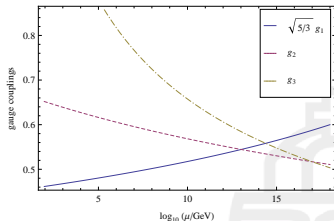


This gives [CCM 2007]

$$167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}$$

Three problems

- 1 This prediction is **falsified** by the measured value.
- 2 In the Standard Model there is not the **presumed unification**.
- 3 There is a problem with the low value of m_h , making the Higgs vacuum un/metastable [Elias-Miro et al. 2011].



Beyond the SM with noncommutative geometry

A solution to the above three problems?

- The matrix coordinates of the Standard Model arise naturally as a restriction of the following **coordinates**

$$\hat{x}^\mu(p) = (q_R^\mu(p), q_L^\mu(p), m^\mu(p)) \in \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

corresponding to a **Pati–Salam unification**:

$$U(1)_Y \times SU(2)_L \times SU(3) \rightarrow SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 **fermionic degrees of freedom** are structured as

$$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array} \right) \quad (i = 1, 2, 3)$$

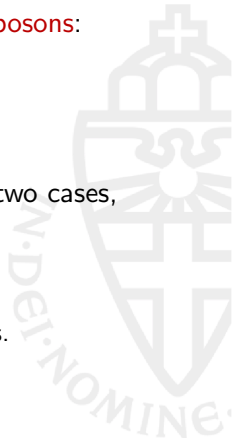
- Again the **finite Dirac operator** is a 96×96 -dimensional matrix (details in [CCS 2013]).

- Inner perturbations of $\not{\partial}_M$ now give **three gauge bosons**:

$$W_R^\mu, \quad W_L^\mu, \quad V^\mu$$

corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.

- For the inner perturbations of $\not{\partial}_F$ we distinguish two cases, depending on the initial form of $\not{\partial}_F$:
 - I The Standard Model $\not{\partial}_F = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$
 - II A more general $\not{\partial}_F$ with zero $\bar{f}_L - f_L$ -interactions.



Scalar sector of the spectral Pati–Salam model

Case I For a SM \not{D}_F , the resulting scalar fields are **composite fields**, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^b$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_J^I	1	1	15

Case II For a more general finite Dirac operator, we have **fundamental scalar fields**:

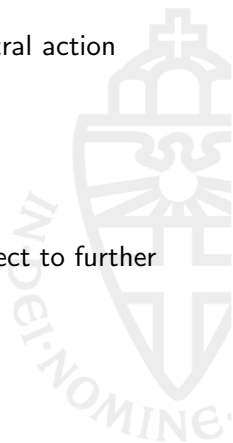
particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	1 + 15
$H_{\dot{a}I}^{bJ} \left\{ \right.$	3	1	10
	1	1	6

As for the Standard Model, we can compute the spectral action which describes the usual **Pati–Salam model** with

- **unification** of the gauge couplings

$$g_R = g_L = g.$$

- A rather involved, fixed **scalar potential**, still subject to further study



Phenomenology of the spectral Pati–Salam model

However, independently from the spectral action, we can analyze the running at one loop of the gauge couplings [CCS 2015]:

- 1 We run the **Standard Model gauge couplings** up to a presumed PS \rightarrow SM symmetry breaking scale m_R
- 2 We take their values as **boundary conditions** to the **Pati–Salam gauge couplings** g_R, g_L, g at this scale via

$$\frac{1}{g_1^2} = \frac{2}{3} \frac{1}{g^2} + \frac{1}{g_R^2}, \quad \frac{1}{g_2^2} = \frac{1}{g_L^2}, \quad \frac{1}{g_3^2} = \frac{1}{g^2},$$

- 3 Vary m_R in a search for a **unification scale** Λ where

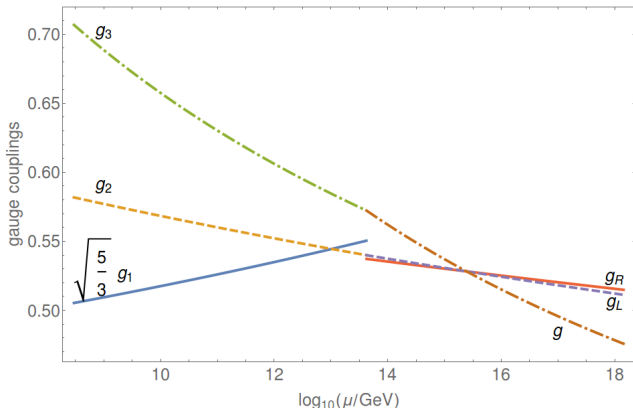
$$g_R = g_L = g$$

which is where the **spectral action** is valid as an **effective theory**.

Phenomenology of the spectral Pati–Salam model

Case I: Standard Model \not{D}_F

For the **Standard Model Dirac operator**, we have found that with $m_R \approx 4.25 \times 10^{13}$ GeV there is unification at $\Lambda \approx 2.5 \times 10^{15}$ GeV:



Phenomenology of the spectral Pati–Salam model

Case I: Standard Model \mathcal{P}_F

In this case, we can also say something about the **scalar particles** that remain after SSB:

	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_1^1 \\ \phi_1^2 \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_2^1 \\ \phi_2^2 \end{pmatrix}$	-1	2	1
σ	0	1	1
η	$-\frac{2}{3}$	1	3

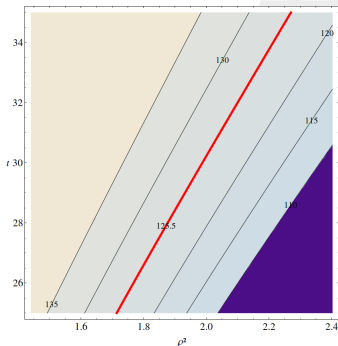
- It turns out that these scalar fields have a **little influence** on the running of the SM-gauge couplings (at one loop).
- This sector contains the **real scalar singlet** σ of [CC 2012].

Stabilization of the Higgs vacuum

- Suppose that the **real scalar singlet** σ is coupled to the Higgs sector in the following way:

$$V(\sigma, h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2) + \frac{1}{24} \lambda_h h^4 + \frac{1}{2} h^2 \sigma^2 + \frac{1}{4} \lambda_\sigma \sigma^4$$

- Instead of the notorious Higgs mass prediction from the nc Standard Model, this real scalar singlet gives a Higgs mass varying with $\rho = m_{\text{top}}/m_\nu$ and the unification scale $t = \log(\Lambda_{\text{GUT}}/M_Z)$



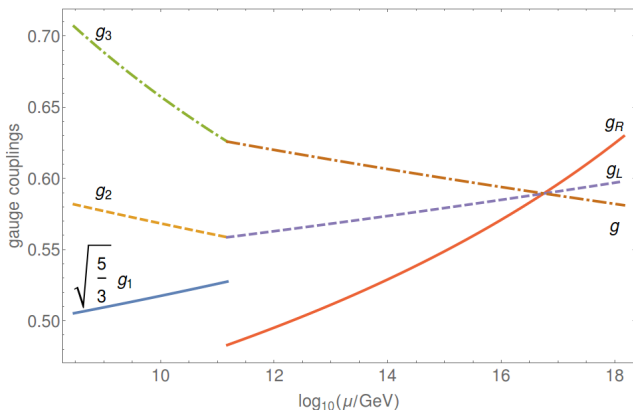
- This allows for $m_h = 125.5 \text{ GeV}$ and $m_\sigma \sim 10^{12} \text{ GeV}$.

Phenomenology of the spectral Pati–Salam model

Case II: General Dirac

For the more general case, we have found that with

$m_R \approx 1.5 \times 10^{11}$ GeV there is unification at $\Lambda \approx 6.3 \times 10^{16}$ GeV:



We have arrived at a **spectral Pati–Salam model** that

- goes beyond the Standard Model
- has a **fixed scalar sector** once the finite Dirac operator has been fixed (only a **few scenarios**)
- exhibits **grand unification** for all of these scenarios (confirmed by [Aydemir–Minic–Sun–Takeuchi 2015])
- the scalar sector has the potential to **stabilize the Higgs vacuum** and allow for a **realistic Higgs mass**.



- 1 No time.
- 2 The spectral action defines an effective theory at a specific (GUT)-scale and is not stable under renormalization.
- 3 What is the link to fuzzy spaces ($M_N(\mathbb{C})$)?
- 4 What about supersymmetry? The MSSM seems incompatible with the spectral action [Beenakker–vdBroek–vS, 2015], but particle content can be captured in noncommutative geometry. Significance?
- 5 $SO(10)$, E_6 -unification?
- 6 Cosmological constant, gravity?

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Pati-Salam Unification. *JHEP* 11 (2013) 132.
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Grand Unification in the Spectral Pati-Salam Model. *JHEP* 11
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