Grand unification in the spectral Pati–Salam model

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Overview

- Spectral geometry
- Matrix algebra and noncommutative geometry
- Semigroup of perturbations
- The spectral Standard Model and Beyond: Grand unfication

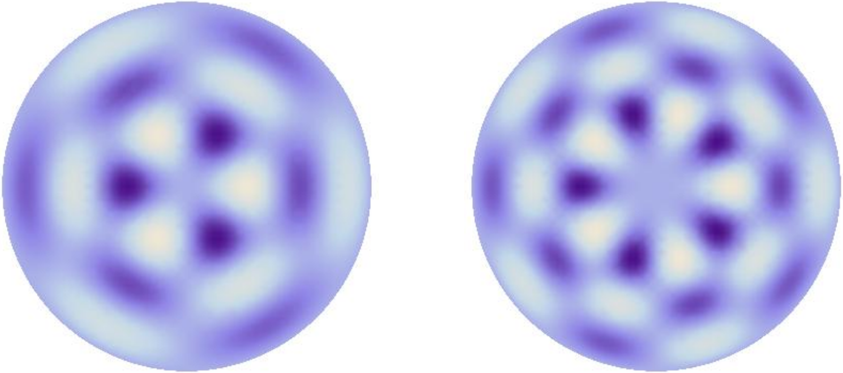
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

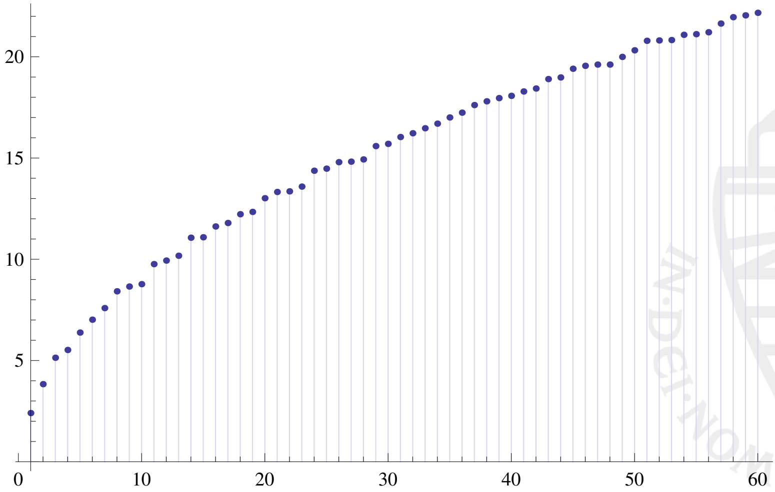
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

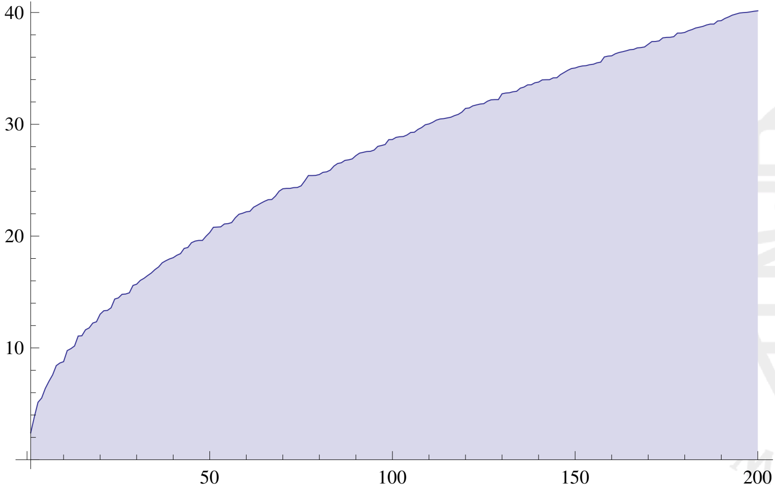
The disc



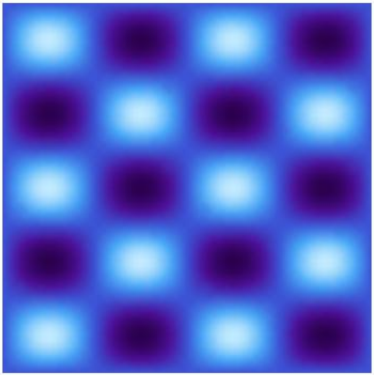
Wave numbers on the disc

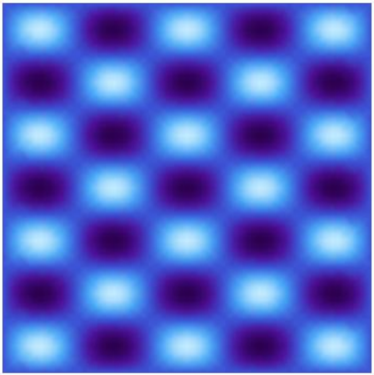


Wave numbers on the disc: high frequencies

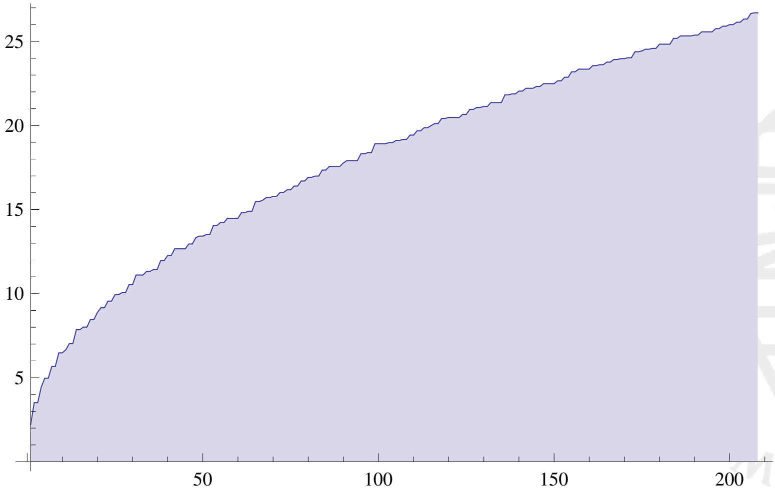


The square



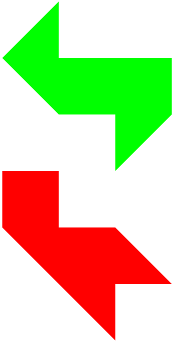


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

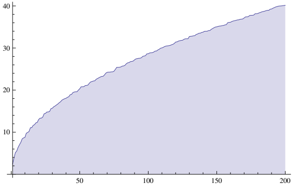
so the answer to Kac's question is no.

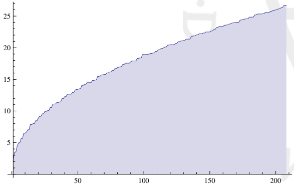
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension n of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n} \Lambda^n$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.

The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.

- The eigenfunctions of $D_{\mathbb{S}^1}$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt$$

  with eigenvalue $n \in \mathbb{Z}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a \frac{\partial}{\partial t_1} + b \frac{\partial}{\partial t_2} \right)^2 = a^2 \frac{\partial^2}{\partial t_1^2} + 2ab \frac{\partial^2}{\partial t_1 \partial t_2} + b^2 \frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

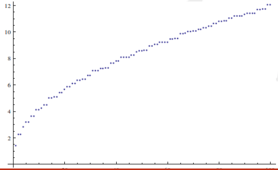
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

- The Dirac operator on $\mathbb{T}^4$ is

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$

- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

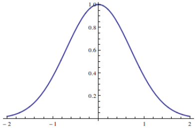
- Reconsider Weyl's estimate, in a smooth version:

$$\mathrm{Tr}\, f\left(\frac{D_M}{\Lambda}\right) = \sum_\lambda f\left(\frac{\lambda}{\Lambda}\right)$$

for a smooth cutoff function $f : \mathbb{R} \to \mathbb{R}$.

- For simplicity, restrict to a Gaussian function

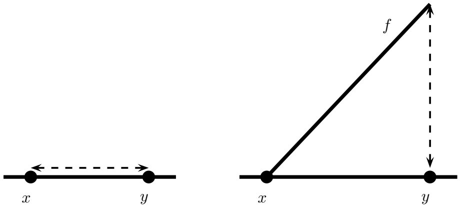
$$f(x) = e^{-x^2}$$



so that we can use heat asymptotics: $\mathrm{Tr}\, e^{-D_M^2/\Lambda^2} \sim \dfrac{\mathrm{Vol}(M)\Lambda^n}{(4\pi)^{n/2}}$

- As said, the geometry of M is not fully determined by spectrum of D_M.
- This can be improved by considering besides D_M also the algebra $C^\infty(M)$ of smooth functions on M, with pointwise product and addition
- In fact, the distance function on M is equal to

$$d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \text{ gradient } f \leq 1 \}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$.
  For example, on the circle we have $[D_{\mathbb{S}^1}, f] = -i \dfrac{df}{dt}$

- Finite space F, discrete topology

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Smooth functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(x,y) = \sup_{f \in C^\infty(F)} \left\{ |f(x) - f(y)| : \|[D_F, f]\| \le 1 \right\}$$

Example: two-point space

$$F = \ _1\bullet \qquad _2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(x,y) = \begin{cases} |c|^{-1} & x \neq y \\ 0 & x = y \end{cases}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \dots, a_N$ are square matrices of size $n_1, n_2, \dots, n_N$.

- Hence we will consider the matrix algebra

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A finite Dirac operator is still given by a hermitian matrix.

The two-point space can be given a noncommutative structure by considering the algebra A_F of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We make the above more dynamical by *perturbing* D_F by matrices in A_F.

Definition

Let A_F be the above algebra of block diagonal matrices (fixed size). The *perturbation semigroup of A_F* *is defined as*

$$\mathrm{Pert}(A_F) := \left\{ \sum_j A_j \otimes B_j \in A_F \otimes A_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\},$$

where ^{t} denotes matrix transpose, $\mathbb{I}$ is the identity matrix in A_F, and $\bar{\ }$ denotes complex conjugation of the matrix entries.

The semigroup law in $\mathrm{Pert}(A_F)$ is given by the matrix product in $A_F \otimes A_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

are called normalization and self-adjointness condition, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j \right) \left(\sum_k A'_k \otimes B'_k \right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $A_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.

- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $A_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C})$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

Perturbation semigroup for all matrix algebras
with Niels Neumann (B.Sc.)

- More generally, consider

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- For direct sums we have

$$\mathrm{Pert}(\mathcal{A} \oplus \mathcal{B}) \cong \mathrm{Pert}(\mathcal{A}) \times \mathrm{Pert}(\mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}^\circ \oplus \mathcal{B} \otimes \mathcal{A}^\circ)^{\mathrm{s.a.}}$$

and we compute that

$$\mathrm{Pert}(M_N(\mathbb{C})) \cong \left\{ \begin{pmatrix} 1 & v & iw \\ 0 & C & iD \\ 0 & iE & F \end{pmatrix} : v, w, C, D, E, F \quad \text{real-valued} \right\}$$

- This is compatible with the decomposition $\mathbb{C}^N \otimes \overline{\mathbb{C}^N} \cong \mathbb{C} \oplus \mathbb{C}^{N^2-1}$ into irreps of $U(N)$.
- Similar decompositions can be shown to hold for $\mathrm{Pert}(M_N(\mathbb{R}))$ and irreps of $O(N)$, and $\mathrm{Pert}(M_N(\mathbb{H}))$ and irreps of $Sp(N)$.

Example: perturbation semigroup of a manifold

- The perturbation semigroup can be defined for any involutive unital associative algebra A, in particular for $C^\infty(M)$.
- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two-variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\},$$

Structure of $\mathrm{Pert}(A_F)$

Proposition

Let $\mathcal{U}(A_F)$ be the unitary block diagonal matrices in A_F. This space forms a group which is a subgroup of the semigroup $\mathrm{Pert}(A_F)$ via $U \mapsto U \otimes \overline{U}$.

This is in agreement with the results for matrix algebras, for which

$$\mathcal{U}(M_N(\mathbb{R})) = O(N); \quad \mathcal{U}(M_N(\mathbb{C})) = U(N); \quad \mathcal{U}(M_N(\mathbb{H})) = Sp(N).$$

- Action of $\mathrm{Pert}(A_F)$ on hermitian matrices D_F:

$$D_F \mapsto \sum_j A_j D_F B_j^t$$

- This action is compatible with the semigroup law, since

$$\sum_{j,k} (A_j A'_k) D_F (B_j B'_k)^t = \sum_j A_j \left(\sum_k A'_k D_F (B'_k)^t \right) (B_j)^t$$

- The restriction of this action to the unitary group $\mathcal{U}(A_F)$ gives

$$D \mapsto UDU^*.$$

Perturbations on noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$:

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- We may call ϕ_1 and ϕ_2 the Higgs field.
- Indeed, the group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x}; \qquad (\mu = 1 \ldots, n),$$

where $f \in C^\infty(M \times M)$ is such that $f(x,x) = 1$ and $\overline{f(x,y)} = f(y,x)$.

- In physics, one writes

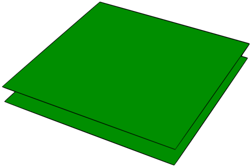
$$A_\mu := \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x}$$

which turns out to be the electromagnetic potential

- Combine (4d) Riemannian spin manifold M with finite noncommutative space F:

$$M \times F$$

- F is internal space at each point of M



- Described by matrix-valued functions on M: algebra $C^\infty(M, A_F)$

Dirac operator on $M \times F$

- Recall the form of D_M:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

- Dirac operator on $M \times F$ is the combination

$$D_{M\times F} = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

- The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F:

$$D_{M\times F}^2 = D_M^2 + D_F^2$$

- Using this, we can expand:

$$\mathrm{Tr}\ e^{-D_{M\times F}^2/\Lambda^2} = \frac{\mathrm{Vol}(M)\Lambda^4}{(4\pi)^2}\mathrm{Tr}\ \left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

The spectral Standard Model

Chamseddine–Connes–Marcolli, 2007

Describe $M \times F_{SM}$

- Algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).
- Dirac operator $D_{M\times F} = D_M + \gamma_5 D_F$ where

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

is a 96×96-dimensional hermitian matrix where 96 is:

$$3 \times 2 \times (\underline{2} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{2} \otimes \underline{3} + \underline{1} \otimes \underline{3} + \underline{1} \otimes \underline{3})$$

3 ↑ families; 2 ↑ anti-particles; $\underline{2} \otimes \underline{1}$: (ν_L, e_L); $\underline{1} \otimes \underline{1}$: ν_R; $\underline{1} \otimes \underline{1}$: e_R; $\underline{2} \otimes \underline{3}$: (u_L, d_L); $\underline{1} \otimes \underline{3}$: u_R; $\underline{1} \otimes \underline{3}$: d_R

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \qquad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

We now find that

- Inner perturbations of $D_M \rightsquigarrow D_M + \gamma^\mu A_\mu$ give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

  corresponding to hypercharge, weak and strong interaction.

- Inner perturbations of D_F give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

  corresponding to SM-Higgs field. Similarly for Y_u, Y_d.

Dynamics and interactions

If we reconsider the spectral action:

$$\mathrm{Tr}\ e^{-D_{M\times F}^2/\Lambda^2} \sim \left(c_4 \Lambda^4 \mathrm{Vol}(M) + c_0 \int F_{\mu\nu} F^{\mu\nu} \right) \left(1 - \frac{|\phi|^2}{\Lambda^2} + \frac{|\phi|^4}{2\Lambda^4} \right) + \cdots$$

we observe [CCM 2007]:

- The coupling constants of hypercharge, weak and strong interaction are expressed in terms of the single constant c_0 which implies

$$\boxed{g_3^2 = g_2^2 = \frac{5}{3} g_1^2}$$

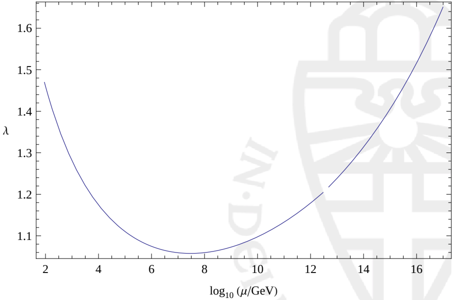
  In other words, there should be grand unification.
- Moreover, the quartic Higgs coupling λ is related via

$$\lambda \approx 24 \frac{3+\rho^4}{(3+\rho^2)^2} g_2^2; \qquad \rho = \frac{m_\nu}{m_{\mathrm{top}}}$$

This can be used to derive predictions as follows:

- Interpret the spectral action as an effective field theory at $\Lambda_{\mathrm{GUT}} \approx 10^{13} - 10^{16}$ GeV.
- Run the quartic coupling constant λ to SM-energies to predict

$$m_h^2 = \frac{4\lambda M_W^2}{3g_2^2}$$

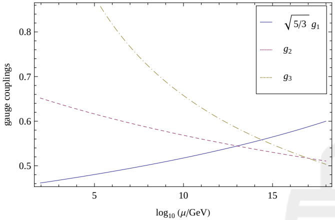


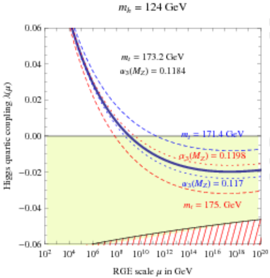
This gives [CCM 2007]

$$\boxed{167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}}$$

Three problems

1. This prediction is falsified by the now measured value.
2. In the Standard Model there is not the presumed grand unification.
3. There is a problem with the low value of m_h, making the Higgs vacuum un/metastable [Elias-Miro et al. 2011].





- The finite algebra of the Standard Model arises naturally as a subalgebra:

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

corresponding to a Pati–Salam unification:

$$U(1)_Y \times SU(2)_L \times SU(3) \to SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 fermionic degrees of freedom are structured as

$$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array} \right) \qquad (i = 1, 2, 3)$$

- Again the finite Dirac operator is a 96×96-dimensional matrix (details in [CCS 2013]).

Inner perturbations

- Inner perturbations of D_M now give three gauge bosons:

$$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

  corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.
- For the inner perturbations of D_F we distinguish two cases, depending on the initial form of D_F:
  - I The Standard Model $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$
  - II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^{b}$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_J^I	1	1	15

II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1 + 15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

Action functional

As for the Standard Model, we can compute the spectral action which describes the usual Pati–Salam model with

- unification of the gauge couplings

$$g_R = g_L = g.$$

- A rather involved, fixed scalar potential, still subject to further study

Phenomenology of the spectral Pati–Salam model
Chamseddine–Connes–vS, 2015

However, independently from the spectral action, we can analyze the running at one loop of the gauge couplings:

1. We run the Standard Model gauge couplings up to a presumed PS $\to$ SM symmetry breaking scale m_R
2. We take their values as boundary conditions to the Pati–Salam gauge couplings g_R, g_L, g at this scale via

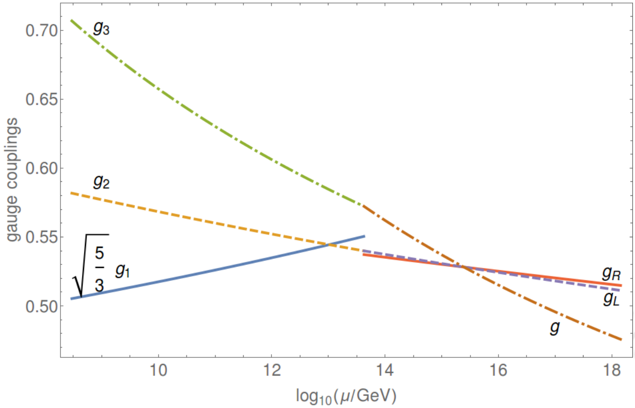
$$\frac{1}{g_1^2} = \frac{2}{3}\frac{1}{g^2} + \frac{1}{g_R^2}, \qquad \frac{1}{g_2^2} = \frac{1}{g_L^2}, \qquad \frac{1}{g_3^2} = \frac{1}{g^2},$$

3. Vary m_R in a search for a unification scale Λ where

$$g_R = g_L = g$$

which is where the spectral action is valid as an effective theory.

For the Standard Model Dirac operator, we have found that with $m_R \approx 4.25 \times 10^{13}\ \mathrm{GeV}$ there is unification at $\Lambda \approx 2.5 \times 10^{15}\ \mathrm{GeV}$:



In this case, we can also say something about the scalar particles that remain after SSB:

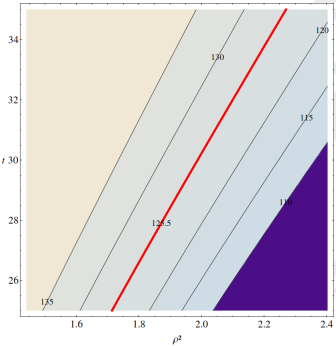
	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_{\dot{1}}^1 \\ \phi_{\dot{1}}^2 \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_{\dot{2}}^1 \\ \phi_{\dot{2}}^2 \end{pmatrix}$	-1	2	1
σ	0	1	1
η	$-\frac{2}{3}$	1	3

- It turns out that these scalar fields have a little influence on the running of the SM-gauge couplings (at one loop).
- This sector contains the real scalar singlet σ of [CC 2012].

- Suppose that the real scalar singlet σ is coupled to the Higgs sector in the following way:

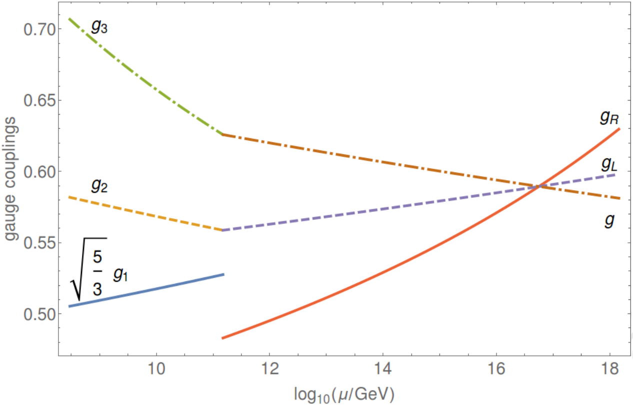
$$V(\sigma,h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2+\sigma^2) + \frac{1}{24}\lambda_h h^4 + \frac{1}{2} h^2\sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4$$

- Instead of the notorious Higgs mass prediction from the nc Standard Model, this real scalar singlet gives a Higgs mass varying with $\rho = m_{\text{top}}/m_\nu$ and the unification scale $t = \log(\Lambda_{\text{GUT}}/M_Z)$



- This allows for $m_h = 125.5\text{GeV}$ and $m_\sigma \sim 10^{12}\text{GeV}$.

For the more general case, we have found that with $m_R \approx 1.5 \times 10^{11}$ GeV there is unification at $\Lambda \approx 6.3 \times 10^{16}$ GeV:



Conclusion

With our walk through the noncommutative garden, we have arrived at a spectral Pati–Salam model that

- goes beyond the Standard Model
- has a fixed scalar sector once the finite Dirac operator has been fixed (only a few scenarios)
- exhibits grand unification for all of these scenarios (confirmed by [Aydemir–Minic–Sun–Takeuchi 2015])
- the scalar sector has the potential to stabilize the Higgs vacuum and allow for a realistic Higgs mass.

Further reading

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Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

Grand Unification in the Spectral Pati-Salam Model. *JHEP* 11 (2015) 011 [arXiv:1507.08161]

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The spectral model of particle physics. Nieuw Archief voor de Wiskunde 5/15 (2014) 240–250.

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and also: `http://www.noncommutativegeometry.nl`