Noncommutative Geometry and Particle Physics

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Preface

The seeds of this book have been planted in the far east, where I wrote lecture notes for international schools in Tianjin, China in 2007 and in Bangkok, Thailand in 2011. I then realized that an up-to-date text for beginning noncommutative geometers on the applications of this rather new mathematical field to particle physics was missing in the literature.

This made me decide to transform my notes into the form of a book. Besides the given challenge inherent in such a project, this was not made easy because of recent, rapid developments in the field, making it difficult to choose what to include and to decide where to stop in my treatment. The current state of affairs is at least touched upon in the final chapter of this book, where I discuss the latest particle physics models in noncommutative geometry, and compare them to the latest experimental findings. With this, I hope to have provided a path that starts with the basic principles of noncommutative geometry and leads to the forefront of research in noncommutative geometry and particle physics.

The intended audience consists of mathematicians with some knowledge of particle physics, and of theoretical physicists with some mathematical background. Concerning the level of this textbook, for mathematicians I assume prerequisites on gauge theories at the level of *e.g.* [**32, 19**], and recommend to first read the book [**72**] to really appreciate the last few chapters of this book on particle physics/the Standard Model. For physicists, I assume knowledge of some basic algebra, Hilbert space and operator theory (*e.g.* [**185**, Chapter 2]), and Riemannian geometry (*e.g.* [**116, 155**]). This makes the book particularly suitable for a starting PhD student, after a master degree in mathematical/theoretical physics including the above background.

I would like to thank the organizers and participants of the aforementioned schools for their involvement and their feedback. This also applies to the MRI-Masterclass in Utrecht in 2010 and the Conference on index theory in Bogotá in 2008, where Chapter 5 finds its roots. Much feedback on previous drafts was gratefully received from students in my class on noncommutative geometry in Nijmegen: Bas Jordans, Joey van der Leer and Sander Uijlen. I thank my students and co-authors Jord Boeijink, Thijs van den Broek and Koen van den Dungen for allowing me to transcribe part of our results in the present book form. Simon Brain, Alan Carey and Adam Rennie are gratefully acknowledged for their feedback and suggested corrections. Strong motivation to writing this book was given to me by my co-author Matilde Marcolli. I thank Gerard Bäuerle, Gianni Landi and Klaas Landsman for having been my main tutors in writing, and Klaas in particular for a careful final proofreading. I also thank Aldo Rampioni at Springer for his help and guidance. I thank Alain Connes for his inspiration and enthousiasm for the field, without whose work this book could of course not have been written.

I am thankful to my family and friends for their continuous love and support. My deepest gratitude goes to Mathilde for being my companion in life, and to Daniël for making sure that the final stages of writing were frequently, and happily, interrupted.

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Contents

CHAPTER 1

Introduction

Ever since the early days of noncommutative geometry it has become clear that this field of mathematics has close ties with physics, and with gauge theories in particular. In fact, non-abelian gauge theories, and even more prominently, the Standard Model of particle physics, were a guiding principle in the formulation of noncommutative manifolds in [**62, 63**].

For one thing, noncommuting operators appear naturally in quantum mechanics. As a matter of fact, there is a rather direct path from experimentally measured atomic spectra to Heisenberg's matrix mechanics which is one of the motivating examples of noncommutative geometry [**60**, Section I.1].

In the other direction, it turns out that the main technical device in noncommutative geometry, a *spectral triple*, naturally gives rise to a gauge theory. This holds in full generality, but the great potential of the noncommutative approach, at least in particle physics, becomes really visible when specific examples are considered that in fact correspond to familiar gauge theories arising in physics. This is crowned by the derivation [**54**] of the full Standard Model of particle physics together with all its subtleties, including the Higgs field, the spontaneous symmetry breaking mechanism, neutrino mixing, see-saw mechanism, *et cetera*.

It is the goal of this book to explore this path, and, starting with the basics, to work towards applications in particle physics, notably to the Standard Model of elementary particles.

The first ingredient of a spectral triple is an *involutive* or $*$-*algebra* $\mathcal{A}$ of operators in a Hilbert space $\mathcal{H}$, with the involution given by the hermitian adjoint of an operator. This immediately gives rise to a *gauge group* $\mathfrak{G}$ determined by the unitary elements in $\mathcal{A}$. In general, if $\mathcal{A}$ is noncommutative, then this group is non-abelian.

The *gauge fields* arise from a second, purely spectral data, in the guise of a *self-adjoint operator* D in $\mathcal{H}$, satisfying suitable conditions (*cf.* Definition 4.30 below). The operator D is modeled on the Dirac operator on a Riemannian spin manifold M, an elliptic first-order differential operator whose square coincides, up to a scalar term, with the Laplacian.

A key role will be played by the spectrum of D, assumed discrete; we will list its eigenvalues (with multiplicities) as $\{\lambda_n\}_{n\in\mathbb{Z}}$. The gauge group $\mathfrak{G}$ acts on D by conjugation with a unitary operator, $D \mapsto UDU^*$. Unitarity guarantees invariance of the spectrum under such a gauge transformation.

Hence a *spectral invariant* is in particular *gauge invariant*, and it is natural to define the so-called *spectral action* as [**49, 50**]

$$\sum_{n\in\mathbb{Z}} f\left(\frac{\lambda_n}{\Lambda}\right).$$

Here the function f is a suitable cutoff function that makes the outcome of the sum finite, and Λ is a real cutoff parameter. The spectral action is interpreted as an action functional that describes the dynamics and interactions of the gauge fields constituting D.

The *fermionic fields* that are associated to a spectral triple are simply vectors ψ in the given Hilbert space, and their natural invariant is the *fermionic action*:

$$(\psi, D\psi).$$

The previous paragraphs sketch the derivation of a generalized gauge theory from any spectral triple. When one restrict to a particular class of spectral triples, this leads to ordinary gauge theory defined on a manifold M in terms of vector bundles and connections. The idea is very simple, essentially dating back to [**59**]: one considers the noncommutative space $M \times F$ given by the product of M with a finite, noncommutative space F. The space F gives rise to the internal, gauge degrees of freedom. In fact, it is described by a finite-dimensional algebra of matrices, for which the gauge group becomes a matrix Lie group, such as $SU(N)$. The self-adjoint operator D_F is given by a hermitian matrix. Combined with the background manifold M, these objects are turned into global ones: $\mathcal{A}$ consists of the sections of a bundle of matrix algebras, and D is a combination of D_F and the Dirac operator on M (assumed to be a Riemannian spin manifold). The operator D is found to be parametrized by gauge fields and scalar fields in suitable representations of the gauge group $\mathfrak{G}$. The fermionic fields ψ are sections of a spinor bundle on which D acts as a linear differential operator, minimally coupled to the gauge fields.

As we already said, the spectral action is manifestly gauge invariant, and for this latter class of examples it describes a scalar gauge theory for the group $\mathfrak{G}$. As a bonus, it is minimally coupled to (Euclidean) gravity, in that the gravitational degrees of freedom are present as a background field in the Dirac operator on M. Moreover, the fermionic action then gives the usual coupling of the fermionic fields to the gauge, scalar and gravitational fields.

In this respect, one of the great achievements of noncommutative geometry is the derivation of the full Standard Model of particle physics from a noncommutative space $M \times F_{SM}$ [**54**]. In fact, from this geometric Ansatz one obtains the Standard Model gauge fields, the scalar Higgs field, and the full fermionic content of the Standard Model. Moreover, the spectral and fermionic action on $M \times F_{SM}$ give the full Lagrangian of the Standard Model, including (amongst other benefits) both the Higgs spontaneous symmetry breaking mechanism and minimal coupling to gravity. In addition, the spectral action introduces relations between the coupling constants and the masses of the Standard Model. This allows one to derive physical predictions such as the Higgs mass, finally bringing us back to experiment.

This book is divided into two parts. Part 1 presents the mathematical basics of noncommutative geometry and discusses the local index formula as a mathematical application. As a stand alone, it may be used as a first introduction to noncommutative geometry.

The second part starts in the same mathematical style, where in the first two chapters we analyze the structure of a gauge theory associated to any spectral triple. Comparable to a kaleidoscope, we then focus on a specific class of examples, and within this class select the physically relevant models. In the last two chapters this culminates in the derivation of the full Standard Model of particle physics. All these examples heavily exploit the results from Part 1. Hence the reader who is already somewhat familiar with noncommutative geometry, but is interested in the gauge-theoretical aspects, may want to skip Part 1 and jump immediately to the second part.

Let us quickly go through the contents of each of the chapters. Chapter 2 and 3 present a 'light' version of noncommutative geometry, restricting ourselves to *finite* noncommutative spaces. In other words, we here only consider finite-dimensional spectral triples and avoid technical complications that arise in the general case. Besides the pedagogical advantage, these finite spaces will in fact turn out to be crucial to the physical applications of the later chapters, where they describe the aforementioned internal space F.

Thus, in Chapter 2 we start with finite discrete topological spaces and replace them by matrix algebras. The question whether this procedure can be reversed leads naturally to the notion of Morita equivalence between matrix algebras. The next step is the translation of a metric structure into a symmetric matrix, motivating the definition of a finite spectral triple. We discuss Morita equivalence for spectral triples and conclude with a diagrammatic classification of finite spectral triples.

In Chapter 3 we enrich finite spectral triples with a real structure and discuss Morita equivalences in this context. We give a classification of finite real spectral triples based on Krajewski diagrams [**127**] and relate this to the classification of irreducible geometries in [**51**].

Chapter 4 introduces spectral triples in full generality. Starting with some background on Riemannian spin geometry, we motivate the general definition of a real spectral triple by the Dirac operator on a compact Riemannian spin manifold.

As a first application of spectral triples, we present a proof of the local index formula of Connes and Moscovici [**66**] in Chapter 5, following Higson's proof [**109**].

In the second part of this book we start to build gauge theories from real spectral triples. Chapter 6 takes a very general approach and associates a gauge group and a set of gauge fields to any real spectral triple. An intriguing localization result can be formulated in terms of a bundle of C^*-algebras on a background topological space. The gauge group acts fiberwise on this bundle and the gauge fields appear as sections thereof.

Maintaining the same level of generality, we introduce gauge invariant quantities in Chapter 7, to wit the spectral action, the topological spectral action (which is closely related to the above index), and the fermionic action [**49, 50**]. We discuss two possible ways to expand the spectral action, either asymptotically in terms of the cutoff Λ, or perturbatively in terms of the gauge fields parametrizing D.

In Chapter 8 we introduce the important class of examples alluded to before, *i.e.* noncommutative spaces of the form $M \times F$ with F finite. Here, Chapters 2 and 3 prove their value in the description of F. Following [**186**] we analyze the structure of the gauge group $\mathfrak{G}_F$ for this class of examples, and determine the gauge fields and scalar fields as well as the corresponding gauge transformations. Using heat kernel methods, we obtain an asymptotic expansion for the spectral action on $M \times F$ in terms of local formulas (on M). We conclude that the spectral action describes the dynamics and interactions of a scalar gauge theory for the group $\mathfrak{G}_F$, minimally coupled to gravity. This general form of the spectral action on $M \times F$ will be heavily used in the remainder of this book.

As a first simple example we treat abelian gauge theory in Chapter 9, for which the gauge group $\mathfrak{G}_F \simeq U(1)$. Following [**187**] we describe how to obtain the Lagrangian of electrodynamics from the spectral action.

The next step is the derivation of non-abelian Yang–Mills gauge theory from noncommutative geometry, which we discuss in Chapter 10. We obtain topologically non-trivial gauge configurations by working with algebra bundles, essentially replacing the above direct product $M \times F$ by a fibered product [**35**].

Chapter 11 contains the derivation of the Standard Model of particle physics from a noncommutative manifold $M \times F_{SM}$, first obtained in [**54**]. We apply our results from Chapter 8 to obtain the Standard Model gauge group and gauge fields, and the scalar Higgs field. Moreover, the computation of the spectral action can be applied to this example and yields the full Lagrangian of the Standard Model, including Higgs spontaneous symmetry breaking and minimally coupled to gravity. We also give a detailed discussion on the fermionic action.

The phenomenology of the noncommutative Standard Model is discussed in Chapter 12. Indeed, the spectral action yields relations between the coupling constants and masses of the Standard Model, from which physical predictions can be derived. Here, we adopt the well-known renormalization group equations of the Standard Model to run the couplings to the relevant energy scale. This gives the notorious prediction for the Higgs mass at the order of 170 GeV. As this is at odds with the experiments at the Large Hadron Collider at CERN, we give a careful analysis of the hypotheses used in the derivation of the Standard Model Lagrangian from noncommutative geometry. We argue that if we drop some of these hypotheses, noncommutative geometry can guide us to go beyond the Standard Model. In particular, we will discuss a recently proposed model [**53, 56, 55**] that enlarges the particle content of the Standard Model by a real scalar singlet. We conclude by showing that this noncommutative model is indeed compatible with the experimentally measured Higgs mass.

In order not to interrupt the text too much, I have chosen to collect background information and references to the literature as ‘Notes’ at the end of each Chapter.

Part 1

Noncommutative geometric spaces

CHAPTER 2

Finite noncommutative spaces

In this chapter (and the next) we consider only finite discrete topological spaces. However, we will stretch their usual definition, which is perhaps geometrically not so interesting, to include the more intriguing finite *noncommutative* spaces. Intuitively, this means that each point has some internal structure, described by a particular noncommutative algebra. With such a notion of finite noncommutative spaces, we search for the appropriate notion of maps between, and (geo)metric structure on such spaces, and arrive at a diagrammatic classification of such finite noncommutative geometric spaces. Our exposition of the finite case already gives a good first impression of what noncommutative geometry has in store, whilst having the advantage that it avoids technical complications that might obscure such a first tour through noncommutative geometry. The general case is subsequently treated in Chapter 4.

2.1. Finite spaces and matrix algebras

Consider a finite topological space X consisting of N points (equipped with the discrete topology):

$$1\bullet \qquad 2\bullet \qquad \cdots\cdots \qquad N\bullet$$

The first step towards a noncommutative geometrical description is to trade spaces for their corresponding function algebras.

DEFINITION 2.1. *A (complex, unital)* algebra *is a vector space A (over $\mathbb{C}$) with a bilinear associative product $A \times A \to A$ denoted by $(a, b) \mapsto ab$ (and a unit 1 satisfying $1a = a1 = a$ for all $a \in A$).*

A $*$-algebra *(or,* involutive algebra*) is an algebra A together with a conjugate-linear map (the* involution*) $* : A \to A$ such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$.*

In this book, we restrict to unital algebras, and simply refer to them as algebras.

In the present case, we consider the $*$-algebra $C(X)$ of $\mathbb{C}$-valued functions on the above finite space X. It is equipped with a pointwise linear structure,

$$(f + g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda(f(x)),$$

for any $f, g \in C(X), \lambda \in \mathbb{C}$ and for any point $x \in X$, and with pointwise multiplication

$$fg(x) = f(x)g(x).$$

There is an involution given by complex conjugation at each point:

$$f^*(x) = \overline{f(x)}.$$

The C in $C(X)$ stands for continuous and, indeed, any $\mathbb{C}$-valued function on a finite space X with the discrete topology is automatically continuous.

The $*$-algebra $C(X)$ has a rather simple structure: it is isomorphic to the $*$-algebra $\mathbb{C}^N$ with each complex entry labeling the value the function takes at the corresponding point, with the involution given by complex conjugation of each entry. A convenient way to encode the algebra $C(X) \simeq \mathbb{C}^N$ is in terms of diagonal $N \times N$ matrices, representing a function $f : X \to \mathbb{C}$ as

$$f \rightsquigarrow \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f(N) \end{pmatrix}.$$

Hence, pointwise multiplication then simply becomes matrix multiplication, and the involution is given by hermitian conjugation.

If $\phi : X_1 \to X_2$ is a map of finite discrete spaces, then there is a corresponding map from $C(X_2) \to C(X_1)$ given by pullback:

$$\phi^* f = f \circ \phi \in C(X_1); \qquad (f \in C(X_2)).$$

Note that the pullback ϕ^* is a $*$-homomorphism (or, $*$-algebra map) under the pointwise product, in that

$$\phi^*(fg) = \phi^*(f)\phi^*(g), \qquad \phi^*(\overline{f}) = \overline{\phi^*(f)}, \qquad \phi^*(\lambda f + g) = \lambda\phi^*(f) + \phi^*(g).$$

For example, let X_1 be the space consisting of three points, and X_2 the space consisting of two points. If a map $\phi : X_1 \to X_2$ is defined according to the following diagram,

$$\begin{tikzcd} X_1 : & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[dl] \\ X_2 : & \bullet & \bullet & \end{tikzcd}$$

then

$$\phi^* : \mathbb{C}^2 \simeq C(X_2) \to \mathbb{C}^3 \simeq C(X_1)$$

is given by

$$(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2, \lambda_2).$$

Exercise 2.1. *Show that $\phi : X_1 \to X_2$ is an injective (surjective) map of finite spaces if and only if $\phi^* : C(X_2) \to C(X_1)$ is surjective (injective).*

Definition 2.2. *A (complex)* matrix algebra *A is a direct sum*

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some positive integers n_i and N. The involution on A is given by hermitian conjugation, and we simply refer to the $*$-algebra A with this involution as a matrix algebra.

Hence, we have associated a matrix algebra $C(X)$ to the finite space X, which behaves naturally with respect to maps between topological spaces and $*$-algebras. A natural question is whether this procedure can be inverted. In other words, given a matrix algebra A, can we obtain a finite discrete space X such that $A \simeq C(X)$? Since $C(X)$ is always commutative but matrix algebras need not be, we quickly arrive at the conclusion that the answer is negative. This can be resolved in two ways:

(1) Restrict to *commutative* matrix algebras.
(2) Allow for more morphisms (and consequently, more isomorphisms) between matrix algebras, *e.g.* by generalizing $*$-homomorphisms.

Before explaining each of these options, let us introduce some useful definitions concerning representations of finite-dimensional $*$-algebras (which are not necessarily commutative) which moreover extend in a straightforward manner to the infinite-dimensional case (*cf.* Definitions 4.26 and 4.27). We first need the prototypical example of a $*$-algebra.

EXAMPLE 2.3. *Let H be an (finite-dimensional) inner product space, with inner product $(\cdot,\cdot) \to \mathbb{C}$. We denote by $L(H)$ the* $*$-algebra of operators *on H with product given by composition and the involution is given by mapping an operator T to its adjoint T^*.*

Note that $L(H)$ is a normed vector space: for $T \in L(H)$ we set

$$\|T\| = \sup_{h\in H}\{(Th, Th) : (h, h) \leq 1\}.$$

Equivalently, $\|T\|$ is given by the square root of the largest eigenvalue of T^*T.

DEFINITION 2.4. *A* representation *of a finite-dimensional $*$-algebra A is a pair (H, π) where H is a (finite-dimensional, complex) inner product space and π is a $*$-algebra map*

$$\pi : A \to L(H).$$

A representation (H, π) is called irreducible *if $H \neq 0$ and the only subspaces in H that are left invariant under the action of A are $\{0\}$ or H.*

We will also refer to a finite-dimensional inner product space as a *finite-dimensional Hilbert space.*

EXAMPLE 2.5. *Consider $A = M_n(\mathbb{C})$. The defining representation is given by $H = \mathbb{C}^n$ on which A acts by left matrix multiplication; hence it is irreducible. An example of a reducible representation is $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in M_n(\mathbb{C})$ acting in block-form:*

$$a \in M_n(\mathbb{C}) \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in L(\mathbb{C}^n \oplus \mathbb{C}^n) \simeq M_{2n}(\mathbb{C})$$

which therefore decomposes as the direct sum of two copies of the defining representation. See also Lemma 2.15 below.

EXERCISE 2.2. *Given a representation (H, π) of a $*$-algebra A, the* **commutant** *$\pi(A)'$ of $\pi(A)$ is defined as*

$$\pi(A)' = \{T \in L(H) : \pi(a)T = T\pi(a) \text{ for all } a \in A\}.$$

(1) Show that $\pi(A)'$ is also a $*$-algebra.
(2) Show that a representation (H, π) of A is irreducible if and only if the commutant $\pi(A)'$ of $\pi(A)$ consists of multiples of the identity.

Definition 2.6. *Two representations (H_1, π_1) and (H_2, π_2) of a $*$-algebra A are* unitarily equivalent *if there exists a unitary map $U : H_1 \to H_2$ such that*

$$\pi_1(a) = U^* \pi_2(a) U.$$

Definition 2.7. *The* structure space $\widehat{A}$ *of A is the set of all unitary equivalence classes of irreducible representations of A.*

We end this section with an illustrative exercise on passing from representations of a $*$-algebra to matrices over that $*$-algebra.

Exercise 2.3. *(1) If A is a unital $*$-algebra, show that the $n \times n$-matrices $M_n(A)$ with entries in A form a unital $*$-algebra.*
(2) Let $\pi : A \to L(H)$ be a representation of a $*$-algebra A and set $H^n = H \oplus \cdots \oplus H$ (n copies). Show that the following defines a representation $\widetilde{\pi} : M_n(A) \to L(H^n)$ of $M_n(A)$:

$$\widetilde{\pi}\left((a_{ij})\right) = \left(\pi(a_{ij})\right); \qquad ((a_{ij}) \in M_n(A)).$$

(3) Let $\widetilde{\pi} : M_n(A) \to L(H^n)$ be a representation of the $*$-algebra $M_n(A)$. Show that the following defines a representation $\pi : A \to L(H^n)$ of the $*$-algebra A:

$$\pi(a) = \widetilde{\pi}\left(a \mathbb{I}_n\right)$$

where $\mathbb{I}_n$ is the identity in $M_n(A)$.

2.1.1. Commutative matrix algebras. We now explain how option (1) on page 9 above resolves the question raised by constructing a space from a commutative matrix algebra A. A natural candidate for such a space is, of course, the structure space $\widehat{A}$, which we now determine. Note that any commutative matrix algebra is of the form $A \simeq \mathbb{C}^N$, for which by Exercise 2.2(2) any irreducible representation is given by a map of the form

$$\pi_i : (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C}$$

for some $i = 1, \ldots, N$. We conclude that $\widehat{A} \simeq \{1, \ldots, N\}$.

We conclude that there is a *duality* between finite spaces and commutative matrix algebras. This is nothing but a finite-dimensional version of *Gelfand duality* (see Theorem 4.28 below) between compact Hausdorff topological spaces and unital commutative C^*-algebras. In fact, we will see later (Proposition 4.25) that any finite-dimensional C^*-algebra is a matrix algebra, which reduces Gelfand duality to the present finite-dimensional duality.

2.1.2. Finite spaces and matrix algebras. The above trade of finite discrete spaces for finite-dimensional commutative $*$-algebras does not really make them any more interesting, for the $*$-algebra is always of the form $\mathbb{C}^N$. A more interesting perspective is given by the noncommutative alternative, *viz.* option (2) on page 9. We thus aim for a duality between finite spaces and *equivalence classes* of matrix algebras. These equivalence classes are described by a generalized notion of isomorphisms between matrix algebras, also known as Morita equivalence.

Let us first recall the notion of an algebra (bi)module.

DEFINITION 2.8. *Let A, B be algebras (not necessarily matrix algebras). A* left A-module *is a vector space E that carries a left representation of A, i.e. there is a bilinear map $A \times E \ni (a, e) \mapsto a \cdot e \in E$ such that*

$$(a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \qquad (a_1, a_2 \in A, e \in E).$$

Similarly, a right B-module *is a vector space F that carries a right representation of B, i.e. there is a bilinear map $F \times B \ni (f, b) \mapsto f \cdot b \in F$ such that*

$$f \cdot (b_1 b_2) = (f \cdot b_1) \cdot b_2; \qquad (b_1, b_2 \in B, f \in F).$$

*Finally, an $A-B$-*bimodule *E is both a left A-module and a right B-module, with mutually commuting actions:*

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \qquad (a \in A, b \in B, e \in E).$$

When no confusion can arise, we will also write ae instead of $a \cdot e$ to denote the left module action.

There is a natural notion of (left) **A-module homomorphism** as a linear map $\phi : E \to F$ that respect the representation of A:

$$\phi(a \cdot e) = a \cdot \phi(e); \qquad (a \in A, e \in E).$$

Similarly for right modules and bimodules.

We introduce the following notation:

- ${}_A E$ for a left A-module E;
- F_B for a right B-module F;
- ${}_A E_B$ for an $A - B$-bimodule E.

EXERCISE 2.4. *Check that a representation $\pi : A \to L(H)$ of a $*$-algebra A (cf. Defn. 2.4) turns H into a left A-module ${}_A H$.*

EXERCISE 2.5. *Show that A is itself an $A - A$-bimodule ${}_A A_A$, with left and right actions given by the product in A.*

If E is a right A-module, and F is a left A-module, we can form the *balanced tensor product*:

$$E \otimes_A F := E \otimes F / \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A, e_i \in E, f_i \in F \right\}.$$

In other words, the quotient imposes A-linearity of the tensor product, *i.e.* in $E \otimes_A F$ we have

$$ea \otimes_A f = e \otimes_A af; \qquad (a \in A, e \in E, f \in F).$$

DEFINITION 2.9. *Let A, B be matrix algebras. A* Hilbert bimodule *for the pair (A, B) is given by an $A - B$-bimodule E together with a B-valued inner product $\langle \cdot, \cdot \rangle_E : E \times E \to B$ satisfying*

$$\langle e_1, a \cdot e_2 \rangle_E = \langle a^* \cdot e_1, e_2 \rangle_E; \qquad (e_1, e_2 \in E, a \in A),$$
$$\langle e_1, e_2 \cdot b \rangle_E = \langle e_1, e_2 \rangle_E b; \qquad \langle e_1, e_2 \rangle_E^* = \langle e_2, e_1 \rangle_E; \qquad (e_1, e_2 \in E, b \in B),$$
$$\langle e, e \rangle_E \geq 0 \text{ with equality if and only if } e = 0; \qquad (e \in E).$$

The set of Hilbert bimodules for (A, B) will be denoted by $\mathsf{KK}_f(A, B)$.

In the following, we will also write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_E$, unless confusion might arise.

EXERCISE 2.6. *Check that a representation $\pi : A \to L(H)$ (cf. Defn. 2.4 and Exc. 2.4) of a matrix algebra A turns H into a Hilbert bimodule for $(A, \mathbb{C})$.*

EXERCISE 2.7. *Show that the $A-A$-bimodule given by A itself (cf. Exc. 2.5) is an element in $\mathsf{KK}_f(A, A)$ by establishing that the following formula defines an A-valued inner product $\langle \cdot, \cdot \rangle_A : A \times A \to A$:*

$$\langle a, a' \rangle_A = a^* a'; \qquad (a, a' \in A).$$

EXAMPLE 2.10. *More generally, let $\phi : A \to B$ be a $*$-algebra homomorphism between matrix algebras A and B. From it, we can construct a Hilbert bimodule E_ϕ in $\mathsf{KK}_f(A, B)$ as follows. Let E_ϕ be B as a vector space with the natural right B-module structure and inner product (cf. Exc. 2.7), but with A acting on the left via the homomorphism ϕ:*

$$a \cdot b = \phi(a) b; \qquad (a \in A, b \in E_\phi).$$

DEFINITION 2.11. *The* Kasparov product *$F \circ E$ between Hilbert bimodules $E \in \mathsf{KK}_f(A, B)$ and $F \in \mathsf{KK}_f(B, C)$ is given by the balanced tensor product*

$$F \circ E := E \otimes_B F; \qquad (E \in \mathsf{KK}_f(A, B), F \in \mathsf{KK}_f(B, C)),$$

so that $F \circ E \in \mathsf{KK}_f(A, C)$, with C-valued inner product given on elementary tensors by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F, \tag{2.1.1}$$

and extended linearly to all of $E \otimes F$.

Note that this product is associative up to isomorphism.

EXERCISE 2.8. *Show that the association $\phi \rightsquigarrow E_\phi$ from Example 2.10 is* natural *in the sense that*

(1) $E_{\mathrm{id}_A} \simeq A \in \mathsf{KK}_f(A, A)$,
(2) for $*$-algebra homomorphisms $\phi : A \to B$ and $\psi : B \to C$ we have an isomorphism

$$E_\psi \circ E_\phi \equiv E_\phi \otimes_B E_\psi \simeq E_{\psi \circ \phi} \in \mathsf{KK}_f(A, C),$$

that is, as $A - C$-bimodules.

EXERCISE 2.9. . *In the above definition:*

(1) Check that $E \otimes_B F$ is an $A - C$-bimodule.
(2) Check that $\langle \cdot, \cdot \rangle_{E \otimes_B F}$ defines a C-valued inner product.
(3) Check that $\langle a^*(e_1 \otimes f_1), e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle e_1 \otimes f_1, a(e_2 \otimes f_2) \rangle_{E \otimes_B F}$.

Conclude that $F \circ E$ is indeed an element of $\mathsf{KK}_f(A, C)$.

Let us consider the Kasparov product with the Hilbert bimodule for (A, A) given by A itself (*cf.* Exercise 2.7). Then, since for $E \in \mathsf{KK}_f(A, B)$ we have $E \circ A = A \otimes_A E \simeq E$, the bimodule ${}_A A_A$ is the identity element with respect to the Kasparov product (up to isomorphism). This motivates the following definition.

DEFINITION 2.12. *Two matrix algebras A and B are called* Morita equivalent *if there exist elements $E \in \mathsf{KK}_f(A,B)$ and $F \in \mathsf{KK}_f(B,A)$ such that*

$$E \otimes_B F \simeq A, \qquad F \otimes_A E \simeq B,$$

where $\simeq$ denotes isomorphism as Hilbert bimodules.

If A and B are Morita equivalent, then the representation theories of both matrix algebras are equivalent. More precisely, if A and B are Morita equivalent, then a right A-module is sent to a right B-module by tensoring with $\_ \otimes_A E$ for an invertible element E in $\mathsf{KK}_f(A,B)$.

EXAMPLE 2.13. *As seen in Exercises 2.4 and 2.6, the vector space $E = \mathbb{C}^n$ is an $M_n(\mathbb{C}) - \mathbb{C}$-bimodule; with the standard $\mathbb{C}$-valued inner product it becomes a Hilbert module for $(M_n(\mathbb{C}), \mathbb{C})$. Similarly, the vector space $F = \mathbb{C}^n$ is a $\mathbb{C} - M_n(\mathbb{C})$-bimodule by right matrix multiplication. An $M_n(\mathbb{C})$-valued inner product is given by*

$$\langle v_1, v_2 \rangle = \overline{v}_1 v_2^t \in M_n(\mathbb{C}).$$

We determine the Kasparov products of these Hilbert bimodules as

$$E \otimes_{\mathbb{C}} F \simeq M_n(\mathbb{C}); \qquad F \otimes_{M_n(\mathbb{C})} E \simeq \mathbb{C}.$$

In other words, $E \in \mathsf{KK}_f(M_n(\mathbb{C}), \mathbb{C})$ and $F \in \mathsf{KK}_f(\mathbb{C}, M_n(\mathbb{C}))$ are each other's inverse with respect to the Kasparov product. We conclude that $M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent.

This observation leads us to our first little result.

THEOREM 2.14. *Two matrix algebras are Morita equivalent if and only if their structure spaces are isomorphic as finite discrete spaces, i.e. have the same cardinality.*

PROOF. Let A and B be Morita equivalent. Thus there exists Hilbert bimodules ${}_A E_B$ and ${}_B F_A$ such that

$$E \otimes_B F \simeq A, \qquad F \otimes_A E \simeq B.$$

If $[(\pi_B, H)] \in \widehat{B}$ then we can define a representation π_A by setting

$$\pi_A : A \to L(E \otimes_B H); \quad \pi_A(a)(e \otimes v) = ae \otimes v. \tag{2.1.2}$$

Vice versa, we construct $\pi_B : B \to L(F \otimes_A W)$ from $[(\pi_A, W)] \in \widehat{A}$ by setting $\pi_B(b)(f \otimes w) = bf \otimes w$ and these two maps are one another's inverse. Thus, $\widehat{A} \simeq \widehat{B}$ (see Exercise 2.10 below).

For the converse, we start with a basic result on irreducible representations of $M_n(\mathbb{C})$.

LEMMA 2.15. *The matrix algebra $M_n(\mathbb{C})$ has a unique irreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.*

PROOF. It is clear from Exercise 2.2 that $\mathbb{C}^n$ is an irreducible representation of $A = M_n(\mathbb{C})$. Suppose H is irreducible and of dimension K, and define a linear map

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{K \text{ copies}} \to H^*; \qquad \phi(a_1, \ldots, a_K) \to e^1 \circ a_1^t + \cdots + e^K \circ a_K^t$$

in terms of a basis $\{e^1, \ldots e^K\}$ of the dual vector space H^*. Here $v \circ a$ denotes pre-composition of $v \in H^*$ with $a \in A$, acting on H. This is a morphism of $M_n(\mathbb{C})$-modules, provided a matrix a acts on the dual vector space H^* by sending $v \mapsto v \circ a^t$. It is also surjective, so that the dual map $\phi^* : H \to (A^K)^*$ is injective. Upon identifying $(A^K)^*$ with A^K as A-modules, and noting that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as A-modules, it follows that H is a submodule of $A^K \simeq \oplus^{nK} \mathbb{C}^n$. By irreducibility $H \simeq \mathbb{C}^n$. □

Now, if A, B are matrix algebras of the following form

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \qquad B = \bigoplus_{j=1}^{M} M_{m_j}(\mathbb{C}),$$

then $\widehat{A} \simeq \widehat{B}$ implies that $N = M$. Then, define

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i},$$

with A acting by block-diagonal matrices on the first tensor and B acting in a similar way by right matrix multiplication on the second leg of the tensor product. Also, set

$$F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i},$$

with B now acting on the left and A on the right. Then, as above,

$$\begin{aligned} E \otimes_B F &\simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left(\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i} \right) \otimes \mathbb{C}^{n_i} \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i} \simeq A, \end{aligned}$$

and similarly we obtain $F \otimes_A E \simeq B$, as required. □

EXERCISE 2.10. *Fill in the gaps in the above proof:*

(a) Show that the representation π_A defined by (2.1.2) *is irreducible if and only if π_B is.*

(b) Show that the association of the class $[\pi_A]$ to $[\pi_B]$ through (2.1.2) *is independent of the choice of representatives π_A and π_B.*

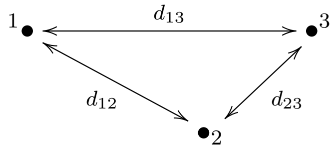
We conclude that there is a duality between finite spaces and Morita equivalence classes of matrix algebras. By replacing $*$-homomorphisms $A \to B$ by Hilbert bimodules for (A, B), we introduce a much richer structure at the level of morphisms between matrix algebras. For example, any finite-dimensional inner product space defines an element in $\mathsf{KK}_f(\mathbb{C}, \mathbb{C})$, whereas there is only one map from the corresponding structure space consisting of one point to itself. When combined with Exercise 2.10 we conclude that

Hilbert bimodules form a proper extension of the $*$-morphisms between matrix algebras.

2.2. Noncommutative geometric finite spaces

Consider again a finite space X, described as the structure space of a matrix algebra A. We would like to introduce some geometry on X and, in particular, a notion of a metric on X.

Thus, the question we want to address is how we can (algebraically) describe distances between the points in X, say, as embedded in a metric space. Recall that a metric on a finite discrete space X is given by an array $\{d_{ij}\}_{i,j\in X}$ of real non-negative entries, indexed by a pair of elements in X and requiring that $d_{ij} = d_{ji}$, $d_{ij} \leq d_{ik} + d_{kj}$, and $d_{ij} = 0$ if and only if $i = j$:



Example 2.16. *If X is embedded in a metric space (e.g. Euclidean space), it can be equipped with the induced metric.*

Example 2.17. *The* **discrete metric** *on the discrete space X is given by:*

$$d_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

In the commutative case, we have the following remarkable result, which completely characterizes the metric on X in terms of linear algebraic data. It is the key result towards a *spectral* description of finite geometric spaces.

Theorem 2.18. *Let d_{ij} be a metric on the space X of N points, and set $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$, so that $\widehat{A} \simeq X$. Then there exists a representation π of A on a finite-dimensional inner product space H and a symmetric operator D on H such that*

$$d_{ij} = \sup_{a\in A} \{|a(i) - a(j)| : \|[D, \pi(a)]\| \leq 1\}. \tag{2.2.1}$$

Proof. We claim that this would follow from the equality

$$\|[D, \pi(a)]\| = \max_{k\neq l} \left\{ \frac{1}{d_{kl}} |a(k) - a(l)| \right\}. \tag{$*$}$$

Indeed, if this holds, then

$$\sup_a \{|a(i) - a(j)| : \|[D, a]\| \leq 1\} \leq d_{ij}.$$

The reverse inequality follows by taking $a \in A$ for fixed i, j to be $a(k) = d_{ik}$. Then, we find $|a(i) - a(j)| = d_{ij}$, while $\|[D, \pi(a)]\| \leq 1$ for this a follows from the reverse triangle inequality for d_{ij}:

$$\frac{1}{d_{kl}} |a(k) - a(l)| = \frac{1}{d_{kl}} |d_{ik} - d_{il}| \leq 1.$$

We prove $(*)$ by induction on N. If $N = 2$, then on $H = \mathbb{C}^2$ we define a representation $\pi : A \to L(H)$ and a hermitian matrix D by

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & (d_{12})^{-1} \\ (d_{12})^{-1} & 0 \end{pmatrix}.$$

It follows that $\|[D, a]\| = (d_{12})^{-1}|a(1) - a(2)|$.

Suppose then that $(*)$ holds for N, with representation π_N of $\mathbb{C}^N$ on an inner product space H_N and symmetric operator D_N; we will show that it also holds for $N + 1$. We define

$$H_{N+1} = H_N \oplus \bigoplus_{i=1}^{N} H_N^i$$

with $H_N^i := \mathbb{C}^2$. Imitating the above construction in the case $N = 2$, we define the representation π_{N+1} by

$$\begin{aligned} \pi_{N+1}(a(1), \dots, a(N+1)) &= \pi_N(a(1), \dots, a(N)) \\ &\oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a(N) & 0 \\ 0 & a(N+1) \end{pmatrix}, \end{aligned}$$

and define the operator D_{N+1} by

$$\begin{aligned} D_{N+1} = D_N &\oplus \begin{pmatrix} 0 & (d_{1(N+1)})^{-1} \\ (d_{1(N+1)})^{-1} & 0 \end{pmatrix} \\ &\oplus \cdots \oplus \begin{pmatrix} 0 & (d_{N(N+1)})^{-1} \\ (d_{N(N+1)})^{-1} & 0 \end{pmatrix}. \end{aligned}$$

It follows by the induction hypothesis that $(*)$ holds for $N + 1$. □

EXERCISE 2.11. *Make the above proof explicit for the case $N = 3$. In other words, compute the metric of* (2.2.1) *on the space of three points from the set of data $A = \mathbb{C}^3$, $H = (\mathbb{C}^2)^{\oplus 3}$ with representation $\pi : A \to L(H)$ given by*

$$\pi(a(1), a(2), a(3)) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & \\ & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & \\ & a(3) \end{pmatrix},$$

and hermitian matrix

$$D = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix},$$

with $x_1, x_2, x_3 \in \mathbb{R}$.

EXERCISE 2.12. *Compute the metric on the space of three points given by formula* (2.2.1) *for the set of data $A = \mathbb{C}^3$ acting in the defining representation on $H = \mathbb{C}^3$, and*

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for some non-zero $d \in \mathbb{R}$.

Even though the above translation of the metric on X into algebraic data assumes commutativity of A, the distance formula itself can be extended to the case of a noncommutative matrix algebra A. In fact, suppose we are given a $*$-algebra representation of A on an inner product space, together with a symmetric operator D on H. Then we can define a metric on the structure space $\widehat{A}$ by

$$d_{ij} = \sup_{a \in A} \{ |\operatorname{Tr} a(i) - \operatorname{Tr} a(j)| : \|[D, a]\| \leq 1 \}, \tag{2.2.2}$$

where i labels the matrix algebra $M_{n_i}(\mathbb{C})$ in the decomposition of A. This distance formula is a special case of Connes' distance formula (see Note 12 on Page 60) on the structure space of A.

Exercise 2.13. *Show that the d_{ij} in (2.2.2) is a metric (actually, an* **extended metric**, *taking values in $[0, \infty]$) on $\widehat{A}$ by establishing that*

$$d_{ij} = 0 \iff i = j, \qquad d_{ij} = d_{ji}, \qquad d_{ij} \leq d_{ik} + d_{kj}.$$

This suggests that the above structure consisting of a matrix algebra A, a finite-dimensional representation space H, and a hermitian matrix D provides the data needed to capture a metric structure on the finite space $X = \widehat{A}$. In fact, in the case that A is commutative, the above argument combined with our finite-dimensional Gelfand duality of Section 2.1.1 is a reconstruction theorem. Indeed, we reconstruct a given metric space (X, d) from the data (A, H, D) associated to it.

We arrive at the following definition, adapted to our finite-dimensional setting.

Definition 2.19. *A* finite spectral triple *is a triple (A, H, D) consisting of a unital $*$-algebra A represented faithfully on a finite-dimensional Hilbert space H, together with a symmetric operator $D : H \to H$.*

We do not demand that A is a matrix algebra, since this turns out to be automatic:

Lemma 2.20. *If A is a unital $*$-algebra that acts faithfully on a finite-dimensional Hilbert space, then A is a matrix algebra of the form*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

Proof. Since A acts faithfully on a Hilbert space it is a $*$-subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C})$; the only such subalgebras are themselves matrix algebras. □

Unless we want to distinguish different representations of A on H, the above representation will usually be implicitly assumed, thus considering elements $a \in A$ as operators on H.

Example 2.21. *Let $A = M_n(\mathbb{C})$ act on $H = \mathbb{C}^n$ by matrix multiplication, with the standard inner product. A symmetric operator on H is represented by a hermitian $n \times n$ matrix.*

We will loosely refer to D as a **finite Dirac operator**, as its infinite-dimensional analogue on Riemannian spin manifolds is the usual Dirac operator (see Chapter 4). In the present case, we can use it to introduce a 'differential geometric structure' on the finite space X that is related to the notion of **divided difference**. The latter is given, for each pair of points $i, j \in X$, by

$$\frac{a(i) - a(j)}{d_{ij}}.$$

Indeed, these divided differences appear precisely as the entries of the commutator $[D, a]$ for the operator D as in Theorem 2.18.

EXERCISE 2.14. *Use the explicit form of D in Theorem 2.18 to confirm that the commutator of D with $a \in C(X)$ is expressed in terms of the above divided differences.*

We will see later that in the continuum case, the commutator $[D, \cdot]$ corresponds to taking derivatives of functions on a manifold.

DEFINITION 2.22. *Let (A, H, D) be a finite spectral triple. The A-bimodule of* Connes' differential one-forms *is given by*

$$\Omega^1_D(A) := \left\{ \sum_k a_k[D, b_k] : a_k, b_k \in A \right\}.$$

Consequently, there is a map $\mathrm{d} : A \to \Omega^1(A)$, given by $\mathrm{d}(\cdot) = [D, \cdot]$.

EXERCISE 2.15. *Verify that* d *is a derivation of a $*$-algebra, in that:*

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b); \qquad \mathrm{d}(a^*) = -\mathrm{d}(a)^*.$$

EXERCISE 2.16. *Verify that $\Omega^1_D(A)$ is an A-bimodule by rewriting the operator $a(a_k[D, b_k])b$ $(a, b, a_k, b_k \in A)$ as $\sum_k a'_k[D, b'_k]$ for some $a'_k, b'_k \in A$.*

As a first little result —though with an actual application to matrix models in physics— we compute Connes' differential one-forms for the above Example 2.21.

LEMMA 2.23. *Let $(A, H, D) = (M_n(\mathbb{C}), \mathbb{C}^n, D)$ be the finite spectral triple of Example 2.21 with D a hermitian $n \times n$ matrix. If D is not a multiple of the identity, then $\Omega^1_D(A) \simeq M_n(\mathbb{C})$.*

PROOF. We may assume that D is a diagonal matrix: $D = \sum_i \lambda_i e_{ii}$ in terms of real numbers λ_i (not all equal) and the standard basis $\{e_{ij}\}$ of $M_n(\mathbb{C})$. For fixed i, j choose k such that $\lambda_k \neq \lambda_j$. Then

$$\left(\frac{1}{\lambda_k - \lambda_j} e_{ik} \right) [D, e_{kj}] = e_{ij}.$$

Hence, since $e_{ik}, e_{kj} \in M_n(\mathbb{C})$, any basis vector $e_{ij} \in \Omega^1_D(A)$. Since also $\Omega^1_D(A) \subset L(\mathbb{C}^n) \simeq M_n(\mathbb{C})$, the result follows. $\square$

EXERCISE 2.17. *Consider the following finite spectral triple:*

$$\left(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \right),$$

with $\lambda \neq 0$. Show that the corresponding space of differential one-forms $\Omega^1_D(A)$ is isomorphic to the vector space of all off-diagonal 2×2 matrices.

2.2.1. Morphisms between finite spectral triples. In a spectral triple (A, H, D) both the $*$-algebra A and a finite Dirac operator D act on the inner product space H. Hence, the most natural notion of equivalence between spectral triples is that of unitary equivalence.

Definition 2.24. *Two finite spectral triples* (A_1, H_1, D_1) *and* (A_2, H_2, D_2) *are called* unitarily equivalent *if* $A_1 = A_2$ *and if there exists a unitary operator* $U : H_1 \to H_2$ *such that*

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a); \qquad (a \in A_1), \\ UD_1U^* &= D_2. \end{aligned}$$

Exercise 2.18. *Show that unitary equivalence of spectral triples is an equivalence relation.*

Remark 2.25. *A special type of unitary equivalence is given by the unitaries in the matrix algebra* A *itself. Indeed, for any such unitary element* u *the spectral triples* (A, H, D) *and* (A, H, uDu^*) *are unitarily equivalent. Another way of writing* uDu^* *is* $D + u[D, u^*]$*, so that this type of unitary equivalence effectively adds a differential one-form to* D*.*

Following the spirit of our extended notion of morphisms between algebras, we might also deduce a notion of "equivalence" coming from Morita equivalence of the corresponding matrix algebras. Namely, given a Hilbert bimodule E in $\mathsf{KK}_f(B, A)$, we can try to construct a finite spectral triple on B starting from a finite spectral triple on A. This transfer of metric structure is accomplished as follows. Let (A, H, D) be a spectral triple; we construct a new spectral triple (B, H', D'). First, we define a vector space

$$H' = E \otimes_A H,$$

which inherits a left action of B from the B-module structure of E. Also, it is an inner product space, with $\mathbb{C}$-valued inner product given as in (2.1.1).

The naive choice of a symmetric operator D' given by $D'(e \otimes \xi) = e \otimes D\xi$ will not do, because it does not respect the ideal defining the balanced tensor product over A, being generated by elements of the form

$$ea \otimes \xi - e \otimes a\xi; \qquad (e \in E, a \in A, \xi \in H).$$

A better definition is

$$D'(e \otimes \xi) = e \otimes D\xi + \nabla(e)\xi, \tag{2.2.3}$$

where $\nabla : E \to E \otimes_A \Omega^1_D(A)$ is some map that satisfies the **Leibniz rule**

$$\nabla(ea) = \nabla(e)a + e \otimes [D, a]; \qquad (e \in E, a \in A). \tag{2.2.4}$$

Indeed, this is precisely the property that is needed to make D' a well-defined operator on the balanced tensor product $E \otimes_A H$:

$$D'(ea \otimes \xi - e \otimes a\xi) = ea \otimes D\xi + \nabla(ea)\xi - e \otimes D(a\xi) - \nabla(e)a\xi = 0.$$

A map $\nabla : E \to E \otimes_A \Omega^1_D(A)$ that satisfies Equation (2.2.4) is called a **connection** on the right A-module E associated to the derivation $\mathrm{d} : a \mapsto [D, a]$ $(a \in A)$.

THEOREM 2.26. *If (A,H,D) is a finite spectral triple and $E \in \mathsf{KK}_f(B,A)$, then (in the above notation) $(B, E\otimes_A H, D')$ is a finite spectral triple, provided that ∇ satisfies the compatibility condition*

$$\langle e_1, \nabla e_2\rangle_E - \langle \nabla e_1, e_2\rangle_E = \mathrm{d}\langle e_1, e_2\rangle_E; \qquad (e_1, e_2 \in E). \tag{2.2.5}$$

PROOF. We only need to show that D' is a symmetric operator. Indeed, for $e_1, e_2 \in E$ and $\xi_1, \xi_2 \in H$ we compute

$$\begin{aligned}\langle e_1\otimes\xi_1, D'(e_2\otimes\xi_2)\rangle_{E\otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2\rangle_E \xi_2\rangle_H + \langle \xi_1, \langle e_1, e_2\rangle_E D\xi_2\rangle_H \\ &= \langle \xi_1, \langle \nabla e_1, e_2\rangle_E\xi_2\rangle_H + \langle \xi_1, \mathrm{d}\langle e_1, e_2\rangle_E \xi_2\rangle_H \\ &\quad + \langle D\xi_1, \langle e_1, e_2\rangle_E\xi_2\rangle_H - \langle \xi_1, [D, \langle e_1, e_2\rangle_E]\xi_2\rangle_H \\ &= \langle D'(e_1\otimes\xi_1), e_2\otimes\xi_2\rangle_{E\otimes_A H},\end{aligned}$$

using the stated compatibility condition and the fact that D is symmetric. □

Theorem 2.26 is our finite-dimensional analogue of Theorem 6.15, to be obtained below.

EXERCISE 2.19. *Let ∇ and ∇' be two connections on a right A-module E. Show that their difference $\nabla - \nabla'$ is a right A-linear map $E \to E\otimes_A \Omega^1_D(A)$.*

EXERCISE 2.20. *In this exercise, we consider the case that $B = A$ and also $E = A$. Let (A,H,D) be a spectral triple, we determine (A, H', D').*

(1) Show that the derivation $\mathrm{d}(\cdot) = [D,\cdot] : A \to A\otimes_A \Omega^1_D(A) = \Omega^1_D(A)$ is a connection on A considered a right A-module.
(2) Upon identifying $A\otimes_A H \simeq H$, what is the operator D' of Equation (2.2.3) when the connection ∇ on A is given by d as in (1)?
(3) Use (1) and (2) of this exercise to show that any connection $\nabla : A \to A\otimes_A \Omega^1_D(A)$ is given by

$$\nabla = \mathrm{d} + \omega,$$

with $\omega \in \Omega^1_D(A)$.
(4) Upon identifying $A\otimes_A H \simeq H$, what is the operator D' of Equation (2.2.3) with the connection on A given as $\nabla = \mathrm{d} + \omega$.

If we combine the above Exercise 2.20 with Lemma 2.23, we see that $\nabla = \mathrm{d} - D$ is an example of a connection on $M_N(\mathbb{C})$ (as a module over itself and with $\omega = -D$), since $\Omega^1_D(A) \simeq M_N(\mathbb{C})$. Hence, for this choice of connection the new finite spectral triple as constructed in Theorem 2.26 is given by $(M_N(\mathbb{C}), \mathbb{C}^N, D' = 0)$. So, Morita equivalence of algebras does not carry over to an equivalence relation on spectral triples. Indeed, we now have $\Omega^1_{D'}(M_N(\mathbb{C})) = 0$, so that no non-zero D can be generated from this spectral triple and the symmetry of this relation fails.

2.3. Classification of finite spectral triples

Here we classify finite spectral triples on A modulo unitary equivalence, in terms of so-called **decorated graphs**.

DEFINITION 2.27. *A* graph *is an ordered pair $(\Gamma^{(0)}, \Gamma^{(1)})$ consisting of a set $\Gamma^{(0)}$ of* vertices *and a set $\Gamma^{(1)}$ of pairs of vertices (called* edges*).*

$$\begin{array}{ccccccc} \mathbf{n}_1 & \cdots & \mathbf{n}_i & \cdots & \mathbf{n}_j & \cdots & \mathbf{n}_N \\ \circ & & \circ & & \circledcirc & & \circ \end{array}$$

FIGURE 2.1. A node at $\mathbf{n}_i$ indicates the presence of the summand $\mathbb{C}^{n_i}$; the double node at $\mathbf{n}_j$ indicates the presence of the summand $\mathbb{C}^{n_j} \oplus \mathbb{C}^{n_j}$ in H.

We allow edges of the form $e = (v, v)$ for any vertex v, that is, we allow loops at any vertex.

Consider then a finite spectral triple (A, H, D); let us determine the structure of all three ingredients and construct a graph from it.

The algebra: We have already seen in Lemma 2.20 that

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some $n_1, \ldots, n_N$. The structure space of A is given by $\widehat{A} \simeq \{1, \ldots, N\}$ with each integer $i \in \widehat{A}$ corresponding to the equivalence classes of the representation of A on $\mathbb{C}^{n_i}$. If we label the latter equivalence class by $\mathbf{n}_i$ we can also identify $\widehat{A} \simeq \{\mathbf{n}_1, \ldots, \mathbf{n}_N\}$.

The Hilbert space: Any finite-dimensional faithful representation H of such a matrix algebra A is completely reducible (*i.e.* a direct sum of irreducible representations).

EXERCISE 2.21. *Prove this result for any $*$-algebra by establishing that the complement $W^\perp$ of an A-submodule $W \subset H$ is also an A-submodule of H.*

Combining this with the proof of Lemma 2.15, we conclude that the finite-dimensional Hilbert space representation H of A has a decomposition into irreducible representations, which we write as

$$H \simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes V_i,$$

with each V_i a vector space; we will refer to the dimension of V_i as the **multiplicity** of the representation labeled by $\mathbf{n}_i$ and to V_i itself as the **multiplicity space**. The above isomorphism is given by a unitary map.

To begin the construction of our decorated graph, we indicate the presence of a summand $\mathbf{n}_i$ in H by drawing a node at position $\mathbf{n}_i \in \widehat{A}$ in a diagram based on the structure space $\widehat{A}$ of the matrix algebra A (see Figure 2.1 for an example). Multiple nodes at the same position represent multiplicities of the representations in H.

The finite Dirac operator: Corresponding to the above decomposition of H we can write D as a sum of matrices

$$D_{ij} : \mathbb{C}^{n_i} \otimes V_i \to \mathbb{C}^{n_j} \otimes V_j,$$

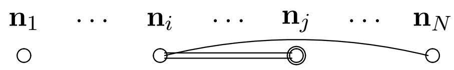


FIGURE 2.2. The edges between the nodes $\mathbf{n}_i$ and $\mathbf{n}_j$, and $\mathbf{n}_i$ and $\mathbf{n}_N$ represent non-zero operators $D_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j} \otimes \mathbb{C}^2$ (multiplicity 2) and $D_{iN} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_N}$, respectively. Their adjoints give the operators D_{ji} and D_{Ni}.

restricted to these subspaces. The condition that D is symmetric implies that $D_{ij} = D_{ji}^*$. In terms of the above diagrammatic representation of H, we express a non-zero D_{ij} and D_{ji} as a (multiple) edge between the nodes $\mathbf{n}_i$ and $\mathbf{n}_j$ (see Figure 2.2 for an example).

Another way of putting this is as follows, in terms of decorated graphs.

DEFINITION 2.28. *A* Λ-decorated graph *is given by an ordered pair* (Γ, Λ) *of a finite graph* Γ *and a finite set* Λ *of positive integers, with a labeling:*

- *of the vertices* $v \in \Gamma^{(0)}$ *by elements* $n(v) \in \Lambda$*;*
- *of the edges* $e = (v_1, v_2) \in \Gamma^{(1)}$ *by operators* $D_e : \mathbb{C}^{n(v_1)} \to \mathbb{C}^{n(v_2)}$ *and its conjugate-transpose* $D_e^* : \mathbb{C}^{n(v_2)} \to \mathbb{C}^{n(v_1)}$ *,*

so that $n(\Gamma^{(0)}) = \Lambda$*.*

The operators D_e between vertices that are labeled by $\mathbf{n}_i$ and $\mathbf{n}_j$, respectively, add up to the above D_{ij}. Explicitly,

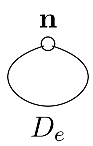
$$D_{ij} = \sum_{\substack{e=(v_1,v_2)\\ n(v_1)=\mathbf{n}_i \\ n(v_2)=\mathbf{n}_j}} D_e,$$

so that also $D_{ij}^* = D_{ji}$. Thus we have proved the following result.

THEOREM 2.29. *There is a one-to-one correspondence between finite spectral triples modulo unitary equivalence and* Λ*-decorated graphs, given by associating a finite spectral triple* (A, H, D) *to a* Λ*-decorated graph* (Γ, Λ) *in the following way:*

$$A = \bigoplus_{n \in \Lambda} M_n(\mathbb{C}), \qquad H = \bigoplus_{v \in \Gamma^{(0)}} \mathbb{C}^{n(v)}, \qquad D = \sum_{e \in \Gamma^{(1)}} D_e + D_e^*.$$

EXAMPLE 2.30. *The following* Λ*-decorated graph*



corresponds to the spectral triple $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$ *of Example 2.21.*

EXERCISE 2.22. *Draw the* Λ*-decorated graph corresponding to the spectral triple*

$$\left(A = \mathbb{C}^3, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \overline{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right); \qquad (\lambda \neq 0).$$

Exercise 2.23. *Use Λ-decorated graphs to classify all finite spectral triples (modulo unitary equivalence) on the matrix algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$.*

Exercise 2.24. *Suppose that (A_1, H_1, D_1) and (A_2, H_2, D_2) are two finite spectral triples. We consider their direct sum and tensor product and give the corresponding Λ-decorated graphs.*

(1) *Show that $(A_1 \oplus A_2, H_1 \oplus H_2, (D_1, D_2))$ is a finite spectral triple.*
(2) *Describe the Λ-decorated graph of this* direct sum spectral triple *in terms of the Λ-decorated graphs of the original spectral triples.*
(3) *Show that $(A_1 \otimes A_2, H_1 \otimes H_2, D_1 \otimes 1 + 1 \otimes D_2)$ is a finite spectral triple.*
(4) *Describe the Λ-decorated graph of this* tensor product spectral triple *in terms of the Λ-decorated graphs of the original spectral triples.*

Notes

Section 2.1. Finite spaces and matrix algebras

1. The notation KK_f in Definition 2.9 is chosen to suggest a close connection to Kasparov's bivariant KK-theory [**121**], here restricted to the finite-dimensional case. In fact, in the case of matrix algebras the notion of a *Kasparov module* for a pair of C^*-algebras (A, B) (*cf.* [**30**, Section 17.1] for a definition) coincides (up to homotopy) with that of a Hilbert bimodule for (A, B) (*cf.* [**132**, Section IV.2.1] for a definition).

2. Definition 2.12 agrees with the notion of equivalence between arbitrary rings introduced by Morita [**153**]. Moreover, it is a special case of strong Morita equivalence between C^*-algebras as introduced by Rieffel [**166**].

3. Theorem 2.14 is a special case of a more general result on the structure spaces of Morita equivalent C^*-algebras (see *e.g.* [**164**, Section 3.3]).

Section 2.2. Noncommutative geometric finite spaces

4. Theorem 2.18 can be found in [**112**].

5. The reconstruction theorem mentioned in the text before Definition 2.19 is a special case, to wit the finite-dimensional case, of a result by Connes [**69**] on a reconstruction of Riemannian (spin) manifolds from so-called spectral triples (*cf.* Definition 4.30 and Note 13 on Page 60 below).

6. A complete proof of Lemma 2.20 can be found in [**90**, Theorem 3.5.4].

7. For a complete exposition on differential algebras, connections on modules, *et cetera*, we refer to [**131**, Chapter 8] and [**3**] and references therein.

8. The failure of Morita equivalence to induce an equivalence between spectral triples was noted in [**65**, Remark 1.143] (see also [**192**, Remark 5.1.2]). This suggests that it is better to consider Hilbert bimodules as *correspondences* rather than equivalences, as was already suggested by Connes and Skandalis in [**67**] and also appeared in the applications of noncommutative geometry to number theory (*cf.* [**65**, Chapter 4.3]) and quantization [**133**]. This forms the starting point for a categorical description of (finite) spectral triples themselves. As objects the category has finite spectral triples (A, H, D), and as morphisms it has pairs (E, ∇) as above. This category is the topic of [**150, 151**], working in the more general setting of spectral triples, hence requiring much more analysis as compared to our finite-dimensional case. The category of finite spectral triples plays a crucial role in the noncommutative generalization of spin networks in [**145**].

CHAPTER 3

Finite real noncommutative spaces

In this chapter, we will enrich the finite noncommutative spaces as analyzed in the previous chapter with a *real structure*. For one thing, this makes the definition of a finite spectral triple more symmetric by demanding the inner product space H be an $A-A$-bimodule, rather than just a left A-module. The implementation of this bimodule structure by an anti-unitary operator has close ties with the Tomita–Takesaki theory of Von Neumann algebras, as well as with physics through charge conjugation, as will become clear in the applications in the later chapters of this book. Our exposition includes a diagrammatic classification of finite real spectral triples for all so-called KO-dimensions, and also identifies the irreducible finite geometries among them.

3.1. Finite real spectral triples

First, the structure of a finite spectral triple can be enriched by introducing a $\mathbb{Z}_2$-grading γ on H, *i.e.* $\gamma^* = \gamma, \gamma^2 = 1$, demanding that A is *even* and D is *odd* with respect to this grading:

$$\gamma D = -D\gamma, \qquad \gamma a = a\gamma; \qquad (a \in A).$$

Next, there is a more symmetric refinement of the notion of finite spectral triple in which H is an $A-A$-bimodule, rather than just a left A-module. Recall that an anti-unitary operator is an invertible operator $J : H \to H$ that satisfies $\langle J\xi_1, J\xi_2\rangle = \langle \xi_2, \xi_1\rangle$ for all $\xi_1, \xi_2 \in H$.

Definition 3.1. *A* finite real spectral triple *is given by a finite spectral triple (A, H, D) and an anti-unitary operator $J : H \to H$ called* real structure*, such that $a^\circ := Ja^*J^{-1}$ is a right representation of A on H, i.e. $(ab)^\circ = b^\circ a^\circ$. We also require that*

$$[a, b^\circ] = 0, \qquad [[D, a], b^\circ] = 0, \tag{3.1.1}$$

for all $a, b \in A$. Moreover, we demand that J, D and (in the even case) γ satisfy the commutation relations:

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon''\gamma J.$$

for numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$. These signs determine the KO-dimension *k (modulo 8) of the finite real spectral triple $(A, H, D; J, \gamma)$ defined according to Table 3.1.*

The signs in Table 3.1 are motivated by the classification of Clifford algebras, see Section 4.1 below. The two conditions in (3.1.1) are called the **commutant property**, and the **first-order** or **order one condition**, respectively. They imply that the left action of an element in A and $\Omega^1_D(A)$

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

TABLE 3.1. The KO-dimension k of a real spectral triple is determined by the signs $\{\epsilon, \epsilon', \epsilon''\}$ appearing in $J^2 = \varepsilon, JD = \varepsilon' DJ$ and $J\gamma = \varepsilon''\gamma J$.

commutes with the right action of A. This is equivalent to the commutation between the right action of A and $\Omega^1_D(A)$ with the left action of A.

REMARK 3.2. *The so-called* opposite algebra A° *is defined to be equal to A as a vector space but with opposite product* $\circ$*:*

$$a \circ b := ba.$$

Thus, $a^\circ = Ja^*J^{-1}$ *defines a left representation of* A° *on* H*:* $(a \circ b)^\circ = a^\circ b^\circ$*.*

EXAMPLE 3.3. *Consider the matrix algebra* $M_N(\mathbb{C})$*, acting on the inner product space* $H = M_N(\mathbb{C})$ *by left matrix multiplication, and with inner product given by the Hilbert–Schmidt inner product:*

$$\langle a, b\rangle = \operatorname{Tr} a^* b.$$

Define

$$\gamma(a) = a, \qquad J(a) = a^*; \qquad (a \in H).$$

Since D *must be odd with respect to the grading* γ*, it vanishes identically.*

EXERCISE 3.1. *In the previous example, show that the right action of* $M_N(\mathbb{C})$ *on* $H = M_N(\mathbb{C})$ *as defined by* $a \mapsto a^\circ$ *is given by right matrix multiplication.*

The following exercises are inspired by Tomita–Takesaki theory of Von Neumann algebras.

EXERCISE 3.2. *Let* $A = \bigoplus_i M_{n_i}(\mathbb{C})$ *be a matrix algebra, which is represented on a vector space* $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$*, i.e. is such that the irreducible representation* $\mathbf{n}_i$ *has multiplicity* m_i*.*

(1) *Show that the commutant* A' *of* A *is isomorphic to* $\bigoplus_i M_{m_i}(\mathbb{C})$*. As a consequence, the double commutant coincides with* A*, that is to say* $A'' \simeq A$*.*

We say that $\xi \in H$ *is a* **cyclic vector** *for* A *if*

$$A\xi := \{a\xi : a \in A\} = H.$$

We call $\xi \in H$ *a* **separating vector** *for* A *if*

$$a\xi = 0 \implies a = 0; \qquad (a \in A).$$

(2) *Show that if* ξ *is a separating vector for the action of* A*, it is cyclic for the action of* A'*. (*Hint*: Assume* ξ *is not cyclic for the action of* A' *and try to derive a contradiction).*

EXERCISE 3.3. *Suppose that $(A, H, D = 0)$ is a finite spectral triple such that H possesses a cyclic and separating vector ξ for A.*

(1) Show that the formula $S(a\xi) = a^*\xi$ defines an anti-linear operator $S : H \to H$.
(2) Show that S is invertible.
(3) Let $J : H \to H$ be the operator appearing in the polar decomposition $S = J\Delta^{1/2}$ of S with $\Delta = S^*S$. Show that J is an anti-unitary operator.

Conclude that $(A, H, D = 0; J)$ is a finite real spectral triple. Can you find such an operator J in the case of Exercise 3.2?

3.1.1. Morphisms between finite real spectral triples. We are now going to extend the notion of unitary equivalence (*cf.* Definition 2.24) to finite real spectral triples.

DEFINITION 3.4. *We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$* unitarily equivalent *if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$ such that*

$$U\pi_1(a)U^* = \pi_2(a); \qquad (a \in A_1),$$
$$UD_1U^* = D_2, \qquad U\gamma_1U^* = \gamma_2, \qquad UJ_1U^* = J_2.$$

Building on our discussion in Section 2.2.1, we can also extend Morita equivalence to finite real spectral triples. Namely, given a Hilbert bimodule E for (B, A), we will construct a finite real spectral triple $(B, H', D'; J', \gamma')$ on B, starting from a finite real spectral triple $(A, H, D; J, \gamma)$ on A.

DEFINITION 3.5. *Let E be a $B-A$-bimodule. The* conjugate module E° *is given by the $A - B$-bimodule*

$$E^\circ = \{\overline{e} : e \in E\},$$

with $a \cdot \overline{e} \cdot b = \overline{b^* \cdot e \cdot a^*}$ for any $a \in A, b \in B$.

This implies for any $\lambda \in \mathbb{C}$ that $\overline{\lambda e} = \overline{\lambda}\overline{e}$, which explains the suggestive notation $\overline{e}$ for the elements of E°. The bimodule E° is not quite a Hilbert bimodule for (A, B), since we do not have a natural B-valued inner product. However, there is a A-valued inner product on the left A-module E° given by

$$\langle \overline{e}_1, \overline{e}_2 \rangle = \langle e_2, e_1 \rangle; \qquad (e_1, e_2 \in E).$$

As opposed to the inner product in Definition 2.9, this inner product is left A-linear: $\langle a\overline{e}_1, \overline{e}_2 \rangle = a\langle \overline{e}_1, \overline{e}_2 \rangle$ for all $a \in A$, as can be easily checked.

EXERCISE 3.4. *Show that E° is a Hilbert bimodule for (B°, A°).*

Let us then start the construction of a finite real spectral triple on B by setting

$$H' := E \otimes_A H \otimes_A E^\circ.$$

There is a ($\mathbb{C}$-valued) inner product on H' given by combining the A-valued inner products on E, E° with the $\mathbb{C}$-valued inner product on H, much as in (2.1.1). The action of B on H' is given by

$$b(e_1 \otimes \xi \otimes \overline{e}_2) = (be_1) \otimes \xi \otimes \overline{e}_2, \tag{3.1.2}$$

using just the $B-A$-bimodule structure of E. In addition, there is a right action of B on H' defined by acting on the right on the component E°. In fact, it is implemented by the following anti-unitary,

$$J'(e_1 \otimes \xi \otimes \overline{e}_2) = e_2 \otimes J\xi \otimes \overline{e}_1,$$

i.e. $b^\circ = J'b^*(J')^{-1}$ with $b^* \in B$ acting on H' according to (3.1.2).

Moreover, there is a finite Dirac operator given in terms of the connection $\nabla : E \to E \otimes_A \Omega^1_D(A)$ as in Section 2.2.1. First, we need the result of the following exercise.

Exercise 3.5. *Let $\nabla : E \to E \otimes_A \Omega^1_D(A)$ be a right connection on E and consider the following anti-linear map*

$$\tau : E \otimes_A \Omega^1_D(A) \to \Omega^1_D(A) \otimes_A E^\circ;$$
$$e \otimes \omega \mapsto -\omega^* \otimes \overline{e}.$$

Show that the map $\overline{\nabla} : E^\circ \to \Omega^1_D(A) \otimes_A E^\circ$ defined by $\overline{\nabla}(\overline{e}) = \tau \circ \nabla(e)$ is a left connection, i.e. show that it satisfies the left Leibniz rule:

$$\overline{\nabla}(a\overline{e}) = [D, a] \otimes \overline{e} + a\overline{\nabla}(\overline{e}).$$

The connections ∇ and $\overline{\nabla}$ give rise to a Dirac operator on $E \otimes_A H \otimes_A E^\circ$:

$$D'(e_1 \otimes \xi \otimes \overline{e}_2) = (\nabla e_1)\xi \otimes \overline{e}_2 + e_1 \otimes D\xi \otimes \overline{e}_2 + e_1 \otimes \xi(\overline{\nabla}\overline{e}_2).$$

The right action of $\omega \in \Omega^1_D(A)$ on $\xi \in H$ is then defined by $\xi \mapsto \epsilon' J\omega^* J^{-1}\xi$.

Finally, for even spectral triples one defines a grading on $E \otimes_A H \otimes_A E^\circ$ by $\gamma' = 1 \otimes \gamma \otimes 1$.

Theorem 3.6. *Suppose $(A, H, D; J, \gamma)$ is a finite real spectral triple of KO-dimension k, and let $\nabla : E \to E \otimes_A \Omega^1_D(A)$ be a compatible connection (cf. Equation (2.2.5)). Then $(B, H', D'; J', \gamma')$ is a finite real spectral triple of KO-dimension k.*

Proof. The only non-trivial thing to check is that the KO-dimension is preserved. In fact, one readily checks that $(J')^2 = 1 \otimes J^2 \otimes 1 = \epsilon$ and $J'\gamma' = \epsilon''\gamma' J'$. Also,

$$\begin{aligned} J'D'(e_1 \otimes \xi \otimes \overline{e}_2) &= J'\left((\nabla e_1)\xi \otimes \overline{e}_2 + e_1 \otimes D\xi \otimes \overline{e}_2 + e_1 \otimes \xi(\tau\nabla e_2)\right) \\ &= \epsilon' D'(e_2 \otimes J\xi \otimes \overline{e}_1) \equiv \epsilon' D'J'(e_1 \otimes \xi \otimes \overline{e}_2), \end{aligned}$$

where we have used $J'(e_1 \otimes J\omega J^{-1}\xi \otimes \overline{e}_2) = e_2 \otimes \omega J\xi \otimes \overline{e}_1$. □

3.2. Classification of finite real spectral triples

In this section, we classify all finite real spectral triples $(A, H, D; J, \gamma)$ modulo unitary equivalence using **Krajewski diagrams**. These play a similar role for finite real spectral triples as Dynkin diagrams do for simple Lie algebras. Moreover, they extend our Λ-decorated graphs of the previous chapter to the case of real spectral triples.

The algebra: First, we already know from our classification of finite spectral triples in Section 2.3 that

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some $n_1, \ldots, n_N$. Thus, the structure space of A is again given by $\widehat{A} = \{\mathbf{n}_1, \ldots, \mathbf{n}_N\}$ where $\mathbf{n}_i$ denotes the irreducible representation of A on $\mathbb{C}^{n_i}$.

The Hilbert space: As before, the irreducible, faithful representations of $A = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$ are given by corresponding direct sums:

$$\bigoplus_{i=1}^{N} \mathbb{C}^{n_i}$$

on which A acts by left block-diagonal matrix multiplication.

Now, besides the representation of A, there should also be a representation of A° on H which commutes with that of A. In other words, we are looking for the irreducible representations of $A \otimes A^\circ$. If we denote the unique irreducible representation of $M_n(\mathbb{C})^\circ$ by $\mathbb{C}^{n\circ}$, this implies that any irreducible representation of $A \otimes A^\circ$ is given by a summand in

$$\bigoplus_{i,j=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}.$$

Consequently, any finite-dimensional Hilbert space representation of A has a decomposition into irreducible representations

$$H = \bigoplus_{i,j=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij},$$

with V_{ij} a vector space; we will refer to the dimension of V_{ij} as the **multiplicity** of the representation $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$.

The integers $\mathbf{n}_i$ and $\mathbf{n}_j^\circ$ form the grid of a diagram (*cf.* Figure 3.1 for an example). Whenever there is a node at the coordinates $(\mathbf{n}_i, \mathbf{n}_j^\circ)$, the representation $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$ is present in the direct sum decomposition of H. Multiplicities are indicated by multiple nodes.

EXAMPLE 3.7. *Consider the algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$. The irreducible representations of A are given by $\mathbf{1}$ and $\mathbf{2}$. The two diagrams*

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{2} \\ \mathbf{1}^\circ & \circ & \\ \mathbf{2}^\circ & & \circ \end{array} \qquad\qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{2} \\ \mathbf{1}^\circ & \circ & \circ \\ \mathbf{2}^\circ & & \end{array}$$

correspond to $H_1 = \mathbb{C} \oplus M_2(\mathbb{C})$ and $H_2 = \mathbb{C} \oplus \mathbb{C}^2$, respectively. We have used the fact that $\mathbb{C}^2 \otimes \mathbb{C}^{2\circ} \simeq M_2(\mathbb{C})$. The left action of A on H_1 is given by the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix},$$

with $a \in M_2(\mathbb{C})$ acting on $M_2(\mathbb{C}) \subset H_1$ by left matrix multiplication. The right action of A on H_1 corresponds to the same matrix acting by right matrix multiplication.

On H_2, the left action of A is given by matrix multiplication by the above matrix on vectors in $\mathbb{C} \oplus \mathbb{C}^2$. However, the right action of $(\lambda, a) \in A$ is given by scalar multiplication with λ on all of H_2.

$$\begin{array}{c|ccccccc}
 & \mathbf{n}_1 & \cdots & \mathbf{n}_i & \cdots & \mathbf{n}_j & \cdots & \mathbf{n}_N \\
\hline
\mathbf{n}_1^\circ & \circ & & & & & & \\
\vdots & & & & & & & \\
\mathbf{n}_i^\circ & & & \circledcirc & & & & \\
\vdots & & & & & & & \\
\mathbf{n}_j^\circ & & & \circ & & & & \circ \\
\vdots & & & & & & & \\
\mathbf{n}_N^\circ & & & & & & & \circ
\end{array}$$

FIGURE 3.1. A node at $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ indicates the presence of the summand $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\circ}$ in H; the double node indicates the presence of $(\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i\circ}) \oplus (\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i\circ})$ in H.

The real structure: Before turning to the finite Dirac operator D, we exploit the presence of a real structure $J : H \to H$ in the diagrammatic approach started above.

EXERCISE 3.6. *Let J be an anti-unitary operator on a finite-dimensional Hilbert space. Show that J^2 is a unitary operator.*

LEMMA 3.8. *Let J be an anti-unitary operator on a finite-dimensional Hilbert space H with $J^2 = \pm 1$.*

(1) If $J^2 = 1$ then there is an orthonormal basis $\{e_k\}$ of H such that $Je_k = e_k$.
(2) If $J^2 = -1$ then there is an orthonormal basis $\{e_k, f_k\}$ of H such that $Je_k = f_k$ (and, consequently, $Jf_k = -e_k$).

PROOF. (1) Take any $v \in H$ and set

$$e_1 := \begin{cases} c(v + Jv) & \text{if } Jv \neq -v \\ iv & \text{if } Jv = -v, \end{cases}$$

with c a normalization constant. Then $J(v + Jv) = Jv + J^2 v = v + Jv$ and $J(iv) = -iJv = iv$ in the two respective cases, so that $Je_1 = e_1$.

Next, take a vector v' that is orthogonal to e_1. Then

$$(e_1, Jv') = (J^2 v', Je_1) = (v', Je_1) = (v', e_1) = 0,$$

so that also $Jv' \perp e_1$. As before, we set

$$e_2 := \begin{cases} c(v' + Jv') & \text{if } Jv' \neq -v' \\ iv' & \text{if } Jv' = -v', \end{cases}$$

which by the above is orthogonal to e_1. Continuing in this way gives a basis $\{e_k\}$ for H with $Je_k = e_k$.

(2) Take any $v \in H$ and set $e_1 = cv$ with c a normalization constant. Then $f_1 = Je_1$ is orthogonal to e_1, since

$$(f_1, e_1) = (Je_1, e_1) = -(Je_1, J^2 e_1) = -(Je_1, e_1) = -(f_1, e_1).$$

Next, take another $v' \perp e_1, f_1$ and set $e_2 = c'v'$. As before, $f_2 := Je_2$ is orthogonal to e_2, and also to e_1 and f_1:

$$\begin{aligned}(e_1, f_2) &= (e_1, Je_2) = -(J^2 e_1, Je_2) = -(e_2, Je_1) = -(e_2, f_1) = 0,\\ (f_1, f_2) &= (Je_1, Je_2) = (e_2, e_1) = 0.\end{aligned}$$

Continuing in this way gives a basis $\{e_k, f_k\}$ for H with $Je_k = f_k$. □

We will now apply these results to the anti-unitary operator given by a real structure on a spectral triple. Recall that in this case, $J : H \to H$ implements a right action of A on H, via

$$a^\circ = Ja^* J^{-1}$$

satisfying $[a, b^\circ] = 0$. Together with the block-form of A, this implies that

$$J(a_1^* \oplus \cdots \oplus a_N^*) = (a_1^\circ \oplus \cdots \oplus a_N^\circ)J.$$

We conclude that the Krajewski diagram for a real spectral triple must be symmetric along the diagonal, J mapping each subspace $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ bijectively to $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}$.

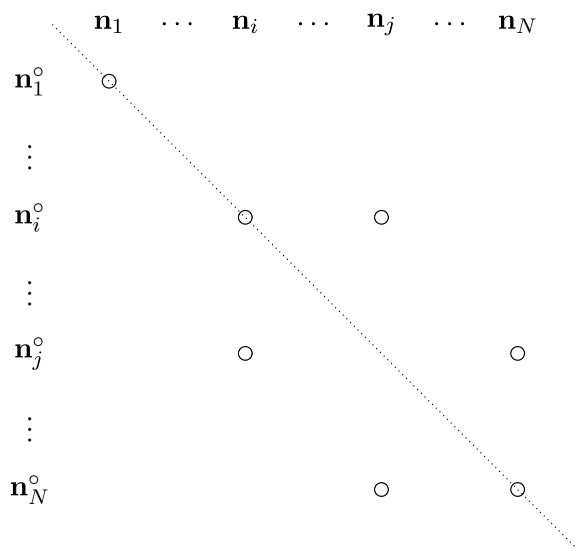


FIGURE 3.2. The presence of the real structure J implies a symmetry in the diagram along the diagonal.

PROPOSITION 3.9. *Let J be a real structure on a finite real spectral triple $(A, H, D; J)$.*

(1) If $J^2 = 1$ (KO-dimension 0,1,6,7) then there is an orthonormal basis $\{e_k^{(ij)}\}$ $(i,j = 1,\dots,N, k = 1,\dots,\dim V_{ij})$ with $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ such that

$$Je_k^{(ij)} = e_k^{(ji)}; \qquad (i,j = 1,\dots,N; k = 1,\dots,\dim V_{ij}).$$

(2) If $J^2 = -1$ (KO-dimension 2,3,4,5) then there is an orthonormal basis $\{e_k^{(ij)}, f_k^{(ji)}\}$ $(i \le j = 1,\dots,N, k = 1,\dots,\dim V_{ij})$ with $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$, $f_k^{(ji)} \in \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}$ and such that

$$Je_k^{(ij)} = f_k^{(ji)}; \qquad (i \le j = 1,\dots,N; k = 1,\dots,\dim V_{ij}).$$

Proof. We imitate the proof Lemma 3.8.
(1) If $i \neq j$, take $v \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ and set $e_1^{(ij)} = cv$. Then, by the above observation, $e_1^{(ji)} = Je_1^{(ij)}$ is an element in $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}$. Next, take $v' \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ with $v' \perp v$ and apply the same procedure to obtain $e_2^{(ij)}$ and $e_2^{(ji)}$. Continuing in this way gives an orthonormal basis $\{e_k^{(ij)}\}$ for $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$, and an orthonormal basis $\{e_k^{(ji)}\}$ for $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}$ which satisfy $Je_k^{(ij)} = e_k^{(ji)}$.

If $i = j$, then Lemma 3.8(1) applies directly to the anti-unitary operator given by J restricted to $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ii}$.

(2) can be proved along the same lines. □

Note that this result implies that in the case of KO-dimension 2, 3, 4 and 5, the diagonal $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ii}$ needs to have even multiplicity.

The finite Dirac operator: Corresponding to the above decomposition of H we can write D as a sum of matrices

$$D_{ij,kl} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \to \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l \circ} \otimes V_{kl},$$

restricted to these subspaces. The condition $D^* = D$ implies that $D_{kl,ij} = D_{ij,kl}^*$. In terms of the above diagrammatic representation of H, we express a non-zero $D_{ij,kl}$ as a line between the nodes $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ and $(\mathbf{n}_k, \mathbf{n}_l^\circ)$. Instead of drawing directed lines, we draw a single undirected line, capturing both $D_{ij,kl}$ and its adjoint $D_{kl,ij}$.

Lemma 3.10. *The condition $JD = \pm DJ$ and the order one condition given by $[[D,a],b^\circ] = 0$ forces the lines in the diagram to run only vertically or horizontally (or between the same node), thereby maintaining the diagonal symmetry between the nodes in the diagram.*

Proof. The condition $JD = \pm DJ$ easily translates into a commuting diagram:

$$\begin{tikzcd}
\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \arrow[r, "D"'] \arrow[d, "J"'] & \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l \circ} \otimes V_{kl} \arrow[d, "J"'] \\
\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji} \arrow[r, "\pm D"'] & \mathbb{C}^{n_l} \otimes \mathbb{C}^{n_k \circ} \otimes V_{lk}
\end{tikzcd}$$

thus relating $D_{ij,kl}$ to $D_{ji,lk}$, maintaining the diagonal symmetry.

If we write the order one condition $[[D,a],b^\circ]=0$ for diagonal elements $a=\lambda_1\mathbb{I}_{n_1}\oplus\cdots\oplus\lambda_N\mathbb{I}_{n_N}\in A$ and $b=\mu_1\mathbb{I}_{n_1}\oplus\cdots\oplus\mu_N\mathbb{I}_{n_N}\in A$ with $\lambda_i,\mu_i\in\mathbb{C}$, we compute

$$D_{ij,kl}(\lambda_i-\lambda_k)(\overline{\mu}_j-\overline{\mu}_l)=0,$$

for all $\lambda_i,\mu_j\in\mathbb{C}$. As a consequence, $D_{ij,kl}=0$ whenever $i\neq k$ or $j\neq l$. $\square$

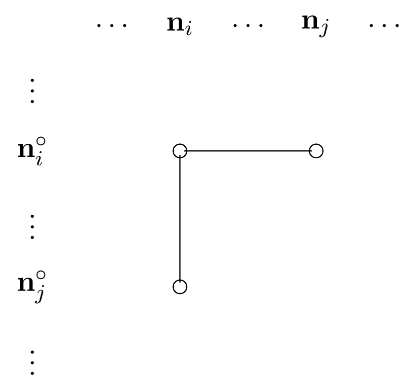


FIGURE 3.3. The lines between two nodes represent a non-zero $D_{ii,ji}:\mathbb{C}^{n_i}\otimes\mathbb{C}^{n_i^\circ}\to\mathbb{C}^{n_j}\otimes\mathbb{C}^{n_i^\circ}$, as well as its adjoint $D_{ji,ii}:\mathbb{C}^{n_j}\otimes\mathbb{C}^{n_i^\circ}\to\mathbb{C}^{n_i}\otimes\mathbb{C}^{n_i^\circ}$. The non-zero components $D_{ii,ij}$ and $D_{ij,ii}$ are related to $\pm D_{ii,ji}$ and $\pm D_{ji,ii}$, respectively, according to $JD=\pm DJ$.

Grading: Finally, if there is a grading $\gamma:H\to H$, then each node in the diagram gets labeled by a plus or minus sign. The rules are that:

- D connects nodes with different signs;
- If the node $(\mathbf{n}_i,\mathbf{n}_j^\circ)$ has sign $\pm$, then the node $(\mathbf{n}_j,\mathbf{n}_i^\circ)$ has sign $\pm\epsilon''$, according to $J\gamma=\epsilon''\gamma J$.

Finally, we arrive at a diagrammatic classification of finite real spectral triples of any KO-dimension.

DEFINITION 3.11. *A* Krajewski diagram *of KO-dimension k is given by an ordered pair (Γ,Λ) of a finite graph Γ and a finite set Λ of positive integers with a labeling:*

- *of the vertices $v\in\Gamma^{(0)}$ by elements $\iota(v)=(n(v),m(v))\in\Lambda\times\Lambda$, where the existence of an edge from v to v' implies that either $n(v)=n(v')$, $m(v)=m(v')$, or both;*
- *of the edges $e=(v_1,v_2)\in\Gamma^{(1)}$ by non-zero operators:*

$$D_e:\mathbb{C}^{n(v_1)}\to\mathbb{C}^{n(v_2)} \qquad \text{if} \qquad m(v_1)=m(v_2);$$
$$D_e:\mathbb{C}^{m(v_1)}\to\mathbb{C}^{m(v_2)} \qquad \text{if} \qquad n(v_1)=n(v_2),$$

  and their adjoints D_e^*,

together with an involutive graph automorphism $j:\Gamma\to\Gamma$ so that the following conditions hold:

(1) every row or column in $\Lambda\times\Lambda$ has non-empty intersection with $\iota(\Gamma)$;
(2) for each vertex v we have $(n(j(v)) = m(v)$;
(3) for each edge e we have $D_e = \epsilon' D_{j(e)}$;
(4) if the KO-dimension k is even, then the vertices are additionally labeled by ± 1 and the edges only connect opposite signs. The signs at v and $j(v)$ differ by a factor ϵ, according to the table of Definition 3.1;
(5) if the KO-dimension is 2,3,4,5 then the inverse image under ι of the diagonal elements in $\Lambda\times\Lambda$ contains an even number of vertices of Γ.

Note that this definition allows for different vertices of Γ to be labeled by the same element in $\Lambda \times \Lambda$; this accounts for the multiplicities appearing in V_{ij} that we have encountered before.

This indeed gives rise to a diagram of the above type, by putting a node at position $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ for each vertex carrying the label $(\mathbf{n}_i, \mathbf{n}_j) \in \Lambda \times \Lambda$. The notation $\mathbf{n}_j^\circ$ instead of $\mathbf{n}_j$ is just for a convenient diagrammatic exposition. The operators D_e between vertices that are labeled by $(\mathbf{n}_i, \mathbf{n}_j)$ and $(\mathbf{n}_k, \mathbf{n}_l)$, respectively, add up to the above $D_{ij,kl}$. Explicitly,

$$D_{ij,kl} = \sum_{\substack{e=(v_1,v_2)\in\Gamma^{(1)}\\ \iota(v_1)=(\mathbf{n}_i,\mathbf{n}_j)\\ \iota(v_2)=(\mathbf{n}_k,\mathbf{n}_l)}} D_e,$$

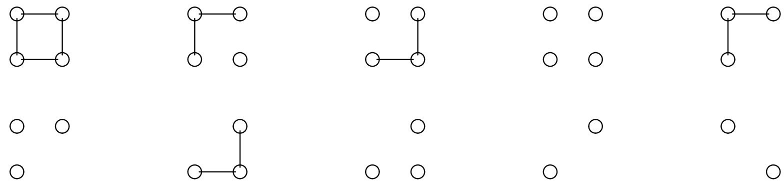
so that indeed $D_{ij,kl}^* = D_{kl,ij}$. Moreover, the only non-zero entries $D_{ij,kl}$ will appear when $i = k$, or $j = l$, or both. Thus, we have shown

THEOREM 3.12. *There is a one-to-one correspondence between finite real spectral triples of KO-dimension k modulo unitary equivalence and Krajewski diagrams of KO-dimension k. Specifically, one associates a real spectral triple $(A, H, D; J, \gamma)$ to a Krajewski diagram in the following way:*

$$\begin{aligned} A &= \bigoplus_{n\in\Lambda} M_n(\mathbb{C}); \\ H &= \bigoplus_{v\in\Gamma^{(0)}} \mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)\circ}; \\ D &= \sum_{e\in\Gamma^{(1)}} D_e + D_e^*. \end{aligned}$$

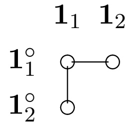
Moreover, the real structure $J : H \to H$ is given as in Proposition 3.9, with the basis dictated by the graph automorphism $j : \Gamma \to \Gamma$. Finally, a grading γ on H is defined by setting γ to be ± 1 on $\mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)\circ} \subset H$ according to the labeling by ± 1 of the vertex v.

EXAMPLE 3.13. *Consider the case $A = \mathbb{C} \oplus \mathbb{C}$. There are ten possible Krajewski diagrams in KO-dimension 0 with multiplicities less than or equal to 1: in terms of $\widehat{A} = \{\mathbf{1}_1, \mathbf{1}_2\}$, we have*



where the diagonal vertices are labeled with a plus sign, and the off-diagonal vertices with a minus sign.

Let us consider the last diagram in the top row in more detail and give the corresponding spectral triple:



First, the inner product space is $H = \mathbb{C}^3$, where we choose the middle copy of $\mathbb{C}$ to correspond to the node on the diagonal. The edges indicate that there are non-zero components of D that map between the first two copies of $\mathbb{C}$ in H and between the second and third copy of $\mathbb{C}$. In other words,

$$D = \begin{pmatrix} 0 & \lambda & 0 \\ \overline{\lambda} & 0 & \mu \\ 0 & \overline{\mu} & 0 \end{pmatrix}$$

for some $\lambda, \mu \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$ that are the given labels on the two edges. In this basis,

$$\gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, J is given by the matrix K composed with complex conjugation on H, where

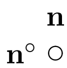
$$K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From this it is clear that we indeed have

$$D\gamma = -\gamma D; \qquad DJ = JD; \qquad J\gamma = \gamma J.$$

EXERCISE 3.7. *Use the ten Krajewski diagrams of the previous example to show that on $A = \mathbb{C} \oplus \mathbb{C}$ a finite real spectral triple of KO-dimension 6 with* $\dim H \leq 4$ *must have vanishing finite Dirac operator.*

EXAMPLE 3.14. *Consider $A = M_n(\mathbb{C})$ so that $\widehat{A} = \{\mathbf{n}\}$. We then have a Krajewski diagram*



The node can be labeled only by either plus or minus one, the choice being irrelevant. This means that $H = \mathbb{C}^n \otimes \mathbb{C}^{n\circ} \simeq M_n(\mathbb{C})$ with γ the trivial grading. The operator J is a combination of complex conjugation and the flip on $\mathbf{n} \otimes \mathbf{n}^\circ$: this translates to $M_n(\mathbb{C})$ as taking the matrix adjoint. Moreover, *since the single node has label* ± 1*, there are no non-zero Dirac operators. Hence, the finite real spectral triple of this diagram corresponds to*

$$(A = M_n(\mathbb{C}), H = M_n(\mathbb{C}), D = 0; J = (\cdot)^*, \gamma = 1),$$

and was encountered already in Exercise 3.3.

3.3. Real algebras and Krajewski diagrams

Thus far, we have considered finite spectral triples on complex algebras. In practice, it is useful to allow real $*$-algebras in Definition 2.19 as well.

DEFINITION 3.15. *A* real algebra *is a vector space* A *over* $\mathbb{R}$ *with a bilinear associative product* $A \times A \to A$ *denoted by* $(a, b) \mapsto ab$ *and a unit* 1 *satisfying* $1a = a1 = a$ *for all* $a \in A$.

A real $*$-algebra *(or,* involutive algebra*) is a real algebra* A *together with a real linear map (the* involution*)* $* : A \to A$ *such that* $(ab)^* = b^*a^*$ *and* $(a^*)^* = a$ *for all* $a, b \in A$.

EXAMPLE 3.16. *A particularly interesting example in this context is given by* $\mathbb{H}$*, the real* $*$*-algebra of quaternions, defined as a real subalgebra of* $M_2(\mathbb{C})$*:*

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

This is indeed closed under multiplication. As a matter of fact, $\mathbb{H}$ *consists of those matrices in* $M_2(\mathbb{C})$ *that commute with the operator* I *defined by*

$$I \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}.$$

The involution is inherited from $M_2(\mathbb{C})$ *and is given by hermitian conjugation.*

EXERCISE 3.8.
(1) *Show that* $\mathbb{H}$ *is a real* $*$*-algebra which contains a real subalgebra isomorphic to* $\mathbb{C}$.
(2) *Show that* $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ *as complex* $*$*-algebras.*
(3) *Show that* $M_k(\mathbb{H})$ *is a real* $*$*-algebra for any integer* k.
(4) *Show that* $M_k(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2k}(\mathbb{C})$ *as complex* $*$*-algebras.*

When considering Hilbert space representations of a real $*$-algebra, one must be careful, because the Hilbert space will be assumed to be a complex space.

DEFINITION 3.17. *A* representation *of a finite-dimensional real* $*$*-algebra* A *is a pair* (H, π) *where* H *is a (finite-dimensional, complex) Hilbert space and* π *is a real-linear* $*$*-algebra map*

$$\pi : A \to L(H).$$

Also, although there is a great deal of similarity, we stress that the definition of the real structure J in Definition 2.19 is not related to the algebra A being real or complex.

EXERCISE 3.9. *Show that there is a one-to-one correspondence between Hilbert space representations of a real* $*$*-algebra* A *and complex representations of its complexification* $A \otimes_{\mathbb{R}} \mathbb{C}$*. Conclude that the unique irreducible (Hilbert space) representation of* $M_k(\mathbb{H})$ *is given by* $\mathbb{C}^{2k}$.

LEMMA 3.18. *Suppose that a real $*$-algebra A is represented faithfully on a finite-dimensional Hilbert space H through a real-linear $*$-algebra map $\pi : A \to L(H)$. Then A is a matrix algebra:*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{F}_i),$$

where $\mathbb{F}_i = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, depending on i.

PROOF. The representation π allows to consider A as a real $*$-subalgebra of $M_{\dim H}(\mathbb{C})$, hence $A + iA$ can be considered a complex $*$-subalgebra of $M_{\dim H}(\mathbb{C})$. Thus $A + iA$ is a matrix algebra, and we may restrict to the case $A+iA = M_k(\mathbb{C})$ for some $k \geq 1$. Note that $A \cap iA$ is a two-sided $*$-ideal in $M_k(\mathbb{C})$. As such, it must be either the whole of $M_k(\mathbb{C})$, or zero. In the first case, $A + iA = A \cap iA$ so that $A = M_k(\mathbb{C})$. If $A \cap iA = \{0\}$, then we can uniquely write any element in $M_k(\mathbb{C})$ as $a + ib$ with $a, b \in A$. Moreover, A is the fixed point algebra of the anti-linear automorphism α of $M_k(\mathbb{C})$ given by $\alpha(a + ib) = a - ib$ $(a, b \in A)$. We can implement α by an anti-linear isometry I on $\mathbb{C}^k$ such that $\alpha(x) = IxI^{-1}$ for all $x \in M_k(\mathbb{C})$. Since $\alpha^2 = 1$, the operator I^2 commutes with $M_k(\mathbb{C})$ and is therefore proportional to a complex scalar. Together with I^2 being an isometry, this implies that $I^2 = \pm 1$ and that A is precisely the commutant of I. We now once again use Lemma 3.8 to conclude that

- If $I^2 = 1$, then there is a basis $\{e_i\}$ of $\mathbb{C}^k$ such that $Ie_i = e_i$. Since a matrix in $M_k(\mathbb{C})$ that commutes with I must have real entries, this gives
$$A = M_k(\mathbb{R}).$$
- If $I^2 = -1$, then there is a basis $\{e_i, f_i\}$ of $\mathbb{C}^k$ such that $Ie_i = f_i$ (and thus k is even). Since a matrix in $M_k(\mathbb{C})$ that commutes with I must be a $k/2 \times k/2$-matrix with quaternionic entries, we obtain
$$A = M_{k/2}(\mathbb{H}).$$ □

We now reconsider the diagrammatic classification of finite spectral triples, with real $*$-algebras represented faithfully on a Hilbert space. In fact, as far as the decomposition of H into irreducible representations is concerned, we can replace A by the complex $*$-algebra

$$A + iA \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}).$$

Thus, the Krajewski diagrams in Definition 3.11 classify such finite real spectral triples as well as long as we take the $\mathbb{F}_i$ for each i into account. That is, we enhance the set Λ to be

$$\Lambda = \{\mathbf{n}_1\mathbb{F}_1, \ldots, \mathbf{n}_N\mathbb{F}_N\},$$

reducing to the previously defined Λ when all $\mathbb{F}_i = \mathbb{C}$.

3.4. Classification of irreducible geometries

We now classify *irreducible* finite real spectral triples of KO-dimension 6. This leads to a remarkably concise list of spectral triples, based on the matrix algebras $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ for some N.

DEFINITION 3.19. *A finite real spectral triple $(A, H, D; J, \gamma)$ is called* irreducible *if the triple (A, H, J) is irreducible. More precisely, we demand that:*

(1) The representations of A and J in H are irreducible;
(2) The action of A on H has a separating vector (cf. Exercise 3.2).

THEOREM 3.20. *Let $(A, H, D; J, \gamma)$ be an irreducible finite real spectral triple of KO-dimension 6. Then there exists a positive integer N such that $A \simeq M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$.*

PROOF. Let $(A, H, D; J, \gamma)$ be an arbitrary finite real spectral triple, corresponding to *e.g.* the Krajewski diagram of Figure 3.2. Thus, as in Section 2.3 we have

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \qquad H = \bigoplus_{i,j=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij},$$

with V_{ij} corresponding to the multiplicities as before. Now each $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ is an irreducible representation of A, but in order for H to support a real structure $J : H \to H$ we need both $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ and $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i}$ to be present in H. Moreover, Lemma 3.8 with $J^2 = 1$ assures that already with multiplicities $\dim V_{ij} = 1$ there exists such a real structure. Hence, the irreducibility condition (1) above yields

$$H = \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j} \oplus \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i},$$

for some $i, j \in \{1, \ldots, N\}$. Or, as a Krajewski diagram:

$$\begin{array}{ccc} & \mathbf{n}_i & \mathbf{n}_j \\ \mathbf{n}_i^\circ & & \circ \\ \mathbf{n}_j^\circ & \circ & \end{array}$$

Then, let us consider condition (2) on the existence of a separating vector. Note first that the representation of A in H is faithful only if $A = M_{n_i}(\mathbb{C}) \oplus M_{n_j}(\mathbb{C})$. Second, the stronger condition of a separating vector ξ then implies $n_i = n_j$, as it is equivalent to $A'\xi = H$ for the commutant A' of A in H (see Exercise 3.2). Namely, since $A' = M_{n_j}(\mathbb{C}) \oplus M_{n_i}(\mathbb{C})$ with $\dim A' = n_i^2 + n_j^2$, and $\dim H = 2n_i n_j$ we find the desired equality $n_i = n_j$. $\square$

With the complex finite-dimensional algebras A given by $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$, the additional demand that H carries a symplectic structure $I^2 = -1$ yields real algebras of which A is the complexification (as in the proof of Lemma 3.18). In view of Exercise 3.8(4) we see that this requires $N = 2k$ so that one naturally considers triples (A, H, J) for which $A = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$ and $H = \mathbb{C}^{2(2k)^2}$. The case $k = 2$ will come back in the final Chapter 11 as the relevant one to consider in particle physics applications that go beyond the Standard Model.

Notes

Section 3.1. Finite real spectral triples

1. The operator D in Definition 3.1 is a first-order differential operator on the bimodule H in the sense of [**83**].

2. Exercises 3.2 and 3.3 develop Tomita–Takesaki theory for matrix algebras, considered as finite-dimensional Von Neumann algebras. For a complete treatment of this theory for general Von Neumann algebras, we refer to *e.g.* [**184**].

Section 3.2. Classification of finite real spectral triples

3. Krajewski's work on the classification of all finite real spectral triples $(A, H, D; J, \gamma)$ modulo unitary equivalence (based on a suggestion in [**63**]) is published in [**127**]. Similar results were obtained independently in [**158**]. We have extended Krajewski's work — which is in KO-dimension 0— to any KO-dimension. The classification of finite real spectral triples (but without Krajewski diagrams) is also the subject of [**42**]. The KO-dimension 6 case —which is of direct physical interest as we will see below in Chapter 11— was also handled in [**118**].

4. Lemma 3.8 is based on [**197**], where Wigner showed that anti-unitary operators on finite-dimensional Hilbert spaces can be written in a normal form. His crucial observation is that J^2 is unitary, allowing for a systematic study of a normal form of J for each of the eigenvalues of J^2 (these eigenvalues form a discrete subset of the complex numbers of modulus one). In our case of interest, J is a real structure on a spectral triple (as in Definition 3.1), so that $J^2 = \pm 1$.

5. In the labelling of the nodes in a Krajewski diagram with $\pm$-signs, it is important whether or not we adopt the so-called *orientation axiom* [**63**]. In the finite-dimensional case, this axiom demands that the grading γ can be implemented by elements $x_i, y_i \in A$ as $\gamma = \sum_i x_i y_i^\circ$. Hence, this is completely dictated by the operator J and the representation of A. In terms of our diagrams, this translates to the fact that the grading of a node only depends on the label $(\mathbf{n}_i, \mathbf{n}_j^\circ)$. In this book, we will not assume the orientation axiom.

Section 3.4. Classification of irreducible geometries

6. Finite irreducible geometries have been classified by Chamseddine and Connes in [**51**], using different methods. We here confront their result with the above approach to finite spectral triples using Krajewski diagrams and find that they are compatible.

CHAPTER 4

Noncommutative Riemannian spin manifolds

We now extend our treatment of noncommutative geometric spaces from the finite case to the continuum. This generalizes spin manifolds to the noncommutative world. The resulting spectral triples form the key technical device in noncommutative geometry, and in the physical applications of Part 2 of this book in particular.

We start with a treatment of Clifford algebras, as a preparation for the definition of a spin structure on a Riemannian manifold, and end with a definition of its noncommutative generalization.

4.1. Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C}$ or $\mathbb{H})$, equipped with a quadratic form $Q : V \to \mathbb{F}$, *i.e.*

$$Q(\lambda v) = \lambda^2 Q(v); \qquad (\lambda \in \mathbb{F}, v \in V),$$
$$Q(v+w) + Q(v-w) = 2Q(v) + 2Q(w); \qquad (v, w \in V).$$

Definition 4.1. *For a quadratic form Q on V, the* Clifford algebra $\mathrm{Cl}(V,Q)$ *is the algebra generated (over $\mathbb{F}$) by the vectors $v \in V$ and with unit 1 subject to the relation*

$$v^2 = Q(v)1. \tag{4.1.1}$$

Note that the Clifford algebra $\mathrm{Cl}(V,Q)$ is $\mathbb{Z}_2$-graded, with grading χ given by

$$\chi(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k,$$

which is indeed compatible with relation (4.1.1). Accordingly, we decompose

$$\mathrm{Cl}(V,Q) =: \mathrm{Cl}^0(V,Q) \oplus \mathrm{Cl}^1(V,Q)$$

into an even and odd part.

Exercise 4.1. *Show that in $\mathrm{Cl}(V,Q)$ we have*

$$vw + wv = 2g_Q(v,w),$$

where g_Q is the pairing $V \times V \to \mathbb{F}$ associated to Q, given by

$$g_Q(v,w) = \frac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right).$$

We also introduce the following convenient notation for the Clifford algebras for the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with the standard quadratic form $Q_n(x_1,\ldots,x_n) = x_1^2 + \cdots x_n^2$:

$$\begin{aligned}\mathrm{Cl}_n^+ &:= \mathrm{Cl}(\mathbb{R}^n, Q_n);\\ \mathrm{Cl}_n^- &:= \mathrm{Cl}(\mathbb{R}^n, -Q_n);\\ \mathbb{C}\mathrm{l}_n &:= \mathrm{Cl}(\mathbb{C}^n, Q_n).\end{aligned}$$

Both Cl_n^+ and Cl_n^- are algebras over $\mathbb{R}$ generated by $e_1,\ldots,e_n$ with relations

$$e_ie_j + e_je_i = \pm 2\delta_{ij}, \tag{4.1.2}$$

for all $i,j = 1,\ldots,n$. Moreover, the even part $(\mathrm{Cl}_n^\pm)^0$ of $\mathrm{Cl}_n^\pm$ consists of products of an even number of e_i's, and the odd part $(\mathrm{Cl}_n^\pm)^1$ of products of an odd number of e_i's.

The Clifford algebra $\mathbb{C}\mathrm{l}_n$ is the complexification of both Cl_n^+ and Cl_n^-, and is therefore generated over $\mathbb{C}$ by the same $e_1,\ldots,e_n$ satisfying (4.1.2).

Exercise 4.2.
(1) *Check that Equation* (4.1.2) *indeed corresponds to the defining relations in* $\mathrm{Cl}_n^\pm$.
(2) *Show that the elements* $e_{i_1}\cdots e_{i_r}$ *with* $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ *form a basis for* $\mathrm{Cl}_n^\pm$.
(3) *Conclude that* $\dim_{\mathbb{R}} \mathrm{Cl}_n^\pm = 2^n$ *and, accordingly,* $\dim_{\mathbb{C}} \mathbb{C}\mathrm{l}_n = 2^n$.
(4) *Find an isomorphism* $\mathrm{Cl}(\mathbb{C}^n, Q_n) \simeq \mathrm{Cl}(\mathbb{C}^n, -Q_n)$ *as Clifford algebras.*

Proposition 4.2. *The even part* $(\mathrm{Cl}_{n+1}^-)^0$ *of* Cl_{n+1}^- *is isomorphic to* Cl_n^-.

Proof. We construct a map $\Psi : \mathrm{Cl}_n^- \mapsto (\mathrm{Cl}_{n+1}^-)^0$ given on generators by

$$\Psi(e_i) = e_{n+1}e_i. \tag{4.1.3}$$

Indeed, for $i,j = 1,\ldots,n$ we have

$$\Psi(e_i)\Psi(e_j) + \Psi(e_j)\Psi(e_i) = e_ie_j + e_je_i = -2\delta_{ij} = \Psi(-2\delta_{ij}),$$

using $e_ie_{r+1} = -e_{r+1}e_i$ and $e_{r+1}e_{r+1} = -1$. Thus, Ψ extends to a homomorphism $\mathrm{Cl}_n^- \mapsto (\mathrm{Cl}_{n+1}^-)^0$. Moreover, since Ψ sends basis vectors in Cl_n^- to basis vectors in $(\mathrm{Cl}_{n+1}^-)^0$ and the dimensions of Cl_n^- and $(\mathrm{Cl}_{n+1}^-)^0$ coincide, it is an isomorphism. □

Exercise 4.3. *Show that the same expression* (4.1.3) *induces an isomorphism from* Cl_n^- *to the even part* $(\mathrm{Cl}_{n+1}^+)^0$ *and conclude that* $(\mathrm{Cl}_{n+1}^+)^0 \simeq (\mathrm{Cl}_{n+1}^-)^0$.

Next, we compute the Clifford algebras $\mathrm{Cl}_n^\pm$ and $\mathbb{C}\mathrm{l}_n$. We start with a recursion relation:

Proposition 4.3. *For any* $k \geq 1$ *we have*

$$\begin{aligned}\mathrm{Cl}_k^+ \otimes_{\mathbb{R}} \mathrm{Cl}_2^- &\simeq \mathrm{Cl}_{k+2}^-,\\ \mathrm{Cl}_k^- \otimes_{\mathbb{R}} \mathrm{Cl}_2^+ &\simeq \mathrm{Cl}_{k+2}^+.\end{aligned}$$

PROOF. The map $\Psi : \mathrm{Cl}^-_{k+2} \to \mathrm{Cl}^+_k \otimes_{\mathbb{R}} \mathrm{Cl}^-_2$ given on generators by

$$\Psi(e_i) = \begin{cases} 1 \otimes e_i & i = 1, 2 \\ e_{i-2} \otimes e_1 e_2 & i = 3, \ldots, n \end{cases}$$

extends to the desired isomorphism. □

Let us compute some of the Clifford algebras in lowest dimensions.

PROPOSITION 4.4.

$$\mathrm{Cl}^+_1 \simeq \mathbb{R} \oplus \mathbb{R}, \qquad \mathrm{Cl}^-_1 \simeq \mathbb{C},$$
$$\mathrm{Cl}^+_2 \simeq M_2(\mathbb{R}), \qquad \mathrm{Cl}^-_2 \simeq \mathbb{H}.$$

PROOF. The Clifford algebra Cl^+_1 is generated (over $\mathbb{R}$) by 1 and e_1 with relation $e_1^2 = 1$. We map Cl^+_1 linearly to the algebra $\mathbb{R} \oplus \mathbb{R}$ by sending

$$1 \mapsto (1, 1), \qquad e_1 \mapsto (1, -1).$$

A dimension count shows that this map is a bijection.

The Clifford algebra Cl^+_2 is generated by $1, e_1, e_2$ with relations

$$e_1^2 = 1, \qquad e_2^2 = 1, \qquad e_1 e_2 = -e_2 e_1.$$

A bijective map $\mathrm{Cl}^+_2 \xrightarrow{\sim} M_2(\mathbb{R})$ is given on generators by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We leave the remaining Cl^-_1 and Cl^+_2 as an illustrative exercise to the reader. □

EXERCISE 4.4. *Show that* $\mathrm{Cl}^-_1 \simeq \mathbb{C}$ *and* $\mathrm{Cl}^-_2 \simeq \mathbb{H}$.

Combining the above two Propositions, we derive Table 4.1 for the Clifford algebras $\mathrm{Cl}^{\pm}_n$ and $\mathbb{C}\mathrm{l}_n$ for $n = 1, \ldots, 8$. For instance,

$$\mathrm{Cl}^+_3 \simeq \mathrm{Cl}^-_1 \otimes_{\mathbb{R}} \mathrm{Cl}^+_2 \simeq \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R}) \simeq M_2(\mathbb{C})$$

and

$$\mathrm{Cl}^+_4 \simeq \mathrm{Cl}^-_2 \otimes_{\mathbb{R}} \mathrm{Cl}^+_2 \simeq \mathbb{H} \otimes_{\mathbb{R}} M_2(\mathbb{R}) \simeq M_2(\mathbb{H})$$

and so on. In particular, we have

$$\mathrm{Cl}^+_n \otimes \mathrm{Cl}^+_4 \simeq \mathrm{Cl}_{n+4}$$

and

$$\mathrm{Cl}^+_{n+8} \simeq \mathrm{Cl}^+_n \otimes \mathrm{Cl}^+_8 \,.$$

With $\mathrm{Cl}^+_8 \simeq M_{16}(\mathbb{R})$ we conclude that Cl^+_{k+8} is *Morita equivalent* to Cl^+_k (*cf.* Theorem 2.14). Similarly, Cl^-_{k+8} is Morita equivalent to Cl^-_k. Thus, in this sense Table 4.1 has periodicity eight and we have determined $\mathrm{Cl}^{\pm}_n$ for all n.

For the complex Clifford algebras, there is a periodicity of two:

$$\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}_2 \simeq \mathbb{C}\mathrm{l}_{n+2},$$

so that with $\mathbb{C}\mathrm{l}_2 \simeq M_2(\mathbb{C})$ we find that $\mathbb{C}\mathrm{l}_n$ is Morita equivalent to $\mathbb{C}\mathrm{l}_{n+2}$.

The (semi)simple structure of $\mathbb{C}\mathrm{l}_n$ is further clarified by

n	Cl_n^+	Cl_n^-	$\mathbb{C}\mathrm{l}_n$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$
2	$M_2(\mathbb{R})$	$\mathbb{H}$	$M_2(\mathbb{C})$
3	$M_2(\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$
5	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$
6	$M_4(\mathbb{H})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$
7	$M_8(\mathbb{C})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$

TABLE 4.1. Clifford algebras $\mathrm{Cl}_n^\pm$ and their complexifications $\mathbb{C}\mathrm{l}_n$ for $n = 1, \ldots, 8$.

DEFINITION 4.5. *The* chirality operator γ_{n+1} *in* $\mathbb{C}\mathrm{l}_n$ *is defined as the element*

$$\gamma_{n+1} = (-i)^m e_1 \cdots e_n,$$

where $n = 2m$ or $n = 2m+1$, depending on whether n is even or odd.

EXERCISE 4.5. *Show that*

(1) *if $n = 2m$ is even, then γ_{n+1} generates the center of $\mathbb{C}\mathrm{l}_n$,*
(2) *if $n = 2m+1$ is odd, then γ_{n+1} lies in the odd part $\mathbb{C}\mathrm{l}_{2k+1}^1$, and the center of $\mathbb{C}\mathrm{l}_n$ is generated by 1 and γ_{n+1}.*

4.1.1. Representation theory of Clifford algebras. We determine the irreducible representations of the Clifford algebras $\mathrm{Cl}_n^\pm$ and $\mathbb{C}\mathrm{l}_n$. Let us start with the complex Clifford algebras.

PROPOSITION 4.6. *The irreducible representations of $\mathbb{C}\mathrm{l}_n$ are given as*

$$\mathbb{C}^{2^m}; \qquad (n = 2m),$$
$$\mathbb{C}^{2^m}, \mathbb{C}^{2^m}; \qquad (n = 2m+1).$$

PROOF. Since the $\mathbb{C}\mathrm{l}_n$ are matrix algebras we can invoke Lemma 2.15 to conclude that in the even-dimensional case the irreducible representation of $\mathbb{C}\mathrm{l}_{2m} \simeq M_{2^m}(\mathbb{C})$ is given by the defining representation $\mathbb{C}^{2^m}$. In the odd-dimensional case we have

$$\mathbb{C}\mathrm{l}_{2m+1} \simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}),$$

so that the irreducible representations are given by two copies of $\mathbb{C}^{2^m}$, corresponding to the two summands in this matrix algebra. □

For the real Clifford algebras $\mathrm{Cl}_n^\pm$ we would like to obtain the irreducible representations from those just obtained for the complexification $\mathbb{C}\mathrm{l}_n \simeq \mathrm{Cl}_n^\pm \otimes_{\mathbb{R}} \mathbb{C}$. As $\mathrm{Cl}_n^\pm$ are matrix algebras over $\mathbb{R}$ and $\mathbb{H}$, this leads us to the following possibilities:

(1) Restrict an (irreducible) representation of $\mathbb{C}\mathrm{l}_n$ to a real subspace, stable under $\mathrm{Cl}_n^\pm$;
(2) Extend an (irreducible) representation of $\mathbb{C}\mathrm{l}_n$ to a quaternionic space, carrying a representation of $\mathrm{Cl}_n^\pm$.

This is very similar to our approach to real algebras in Section 3.3. In fact, we will use an anti-linear map $J_n^\pm$ on the representation space, furnishing it with a real ($(J_n^\pm)^2 = 1$) or quaternionic structure ($(J_n^\pm)^2 = -1$) to select the real subalgebra $\mathrm{Cl}_n^\pm \subset \mathbb{C}\mathrm{l}_n$. For the even-dimensional case we search for operators $J_{2m}^\pm$ such that on the irreducible $\mathbb{C}\mathrm{l}_{2m}$-representations $\mathbb{C}^{2^m}$ we have

$$\mathrm{Cl}_{2m}^\pm \simeq \left\{ a \in \mathbb{C}\mathrm{l}_{2m} : [J_{2m}^\pm, a] = 0 \right\}. \tag{4.1.4}$$

The odd case is slightly more subtle, as only the even part $(\mathrm{Cl}_n^\pm)^0$ of $\mathrm{Cl}_n^\pm$ can be recovered in this way:

$$(\mathrm{Cl}_{2m+1}^\pm)^0 \simeq \left\{ a \in \mathbb{C}\mathrm{l}_{2m+1}^0 : [J_{2m+1}^\pm, a] = 0 \right\}. \tag{4.1.5}$$

PROPOSITION 4.7. *For any $m \geq 1$ there exist anti-linear operators $J_{2m}^\pm : \mathbb{C}^{2^m} \to \mathbb{C}^{2^m}$ and $J_{2m+1}^\pm : \mathbb{C}^{2^m} \to \mathbb{C}^{2^m}$ such that the Equations* (4.1.4) *and* (4.1.5) *hold.*

PROOF. From Proposition 4.2 and Exercise 4.3 we see that $(\mathrm{Cl}_{2m+1}^\pm)^0 \simeq \mathrm{Cl}_{2m}^-$ and $(\mathbb{C}\mathrm{l}_{2m+1})^0 \simeq \mathbb{C}\mathrm{l}_{2m}$ so that the odd case follows from the even case.

By periodicity we can further restrict to construct only $J_{2m}^\pm$ for $m = 1,2,3,4$. For $m = 1$ we select the real form $\mathrm{Cl}_2^+ \simeq M_2(\mathbb{R})$ in $\mathbb{C}\mathrm{l}_2 \simeq M_2(\mathbb{C})$ as the commutant of J_2^+ with

$$\begin{aligned} J_2^+ &: \mathbb{C}^2 \to \mathbb{C}^2; \\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &\mapsto \begin{pmatrix} \overline{v_1} \\ \overline{v_2} \end{pmatrix}. \end{aligned}$$

Instead, as in Example 3.16, $\mathrm{Cl}_2^- \simeq \mathbb{H}$ can be identified as a real subalgebra $\mathbb{C}\mathrm{l}_2 \simeq M_2(\mathbb{C})$ with the commutant of J_2^-, where

$$\begin{aligned} J_2^- &: \mathbb{C}^2 \to \mathbb{C}^2; \\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &\mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}. \end{aligned}$$

For $m = 2$ the sought-for operator $J_4^+ \equiv J_4^-$ on $\mathbb{C}^4$ is given by $J_2^- \oplus J_2^-$.

For $m = 3$ we set $J_6^+ = (J_2^-)^{\oplus 4}$ to select $\mathrm{Cl}_6^+ \simeq M_4(\mathbb{H})$ inside $\mathbb{C}\mathrm{l}_6$, and $J_6^- = (J_2^+)^{\oplus 4}$ to select $\mathrm{Cl}_6^- \simeq M_8(\mathbb{R})$.

Finally, for $m = 4$ the operator $J_8^+ \equiv J_8^- := (J_2^+)^{\oplus 8}$ selects the two isomorphic real forms $\mathrm{Cl}_8^\pm \subset \mathbb{C}\mathrm{l}_8$. □

The signs for the squares $(J_n^\pm)^2$ are listed in Table 4.2. The isomorphisms between the odd- and even-dimensional cases are illustrated by the fact that

$$(J_{2m+1}^\pm)^2 = (J_{2m}^-)^2.$$

with periodicity eight. We also indicated the commutation between $J_n^\pm$ and odd elements in $\mathrm{Cl}_n^\pm$ and between $J_n^\pm$ and the chirality operator γ_{n+1}. For the derivation of the former note that for n even $J_n^\pm$ commutes with all elements in $\mathrm{Cl}_n^\pm$, whereas for n odd we follow the proof of Proposition 4.7:

- $n = 1$: J_1^- is equal to J_0^-, which is given by $J_0^-(z) = \overline{z}$ for $z \in \mathbb{C}$, and (4.1.5) selects $(\mathrm{Cl}_1^-)^0 \simeq \mathbb{R}$ in $\mathrm{Cl}_1^- \simeq \mathbb{C}$. Thus, the remaining part $(\mathrm{Cl}_1^-)^1 \simeq i\mathbb{R}$ so that odd elements $x \in (\mathrm{Cl}_1^-)^1$ anti-commute with J_1^-.

n	1	2	3	4	5	6	7	8
$(J_n^+)^2 = \pm 1$	1	1	−1	−1	−1	−1	1	1
$(J_n^-)^2 = \pm 1$	1	−1	−1	−1	−1	1	1	1
$J_n^- x = (\pm 1) x J_n^-, \quad x$ odd	−1	1	1	1	−1	1	1	1
$J_n^- \gamma_{n+1} = (\pm 1)\gamma_{n+1} J_n^-$		−1		1		−1		1

TABLE 4.2. The real and quaternionic structures on the irreducible representations of $\mathbb{C}\mathrm{l}_n$ that select $\mathrm{Cl}_n^\pm$ via (4.1.4) for n even and $(\mathrm{Cl}_n^\pm)^0$ via (4.1.5) for n odd. For later reference, we also indicated the commutation or anti-commutation of J_n^- with the chirality operator γ_{n+1} defined in Definition 4.5 and odd elements in $(\mathrm{Cl}_n^\pm)^1 \subset \mathrm{Cl}_n^\pm$.

- $n = 3$: J_3^- is equal to J_2^-, which is given by the standard quaternionic structure on $\mathbb{C}^2$. It then follows that all of $\mathrm{Cl}_3^- \simeq \mathbb{H} \oplus \mathbb{H}$ commutes with J_3^-.
- $n = 5$: in this case J_5^- is equal to J_4^-, which is two copies of J_2^-. This selects $(\mathrm{Cl}_5^-)^0 \simeq M_2(\mathbb{H})$ in $\mathrm{Cl}_5^- \simeq M_4(\mathbb{C})$. Again, the remaining part $(\mathrm{Cl}_5^-)^1 \simeq iM_2(\mathbb{H})$ so that odd elements $x \in (\mathrm{Cl}_5^-)^0$ anti-commute with J_5^-.
- $n = 7$: J_7^- is equal to J_6^-, which is given by component-wise complex conjugation of vectors in $\mathbb{C}^8$. It follows that all of $\mathrm{Cl}_7^- \simeq M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$ commutes with J_6^-.

Finally, in the even case $n = 2m$ the (anti)-commutation between the chirality operator γ_{n+1} and the *anti-linear* operator J_n^- depends only on the power of the factor i^m. Indeed, the even product of e_i's in Definition 4.5 already commutes with J_n^-, so that the signs $(-1)^m$ for $n = 2m$ follow from

$$J_n^- i^m = (-i)^m J_n^-.$$

The last three rows of Table 4.2 give precisely the sign table that appears for real spectral triples below, where n is the corresponding KO-dimension, and hence coincide with Table 3.1 of Definition 3.1. We will now slowly move to the spin manifold case, tracing KO-dimension back to its historical roots.

4.2. Riemannian spin geometry

We here give a concise introduction to Riemannian spin manifolds and work towards a Dirac operator. For convenience, we restrict to compact manifolds.

4.2.1. Spin manifolds. The definition of Clifford algebras can be extended to Riemannian manifolds, as we will now explain. First, for completeness we recall the definition of a Riemannian metric on a manifold.

DEFINITION 4.8. *A* Riemannian metric *on a manifold M is a symmetric bilinear form on vector fields* $\Gamma(TM)$

$$g : \Gamma(TM) \times \Gamma(TM) \to C(M)$$

such that

(1) $g(X,Y)$ *is a real function if* X *and* Y *are real vector fields;*
(2) g *is* $C(M)$*-bilinear:*

$$g(fX,Y) = g(X,fY) = fg(X,Y); \qquad (f \in C(M));$$

(3) $g(X,X) \geq 0$ *for all real vector fields* X *and* $g(X,X) = 0$ *if and only if* $X = 0$.

The non-degeneracy condition (3) allows us to identify $\Gamma(TM)$ with $\Omega^1_{\mathrm{dR}}(M) = \Gamma(T^*M)$.

A Riemannian metric g on M gives rise to a **distance function** on M, given as an infimum of path lengths

$$d_g(x,y) = \inf_\gamma \left\{ \int_0^1 \sqrt{g(\dot\gamma(t),\dot\gamma(t))}dt : \gamma(0) = x, \gamma(1) = y \right\}. \tag{4.2.1}$$

Moreover, the inner product that g defines on the fibers T_xM of the tangent bundle allows us to define Clifford algebras at each point in M as follows. With the inner product at $x \in M$ given explicitly by $g_x(X_x, Y_x) := g(X,Y)|_x$ we consider the quadratic form on T_xM defined by

$$Q_g(X_x) = g_x(X_x, X_x).$$

We can then apply the construction of the Clifford algebra of the previous section to each fiber of the tangent bundle. At each point $x \in M$ this gives rise to $\mathrm{Cl}(T_xM, Q_g)$ and its complexification $\mathbb{C}\mathrm{l}(T_xM, Q_g)$. When x varies, these Clifford algebras combine to give a bundle of algebras.

Definition 4.9. *The* Clifford algebra bundle $\mathrm{Cl}^+(TM)$ *is the bundle of algebras* $\mathrm{Cl}(T_xM, Q_g)$*, with the transition functions inherited from* TM*. Namely, transition functions on the tangent bundle are given for open* $U, V \subset M$ *by* $t_{UV} : U \cap V \to SO(n)$ *where* $n = \dim M$*. Their action on each fiber* T_xM *can be extended to* $\mathrm{Cl}(T_xM, Q_x)$ *by*

$$v_1 v_2 \cdots v_k \mapsto t_{UV}(v_1) \cdots t_{UV}(v_k); \qquad (v_1, \ldots, v_k \in T_xM).$$

The algebra of continuous real-valued sections of $\mathrm{Cl}^+(TM)$ *will be denoted by* $\mathrm{Cliff}^+(M) = \Gamma(\mathrm{Cl}^+(TM))$.

Similarly, replacing Q_g *by* $-Q_g$*, we define* $\mathrm{Cliff}^-(M)$ *as the space of sections of* $\mathrm{Cl}^-(TM)$.

Finally, we define the complexified algebra

$$\mathbb{C}\mathrm{liff}(M) := \mathrm{Cliff}^+(M) \otimes_{\mathbb{R}} \mathbb{C},$$

consisting of continuous sections of the bundle of complexified algebras $\mathbb{C}\mathrm{l}(TM)$*, which is defined in a similar manner.*

Let us determine local expressions for the algebra $\mathrm{Cliff}^+(M)$. If $\{x^\mu\}_{\mu=1}^n$ are local coordinates on a chart U of M, the algebra of sections of $\mathrm{Cliff}^+(M)|_U$ is generated by γ_μ with relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \tag{4.2.2}$$

with $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$. After choosing an orthonormal basis for $\Gamma(TM)|_U$ with respect to the metric g, at a point of U this relation reduces precisely to the relation (4.1.2).

Let us see if we can import more of the structure for Clifford algebras explored so far to the setting of a Riemannian manifold. First, recall that

$$\mathbb{C}\mathrm{l}_{2m} \cong M_{2^m}(\mathbb{C}), \qquad \mathbb{C}\mathrm{l}^0_{2m+1} \cong M_{2^m}(\mathbb{C}).$$

Another way of phrasing this is to say that the (even parts of the) Clifford algebras $\mathbb{C}\mathrm{l}_n$ are endomorphism algebras $\mathrm{End}(\mathbb{C}^{2^m})$. The natural question that arises in the setting of Riemannian manifolds is whether or not this holds for all fibers of the Clifford algebra bundle, in which case it would extend to a global isomorphism of algebra bundles.

Definition 4.10. *A Riemannian manifold is called* spin^c *if there exists a vector bundle $S \to M$ such that there is an algebra bundle isomorphism*

$$\mathbb{C}\mathrm{l}(TM) \simeq \mathrm{End}(S) \quad (M \textit{ even-dimensional}),$$
$$\mathbb{C}\mathrm{l}(TM)^0 \simeq \mathrm{End}(S) \quad (M \textit{ odd-dimensional}).$$

The pair (M, S) is called a spin^c structure *on M.*

If a spin^c structure (M, S) exists we refer to S as the **spinor bundle** and the sections in $\Gamma(S)$ as **spinors**. Using the metric and the action of $\mathrm{Cliff}^+(M)$ by endomorphisms on $\Gamma(S)$ we introduce the following notion.

Definition 4.11. *Let (M, S) be a spin^c structure on M.* Clifford multiplication *is defined by the linear map*

$$c : \Omega^1_{\mathrm{dR}}(M) \times \Gamma(S) \to \Gamma(S);$$
$$(\omega, \psi) \mapsto \omega^\# \cdot \psi,$$

where $\omega^\#$ is the vector field in $\Gamma(TM)$ corresponding to the one-form $\omega \in \Omega^1_{\mathrm{dR}}(M)$ via the metric g. This vector field acts as an endomorphism on $\Gamma(S)$ via the embedding $\Gamma(TM) \hookrightarrow \mathrm{Cliff}^+(M) \subset \Gamma\,\mathrm{End}(S)$.

In local coordinates on $U \subset M$, we can write $\omega|_U = \omega_\mu dx^\mu$ with $\omega_\mu \in C(U)$ so that Clifford multiplication can be written as

$$c(\omega)\psi|_U \equiv c(\omega, \psi)|_U = \omega_\mu(\gamma^\mu \psi)|_U; \qquad (\psi \in \Gamma(S)),$$

with $\gamma^\mu = g^{\mu\nu}\gamma_\nu$ and γ_ν as in (4.2.2) but now represented as endomorphisms on the fibers of S. The appearance of γ^μ comes from the identification of the basis covector $dx^\mu \in \Omega^1_{\mathrm{dR}}(M)|_U$ with the basis vector $\partial_\mu \in \Gamma(TM)|_U$ using the metric, which is then embedded in $\mathrm{Cliff}^+(M)$. That is, we have

$$dx^\mu_p = g(\partial_\mu, \cdot)_p$$

as (non-degenerate) maps from T_pM to $\mathbb{C}$ with $p \in U \subset M$.

Recall that if M is compact, then any vector bundle carries a continuously varying inner product on its fibers,

$$\langle \cdot, \cdot \rangle : \Gamma(S) \times \Gamma(S) \to C(M).$$

Exercise 4.6. *Use a partition of unity argument to show that any vector bundle on a compact manifold M admits a continuously varying inner product on its fibers.*

DEFINITION 4.12. *The* Hilbert space of square-integrable spinors $L^2(S)$ *is defined as the completion of* $\Gamma(S)$ *in the norm corresponding to the inner product*

$$(\psi_1, \psi_2) = \int_M \langle \psi_1, \psi_2 \rangle (x) \sqrt{g} dx,$$

where $\sqrt{g}dx$ *is the Riemannian volume form.*

Recall that in the previous subsection we selected the real Clifford algebras $\mathrm{Cl}_n^\pm$ as subalgebras in $\mathbb{C}\mathrm{l}_n$ that commute with a certain anti-linear operator $J_n^\pm$. We now try to select $\mathrm{Cliff}^\pm(M) \subset \mathbb{C}\mathrm{liff}(M)$, considered as endomorphisms on $\Gamma(S)$, through a globally-defined operator $J_M : \Gamma(S) \to \Gamma(S)$, so that

$$(J_M \psi)(x) = J_n^\pm(\psi(x)),$$

for any section $\psi \in \Gamma(S)$, where $n = \dim M$. Such a global operator does not always exist: this gives rise to the notion of a spin manifold. It is conventional to work with J_n^- to select $\mathrm{Cliff}^-(M) \subset \mathbb{C}\mathrm{liff}(M)$, making our sign Table 4.2 fit with the usual definition of KO-dimension in noncommutative geometry.

DEFINITION 4.13. *A Riemannian spin*^{c} *manifold is called* spin *if there exists an anti-unitary operator* $J_M : \Gamma(S) \to \Gamma(S)$ *such that:*

(1) J_M *commutes with the action of real-valued continuous functions on* $\Gamma(S)$*;*
(2) J_M *commutes with* $\mathrm{Cliff}^-(M)$ *(or with* $\mathrm{Cliff}^-(M)^0$ *in the odd case).*

We call the pair (S, J_M) *a* spin structure *on* M *and refer to the operator* J_M *as the* charge conjugation*.*

If the manifold M is even dimensional, we can define a grading

$$(\gamma_M \psi)(x) = \gamma_{n+1}(\psi(x)); \qquad (\psi \in \Gamma(S)).$$

Then, the sign rules of Table 4.2 for the square of J_n^- and the (anti)-commutation of J_n^- with γ_{n+1} and odd elements in Cl_n^- hold in each fiber of $\Gamma(S)$. Hence, we find that also globally

$$J_M^2 = \epsilon, \qquad J_M x = \epsilon' x J_M; \quad (x \in (\mathrm{Cliff}^-(M))^1, \qquad J_M \gamma_M = \epsilon'' \gamma_M J_M,$$

with $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ being the signs in Table 4.2 with $n = \dim M$ modulo eight. This will be crucial for our definition of a real spectral triple in the next section, where these signs determine the KO-dimension of a noncommutative Riemannian spin manifold.

4.2.2. Spin connection and Dirac operator. The presence of a spin structure on a Riemannian manifold allows for the construction of a first-order differential operator that up to a scalar term squares to the Laplacian associated to g. This is the same operator that Dirac searched for (with success) in his attempt to replace the Schrödinger equation by a more general covariant differential equation in Minkowski space. The Dirac operator that we will describe below is the analogue for Riemannian spin manifolds of Dirac's operator on flat Minkowski space. In order to allow for differentiation, we will restrict to smooth, rather than continuous sections.

DEFINITION 4.14. *A* connection *on a vector bundle $E \to M$ is given by a $\mathbb{C}$-linear map on the space of smooth sections:*

$$\nabla : \Gamma^\infty(E) \to \Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$$

that satisfies the Leibniz rule

$$\nabla(f\eta) = f\nabla(\eta) + df \otimes \eta; \qquad (f \in C^\infty(M), \eta \in \Gamma^\infty(E)).$$

The curvature Ω^E *of ∇ is defined by the $C^\infty(M)$-linear map*

$$\Omega^E := \nabla^2 : \Gamma^\infty(E) \to \Omega^2(M) \otimes_{C^\infty(M)} \Gamma^\infty(E).$$

Finally, if $\langle \cdot, \cdot \rangle$ is a smoothly varying (i.e. $C^\infty(M)$-valued) inner product on $\Gamma^\infty(E)$, a connection is said to be hermitian*, or* compatible *if*

$$\langle \nabla\eta, \eta' \rangle + \langle \eta, \nabla\eta' \rangle = d\langle \eta, \eta' \rangle; \qquad (\eta, \eta' \in \Gamma^\infty(E)).$$

Equivalently, when evaluated on a vector field $X \in \Gamma^\infty(TM)$ a connection gives rise to a map

$$\nabla_X : \Gamma^\infty(E) \to \Gamma^\infty(E).$$

More precisely, the relation with the above definition is given by

$$\nabla_X(\eta) := \nabla(\eta)(X),$$

for all $X \in \Gamma^\infty(TM)$ and $\eta \in \Gamma^\infty(E)$. The corresponding curvature then becomes

$$\Omega^E(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}; \qquad (X, Y \in \Gamma^\infty(TM)), \tag{4.2.3}$$

i.e. it is a measure of the defect of ∇ to be a Lie algebra map.

EXAMPLE 4.15. *Consider the tangent bundle $TM \to M$ on a Riemannian manifold (M, g). A classical result is that there is a unique connection on TM that is compatible with the inner product g on $\Gamma(TM)$, i.e.*

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = X(\langle Y, Z \rangle)$$

and that is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y]; \qquad (X, Y \in \Gamma^\infty(TM)).$$

This connection is called the Levi–Civita connection *and can be written in local coordinates $\{x^\mu\}_{\mu=1}^n$ on a chart $U \subset M$ as $\nabla(\partial_\nu) = \Gamma^\kappa_{\mu\nu} dx^\mu \otimes \partial_\kappa$, or*

$$\nabla_{\partial_\mu}(\partial_\nu) = \Gamma^\kappa_{\mu\nu}\partial_\kappa.$$

The $C^\infty(U)$-valued coefficients $\Gamma^\kappa_{\mu\nu}$ are the so-called Christoffel symbols *and torsion-freeness corresponds to the symmetry $\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$.*

Recall also the definition of the Riemannian curvature tensor *on (M, g) as the curvature of the Levi–Civita connection, i.e.*

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \Gamma(\mathrm{End}\, TM),$$

which is indeed a $C^\infty(M)$-linear map. Locally, we have for its components

$$R_{\mu\nu\kappa\lambda} := g(\partial_\mu, R(\partial_\kappa, \partial_\lambda)\partial_\nu).$$

The contraction $R_{\nu\lambda} := g^{\mu\kappa} R_{\mu\nu\kappa\lambda}$ is called the Ricci tensor*, and the subsequent contraction $s := g^{\nu\lambda} R_{\nu\lambda} \in C^\infty(M)$ is the* scalar curvature.

Similar results hold for the cotangent bundle, with the unique, compatible, torsion-free connection thereon related to the above via the metric g.

DEFINITION 4.16. *If ∇^E is a connection on a vector bundle E, the* Laplacian *associated to ∇^E is the second order differential operator on E defined by*

$$\Delta^E := -\operatorname{Tr}_g(\nabla \otimes 1 + 1 \otimes \nabla^E) \circ \nabla^E : \Gamma^\infty(E) \to \Gamma^\infty(E),$$

where

$$\nabla \otimes 1 + 1 \otimes \nabla^E : \Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Gamma^\infty(E) \\ \to \Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$$

is the combination of the Levi–Civita connection on the cotangent bundle with the connection ∇^E and Tr_g is the trace associated to g mapping $\Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Omega^1_{\mathrm{dR}}(M) \to C^\infty(M)$.

Locally, we find

$$\Delta^E = -g^{\mu\nu}(\nabla^E_\mu \nabla^E_\nu - \Gamma^\kappa_{\mu\nu}\nabla^E_\kappa).$$

If M is a Riemannian spin^c manifold, then the above Levi–Civita connection can be lifted to the spinor bundle. First, choose a local orthonormal basis for $TM|_U$:

$$\{E_1, \ldots, E_n\} \text{ for } \Gamma(TM)|_U : \qquad g(E_a, E_b) = \delta_{ab}.$$

The corresponding dual orthonormal basis of $T^*M|_U$ is denoted by θ^a. We can then write the Christoffel symbols in this basis, namely by

$$\nabla E_a =: \widetilde{\Gamma}^b_{\mu a} dx^\mu \otimes E_b$$

on vector fields, and on one-forms by

$$\nabla \theta^b = -\widetilde{\Gamma}^b_{\mu a} dx^\mu \otimes \theta^a.$$

The local orthonormal basis for $TM|_U$ allows us to write Clifford relations for (globally) fixed matrices γ^a:

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}; \qquad (a, b = 1, \ldots, n). \tag{4.2.4}$$

DEFINITION 4.17. *Let M be a spin^c manifold. The* spin connection *∇^S on the spinor bundle $S \to M$ is given as the lift of the Levi–Civita connection to the spinor bundle, written locally as*

$$\nabla^S_\mu \psi(x) = \left(\partial_\mu - \frac{1}{4}\widetilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b\right)\psi(x).$$

PROPOSITION 4.18. *If M is a spin manifold and J_M is the corresponding anti-unitary operator on $\Gamma(S)$, then the spin connection commutes with J_M.*

PROOF. Observe that the product $\gamma^a\gamma^b = -(i\gamma^a)(i\gamma^b)$ is in the even part of the Clifford algebra Cl^-_n, since

$$(i\gamma^a)(i\gamma^b) + (i\gamma^b)(i\gamma^a) = -2\delta^{ab}.$$

Since by definition the operator J^-_n commutes with the even elements in Cl^-_n acting fiberwise on the spinor bundle, the result follows. □

DEFINITION 4.19. *Let M be a spin manifold, with spin structure (S, J_M). The* Dirac operator D_M *is the composition of the spin connection on S with Clifford multiplication of Definition 4.11:*

$$D_M : \Gamma^\infty(S) \xrightarrow{\nabla^S} \Omega^1_{\mathrm{dR}}(M) \otimes_{C^\infty(M)} \Gamma^\infty(S) \xrightarrow{-ic} \Gamma^\infty(S).$$

In local coordinates, we have

$$D_M\psi(x) = -i\gamma^\mu \left(\partial_\mu - \frac{1}{4}\widetilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b \right) \psi(x).$$

The final result from this subsection forms the starting point for an operator-algebraic formulation of noncommutative Riemannian spin manifolds.

THEOREM 4.20. *The operator D_M is self-adjoint on $L^2(S)$ with compact resolvent $(i+D)^{-1}$, and has bounded commutators with elements in $C^\infty(M)$. In fact*

$$[D_M, f] = -ic(df),$$

so that $\|[D_M, f]\| = \|f\|_{Lip}$ is the Lipschitz (semi)-norm of f:

$$\|f\|_{Lip} = \sup_{x \neq y} \left\{ \frac{f(x) - f(y)}{d_g(x, y)} \right\}.$$

PROOF. See Note 7 on Page 60. □

4.2.3. Lichnerowicz formula. Let us come back to the original motivation of Dirac, which was to find an operator whose square is the Laplacian. Up to a scalar this continues to hold for the Dirac operator on a Riemannian spin manifold, a result that will turn out to be very useful later on in our physical applications. For this reason we include it here with proof.

THEOREM 4.21. *Let (M, g) be a Riemannian spin manifold with Dirac operator D_M. Then*

$$D_M^2 = \Delta^S + \frac{1}{4}s,$$

in terms of the Laplacian Δ^S associated to the spin connection ∇^S and the scalar curvature s.

PROOF. We exploit the local expressions for D_M, Δ^S and s, as the above formula is supposed to hold in each chart that trivializes S. With $D_M = -i\gamma^\mu\nabla^S_\mu$ we compute

$$\begin{aligned} D_M^2 &= -\gamma^\mu\nabla^S_\mu\gamma^\nu\nabla^S_\nu = -\gamma^\mu\gamma^\nu\nabla^S_\mu\nabla^S_\nu - \gamma^\mu c(\nabla_\mu dx^\kappa)\nabla^S_\kappa \\ &= -\gamma^\mu\gamma^\nu(\nabla^S_\mu\nabla^S_\nu - \Gamma^\kappa_{\mu\nu}\nabla^S_\kappa). \end{aligned}$$

We then use the Clifford relations (4.2.2) to write $\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + g^{\mu\nu}$, and combine this with torsion freedom $\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$ to obtain

$$D_M^2 = -g^{\mu\nu}(\nabla^S_\mu\nabla^S_\nu - \Gamma^\kappa_{\mu\nu}\nabla^S_\kappa) - \frac{1}{2}[\gamma^\mu, \gamma^\nu]\nabla^S_\mu\nabla^S_\nu \equiv \Delta^S - \frac{1}{2}\gamma^\mu\gamma^\nu R^S(\partial_\mu, \partial_\nu),$$

in terms of the Laplacian for ∇^S on S and the curvature R^S thereof. The latter is given by $-\frac{1}{4}R_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda$, as one can easily compute from the explicit local form of ∇^S in Definition 4.17. Thus,

$$D_M^2 = \Delta^S - \frac{1}{8}R_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda.$$

Using the cyclic symmetry of the Riemann curvature tensor in the last three indices, and the Clifford relations (4.2.2) we find that the second term on the right-hand side is equal to $\frac{1}{4}R_{\nu\lambda}g^{\nu\lambda} = \frac{1}{4}s$, in terms of the scalar curvature defined in Example 4.15. □

4.3. Noncommutative Riemannian spin manifolds: spectral triples

This section introduces the main technical device that generalizes Riemannian spin geometry to the noncommutative world. The first step towards noncommutative manifolds is to arrive at an algebraic characterization of topological spaces. This is accomplished by **Gelfand duality**, giving a one-to-one correspondence between compact Hausdorff topological spaces and commutative C^*-algebras. Let us recall some definitions.

Definition 4.22. *A* C^*-algebra A *is a (complex)* $*$*-algebra (Definition 2.1) that is complete with respect to a multiplicative norm (i.e.* $\|ab\| \leq \|a\|\|b\|$ *for all* $a, b \in A$*) that satisfies the* C^**-property:*

$$\|a^*a\| = \|a\|^2.$$

Example 4.23. *The key example of a commutative* C^**-algebra is the algebra* $C(X)$ *for a compact topological space* X*. Indeed, uniform continuity is captured by the norm*

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

and involution defined by $f^*(x) = \overline{f(x)}$*. This indeed satisfies* $\|f^*f\| = \|f\|^2$*.*

Example 4.24. *Another key example where* A *is noncommutative is given by the* $*$*-algebra of bounded operators* $\mathcal{B}(\mathcal{H})$ *on a Hilbert space* $\mathcal{H}$*, equipped with the operator norm.*

The following result connects with the matrix algebras of Chapter 2.

Proposition 4.25. *If* A *is a finite-dimensional* C^**-algebra, then it is isomorphic to a matrix algebra:*

$$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}).$$

Proof. See Note 9 on Page 60. □

In Chapter 2 we defined the structure space of a $*$-algebra A to consist of (equivalence classes of) irreducible representations of A. Let us extend this definition to C^*-algebras.

Definition 4.26. *A* representation *of a C^*-algebra A is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a $*$-algebra map*

$$\pi : A \to \mathcal{B}(\mathcal{H}).$$

A representation $(\mathcal{H}, \pi)$ is irreducible *if $H \neq 0$ and the only closed subspaces in $\mathcal{H}$ that are left invariant under the action of A are $\{0\}$ and $\mathcal{H}$.*

Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of a C^*-algebra A are unitarily equivalent *if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\pi_1(a) = U^* \pi_2(a) U.$$

Definition 4.27. *The* structure space $\widehat{A}$ *of a C^*-algebra A is the set of all unitary equivalence classes of irreducible representations of A.*

In Chapter 4 we considered the commutative matrix algebra $\mathbb{C}^N$ whose structure space was the finite topological space consisting of N points. Let us sketch the generalization to compact Hausdorff topological spaces, building towards **Gelfand duality**. As a motivating example, we consider the C^*-algebra $C(X)$ for a compact Hausdorff topological space X (*cf.* Example 4.23). As this C^*-algebra is commutative, a standard argument shows that any irreducible representation π of $C(X)$ is one-dimensional. In fact, any such π is equivalent to the evaluation map ev_x at some point x of X, given by

$$\begin{aligned} \mathrm{ev}_x : C(X) &\to \mathbb{C}; \\ f &\mapsto f(x). \end{aligned}$$

Being a one-dimensional representation, ev_x is automatically an irreducible representation. It follows that the structure space of $C(X)$ is given by the set of points of X. But more is true, as the topology of X is also captured by the structure space. Namely, since in the commutative case the irreducible representations are one-dimensional $\pi : A \to \mathbb{C}$ the structure space can be equipped with the **weak $*$-topology**. That is to say, for a sequence $\{\pi_n\}_n$ in $\widehat{A}$, π_n converges weakly to π if $\pi_n(a) \to \pi(a)$ for all $a \in A$.

We state the main result, generalizing our finite-dimensional version of Section 2.1.1 to the infinite-dimensional setting.

Theorem 4.28 (Gelfand duality). *The structure space $\widehat{A}$ of a commutative unital C^*-algebra A is a compact Hausdorff topological space, and $A \simeq C(\widehat{A})$ via the* Gelfand transform

$$a \in A \mapsto \widehat{a} \in \widehat{A}; \qquad \widehat{a}(\pi) = \pi(a).$$

Moreover, for any compact Hausdorff topological space X we have

$$\widehat{C(X)} \simeq X.$$

Proof. See Note 10 on Page 60. □

The next milestone which we need to reach noncommutative Riemannian spin geometry is the translation of the Riemannian distance (4.2.1) on a compact Riemannian spin manifold into functional analytic data. Indeed, we will give an alternative formula as a *supremum* over functions in $C^\infty(M)$. The translation from points in M to functions on M is accomplished by

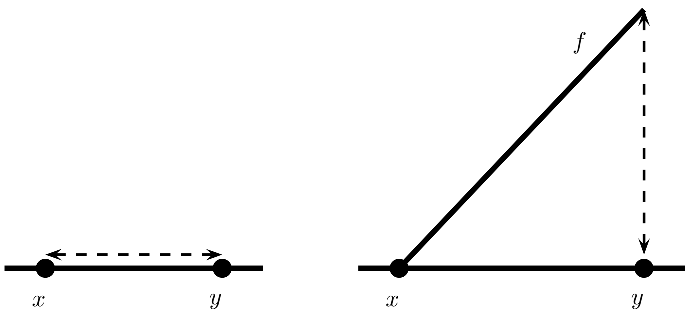


FIGURE 4.1. The translation of the distance between points x, y in M to a formulation in terms of functions of slope ≤ 1.

imposing that the gradient of the functions is less than 1 (see Figure 4.1). This is a continuous analogue of Theorem 2.18.

PROPOSITION 4.29. *Let M be a Riemannian spin^c-manifold with Dirac operator D_M. The following formula defines a distance between points in $\widehat{C(M)} \simeq M$:*

$$d(x,y) = \sup_{f \in C^\infty(M)} \left\{ |f(x) - f(y)| : \|[D_M, f]\} \leq 1 \right\}.$$

Moreover, this distance function d coincides with the Riemannian distance function d_g.

PROOF. First, note that the relation $\|f\|_{\mathrm{Lip}} = \|[D_M, f]\| \leq 1$ (*cf.* Theorem 4.19) already ensures that $d(x,y) \leq d_g(x,y)$. For the opposite inequality we fix $y \in M$ and consider the function $f_{g,y}(z) = d_g(z,y)$. Then $\|f_{g,y}\|_{\mathrm{Lip}} \leq 1$ and

$$d(x,y) \geq |f_{g,y}(x) - f_{g,y}(y)| = d_g(x,y),$$

as required. □

Thus, we have reconstructed the Riemannian distance on M from the algebra $C^\infty(M)$ of functions on M and the Dirac operator D_M, both acting in the Hilbert space $L^2(S)$ of square-integrable operators. Note that the triple $(C^\infty(M), L^2(S), D_M)$ consists of mere functional analytical, or 'spectral' objects, instead of geometrical. Upon allowing for noncommutative algebras as well, we arrive at the following spectral data required to describe a noncommutative Riemannian spin manifold.

DEFINITION 4.30. *A* spectral triple *$(\mathcal{A}, \mathcal{H}, D)$ is given by a unital $*$-algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D in $\mathcal{H}$ such that the resolvent $(i + D)^{-1}$ is a compact operator and $[D, a]$ is bounded for each $a \in \mathcal{A}$.*

A spectral triple is even *if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}_2$-grading γ such that $\gamma a = a\gamma$ and $\gamma D = -D\gamma$.*

A real structure *of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$ on a spectral triple is an anti-linear isometry $J : \mathcal{H} \to \mathcal{H}$ such that*

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon'' \gamma J \quad \textit{(even case)},$$

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

TABLE 4.3. The KO-dimension n of a real spectral triple is determined by the signs $\{\epsilon, \epsilon', \epsilon''\}$ appearing in $J^2 = \varepsilon, JD = \varepsilon' DJ$ and $J\gamma = \varepsilon''\gamma J$.

where the numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are given as a function of n modulo 8, as they appear in Table 4.3.

Moreover, with $b^0 = Jb^* J^{-1}$ we impose the commutant property *and the* order one condition*:*

$$[a, b^0] = 0, \qquad [[D, a], b^0] = 0; \qquad (a, b \in \mathcal{A}). \tag{4.3.1}$$

A spectral triple with a real structure is called a real spectral triple*.*

REMARK 4.31. *The notation $(\mathcal{A}, \mathcal{H}, D)$ is chosen to distinguish a general spectral triple from the finite spectral triples considered in Chapter 2 and 3, which were denoted as (A, H, D).*

The basic example of a spectral triple is the **canonical triple** associated to a compact Riemannian spin manifold:

- $\mathcal{A} = C^\infty(M)$, the algebra of smooth functions on M;
- $\mathcal{H} = L^2(S)$, the Hilbert space of square integrable sections of a spinor bundle $S \to M$;
- $D = D_M$, the Dirac operator associated to the Levi–Civita connection lifted to the spinor bundle.

The real structure J is given by the charge conjugation J_M of Definition 4.13. If the manifold is even dimensional then there is a grading on $\mathcal{H}$, defined just below Definition 4.13. Since the signs in the above table coincide with those in Table 4.2, the KO-dimension of the canonical triple coincides with the dimension of M.

EXAMPLE 4.32. *The tangent bundle of the circle $\mathbb{S}^1$ is trivial and has one-dimensional fibers, so that spinors are given by ordinary functions on $\mathbb{S}^1$. Moreover, the Dirac operator $D_{\mathbb{S}^1}$ is given by $-id/dt$ where $t \in [0, 2\pi)$, acting on $C^\infty(\mathbb{S}^1)$ (which is a core for $D_{\mathbb{S}^1}$). The eigenfunction of $D_{\mathbb{S}^1}$ are the exponential function e^{int} with eigenvalues $n \in \mathbb{Z}$. As such, $(i + D_{\mathbb{S}^1})^{-1}$ is a compact operator. Moreover $[D_{\mathbb{S}^1}, f] = df/dt$ is bounded. Summarizing, we have the following spectral triple:*

$$\left(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), -i\frac{d}{dt} \right).$$

Note that the supremum norm of a function $f \in C^\infty(\mathbb{S}^1)$ coincides with the operator norm of f considered as multiplication operator on $L^2(\mathbb{S}^1)$. A real structure is given by complex conjugation on $L^2(\mathbb{S}^1)$, making the above a real spectral triple of KO-dimension 1.

EXAMPLE 4.33. *Since the tangent bundle of the torus* $\mathbb{T}^2$ *is trivial, we have* $\mathrm{Cliff}(\mathbb{T}^2) \simeq C(\mathbb{T}^2) \otimes \mathbb{C}\mathrm{l}_2$. *As a consequence, the spinor bundle is trivial,* $S = \mathbb{T}^2 \times \mathbb{C}^2$, *and* $L^2(S) = L^2(\mathbb{T}^2) \otimes \mathbb{C}^2$. *The generators* γ^1 *and* γ^2 *are given by*

$$\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which satisfy (4.2.4). *The chirality operator is then given by*

$$\gamma_{\mathbb{T}^2} = -i\gamma^1\gamma^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the real structure $J_{\mathbb{T}^2}$ *that selects* $\mathrm{Cl}_2^- \subset \mathbb{C}\mathrm{l}_2$ *is*

$$J_{\mathbb{T}^2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}.$$

Finally, the Dirac operator on $\mathbb{T}^2$ *is*

$$D_{\mathbb{T}^2} = -i\gamma^\mu \partial_\mu = \begin{pmatrix} 0 & -\partial_1 - i\partial_2 \\ \partial_1 - i\partial_2 & 0 \end{pmatrix}.$$

The eigenspinors of $D_{\mathbb{T}^2}$ *are given by the vectors*

$$\phi^{\pm}_{n_1,n_2}(t_1,t_2) := \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(n_1t_1+n_2t_2)} \\ \pm \frac{in_1+n_2}{\sqrt{n_1^2+n_2^2}} e^{i(n_1t_1+n_2t_2)} \end{pmatrix}; \qquad (n_1, n_2 \in \mathbb{Z}),$$

with eigenvalues $\pm\sqrt{n_1^2+n_2^2}$. *Again, this ensures that* $(i + D_{\mathbb{T}^2})^{-1}$ *is a compact operator. For the commutator with a function* $f \in C^\infty(\mathbb{T}^2)$ *we compute*

$$[D_{\mathbb{T}^2}, f] = \begin{pmatrix} 0 & -\partial_1 f - i\partial_2 f \\ \partial_1 f - i\partial_2 f & 0 \end{pmatrix},$$

which is bounded because $\partial_1 f$ *and* $\partial_2 f$ *are bounded. The signs in the commutation between* $J_{\mathbb{T}^2}, D_{\mathbb{T}^2}$ *and* $\gamma_{\mathbb{T}^2}$ *makes the following a spectral triple of KO-dimension 2:*

$$\left(C^\infty(\mathbb{T}^2), L^2(\mathbb{T}^2) \otimes \mathbb{C}^2, D_{\mathbb{T}^2}; J_{\mathbb{T}^2}, \gamma_{\mathbb{T}^2}\right).$$

Other examples are given by finite spectral triples, discussed at length —and classified— in Chapter 2. Indeed, the compact resolvent condition is automatic in finite-dimensional Hilbert spaces; similarly, any operator such as $[D, a]$ is bounded as in this case also D is a bounded operator.

Definition 2.24 encountered before in the context of finite spectral triples can be translated *verbatim* to the general case:

DEFINITION 4.34. *Two spectral triples* $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ *and* $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ *are called* unitarily equivalent *if* $\mathcal{A}_1 = \mathcal{A}_2$ *and if there exists a unitary operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a); \qquad (a \in \mathcal{A}_1), \\ UD_1U^* &= D_2, \end{aligned}$$

where we have explicitly indicated the representations π_i *of* $\mathcal{A}_i$ *on* $\mathcal{H}_i$ *(*$i = 1, 2$*).*

Corresponding to the direct product of manifolds, one can take the product of spectral triples as follows (see also Exercise 2.24). Suppose that $(\mathcal{A}_1, \mathcal{H}_1, D_1; \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2; \gamma_2, J_2)$ are even real spectral triples, then we define the **product spectral triple** by

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2;\\
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2;\\
D &= D_1 \otimes 1 + \gamma_1 \otimes D_2;\\
\gamma &= \gamma_1 \otimes \gamma_2;\\
J &= J_1 \otimes J_2.
\end{aligned}$$

If $(\mathcal{A}_2, \mathcal{H}_2, D_2; J_2)$ is odd, then we can still form the product when we leave out γ. Note that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$, since the cross-terms vanish due to the fact that $\gamma_1 D_1 = -D_1\gamma_1$.

EXAMPLE 4.35. *In the physical applications later in this book (Chapter 8 and afterwards) we are mainly interested in* almost-commutative manifolds *which are defined as products of a Riemannian spin manifold M with a finite noncommutative space F. More precisely, we will consider*

$$M \times F := (C^\infty(M), L^2(S), D_M; J_M, \gamma_M) \otimes (A_F, H_F, D_F; J_F, \gamma_F),$$

with $(A_F, H_F, D_F; J_F, \gamma_F)$ as in Definition 2.19. Note that this can be identified with:

$$M\times F = (C^\infty(M, A_F), L^2(S\otimes(M\times H_F)), D_M\otimes 1+\gamma_M\otimes D_F; J_M\otimes J_F, \gamma_M\otimes\gamma_F),$$

in terms of the trivial vector bundle $M \times H_F$ on M.

Returning to the general case, any spectral triple gives rise to a differential calculus. This generalizes our previous Definition 2.22 for the finite-dimensional case. Again, we focus only on differential one-forms, as this is sufficient for our applications to gauge theory later on.

DEFINITION 4.36. *The $\mathcal{A}$-bimodule of Connes' differential one-forms is given by*

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_k a_k[D, b_k] : a_k, b_k \in \mathcal{A} \right\},$$

and the corresponding derivation $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ *is given by* $\mathrm{d} = [D, \cdot]$.

EXERCISE 4.7.
(1) *In the case of a Riemannian spin manifold M, verify that we can identify $\Omega^1_{D_M}(C^\infty(M)) \simeq \Omega^1_{\mathrm{dR}}(M)$, the usual De Rham differential one-forms.*
(2) *In the case of an almost-commutative manifold $M \times F$, verify that we have*

$$\Omega^1_{D_M\otimes 1+\gamma_M\otimes D_F}(C^\infty(M, A_F)) \simeq \Omega^1_{\mathrm{dR}}(M, A_F) \oplus C^\infty(M, \Omega^1_{D_F}(A_F)).$$

4.3.1. Commutative subalgebra. In general, given a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$ we can construct a spectral triple on some commutative subalgebra of $\mathcal{A}$, derived from this data. Indeed, set

$$\mathcal{A}_J := \{a \in A : aJ = Ja^*\}.$$

As we will see shortly, this is a complex subalgebra, contained in the center of $\mathcal{A}$ (and hence commutative). Later, in Chapter 8, this subalgebra will turn out to be very useful in the description of the gauge group associated to any real spectral triple.

PROPOSITION 4.37. *Let $(\mathcal{A}, \mathcal{H}, D; J)$ be a real spectral triple. Then*

(1) $\mathcal{A}_J$ defines an involutive commutative complex subalgebra of the center of $\mathcal{A}$.
(2) $(\mathcal{A}_J, \mathcal{H}, D; J)$ is a real spectral triple.
(3) Any $a \in \mathcal{A}_J$ commutes with the algebra generated by the sums $\sum_j a_j[D, b_j] \in \Omega^1_D(\mathcal{A})$ with $a_j, b_j \in \mathcal{A}$.

PROOF. (1) If $a \in \mathcal{A}_J$ then also $Ja^*J^{-1} = (JaJ^{-1})^* = a$, since J is isometric. Hence, $\mathcal{A}_J$ is involutive. Moreover, for all $a \in \mathcal{A}_J$ and $b \in \mathcal{A}$ we have $[a, b] = [Ja^*J^{-1}, b] = 0$ by the commutant property (4.3.1). Thus, $\mathcal{A}_J$ is in the center of $\mathcal{A}$.

(2) Since $\mathcal{A}_J$ is a subalgebra of $\mathcal{A}$, all conditions for a spectral triple are automatically satisfied.

(3) This follows from the order-one condition (4.3.1):

$$[a, [D, b]] = [Ja^*J^{-1}, [D, b]] = 0,$$

for $a \in \mathcal{A}_J$ and $b \in \mathcal{A}$. □

EXAMPLE 4.38. *In the case of a Riemannian spin manifold M with real structure J_M given by charge conjugation, one checks that*

$$C^\infty(M)_{J_M} = C^\infty(M).$$

More generally, under suitable conditions on the triple $(\mathcal{A}, \mathcal{H}, D; J)$ the spectral triple $(\mathcal{A}_J, \mathcal{H}, D)$ is a so-called commutative spin geometry. Then, Connes' Reconstruction Theorem (*cf.* Note 13 on Page 60) establishes the existence of a compact Riemannian spin manifold M such that there is an isomorphism $(\mathcal{A}_J, \mathcal{H}, D) \simeq (C^\infty(M), L^2(S \otimes E), D_E)$. The spinor bundle $S \to M$ is twisted by a vector bundle $E \to M$ and the twisted Dirac operator is of the form $D_E = D_M + \rho$ with $\rho \in \Gamma^\infty(\mathrm{End}(S \otimes E))$.

Notes

Section 4.1. Clifford algebras

1. In our treatment of Clifford algebras, we stay close to the seminal paper by Atiyah, Bott and Shapiro [**8**], but also refer to the standard textbook [**135**] and the book [**103**, Chapter 5]. We also take inspiration from the lecture notes [**190**] and [**134**].

2. The definition of a quadratic form given here is equivalent with the usual definition, which states that Q is a quadratic form if $Q(v) = S(v, v)$ for some symmetric bilinear form S (*cf.* Exercise 4.1). This is shown by Jordan and von Neumann in [**115**].

3. The periodicity eight encountered for the real Clifford algebras $\mathrm{Cl}^\pm_k$ is closely related to the eightfold periodicity of KO-theory [**6**]. The periodicity two encountered for the complex Clifford algebras $\mathbb{C}\mathrm{l}_n$ is closely related to Bott periodicity in K-theory [**14**].

Section 4.2. Riemannian spin geometry

4. A standard textbook on Riemannian geometry is [**116**]. For a complete treatment of Riemannian spin manifolds we refer to *e.g.* [**135, 27**]. A noncommutative approach to (commutative) spin geometry can be found in [**103**, Chapter 9] or [**189, 190**].

5. In Definition 4.10 a Riemannian manifold is said to be spin^c if $\mathbb{C}\mathrm{l}(TM) \simeq \mathrm{End}(S)$ (even case). Glancing back at Chapter 2 we see that $\mathbb{C}\mathrm{l}_n$ is Morita equivalent to $\mathbb{C}$ (n even). With Definition 6.9 of the next Chapter, we conclude that a manifold is spin^c precisely if $\mathbb{C}\mathrm{liff}(M)$ is Morita equivalent to $C(M)$. This is the algebraic approach to spin^c manifolds laid out in [**103**, Section 9.2].

6. Just as for the Levi–Civita connection on the tangent bundle, there is a uniqueness result for the spin connection on spin manifolds, under the condition that ∇^S_X commutes with J_M for real vector fields X and that

$$\nabla^S(c(\omega)\psi) = c(\nabla\omega)\psi + c(\omega)\nabla^S\psi; \qquad (\omega \in \Omega^1_{\mathrm{dR}}(M), \psi \in \Gamma^\infty(S)),$$

where ∇ is the Levi–Civita connection on one-forms. See for example [**103**, Theorem 9.8].

7. A proof of Theorem 4.20 can be found in [**60**, Section VI.1] (see also [**103**, Theorem 11.1].

Section 4.3. Noncommutative Riemannian spin manifolds: spectral triples

8. A complete treatment of C^*-algebras, their representation theory and Gelfand duality can be found in [**31**] or [**183**].

9. A proof of Lemma 4.25 can be found in [**183**, Theorem 11.2].

10. A proof of Theorem 4.28 can be found in *e.g.* [**31**, Theorem II.2.2.4] or [**183**, Theorem 3.11].

11. Spectral triples were introduced by Connes in the early 1980s. See [**60**, Section IV.2.δ] (where they were called unbounded K-cycles) and [**62**].

12. The distance formula appearing in Proposition 4.29, as well as the proof of this Proposition can be found in [**60**, Sect. VI.1]. Moreover, it extends to a distance formula on the state space $S(A)$ of a C^*-algebra A as follows. Recall that a linear functional $\omega : A \to \mathbb{C}$ is a state if it is positive $\omega(a^*a) > 0$ for all non-zero $a \in A$, and such that $\omega(1) = 1$. One then defines a distance function on $S(A)$ by [**62**]

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| : \|[D, a]\| \leq 1\}.$$

It is noted in [**167, 74**] that this distance formula, in the case of locally compact complete manifolds, is in fact a reformulation of the Wasserstein distance in the theory of optimal transport. We also refer to [**75, 147, 148**].

13. Proposition 4.29 establishes that from the canonical triple on a Riemannian spin manifold M one can reconstruct the Riemannian distance on M. As a matter of fact, there is a reconstruction theorem for the smooth manifold structure of M as well [**69**]. It states that if $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$ is a real spectral triple with $\mathcal{A}$ commutative, then under suitable conditions [**63**] there is a Riemannian spin manifold (M, g) with spin structure (S, J_M) such that $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$ is given by $(C^\infty(M), L^2(S), D_M; J_M, \gamma_M)$ (see also the discussion in [**103**, Section 11.4]).

14. Real spectral triples as defined in Definition 4.30 are noncommutative generalization of Riemannian spin manifolds. An immediate question that arises is whether noncommutative generalizations of Riemannian spin^c manifolds, or even just Riemannian manifolds can be defined. In fact, building on the algebraic approach to defining spin^c manifolds as in [**103**] (as also adopted above) the authors [**140**] introduce such noncommutative analogues. For earlier attempts, refer to [**94**].

15. Products of spectral triples are described in detail in [**188**], and generalized to include the odd case as well in [**73**].

16. The differential calculi that are associated to any spectral triple are explained in [**60**, Section VI.1] (see also [**131**, Chapter 7]).

17. The definition of the commutative subalgebra $\mathcal{A}_J$ in Section 4.3.1 is quite similar to the definition of a subalgebra of $\mathcal{A}$ defined in [**54**, Prop. 3.3] (*cf.* [**65**, Prop. 1.125]), which is the *real* commutative subalgebra in the center of $\mathcal{A}$ consisting of elements for which $aJ = Ja$. Following [**186**] we propose a similar but different definition, since this subalgebra will turn out to be very useful for the description of the gauge group associated to any real spectral triple.

CHAPTER 5

The local index formula in noncommutative geometry

In this chapter we present a proof of the Connes–Moscovici index formula, expressing the index of a (twisted) operator D in a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ by a local formula. First, we illustrate the contents of this chapter in the context of two examples in the odd and even case: the index on the circle and on the torus.

5.1. Local index formula on the circle and on the torus

5.1.1. The winding number on the circle. Consider the canonical triple on the circle (Example 4.32):

$$\left(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), D_{\mathbb{S}^1} = -i\frac{d}{dt}\right).$$

The eigenfunctions of $D_{\mathbb{S}^1}$ are given for any $n \in \mathbb{Z}$ by $e_n(t) = e^{int}$, where $t \in [0, 2\pi)$. Indeed, $D_{\mathbb{S}^1} e_n = n e_n$ and $\{e_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{S}^1)$. We denote the projection onto the non-negative eigenspace of $D_{\mathbb{S}^1}$ by P, *i.e.*

$$Pe_n = \begin{cases} e_n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is equivalent to defining $P = (1+F)/2$, where $F = D_{\mathbb{S}^1}|D_{\mathbb{S}^1}|^{-1}$ (defined to be $+1$ on $\ker D_{\mathbb{S}^1}$). Concretely, F is the **Hilbert transform**:

$$F\left(\sum_{n\in\mathbb{Z}} \psi_n e_n(t)\right) = -\sum_{n<0} \psi_n e_n + \sum_{n\geq 0} \psi_n e_n,$$

with complex coefficients ψ_n $(n \in \mathbb{Z})$.

Let u be a unitary in $C^\infty(\mathbb{S}^1)$, say $u = e_m$ for some $m \in \mathbb{Z}$. The index we are interested in is given by the difference between the dimensions of the kernel and cokernel of $PuP : PL^2(\mathbb{S}^1) \to PL^2(\mathbb{S}^1)$:

$$\operatorname{index} PuP = \dim\ker PuP - \dim\ker Pu^*P.$$

Indeed, $\operatorname{Im} T^\perp = \ker T^*$ for any bounded operator. We wish to write this index as a local, integral expression. First, we check that the index is well defined by noting that PuP has finite-dimensional kernel and cokernel. In fact, the kernel of PuP (with $u = e_m$) consists of $\psi = \sum_{n\geq 0} \psi_n e_n \in PL^2(\mathbb{S}^1)$ such that

$$P\left(\sum_{n\geq 0} \psi_n e_{m+n}\right) = 0.$$

In other words, the kernel of PuP consists of linear combinations of the vectors $e_0, \ldots, e_{-m+1}$ for $m < 0$. We conclude that $\dim\ker PuP = m$ if $m < 0$. If $m > 0$ then this dimension is zero, but in that case $\dim\ker Pu^*P = m$. In both cases, and also in the remaining case $m = 0$, for $u = e_m$ we find that

$$\operatorname{index} PuP = -m.$$

EXERCISE 5.1. *In this exercise we show that* $\operatorname{index} PuP$ *is well defined for any unitary* $u \in C^\infty(\mathbb{S}^1)$.

(1) *Show that* $[F, e_m]$ *is a compact operator for any* $m \in \mathbb{Z}$.
(2) *Show that* $[F, f]$ *is a compact operator for any function* $f = \sum_n f_n e_n \in C^\infty(\mathbb{S}^1)$ *(convergence is in sup-norm).*
(3) *Atkinson's Theorem states that an operator is Fredholm (i.e. has finite kernel and cokernel) if it is invertible modulo compact operators. Use this to show that* PuP *is a Fredholm operator.*

On the other hand, we can compute the following zeta function given by the trace (taken for simplicity over the complement of $\ker D_{\mathbb{S}^1}$):

$$\operatorname{Tr}\left(u^*[D_{\mathbb{S}^1}, u]|D_{\mathbb{S}^1}|^{-2z-1}\right) = m \operatorname{Tr}|D_{\mathbb{S}^1}|^{-2z-1} = 2m\zeta(1+2z),$$

since $[D_{\mathbb{S}^1}, u] = mu$ for $u = e_m$. Here $\zeta(s)$ is the well-known *Riemann zeta function.* Since $\zeta(s)$ has a pole at $s = 1$, we conclude that

$$\operatorname{index} PuP = -\operatorname{res}_{z=0} \operatorname{Tr}\left(u^*[D_{\mathbb{S}^1}, u]|D_{\mathbb{S}^1}|^{-2z-1}\right).$$

This is a manifestation of the noncommutative index formula in the simple case of the circle, expressing the winding number m (*cf.* Figure 5.1) of the unitary $u = e_m$ as a 'local' expression. In fact,

$$\operatorname{res}_{z=0} \operatorname{Tr}\left(u^*[D_{\mathbb{S}^1}, u]|D_{\mathbb{S}^1}|^{-2z-1}\right) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} u^* du,$$

as one can easily check. The right-hand side is indeed a local integral expression for the (global) index of PuP.

In this chapter, we generalize this formula to any (odd) spectral triple, translating this locality to the appropriate algebraic notion, namely, in terms of cyclic cocycles.

EXERCISE 5.2. *Prove the following index formula, for a unitary* $u = e_m$*, say, with* $m < 0$*:*

$$\operatorname{index} PuP = -\frac{1}{4}\operatorname{Tr} F[F, u^*][F, u].$$

5.1.2. The winding number on the torus. The same winding number —now in one of the two circle directions— can also be obtained as an index on the two-dimensional torus, as we will now explain.

Consider the even canonical triple on the 2-dimensional torus (Example 4.33):

$$\left(C^\infty(\mathbb{T}^2), L^2(\mathbb{T}^2) \otimes \mathbb{C}^2, D_{\mathbb{T}^2} = \begin{pmatrix} 0 & -\partial_1 - i\partial_2 \\ \partial_1 - i\partial_2 & 0 \end{pmatrix}\right).$$

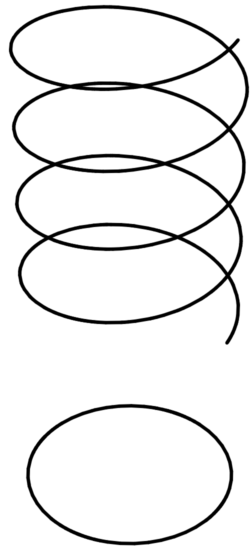


FIGURE 5.1. The map $e_m : t \in [0, 2\pi) \mapsto e^{imt}$ winds m times around the circle; this winding number is (minus) the index of the operator Pe_mP.

The eigenspinors of $D_{\mathbb{T}^2}$ are given by the vectors

$$\phi^{\pm}_{n_1,n_2}(t_1,t_2) := \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(n_1t_1+n_2t_2)} \\ \pm \frac{in_1+n_2}{\sqrt{n_1^2+n_2^2}} e^{i(n_1t_1+n_2t_2)} \end{pmatrix}; \qquad (n_1, n_2 \in \mathbb{Z}),$$

with eigenvalues $\pm\sqrt{n_1^2+n_2^2}$.

Instead of unitaries, we now consider projections $p \in C^\infty(\mathbb{T}^2)$ or rather, projections in matrix algebras with entries in $C^\infty(\mathbb{T}^2)$. Indeed, there are no non-trivial projections p in $C(\mathbb{T}^2)$: a continuous function with the property $p^2 = p$ is automatically 0 or 1. Thus, we consider the following class of projections in $M_2(C^\infty(\mathbb{T}^2))$:

$$p = \begin{pmatrix} f & g + hU^* \\ g + hU & 1 - f \end{pmatrix}, \tag{5.1.1}$$

where f, g, h are real-valued (periodic) functions of the first variable t_1, and U is a unitary depending only on the second variable t_2, say $U(t_2) = e_m(t_2)$. The projection property $p^2 = p$ translates into the two conditions

$$gh = 0, \qquad g^2 + h^2 = f - f^2.$$

A possible solution of these relations is given by

$$0 \leq f \leq 1 \quad \text{such that } f(0) = 1, \quad f(\pi) = 0,$$

and then $g = \chi_{[0,\pi]}\sqrt{f - f^2}$ and $h = \chi_{[\pi,2\pi]}\sqrt{f - f^2}$, where χ_X is the indicator function for the set X (see Figure 5.2).

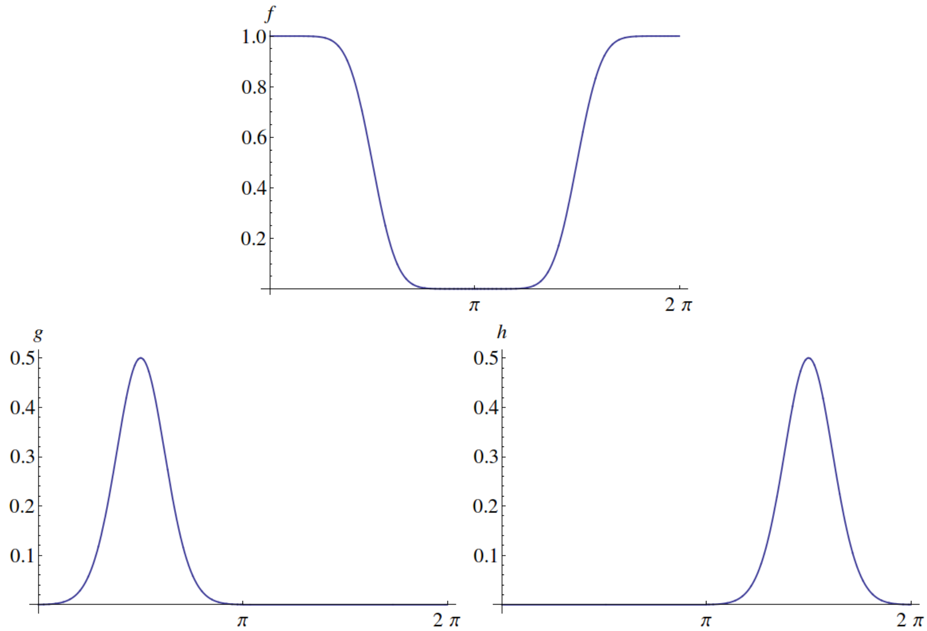


FIGURE 5.2. Functions f, g, h that ensure that p in (5.1.1) is a projection.

The Fredholm operator we would like to compute the index of is $p(D_{\mathbb{T}^2} \otimes \mathbb{I}_2)p$, acting on the doubled spinor Hilbert space $L^2(S) \otimes \mathbb{C}^2 \simeq L^2(\mathbb{T}^2) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. This doubling is due to the fact that we take a 2×2 matricial projection. To avoid notation cluttery, we will simply write $D_{\mathbb{T}^2}$ for $D_{\mathbb{T}^2} \otimes \mathbb{I}_2$.

The local index formula which we would like to illustrate on the torus is

$$\operatorname{index} pD_{\mathbb{T}^2}p = -\operatorname{res}_{z=0} \operatorname{Tr}\left(\gamma\left(p - \tfrac{1}{2}\right)[D_{\mathbb{T}^2}, p][D_{\mathbb{T}^2}, p]|D_{\mathbb{T}^2}|^{-2-2z}\right),$$

where the trace is both over the matrix indices of p and over the spinor indices.

PROPOSITION 5.1. *With $U(t_2) = e_m(t_2)$ and p of the above form, we have*

$$\operatorname{res}_{z=0} \operatorname{Tr}\left(\gamma\left(p - \tfrac{1}{2}\right)[D_{\mathbb{T}^2}, p][D_{\mathbb{T}^2}, p]|D_{\mathbb{T}^2}|^{-2-2z}\right) = m.$$

PROOF. We first prove the following formula, which holds for any $F \in C^\infty(\mathbb{T}^2)$:

$$\operatorname{Tr} F|D_{\mathbb{T}^2}|^{-2s} = 2F(0,0)\zeta_E(s), \tag{5.1.2}$$

where the trace is over spinor indices, and where ζ_E is the Epstein zeta function, defined by

$$\zeta_E(s) = \sum_{n_1, n_2 \in \mathbb{Z}} (n_1^2 + n_2^2)^{-s}.$$

Equality (5.1.2) will be proved in Exercise 5.3. Since ζ_E has a pole at $s = 1$ with residue π, we conclude that

$$\operatorname{res}_{z=0} \operatorname{Tr} F|D|^{-2-2z} = 2\pi F(0,0).$$

Returning to the claimed equality, we compute the trace over spinor indices:

$$\begin{aligned}\operatorname{Tr}\gamma\left(p-\tfrac{1}{2}\right)[D_{\mathbb{T}^2},p]^2 &= \operatorname{Tr}(p-\tfrac{1}{2})\begin{pmatrix}0 & -\partial_1 p - i\partial_2 p\\ \partial_1 p - i\partial_2 p & 0\end{pmatrix}^2\\ &= 2i(p-\tfrac{1}{2})\left(\partial_1 p\partial_2 p - \partial_2 p\partial_1 p\right).\end{aligned}$$

Since g and h in (5.1.1) have disjoint support, $g'h=0$, we have

$$\partial_1 p\partial_2 p = -\partial_2 p\partial_1 p = -im\begin{pmatrix}-hh' & f'hU^*\\ f'hU & hh'\end{pmatrix}.$$

Hence, taking the remaining trace over the indices of the projection, we find

$$\operatorname{Tr} 2i(p-\tfrac{1}{2})\left(\partial_1 p\partial_2 p - \partial_2 p\partial_1 p\right) = 4m\left(-2fhh' + hh' + 2f'h^2\right).$$

Inserting this back in (5.1.2) we see that we have to determine the value of $-2fhh'+hh'+2f'h^2$ at 0 or, equivalently, integrate this expression over the circle. A series of partial integrations yields

$$\left(-2fhh' + hh' + 2f'h^2\right)(0) = \frac{1}{2\pi}\int -2fhh' + hh' + 2f'h^2 = \frac{1}{2\pi}\int 3f'h^2.$$

Inserting the explicit expression of h, we easily determine

$$\int f'h^2 = \int_\pi^{2\pi}(f-f^2)f' = \int_0^1 (x-x^2)dx = \frac{1}{6}.$$

Combining all coefficients, including the residue of Epstein's zeta function, we finally find

$$\operatorname{res}_{z=0}\operatorname{Tr}\left(\gamma\left(p-\tfrac{1}{2}\right)[D_{\mathbb{T}^2},p]^2|D_{\mathbb{T}^2}|^{-2-2z}\right) = 4m\frac{3}{2\pi}\frac{1}{6}\pi = m,$$

as required. □

Thus, we recover the winding number of the unitary U, winding m times around one of the circle directions in $\mathbb{T}^2$, just as in the previous subsection. The case $m=2$ is depicted in Figure 5.3; it shows the winding of the range of p in $\mathbb{C}^2$ at $t_1 = 3\pi/4$ and with t_2 varying from 0 to 2π.

The fact that the index of $pD_{\mathbb{T}^2}p$ is also equal to (minus) this winding number is highly non-trivial and much more difficult to prove. Therefore, already this simple example illustrates the power of the Connes–Moscovici index formula, expressing the index by a local formula. We will now proceed and give a proof of the local index formula for any spectral triple.

Exercise 5.3. *Prove Equation* (5.1.2) *and show that for any function* $F \in C^\infty(\mathbb{T}^2)$ *we have*

$$\operatorname{Tr} F|D_{\mathbb{T}^2}|^{-2s} = \frac{\zeta_E(s)}{\pi}\int_{\mathbb{T}^2} F.$$

5.2. Hochschild and cyclic cohomology

We introduce cyclic cohomology, which can be seen as a noncommutative generalization of De Rham homology.

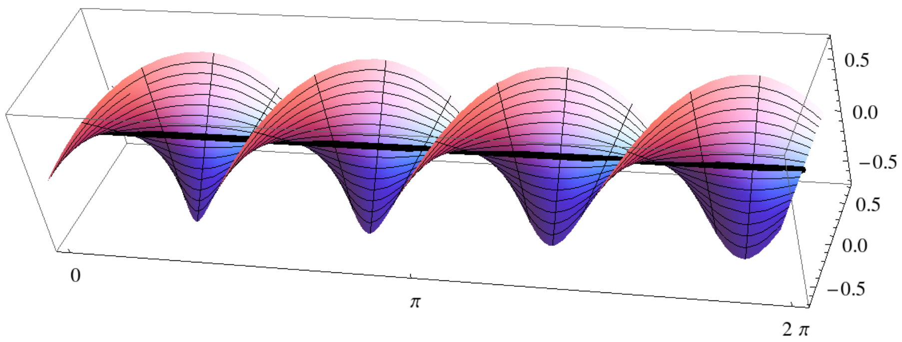


FIGURE 5.3. Winding twice around one of the circle directions on the torus. Let the range of the projection p be $v(t_1, t_2)s$ with $s \in \mathbb{C}$ and $v(t_1, t_2) \in \mathbb{C}^2$ varies with $(t_1, t_2) \in \mathbb{T}^2$. We have drawn the real and imaginary parts of the first component $v_1(t_1 = 3\pi/4, t_2)s$ with $0 \leq t_2 \leq 2\pi$ and $-1 \leq s \leq 1$. The other component $v_2(t_1 = 3\pi/4, t_2)$ is constant.

DEFINITION 5.2. *If $\mathcal{A}$ is an algebra, we define the space of n-*cochains*, denoted by $C^n(\mathcal{A})$, as the space of $(n+1)$-linear functionals on $\mathcal{A}$ with the property that if $a^j = 1$ for some $j \geq 1$, then $\phi(a^0, \ldots, a^n) = 0$. Define operators $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ and $B : C^{n+1}(\mathcal{A}) \to C^n(\mathcal{A})$ by*

$$
\begin{aligned}
b\phi(a^0, a^1, \ldots, a^{n+1}) &:= \sum_{j=0}^{n} (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) \\
&\qquad + (-1)^{n+1} \phi(a^{n+1} a^0, a^1, \ldots, a^n), \\
B\phi(a^0, a^1, \ldots, a^n) &:= \sum_{j=0}^{n} (-1)^{nj} \phi(1, a^j, a^{j+1}, \ldots, a^{j-1}).
\end{aligned}
$$

EXERCISE 5.4. *Show that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$.*

This means that a cochain which is in the image of b is also in the kernel of b, and similarly for B. We say that b and B define *complexes* of cochains

$$
\cdots \xrightarrow{b} C^n(\mathcal{A}) \xrightarrow{b} C^{n+1}(\mathcal{A}) \xrightarrow{b} \cdots
$$

$$
\cdots \xleftarrow{B} C^n(\mathcal{A}) \xleftarrow{B} C^{n+1}(\mathcal{A}) \xleftarrow{B} \cdots,
$$

where the maps have the (complex) defining property that composing them gives zero: $b \circ b = 0 = B \circ B$. This property of b and B being a differential is a crucial ingredient in *cohomology*, where so-called *cohomology groups* are defined as the quotients of the kernel by the image of the differential. In our case, we have

Definition 5.3. *The* Hochschild cohomology *of $\mathcal{A}$ is given by the quotients*

$$HH^n(\mathcal{A}) = \frac{\ker b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})}{\operatorname{Im} b : C^{n-1}(\mathcal{A}) \to C^n(\mathcal{A})}; \qquad (n \geq 0).$$

Elements in $\ker b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ *are called* Hochschild n-cocycles*, and elements in* $\operatorname{Im} b : C^{n-1}(\mathcal{A}) \to C^n(\mathcal{A})$ *are called* Hochschild n-coboundaries.

Exercise 5.5.
1. *Characterize the cohomology group $HH^0(\mathcal{A})$ for any algebra $\mathcal{A}$.*
2. *Compute $HH^n(\mathbb{C})$ for any $n \geq 0$.*
3. *Establish the following functorial property of HH^n: if $\psi : \mathcal{A} \to \mathcal{B}$ is an algebra map, then there is a homomorphism of groups $\psi^* : HH^n(\mathcal{B}) \to HH^n(\mathcal{A})$.*

Example 5.4. *Let M be a compact n-dimensional manifold without boundary. The following expression defines an n-cochain on $\mathcal{A} = C^\infty(M)$:*

$$\phi(f_0, f_1, \ldots, f_n) = \int_M f_0 df_1 \cdots df_n.$$

In fact, one can compute that $b\phi = 0$ so that this is an n-cocycle which defines a class in the Hochschild cohomology group $HH^n(C^\infty(M))$.

Exercise 5.6. *Check that $b\phi = 0$ in the above example.*

Next, we turn our attention to the differential B, and its compatibility with b. Namely, b and B define a so-called *double complex*:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^2(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^1(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^0(\mathcal{A}) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^1(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^0(\mathcal{A}) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(\mathcal{A}) \arrow[r, "B"] \arrow[u, "b"] & C^0(\mathcal{A}) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(\mathcal{A}) \arrow[u, "b"] & & &
\end{tikzcd}$$

The *totalization* of this double complex by definition consists of the even and odd cochains:

$$C^{\mathrm{ev}}(\mathcal{A}) = \bigoplus_k C^{2k}(\mathcal{A});$$

$$C^{\mathrm{odd}}(\mathcal{A}) = \bigoplus_k C^{2k+1}(\mathcal{A}),$$

and these also form a complex, now with differential $b + B$:

$$\cdots \xrightarrow{b+B} C^{\mathrm{ev}}(\mathcal{A}) \xrightarrow{b+B} C^{\mathrm{odd}}(\mathcal{A}) \xrightarrow{b+B} C^{\mathrm{ev}}(\mathcal{A}) \xrightarrow{b+B} \cdots$$

DEFINITION 5.5. *The* periodic cyclic cohomology *of $\mathcal{A}$ is the cohomology of the totalization of this complex. That is, the even and odd cyclic cohomology groups are given by*

$$HCP^{\mathrm{ev}}(\mathcal{A}) = \frac{\ker b + B : C^{\mathrm{ev}}(\mathcal{A}) \to C^{\mathrm{odd}}(\mathcal{A})}{\operatorname{Im} b + B : C^{\mathrm{odd}}(\mathcal{A}) \to C^{\mathrm{ev}}(\mathcal{A})},$$
$$HCP^{\mathrm{odd}}(\mathcal{A}) = \frac{\ker b + B : C^{\mathrm{odd}}(\mathcal{A}) \to C^{\mathrm{ev}}(\mathcal{A})}{\operatorname{Im} b + B : C^{\mathrm{ev}}(\mathcal{A}) \to C^{\mathrm{odd}}(\mathcal{A})}.$$

Elements in $\ker b + B$ *are called (even or odd)* (b, B)-cocycles*, and elements in* $\operatorname{Im} b + B$ *are called (even or odd)* (b, B)-coboundaries.

Explicitly, an even (b, B)-cocycle is given by a sequence

$$(\phi_0, \phi_2, \phi_4, \ldots),$$

where $\phi_{2k} \in C^{2k}(\mathcal{A})$, and

$$b\phi_{2k} + B\phi_{2k+2} = 0,$$

for all $k \geq 0$. Note that only finitely many ϕ_{2k} are non-zero.
Similarly, an odd (b, B)-cocycle is given by a sequence

$$(\phi_1, \phi_3, \phi_5, \ldots),$$

where $\phi_{2k+1} \in C^{2k+1}(\mathcal{A})$ and

$$b\phi_{2k+1} + B\phi_{2k+3} = 0,$$

for all $k \geq 0$, and also $B\phi_1 = 0$. Again, only finitely many ϕ_{2k+1} are non-zero.
The following result allows us to evaluate an even (odd) (b, B)-cocycle on a projection (unitary) in a given $*$-algebra $\mathcal{A}$.

PROPOSITION 5.6. *Let $\mathcal{A}$ be a unital $*$-algebra.*

- *If $\phi = (\phi_1, \phi_3, \ldots)$ is an odd (b, B)-cocycle for $\mathcal{A}$, and u is an unitary in $\mathcal{A}$, then the quantity*

$$\langle \phi, u \rangle := \frac{1}{\Gamma(\frac{1}{2})} \sum_{k=0}^{\infty} (-1)^{k+1} k! \phi_{2k+1}(u^*, u, \ldots, u^*, u)$$

  only depends on the class of ϕ in $HCP^{\mathrm{odd}}(\mathcal{A})$.
- *If $\phi = (\phi_0, \phi_2, \ldots)$ is an even (b, B)-cocycle for $\mathcal{A}$, and p is an projection in $\mathcal{A}$, then the quantity*

$$\langle \phi, p \rangle := \phi_0(p) + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \phi_{2k}(p - \tfrac{1}{2}, p, p, \ldots, p)$$

  only depends on the class of ϕ in $HCP^{\mathrm{ev}}(\mathcal{A})$.

PROOF. We show that $\langle (b+B)\Theta, u \rangle = 0$ for any even cochain $(\Theta_0, \Theta_2, \ldots)$ and that $\langle (b + B)\Theta, e \rangle = 0$ for any odd cochain $(\Theta_1, \Theta_3, \ldots)$.
The former equation would follow from

$$(-1)^{k+1} k! b\Theta_{2k}(u^*, u, \ldots, u^*, u) + (-1)^k (k-1)! B\Theta_{2k}(u^*, u, \ldots, u^*, u) = 0,$$

for any $k \geq 0$. Using the definition of b and B, we compute that indeed:

$$(-1)^{k+1}k!\left[\Theta_{2k}(1,u^*,u,\ldots,u^*,u)+(-1)^{2k+1}\Theta_{2k}(1,u,u^*,\ldots,u,u^*)\right]$$
$$+(-1)^k(k-1)!\left[k\Theta_{2k}(1,u^*,u,\ldots,u^*,u)-k\Theta_{2k}(1,u,u^*,\ldots,u,u^*)\right]=0.$$

The second claim would follow from

$$(-1)^{k+1}\frac{(2k+2)!}{(k+1)!}b\Theta_{2k+1}(p-\tfrac{1}{2},p,\ldots,p)+(-1)^k\frac{(2k)!}{k!}B\Theta_{2k+1}(p-\tfrac{1}{2},p,\ldots,p)=0,$$

for any $k \geq 1$, and indeed

$$-2b\Theta_1(p-\tfrac{1}{2},p,p)+B\Theta_1(p)=0.$$

Let us start with the latter, for which we compute

$$-2\left[2\Theta_1(p-\tfrac{1}{2}p,p)-\Theta_1(p-\tfrac{1}{2},p)\right]+\Theta_1(1,p)=$$
$$-2\Theta_1(p,p)+2\Theta_1(p,p)-\Theta_1(1,p)+\Theta_1(1,p)=0.$$

The same trick applies also to the first expression, for any $k \geq 1$:

$$(-1)^{k+1}\frac{(2k+2)!}{(k+1)!}\left[2\Theta_{2k+1}(p-\tfrac{1}{2}p,p,\ldots,p)-\Theta_{2k+1}(p-\tfrac{1}{2},p,\ldots,p)\right]$$
$$+(-1)^k\frac{(2k)!}{k!}\left[(2k+1)\Theta_{2k+1}(1,p,\ldots,p)\right]=0,$$

which follows directly from the identity

$$\frac{1}{2}\frac{(2k+2)!}{(k+1)!}-(2k+1)\frac{(2k)!}{k!}=0. \qquad \square$$

Exercise 5.7. *Let $\phi \in C^k(\mathcal{A})$ be a b-cocycle (i.e. $b\phi = 0$) that also satisfies the following condition:*

$$\phi(a^0,a^1,\ldots a^k)=(-1)^k\phi(a^k,a^0,a^1,\ldots a^{k-1}),$$

for all $a^0,a^1,\ldots a^k \in \mathcal{A}$. Show that $(0,\ldots,0,\phi,0,\ldots)$ (with ϕ at the k'th position) is a (b,B)-cocycle.

Exercise 5.8. *In the example of the circle, show that the odd cochain $(\phi^1,0,\ldots)$ on $C^\infty(\mathbb{S}^1)$ with (cf. Exc. (5.2))*

$$\phi^1(f^0,f^1)=\operatorname{Tr} F[F,f^0][F,f^1]; \qquad (f^0,f^1\in C^\infty(\mathbb{S}^1)),$$

is an odd (b,B)-cocycle.

5.3. Abstract differential calculus

Starting with a spectral triple, we now introduce a differential calculus. In the case of the canonical triple of a spin manifold M, this will agree with the usual differential calculus on M.

Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple; we assume that D is invertible. We introduce *Sobolev spaces* $\mathcal{H}^s$ as follows:

$$\mathcal{H}^s := \operatorname{Dom}|D|^s; \qquad (s\in\mathbb{R}).$$

These spaces are naturally normed by

$$\|\xi\|_s^2=\|\xi\|^2+\||D|^s\xi\|^2,$$

and are complete in this norm. Moreover, for $s>t$ the inclusion $\mathcal{H}^s\to\mathcal{H}^t$ is continuous.

EXERCISE 5.9. *Prove this last statement.*

Obviously $\mathcal{H}^0 = \mathcal{H}$, while at the other extreme we have the intersection

$$\mathcal{H}^\infty := \bigcap_{s \geq 0} \mathcal{H}^s.$$

DEFINITION 5.7. *For each $r \in \mathbb{R}$ we define operators of* analytic order $\leq r$ *to be operators in $\mathcal{H}^\infty$ that extend to bounded operators from $\mathcal{H}^s$ to $\mathcal{H}^{s-r}$ for all $s \geq 0$. We denote the space of such operators by op^r.*

In order to find interesting differential operators coming from our spectral triple, we introduce some smoothness conditions. The first is that the spectral triple is **finitely summable**, *i.e.* there exists p so that $|D|^{-p}$ is a trace class operator.

DEFINITION 5.8. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called* regular *if $\mathcal{A}$ and $[D, \mathcal{A}] = \{[D, a] : a \in \mathcal{A}\}$ belong to the smooth domain of $\delta(\cdot) = [|D|, \cdot]$. That is, for each $k \geq 0$ the operators $\delta^k(a)$ and $\delta^k([D, a])$ are bounded.*

We will denote by $\mathcal{B}$ the algebra generated by $\delta^k(a)$, $\delta^k([D, a])$ for all $a \in \mathcal{A}$ and $k \geq 0$.

DEFINITION 5.9. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely-summable regular spectral triple. The* dimension spectrum Sd *is the subset of $\{z \in \mathbb{C} : \Re(z) \geq 0\}$ of singularities of the analytic functions*

$$\zeta_b(z) = \operatorname{Tr} b|D|^{-z}; \qquad (b \in \mathcal{B}).$$

We say the the dimension spectrum is simple *when the functions ζ_b have at most simple poles.*

In our treatment we restrict to finitely-summable, regular spectral triples with simple dimension spectrum and for which there is a finite number of poles in Sd.

LEMMA 5.10. *The algebra $\mathcal{B}$ maps $\mathcal{H}^\infty$ to itself.*

PROOF. This follows by induction from the identity

$$\begin{aligned} \|T\xi\|_s^2 &= \|T\xi\|^2 + \||D|^s T\xi\|^2 \\ &= \|T\|^2\|\xi\|^2 + \left(\||D|^{s-1}\delta(T)\xi\| + \||D|^{s-1}T|D|\xi\|\right)^2, \end{aligned}$$

for any operator T in the smooth domain of δ and any $s \geq 0$. $\square$

We will regard the elements in $\mathcal{B}$ as pseudodifferential operators of order 0, according to the following definition.

DEFINITION 5.11. *A* pseudodifferential operator of order $k \in \mathbb{Z}$ *associated to a regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a finite sum:*

$$b_k|D|^k + b_{k-1}|D|^{k-1} + \cdots,$$

where $b_k, b_{k-1}, \dots \in \mathcal{B}$. We denote the space of pseudodifferential operators of order k by $\Psi^k(\mathcal{A}, \mathcal{H}, D)$, or simply $\Psi^k(\mathcal{A})$.

LEMMA 5.12. *The subspaces $\Psi^k(\mathcal{A})$ ($k \in \mathbb{Z}$) furnish a $\mathbb{Z}$-filtration on the algebra $\Psi(\mathcal{A})$ of pseudodifferential operators.*

PROOF. This follows directly from the expression:

$$b_1|D|^{k_1} \cdot b_2|D|^{k_2} = \sum_{j=0}^{k_1} \binom{k_1}{j} b_1 \delta^j(b_2)|D|^{k_1+k_2-j}. \qquad \square$$

On this algebra, the map $\delta(\cdot) = [|D|, \cdot]$ acts as a derivation, preserving the filtration. For any operator T in $\mathcal{H}$ we also define the following (iterated) derivation,

$$\nabla(T) = [D^2, T]; \qquad T^{(k)} := \nabla^k(T).$$

EXERCISE 5.10. *Prove that for any $P \in \Psi(\mathcal{A})$ we have*

$$\nabla(P) = 2\delta(P)|D|^2 + \delta^2(P).$$

Conclude that $\nabla : \Psi^k(\mathcal{A}) \to \Psi^{k+1}(\mathcal{A})$.

PROPOSITION 5.13. *Let $P \in \Psi^k(\mathcal{A})$. Then $P : \mathcal{H}^{s+k} \to \mathcal{H}^s$ is a continuous map. Hence, such a P has analytic order $\leq k$ and we have $\Psi^k(\mathcal{A}) \subset \mathrm{op}^k$.*

Using this abstract pseudodifferential calculus, we now introduce the functionals of relevance for the index formula.

DEFINITION 5.14. *Let $(\mathcal{A}, \mathcal{H}, D; \gamma)$ be a regular spectral triple. For pseudodifferential operators $X^0, X^1, \dots X^p \in \Psi(\mathcal{A})$ and $\Re(z) \gg 0$ define*

$$\langle X^0, X^1, \dots, X^p \rangle_z = \\ (-1)^p \frac{\Gamma(z)}{2\pi i} \operatorname{Tr}\left(\int \lambda^{-z} \gamma X^0 (\lambda - D^2)^{-1} X^1 (\lambda - D^2)^{-1} \cdots X^p (\lambda - D^2)^{-1} d\lambda \right).$$

Let us show that this expression is well defined, *i.e.* that the integral is actually trace class. We first practice with this expression in a special case.

EXERCISE 5.11. *Assume that $X^j \in \Psi^{k_j}(\mathcal{A})$ commutes with D for all $j = 0, \dots, p$.*

(1) Use Cauchy's integral formula to show that

$$\langle X^0, X^1, \dots, X^p \rangle_z = \frac{\Gamma(z+p)}{p!} \operatorname{Tr}(\gamma X^0 \cdots X^p |D|^{-2z-2p}).$$

(2) Show that this expression extends to a meromorphic function on $\mathbb{C}$.

This exercise suggests that, in the general case, we move all terms $(\lambda - D^2)^{-1}$ in $\langle X^0, X^1, \dots, X^p \rangle_z$ to the right. This we will do in the remainder of this section. First, we need the following result.

LEMMA 5.15. *Let $X \in \Psi^q(\mathcal{A})$ and let $n > 0$. Then for any positive integer k, we have*

$$\begin{aligned}(\lambda - D^2)^{-n} X = X(\lambda - D^2)^{-n} &+ nX^{(1)}(\lambda - D^2)^{-(n+1)} \\ &+ \frac{n(n+1)}{2} X^{(2)}(\lambda - D^2)^{-(n+2)} + \cdots \\ &+ \frac{n(n+1)\cdots(n+k)}{k!} X^{(k)}(\lambda - D^2)^{-(n+k)} + R_k,\end{aligned}$$

where the remainder R_k is of analytic order $q - 2n - k - 1$ or less.

PROOF. This follows by repeatedly applying the formula

$$\begin{aligned}(\lambda - D^2)^{-1} X &= X(\lambda - D^2)^{-1} + [(\lambda - D^2)^{-1}, X] \\ &= X(\lambda - D^2)^{-1} + (\lambda - D^2)^{-1}[D^2, X](\lambda - D^2)^{-1}.\end{aligned}$$

This yields an asymptotic expansion

$$(\lambda - D^2)^{-1} X \sim \sum_{i \geq 0} X^{(i)} (\lambda - D^2)^{-1-i},$$

so that for each $m \ll 0$ every sufficiently large finite partial sum agrees with the left-hand side up to an operator of analytic order m or less. Indeed, truncating the above sum at $i = k$, we find that the remainder is

$$(\lambda - D^2)^{-1} X^{(k+1)} (\lambda - D^2)^{-1-k},$$

which is of analytic order $-2 + (q + k + 1) - 2(k + 1) = q - k - 3$ or less.

More generally for any positive integer n one has:

$$(\lambda - D^2)^{-n} X \sim \sum_{k \geq 0} (-1)^k \binom{-n}{k} X^{(k)} (\lambda - D^2)^{-n-k}.$$

Estimates similar to those above show that the remainder has the claimed analytic order. □

We now arrive at the final result of this section which will form the main ingredient in the next section, where we will introduce the (b, B)-cocycles relevant for the index formula.

PROPOSITION 5.16. *The expression $\langle X^0, \ldots, X^p \rangle_z$ in Definition 5.14 seen as a function of z extends meromorphically to $\mathbb{C}$.*

PROOF. We use Lemma 5.15 to bring all $(\lambda - D^2)^{-1}$ to the right. We first introduce the combinatorial quantities:

$$c(k_1, \ldots, k_j) = \frac{(k_1 + \cdots + k_j + j)!}{k_1! \cdots k_j! (k_1 + 1) \cdots (k_1 + \cdots k_j + j)},$$

for non-negative integers $k_1, \ldots, k_j$. These satisfy

$$c(k_1, \ldots, k_j) = c(k_1, \ldots, k_{j-1}) \frac{(k_1 + \cdots + k_{j-1} + j) \cdots (k_1 + \cdots + k_j + j - 1)}{k_j!},$$

while $c(k_1) = 1$ for all k_1.

From Lemma 5.15 we know that there is the following asymptotic expansion:

$$(\lambda - D^2)^{-1} X^1 \sim \sum_{k_1 \geq 0} c(k_1) X^{1(k_1)} (\lambda - D^2)^{-k_1}.$$

Then, in the subsequent step we find

$$\begin{aligned}(\lambda - D^2)^{-1} X^1 (\lambda - D^2)^{-1} X^2 &\sim \sum_{k_1 \geq 0} c(k_1) X^{1(k_1)} (\lambda - D^2)^{-(k_1+2)} X^2 \\ &\sim \sum_{k_1, k_2 \geq 0} c(k_1, k_2) X^{1(k_1)} X^{2(k_2)} (\lambda - D^2)^{-(k_1+k_2+2)},\end{aligned}$$

and finally

$$(\lambda - D^2)^{-1}X^1 \cdots (\lambda - D^2)^{-1}X^p \sim \sum_{k\geq 0} c(k) X^{1(k_1)} \cdots X^{p(k_p)} (\lambda - D^2)^{-(|k|+p)},$$

where $k = (k_1, \ldots, k_p)$ is a multi-index and $|k| = k_1 + \ldots + k_p$.

Multiplying this with γX_0 and integrating as in Definition 5.14, this yields

$$\begin{aligned}
(-1)^p \frac{\Gamma(z)}{2\pi i} &\int \lambda^{-z} \gamma X^0 (\lambda - D^2)^{-1} X^1 \cdots (\lambda - D^2)^{-1} X^p (\lambda - D^2)^{-1} d\lambda \\
&\sim \sum_{k \geq 0} c(k) \gamma X^0 X^{1(k_1)} \cdots X^{p(k_p)} (-1)^p \frac{\Gamma(z)}{2\pi i} \int \lambda^{-z} (\lambda - D^2)^{-(|k|+p+1)} d\lambda \\
&= \sum_{k \geq 0} c(k) \gamma X^0 X^{1(k_1)} \cdots X^{p(k_p)} (-1)^p \Gamma(z) \binom{-z}{|k|+p} |D|^{-2(z+|k|+p)},
\end{aligned}$$

where we have used the integral formula, valid for real λ_0:

$$\frac{1}{2\pi i} \int \frac{\lambda^{-z}}{(\lambda - \lambda_0)^{N+1}} d\lambda = \binom{-z}{N} \lambda_0^{-(N+z)}. \tag{5.3.1}$$

Finally, using the functional equation for the gamma function,

$$(-1)^p \Gamma(z) \binom{-z}{|k|+p} = (-1)^{|k|} \frac{\Gamma(z+p+|k|)}{(|k|+p)!},$$

we obtain an asymptotic expansion

$$\begin{aligned}
\langle X_0, \ldots, X_p \rangle_z \sim \sum_{k \geq 0} &(-1)^{|k|} \frac{\Gamma(z+p+|k|)}{(|k|+p)!} c(k) \\
&\times \operatorname{Tr}\left(\gamma X^0 X^{1(k_1)} \cdots X^{p(k_p)} |D|^{-2(z+|k|+p)} \right).
\end{aligned} \tag{5.3.2}$$

As $|k|$ becomes large the remainder in the truncated expansion on the right-hand side becomes trace class. □

EXERCISE 5.12. *Use Cauchy's integral formula to prove Equation* (5.3.1).

5.4. Residues and the local (b, B)-cocycle

In this section we derive even and odd (b, B)-cocycles on a given algebra $\mathcal{A}$ from the functionals $\langle X^0, X^1, \ldots, X^p \rangle_z$ defined in the previous section. First, we derive some useful relations between them. We denote the $\mathbb{Z}_2$-grading of an operator X by $(-1)^X$, according to the grading γ on $\mathcal{H}$. Moreover, for such an operator X we denote the graded commutator by $[D, X] = DX - (-1)^X XD$. Note that with these conventions we have

$$[D, [D, T]] = [D^2, T] \equiv \nabla(T),$$

for any even operator T.

LEMMA 5.17. *The meromorphic functions $\langle X^0, \dots, X^p \rangle_z$ satisfy the following functional equations:*

(a) $$\langle X^0, \dots, X^p \rangle_z = (-1)^{X^p} \langle X^p, X^0, \dots, X^{p-1} \rangle_z;$$

(b) $$\langle X^0, \dots, X^p \rangle_{z+1} = \sum_{j=0}^{p} \langle X^0, \dots, X^{j-1}, 1, X^j, \dots, X^p \rangle_z;$$

(c) $$\langle X^0, \dots, [D^2, X^j], \dots, X^p \rangle_z = \langle X^0, \dots, X^{j-1} X^j, \dots, X^p \rangle_z - \langle X^0, \dots, X^j X^{j+1}, \dots, X^p \rangle_z;$$

(d) $$\sum_{j=0}^{p} (-1)^{X^0 \dots X^{j-1}} \langle X^0, \dots, [D, X^j], \dots, X^p \rangle_z = 0.$$

PROOF. (a) follows directly from the property of the trace in $\langle X^0, \dots, X^p \rangle_z$, taking into account the commutation of X^p with the grading γ. For (b), note that the integral of the following expression vanishes:

$$\begin{aligned} \frac{d}{d\lambda} \left(\lambda^{-z} X^0 (\lambda - D^2)^{-1} \cdots X^p (\lambda - D^2)^{-1} \right) \\ = -z \lambda^{-z-1} X^0 (\lambda - D^2)^{-1} \cdots X^p (\lambda - D^2)^{-1} \\ - \sum_{j=0}^{p} \lambda^{-z} X^0 (\lambda - D^2)^{-1} \cdots X^j (\lambda - D^2)^{-1} \cdots X^p (\lambda - D^2)^{-1}. \end{aligned}$$

Equation (c) follows from

$$(\lambda - D^2)^{-1} [D^2, X^j] (\lambda - D^2)^{-1} = (\lambda - D^2)^{-1} X^j - X^j (\lambda - D^2)^{-1}.$$

Finally, (d) is equivalent to

$$\operatorname{Tr} \gamma \left[D, \int \lambda^{-z} X^0 (\lambda - D^2)^{-1} \cdots X^p (\lambda - D^2)^{-1} d\lambda \right] = 0,$$

which is the supertrace of a (graded) commutator. □

DEFINITION 5.18. *For any $p \geq 0$, define a $(p+1)$-linear functional on $\mathcal{A}$ with values in the meromorphic functions on $\mathbb{C}$ by*

$$\Psi_p(a^0, \dots, a^p) = \langle a^0, [D, a^1], \dots, [D, a^p] \rangle_{s - \frac{p}{2}}.$$

PROPOSITION 5.19. *The even (b, B)-cochain $\Psi = (\Psi_0, \Psi_2, \dots)$ is an (improper) even (b, B)-cocycle in the sense that*

$$b\Psi_{2k} + B\Psi_{2k+2} = 0.$$

Similarly, the odd (b, B)-cochain $\Psi = (\Psi_1, \Psi_3, \dots)$ is an (improper) odd (b, B)-cocycle.

PROOF. It follows from the definition of B and a subsequent application of (a) and (b) of Lemma 5.17 that

$$\begin{aligned} B\Psi_{2k+2}(a^0, \ldots, a^{2k+1}) &= \sum_{j=0}^{2k+1} (-1)^j \langle 1, [D, a^j], \ldots, [D, a^{j-1}] \rangle_{s-(k+1)} \\ &= \sum_{j=0}^{2k+1} \langle [D, a^0], \ldots, [D, a^{j-1}], 1, [D, a^j], \ldots [D, a^{2k+1}] \rangle_{s-(k+1)} \\ &= \langle [D, a^0], \ldots, [D, a^{2k+1}] \rangle_{s-k}. \end{aligned}$$

Also, from the definition of b and the Leibniz rule

$$[D, a^j a^{j+1}] = a^j [D, a^{j+1}] + [D, a^j] a^{j+1}$$

it follows that

$$\begin{aligned} b\Psi_{2k}(a^0, \ldots, a^{2k+1}) = {} & \langle a^0 a^1, [D, a^2], \ldots [D, a^{2k+1}] \rangle_{s-k} \\ & - \langle a^0, a^1 [D, a^2], \ldots [D, a^{2k+1}] \rangle_{s-k} \\ & - \langle a^0, [D, a^1] a^2, \ldots [D, a^{2k+1}] \rangle_{s-k} \\ & + \langle a^0, [D, a^1], a^2 [D, a^3], \ldots [D, a^{2k+1}] \rangle_{s-k} \\ & + \langle a^0, [D, a^1], [D, a^2] a^3, \ldots [D, a^{2k+1}] \rangle_{s-k} \\ & - \cdots \\ & - \langle a^{2k+1} a^0, [D, a^1], \ldots [D, a^{2k}] \rangle_{s-k}, \end{aligned}$$

which, by Lemma 5.17(c), becomes

$$\sum_{j=0}^{2k+1} (-1)^{j-1} \langle a^0, [D, a^1], \ldots, [D^2, a^j], \ldots, [D, a^{2k+1}] \rangle_{s-k}.$$

Combining these expressions for $B\Psi_{2k+2}$ and $b\Psi_{2k}$ and writing $X^0 = a^0$, and $X^j = [D, a^j]$ for $j \geq 1$, we obtain

$$\begin{aligned} & B\Psi_{2k+2}(a^0, \ldots, a^{2k+1}) + b\Psi_{2k}(a^0, \ldots, a^{2k+1}) \\ & \qquad = \sum_{j=0}^{2k+1} (-1)^{X^0 \cdots X^j} \langle X^0, \ldots, [D, X^j], \ldots, X^{2k+1} \rangle_{s-k}, \end{aligned}$$

which vanishes because of Lemma 5.17(d).

In the odd case, a similar argument shows that $b\Psi_{2k-1} + B\Psi_{2k+1} = 0$. □

The above cocycles have been termed *improper* because all Ψ_p might be non-zero, on top of which (rather than in $\mathbb{C}$) they take values in the field of meromorphic functions on $\mathbb{C}$. By taking residues of the meromorphic functions Ψ_p we obtain a *proper* even or odd (b, B)-cocycle. This is the residue cocycle that was introduced by Connes and Moscovici.

THEOREM 5.20. *For any $p \geq 0$ and all $a^0, \ldots, a^p \in \mathcal{A}$ the following formulas define an even or odd (b, B)-cocycle:*

$$\mathrm{res}_{s=0} \Psi_0(a^0) = \mathrm{Tr}\, \gamma a^0 |D|^{-2s} |_{s=0},$$

and

$$\begin{aligned}
&\mathrm{res}_{s=0}\Psi_p(a^0,\dots,a^p)\\
&\qquad=\sum_{k\geq 0} c_{p,k}\mathrm{res}_{s=0}\,\mathrm{Tr}\left(\gamma a^0[D,a^1]^{(k_1)}\cdots[D,a^p]^{(k_p)}|D|^{-p-2|k|-2s}\right),
\end{aligned}$$

for $p\geq 1$*, where the constants* $c_{p,k}$ *are given in terms of the (non-negative) multi-indices* $(k_1,\dots,k_p)$ *by*

$$c_{p,k}:=\frac{(-1)^{|k|}}{k!}\frac{\Gamma(|k|+\frac{p}{2})}{(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots k_p+p)}.$$

PROOF. We use the asymptotic expansion (5.3.2). Indeed, setting $z=s-\frac{p}{2}$ in that expression and taking residues at $s=0$ gives the desired expansion, with the coefficients $c_{p,k}$ appearing because

$$c_{p,k}\equiv(-1)^{|k|}\Gamma(|k|+\frac{p}{2})\frac{c(k)}{(p+|k|)!}.$$

□

5.5. The local index formula

Let $(\mathcal{A},\mathcal{H},D)$ be a regular spectral triple, as above. The local index formula expresses the index of twisted Dirac operators in terms of cocycles in the (b,B) bicomplex, which are easier to compute. We are interested in the indices of the following two Fredholm operators.

Suppose that $(\mathcal{A},\mathcal{H},D)$ is even. If $p\in\mathcal{A}$ is a projection, then $D_p=pDp$ is a Fredholm operator on the Hilbert space $\mathcal{H}$. This follows from the fact that D_p is essentially a finite-dimensional extension of the Fredholm operator D. We are interested in the index of this so-called twisted Dirac operator D_p.

In case that $(\mathcal{A},\mathcal{H},D)$ is an odd spectral triple, we take a unitary $u\in\mathcal{A}$ and define $D_u=PuP$, where $P=\frac{1}{2}(1+\mathrm{Sign}\,D)$. Again, D_u is a Fredholm operator on $\mathcal{H}$ and we are interested in the index of D_u.

THEOREM 5.21. *Let* $(\mathcal{A},\mathcal{H},D)$ *be a regular spectral triple with simple and finite dimension spectrum* Sd *and let* $\mathrm{res}_{s=0}\Psi$ *be the (even or odd)* (b,B)*-cocycle derived previously.*

- *If* $(\mathcal{A},\mathcal{H},D)$ *is even and* p *is a projection in* $\mathcal{A}$*, then*
$$\mathrm{index}\,D_p=\langle\mathrm{res}_{s=0}\Psi,p\rangle.$$
- *If* $(\mathcal{A},\mathcal{H},D)$ *is odd and* u *is a unitary in* $\mathcal{A}$*, then*
$$\mathrm{index}\,D_u=\langle\mathrm{res}_{s=0}\Psi,u\rangle.$$

REMARK 5.22. *Sometimes a projection or a unitary is given in* $M_N(\mathcal{A})$ *instead of* $\mathcal{A}$*. The above result can be extended easily to this case, namely by constructing a spectral triple on* $M_N(\mathcal{A})$ *and doing the index computation there. Indeed, it would follow from Theorem 6.15 that if* $(\mathcal{A},\mathcal{H},D)$ *is a spectral triple, then so is* $(M_N(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^N,D\otimes\mathbb{I}_N)$*.*

PROOF OF THEOREM 5.21. We will prove the even case in two steps (for the odd case see Note 13 on Page 81),

(1) the Atiyah–Bott formula for the index:

$$\operatorname{index} D_p = \operatorname{res}_{s=0}\Gamma(s)\operatorname{Tr}\gamma|D_p|^{-2s}.$$

(2) Change the representative of the class $\operatorname{res}_{s=0}\Psi$ in $HCP^{\mathrm{ev}}(\mathcal{A})$ to reduce to the case that D commutes with p, so that

$$\langle \operatorname{res}_{s=0}\Psi, p\rangle = \operatorname{res}_{s=0}\Gamma(s)\operatorname{Tr}\gamma p|D|^{-2s}.$$

For (1) let us first prove another well-known formula.

LEMMA 5.23 (McKean–Singer formula). *Let $(\mathcal{A},\mathcal{H},D)$ be an even spectral triple. Then*

$$\operatorname{index} D = \operatorname{Tr}\gamma e^{-tD^2}.$$

PROOF. Since D is odd with respect to γ, its spectrum lies symmetrically around 0 in $\mathbb{R}$, including multiplicities. If we denote the λ-eigenspace in $\mathcal{H}$ by $\mathcal{H}_\lambda$ we therefore have $\dim\mathcal{H}_\lambda = \dim\mathcal{H}_{-\lambda}$ for any non-zero eigenvalue λ. Including also the kernel of D, we have

$$\operatorname{Tr}\gamma e^{-tD^2} = \sum_{\lambda>0}(\dim\mathcal{H}_\lambda - \dim\mathcal{H}_{-\lambda})\, e^{-t\lambda^2} + \operatorname{Tr}_{\mathcal{H}_0}\gamma = \operatorname{Tr}_{\ker D}\gamma,$$

which is nothing but the index of D. □

Note that the McKean–Singer formula tells us in particular that $\operatorname{Tr}\gamma e^{-tD^2}$ does not depend on t. Using the integral formula of the gamma function, we can write:

$$\operatorname{Tr}\gamma|D|^{-2s} = \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\gamma e^{-tD^2}t^{s-1}dt.$$

We analyze the behaviour of the right-hand side as $s\to 0$. For this, we use

$$\frac{1}{\Gamma(s)} \sim s, \qquad s\to 0.$$

Thus, only the pole part of the above integral contributes to the zeta function evaluated at $s=0$. This is given by

$$\int_0^1 \operatorname{Tr}\gamma e^{-tD^2}t^{s-1}dt = \frac{1}{s}\operatorname{index} D,$$

where we have used the McKean–Singer formula. The remaining integral from 1 to ∞ gives an entire function of s, because by finite summability the eigenvalues of D grow as $j^{1/p}$ for some $p>0$. In other words,

$$\operatorname{index} D = \operatorname{Tr}\gamma|D|^{-2s}|_{s=0},$$

which proves (1).

Let us then continue with (2). Consider the family of operators

$$D_t = D + t[p,[D,p]]; \qquad (t\in[0,1]).$$

We have $D_0 = D$ and $D_1 = pDp + (1-p)D(1-p)$ so that $[D_1,p]=0$. Moreover, $\operatorname{index} D_t$ depends continuously on t, and (being an integer) it is therefore constant in t.

Next, we consider a family of improper cocycles Ψ^t which are defined by replacing D by D_t in Definition 5.18 .

LEMMA 5.24. *The derivative of Ψ^t is an (improper) even cyclic coboundary, i.e. there exists a cochain Θ^t such that*

$$\frac{d}{dt}\Psi^t_p + B\Theta^t_{p+1} + b\Theta^t_{p-1} = 0,$$

which is explicitly given by

$$\Theta^t_p(a^0,\ldots,a^p) = \sum_{j=0}^{p}(-1)^{j-1}\langle a^0,\ldots[D,a^j],\dot{D},[D,a^{j+1}],\ldots[D,a^p]\rangle_{s-\frac{p+1}{2}},$$

with $\dot{D} = \frac{d}{dt}D_t \equiv [p,[D,p]]$.

PROOF. Imitating the proof of Proposition 5.19 one can show the following identity (see also Note 15 on Page 81).

$$\begin{aligned}
&B\Theta^t_{2k+1}(a^0,\ldots,a^{2k}) + b\Theta^t_{2k-1}(a^0,\ldots,a^{2k})\\
&\qquad = -\sum_{j=0}^{2k}\langle a^0,[D,a^1],\ldots[D,a^j],[D,\dot{D}],\ldots,[D,a^{2k}]\rangle_{s-k}\\
&\qquad\qquad -\sum_{j=1}^{2k}\langle a^0,[D,a^1],\ldots[\dot{D},a^j],,\ldots,[D,a^{2k}]\rangle_{s-k}.
\end{aligned}$$

The fact that $\frac{d}{dt}\Psi^t$ coincides with the right-hand side follows from

$$\frac{d}{dt}(\lambda - D_t^2)^{-1} = (\lambda - D_t^2)^{-1}\left(D\dot{D} + \dot{D}D\right)(\lambda - D_t^2)^{-1}. \qquad \square$$

Continuing the proof of the theorem, we integrate the resulting coboundary to obtain

$$B\int_0^1 \Theta^t_{2k+1}dt + b\int_0^1 \Theta^t_{2k-1}dt = \Psi^0_{2k} - \Psi^1_{2k}.$$

In other words, $\mathrm{res}_{s=0}\Psi^0$ and $\mathrm{res}_{s=0}\Psi^1$ define the same class in even cyclic cohomology $HCP^{\mathrm{ev}}(\mathcal{A})$. So, with the help of Proposition 5.6, we can compute $\langle \mathrm{res}_{s=0}\Psi, p\rangle$ using Ψ^1 instead of $\Psi^0 \equiv \Psi$, with the advantage that D_1 commutes with p. Indeed, this implies that

$$\Psi^1_{2k}(p-\tfrac{1}{2},p,\ldots,p) = 0,$$

for all $k \geq 1$, so that

$$\begin{aligned}
\langle \mathrm{res}_{s=0}\Psi^1, p\rangle &\equiv \mathrm{res}_{s=0}\Psi^1_0(p) + \sum_{k\geq 1}(-1)^k\frac{(2k)!}{k!}\mathrm{res}_{s=0}\Psi^1_{2k}(p-\tfrac{1}{2},p,\ldots,p)\\
&= \mathrm{res}_{s=0}\Psi^1_0(p)\\
&= \mathrm{res}_{s=0}\Gamma(s)\,\mathrm{Tr}\,\gamma p|D_1|^{-2s}.
\end{aligned}$$

This completes the proof of Theorem 5.21, as by the Atiyah–Bott formula the latter expression is the index of D_p. $\square$

Notes

1. The local index formula was obtained by Connes and Moscovici in [**66**]. In our proof of the local index formula, we closely follow Higson [**109**]. More general proofs have been obtained in [**46, 47, 48**], see Note 12 of this Chapter.

Section 5.1. Local index formula on the circle and on the torus

2. The Theorem of Atkinson that appears in Exercise 5.1 can be found in [**160**, Proposition 3.3.11].

3. The index formula on the circle of Exercise 5.2 is a special case of [**58**, Theorem 5].

4. In Section 5.1.2 we follow [**141**], where a class of projections on the torus was constructed, much inspired by the so-called Powers–Rieffel projections on the noncommutative torus [**165**].

5. The zeta function ζ_E that appears in (5.1.2) is a special case of an Epstein zeta function, introduced and analyzed in [**87**]. It turns out that ζ_E has a pole at $s = 1$ with residue π. That (5.1.2) holds also follows from the general result [**66**, Theorem I.2].

Section 5.2. Hochschild and cyclic cohomology

6. In [**58**] Connes introduced cyclic cohomology as a noncommutative generalization of De Rham homology, and showed that for the algebra $C^\infty(M)$ cyclic cohomology indeed reduces to De Rham homology. Besides the original article there are many texts in which this is worked out in full detail (*e.g.* [**60, 103, 123, 139**]).

7. Example 5.4 is a special case of the fact that $HH^k(C^\infty(M)) \simeq \Omega_k(M)$, the space of De Rham k-currents. The latter are by definition continuous linear forms on the space of De Rham differential k-forms $\Omega^k_{\mathrm{dR}}(M)$. This isomorphism is proved in [**58**].

8. Proposition 5.6 was established in [**60**]. The statement can be slightly enhanced. Namely, the quantities in Proposition 5.6 also only depend on the classes of u and p in the (odd and even) K-theory of $\mathcal{A}$. We refer to [**60**, Section IV.1.γ] for more details.

9. Originally, Connes introduced cyclic cohomology by means of cocycles satisfying such a *cyclic* condition, explaining the terminology. It turns out that this is equivalent to taking an even/odd cocycle in the (b, B)-bicomplex. For more details we refer to [**58**, Theorem II.40] (or [**60**, Theorem III.1.29]).

Section 5.3. Abstract differential calculus

10. In our development of an abstract differential calculus we closely follow Connes and Moscovici [**66**]. In the case of the canonical triple of a spin manifold M, this will reproduce (part of) the usual differential calculus on M. We refer to [**109**] for a more detailed treatment. Note that the hypothesis that D is invertible can be removed, as described in [**109**, Section 6.1].

11. The notion of finite summability for spectral triples was introduced in [**60**, Section IV.2.γ] (see also [**103**, Definition 10.8]).

12. Even though we restrict to finitely-summable, regular spectral triples with simple dimension spectrum and for which there is a finite number of poles in Sd, the index formula can be proved in the presence of essential and infinitely many singularities as well [**46, 47, 48**].

Section 5.5. The local index formula

13. In our proof of Theorem 5.21 we follow Higson [**109**]. For the odd case, we refer to the original paper by Connes and Moscovici [**66**] (see also the more general [**46**]).

14. The McKean–Singer formula is due to [**149**].

15. For more details on the 'transgression formula' that is essential in the proof of Lemma 5.24 we refer to the discussion resulting in [**103**, Eq. 10.40].

16. It is noted in [**66**, Remark II.1] that if $(\mathcal{A}, \mathcal{H}, D)$ is the canonical triple associated to a Riemannian spin manifold M, then the local index formula of Connes and Moscovici reduces to the celebrated Atiyah–Singer index theorem for the Dirac operator [**12, 13**]. Namely, the operator D_p is then the Dirac operator with coefficients in a vector bundle $E \to M$. The latter is defined as a subbundle of the trivial bundle $M \times \mathbb{C}^N$ using the projection $p \in M_N(C(M))$: one sets the fiber to be $E_x = p(x)\mathbb{C}^N$ at each point $x \in M$. We then have

$$\operatorname{index} D_p = (2\pi i)^{-\frac{n}{2}} \int_M \hat{A}(R) \wedge \operatorname{ch}(E),$$

where $\hat{A}(R)$ is the $\hat{A}$-form of the Riemannian curvature of M and $\operatorname{ch}(E)$ is the Chern character of the vector bundle E (*cf.* [**27**]). The proof exploits Getzler's symbol calculus [**97, 98, 99**], as in [**33**]. See also [**162**].

Part 2

Noncommutative geometry and gauge theories

CHAPTER 6

Gauge theories from noncommutative manifolds

In this Chapter we demonstrate how every noncommutative (Riemannian spin) manifold, *viz.* every spectral triple, gives rise to a gauge theory in a generalized sense. We derive so-called inner fluctuations via Morita equivalences and interpret these as generalized gauge fields. This is quite similar to the construction in the finite case in Chapters 2 and 3. We then interpret our generalized gauge theory in terms of a C^*-bundle on which the gauge group acts by vertical automorphisms.

6.1. 'Inner' unitary equivalences as the gauge group

In Chapter 2 we already noticed the special role played by the unitary elements in the matrix algebras, and how they give rise to equivalences of finite noncommutative spaces (*cf.* Remark 2.25). We now extend this to general real spectral triples $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$.

DEFINITION 6.1. *A* $*$-automorphism *of a* $*$*-algebra* $\mathcal{A}$ *is a linear invertible map* $\alpha : \mathcal{A} \to \mathcal{A}$ *that satisfies*

$$\alpha(ab) = \alpha(a)\alpha(b), \qquad \alpha(a^*) = \alpha(a)^*.$$

We denote the group of automorphisms of the $*$*-algebra* $\mathcal{A}$ *by* $\mathrm{Aut}(\mathcal{A})$.

An automorphism α *is called* inner *if it is of the form* $\alpha(a) = uau^*$ *for some element* $u \in \mathcal{U}(\mathcal{A})$ *where*

$$\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} : uu^* = u^*u = 1\}$$

is the group of unitary elements in $\mathcal{A}$*. The group of inner automorphisms is denoted by* $\mathrm{Inn}(\mathcal{A})$.

The group of outer automorphisms *of* $\mathcal{A}$ *is defined by the quotient*

$$\mathrm{Out}(\mathcal{A}) := \mathrm{Aut}(\mathcal{A}) / \mathrm{Inn}(\mathcal{A}).$$

Note that $\mathrm{Inn}(\mathcal{A})$ is indeed a normal subgroup of $\mathrm{Aut}(\mathcal{A})$ since

$$\beta \circ \alpha_u \circ \beta^{-1}(a) = \beta\big(u\beta^{-1}(a)u^*\big) = \beta(u)a\beta(u)^* = \alpha_{\beta(u)}(a),$$

for any $\beta \in \mathrm{Aut}(\mathcal{A})$.

An inner automorphism α_u is completely determined by the unitary element $u \in \mathcal{U}(\mathcal{A})$, but not in a unique manner. In other words, the map $\phi\colon \mathcal{U}(\mathcal{A}) \to \mathrm{Inn}(\mathcal{A})$ given by $u \mapsto \alpha_u$ is surjective, but not injective. The kernel is given by $\ker(\phi) = \{u \in \mathcal{U}(\mathcal{A}) \mid uau^* = a,\ a \in \mathcal{A}\}$. In other words, $\ker \phi = \mathcal{U}(Z(\mathcal{A}))$ where $Z(\mathcal{A})$ is the center of $\mathcal{A}$. We conclude that the group of inner automorphisms is given by the quotient

$$\mathrm{Inn}(\mathcal{A}) \simeq \mathcal{U}(\mathcal{A})/\mathcal{U}(Z(\mathcal{A})). \tag{6.1.1}$$

This can be summarized by the following exact sequences:

$$1 \longrightarrow \mathrm{Inn}(\mathcal{A}) \longrightarrow \mathrm{Aut}(\mathcal{A}) \longrightarrow \mathrm{Out}(\mathcal{A}) \longrightarrow 1,$$

$$1 \longrightarrow \mathcal{U}(Z(\mathcal{A})) \longrightarrow \mathcal{U}(\mathcal{A}) \longrightarrow \mathrm{Inn}(\mathcal{A}) \longrightarrow 1.$$

EXAMPLE 6.2. *If $\mathcal{A}$ is a commutative $*$-algebra, then there are no non-trivial inner automorphisms since $Z(\mathcal{A}) = \mathcal{A}$. Moreover, if $\mathcal{A} = C^\infty(X)$ with X a smooth compact manifold, then $\mathrm{Aut}(\mathcal{A}) \simeq \mathrm{Diff}(X)$, the group of diffeomorphisms of X. Explicitly, a diffeomorphism $\phi : X \to X$ yields an automorphisms by pullback of a function f:*

$$\phi^*(f)(x) = f(\phi(x)); \qquad (x \in X).$$

Compare this with the discussion in the case of finite discrete topological spaces in Section 2.1. More generally, there is a continuous version of the above group isomorphism, relating $\mathrm{Aut}(C(X))$ one-to-one to homeomorphisms of X. This follows from functoriality of Gelfand duality. Namely, the Gelfand transform in Theorem 4.28 naturally extends to homomorphisms between commutative unital C^*-algebras, mapping these to homeomorphism between the corresponding structure spaces.

The fact that all automorphisms of $C^\infty(X)$ come from a diffeomorphism of X can be seen as follows. Consider a smooth family $\{\alpha_t\}_{t\in[0,1]}$ of automorphisms of $C^\infty(X)$ from $\alpha_{t=0} = \mathrm{id}$ to $\alpha_{t=1} = \alpha$. The derivative at $t = 0$ of this family, $\dot\alpha := \mathrm{d}\alpha_t/\mathrm{d}t|_{t=0}$, is a $*$-algebra derivation, since

$$\dot\alpha(f_1 f_2) = \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t(f_1 f_2)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t(f_1)\alpha_t(f_2)|_{t=0} = \dot\alpha(f_1)f_2 + f_1\dot\alpha(f_2).$$

As such, $\dot\alpha$ corresponds to a smooth vector field on X and the end point $\phi_{t=1}$ of the flow ϕ_t of this vector field is the sought-for diffeomorphism of X. Its pullback $\phi^*_{t=1}$ on smooth functions coincides with the automorphism $\alpha_{t=1} = \alpha$.

EXAMPLE 6.3. *At the other extreme, we consider an example where all automorphisms are inner. Let $\mathcal{A} = M_N(\mathbb{C})$ and let u be an element in the unitary group $U(N)$. Then u acts as an automorphism on $a \in M_N(\mathbb{C})$ by sending $a \mapsto uau^*$. If $u = \lambda\mathbb{I}_N$ is a multiple of the identity with $\lambda \in U(1)$, this action is trivial, hence the group of automorphisms of $\mathcal{A}$ is the projective unitary group $PU(N) = U(N)/U(1)$, in concordance with* (6.1.1).

The fact that all automorphisms are inner follows from the following observation. First, any $*$-algebra map $\alpha : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ can be considered a representation of $\mathcal{A}$ on $\mathbb{C}^N$. As the unique irreducible representation space of $M_N(\mathbb{C})$ is given by the defining representation (Lemma 2.15) we conclude that the representation α is unitarily equivalent to the defining representation on $\mathbb{C}^N$. Hence, $\alpha(a) = uau^*$ with $u \in U(N)$.

EXERCISE 6.1. *Show that* $\mathrm{Aut}(M_N(\mathbb{C}) \oplus M_N(\mathbb{C})) \simeq (U(N) \times U(N)) \rtimes S_2$ *with the symmetric group S_2 acting by permutation on the two copies of $U(N)$.*

Inner automorphisms α_u not only act on the $*$-algebra $\mathcal{A}$, via the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ they also act on the Hilbert space $\mathcal{H}$ present in the spectral triple. In fact, with $U = \pi(u)J\pi(u)J^{-1}$, the unitary u induces a unitary equivalence of real spectral triples in the sense of Definition 3.4, as the following exercise shows.

EXERCISE 6.2. *Use Definition 4.30 to establish the following transformation rules for a unitary $U = \pi(u) J \pi(u) J^{-1}$ with $u \in \mathcal{U}(\mathcal{A})$:*

$$
\begin{aligned}
U\pi(a)U^* &= \pi \circ \alpha_u(a); \\
U\gamma &= \gamma U; \\
UJU^* &= J.
\end{aligned}
\tag{6.1.2}
$$

We conclude that an inner automorphism α_u of $\mathcal{A}$ induces a unitary equivalent spectral triple $(\mathcal{A}, \mathcal{H}, UDU^*; J, \gamma)$, where the action of the $*$-algebra is given by $\pi \circ \alpha_u$. Note that the grading and the real structure are left unchanged under these 'inner' unitary equivalences; only the operator D is affected by the unitary transformation. For the latter, we compute, using (4.3.1),

$$
D \mapsto UDU^* = D + u[D, u^*] + \epsilon' J u[D, u^*] J^{-1}, \tag{6.1.3}
$$

where as before we have suppressed the representation π. We recognize the extra terms as *pure gauge* fields udu^* in the space of Connes' differential one-forms $\Omega^1_D(\mathcal{A})$ of Definition 4.36. This motivates the following definition

DEFINITION 6.4. *The* gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ *of the spectral triple is*

$$
\mathfrak{G}(\mathcal{A}, \mathcal{H}; J) := \left\{ U = uJuJ^{-1} \mid u \in \mathcal{U}(\mathcal{A}) \right\}.
$$

Recall (from Section 4.3.1) the construction of a complex subalgebra $\mathcal{A}_J$ in the center of $\mathcal{A}$ from a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$, given by

$$
\mathcal{A}_J := \{ a \in \mathcal{A} : aJ = Ja^* \}.
$$

PROPOSITION 6.5. *There is a short exact sequence of groups*

$$
1 \to \mathcal{U}(\mathcal{A}_J) \to \mathcal{U}(\mathcal{A}) \to \mathfrak{G}(\mathcal{A}, \mathcal{H}; J) \to 1.
$$

Moreover, there is a surjective map $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J) \to \mathrm{Inn}(\mathcal{A})$.

PROOF. Consider the map $\mathrm{Ad}\colon \mathcal{U}(\mathcal{A}) \to \mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ given by $u \mapsto uJuJ^{-1}$. This map Ad is a group homomorphism, since the commutation relation $[u, JvJ^{-1}] = 0$ of (4.3.1) implies that

$$
\mathrm{Ad}(v)\,\mathrm{Ad}(u) = vJvJ^{-1}uJuJ^{-1} = vuJvuJ^{-1} = \mathrm{Ad}(vu).
$$

By definition Ad is surjective, and $\ker(\mathrm{Ad}) = \{u \in \mathcal{U}(\mathcal{A}) \mid uJuJ^{-1} = 1\}$. The relation $uJuJ^{-1} = 1$ is equivalent to $uJ = Ju^*$ which is the defining relation of the commutative subalgebra $\mathcal{A}_J$. This proves that $\ker(\mathrm{Ad}) = \mathcal{U}(\mathcal{A}_J)$. The map $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J) \to \mathrm{Inn}(\mathcal{A})$ is given by (6.1.2), from which surjectivity readily follows. $\square$

COROLLARY 6.6. *If $\mathcal{U}(\mathcal{A}_J) = \mathcal{U}(Z(\mathcal{A}))$, then $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J) \simeq \mathrm{Inn}(\mathcal{A})$.*

PROOF. This is immediate from the above Proposition and (6.1.1). $\square$

We summarize this by the following sequence, which is exact in the horizontal direction:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{U}(\mathcal{A}_J) \arrow[r] \arrow[d] & \mathcal{U}(\mathcal{A}) \arrow[r] \arrow[d, equal] & \mathfrak{G}(\mathcal{A},\mathcal{H};J) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathcal{U}(Z(\mathcal{A})) \arrow[r] & \mathcal{U}(\mathcal{A}) \arrow[r] & \mathrm{Inn}(\mathcal{A}) \arrow[r] & 1 \end{tikzcd}$$

6.1.1. The gauge algebra. A completely analogous discussion applies to the definition of a gauge Lie algebra, where instead of automorphisms we now take (inner and outer) **derivations** of $\mathcal{A}$. The following definition essentially gives the infinitesimal version of $\mathfrak{G}(\mathcal{A},\mathcal{H};J)$.

Definition 6.7. *The* gauge Lie algebra $\mathfrak{g}(\mathcal{A},\mathcal{H};J)$ *of the spectral triple is*

$$\mathfrak{g}(\mathcal{A},\mathcal{H};J) := \left\{ T = X + JXJ^{-1} \mid X \in \mathfrak{u}(\mathcal{A}) \right\},$$

where $\mathfrak{u}(\mathcal{A})$ *consists of the skew-hermitian elements in* $\mathcal{A}$.

One easily checks using the commutant property,

$$[T,T'] = [X,X'] + J[X,X']J^{-1},$$

so that $\mathfrak{g}(\mathcal{A},\mathcal{H};J)$ is indeed a Lie algebra.

Proposition 6.8. *There is a short exact sequence of Lie algebras*

$$0 \to \mathfrak{u}(\mathcal{A}_J) \to \mathfrak{u}(\mathcal{A}) \to \mathfrak{g}(\mathcal{A},\mathcal{H};J) \to 0.$$

There are also inner derivations of $\mathcal{A}$ that are of the form $a \to [X,a]$; these form a Lie subalgebra $\mathrm{Der}_{\mathrm{Inn}}(\mathcal{A})$ of the Lie algebra of all derivations $\mathrm{Der}(\mathcal{A})$. If $\mathfrak{u}(\mathcal{A}_J) = \mathfrak{u}(Z(\mathcal{A}))$ then

$$\mathfrak{g}(\mathcal{A},\mathcal{H};J) \simeq \mathrm{Der}_{\mathrm{Inn}}(\mathcal{A}),$$

which essentially is the infinitesimal version of Corollary 6.6.

Exercise 6.3. *Show that* $\mathrm{Der}(M_N(\mathbb{C})) \simeq su(N)$ *as Lie algebras.*

6.2. Morita self-equivalences as gauge fields

We have seen that a non-abelian gauge group appears naturally when the unital $*$-algebra $\mathcal{A}$ in a real spectral triple is noncommutative. Moreover, noncommutative algebras allow for a more general – and in fact more natural – notion of equivalence than automorphic equivalence, namely Morita equivalence. We have already seen this in Chapter 2. Indeed, let us imitate the construction in Theorem 2.26 and Theorem 3.6 and see if we can lift Morita equivalence to the level of spectral triples in this more general setting.

Let us first recall some of the basic definitions. We keep working in the setting of unital algebras, which greatly simplifies matters (See Note 4 on Page 97).

6.2.1. Morita equivalence. Recall Definition 2.8 of algebra modules. For two right $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ we denote the space of right **$\mathcal{A}$-module homomorphisms** by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F})$, *i.e.*

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{F}) := \{\phi : \mathcal{E} \to \mathcal{F} : \phi(\eta a) = \phi(\eta) a \text{ for all } \eta \in \mathcal{E}, a \in A\}. \tag{6.2.1}$$

We also write $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) := \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E})$ for the **algebra of right $\mathcal{A}$-module endomorphisms** of $\mathcal{E}$.

Definition 6.9. *Two unital algebras $\mathcal{A}$ and $\mathcal{B}$ are called* Morita equivalent *if there exists a $\mathcal{B}-\mathcal{A}$-bimodule $\mathcal{E}$ and an $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{F}$ such that*

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}, \qquad \mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A},$$

as $\mathcal{B}$ and $\mathcal{A}$-bimodules, respectively.

Exercise 6.4. *Taking inspiration from Exercise 2.9, show that Morita equivalence is an equivalence relation.*

Exercise 6.5. *Define $\mathcal{A}^N = \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ (N copies) as an $\mathcal{A}-M_N(\mathcal{A})$-bimodule.*

(1) *Show that $\mathcal{A}^N \otimes_{\mathcal{A}} \mathcal{A}^N \simeq M_N(\mathcal{A})$, as $M_N(\mathcal{A})-M_N(\mathcal{A})$-bimodules.*
(2) *Show that $\mathcal{A}^N \otimes_{M_N(\mathcal{A})} \mathcal{A}^N \simeq \mathcal{A}$, so that $M_N(\mathcal{A})$ is Morita equivalent to $\mathcal{A}$.*

A convenient characterisation of Morita equivalent algebras is given by the concept of endomorphism algebras of so-called finitely generated projective modules, as we now explain.

Definition 6.10. *A right $\mathcal{A}$-module is called* finitely generated projective *(or, briefly,* finite projective*) if there is an idempotent $p = p^2$ in $M_N(\mathcal{A})$ for some N such that $\mathcal{E} \simeq p\mathcal{A}^N$.*

Lemma 6.11. *A right $\mathcal{A}$-module is finitely generated projective if and only if*

$$\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \simeq \mathcal{E} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A}).$$

Proof. First note that the right-hand side can be considered to be a two-sided ideal in $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$. Namely, we consider an element $\eta \otimes_{\mathcal{A}} \phi$ in $\mathcal{E} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ as an element in $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ by mapping

$$\xi \mapsto \eta\phi(\xi); \qquad (\xi \in \mathcal{E}).$$

That this map is injective and that its image forms an ideal in $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is readily checked. Hence, the above isomorphism is equivalent to the existence of an element in $\mathcal{E} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ that acts as the identity map $\mathrm{id}_{\mathcal{E}}$ on $\mathcal{E}$.

Suppose that $\mathcal{E}$ is finite projective, $\mathcal{E} \simeq p\mathcal{A}^N$ for some idempotent $p \in M_N(\mathcal{A})$. We identify two maps

$$\lambda : \mathcal{E} \to \mathcal{A}^N,$$
$$\rho : \mathcal{A}^N \to \mathcal{E},$$

which are injective and surjective, respectively. These maps are related to the identification of $\mathcal{E}$ with a direct summand of $\mathcal{A}^N$, via the obvious direct sum decomposition $\mathcal{A}^N = p\mathcal{A}^N \oplus (1-p)\mathcal{A}^N$. Namely, λ identifies $\mathcal{E}$ with $p\mathcal{A}^N \subset \mathcal{A}^N$, whereas ρ projects $\mathcal{A}^N$ onto the direct summand $p\mathcal{A}^N$ and then identifies it with $\mathcal{E}$. Let us write λ_k for the k'th component of λ mapping $\mathcal{E}$ to $\mathcal{A}^N$; thus, $\lambda_k : \mathcal{E} \to \mathcal{A}$ is right $\mathcal{A}$-linear for any $k = 1, \ldots, N$. We write $\rho_k := \rho(e_k) \in \mathcal{E}$, where $\{e_k\}_{k=1}^N$ is the standard basis of $\mathcal{A}^N$. The composition $\sum_{k=1}^N \rho_k \otimes \lambda_k$ then acts as the identity operator on $\mathcal{E}$.

Conversely, suppose $\mathrm{id}_\mathcal{E}$ can be written as a finite sum

$$\sum_{k=1}^{N} \rho_k \otimes \lambda_k \in \mathcal{E} \otimes_\mathcal{A} \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}). \tag{6.2.2}$$

Reversing the construction in the previous paragraph, we are now going to define an idempotent $p \in M_N(\mathcal{A})$ such that $\mathcal{E} \simeq p\mathcal{A}^N$. Thus, we define maps

$$\begin{aligned} \lambda : \mathcal{E} \to \mathcal{A}^N; \qquad & \eta \mapsto (\lambda_1(\eta), \ldots, \lambda_N(\eta)), \\ \rho : \mathcal{A}^N \to \mathcal{E}; \qquad & (a_1, \ldots, a_N) \mapsto \rho_1 a_1 + \cdots + \rho_N a_N. \end{aligned}$$

From their very definition, these maps satisfy $\rho \circ \lambda = \mathrm{id}_\mathcal{E}$, so that $p = \lambda \circ \rho$ is the sought-for idempotent in $M_N(\mathcal{A})$. □

EXERCISE 6.6. *In this exercise we are going to analyze the ambiguity due to the balanced tensor product that appears in the decomposition* (6.2.2) *of* $\mathrm{id}_\mathcal{E}$.

(1) *If* $\mathcal{E} = \mathcal{A}$ *then* $\mathrm{id}_\mathcal{E} = 1 \otimes 1 \subset \mathcal{E} \otimes_\mathcal{A} \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$ *but also*
$$\mathrm{id}_\mathcal{E} = 1 \otimes 1 + a \otimes 1 + 1 \otimes (-a),$$
for any $a \in \mathcal{A}$. *Show that the projection corresponding to the latter decomposition of* $\mathrm{id}_\mathcal{E}$ *is*
$$p = \begin{pmatrix} 1 & 1 & -a \\ a & a & -a^2 \\ 1 & 1 & -a \end{pmatrix}.$$
(2) *Show that there is a similarity transformation* S *such that*
$$SpS^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Therefore, the projection corresponding to $\mathrm{id}_\mathcal{E} = 1 \otimes 1$ *appears as the first diagonal entry, and we can conclude that both decompositions give isomorphic projective modules* $p\mathcal{A}^3 \simeq \mathcal{A}$.
(3) *Extend this argument to any finite projective* $\mathcal{E}$ *to show that the construction of a projection* p *from* (6.2.2) *is well defined.*

PROPOSITION 6.12. *Two unital algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are Morita equivalent if and only if* $\mathcal{B} \simeq \mathrm{End}_\mathcal{A}(\mathcal{E})$, *with* $\mathcal{E}$ *a finite projective* $\mathcal{A}$*-module.*

PROOF. If $\mathcal{B} \simeq \mathrm{End}_\mathcal{A}(\mathcal{E})$ for some finite projective $\mathcal{E}$, then $\mathcal{F} = \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$ is the required $\mathcal{A}-\mathcal{B}$-bimodule implementing the desired Morita equivalence, with bimodule structure given by

$$(a \cdot \phi \cdot b)(\eta) = a\phi(b \cdot \eta); \qquad (\phi \in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})). \tag{6.2.3}$$

The property $\mathcal{E} \otimes_\mathcal{A} \mathcal{F} \simeq \mathcal{B}$ follows from Lemma 6.11, and the isomorphism $\mathcal{F} \otimes_\mathcal{B} \mathcal{E} \simeq \mathcal{A}$ is implemented by the evaluation map, that is,

$$(\phi \otimes \eta) \in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}) \otimes_\mathcal{B} \mathcal{E} \mapsto \phi(\eta) \in \mathcal{A}.$$

Conversely, suppose $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. If $\mathcal{B} \simeq \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$, then $\mathcal{B} \simeq \mathrm{End}_{\mathcal{B}}(\mathcal{B}) \simeq \mathrm{End}_{\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})$, and there is an algebra map

$$\begin{aligned} \mathrm{End}_{\mathcal{A}}(\mathcal{E}) &\to \mathrm{End}_{\mathcal{B}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}); \\ \phi &\mapsto \phi \otimes 1_{\mathcal{F}}. \end{aligned}$$

On the other hand, $\mathrm{End}_{\mathcal{A}}(\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}) \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, and there is an algebra map

$$\begin{aligned} \mathrm{End}_{\mathcal{B}}(\mathcal{B}) &\to \mathrm{End}_{\mathcal{A}}(\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}); \\ \phi' &\mapsto \phi' \otimes 1_{\mathcal{E}}. \end{aligned}$$

Identifying $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}$ and $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A}$, one readily checks that these two maps are each other's inverses. This shows that $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

Finally, the fact that the right $\mathcal{A}$-module $\mathcal{E}$ is finitely generated and projective follows *mutatis mutandis* from the proof of Lemma 6.11, after realizing that the isomorphism $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A}$ associates an element in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ to any element in $\mathcal{F}$. □

EXERCISE 6.7. *Show that* (6.2.3) *is a well-defined* $\mathcal{A}-\mathcal{B}$*-bimodule structure on* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$*, i.e. show that it respects the right* $\mathcal{A}$*-linearity of the map* $\phi : \mathcal{E} \to \mathcal{A}$*.*

We conclude this subsection by specializing from algebras to $*$-algebras. The above results on Morita equivalence still hold, with the additional requirement that in the definition of finite projectivity the idempotent $p \in M_N(\mathbb{C})$ needs to be self-adjoint: $p^* = p$. That is to say, p is an **orthogonal projection**.

As in Definition 3.5, we define the **conjugate module** $\mathcal{E}^\circ$ to a right $\mathcal{A}$-module $\mathcal{E}$ as

$$\mathcal{E}^\circ = \{\overline{\xi} : \xi \in \mathcal{E}\},$$

equipped with a left $\mathcal{A}$ action defined by $a\overline{\xi} = \overline{\xi a^*}$ for any $a \in \mathcal{A}$.

PROPOSITION 6.13. *If* $\mathcal{A}$ *is a* $*$*-algebra and* $\mathcal{E}$ *is a finite projective right* $\mathcal{A}$*-module, then we can identify* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ *as a left* $\mathcal{A}$*-module with the conjugate module* $\mathcal{E}^\circ$*,*

PROOF. If $\mathcal{E} \simeq p\mathcal{A}^N$ then $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \simeq pM_N(\mathcal{A})p$, as one can easily show using the maps λ and ρ from the first part of the proof of Lemma 6.11. Hence $\mathcal{E} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \simeq pM_N(\mathcal{A})p$. But also $p\mathcal{A}^N \otimes_{\mathcal{A}} \mathcal{A}^N p \simeq pM_N(\mathcal{A})p$ (*cf.* Exercise 6.5), so $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \simeq \mathcal{A}^N p$ as left $\mathcal{A}$-modules. We now show that $\mathcal{E}^\circ \simeq \mathcal{A}^N p$ as well.

For that, write $\xi \in \mathcal{E} \simeq p\mathcal{A}^N$ as a column vector:

$$\xi = \begin{pmatrix} \sum_{j=1}^N p_{1j} a_j \\ \vdots \\ \sum_{j=1}^N p_{Nj} a_j \end{pmatrix}.$$

The corresponding element $\overline{\xi}$ in $\mathcal{E}^\circ$ is identified with

$$\begin{pmatrix} \sum_{j=1}^N a_j^* p_{j1} & \cdots & \sum_{j=1}^N a_j^* p_{jN} \end{pmatrix},$$

written as a row vector in $\mathcal{A}^N p$. Note that the relation between ξ and this row vector is essentially given by the involution on $\mathcal{A}^N$, exploiting the self-adjointness of p, that is, $p_{ji}^* = p_{ij}$. Consequently, the element $a\overline{\xi} = \overline{\xi a^*}$ is mapped to

$$a \begin{pmatrix} \sum_{j=1}^N a_j^* p_{j1} & \cdots & \sum_{j=1}^N a_j^* p_{jN} \end{pmatrix},$$

as required. □

PROPOSITION 6.14. *Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{E}$ a finite projective right $\mathcal{A}$-module. Then there exists a* hermitian structure *on $\mathcal{E}$, that is to say, there is a pairing $\langle \cdot, \cdot \rangle_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ on $\mathcal{E}$ that satisfies (as in Definition 2.9)*

$$\begin{aligned}
&\langle \eta_1, \eta_2 \cdot a \rangle_{\mathcal{E}} = \langle \eta_1, \eta_2 \rangle_{\mathcal{E}} a; \qquad (\eta_1, \eta_2 \in \mathcal{E}, a \in \mathcal{A}),\\
&\langle \eta_1, \eta_2 \rangle_{\mathcal{E}}^* = \langle \eta_2, \eta_1 \rangle_{\mathcal{E}}; \qquad (\eta_1, \eta_2 \in \mathcal{E}),\\
&\langle \eta, \eta \rangle_{\mathcal{E}} \geq 0, \ \textit{with equality if and only if } \eta = 0; \qquad (\eta \in \mathcal{E}).
\end{aligned}$$

PROOF. On $\mathcal{A}^N$ we have a hermitian structure given by

$$\langle \eta, \xi \rangle = \sum_{j=1}^N \eta_j^* \xi_j,$$

which satisfies the above properties. By restriction to $p\mathcal{A}^N$ we then obtain a hermitian structure on $\mathcal{E} \simeq p\mathcal{A}^N$. □

6.2.2. Morita equivalence and spectral triples. For a given spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and for a given finite projective right $\mathcal{A}$-module $\mathcal{E}$, we try to construct another spectral triple $(\mathcal{B}, \mathcal{H}', D')$ where $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. This generalizes the finite-dimensional constructions of Chapters 2 and 3. Naturally,

$$\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

carries an action of $\phi \in \mathcal{B}$:

$$\phi(\eta \otimes \psi) = \phi(\eta) \otimes \psi; \qquad (\eta \in \mathcal{E}, \psi \in \mathcal{H}).$$

Moreover, by finite projectivity of $\mathcal{E}$, $\mathcal{H}'$ is a Hilbert space. Indeed, we have

$$\mathcal{H}' \simeq p\mathcal{A}^N \otimes_{\mathcal{A}} \mathcal{H} \simeq p\mathcal{H}^N,$$

and since p is an orthogonal projection it has closed range.

However, the naive choice of an operator D' by $D'(\eta \otimes \psi) = \eta \otimes D\psi$ will not do, because it does not respect the ideal defining the tensor product over $\mathcal{A}$, which is generated by elements of the form

$$\eta a \otimes \psi - \eta \otimes a\psi; \qquad (\eta \in \mathcal{E}, a \in \mathcal{A}, \psi \in \mathcal{H}).$$

A better definition is

$$D'(\eta \otimes \psi) = \eta \otimes D\psi + \nabla(\eta)\psi.$$

where $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$ is a **connection** associated to the derivation $\mathrm{d} : a \mapsto [D, a]$ $(a \in \mathcal{A})$. This means that ∇ is a linear map that satisfies the Leibniz rule:

$$\nabla(\eta a) = (\nabla \eta) a + \eta \otimes_{\mathcal{A}} \mathrm{d}a; \qquad (\eta \in \mathcal{E}, a \in \mathcal{A}).$$

EXERCISE 6.8. *(1) Let ∇ and ∇' be two connections on a right $\mathcal{A}$-module $\mathcal{E}$. Show that their difference $\nabla - \nabla'$ is a right $\mathcal{A}$-linear map $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$.*

(2) Show that the following map defines a connection on $\mathcal{E} = p\mathcal{A}^N$:

$$\nabla = p \circ \mathrm{d},$$

with d acting on each copy of $\mathcal{A}$ as the commutator $[D, \cdot]$. This connection is referred to as the **Grassmann connection** *on $\mathcal{E}$.*

THEOREM 6.15. *If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and ∇ is a connection on a finite projective right $\mathcal{A}$-module $\mathcal{E}$, then $(\mathcal{B}, \mathcal{H}', D')$ is a spectral triple, provided that ∇ is a hermitian connection, i.e. provided that*

$$\langle \eta_1, \nabla\eta_2\rangle_{\mathcal{E}} - \langle \nabla\eta_1, \eta_2\rangle_{\mathcal{E}} = \mathrm{d}\langle \eta_1, \eta_2\rangle_{\mathcal{E}}; \qquad (\eta_1, \eta_2 \in \mathcal{E}). \tag{6.2.4}$$

PROOF. Suppose $\mathcal{E} = p\mathcal{A}^N$, so that $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \simeq pM_N(\mathcal{A})p$ and $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \simeq p\mathcal{H}^N$. The boundedness of the action of $\mathcal{B}$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ then follows directly from the boundedness of the action of $\mathcal{A}$ on $\mathcal{H}$. Similarly, for $\phi \in \mathcal{B}$ the commutator $[D, \phi]$ can be regarded as a matrix with entries of the form $[D, a]$ with $a \in \mathcal{A}$. These commutators are all bounded, so that $[D, \phi]$ is bounded. Let us prove compactness of the resolvent. By Exercise 6.8 any connection can be written as $\nabla = p \circ [D, \cdot] + \omega$ for a right $\mathcal{A}$-linear map $\omega : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$. Hence, after making the above identifications we see that the operator $\nabla \otimes 1 + 1 \otimes D$ coincides with $pDp + \omega$. The action of ω is as a bounded operator, which by (6.2.4) is self-adjoint. Moreover, it is given by a matrix acting on $p\mathcal{H}^N$ with entries in $\Omega^1_D(\mathcal{A})$. Since for any self-adjoint operator T we have

$$(i + T + \omega)^{-1} = (i + T)^{-1} \left(1 - \omega(i + T + \omega)^{-1}\right),$$

with $\left(1 - \omega(i + T + \omega)^{-1}\right)$ bounded, compactness of the resolvent of $pDp+\omega$ would follow from compactness of $(ip + pDp)^{-1}$ (note that p is the identity on the Hilbert space $p\mathcal{H}^N$). The required compactness property is a consequence of the identity

$$(ip + pDp)p(i + D)^{-1}p = p[i + D, p](i + D)^{-1}p + p.$$

Indeed, when multiplied on the left with $(ip+pDp)^{-1}$ we find that on $p\mathcal{H}^N$:

$$(ip + pDp)^{-1} = p(i + D)^{-1}p - (ip + pDp)^{-1}p[D, p](i + D)^{-1}p,$$

which is compact since $(i + D)^{-1}$ is compact by definition of a spectral triple. □

Analogously, for a given real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ we define another real spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$ by setting

$$\mathcal{H}' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ.$$

Then, $\phi \in \mathcal{B}$ acts on $\mathcal{H}'$ by

$$\phi(\eta \otimes \psi \otimes \overline{\xi}) = \phi(\eta) \otimes \psi \otimes \overline{\xi},$$

and the operator D' and J' may be defined by

$$D'(\eta \otimes \psi \otimes \overline{\xi}) = (\nabla\eta)\psi \otimes \overline{\xi} + \eta \otimes D\psi \otimes \overline{\xi} + \eta \otimes \psi(\overline{\nabla\xi}),$$
$$J'(\eta \otimes \psi \otimes \overline{\xi}) = \xi \otimes J\psi \otimes \overline{\eta}.$$

Finally, for even spectral triples one defines a grading γ' on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}^\circ$ by $\gamma' = 1 \otimes \gamma \otimes 1$. We have therefore proved:

THEOREM 6.16. *If $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$ is a real spectral triple and ∇ is a hermitian connection, then $(\mathcal{B}, \mathcal{H}', D'; J', \gamma')$ is a real spectral triple.*

We now focus on **Morita self-equivalences**, for which $\mathcal{B} = \mathcal{A}$ and $\mathcal{E} = \mathcal{A}$ so that $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \simeq \mathcal{A}$. Let us look at connections

$$\nabla : \mathcal{A} \to \Omega^1_D(\mathcal{A}).$$

Clearly, by the Leibniz rule we must have $\nabla = \mathrm{d} + \omega$ (see also Exercise 6.8), where $\omega = \nabla(1) = \sum_j a_j [D, b_j]$ is a generic element in $\Omega^1_D(\mathcal{A})$ acting as a bounded operator on $\mathcal{H}$. Similarly, $\psi \overline{\nabla} \overline{a} = (\epsilon' J \mathrm{d} a J^{-1} + \epsilon' J \omega a J^{-1})\psi$. Since $\mathcal{H}' \simeq \mathcal{H}$, under this identification we have,

$$D'(\psi) \equiv D'(1 \otimes \psi \otimes \overline{1}) = \nabla(1)\psi + \psi \overline{\nabla}(\overline{1}) + D\psi = D\psi + \omega\psi + \epsilon' J \omega J^{-1} \psi.$$

In other words, D is 'innerly perturbed' by the given Morita self-equivalence to

$$D_\omega := D + \omega + \epsilon' J \omega J^{-1},$$

where $\omega^* = \omega \in \Omega^1_D(\mathcal{A})$ is called a **gauge field**, alternatively called an **inner fluctuation** of the operator D, since it is the algebra $\mathcal{A}$ that — through Morita self-equivalences— generates the field ω.

PROPOSITION 6.17. *A unitary equivalence of a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$ as implemented by $U = uJuJ^{-1}$ with $u \in \mathcal{U}(\mathcal{A})$ (discussed before Definition 6.4) is a special case of a Morita self-equivalence, arising by taking $\omega = u[D, u^*]$.*

PROOF. This follows upon inserting $\omega = u[D, u^*]$ in the above formula for D_ω, yielding (6.1.3). □

In the same way there is an action of the unitary group $\mathcal{U}(\mathcal{A})$ on the new spectral triple $(\mathcal{A}, \mathcal{H}, D_\omega)$ by unitary equivalences. Recall that $U = uJuJ^{-1}$ acts on D_ω by conjugation:

$$D_\omega \mapsto U D_\omega U^*. \tag{6.2.5}$$

This is equivalent to

$$\omega \mapsto u\omega u^* + u[D, u^*],$$

which is the usual rule for a gauge transformation on a gauge field.

6.3. Localization

Recall (from Section 4.3.1) the construction of a complex subalgebra $\mathcal{A}_J$ in the center of $\mathcal{A}$ from a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$, given by

$$\mathcal{A}_J := \{a \in \mathcal{A} : aJ = Ja^*\}.$$

As $\mathcal{A}_J$ is commutative, Gelfand duality (Theorem 4.28) ensures the existence of a compact Hausdorff space such that $\mathcal{A}_J \subset C(X)$ as a dense $*$-subalgebra. Indeed, the C^*-completion of $\mathcal{A}_J$ in $\mathcal{B}(\mathcal{H})$ is commutative and hence isomorphic to such a $C(X)$. We consider this space X to be the 'background space' on which $(\mathcal{A}, \mathcal{H}, D; J, \gamma)$ describes a gauge theory, as we now work out in detail.

Heuristically speaking, the above gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ considers only transformations that are 'vertical', or 'purely noncommutative' with respect to X, quotienting out the unitary transformations of the commutative subalgebra $\mathcal{A}_J$. Let us make this precise by identifying a bundle $\mathfrak{B} \to X$ of C^*-algebras such that:

- the space of continuous sections $\Gamma(X, \mathfrak{B})$ forms a C^*-algebra isomorphic to $A = \overline{\mathcal{A}}$, the C^*-completion of $\mathcal{A}$;
- the gauge group acts as bundle automorphisms covering the identity.

Moreover, we search for a bundle of C^*-algebras of which the gauge fields $\omega \in \Omega^1_D(\mathcal{A})$ are sections and on which the gauge group again acts by bundle automorphisms.

We avoid technical complications that might arise from working with dense subalgebras of C^*-algebras, and work with the C^*-algebras A_J and A themselves, as completions of $\mathcal{A}_J$ and $\mathcal{A}$, respectively. First, note that there is an inclusion map $C(X) \simeq A_J \hookrightarrow A$. This means that A is a so-called $C(X)$-algebra, which by definition is a C^*-algebra A with a map from $C(X)$ to the center of A. Indeed, it follows from Proposition 4.37 that A_J is contained in the center of A.

In such a case A is the C^*-algebra of continuous sections of an upper semi-continuous C^*-bundle over X. We will briefly sketch the setup (see Note 7 on Page 98). Recall that a function $f : A \to \mathbb{C}$ is **upper semi-continuous** at $a_0 \in A$ if $\limsup_{a \to a_0} \|f(a)\| \leq \|f(a_0)\|$.

Definition 6.18. *An* upper semi-continuous C^*-bundle *over a compact topological space X is a continuous, open, surjection $\pi : \mathfrak{B} \to X$ together with operations and norms that turn each fiber $\mathfrak{B}_x = \pi^{-1}(x)$ into a C^*-algebra, such that (1) the map $a \mapsto \|a\|$ is upper semi-continuous, (2) all algebraic operations are continuous on $\mathfrak{B}$, (3) if $\{a_i\}$ is a net in $\mathfrak{B}$ such that $\|a_i\| \to 0$ and $\pi(a_i) \to x$ in X, then $a_i \to 0_x$, where 0_x is the zero element in $\mathfrak{B}_x$.*

A (continuous) section *of $\mathfrak{B}$ is a (continuous) map $s : X \to \mathfrak{B}$ such that $\pi(s(x)) = x$.*

A base for the topology on $\mathfrak{B}$ is given by the following collection of open sets:

$$W(s, \mathcal{O}, \epsilon) := \{b \in \mathfrak{B} : \pi(b) \in \mathcal{O} \text{ and } \|b - s(\pi(b))\| < \epsilon\}, \tag{6.3.1}$$

indexed by continuous sections $s \in \Gamma(X, \mathfrak{B})$, open subsets $\mathcal{O} \subset X$ and $\epsilon > 0$.

Proposition 6.19. *The space $\Gamma(X, \mathfrak{B})$ of continuous sections forms a C^*-algebra when it is equipped with the norm*

$$\|s\| := \sup_{x \in X} \|s(x)\|_{\mathfrak{B}_x}.$$

Proof. See Note 7 on Page 98. $\square$

In our case, after identifying $C(X)$ with A_J, we can define a closed two-sided ideal in A by

$$I_x := \{fa : a \in A, f \in C(X), f(x) = 0\}^- . \tag{6.3.2}$$

We think of the quotient C^*-algebra $\mathfrak{B}_x := A/I_x$ as the fiber of A over x and set

$$\mathfrak{B} := \coprod_{x \in X} \mathfrak{B}_x, \tag{6.3.3}$$

with an obvious surjective map $\pi : \mathfrak{B} \to X$. If $a \in A$, then we write $a(x)$ for the image $a + I_x$ of a in $\mathfrak{B}_x$, and we think of a as a section of $\mathfrak{B}$. The fact that all these sections are continuous and that elements in A can be obtained in this way is guaranteed by the following result.

THEOREM 6.20. *The above map $\pi : \mathfrak{B} \to X$ with $\mathfrak{B}$ as in* (6.3.3) *defines an upper semi-continuous C^*-bundle over X. Moreover, there is a $C(X)$-linear isomorphism of A onto $\Gamma(X, \mathfrak{B})$.*

PROOF. See Note 7 on Page 98. □

Having obtained the C^*-algebra A as the space of sections of a C^*-bundle, we are ready to analyze the action of the gauge group on A. Staying at the C^*-algebraic level, we consider the **continuous gauge group**

$$\mathfrak{G}(A, \mathcal{H}; J) \simeq \frac{\mathcal{U}(A)}{\mathcal{U}(A_J)}.$$

This contains the gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ of Definition 6.4 as a dense subgroup in the topology induced by the C^*-norm on A. The next result realizes the gauge group as a group of vertical bundle automorphisms of $\mathfrak{B}$.

PROPOSITION 6.21. *The action α of $\mathfrak{G}(A, \mathcal{H}; J)$ on A by inner C^*-algebra automorphisms induces an action $\widetilde{\alpha}$ of $\mathfrak{G}(A, \mathcal{H}; J)$ on $\mathfrak{B}$ by continuous bundle automorphisms that cover the identity. In other words, for $g \in \mathfrak{G}(A, \mathcal{H}; J)$ we have*

$$\pi(\widetilde{\alpha}_g(b)) = \pi(b); \qquad (b \in \mathfrak{B}).$$

Moreover, under the identification of Theorem 6.20 the induced action $\widetilde{\alpha}^*$ on $\Gamma(X, \mathfrak{B})$ given by

$$\widetilde{\alpha}_g^*(s)(x) = \widetilde{\alpha}_g(s(x))$$

coincides with the action α on A.

PROOF. The action α induces an action on $A/I_x = \pi^{-1}(x)$, since $\alpha_g(I_x) \subset I_x$ for all $g \in \mathfrak{G}(A, \mathcal{H}; J)$. We denote the corresponding action of $\mathfrak{G}(A, \mathcal{H}; J)$ on $\mathfrak{B}$ by $\widetilde{\alpha}$, so that, indeed,

$$\pi(\widetilde{\alpha}_g(b)) = \pi(b); \qquad (b \in \pi^{-1}(x)).$$

Let us also check continuity of this action. In terms of the base $W(s, \mathcal{O}, \epsilon)$ of (6.3.1), we find that

$$\widetilde{\alpha}_g(W(s, \mathcal{O}, \epsilon)) = W(\widetilde{\alpha}_g^*(s), \mathcal{O}, \epsilon),$$

mapping open subsets one-to-one and onto open subsets.

For the second claim, it is enough to check that the action $\widetilde{\alpha}^*$ on the section $s : x \mapsto a + I_x \in \mathfrak{B}_x$, defined by an element $a \in A$, corresponds to the action α on that a. In fact,

$$\widetilde{\alpha}_g^*(s)(x) = \widetilde{\alpha}_g(s(x)) = \alpha_g(a + I_x) = \alpha_g(a) + I_x,$$

which completes the proof. □

At the infinitesimal level, the derivations in the gauge algebra $\mathfrak{g}(\mathcal{A}, \mathcal{H}; J)$ also act vertically on the C^*-bundle $\mathfrak{B}$ defined in (6.3.3), and the induced action on the sections $\Gamma(X, \mathfrak{B})$ agrees with the action of $\mathfrak{g}(\mathcal{A}, \mathcal{H}; J)$ on A.

6.3.1. Localization of gauge fields. Also the gauge fields ω that enter as inner fluctuations of D can be parametrized by sections of some bundle of C^*-algebras. In order for this to be compatible with the vertical action of the gauge group found above, we will write any connection in the form,

$$\nabla = \mathrm{d} + \omega_0 + \omega,$$

where $\mathrm{d} = [D, \cdot]$ and $\omega_0, \omega \in \Omega^1_D(\mathcal{A})$. The action of a gauge transformation on ∇ then induces the following transformation:

$$\omega_0 \mapsto u\omega_0 u^* + u[D, u^*]; \qquad \omega \mapsto u\omega u^*.$$

The C^*-algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ is a $C(X)$-algebra, since $C(X) \simeq A_J$, which according to Proposition 4.37 commutes with both $\mathcal{A}$ and $[D, \mathcal{A}]$. Thus, a similar construction as in the previous subsection establishes the existence of an upper semi-continuous C^*-bundle $\mathfrak{B}_\Omega$ over X, explicitly given by

$$\mathfrak{B}_\Omega = \prod_{x \in X} C^*(\mathcal{A}, [D, \mathcal{A}])/I'_x,$$

where $C^*(\mathcal{A}, [D, \mathcal{A}])$ is the C^*-algebra generated by a and $[D, b]$ for $a, b \in \mathcal{A}$, and I'_x is the two-sided ideal in $C^*(\mathcal{A}, [D, \mathcal{A}])$ generated by I_x that has been defined before (see Equation (6.3.2)). Again, one can show that $\Gamma(X, \mathfrak{B}_\Omega)$ is isomorphic to this C^*-algebra and establish the following result.

PROPOSITION 6.22. *Let $\pi : \mathfrak{B}_\Omega \to X$ be as above.*

(1) The gauge field ω defines a continuous section of $\mathfrak{B}_\Omega$.
(2) The gauge group $\mathfrak{G}(A, \mathcal{H}; J)$ acts fiberwise on this bundle, and the induced action on $\Gamma(X, \mathfrak{B}_\Omega)$ agrees with the action on $C^*(\mathcal{A}, [D, \mathcal{A}])$.

Consequently, if we regard $\omega \in \Omega^1_D(\mathcal{A})$ as a continuous section $\omega(x)$ of $\mathfrak{B}_\Omega$, an element $uJuJ^{-1} \in \mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ acts as

$$\omega(x) \mapsto (u\omega u^*)(x) \equiv u\omega(x)u^*.$$

Notes

Section 6.1. 'Inner' unitary equivalences as the gauge group

1. The interpretation of the inner automorphism group as the gauge group is presented in [**63**].

2. For a precise proof of the isomorphism between $\mathrm{Aut}(C(X))$ and the group of homeomorphisms of X, we refer to [**31**, Theorem II.2.2.6]. For a more detailed treatment of the smooth analogue, we refer to [**103**, Section 1.3].

3. The gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ introduced in Definition 6.4 (following [**63, 65, 186**]) is a natural lift of the group of inner automorphisms of the algebra $\mathcal{A}$, as is proved in Proposition 6.5. Another approach to lifting $\mathrm{Inn}(\mathcal{A})$ to be represented on $\mathcal{H}$ is by *central extensions*; this is described in [**136**].

Section 6.2. Morita self-equivalences as gauge fields

4. For unital algebras algebraic Morita equivalence [**153**] coincides with Rieffel's notion of strong Morita equivalence for C^*-algebras [**166**]. This is proved in [**24**] and explains why we can safely work with algebraic tensor products. We also refer for a more general treatment to *e.g.* [**103**, Section 4.5] and [**131**, Section A.3 and A.4].

5. Besides Morita equivalence, also the more general notion of KK-equivalence can be lifted to spectral triples, but this requires much more analysis [**17, 129, 151**].

6. Theorem 6.15 and Theorem 6.16 are due to Connes in [**63**].

Section 6.3. Localization

7. The notion of $C(X)$-algebra was introduced by Kasparov in [**122**]. Proposition 6.19 and Theorem 6.20 are proved in [**124, 157**] (see also Appendix C in [**198**]). Note that the bundles are in general only upper semi-continuous, and not necessarily continuous. For a discussion of this point, see [**157**].

8. Later, in Chapters 8 to 11 we will work towards physical applications in which the above C^*-bundle is a locally trivial (or, even a globally trivial) $*$-algebra bundle with finite-dimensional fiber. The above generalized gauge theories then become ordinary gauge theories, defined in terms of vector bundles and connections. It would be interesting to study the gauge theories corresponding to the intermediate cases, such as continuous trace C^*-algebras (*cf.* [**164**] for a definition), or the more general KK-fibrations that were introduced in [**84**]. First examples in this direction are studied in [**37**].

CHAPTER 7

Spectral invariants

In the previous chapter we have identified the gauge group canonically associated to any spectral triple and have derived the generalized gauge fields that carry an action of that gauge group. In this chapter we take the next step and search for gauge invariants of these gauge fields, to wit, the spectral action, the topological spectral action and the fermionic action. We derive (asymptotic) expansions of the spectral action.

7.1. Spectral action functional

The simplest *spectral invariant* associated to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by the trace of some function of D. We also allow for inner fluctuations, and more generally consider the operators $D_\omega = D + \omega + \epsilon' J\omega J^{-1}$ with $\omega = \omega^* \in \Omega^1_D(\mathcal{A})$.

Definition 7.1. *Let f be a suitable positive and even function from $\mathbb{R}$ to $\mathbb{R}$. The* spectral action *is defined by*

$$S_b[\omega] := \operatorname{Tr} f(D_\omega/\Lambda), \tag{7.1.1}$$

where Λ is a real cutoff parameter. The minimal condition on the function f is that it makes $f(D_\omega/\Lambda)$ a traceclass operator, requiring sufficiently rapid decay at $\pm\infty$.

The subscript b refers to *bosonic* since in the later physical applications ω will describe bosonic fields.

There is also a *topological spectral action*, which is defined in terms of the grading γ by

$$S_{\mathrm{top}}[\omega] = \operatorname{Tr} \gamma f(D_\omega/\Lambda). \tag{7.1.2}$$

The term 'topological' will be justified below. First, we prove gauge invariance of these functionals.

Theorem 7.2. *The spectral action and the topological spectral action are gauge invariant functionals of the gauge field $\omega \in \Omega^1_D(\mathcal{A})$, assumed to transform under $\operatorname{Ad} u = uJuJ^{-1} \in \mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ as*

$$\omega \mapsto u\omega u^* + u[D, u^*].$$

Proof. By (6.2.5) this is equivalent to $D_\omega \mapsto UD_\omega U^*$ with $U = uJuJ^{-1}$. Since the eigenvalues of $UD_\omega U^*$ coincide with those of D_ω and the (topological) spectral action is defined on the spectrum of D_ω, the result follows. $\square$

Another gauge invariant one can naturally associate to a spectral triple is of a *fermionic* nature, as opposed to the above bosonic spectral action functional. This invariant is given by combining the operator D_ω with a Grassmann vector in the Hilbert space (*cf. Appendix 9.A), as follows.*

DEFINITION 7.3. *The* fermionic action *is defined by*

$$S_f[\omega, \psi] = (J\widetilde{\psi}, D_\omega \widetilde{\psi})$$

with $\widetilde{\psi} \in \mathcal{H}^+_{\mathrm{cl}}$ *where*

$$\mathcal{H}^+_{\mathrm{cl}} = \left\{ \widetilde{\psi} : \psi \in \mathcal{H}^+ \right\}$$

is the set of Grassmann variables in $\mathcal{H}$ *in the* $+1$*-eigenspace of the grading* γ*.*

THEOREM 7.4. *The fermionic action is a gauge invariant functional of the gauge field* ω *and the fermion field* ψ*, the latter transforming under* $\mathrm{Ad}\, u \in \mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ *as*

$$\psi \mapsto uJuJ^{-1}\psi.$$

Moreover, if the KO-dimension of $(\mathcal{A}, \mathcal{H}, D; \gamma, J)$ *is* 2 *modulo* 8*, then* $(\psi, \psi') \mapsto \langle J\psi, D_\omega \psi'\rangle$ *defines a skew-symmetric form on the* $+1$*-eigenspace of* γ *in* $\mathcal{H}$*.*

PROOF. Again, $D_\omega \mapsto UD_\omega U^*$ with $U = uJuJ^{-1}$, whilst $\psi \mapsto U\psi$. The claim then follows from the observation $UJ = JU$.

Skew-symmetry follows from a small computation:

$$\langle J\psi, D\psi'\rangle = -\langle J\psi, J^2 D\psi'\rangle = -\langle JD\psi', \psi\rangle = -\langle DJ\psi', \psi\rangle = -\langle J\psi', D\psi\rangle.$$

where we used Table 4.2 for $DJ = JD$ in KO-dimension 2 modulo 8. □

The above skew-symmetry is in concordance with the Grassmann nature of fermionic fields $\widetilde{\psi}$, guaranteeing that S_f as defined above is in fact non-zero.

7.2. Expansions of the spectral action

We assume that f is given by a Laplace–Stieltjes transform:

$$f(x) = \int_{t>0} e^{-tx^2} d\mu(t),$$

with μ some measure on $\mathbb{R}^+$. Under this assumption, we can expand S_b in two ways: either asymptotically, in powers of Λ, or in powers of the gauge field ω. But first, let us find an expression for the topological spectral action.

PROPOSITION 7.5. *Suppose* f *is of the above form. Then,*

$$S_{\mathrm{top}}[\omega] = f(0)\, \mathrm{index}\, D_\omega.$$

PROOF. This follows from the McKean-Singer formula (Lemma 5.23):

$$\mathrm{index}\, D_\omega = \mathrm{Tr}\, \gamma e^{-tD_\omega^2/\Lambda^2}.$$

Since this expression is independent of Λ and t, an integration over t yields

$$\int_{t>0} d\mu(t) = f(0).$$ □

7.2.1. Asymptotic expansion. The asymptotic expansion of S can be derived from the existence of a **heat kernel expansion** of the form

$$\operatorname{Tr} e^{-tD^2} = \sum_\alpha t^\alpha c_\alpha, \tag{7.2.1}$$

as $t \to 0$. Note that this is written down here for the unperturbed operator D, but similar expressions hold for any bounded perturbation of D, such as D_ω.

LEMMA 7.6. *If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple with simple dimension spectrum (see Definition 5.9), then the heat kernel expansion* (7.2.1) *is valid as an asymptotic expansion as $t \to 0$. Moreover, for $\alpha < 0$ we have*

$$\operatorname{res}_{z=-2\alpha}\zeta_1(z) = \frac{2c_\alpha}{\Gamma(-\alpha)},$$

with $\zeta_b(z) = \operatorname{Tr} b|D|^{-z}$.

PROOF. This follows from the Mellin transform:

$$|D|^{-z} = \frac{1}{\Gamma(z/2)} \int_0^\infty e^{-tD^2} t^{z/2-1}\, dt,$$

or, after inserting the heat kernel expansion,

$$\begin{aligned}
\operatorname{Tr}|D|^{-z} &= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^\infty c_\alpha t^{\alpha+z/2-1}\, dt \\
&= \frac{1}{\Gamma(z/2)} \sum_\alpha \int_0^1 c_\alpha t^{\alpha+z/2-1}\, dt + \text{ holomorphic} \\
&= \sum_\alpha \frac{c_\alpha}{\Gamma(z/2)(\alpha+z/2)} + \text{ holomorphic.}
\end{aligned}$$

Taking residues at $z = -2\alpha$ on both sides gives the desired result. □

Using the Laplace–Stieltjes transform, we now derive an asymptotic expansion of the spectral action in terms of the heat coefficients c_α.

PROPOSITION 7.7. *Under the above conditions, the spectral action is given asymptotically (as $\Lambda \to \infty$) by*

$$\operatorname{Tr} f(D/\Lambda) = \sum_{\beta \in \mathrm{Sd}} f_\beta \Lambda^\beta \frac{2}{\Gamma(\beta/2)} c_{-\frac{1}{2}\beta} + f(0)c_0 + \mathcal{O}(\Lambda^{-1}), \tag{7.2.2}$$

where $f_\beta := \int f(v) v^{\beta-1} dv$ and Sd *is the dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$.*

PROOF. This follows directly after inserting the heat expansion in the Laplace–Stieltjes transform:

$$\operatorname{Tr} f(D/\Lambda) = \sum_\alpha \int_{t>0} t^\alpha \Lambda^\alpha c_\alpha\, d\mu(t). \tag{7.2.3}$$

The terms with $\alpha > 0$ are of order Λ^{-1}; if $\alpha < 0$, then

$$t^\alpha = \frac{1}{\Gamma(\alpha)} \int_{v>0} e^{-tv} v^{-\alpha-1}\, dv.$$

Applying this to the integral (7.2.3) gives

$$\begin{aligned}\Lambda^{-2\alpha}c_\alpha\int_{t>0}t^\alpha\ d\mu(t)&=\Lambda^{-2\alpha}c_\alpha\int_{t>0}\int_{v>0}e^{-tv}v^{-\alpha-1}\ dvd\mu(t)\\&=2\Lambda^{-2\alpha}c_\alpha\int_{t>0}\int_{v>0}e^{-tv^2}v^{-2\alpha-1}\ dvd\mu(t)\\&=2\Lambda^{-2\alpha}c_\alpha\int_{v>0}f(v)v^{-2\alpha-1}\ dv\equiv 2\Lambda^{-2\alpha}c_\alpha f_{-2\alpha},\end{aligned}$$

substituting $v\mapsto v^2$ in going to the second line. Since $c_\alpha=0$ unless $-2\alpha\in\mathrm{Sd}$, we substitute $\beta=-2\alpha$ to obtain (7.2.2). □

COROLLARY 7.8. *For the perturbed operator D_ω we have*

$$S_b[\omega]=\sum_{\beta\in\mathrm{Sd}}f_\beta\Lambda^\beta\mathrm{res}_{z=\beta}\,\mathrm{Tr}\,|D_\omega|^{-z}+f(0)\,\mathrm{Tr}\,|D_\omega|^{-z}\big|_{z=0}+\mathcal{O}(\Lambda^{-1}).$$

7.2.2. Perturbative expansion in the gauge field. Another approach to analyze S_b is given by expanding in ω, rather than in Λ. We first take a closer look at the heat operator e^{-tD^2} and its perturbations.

LEMMA 7.9. *Let ω be a bounded operator and denote $D_\omega=D+\omega$. Then*

$$e^{-t(D_\omega)^2}=e^{-tD^2}-t\int_0^1 ds\ e^{-st(D_\omega)^2}P(\omega)e^{-(1-s)tD^2},$$

with $P(\omega)=D\omega+\omega D+\omega^2$.

PROOF. Note that $e^{-tD_\omega^2}$ is the unique solution of the Cauchy problem

$$\begin{cases}(d_t+D_\omega)\,u(t)=0\\ u(0)=1,\end{cases}$$

with $d_t=d/dt$. Using the fundamental theorem of calculus, we find

$$\begin{aligned}&d_t\left[e^{-tD^2}-\int_0^t dt' e^{-(t-t')D_\omega^2}P(\omega)e^{-t'D^2}\right]\\&\qquad=-D_\omega^2\left(e^{-tD^2}-\int_0^t dt' e^{-(t-t')D_\omega^2}P(\omega)e^{-t'D^2}\right),\end{aligned}$$

showing that the bounded operator $e^{-tD^2}-\int_0^t dt'e^{-(t-t')D_\omega^2}P(\omega)e^{-t'D^2}$ also solves the above Cauchy problem. □

In what follows, we will repeatedly apply this Lemma to obtain a perturbative expansion for $e^{-t(D_\omega)^2}$ in powers of ω in terms of multiple integrals of heat operators. We introduce the following convenient notation, valid for operators $X_0,\dots X_n$:

$$\langle X_0,\dots,X_n\rangle_{t,n}:=t^n\,\mathrm{Tr}\int_{\Delta_n}X_0e^{-s_0tD^2}X_1e^{-s_1tD^2}\cdots X_ne^{-s_ntD^2}d^ns.$$

Here, the standard n-simplex Δ_n is the set of all n-tuples $(t_1,\dots,t_n)$ satisfying $0\le t_1\le\dots\le t_n\le 1$. Equivalently, Δ_n can be given as the set of $n+1$-tuples $(s_0,s_1,\dots,s_n)$ such that $s_0+\dots+s_n=1$ and $0\le s_i\le 1$ for any $i=0,\dots,n$. Indeed, we have $s_0=t_1$, $s_i=t_{i+1}-t_i$ and $s_n=1-t_n$ and, *vice versa*, $t_k=s_0+s_1+\cdots s_{k-1}$.

We recall the notion of Gâteaux derivatives.

Definition 7.10. *The* Gâteaux derivative *of a map $F : X \to Y$ (between locally convex topological vector spaces) at $x \in X$ is defined for $h \in X$ by*

$$F'(x)(h) = \lim_{u \to 0} \frac{F(x + uh) - F(x)}{u}.$$

In general, the map $F'(x)(\cdot)$ is not linear, in contrast with the Fréchet derivative. However, if X and Y are Fréchet spaces, then the Gâteaux derivatives actually defines a linear map $F'(x)(\cdot)$ for any $x \in X$. In this case, higher order derivatives are denoted as F'', F''', *et cetera*, or more conveniently as $F^{(k)}$ for the k'th order derivative. The latter will be understood as a bounded operator from $X \times \cdots \times X$ ($k+1$ copies) to Y, which is linear in the k last variables.

Theorem 7.11 (Taylor's formula with integral remainder). *For a Gâteaux $k+1$-differentiable map $F : X \to Y$ between Fréchet spaces X and Y,*

$$\begin{aligned} F(x) = F(a) + F'(a)(x-a) + \frac{1}{2!}F''(a)(x-a, x-a) + \cdots \\ + \frac{1}{n!}F^{(k)}(a)(x-a, \ldots, x-a) + R_k(x), \end{aligned}$$

for $x, a \in X$, with remainder given by

$$R_k(x) = \frac{1}{k!} \int_0^1 F^{(k+1)}(a + t(x-a))((1-t)h, \ldots, (1-t)h, h)dt.$$

In view of this Theorem, we have the following asymptotic Taylor expansion (around 0) in $\omega \in \Omega^1_D(\mathcal{A})$ for the spectral action $S_b[\omega]$:

$$S_b[\omega] = \sum_{n=0}^{\infty} \frac{1}{n!} S_b^{(n)}(0)(\omega, \ldots, \omega), \tag{7.2.4}$$

provided we make the following

Assumption 1. *For all $\alpha > 0, \beta > 0, \gamma > 0$ and $0 \leq \epsilon < 1$, there exist constants $C_{\alpha\beta\gamma\epsilon}$ such that*

$$\int_{t>0} \operatorname{Tr} t^{\alpha} |D|^{\beta} e^{-t(\epsilon D^2 - \beta)} \, |d\mu(t)| < C_{\alpha\beta\gamma\epsilon}.$$

Proposition 7.12. *If $n = 0, 1, \ldots$ and $\omega \in \Omega^1_D(\mathcal{A})$, then $S_b^{(n)}(0)(\omega, \ldots, \omega)$ exists, and*

$$\begin{aligned} S_b^{(n)}(0)(\omega, \ldots, \omega) = n! \sum_{k=0}^{n} (-1)^k \sum_{\varepsilon_1, \ldots, \varepsilon_k} \langle 1, (1-\varepsilon_1)\{D, \omega\} + \varepsilon_1 \omega^2, \ldots, \\ (1-\varepsilon_k)\{D, \omega\} + \varepsilon_k \omega^2 \rangle_{t,k} \, d\mu(t), \end{aligned}$$

where the sum is over multi-indices $(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k$ such that $\sum_{i=1}^k (1 + \varepsilon_i) = n$.

PROOF. We prove this by induction on n, the case $n = 0$ being trivial. By definition of the Gâteaux derivative and using Lemma 7.9,

$$
\begin{aligned}
S_b^{(n+1)}(0)(\omega,\dots,\omega) = n! \sum_{k=0}^{n} \sum_{\varepsilon_1,\dots,\varepsilon_k} \Bigg[&\sum_{i=1}^{k} (-1)^{k+1} \langle 1, (1-\varepsilon_1)\{D,\omega\} + \varepsilon_1\omega^2, \\
&\qquad \dots, \underset{i}{\{D,\omega\}}, \dots, (1-\varepsilon_k)\{D,\omega\} + \varepsilon_k\omega^2 \rangle_{t,k+1} \\
&+ \sum_{i=1}^{k} (-1)^{k} \langle 1, (1-\varepsilon_1)\{D,\omega\} + \varepsilon_1\omega^2, \dots, 2(1-\varepsilon_i)\omega^2, \\
&\qquad \dots, (1-\varepsilon_k)\{D,\omega\} + \varepsilon_k\omega^2 \rangle_{t,k} \Bigg] \, d\mu(t).
\end{aligned}
$$

The first sum corresponds to a multi-index $\vec{\varepsilon}\,' = (\varepsilon_1,\dots,\varepsilon_{i-1},0,\varepsilon_i,\dots,\varepsilon_k)$, the second corresponds to $\vec{\varepsilon}\,' = (\varepsilon_1,\dots,\varepsilon_i+1,\dots,\varepsilon_k)$ if $\varepsilon_i = 0$, counted with a factor of 2. In both cases, we compute that $\sum_j (1+\varepsilon_j') = n+1$. In other words, the induction step from n to $n+1$ corresponds to inserting in a sequence of 0's and 1's (of, say, length k) either a zero at any of the $k+1$ places, or replacing a 0 by a 1 (with the latter counted twice). In order to arrive at the right combinatorial coefficient $(n+1)!$, we have to show that any $\vec{\varepsilon}\,'$ satisfying $\sum_i (1+\varepsilon_i') = n+1$ appears in precisely $n+1$ ways from $\vec{\varepsilon}$ that satisfy $\sum_i (1+\varepsilon_i) = n$. If $\vec{\varepsilon}\,'$ has length k, it contains $n+1-k$ times 1 as an entry and, consequently, $2k-n-1$ a 0. This gives (with the double counting for the 1's) for the number of possible $\vec{\varepsilon}$:

$$2(n+1-k) + 2k - n - 1 = n+1,$$

as claimed. This completes the proof. □

EXAMPLE 7.13.

$$
\begin{aligned}
S_b^{(1)}(0)(\omega) &= \int \Big(-\langle 1, \{D,\omega\}\rangle_{t,1} \Big) \, d\mu(t), \\
S_b^{(2)}(0)(\omega,\omega) &= 2 \int \Big(-\langle 1, \omega^2 \rangle_{t,1} + \langle 1, \{D,\omega\}, \{D,\omega\}\rangle_{t,2} \Big) \, d\mu(t), \\
S_b^{(3)}(0)(\omega,\omega,\omega) &= 3! \int \Big(\langle 1, \omega^2, \{D,\omega\}\rangle_{t,2} + \langle 1, \{D,\omega\}, \omega^2 \rangle_{t,2} \\
&\qquad - \langle 1, \{D,\omega\}, \{D,\omega\}, \{D,\omega\}\rangle_{t,3} \Big) \, d\mu(t).
\end{aligned}
$$

7.2.2.1. *Taylor expansion of the spectral action.* We fix a complete set of eigenvectors $\{\psi_n\}_n$ of D with eigenvalues $\lambda_n \in \mathbb{R}$, respectively, forming an orthonormal basis for $\mathcal{H}$. We also write $\omega_{mn} := (\psi_m, \omega\psi_n)$ for the matrix coefficients of ω with respect to this orthonormal basis. Recall from Appendix 7.A the notion of divided difference $f[x_0, x_1, \dots, x_n]$ of a function $f : \mathbb{R} \to \mathbb{R}$.

THEOREM 7.14. *If f satisfies Assumption 1 and $\omega \in \Omega_D^1(\mathcal{A})$, then*

$$S_b^{(n)}(0)(\omega,\dots,\omega) = n! \sum_{i_1,\dots,i_n} \omega_{i_n i_1}\omega_{i_1 i_2} \cdots \omega_{i_{n-1} i_n} f[\lambda_{i_p}, \lambda_{i_1}, \dots, \lambda_{i_n}].$$

PROOF. Proposition 7.12 gives us an expression for $S_b^{(n)}$ in terms of the brackets $\langle \cdots \rangle_t$. For these we compute:

$$(-1)^k \langle 1, (1-\varepsilon_1)\{D,\omega\} + \varepsilon_1 \omega^2, \ldots, (1-\varepsilon_k)\{D,\omega\} + \varepsilon_k \omega^2 \rangle_{t,k} \, d\mu(t)$$
$$= (-1)^k \sum_{i_0 = i_k, i_1, \ldots, i_k} \int_{\Delta_k} \left(\prod_{j=1}^k \left((1-\varepsilon_j)(\lambda_{i_{j-1}} - \lambda_{i_j})\omega + \varepsilon_j \omega^2 \right)_{i_{j-1} i_j} \right) \times e^{-(s_0 t \lambda_{i_0}^2 + \cdots + s_k t \lambda_{i_k}^2)} d^k s d\mu(t)$$
$$= \sum_{i_0 = i_k, i_1, \ldots, i_k} \left(\prod_{j=1}^k \left((1-\varepsilon_j)(\lambda_{i_{j-1}} - \lambda_{i_j})\omega + \varepsilon_j \omega^2 \right)_{i_{j-1} i_j} \right) g[\lambda_{i_0}^2, \ldots, \lambda_{i_k}^2].$$

Glancing back at Proposition 7.19, we are finished if we establish a one-to-one relation between the order index sets $I = \{0 = i_0 < i_1 < \cdots < i_k = n\}$ such that $i_{j-1} - i_j \leq 2$ for all $1 \leq j \leq k$ and the multi-indices $(\varepsilon_1, \ldots, \varepsilon_k) \in \{0,1\}^k$ such that $\sum_{i=1}^k (1+\varepsilon_i) = n$. If I is such an index set, we define a multi-index

$$\varepsilon_j = \begin{cases} 0 & \text{if } \{i_j - 1, i_j\} \subset I, \\ 1 & \text{otherwise.} \end{cases}$$

Indeed, $i_j = i_{j-1} + 1 + \varepsilon_j$, so that

$$\sum_{i=1}^k (1+\varepsilon_i) = i_0 + \sum_{i=1}^k (1+\varepsilon_i) = i_k = n.$$

It is now clear that, *vice-versa*, if ε is as above, we define

$$I = \{0 = i_0 < i_1 < \cdots < i_k = n\}$$

by $i_j = i_{j-1} + 1 + \varepsilon_j$, and starting with $i_0 = 0$. □

COROLLARY 7.15. *If $n \geq 0$ and $\omega \in \Omega_D^1(\mathcal{A})$, then*

$$S_b^{(n)}(0)(\omega, \ldots, \omega) = (n-1)! \sum_{i_1, \ldots i_n} \omega_{i_1 i_2} \cdots \omega_{i_n i_1} f'[\lambda_{i_1}, \ldots, \lambda_{i_n}].$$

Consequently,

$$S_b[\omega] = \sum_{n=0}^\infty \frac{1}{n} \sum_{i_1, \ldots i_n} \omega_{i_1 i_2} \cdots \omega_{i_n i_1} f'[\lambda_{i_1}, \ldots, \lambda_{i_n}].$$

An interesting consequence is the following.

COROLLARY 7.16. *If $n \geq 0$ and $\omega \in \Omega_D^1(\mathcal{A})$ and if f' has compact support, then*

$$S_b^{(n)}(0)(\omega, \ldots, \omega) = \frac{(n-1)!}{2\pi i} \operatorname{Tr} \oint f'(z) \omega (z-D)^{-1} \cdots \omega (z-D)^{-1},$$

where the contour integral encloses the intersection of the spectrum of D with $\operatorname{supp} f'$.

PROOF. This follows directly from Cauchy's formula for divided differences (see Note 13 on Page 108):

$$g[x_0, \ldots x_n] = \frac{1}{2\pi i} \oint \frac{g(z)}{(z-x_0)\cdots(z-x_n)} dz,$$

with the contour enclosing the points x_i. □

7.A. Divided differences

We recall the definition of and some basic results on divided differences.

DEFINITION 7.17. *Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0, x_1, \dots x_n$ be distinct points in $\mathbb{R}$. The* divided difference of order n *is defined by the recursive relations*

$$f[x_0] = f(x_0),$$
$$f[x_0, x_1, \dots x_n] = \frac{f[x_1, \dots x_n] - f[x_0, x_1, \dots x_{n-1}]}{x_n - x_0}.$$

On coinciding points we extend this definition as the usual derivative:

$$f[x_0, \dots, x \dots, x \dots x_n] := \lim_{u \to 0} f[x_0, \dots, x + u \dots, x \dots x_n].$$

Finally, as a shorthand notation, for an index set $I = \{i_1, \dots, i_n\}$ we write

$$f[x_I] = f[x_{i_1}, \dots, x_{i_n}].$$

Also note the following useful representation:

PROPOSITION 7.18. *For any $x_0, \dots, x_n \in \mathbb{R}$,*

$$f[x_0, x_1, \dots, x_n] = \int_{\Delta_n} f^{(n)}\left(s_0 x_0 + s_1 x_1 + \dots + s_n x_n\right) d^n s.$$

PROOF. See Note 12 on Page 108. □

EXERCISE 7.1. *Prove Proposition 7.18 and show that it implies*

$$\sum_{i=0}^{n} f[x_0, \dots, x_i, x_i, \dots, x_n] = f'[x_0, x_1, \dots, x_n].$$

PROPOSITION 7.19. *For any $x_1, \dots x_n \in \mathbb{R}$ for $f(x) = g(x^2)$ we have,*

$$f[x_0, \cdots, x_n] = \sum_I \left(\prod_{\{i-1,i\} \subset I} (x_i + x_{i+1}) \right) g[x_I^2],$$

where the sum is over all ordered index sets $I = \{0 = i_0 < i_1 < \dots < i_k = n\}$ such that $i_j - i_{j-1} \leq 2$ for all $1 \leq j \leq k$ (i.e. there are no gaps in I of length greater than 1).

PROOF. This follows from the chain rule for divided differences (see Note 13 on Page 108): if $f = g \circ \phi$, then

$$f[x_0, \dots x_n] = \sum_{k=1}^{n} \sum_{0 = i_0 < i_1 < \dots < i_k = n} g[\phi(x_{i_0}), \dots, \phi(x_{i_k})] \prod_{j=0}^{k-1} \phi[x_{i_j}, \dots, x_{i_{j+1}}].$$

For $\phi(x) = x^2$ we have $\phi[x, y] = x + y$, $\phi[x, y, z] = 1$ and all higher divided differences are zero. Thus, if $i_{j+1} - i_j > 2$ then $\phi[x_{i_j}, \dots, x_{i_{j+1}}] = 0$. In the remaining cases one has

$$\phi[x_{i_j}, \dots, x_{i_{j+1}}] = \begin{cases} x_{i_j} + x_{i_{j+1}} & \text{if} \quad i_{j+1} - i_j = 1 \\ 1 & \text{if} \quad i_{j+1} - i_j = 2, \end{cases}$$

and in the above summation this selects precisely the index sets I. □

EXAMPLE 7.20. *For the first few terms, we have*

$$\begin{aligned} f[x_0,x_1] &= (x_0+x_1)g[x_0^2,x_1^2], \\ f[x_0,x_1,x_2] &= (x_0+x_1)(x_1+x_2)g[x_0^2,x_1^2,x_2^2] + g[x_0^2,x_2^2], \\ f[x_0,x_1,x_2,x_3] &= (x_0+x_1)(x_1+x_2)(x_2+x_3)g[x_0^2,x_1^2,x_2^2,x_3^2] \\ &\quad + (x_2+x_3)g[x_0^2,x_2^2,x_3^2] + (x_0+x_1)g[x_0^2,x_1^2,x_3^2]. \end{aligned}$$

Notes

Section 7.1. Spectral action functional

1. The spectral action principle was introduced by Chamseddine and Connes in [**49, 50**].

2. Note that we have put two restrictions on the fermions in the fermionic action S_f of Definition 7.3. The first is that we restrict ourselves to even vectors in $\mathcal{H}^+$, instead of considering all vectors in $\mathcal{H}$. The second restriction is that we do not consider the inner product $\langle J\widetilde{\psi}', D_\omega\widetilde{\psi}\rangle$ for two independent vectors ψ and ψ', but instead use the same vector ψ on both sides of the inner product. Each of these restrictions reduces the number of degrees of freedom in the fermionic action by a factor of 2, yielding a factor of 4 in total. It is precisely this approach that solves the problem of fermion doubling pointed out in [**138**] (see also the discussion in [**65**, Ch. 1, Sect. 16.3]). We shall discuss this in more detail in Chapter 9 and Chapter 11, where we calculate the fermionic action for electrodynamics and the Standard Model, respectively.

Section 7.2. Expansions of the spectral action

3. For a complete treatment of the Laplace–Stieltjes transform, see [**196**].

4. Lemma 7.6 appeared as [**65**, Lemma 1.144].

5. Corollary 7.8 is [**65**, Theorem 1.145]. An analysis of the term $\operatorname{Tr}|D_\omega|^{-z}\big|_{z=0}$ therein, including a perturbative expansion in powers of ω has been obtained in [**70**].

6. Section 7.2.2 is based on [**178**].

7. The notation $\langle X_0,\dots,X_n\rangle_{t,n}$ should not be confused with the zeta functions $\langle X_0,\dots,X_n\rangle_z$ introduced in Chapter 5. However, they are related through the formula

$$\langle X_0,\dots,X_n\rangle_{t,n} = \frac{(-1)^p}{2\pi i}\operatorname{Tr}\int e^{-t\lambda}X_0(\lambda-D^2)^{-1}X_1\cdots A^n(\lambda-D^2)^{-1}d\lambda.$$

Multiplying this expression by t^{z-1} and integrating over t eventually yields $\langle X_0,\dots,X_n\rangle_z$. For details, we refer to [**109**, Appendix A].

8. For more details on Gâteaux derivatives, we refer to [**105**]. For instance, that the Gâteaux derivative of a linear map F between Fréchet spaces is a linear map $F'(x)(\cdot)$ for any $x\in X$ is shown in [**105**, Theorem 3.2.5].

9. The expansion in Equation 7.2.4 is asymptotic in the sense that the partial sums $\sum_{n=0}^N \frac{1}{n!}S_b^{(n)}(0)(\omega,\dots,\omega)$ can be estimated to differ from $S_b[\omega]$ by $\mathcal{O}(\|\omega\|^{N+1})$. This is made precise in [**178**].

10. Theorem 7.14 was proved in [**178**]. A similar result was obtained in finite dimensions in [**106**] and in a different setting in [**171**]. Corollary 7.16 was obtained at first order for bounded operators [**101**].

11. There is a close connection between the spectral action, the Krein spectral shift function [**137, 128**], as well as the spectral flow of Atiyah and Lusztig [**9, 10, 11**]. One way to see this is from Theorem 7.11, where we can control the asymptotic expansion of the spectral action using the remainder terms R_k. In [**171**] these terms are analyzed and related to a spectral shift formula [**137, 128**] (see also the book [**200**] and the review [**29**], and references therein). In fact, under the assumption that f has compact support, the first rest term $S_b[\omega] - S_b^{(0)}(0)$ becomes

$$\operatorname{Tr} f(D+\omega) - \operatorname{Tr} f(D) = \int_{\mathbb{R}} f(x) d(\operatorname{Tr} E_{D+\omega}(x)) - \int_{\mathbb{R}} f(x) d(\operatorname{Tr} E_D(x)),$$

where $E_{D+\omega}$ and E_D are the spectral projections of $D+\omega$ and D, respectively. After a partial integration, we then obtain [**171**, Theorem 3.9]

$$(*) \qquad \operatorname{Tr} f(D+\omega) - \operatorname{Tr} f(D) = \int_{\mathbb{R}} f'(x)\xi(x)dx,$$

where

$$\xi(x) = \operatorname{Tr}(E_{D+\omega}(x) - E_D(x))$$

is the so-called *spectral shift function*. Moreover, it turns out that the higher-order rest terms are related to higher-order spectral shift functions [**126, 163**].

Let us also briefly describe the intriguing connection between the spectral shift function and the local index formula of Chapter 5. In fact, [**44**] (using a result from [**161**, Appendix B]) relates the index of PuP which appears in the odd local index formula (Theorem 5.21) to the *spectral flow* $\operatorname{sf}(\{D_t\})$ of the family $D_t = (1-t)D + tuDu^* = D + tu[D, u^*]$ for $0 \leq t \leq 1$. Roughly speaking, the spectral flow of such a family of operators is given by the net number of eigenvalues of D_t that pass through 0 in the positive direction when t runs from 0 to 1. One then has

$$\operatorname{index} PuP = \operatorname{sf}(\{D_t\}_{t\in[0,1]}).$$

The connection between spectral flow and the spectral shift function was first hinted at in [**154**] and has been worked out in [**16, 15**]. Essentially, these latter papers build on the observation that the spectral flow from $D_0 - x$ to $D_1 - x$ for any real number x is equal to the spectral shift function $\xi(x)$ defined above in terms of the spectral projections of D_0 and D_1. Note that for a path connecting D and the unitarily equivalent operator uDu^* the spectral shift function is a constant. In fact, since D and uDu^* have identical spectrum, the left-hand side of $(*)$ vanishes. Integration by parts on the right-hand side then ensures that ξ is constant (and in fact equal to the above index).

Eventually, a careful analysis of the spectral flow [**46**] (and [**47**] for the even case) allows one to prove the local index formula in the much more general setting of semi-finite spectral triples [**44, 25, 43, 45**].

Another encounter of spectral shift and spectral flow is in the computation of the index of the operator $d/dt + A(t)$ with $A(t)$ a suitable family of perturbations ($t \in \mathbb{R}$). In fact, they were the operators studied by Atiyah, Patodi and Singer in [**9, 10, 11**]. The index of $d/dt + A(t)$ can be expressed in terms of the spectral flow of $A(t)$ under the assumptions that $A(\pm\infty)$ is boundedly invertible, and that $A(t)$ has discrete spectrum for all $t \in \mathbb{R}$. We refer to [**96**] for a careful historical account, and the extension of this result to relatively trace class perturbations $A(t)$.

12. Proposition 7.18 is due to Hermite [**107**].

13. The chain rule for divided differences is proved in [**92**]. For Cauchy's formula for divided differences, we refer to [**78**, Ch. I.1].

CHAPTER 8

Almost-commutative manifolds and gauge theories

In this chapter we analyze the gauge theories corresponding (in the sense of Chapter 6) to a special class of noncommutative manifolds, to wit *almost-commutative*, or AC manifolds. We will see that this class leads to the usual gauge theories in physics. After identifying the gauge group, the gauge fields and the scalar fields, we compute the spectral action that yields the Lagrangian of physical interest.

8.1. Gauge symmetries of AC manifolds

We consider almost-commutative manifolds $M \times F$ that are the products of a Riemannian spin manifold M with a finite noncommutative space F.

As such, these are reminiscent of the original Kaluza–Klein theories where one considers the product $M \times \mathbb{S}^1$. The crucial difference is that the space F is *finite* so that no extra dimensions appear, while it can have non-trivial (noncommutative) structure.

Definition 8.1. *Let M be a Riemannian spin manifold with canonical triple $(C^\infty(M), L^2(S), D_M; J_M, \gamma_M)$, and let $(A_F, H_F, D_F; J_F, \gamma_F)$ be a finite real spectral triple. The* almost-commutative manifold $M \times F$ *is given by the real spectral triple:*

$$M\times F = (C^\infty(M, A_F), L^2(S\otimes(M\times H_F)), D_M\otimes 1+\gamma_M\otimes D_F; J_M\otimes J_F, \gamma_M\otimes\gamma_F).$$

Recall the definition of the gauge group of a real spectral triple (*cf.* Definition 6.4). In the case of AC manifolds, it is given by

$$\mathfrak{G}(M \times F) := \left\{uJuJ^{-1} : u \in C^\infty(M,\mathcal{U}(A_F))\right\},$$

with $J = J_M\otimes J_F$. Here we have identified $\mathcal{U}(C^\infty(M, A_F)) = C^\infty(M,\mathcal{U}(A_F))$. For the Lie algebra of the gauge group we have

$$\mathfrak{g}(M \times F) := \left\{X + JXJ^{-1} : X \in C^\infty(M, \mathfrak{u}(A_F))\right\}.$$

In the same way, we also obtain the groups $\mathfrak{G}(M)$ and $\mathfrak{G}(F)$. For the canonical triple on the spin manifold M, we have seen in Example 4.38 that $C^\infty(M)_{J_M} = C^\infty(M)$, which means that the group $\mathfrak{G}(M)$ is just the trivial group. For the finite space F, we obtain the *local* gauge group $\mathfrak{G}(F)$. Let us have a closer look at the structure of this local gauge group. We define two subsets of A_F by

$$\mathfrak{H}(F) := \mathcal{U}\big((A_F)_{J_F}\big), \tag{8.1.1a}$$

$$\mathfrak{h}(F) := \mathfrak{u}\big((A_F)_{J_F}\big). \tag{8.1.1b}$$

Note that the group $\mathfrak{H}(F)$ is the counterpart for the finite space F of the group $\mathcal{U}(\mathcal{A}_J)$ in Proposition 6.5, and $\mathfrak{h}(F)$ is its Lie algebra.

PROPOSITION 8.2. *Let M be simply connected. Then the gauge group $\mathfrak{G}(M \times F)$ of an almost-commutative manifold is given by $C^\infty(M, \mathfrak{G}(F))$, where $\mathfrak{G}(F) = \mathcal{U}(A_F)/\mathfrak{H}(F)$ is the gauge group of the finite space. Consequently, the gauge Lie algebra $\mathfrak{g}(M \times F)$ is given by $C^\infty(M, \mathfrak{g}(F))$, where $\mathfrak{g}(F) = \mathfrak{u}(A_F)/\mathfrak{h}(F)$.*

PROOF. This follows from Propositions 6.5 and 6.8, combined with the fact that for the algebra $\mathcal{A} = C^\infty(M, A_F)$ we have $\mathcal{U}(\mathcal{A}) \simeq C^\infty(M,\mathcal{U}(A_F))$, while $\mathcal{U}(\mathcal{A}_J) = C^\infty(M, \mathfrak{H}(F))$. The quotient of the latter two groups is isomorphic to $C^\infty(M, \mathfrak{G}(F))$ if the following homomorphism

$$C^\infty(M,\mathcal{U}(A_F)) \to C^\infty(M,\mathcal{U}(A_F)/\mathfrak{H}(F))$$

is surjective. This happens when M is simply connected, as in that case there exists a global lift from $\mathcal{U}(A_F)/\mathfrak{H}(F)$ to $\mathcal{U}(A_F)$ (see Note 4 on Page 125). □

This is in concordance with the picture derived in Section 6.3, where the gauge group acts fiberwise on a C^*-bundle. Namely, in the case of an almost-commutative manifold we have a globally trivial C^*-bundle $M \times A_F$ for which $\mathcal{A}$ are the (smooth) sections. Since $\mathfrak{G}(M \times F) \simeq C^\infty(M, \mathfrak{G}(F))$, the gauge group is given by sections of the group bundle $M \times \mathfrak{G}(F)$, which then naturally acts fiberwise on the C^*-bundle $M \times A_F$.

Combined with the *outer* automorphisms on $C^\infty(M)$, we arrive at the full symmetry group of an almost-commutative manifold $M \times F$ as a semi-direct product, where the 'internal symmetries' are given by the gauge group $\mathfrak{G}(M \times F)$. Furthermore, we also still have invariance under the group of diffeomorphisms $\mathrm{Diff}(M)$, as in Example 6.2. There exists a group homomorphism $\theta\colon \mathrm{Diff}(M) \to \mathrm{Aut}\big(\mathfrak{G}(M \times F)\big)$ given by

$$\theta(\phi)U := U \circ \phi^{-1},$$

for $\phi \in \mathrm{Diff}(M)$ and $U \in \mathfrak{G}(M \times F)$. Hence, we can describe the *full symmetry group* by the semi-direct product

$$\mathfrak{G}(M \times F) \rtimes \mathrm{Diff}(M).$$

8.1.1. Unimodularity. Suppose that A_F is a complex unital $*$-algebra, conform Definition 2.1. This algebra has a unit 1, and by complex linearity we see that $\mathbb{C}1 \subset (A_F)_{J_F}$. Restricting to unitary elements, we then find that $U(1)$ is a subgroup of $\mathfrak{H}(F)$. Because $\mathfrak{H}(F)$ is commutative, $U(1)$ is then automatically a normal subgroup of $\mathfrak{H}(F)$.

If, on the other hand, A_F is a real algebra, we can only say that $\mathbb{R}1 \subset (A_F)_{J_F}$. Restricting to unitary (*i.e.* in this case orthogonal) elements, we then only obtain the insight that $\{1, -1\}$ is a normal subgroup of $\mathfrak{H}(F)$.

PROPOSITION 8.3. *If A_F is a complex algebra, the gauge group is isomorphic to*

$$\mathfrak{G}(F) \simeq S\mathcal{U}(A_F)/S\mathfrak{H}(F),$$

where

$$
\begin{aligned}
\mathcal{SU}(A_F) &:= \{g \in \mathcal{U}(A_F) \mid \det{}_{H_F} g = 1\}, \\
S\mathfrak{H}(F) &:= \mathcal{SU}(A_F) \cap \mathfrak{H}(F).
\end{aligned}
$$

In this case the gauge algebra is

$$\mathfrak{g}(F) \simeq s\mathfrak{u}(A_F)/s\mathfrak{h}(F),$$

with

$$
\begin{aligned}
s\mathfrak{u}(A_F) &:= \{X \in \mathfrak{u}(A_F) \mid \mathrm{Tr}_{H_F} X = 0\}, \\
s\mathfrak{h}(F) &:= s\mathfrak{u}(A_F) \cap h_F.
\end{aligned}
$$

PROOF. Elements of the quotient $\mathfrak{G}(F) = \mathcal{U}(A_F)/\mathfrak{H}(F)$ are given by the equivalence classes $[u]$ for $u \in \mathcal{U}(A_F)$, subject to the equivalence relation $[u] = [uh]$ for all $h \in \mathfrak{H}(F)$. Similarly, the quotient $\mathcal{SU}(A_F)/S\mathfrak{H}(F)$ consists of classes $[v]$ for $v \in \mathcal{SU}(A_F)$, with the equivalence relation $[v] = [vg]$ for all $g \in S\mathfrak{H}(F)$. We first show that this quotient is well defined, *i.e.* that $S\mathfrak{H}(F)$ is a normal subgroup of $\mathcal{SU}(A_F)$. For this we need to check that $vgv^{-1} \in S\mathfrak{H}(F)$ for all $v \in \mathcal{SU}(A_F)$ and $g \in S\mathfrak{H}(F)$. We already know that $vgv^{-1} \in \mathfrak{H}(F)$, because $\mathfrak{H}(F)$ is a normal subgroup of $\mathcal{U}(A_F)$. We then also see that $\det_{H_F}(vgv^{-1}) = \det_{H_F} g = 1$, so $vgv^{-1} \in S\mathfrak{H}(F)$, and the quotient $\mathcal{SU}(A_F)/S\mathfrak{H}(F)$ is indeed well defined.

As to for the claimed isomorphism, consider the map $\varphi : \mathcal{U}(A_F) \to \mathcal{SU}(A_F)/S\mathfrak{H}(F)$ given by

$$\varphi(u) = [{\lambda_u}^{-1} u],$$

where $\lambda_u \in U(1)$ is an element in $U(1)$ such that ${\lambda_u}^N = \det u$, where N is the dimension of the finite-dimensional Hilbert space H_F.

Since $U(1)$ is a subgroup of $\mathcal{U}(A_F)$ (because we assume A_F to be a complex algebra), we see that indeed ${\lambda_u}^{-1} u \in \mathcal{SU}(A_F)$. Let us also check that φ does not depend on the choice of the N'th root λ_u of $\det u$ we take. Suppose λ'_u is such that ${\lambda'_u}^N = \det u$. We then must have ${\lambda_u}^{-1}\lambda'_u \in \mu_N$, where μ_N is the multiplicative group of the N'th roots of unity. Since $U(1)$ is a subgroup of $\mathfrak{H}(F)$, we see that μ_N is a subgroup of $S\mathfrak{H}(F)$, so $[{\lambda_u}^{-1} u] = [{\lambda'_u}^{-1} u]$, and hence the image of φ is indeed independent of the choice of λ_u.

Next, since $\mathcal{SU}(A_F) \subset \mathcal{U}(A_F)$, the homomorphism φ is clearly surjective. We determine its kernel:

$$\ker \varphi = \left\{u \in \mathcal{U}(A_F) : \lambda_u^{-1} u \in \mathfrak{H}(F)\right\} \simeq \{u \in \mathcal{U}(A_F) : u \in \mathfrak{H}(F)\} \equiv \mathfrak{H}(F),$$

since $\lambda_u \in \mathfrak{H}(F)$. □

The significance of Proposition 8.3 is that in the case of a complex algebra with a complex representation, equivalence classes of the quotient $\mathfrak{G}(F) = \mathcal{U}(A_F)/\mathfrak{H}(F)$ can always be represented (though not uniquely) by elements of $\mathcal{SU}(A_F)$. In that sense, all elements $g \in \mathfrak{G}(F)$ naturally satisfy the so-called **unimodularity condition**, *i.e.* they satisfy

$$\det{}_{H_F} g = 1.$$

In the case of an algebra with a real representation, this is not true and it is natural to impose the unimodularity condition for such representations by hand. We will see later in Chapter 11 how this works in the derivation of the Standard Model from noncommutative geometry.

EXAMPLE 8.4. *Define the so-called* **Yang–Mills finite spectral triple** *(cf. Example 3.14)*

$$F_{YM} = (M_N(\mathbb{C}), M_N(\mathbb{C}), D = 0; J_F = (\cdot)^*, \gamma_F = 1).$$

One easily checks that the commutative subalgebra $(A_F)_{J_F}$ is given by $\mathbb{C}\mathbb{I}_N$. The group $\mathfrak{H}(F)$ of unitary elements of this subalgebra is then equal to the group $U(1)\mathbb{I}_N$. Note that in this case $\mathfrak{H}(F)$ is equal to the subgroup $\mathcal{U}(Z(A_F))$ of $U(N)$ that commutes with the algebra $M_N(\mathbb{C})$. We thus obtain that the gauge group is given by the quotient $\mathfrak{G}(F_{YM}) = U(N)/U(1) =: PU(N)$, which by Example 6.3 is equal to the group of inner automorphisms of $M_N(\mathbb{C})$. As in Proposition 8.3, this group can also be written as $SU(N)/\mu_N$, where the multiplicative group μ_N of N'th roots of unity is the center of $SU(N)$. The Lie algebra $\mathfrak{g}(F_{YM})$ consists of the traceless anti-hermitian matrices, i.e. it is $su(N)$.

The almost-commutative manifold $M \times F_{YM}$ will be referred to as the **Yang–Mills manifold**. *By Proposition 8.2, in the simply connected case the global gauge group $\mathfrak{G}(M \times F_{YM})$ is given by maps $C^\infty(M, PU(N))$, or, equivalently, by the space of smooth sections of the trivial group bundle $M \times PU(N)$.*

EXERCISE 8.1. *In the context of the above example, check that indeed:*

(1) the commutative subalgebra $M_N(\mathbb{C})_{J_F} \simeq \mathbb{C}\mathbb{I}_N$,
(2) $S\mathfrak{H}(F) = \mu_N$, the multiplicative group of N'th roots of unity.

Explain the difference with the case of $M_N(\mathbb{R})$.

8.2. Gauge fields and scalar fields

Let us apply the discussion in Section 6.2 on Morita self-equivalences to the almost-commutative manifold $M \times F$ and see what the corresponding gauge fields look like. For convenience, we restrict ourselves to simply connected manifolds M of dimension $\dim M = 4$ and F of even KO-dimension so that $\epsilon'_F = 1$ in Table 3.1; this is sufficient for the physical applications later on.

Thus, we determine $\Omega^1_D(\mathcal{A})$ for almost-commutative manifolds, much as in Exercise 4.7. The Dirac operator $D = D_M \otimes 1 + \gamma_M \otimes D_F$ consists of two terms, and hence we can also split the inner fluctuation $\omega = a[D, b]$ into two terms. The first term is given by

$$a[D_M \otimes 1, b] = -i\gamma^\mu \otimes a\partial_\mu b =: \gamma^\mu \otimes A_\mu, \tag{8.2.1}$$

where $A_\mu := -ia\partial_\mu b \in i\mathcal{A}$ must be hermitian.[1] The second term yields

$$a[\gamma_M \otimes D_F, b] = \gamma_M \otimes a[D_F, b] =: \gamma_M \otimes \phi, \tag{8.2.2}$$

[1]Note that $i\mathcal{A} = \mathcal{A}$ for complex algebras only.

for hermitian $\phi := a[D_F, b]$. Thus, the inner fluctuations of an even almost-commutative manifold $M \times F$ take the form

$$\omega = \gamma^\mu \otimes A_\mu + \gamma_M \otimes \phi, \tag{8.2.3}$$

for certain hermitian operators $A_\mu \in i\mathcal{A}$ and $\phi \in \Gamma\big(\operatorname{End}(V)\big)$, where V is the trivial vector bundle $V = M \times H_F$.

The 'fluctuated' Dirac operator is given by $D_\omega = D + \omega + \epsilon' J \omega J^{-1}$ (*cf.* Section 6.2.2 above), for which we calculate

$$\gamma^\mu \otimes A_\mu + \epsilon' J \gamma^\mu \otimes A_\mu J^{-1} = \gamma^\mu \otimes \big(A_\mu - J_F A_\mu J_F^{-1}\big) =: \gamma^\mu \otimes B_\mu, \tag{8.2.4}$$

which defines $B_\mu \in \Gamma\big(\operatorname{End}(V)\big)$, and where we have used that $J_M \gamma^\mu J_M^{-1} = -\gamma^\mu$ in dimension 4. Note that if ∇^E denotes the twisted connection on the tensor product bundle $E := S \otimes V$, *i.e.*

$$\nabla^E_\mu = \nabla^S_\mu \otimes 1 + i 1 \otimes B_\mu,$$

we see that we can rewrite

$$D_M \otimes 1 + \gamma^\mu \otimes B_\mu = -i\gamma^\mu \nabla^E_\mu.$$

For the remainder of the fluctuated Dirac operator, we define $\Phi \in \Gamma\big(\operatorname{End}(E)\big)$ by

$$\Phi := D_F + \phi + J_F \phi J_F^{-1}. \tag{8.2.5}$$

The fluctuated Dirac operator of a real even AC-manifold then takes the form

$$D_\omega = D_M \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi = -i\gamma^\mu \nabla^E_\mu + \gamma_M \otimes \Phi. \tag{8.2.6}$$

In Section 8.1 we obtained the local gauge group $\mathfrak{G}(F)$ with Lie algebra $\mathfrak{g}(F)$. For consistency we should now check that the gauge field A_μ arising from the inner fluctuation indeed corresponds to this same gauge group.

The requirement that A_μ is hermitian is equivalent to $(iA_\mu)^* = -iA_\mu$. Since A_μ is of the form $-ia\partial_\mu b$ for $a, b \in \mathcal{A}$ (see (8.2.1)), we see that iA_μ is an element of the algebra $\mathcal{A}$ (also if $\mathcal{A}$ is only a real algebra). Thus we have $A_\mu(x) \in i\,\mathfrak{u}(A_F)$.

The only way in which A_μ appears in D_ω is through the action of $A_\mu - J_F A_\mu J_F^{-1}$. If we take $A'_\mu = A_\mu - a_\mu$ for some $a_\mu \in i\mathfrak{h}(F) = i\,\mathfrak{u}\big((A_F)_{J_F}\big)$ (which commutes with J_F), we see that $A'_\mu - J_F A'_\mu J_F^{-1} = A_\mu - J_F A_\mu J_F^{-1}$. Therefore we may without any loss of generality assume that $A_\mu(x)$ is an element of the quotient $i\mathfrak{g}(F) = i(\mathfrak{u}(A_F)/\mathfrak{h}(F)))$. Since $\mathfrak{g}(F)$ is the Lie algebra of the gauge group $\mathfrak{G}(F)$, we have therefore confirmed that

$$A_\mu \in C^\infty(M, i\mathfrak{g}(F)) \tag{8.2.7}$$

is indeed a gauge field for the local gauge group $\mathfrak{G}(F)$. For the field B_μ found in (8.2.6), we can also write

$$B_\mu = \operatorname{ad}(A_\mu) := A_\mu - J_F A_\mu J_F^{-1}.$$

So, we conclude that B_μ is given by the adjoint action of a gauge field A_μ for the gauge group $\mathfrak{G}(F)$ with Lie algebra $\mathfrak{g}(F)$.

If the finite noncommutative space F has a grading γ_F, the field ϕ satisfies $\phi\gamma_F = -\gamma_F \phi$ and the field Φ satisfies $\Phi\gamma_F = -\gamma_F \Phi$ and $\Phi J_F = J_F \Phi$.

These relations follow directly from the definitions of ϕ and Φ and the commutation relations for D_F according to Definition 3.1.

Using the cyclic property of the trace, it is easy to see that the traces of the fields B_μ, ϕ and Φ over the finite-dimensional Hilbert space H_F vanish identically: for B_μ we find

$$\mathrm{Tr}_{H_F}\left(B_\mu\right) = \mathrm{Tr}_{H_F}\left(A_\mu - J_F A_\mu J_F^{-1}\right) = \mathrm{Tr}_{H_F}\left(A_\mu - A_\mu J_F^{-1} J_F\right) = 0,$$

whereas for the field ϕ we find

$$\mathrm{Tr}_{H_F}\left(\phi\right) = \mathrm{Tr}_{H_F}\left(a[D_F, b]\right) = \mathrm{Tr}_{H_F}\left([b,a]D_F\right).$$

Since the grading commutes with the elements in the algebra and anticommutes with the Dirac operator, it follows that this latter trace also vanishes. It then automatically follows that $\Phi = D_F + \phi + J_F \phi J_F^{-1}$ is traceless too.

EXAMPLE 8.5. *For the Yang–Mills manifold $M \times F_{\mathrm{YM}}$ of Example 8.4 the inner fluctuations take the form $\omega = \gamma^\mu \otimes A_\mu$ for some traceless hermitian field $A_\mu = A_\mu^* \in C^\infty(M, isu(N))$. Since $J_F A_\mu J_F^{-1} m = m A_\mu$ for $m \in M_N(\mathbb{C})$, we see that for the field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ we obtain the action*

$$m \mapsto B_\mu m = A_\mu m - m A_\mu = [A_\mu, m] = (\mathrm{ad}\, A_\mu) m.$$

Thus A_μ is a $PU(N)$ gauge field which acts on the fermions in $L^2(S) \otimes M_N(\mathbb{C})$ in the adjoint representation.

8.2.1. Gauge transformations. Recall from Section 6.2 that an element $U \in \mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ acts on the inner fluctuations as a gauge transformation. In fact, the rule $D_\omega \mapsto U D_\omega U^*$ with $U = uJuJ^{-1}$ can be implemented by

$$u : \omega \mapsto \omega^u := u\omega u^* + u[D, u^*], \tag{8.2.8}$$

so that $U D_\omega U^* = D_{\omega^u}$. In physics, the resulting transformation on the inner fluctuation $\omega \mapsto \omega^u$ will be interpreted as a gauge transformation of the gauge field.

Note that for an element $U = uJuJ^{-1}$ in the gauge group $\mathfrak{G}(M \times F)$, there is an ambiguity in the corresponding transformation of ω. Namely, for $u \in \mathcal{U}(\mathcal{A})$ and $h \in \mathcal{U}(\mathcal{A}_J)$, we can also write $U = uhJuhJ^{-1}$. Replacing u with uh using (4.3.1) we then obtain

$$\omega^{uh} = u\omega u^* + u[D, u^*] + h[D, h^*].$$

However, when considering the total inner fluctuation $\omega^{uh} + J\omega^{uh}J^{-1}$, the extra term $h[D, h^*]$ cancels out:

$$h[D, h^*] + Jh[D, h^*]J^{-1} = h[D, h^*] + [D, h]h^* = [D, hh^*] = 0.$$

Hence the transformation of $D_\omega = D + \omega + J\omega J^{-1}$ is well defined.

For an AC-manifold $M \times F$, by (8.2.3) we have $\omega = \gamma^\mu \otimes A_\mu + \gamma_M \otimes \phi$ and $D = -i\gamma^\mu \nabla_\mu^S \otimes 1 + \gamma_M \otimes D_F$, and, using $[\nabla_\mu^S, u^*] = \partial_\mu u^*$, we thus obtain

$$\begin{aligned} A_\mu &\to u A_\mu u^* - iu\partial_\mu u^*, \\ \phi &\to u\phi u^* + u[D_F, u^*]. \end{aligned} \tag{8.2.9}$$

The first equation is precisely the gauge transformation for a gauge field $A_\mu \in C^\infty(M, i\mathfrak{g}(F))$, as desired. However, the transformation property of the field ϕ is a bit surprising. In the Standard Model, the Higgs field is in the defining representation of the gauge group. The transformation for ϕ derived above, on the other hand, is in the adjoint representation. From the framework of noncommutative geometry this is no surprise, since both bosonic fields A_μ and ϕ are obtained from the inner fluctuations of the Dirac operator, and are thereby expected to transform in a similar manner. Fortunately, for particular choices of the finite space F, the adjoint transformation property of ϕ reduces to that of the defining representation. The key example of this will be discussed in Chapter 11, where we present the derivation of the Standard Model from an almost-commutative manifold.

8.3. The heat expansion of the spectral action

In the remainder of this chapter we shall derive an explicit formula for the bosonic Lagrangian of an almost-commutative manifold $M \times F$ from the spectral action of Definition 7.1. We start by calculating a generalized Lichnerowicz formula for the square of the fluctuated Dirac operator. Subsequently, we show how we can use this formula to obtain an asymptotic expansion of the spectral action in the form of (7.2.1). We explicitly calculate the coefficients in this **heat kernel expansion**, allowing for a derivation of the general form of the Lagrangian for an almost-commutative manifold.

8.3.1. A generalized Lichnerowicz formula. Suppose we have a vector bundle $E \to M$. We say that a second-order differential operator H is a *generalized Laplacian* if it is of the form $H = \Delta^E - F$, where Δ^E is a Laplacian in the sense of Definition 4.16 and $F \in \Gamma(\mathrm{End}(E))$.

Our first task is to show that the fluctuated Dirac operator D_ω on an almost-commutative manifold squares to a generalized Laplacian, $D_\omega^2 = \Delta^E - F$, and then determine F. Before we prove this, let us first have a closer look at some explicit formulas for the fluctuated Dirac operator. Recall from (8.2.6) that we can write

$$D_\omega = -i\gamma^\mu \nabla^E_\mu + \gamma_M \otimes \Phi$$

for the connection $\nabla^E_\mu = \nabla^S_\mu \otimes 1 + 1 \otimes (\partial_\mu + iB_\mu)$ on $E = S \otimes V$, and the scalar field $\Phi \in \Gamma(\mathrm{End}(E))$. Let us evaluate the relations between the connection, its curvature and their adjoint actions. We define the operator D_μ as the adjoint action of the connection ∇^E_μ, i.e. $D_\mu = \mathrm{ad}\left(\nabla^E_\mu\right)$. In other words, we have

$$D_\mu \Phi = [\nabla^E_\mu, \Phi] = \partial_\mu \Phi + i[B_\mu, \Phi]. \tag{8.3.1}$$

We define the curvature $F_{\mu\nu}$ of the gauge field B_μ as usual by

$$F_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]. \tag{8.3.2}$$

Recall the curvature of the connection ∇^E from (4.2.3). Since in local coordinates we have $[\partial_\mu, \partial_\nu] = 0$, we find

$$\begin{aligned}
\Omega^E_{\mu\nu} &= \nabla^E_\mu \nabla^E_\nu - \nabla^E_\nu \nabla^E_\mu \\
&= (\nabla^S_\mu \otimes 1 + i1 \otimes B_\mu)(\nabla^S_\nu \otimes 1 + i1 \otimes B_\nu) \\
&\quad - (\nabla^S_\nu \otimes 1 + i1 \otimes B_\nu)(\nabla^S_\mu \otimes 1 + i1 \otimes B_\mu) \\
&= \Omega^S_{\mu\nu} \otimes 1 + i1 \otimes \partial_\mu B_\nu - i1 \otimes \partial_\nu B_\mu - 1 \otimes [B_\mu, B_\nu].
\end{aligned}$$

Inserting (8.3.2), we obtain the formula

$$\Omega^E_{\mu\nu} = \left[\nabla^E_\mu, \nabla^E_\nu\right] = \Omega^S_{\mu\nu} \otimes 1 + i1 \otimes F_{\mu\nu}. \tag{8.3.3}$$

Next, let us have a look at the commutator $\left[D_\mu, D_\nu\right]$. Using the definition of D_μ and the Jacobi identity, we obtain

$$\begin{aligned}
[D_\mu, D_\nu]\Phi &= \mathrm{ad}\left(\nabla^E_\mu\right) \mathrm{ad}\left(\nabla^E_\nu\right)\Phi - \mathrm{ad}\left(\nabla^E_\nu\right) \mathrm{ad}\left(\nabla^E_\mu\right)\Phi \\
&= \left[\nabla^E_\mu, [\nabla^E_\nu, \Phi]\right] - \left[\nabla^E_\nu, [\nabla^E_\mu, \Phi]\right] \\
&= \left[[\nabla^E_\mu, \nabla^E_\nu], \Phi]\right] = \left[\Omega^E_{\mu\nu}, \Phi\right] = \mathrm{ad}\left(\Omega^E_{\mu\nu}\right)\Phi.
\end{aligned}$$

Since $\Omega^S_{\mu\nu}$ commutes with Φ, we obtain the relation

$$\left[D_\mu, D_\nu\right] = i\,\mathrm{ad}\left(F_{\mu\nu}\right).$$

Note that this relation simply reflects the fact that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a Lie algebra homomorphism.

In local coordinates, the Laplacian is given by

$$\Delta^E = -g^{\mu\nu}\left(\nabla^E_\mu \nabla^E_\nu - \Gamma^\rho{}_{\mu\nu}\nabla^E_\rho\right).$$

We can then calculate the explicit formula

$$\begin{aligned}
\Delta^E &= -g^{\mu\nu}\left(\nabla^E_\mu \nabla^E_\nu - \Gamma^\rho{}_{\mu\nu}\nabla^E_\rho\right) \\
&= \Delta^S \otimes 1 - g^{\mu\nu}\Big(i(\nabla^S_\mu \otimes 1)(1 \otimes B_\nu) + i(1 \otimes B_\mu)(\nabla^S_\nu \otimes 1) \\
&\qquad\qquad - 1 \otimes B_\mu B_\nu - i\Gamma^\rho{}_{\mu\nu} \otimes B_\rho\Big) \\
&= \Delta^S \otimes 1 - 2i(1 \otimes B^\mu)(\nabla^S_\mu \otimes 1) - ig^{\mu\nu}(1 \otimes \partial_\mu B_\nu) \\
&\qquad + 1 \otimes B_\mu B^\mu + ig^{\mu\nu}\Gamma^\rho{}_{\mu\nu} \otimes B_\rho.
\end{aligned} \tag{8.3.4}$$

We are now ready to prove that the fluctuated Dirac operator D_ω of an almost-commutative manifold satisfies the following *generalized Lichnerowicz formula* or *Weitzenböck formula*. First, for the canonical Dirac operator D_M on a compact Riemannian spin manifold M, recall the Lichnerowicz formula of Theorem 4.21:

$$D_M^2 = \Delta^S + \frac{1}{4}s, \tag{8.3.5}$$

where Δ^S is the Laplacian of the spin connection ∇^S, and s is the scalar curvature of M.

PROPOSITION 8.6. *The square of the fluctuated Dirac operator on an almost-commutative manifold is a generalized Laplacian of the form*

$${D_\omega}^2 = \Delta^E - F,$$

where the endomorphism F is given by

$$F = -\frac{1}{4}s \otimes 1 - 1 \otimes \Phi^2 + \frac{1}{2} i\gamma^\mu\gamma^\nu \otimes F_{\mu\nu} - i\gamma_M\gamma^\mu \otimes D_\mu\Phi, \tag{8.3.6}$$

in which D_μ and $F_{\mu\nu}$ are defined in (8.3.1) *and* (8.3.2), *respectively.*

PROOF. Rewriting the formula for D_ω, we have

$$\begin{aligned}
{D_\omega}^2 &= (D_M \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi)^2 \\
&= D_M^2 \otimes 1 + \gamma^\mu\gamma^\nu \otimes B_\mu B_\nu + 1 \otimes \Phi^2 + (D_M\gamma^\mu \otimes 1)(1 \otimes B_\mu) \\
&\quad + (1 \otimes B_\mu)(\gamma^\mu D_M \otimes 1) + (D_M \otimes 1)(\gamma_M \otimes \Phi) + (\gamma_M \otimes \Phi)(D_M \otimes 1) \\
&\quad + (\gamma^\mu \otimes B_\mu)(\gamma_M \otimes \Phi) + (\gamma_M \otimes \Phi)(\gamma^\mu \otimes B_\mu).
\end{aligned}$$

For the first term we use the Lichnerowicz formula of (8.3.5). We rewrite the second term into

$$\begin{aligned}
\gamma^\mu\gamma^\nu \otimes B_\mu B_\nu &= \frac{1}{2}\gamma^\mu\gamma^\nu \otimes (B_\mu B_\nu + B_\nu B_\mu + [B_\mu, B_\nu]) \\
&= 1 \otimes B_\mu B^\mu + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu],
\end{aligned}$$

where we have used the Clifford relation (4.2.2) to obtain the second equality. For the fourth and fifth terms we use the local formula $D_M = -i\gamma^\nu\nabla_\nu^S$ to obtain

$$\begin{aligned}
&(D_M\gamma^\mu \otimes 1)(1 \otimes B_\mu) + (1 \otimes B_\mu)(\gamma^\mu D_M \otimes 1) \\
&\qquad = -(i\gamma^\nu\nabla_\nu^S\gamma^\mu \otimes 1)(1 \otimes B_\mu) - (1 \otimes B_\mu)(\gamma^\mu i\gamma^\nu\nabla_\nu^S \otimes 1).
\end{aligned}$$

Using the identity $[\nabla_\nu^S, c(\alpha)] = c(\nabla_\nu\alpha)$ for the spin connection, we find $[\nabla_\nu^S \otimes 1, (\gamma^\mu \otimes 1)(1 \otimes B_\mu)] = c\big(\nabla_\nu(dx^\mu \otimes B_\mu)\big)$. We thus obtain

$$\begin{aligned}
&(D_M\gamma^\mu \otimes 1)(1 \otimes B_\mu) + (1 \otimes B_\mu)(\gamma^\mu D_M \otimes 1) \\
&\quad = -i(\gamma^\nu \otimes 1)c\big(\nabla_\nu(dx^\mu \otimes B_\mu)\big) \\
&\qquad\qquad - i(\gamma^\nu\gamma^\mu \otimes 1)(1 \otimes B_\mu)(\nabla_\nu^S \otimes 1) - i(1 \otimes B_\mu)(\gamma^\mu\gamma^\nu\nabla_\nu^S \otimes 1) \\
&\quad = -i(\gamma^\nu \otimes 1)c\big(dx^\mu \otimes (\partial_\nu B_\mu) - {\Gamma^\rho}_{\mu\nu} dx^\mu \otimes B_\rho\big) - 2i(1 \otimes B^\nu)(\nabla_\nu^S \otimes 1) \\
&\quad = -i(\gamma^\nu\gamma^\mu \otimes 1)\Big(1 \otimes \partial_\nu B_\mu - {\Gamma^\rho}_{\mu\nu} \otimes B_\rho\Big) - 2i(1 \otimes B^\nu)(\nabla_\nu^S \otimes 1) \\
&\quad = -i(\gamma^\nu\gamma^\mu \otimes 1)(1 \otimes \partial_\nu B_\mu) + ig^{\mu\nu}{\Gamma^\rho}_{\mu\nu} \otimes B_\rho - 2i(1 \otimes B^\nu)(\nabla_\nu^S \otimes 1).
\end{aligned}$$

The sixth and seventh terms are rewritten into

$$\begin{aligned}
&(D_M \otimes 1)(\gamma_M \otimes \Phi) + (\gamma_M \otimes \Phi)(D_M \otimes 1) = -(\gamma_M \otimes 1)\big[D_M \otimes 1, 1 \otimes \Phi\big] \\
&\quad = (\gamma_M \otimes 1)(i\gamma^\mu \otimes \partial_\mu\Phi) = i\gamma_M\gamma^\mu \otimes \partial_\mu\Phi.
\end{aligned}$$

The eighth and ninth terms are rewritten as

$$(\gamma^\mu \otimes B_\mu)(\gamma_M \otimes \Phi) + (\gamma_M \otimes \Phi)(\gamma^\mu \otimes B_\mu) = -\gamma_M\gamma^\mu \otimes [B_\mu, \Phi].$$

Summing all these terms then yields the formula

$$\begin{aligned}
{D_\omega}^2 &= (\Delta^S + \frac{1}{4}s) \otimes 1 + (1 \otimes B_\mu B^\mu) + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu] \\
&\quad + 1 \otimes \Phi^2 - i(\gamma^\nu\gamma^\mu \otimes 1)(1 \otimes \partial_\nu B_\mu) + ig^{\mu\nu}{\Gamma^\rho}_{\mu\nu} \otimes B_\rho \\
&\quad - 2i(1 \otimes B^\nu)(\nabla_\nu^S \otimes 1) + i\gamma_M\gamma^\mu \otimes \partial_\mu\Phi - \gamma_M\gamma^\mu \otimes [B_\mu, \Phi].
\end{aligned}$$

Inserting the formula for Δ^E from (8.3.4), we obtain

$$\begin{aligned}{D_\omega}^2 &= \Delta^E + \frac{1}{4}s\otimes 1 + \frac{1}{2}\gamma^\mu\gamma^\nu \otimes [B_\mu, B_\nu] \\ &\quad + 1\otimes\Phi^2 - i(\gamma^\nu\gamma^\mu\otimes 1)(1\otimes\partial_\nu B_\mu) + ig^{\mu\nu}(1\otimes\partial_\mu B_\nu) \\ &\quad + i\gamma_M\gamma^\mu\otimes\partial_\mu\Phi - \gamma_M\gamma^\mu\otimes[B_\mu,\Phi].\end{aligned}$$

Using (8.3.2), we rewrite

$$\begin{aligned}&-i(\gamma^\nu\gamma^\mu\otimes 1)(1\otimes\partial_\nu B_\mu) + ig^{\mu\nu}(1\otimes\partial_\mu B_\nu) \\ &\quad = -i(\gamma^\nu\gamma^\mu\otimes 1)(1\otimes\partial_\nu B_\mu) + \frac{1}{2}i(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\otimes(\partial_\mu B_\nu) \\ &\quad = -\frac{1}{2}i\gamma^\mu\gamma^\nu\otimes(\partial_\mu B_\nu) + \frac{1}{2}i\gamma^\nu\gamma^\mu\otimes(\partial_\mu B_\nu) \\ &\quad = -\frac{1}{2}i\gamma^\mu\gamma^\nu\otimes F_{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\otimes[B_\mu,B_\nu].\end{aligned}$$

Using (8.3.1), we finally obtain

$${D_\omega}^2 = \Delta^E + \frac{1}{4}s\otimes 1 + 1\otimes\Phi^2 - \frac{1}{2}i\gamma^\mu\gamma^\nu\otimes F_{\mu\nu} + i\gamma_M\gamma^\mu\otimes D_\mu\Phi,$$

from which we can read off formula (8.3.6) for F. □

8.3.2. The heat expansion. Below, we present two important theorems (without proof) which we will need to calculate the spectral action of almost-commutative manifolds. The first of these theorems states that there exists a heat expansion for a generalized Laplacian. The second theorem gives explicit formulas for the first three non-zero coefficients of this expansion. Next, we will show how these theorems can be applied to obtain a perturbative expansion of the spectral action for an almost-commutative manifold, just as in Proposition 7.7.

THEOREM 8.7. *For a generalized Laplacian $H = \Delta^E - F$ on E we have the following asymptotic expansion as $t \to 0$, known as the* heat expansion:

$$\mathrm{Tr}\left(e^{-tH}\right) \sim \sum_{k\geq 0} t^{\frac{k-n}{2}} a_k(H), \tag{8.3.7}$$

where n is the dimension of the manifold, the trace is taken over the Hilbert space $L^2(E)$ and the coefficients of the expansion are given by

$$a_k(H) := \int_M a_k(x,H)\sqrt{g}d^4x, \tag{8.3.8}$$

where $\sqrt{g}d^4x$ denotes the Riemannian volume form. The coefficients $a_k(x,H)$ are called the Seeley-DeWitt coefficients.

PROOF. See Note 6 on Page 126. □

THEOREM 8.8. *For a generalized Laplacian $H = \Delta^E - F$ (as in Theorem 8.7), the Seeley-DeWitt coefficients are given by*

$$\begin{aligned}
a_0(x,H) &= (4\pi)^{-\frac{n}{2}} \operatorname{Tr}(\mathrm{id}),\\
a_2(x,H) &= (4\pi)^{-\frac{n}{2}} \operatorname{Tr}\left(\frac{s}{6} + F\right),\\
a_4(x,H) &= (4\pi)^{-\frac{n}{2}} \frac{1}{360} \operatorname{Tr}\big(-12\Delta s + 5s^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\
&\qquad + 60sF + 180F^2 - 60\Delta F + 30\Omega^E_{\mu\nu}(\Omega^E)^{\mu\nu}\big),
\end{aligned}$$

where this time the traces are taken over the fibre E_x. Here s is the scalar curvature of the Levi-Civita connection ∇, Δ is the scalar Laplacian, and Ω^E is the curvature of the connection ∇^E corresponding to Δ^E. All $a_k(x,H)$ with odd k vanish.

PROOF. See Note 6 on Page 126. □

We saw in Proposition 8.6 that the square of the fluctuated Dirac operator of an almost-commutative manifold is a generalized Laplacian. Applying Theorem 8.7 to ${D_\omega}^2$ in dimension $n = 4$ then yields the heat expansion:

$$\operatorname{Tr}\left(e^{-t{D_\omega}^2}\right) \sim \sum_{k\geq 0} t^{\frac{k-4}{2}} a_k({D_\omega}^2), \tag{8.3.9}$$

where the Seeley-DeWitt coefficients are given by Theorem 8.8. In the following proposition, we use this heat expansion for ${D_\omega}^2$ to obtain an expansion of the spectral action.

PROPOSITION 8.9. *For an almost-commutative manifold $M \times F$ with M of dimension 4, the spectral action given by* (7.1.1) *can be expanded asymptotically (as $\Lambda \to \infty$) as*

$$\operatorname{Tr}\left(f\left(\frac{D_\omega}{\Lambda}\right)\right) \sim a_4({D_\omega}^2)f(0) + 2\sum_{\substack{0\leq k<4\\ k \text{ even}}} f_{4-k}\Lambda^{4-k} a_k({D_\omega}^2)\,\frac{1}{\Gamma\left(\frac{4-k}{2}\right)} + \mathcal{O}(\Lambda^{-1}),$$

where $f_j = \int_0^\infty f(v)v^{j-1}dv$ are the moments of the function f, $j > 0$.

PROOF. Our proof is based on Proposition 7.7. Let g be the function $g(u^2) = f(u)$, so that its Laplace–Stieltjes transform

$$g(v) = \int_0^\infty e^{-sv} d\mu(s).$$

We can then formally write

$$g(t{D_\omega}^2) = \int_0^\infty e^{-st{D_\omega}^2} d\mu(s).$$

We now take the trace and use the heat expansion of ${D_\omega}^2$ to obtain

$$\begin{aligned}
\operatorname{Tr}\left(g(t{D_\omega}^2)\right) &= \int_0^\infty \operatorname{Tr}\left(e^{-st{D_\omega}^2}\right)d\mu(s) \sim \int_0^\infty \sum_{k\geq 0} (st)^{\frac{k-4}{2}} a_k({D_\omega}^2) d\mu(s)\\
&= \sum_{k\geq 0} t^{\frac{k-4}{2}} a_k({D_\omega}^2) \int_0^\infty s^{\frac{k-4}{2}} d\mu(s). \qquad (8.3.10)
\end{aligned}$$

The parameter t is considered to be a formal expansion parameter. From here on, we will drop the terms with $k > 4$. The term with $k = 4$ equals

$$a_4({D_\omega}^2) \int_0^\infty s^0 d\mu(s) = a_4({D_\omega}^2) g(0).$$

We can rewrite the terms with $k < 4$ using the definition of the Γ-function as the analytic continuation of

$$\Gamma(z) = \int_0^\infty r^{z-1} e^{-r} dr, \tag{8.3.11}$$

for $z \in \mathbb{C}$ with $\Re(z) > 0$, and by inserting $r = sv$, we see that (for $k < 4$) we have

$$\Gamma\Big(\frac{4-k}{2}\Big) = \int_0^\infty (sv)^{\frac{4-k}{2}-1} e^{-sv} d(sv) = s^{\frac{4-k}{2}} \int_0^\infty v^{\frac{4-k}{2}-1} e^{-sv} dv.$$

From this, we obtain an expression for $s^{\frac{k-4}{2}}$, which we insert into equation (8.3.10), and then we perform the integration over s to obtain

$$\begin{aligned} \operatorname{Tr}\big(g(t{D_\omega}^2)\big) \sim\ & a_4({D_\omega}^2) f(0) \\ & + \sum_{0\le k<4} t^{\frac{k-4}{2}} a_k({D_\omega}^2) \frac{1}{\Gamma\big(\frac{4-k}{2}\big)} \int_0^\infty v^{\frac{4-k}{2}-1} g(v) dv + \mathcal{O}(\Lambda^{-1}). \end{aligned}$$

Now we choose the function g such that $g(u^2) = f(u)$. We rewrite the integration over v by substituting $v = u^2$ and obtain

$$\int_0^\infty v^{\frac{4-k}{2}-1} g(v) dv = \int_0^\infty u^{4-k-2} g(u^2) d(u^2) = 2\int_0^\infty u^{4-k-1} f(u) du,$$

which by definition equals $2f_{4-k}$. Upon writing $t = \Lambda^{-2}$, we have modulo Λ^{-1},

$$\begin{aligned} \operatorname{Tr}\left(f\Big(\frac{D_\omega}{\Lambda}\Big)\right) &= \operatorname{Tr}\big(g(\Lambda^{-2}{D_\omega}^2)\big) \\ &\sim a_4({D_\omega}^2) f(0) + 2\sum_{0\le k<4} f_{4-k}\Lambda^{4-k} a_k({D_\omega}^2) \frac{1}{\Gamma\big(\frac{4-k}{2}\big)} + \mathcal{O}(\Lambda^{-1}). \end{aligned}$$

Using $a_k({D_\omega}^2) = 0$ for odd k, the claim follows. □

8.4. The spectral action on AC manifolds

In the previous section we obtained a perturbative expansion of the spectral action for an almost-commutative manifold. We now explicitly calculate the coefficients in this expansion, first for the canonical triple (yielding the (Euclidean) Einstein–Hilbert action of General Relativity) for a four-dimensional Riemannian spin manifold M and then for a general almost-commutative manifold $M \times F$.

By Proposition 8.9 we have an asymptotic expansion as $\Lambda \to \infty$:

$$\operatorname{Tr}\left(f\Big(\frac{D_\omega}{\Lambda}\Big)\right) \sim 2f_4\Lambda^4 a_0({D_\omega}^2) + 2f_2\Lambda^2 a_2({D_\omega}^2) + f(0) a_4({D_\omega}^2) + \mathcal{O}(\Lambda^{-1}). \tag{8.4.1}$$

PROPOSITION 8.10. *For the canonical triple $(C^\infty(M), L^2(S), D_M)$, the spectral action is given by:*

$$\mathrm{Tr}\left(f\Big(\frac{D_M}{\Lambda}\Big)\right) \sim \int_M \mathcal{L}_M(g_{\mu\nu})\sqrt{g}d^4x + \mathcal{O}(\Lambda^{-1}), \tag{8.4.2}$$

where the Lagrangian is defined by

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s + \frac{f(0)}{16\pi^2}\Big(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\Big).$$

Here the Weyl tensor $C_{\mu\nu\rho\sigma}$ is given by the traceless part of the Riemann curvature tensor, so that

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\nu\sigma}R^{\nu\sigma} + \frac{1}{3}s^2, \tag{8.4.3}$$

and R^* is related to the Pontryagin class:

$$R^*R^* = s^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{8.4.4}$$

PROOF. We have $n = 4$, and $\mathrm{Tr(id)} = \dim S_x = 4$ where S_x is the fiber of S at some $x \in M$. Inserting this into Theorem 8.8 gives

$$a_0(D_M^2) = \frac{1}{4\pi^2}\int_M \sqrt{g}d^4x.$$

From the Lichnerowicz formula (8.3.5) we see that $F = -\frac{1}{4}s\,\mathrm{id}$, so

$$a_2(D_M^2) = -\frac{1}{48\pi^2}\int_M s\sqrt{g}d^4x.$$

Moreover,

$$5s^2\mathrm{id} + 60sF + 180F^2 = \frac{5}{4}s^2\mathrm{id}.$$

Inserting this into $a_4(D_M^2)$ gives

$$\begin{aligned}a_4(D_M^2) = \frac{1}{16\pi^2}\frac{1}{360}\int_M \mathrm{Tr}\big(3\Delta s\,\mathrm{id} &+ \frac{5}{4}s^2\mathrm{id} - 2R_{\mu\nu}R^{\mu\nu}\mathrm{id}\\ &+ 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\mathrm{id} + 30\Omega^S_{\mu\nu}\Omega^{S\mu\nu}\big)\sqrt{g}d^4x.\end{aligned}$$

The curvature Ω^S of the spin connection is defined as in (4.2.3), and its components are $\Omega^S_{\mu\nu} = \Omega^S(\partial_\mu, \partial_\nu)$. The spin curvature Ω^S is related to the Riemannian curvature tensor by (see Note 7 on Page 126),

$$\Omega^S_{\mu\nu} = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma. \tag{8.4.5}$$

We use this as well as the trace identity

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

to calculate the last term of $a_4(D_M^2)$:

$$\begin{aligned}\mathrm{Tr}(\Omega^S_{\mu\nu}\Omega^{S\mu\nu}) &= \frac{1}{16}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\kappa}\ \mathrm{Tr}(\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\kappa)\\ &= \frac{1}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\kappa}\ (g^{\rho\sigma}g^{\lambda\kappa} - g^{\rho\lambda}g^{\sigma\kappa} + g^{\rho\kappa}g^{\sigma\lambda}) = -\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},\end{aligned} \tag{8.4.6}$$

where the first term in the second line vanishes because of the antisymmetry of $R_{\mu\nu\rho\sigma}$ in ρ and σ, and the other two terms contribute equally. We thus obtain

(8.4.7)
$$a_4(D_M^2) = \frac{1}{16\pi^2}\frac{1}{360}\int_M \left(12\Delta s + 5s^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\sqrt{g}d^4x.$$

We rewrite this into a more convenient form, using (8.4.3) and (8.4.4), which together yield:

$$\begin{aligned}
-&\frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^* \\
&= -\frac{1}{20}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{10}R_{\nu\sigma}R^{\nu\sigma} - \frac{1}{60}s^2 \\
&\quad + \frac{11}{360}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{44}{360}R_{\nu\sigma}R^{\nu\sigma} + \frac{11}{360}s^2 \\
&= \frac{1}{360}\big(-7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8R_{\nu\sigma}R^{\nu\sigma} + 5s^2\big).
\end{aligned}$$

Therefore, we may rewrite (8.4.7) so as to obtain

$$a_4(D_M^2) = \frac{1}{16\pi^2}\int_M \left(\frac{1}{30}\Delta s - \frac{1}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{360}R^*R^*\right)\sqrt{g}d^4x.$$

Inserting the obtained formulas for $a_0(D_M^2)$, $a_2(D_M^2)$ and $a_4(D_M^2)$ into (8.4.1) proves the proposition. □

Remark 8.11. *In general, an expression of the form*

$$as^2 + bR_{\nu\sigma}R^{\nu\sigma} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

for certain constants $a, b, c \in \mathbb{R}$, can always be rewritten in the form $\alpha s^2 + \beta C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \gamma R^*R^*$, for new constants $\alpha, \beta, \gamma \in \mathbb{R}$. One should note here that the term s^2 is not present in the spectral action of the canonical triple as calculated in Proposition 8.10. The only higher-order gravitational term that arises is the conformal gravity term $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$.

Note that alternatively, using only (8.4.4), we could also have written

$$a_4(D_M^2) = \frac{1}{16\pi^2}\frac{1}{30}\int_M \left(\Delta s + s^2 - 3R_{\mu\nu}R^{\mu\nu} - \frac{7}{12}R^*R^*\right)\sqrt{g}d^4x.$$

The integral over Δs only yields a boundary term, so if the manifold M is compact without boundary, we can discard the term with Δs. Furthermore, for a 4-dimensional compact orientable manifold M without boundary, we have the formula

$$\int_M R^*R^*\sqrt{g}dx = 8\pi^2\chi(M),$$

where $\chi(M)$ is Euler characteristic. Hence the term with R^*R^* only yields a topological contribution to the action, which we will also disregard. From here on, we will therefore consider the Lagrangian

$$\mathcal{L}_M(g_{\mu\nu}) = \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \tag{8.4.8}$$

or, which is the same,

$$\mathcal{L}_M(g_{\mu\nu}) = \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s + \frac{f(0)}{480\pi^2}\Big(s^2 - 3R_{\mu\nu}R^{\mu\nu}\Big). \tag{8.4.9}$$

PROPOSITION 8.12. *The spectral action of the fluctuated Dirac operator of an almost-commutative manifold with* $\dim M = 4$ *is given by*

$$\operatorname{Tr}\left(f\Big(\frac{D_\omega}{\Lambda}\Big)\right) \sim \int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}d^4x + \mathcal{O}(\Lambda^{-1}),$$

where

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) := N\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_B(B_\mu) + \mathcal{L}_\phi(g_{\mu\nu}, B_\mu, \Phi).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ *is defined in Proposition 8.10,* N *is the dimension of the finite-dimensional Hilbert space* H_F*, and* $\mathcal{L}_B$ *gives the kinetic term of the gauge field as*

$$\mathcal{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

and $\mathcal{L}_\phi$ *gives a scalar-field Lagrangian including its interactions plus a boundary term as*

(8.4.10)

$$\begin{aligned}\mathcal{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2}\operatorname{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2}\operatorname{Tr}(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta\big(\operatorname{Tr}(\Phi^2)\big)\\ + \frac{f(0)}{48\pi^2}s\operatorname{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2}\operatorname{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big).\end{aligned}$$

PROOF. The proof is very similar to Proposition 8.10, but we now use the formula for ${D_\omega}^2$ given by Proposition 8.6. The trace over the Hilbert space H_F yields an overall factor $N := \operatorname{Tr}(1_{H_F})$, so we have

$$a_0({D_\omega}^2) = Na_0(D_M^2).$$

The square of the Dirac operator now contains three extra terms. The trace of $\gamma_M\gamma^\mu$ vanishes, which follows from cyclicity of the trace and the fact that $\gamma_M\gamma^\mu = -\gamma^\mu\gamma_M$. Since $\operatorname{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $F_{\mu\nu}$ is anti-symmetric, the trace of $\gamma^\mu\gamma^\nu F_{\mu\nu}$ also vanishes. Thus we find that

$$a_2({D_\omega}^2) = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \operatorname{Tr}(\Phi^2)\sqrt{g}d^4x.$$

Furthermore we obtain several new terms from the formula for $a_4({D_\omega}^2)$. First, we calculate

$$\frac{1}{360}\operatorname{Tr}(60sF) = -\frac{1}{6}s\left(Ns + 4\operatorname{Tr}(\Phi^2)\right).$$

The next contribution arises from the trace over F^2, which equals

$$\begin{aligned}F^2 = \frac{1}{16}s^2\otimes 1 + 1\otimes\Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\otimes F_{\mu\nu}F_{\rho\sigma}\\ + \gamma^\mu\gamma^\nu\otimes(D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s\otimes\Phi^2 + \text{ traceless terms.}\end{aligned}$$

Taking the trace then yields

$$\frac{1}{360}\operatorname{Tr}(180F^2) = \frac{N}{8}s^2 + 2\operatorname{Tr}(\Phi^4) + \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2\operatorname{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + s\operatorname{Tr}(\Phi^2).$$

Another contribution arises from $-\Delta F$. Again, we can simply ignore the traceless terms and obtain

$$\frac{1}{360}\operatorname{Tr}(-60\Delta F) = \frac{1}{6}\Delta\left(Ns + 4\operatorname{Tr}(\Phi^2)\right).$$

The final contribution comes from the term $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$, where the curvature Ω^E is given by (8.3.3); we obtain

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu}\otimes 1 - 1\otimes F_{\mu\nu}F^{\mu\nu} + 2i\Omega^S_{\mu\nu}\otimes F^{\mu\nu}.$$

Using (8.4.5), by the anti-symmetry of $R_{\rho\sigma\mu\nu}$ we find

$$\operatorname{Tr}(\Omega^S_{\mu\nu}) = \frac{1}{4}R_{\rho\sigma\mu\nu}\operatorname{Tr}(\gamma^\rho\gamma^\sigma) = \frac{1}{4}R_{\rho\sigma\mu\nu}g^{\rho\sigma} = 0,$$

so the trace over the cross-terms in $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ vanishes. From (8.4.6) we then obtain

$$\frac{1}{360}\operatorname{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{1}{12}\left(-\frac{N}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\right).$$

Gathering all terms, we obtain

$$\begin{aligned}
a_4(x, D_\omega{}^2) = \frac{1}{(4\pi)^2}\frac{1}{360}\Bigg(&-48N\Delta s + 20Ns^2 - 8NR_{\mu\nu}R^{\mu\nu} \\
&+ 8NR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 60s\left(Ns + 4\operatorname{Tr}(\Phi^2)\right) \\
&+ 360\bigg(\frac{N}{8}s^2 + 2\operatorname{Tr}(\Phi^4) + \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
&\qquad + 2\operatorname{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + s\operatorname{Tr}(\Phi^2)\bigg) \\
&+ 60\Delta\left(Ns + 4\operatorname{Tr}(\Phi^2)\right) \\
&- 30\left(\frac{N}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 4\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\right)\Bigg) \\
= \frac{1}{(4\pi)^2}\frac{1}{360}\Bigg(&12N\Delta s + 5Ns^2 - 8NR_{\mu\nu}R^{\mu\nu} \\
&- 7NR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 120s\operatorname{Tr}(\Phi^2) \\
&+ 360\Big(2\operatorname{Tr}(\Phi^4) + 2\operatorname{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big)\Big) \\
&+ 240\Delta\left(\operatorname{Tr}(\Phi^2)\right) + 240\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\Bigg).
\end{aligned}$$

Comparing the first line of the second equality to (8.4.7), we see that

$$a_4(x, D_\omega{}^2) = N a_4(x, D_M^2) + \frac{1}{4\pi^2}\Bigg(\frac{1}{12} s \operatorname{Tr}(\Phi^2) + \frac{1}{2}\operatorname{Tr}(\Phi^4)$$
$$+ \frac{1}{2}\operatorname{Tr}\big((D_\mu\Phi)(D^\mu\Phi)\big) + \frac{1}{6}\Delta\left(\operatorname{Tr}(\Phi^2)\right) + \frac{1}{6}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\Bigg).$$

Inserting these Seeley-DeWitt coefficients into (8.4.1) proves the proposition.

□

Note that the above Lagrangian is indeed gauge invariant. This is of course a consequence of the manifest gauge invariance of the spectral action, which follows from the invariance of the spectrum under unitary transformations.

EXAMPLE 8.13. *Let us return to the Yang–Mills manifold $M \times F_{\mathrm{YM}}$ of Examples 8.4 and 8.5. We have already seen that the inner fluctuations are parametrized by a $PU(N)$ gauge field A_μ, which acts in the adjoint representation $B_\mu = \operatorname{ad} A_\mu$ on the fermions. There is no scalar field ϕ and $\Phi = D_F = 0$. We can insert these fields into the result of Proposition 8.12. The dimension of the Hilbert space $H_F = M_N(\mathbb{C})$ is N^2. We then find that the Lagrangian of the Yang–Mills manifold is given by*

$$\mathcal{L}(g_{\mu\nu}, B_\mu) := N^2 \mathcal{L}_M(g_{\mu\nu}) + \frac{f(0)}{24\pi^2}\mathcal{L}_{YM}(B_\mu).$$

Here $\mathcal{L}_{YM}$ is the Yang–Mills Lagrangian given by

$$\mathcal{L}_{YM}(B_\mu) := \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

where $F_{\mu\nu}$ denotes the curvature of B_μ.

Notes

Section 8.1. Gauge symmetries of AC manifolds

1. Kaluza–Klein theories date back to [**119, 125**].

2. The name almost-commutative manifolds was coined in [**114**], suggesting that the noncommutativity is mild since it is simply given by the matrix product in A_F, pointwise on M. Almost-commutative manifolds essentially already appeared in [**59**], and somewhat later in the work of Connes and Lott [**71**]. Around the same time, a similar structure appeared in a series of papers by Dubois-Violette, Kerner and Madore [**79, 80, 81, 82**], who studied the noncommutative differential geometry for the algebra of functions tensored with a matrix algebra, and its relevance to the description of gauge and scalar Higgs fields. Almost-commutative manifolds were later used by Chamseddine and Connes [**49, 50**], and by Chamseddine, Connes and Marcolli in [**54**] to geometrically describe Yang–Mills theories and the Standard Model of elementary particles, as we will see in the next chapters. We here base our treatment on [**186**].

3. We can regard $C^\infty(M, A_F)$ as the space of smooth sections of a globally trivial $*$-algebra bundle $M \times A_F$. The natural question whether the above definition can be extended to the topologically non-trivial case is addressed in [**40, 41, 34**]. The special case of topologically non-trivial Yang–Mills theories is treated in [**35**] and in the next Chapter.

4. In the proof of Proposition 8.2 we have exploited a lift of group bundles, which exists if the manifold is simply connected. We refer to [**34**] for a careful discussion on this point.

Section 8.3. The heat expansion of the spectral action

5. For more details on generalized Laplacians we refer to [**27**, Sect. 2.1].

6. Theorem 8.7 is proved by Gilkey in [**100**, Sect. 1.7]. Theorem 8.8 can be found as [**100**, Theorem 4.8.16]. For a more physicist-friendly approach, we refer to [**191**]. Note that the conventions used by Gilkey for the Riemannian curvature R are such that $g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}$ is negative for a sphere, in contrast to our own conventions. Therefore we have replaced $s = -R$.

7. The relation (8.4.5) is derived in [**103**, p.395].

8. The derivation of Yang–Mills gauge theory from a noncommutative spin manifold as in Example 8.13 is due to Chamseddine and Connes in [**49, 50**].

CHAPTER 9

The noncommutative geometry of electrodynamics

In the previous chapters we have described the general framework for the description of gauge theories in terms of noncommutative manifolds. The present chapter serves two purposes. First, we describe abelian gauge theories within the framework of noncommutative geometry, which at first sight appears to be a *contradictio in terminis.* Second, in Section 9.2 we show how this example can be modified to provide a description of one of the simplest examples of a field theory in physics, namely electrodynamics. Because of its simplicity, it helps in gaining an understanding of the formulation of gauge theories in terms of almost-commutative manifolds, and as such it provides a first stepping stone towards the derivation of the Standard Model from noncommutative geometry in Chapter 11.

9.1. The two-point space

In this section we discuss one of the simplest finite noncommutative spaces, namely the two-point space $X = \{x, y\}$. Recall from Chapters 2 and 3 that such a space can be described by an even finite real spectral triple:

$$F_X := \left(C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_F\right). \tag{9.1.1}$$

As we require the action of $C(X)$ on the finite-dimensional Hilbert space H_F to be faithful, H_F must at least be 2-dimensional. For now we restrict ourselves to the simplest case, taking $H_F = \mathbb{C}^2$. We use the $\mathbb{Z}_2$-grading γ_F to decompose $H_F = H_F^+ \oplus H_F^- = \mathbb{C} \oplus \mathbb{C}$ into the two eigenspaces $H_F^\pm = \{\psi \in H_F \mid \gamma_F \psi = \pm\psi\}$. The action of $C(X)$ on H_F respects this decomposition, whereas D_F interchanges the two subspaces $H_F^\pm$, say

$$D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix},$$

for some $t \in \mathbb{C}$.

Proposition 9.1. *The finite space F_X of* (9.1.1) *can only have a real structure J_F if $D_F = 0$. In that case, its KO-dimension is 0,2 or 6.*

Proof. The diagonal representation of the algebra $\mathbb{C} \oplus \mathbb{C}$ on $\mathbb{C} \oplus \mathbb{C}$ gives rise to one of the following two Krajewski diagrams (*cf.* Example 3.13):

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circ \\ \mathbf{1}^\circ & \circ & \end{array} \qquad\qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & \circ & \\ \mathbf{1}^\circ & & \circ \end{array}$$

As a Dirac operator D_F that fulfills the first-order condition 3.1.1 (for arbitrary J_F) should connect nodes either vertically or horizontally, we find that $D_F = 0$.

The diagram on the left corresponds to KO-dimension 2 and 6, while the diagram on the right corresponds to KO-dimension 0 and 4. KO-dimension 4 is ruled out because of Lemma 3.8, combined with the fact that $\dim H_F^{\pm} = 1$, which does not allow for a J_F with $J_F^2 = -1$. □

9.1.1. The product space. Let M be a compact 4-dimensional Riemannian spin manifold. We now consider the almost-commutative manifold $M \times F_X$ given by the product of M with the even finite space F_X corresponding to the two-point space (9.1.1). Thus we consider the almost-commutative manifold given by the data

$$M \times F_X := \Big(C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F\Big),$$

where we still need to make a choice for J_F. The algebra of this almost-commutative manifold is given by $C^\infty(M, \mathbb{C}^2) \simeq C^\infty(M) \oplus C^\infty(M)$. By Gelfand duality (Theorem 4.28) this algebra corresponds to the space

$$N := M \times X \simeq M \sqcup M,$$

which consists of the disjoint union of two copies of the space M, so we can write $C^\infty(N) = C^\infty(M) \oplus C^\infty(M)$. We can also decompose the total Hilbert space as $\mathcal{H} = L^2(S) \oplus L^2(S)$. For $a, b \in C^\infty(M)$ and $\psi, \phi \in L^2(S)$, an element $(a, b) \in C^\infty(N)$ then simply acts on $(\psi, \phi) \in \mathcal{H}$ as $(a, b)(\psi, \phi) = (a\psi, b\phi)$.

Remark 9.2. *Let us consider Connes' distance formula (cf. Note 12 on Page 60) on $M \times F_X$. First, as in (2.2.2), on the structure space X of A_F we may write a metric by:*

$$d_{D_F}(x, y) = \sup\left\{|a(x) - a(y)| \colon a \in A_F, \|[D_F, a]\| \leq 1\right\}.$$

Note that now we only have two distinct points x and y in the space X, and we are going calculate the distance between these points. An element $a \in \mathbb{C}^2 = C(X)$ is specified by two complex numbers $a(x)$ and $a(y)$, so a small computation of the commutator with D_F gives

$$[D_F, a] = \big(a(y) - a(x)\big) \begin{pmatrix} 0 & t \\ -\bar{t} & 0 \end{pmatrix}.$$

The norm of this commutator is given by $|a(y) - a(x)|\,|t|$, so $\|[D_F, a]\| \leq 1$ implies $|a(y) - a(x)| \leq \frac{1}{|t|}$. We therefore obtain that the distance between the two points x and y is given by

$$d_{D_F}(x, y) = \frac{1}{|t|}.$$

If there is a real structure J_F, we have $t = 0$ by Proposition 9.1, so in that case the distance between the two points becomes infinite.

Let p be a point in M, and write (p, x) and (p, y) for the two corresponding points in $N = M \times X$. A function $a \in C^\infty(N)$ is then determined by two functions $a_x, a_y \in C^\infty(M)$, given by $a_x(p) := a(p, x)$ and $a_y(p) := a(p, y)$. Now the distance function on N is given by

$$d_{D_M \otimes 1}(n_1, n_2) = \sup\left\{|a(n_1) - a(n_2)| \colon a \in \mathcal{A}, \|[D_M \otimes 1, a]\| \leq 1\right\}.$$

If n_1 and n_2 are points in the same copy of M, for instance, if $n_1 = (p, x)$ and $n_2 = (q, x)$ for points $p, q \in M$, then their distance is determined by $|a_x(p) -$

$a_x(q)|$, for functions $a_x \in C^\infty(M)$ for which $\|[D_M, a_x]\| \leq 1$. Therefore, in this case we recover the geodesic distance on M, i.e.

$$d_{D_M \otimes 1}(n_1, n_2) = d_g(p, q).$$

However, if n_1 and n_2 lie in different copies of M, for instance if, $n_1 = (p, x)$ and $n_2 = (q, y)$, then their distance is determined by $|a_x(p) - a_y(q)|$ for two functions $a_x, a_y \in C^\infty(M)$, such that $\|[D_M, a_x]\| \leq 1$ and $\|[D_M, a_y]\| \leq 1$. However, these requirements yield no restriction on $|a_x(p) - a_y(q)|$, so in this case the distance between n_1 and n_2 is infinite. We find that the space N is given by two disjoint copies of M that are separated by an infinite distance.

It should be noted that the only way in which the distance between the two copies of M could have been finite, is when the commutator $[D_F, a]$ would be nonzero. This same commutator generates the scalar field ϕ of (8.2.2), hence finiteness of the distance is related to the existence of scalar fields.

9.1.2. $U(1)$ gauge theory. We determine the gauge theory that corresponds to the almost-commutative manifold $M \times F_X$. The gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ from Definition 6.4 is given by the quotient $\mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A}_J)$, so if we wish to obtain a nontrivial gauge group, we need to choose J such that $\mathcal{U}(\mathcal{A}_J) \neq \mathcal{U}(\mathcal{A})$. Or, which in view of Example 4.38 is the same, we need to choose J_F so that $\mathcal{U}((\mathcal{A}_F)_{J_F}) \neq \mathcal{U}(\mathcal{A}_F)$. Looking at the form of J_F for the different (even) KO-dimensions (see the proof of Proposition 9.1), we conclude that we need KO-dimension 2 or 6. As we will see in the noncommutative description of the Standard Model in Chapter 11, the correct signature for the internal space is KO-dimension 6. Therefore, we choose to work in KO-dimension 6 as well. The almost-commutative manifold $M \times F_X$ then has KO-dimension $6 + 4 \mod 8 = 2$. This also means that we can use Definition 7.3 to calculate the fermionic action.

Summarizing, we will consider the finite space F_X given by the data

$$F_X := \left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

with C denoting complex conjugation, defining a real even finite space of KO-dimension 6. In the classification of irreducible geometries of Theorem 3.20, this space corresponds to the first case.

PROPOSITION 9.3. *The gauge group $\mathfrak{G}(F)$ of the two-point space is given by $U(1)$.*

PROOF. First, note that $\mathcal{U}(A_F) = U(1) \times U(1)$. We now show that $\mathcal{U}((A_F)_{J_F}) \equiv \mathcal{U}(A_F) \cap (A_F)_{J_F} \simeq U(1)$ so that the quotient $\mathfrak{G}(F) \simeq U(1)$ as claimed. Indeed, for $a \in \mathbb{C}^2$ to be in $(A_F)_{J_F}$ it has to satisfy $J_F a^* J_F = a$. Since

$$J_F a^* J_F^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix},$$

this is the case if and only if $a_1 = a_2$. Thus, $(A_F)_{J_F} \simeq \mathbb{C}$, whose unitary elements form the group $U(1)$, contained in $\mathcal{U}(A_F)$ as the diagonal subgroup. $\square$

In Proposition 8.12 we calculated the spectral action of an almost-commutative manifold. Before we can apply this to the two-point space, we need to find the exact form of the field B_μ. Since we have $(A_F)_{J_F} \simeq \mathbb{C}$, we find $\mathfrak{h}(F) = \mathfrak{u}\big((A_F)_{J_F}\big) \simeq i\mathbb{R}$. From Proposition 8.3 and (8.2.7) we then see that the gauge field

$$A_\mu(x) \in i\mathfrak{g}_F = i\big(\mathfrak{u}(A_F)/(i\mathbb{R})\big) = i\, su(A_F) \simeq \mathbb{R}$$

becomes traceless.

Let us also explicitly derive this $U(1)$ gauge field. An arbitrary hermitian field of the form $A_\mu = -ia\partial_\mu b$ would be given by two $U(1)$ gauge fields $X^1_\mu, X^2_\mu \in C^\infty(M,\mathbb{R})$. However, because A_μ only appears in the combination $A_\mu - J_F A_\mu J_F^{-1}$, we obtain

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X^1_\mu & 0 \\ 0 & X^2_\mu \end{pmatrix} - \begin{pmatrix} X^2_\mu & 0 \\ 0 & X^1_\mu \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F,$$

where we have defined the $U(1)$ gauge field

$$Y_\mu := X^1_\mu - X^2_\mu \in C^\infty(M,\mathbb{R}) = C^\infty(M, i\,u(1)).$$

Thus, the fact that we only have the combination $A_\mu - J_F A_\mu J_F^{-1}$ effectively identifies the $U(1)$ gauge fields on the two copies of M, so that A_μ is determined by only one $U(1)$ gauge field. This ensures that we can take the quotient of the Lie algebra $\mathfrak{u}(A_F)$ with $\mathfrak{h}(F)$. We can then write

$$A_\mu = \frac{1}{2}\begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = \frac{1}{2} Y_\mu \otimes \gamma_F,$$

which yields the same result:

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = 2A_\mu = Y_\mu \otimes \gamma_F. \tag{9.1.2}$$

We summarize:

PROPOSITION 9.4. *The inner fluctuations of the almost-commutative manifold $M \times F_X$ described above are parametrized by a $U(1)$-gauge field Y_μ as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F.$$

The action of the gauge group $\mathfrak{G}(M \times F_X) \simeq C^\infty(M, U(1))$ on D', as in (8.2.8)*, is implemented by*

$$Y_\mu \mapsto Y_\mu - iu\partial_\mu u^*; \qquad (u \in \mathfrak{G}(M \times F_X)).$$

9.2. Electrodynamics

Inspired by the previous section, which shows that one can use the framework of noncommutative geometry to describe a gauge theory with abelian gauge group $U(1)$, we proceed and try to describe the full theory of electrodynamics by an almost-commutative manifold. Our approach provides a unified description of gravity and electromagnetism, albeit at the classical level.

We have seen that the almost-commutative manifold $M \times F_X$ describes a gauge theory with local gauge group $U(1)$, where the inner fluctuations of the Dirac operator provide the $U(1)$ gauge field Y_μ. There appear to be two problems if one wishes to use this model for a description of (classical) electrodynamics. First, by Proposition 9.1, the finite Dirac operator D_F must vanish. However, we want our electrons to be massive, and for this purpose we need a finite Dirac operator that is non-zero.

Second, the Euclidean action for a free Dirac field is of the form

$$S = -\int i\overline{\psi}(\gamma^\mu\partial_\mu - m)\psi d^4x, \tag{9.2.1}$$

where the fields ψ and $\overline{\psi}$ must be considered *independent variables*. Thus, we require that the fermionic action S_f should also yield two *independent* Dirac spinors. Let us write $\{e, \overline{e}\}$ for the set of orthonormal basis vectors of H_F, where e is the basis element of H_F^+ and $\overline{e}$ of H_F^-. Note that on this basis, we have $J_F e = \overline{e}$, $J_F\overline{e} = e$, $\gamma_F e = e$ and $\gamma_F\overline{e} = -\overline{e}$. The total Hilbert space $\mathcal{H}$ is given by $L^2(S)\otimes H_F$. Since by means of γ_M we can also decompose $L^2(S) = L^2(S)^+\oplus L^2(S)^-$, we obtain that the positive eigenspace $\mathcal{H}^+$ of $\gamma = \gamma_M\otimes\gamma_F$ is given by

$$\mathcal{H}^+ = L^2(S)^+\otimes H_F^+ \oplus L^2(S)^-\otimes H_F^-.$$

Consequently, an arbitrary vector $\xi\in\mathcal{H}^+$ can uniquely be written as

$$\xi = \psi_L\otimes e + \psi_R\otimes\overline{e},$$

for two Weyl spinors $\psi_L\in L^2(S)^+$ and $\psi_R\in L^2(S)^-$. One should note here that ξ is completely determined by only one Dirac spinor $\psi := \psi_L + \psi_R$, instead of the required two independent spinors. Thus, the restrictions that are incorporated into the fermionic action of Definition 7.3 in fact constrain the finite space F_X too much.

9.2.1. The finite space. It turns out that both problems sketched above can be simply solved by doubling our finite-dimensional Hilbert space. Essentially, we introduce multiplicities in the Krajewski diagram that appeared in the proof of Proposition 9.1.

Thus, we start with the same algebra $C^\infty(M,\mathbb{C}^2)$ that corresponds to the space $N = M\times X\simeq M\sqcup M$. The finite-dimensional Hilbert space will now be used to describe *four* particles, namely both the left-handed and the right-handed electrons and positrons. We choose the orthonormal basis $\{e_R, e_L, \overline{e_R}, \overline{e_L}\}$ for $H_F = \mathbb{C}^4$, with respect to the standard inner product. The subscript L denotes left-handed particles, and the subscript R denotes right-handed particles, and we have $\gamma_F e_L = e_L$ and $\gamma_F e_R = -e_R$.

We choose J_F such that it interchanges particles with their antiparticles, so $J_F e_R = \overline{e_R}$ and $J_F e_L = \overline{e_L}$. We again choose the real structure such that it has KO-dimension 6, so we have $J_F^2 = 1$ and $J_F\gamma_F = -\gamma_F J_F$. This last relation implies that the element $\overline{e_R}$ is left-handed, whereas $\overline{e_L}$ is right-handed.

The grading γ_F decomposes the Hilbert space H_F into $H_F^+\oplus H_F^-$, where the bases of H_F^+ and H_F^- are given by $\{e_L, \overline{e_R}\}$ and $\{e_R, \overline{e_L}\}$, respectively. Alternatively, we can decompose the Hilbert space into $H_e\oplus H_{\overline{e}}$, where H_e contains the electrons $\{e_R, e_L\}$, and $H_{\overline{e}}$ contains the positrons $\{\overline{e_R}, \overline{e_L}\}$.

The elements $a \in A_F = \mathbb{C}^2$ now act as the following matrix with respect to the basis $\{e_R, e_L, \overline{e_R}, \overline{e_L}\}$:

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}. \tag{9.2.2}$$

Note that this action commutes with the grading, as it should. We can also easily check that $[a, b^0] = 0$ for $b^0 := J_F b^* J_F^{-1}$, since both the left and the right action are given by diagonal matrices. For now, we still take $D_F = 0$, and hence the order one condition is trivially satisfied. We have therefore obtained the following result:

Proposition 9.5. *The data*

$$\left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

define a real even spectral triple of KO-dimension 6*.*

This can be summarized by the following Krajewski diagram, with two nodes (of opposite grading) of multiplicity two:

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

9.2.2. A non-trivial finite Dirac operator. Let us now consider the possibilities for adding a non-zero Dirac operator to the finite space F_{ED}. From the above Krajewski diagram, it can be easily seen that the only possible edges exist between the multiple vertices. That is, the only possible Dirac operator depends on one complex parameter and is given by

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \overline{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{d} \\ 0 & 0 & d & 0 \end{pmatrix}. \tag{9.2.3}$$

From here on, we will consider the finite space F_{ED} given by

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F).$$

9.2.3. The almost-commutative manifold. Taking the product with the canonical triple, the almost-commutative manifold $M \times F_{ED}$ (of KO-dimension 2) under consideration is given by the spectral triple

$$M \times F_{ED} := \big(C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^4, D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F \big). \tag{9.2.4}$$

As in Section 9.1, the algebra decomposes as

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M),$$

and we now decompose the Hilbert space as

$$\mathcal{H} = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\overline{e}}).$$

The action of the algebra on $\mathcal{H}$, given by (9.2.2), is then such that one component of the algebra acts on the electron fields $L^2(S) \otimes H_e$, and the other component acts on the positron fields $L^2(S) \otimes H_{\bar{e}}$.

The derivation of the gauge group for F_{ED} is exactly the same as in Proposition 9.3, so again we have the finite gauge group $\mathfrak{G}(F) \simeq U(1)$. The field $B_\mu := A_\mu - J_F A_\mu J_F^{-1}$ now takes the form

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & -Y_\mu & 0 \\ 0 & 0 & 0 & -Y_\mu \end{pmatrix} \qquad \text{for } Y_\mu(x) \in \mathbb{R}. \tag{9.2.5}$$

Thus, we again obtain a single $U(1)$ gauge field Y_μ, carrying an action of the gauge group $\mathfrak{G}(M \times F_{ED}) \simeq C^\infty(M, U(1))$ (as in Proposition 9.4).

As mentioned before, our space N consists of two copies of M and if $D_F = 0$ the distance between these two copies is infinite (see Remark 9.2). This time we have introduced a non-zero Dirac operator, but it commutes with the algebra, *i.e.* $[D_F, a] = 0$ for all $a \in \mathcal{A}$. Therefore, the distance between the two copies of M is still infinite.

To summarize, the $U(1)$ gauge theory arises from the geometric space $N = M \sqcup M$ as follows. On one copy of M, we have the vector bundle $S \otimes (M \times H_e)$, and on the other copy we have the vector bundle $S \otimes (M \times H_{\bar{e}})$. The gauge fields on each copy of M are identified with each other. The electrons e and positrons $\bar{e}$ are then both coupled to the same gauge field, and as such the gauge field provides an interaction between electrons and positrons. For comparison with Kaluza–Klein theories, note the different role that is played by the internal space.

9.2.4. The spectral action. We are now ready to explicitly calculate the Lagrangian that corresponds to the almost-commutative manifold $M \times F_{ED}$, and we will show that this yields the usual Lagrangian for electrodynamics (on a curved background manifold), as well as a purely gravitational Lagrangian. It consists of the spectral action S_b of Definition 7.1 and the fermionic action S_f of Definition 7.3, which we calculate separately (here and in the next section).

The spectral action for an almost-commutative manifold has been calculated in Proposition 8.12, and we only need to insert the fields B_μ (given by (9.2.5)) and $\Phi = D_F$. We obtain the following result:

PROPOSITION 9.6. *The spectral action of the almost-commutative manifold $M \times F_{ED}$ defined in* (9.2.4) *is given by*

$$\operatorname{Tr}\left(f\left(\frac{D_\omega}{\Lambda}\right)\right) \sim \int_M \mathcal{L}(g_{\mu\nu}, Y_\mu)\sqrt{g}d^4x + O(\Lambda^{-1}),$$

with Lagrangian

$$\mathcal{L}(g_{\mu\nu}, Y_\mu) := 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_Y(Y_\mu) + \mathcal{L}_\phi(g_{\mu\nu}, d).$$

Here $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 8.10; the term $\mathcal{L}_Y$ gives the kinetic term of the $U(1)$ gauge field Y_μ as

$$\mathcal{L}_Y(Y_\mu) := \frac{f(0)}{6\pi^2} Y_{\mu\nu} Y^{\mu\nu},$$

where the curvature $Y_{\mu\nu}$ of the field Y_μ is given by

$$Y_{\mu\nu} := \partial_\mu Y_\nu - \partial_\nu Y_\mu.$$

The scalar potential $\mathcal{L}_\phi$ (ignoring the boundary term) gives two constant terms which add to the cosmological constant, plus an extra contribution to the Einstein–Hilbert action:

$$\mathcal{L}_\phi(g_{\mu\nu}) := -\frac{2f_2\Lambda^2}{\pi^2}|d|^2 + \frac{f(0)}{2\pi^2}|d|^4 + \frac{f(0)}{12\pi^2}s|d|^2,$$

where the constant d originates from (9.2.3)*.*

Proof. The trace over the Hilbert space $\mathbb{C}^4$ yields an overall factor $N = 4$. The field B_μ is given by (9.2.5), and we obtain $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 4Y_{\mu\nu}Y^{\mu\nu}$. Inserting this into Proposition 8.12 provides the Lagrangian $\mathcal{L}_Y$. In addition, we have $\Phi^2 = {D_F}^2 = |d|^2$, and the scalar-field Lagrangian $\mathcal{L}_\phi$ only yields extra numerical contributions to the cosmological constant and the Einstein–Hilbert action. □

9.2.5. The fermionic action. We have written the set of basis vectors of H_F as $\{e_R, e_L, \overline{e_R}, \overline{e_L}\}$, and the subspaces H_F^+ and H_F^- are spanned by $\{e_L, \overline{e_R}\}$ and $\{e_R, \overline{e_L}\}$, respectively. The total Hilbert space $\mathcal{H}$ is given by $L^2(S) \otimes H_F$. Since we can also decompose

$$L^2(S) = L^2(S)^+ \oplus L^2(S)^-$$

by means of γ_M, we obtain for the +1-eigenspace of $\gamma_M \otimes \gamma_F$:

$$\mathcal{H}^+ = L^2(S)^+ \otimes H_F^+ \oplus L^2(S)^- \otimes H_F^-.$$

A spinor $\psi \in L^2(S)$ can be decomposed as $\psi = \psi_L + \psi_R$. Each subspace $H_F^\pm$ is now spanned by two basis vectors. A generic element of the tensor product of two spaces consists of sums of tensor products, so an arbitrary vector $\xi \in \mathcal{H}^+$ can be uniquely written as

$$\xi = \chi_R \otimes e_R + \chi_L \otimes e_L + \psi_L \otimes \overline{e_R} + \psi_R \otimes \overline{e_L}, \tag{9.2.6}$$

for Weyl spinors $\chi_L, \psi_L \in L^2(S)^+$ and $\chi_R, \psi_R \in L^2(S)^-$. Note that this vector $\xi \in \mathcal{H}^+$ is now completely determined by two Dirac spinors $\chi := \chi_L + \chi_R$ and $\psi := \psi_L + \psi_R$.

Proposition 9.7. *The fermionic action of the almost-commutative manifold $M \times F_{ED}$ defined in* (9.2.4)*, is given by*

$$S_f = -i\big(J_M\widetilde{\chi}, \gamma^\mu(\nabla^S_\mu - iY_\mu)\widetilde{\psi}\big) + (J_M\widetilde{\chi}_L, \overline{d}\widetilde{\psi}_L) - (J_M\widetilde{\chi}_R, d\widetilde{\psi}_R).$$

Proof. The fluctuated Dirac operator is given by

$$D_\omega = D_M \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes D_F.$$

An arbitrary $\xi \in \mathcal{H}^+$ has the form of (9.2.6), from which we obtain the following expressions:

$$\begin{aligned}
J\xi &= J_M\chi_R \otimes \overline{e_R} + J_M\chi_L \otimes \overline{e_L} + J_M\psi_L \otimes e_R + J_M\psi_R \otimes e_L,\\
(D_M \otimes 1)\xi &= D_M\chi_R \otimes e_R + D_M\chi_L \otimes e_L + D_M\psi_L \otimes \overline{e_R} + D_M\psi_R \otimes \overline{e_L},\\
(\gamma^\mu \otimes B_\mu)\xi &= \gamma^\mu\chi_R \otimes Y_\mu e_R + \gamma^\mu\chi_L \otimes Y_\mu e_L - \gamma^\mu\psi_L \otimes Y_\mu\overline{e_R} - \gamma^\mu\psi_R \otimes Y_\mu\overline{e_L},\\
(\gamma_M \otimes D_F)\xi &= \gamma_M\chi_L \otimes \overline{d}e_R + \gamma_M\chi_R \otimes de_L + \gamma_M\psi_R \otimes d\overline{e_R} + \gamma_M\psi_L \otimes \overline{d}\overline{e_L}.
\end{aligned}$$

We decompose the fermionic action into the three terms

$$\frac{1}{2}(J\widetilde{\xi}, D_\omega\widetilde{\xi}) = \frac{1}{2}(J\widetilde{\xi}, (D_M \otimes 1)\widetilde{\xi}) + \frac{1}{2}(J\widetilde{\xi}, (\gamma^\mu \otimes B_\mu)\widetilde{\xi}) + \frac{1}{2}(J\widetilde{\xi}, (\gamma_M \otimes D_F)\widetilde{\xi}),$$

and then continue to calculate each term separately. The first term is given by

$$\begin{aligned}\frac{1}{2}(J\widetilde{\xi}, (D_M \otimes 1)\widetilde{\xi}) &= \frac{1}{2}(J_M\widetilde{\chi}_R, D_M\widetilde{\psi}_L) + \frac{1}{2}(J_M\widetilde{\chi}_L, D_M\widetilde{\psi}_R) \\ &\quad + \frac{1}{2}(J_M\widetilde{\psi}_L, D_M\widetilde{\chi}_R) + \frac{1}{2}(J_M\widetilde{\psi}_R, D_M\widetilde{\chi}_L).\end{aligned}$$

Using the facts that D_M changes the chirality of a Weyl spinor, and that the subspaces $L^2(S)^+$ and $L^2(S)^-$ are orthogonal, we can rewrite this term as

$$\frac{1}{2}(J\widetilde{\xi}, (D_M \otimes 1)\widetilde{\xi}) = \frac{1}{2}(J_M\widetilde{\chi}, D_M\widetilde{\psi}) + \frac{1}{2}(J_M\widetilde{\psi}, D_M\widetilde{\chi}).$$

Using the symmetry of the form $(J_M\widetilde{\chi}, D_M\widetilde{\psi})$, we obtain

$$\frac{1}{2}(J\widetilde{\xi}, (D_M \otimes 1)\widetilde{\xi}) = (J_M\widetilde{\chi}, D_M\widetilde{\psi}) = -i(J_M\widetilde{\chi}, \gamma^\mu\nabla^S_\mu\widetilde{\psi}).$$

Note that the factor $\frac{1}{2}$ has now disappeared from the result, which is the reason why this factor had to be included in the definition of the fermionic action. The second term is given by

$$\begin{aligned}\frac{1}{2}(J\widetilde{\xi}, (\gamma^\mu \otimes B_\mu)\widetilde{\xi}) &= -\frac{1}{2}(J_M\widetilde{\chi}_R, \gamma^\mu Y_\mu\widetilde{\psi}_L) - \frac{1}{2}(J_M\widetilde{\chi}_L, \gamma^\mu Y_\mu\widetilde{\psi}_R) \\ &\quad + \frac{1}{2}(J_M\widetilde{\psi}_L, \gamma^\mu Y_\mu\widetilde{\chi}_R) + \frac{1}{2}(J_M\widetilde{\psi}_R, \gamma^\mu Y_\mu\widetilde{\chi}_L).\end{aligned}$$

In a similar manner, we obtain

$$\frac{1}{2}(J\widetilde{\xi}, (\gamma^\mu \otimes B_\mu)\widetilde{\xi}) = -(J_M\widetilde{\chi}, \gamma^\mu Y_\mu\widetilde{\psi}),$$

where we have used the anti-symmetry of the form $(J_M\widetilde{\chi}, \gamma^\mu Y_\mu\widetilde{\psi})$. The third term is given by

$$\begin{aligned}\frac{1}{2}(J\widetilde{\xi}, (\gamma_M \otimes D_F)\widetilde{\xi}) &= \frac{1}{2}(J_M\widetilde{\chi}_R, d\gamma_M\widetilde{\psi}_R) + \frac{1}{2}(J_M\widetilde{\chi}_L, \overline{d}\gamma_M\widetilde{\psi}_L) \\ &\quad + \frac{1}{2}(J_M\widetilde{\psi}_L, \overline{d}\gamma_M\widetilde{\chi}_L) + \frac{1}{2}(J_M\widetilde{\psi}_R, d\gamma_M\widetilde{\chi}_R).\end{aligned}$$

The bilinear form $(J_M\widetilde{\chi}, \gamma_M\widetilde{\psi})$ is again symmetric in the Grassmann variables $\widetilde{\chi}$ and $\widetilde{\psi}$, but we now face the extra complication that two terms contain the parameter d, while the other two terms contain $\overline{d}$. Therefore we are left with two distinct terms:

$$\frac{1}{2}(J\widetilde{\xi}, (\gamma_M \otimes D_F)\widetilde{\xi}) = (J_M\widetilde{\chi}_L, \overline{d}\widetilde{\psi}_L) - (J_M\widetilde{\chi}_R, d\widetilde{\psi}_R). \qquad \square$$

REMARK 9.8. *It is interesting to note that the fermions acquire mass terms without being coupled to a scalar field. However, it seems that we obtain a complex mass parameter d, where we would desire a real parameter m.*

Simply requiring that our result should reproduce (9.2.1)*, we will therefore choose* $d := -im$*, so that*

$$(J_M\widetilde{\chi}_L, \overline{d}\widetilde{\psi}_L) - (J_M\widetilde{\chi}_R, d\widetilde{\psi}_R) = i\big(J_M\widetilde{\chi}, m\widetilde{\psi}\big).$$

The results obtained in this section can now be summarized into the following theorem.

THEOREM 9.9. *The full Lagrangian of the almost-commutative manifold* $M \times F_{ED}$ *as defined in Equation* (9.2.4)*, can be written as the sum of a purely gravitational Lagrangian,*

$$\mathcal{L}_{grav}(g_{\mu\nu}) = 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_\phi(g_{\mu\nu}),$$

and a Lagrangian for electrodynamics,

$$\mathcal{L}_{ED} = -i\Big\langle J_M\widetilde{\chi}, \big(\gamma^\mu(\nabla^S_\mu - iY_\mu) - m\big)\widetilde{\psi}\Big\rangle + \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}.$$

PROOF. The spectral action S_b and the fermionic action S_f are given by Propositions 9.6 and 9.7. This immediately yields $\mathcal{L}_{\text{grav}}$. To obtain $\mathcal{L}_{\text{ED}}$, we need to rewrite the fermionic action S_f as the integral over a Lagrangian. The inner product $(\cdot,\cdot)$ on the Hilbert space $L^2(S)$ is given by

$$(\xi,\psi) = \int_M \langle\xi,\psi\rangle\sqrt{g}d^4x,$$

where the hermitian pairing $\langle\cdot,\cdot\rangle$ is given by the pointwise inner product on the fibres. Choosing $d = -im$ as in Remark 9.8, we can then rewrite the fermionic action into

$$S_f = -\int_M i\Big\langle J_M\widetilde{\chi}, \big(\gamma^\mu(\nabla^S_\mu - iY_\mu) - m\big)\widetilde{\psi}\Big\rangle\sqrt{g}d^4x. \qquad \square$$

9.2.6. Fermionic degrees of freedom. To conclude this chapter, let us make a final remark on the fermionic degrees of freedom in the Lagrangian derived above. We refer the reader to Appendix 9.A for a short introduction to Grassmann variables and Grassmann integration.

As mentioned in Note 2 on Page 107, the number of degrees of freedom of the fermion fields in the fermionic action is related to the restrictions that are incorporated into the definition of the fermionic action. These restrictions make sure that in this case we obtain two independent Dirac spinors in the fermionic action.

In fact, in quantum field theory one would consider the functional integral of e^S over the fields. We hence consider the case that $\mathfrak{A}$ is the antisymmetric bilinear form on $\mathcal{H}^+$ given by

$$\mathfrak{A}(\xi,\zeta) := (J\xi, D_\omega\zeta), \qquad \text{for } \xi,\zeta\in\mathcal{H}^+,$$

and $\mathfrak{A}'$ is the bilinear form on $L^2(S)$ given by

$$\mathfrak{A}'(\chi,\psi) := -i\Big(J_M\chi, \big(\gamma^\mu(\nabla^S_\mu - iY_\mu) - m\big)\psi\Big), \qquad \text{for } \chi,\psi\in L^2(S).$$

We have shown in Proposition 9.7 that for $\xi = \chi_L\otimes e_L + \chi_R\otimes e_R + \psi_R\otimes \overline{e_L} + \psi_L\otimes\overline{e_R}$, where we can define two Dirac spinors by $\chi := \chi_L + \chi_R$ and

$\psi := \psi_L + \psi_R$, we obtain

$$\frac{1}{2}\mathfrak{A}(\xi,\xi) = \mathfrak{A}'(\chi,\psi).$$

Using the Grassmann integrals of (9.A.1) and (9.A.2), we then obtain for the bilinear forms $\mathfrak{A}$ and $\mathfrak{A}'$ the equality

$$\mathrm{Pf}(\mathfrak{A}) = \int e^{\frac{1}{2}\mathfrak{A}(\widetilde{\xi},\widetilde{\xi})} D[\widetilde{\xi}] = \int e^{\mathfrak{A}'(\widetilde{\chi},\widetilde{\psi})} D[\widetilde{\psi},\widetilde{\chi}] = \det(\mathfrak{A}').$$

9.A. Grassmann variables, Grassmann integration and Pfaffians

We will give a short introduction to Grassmann variables, and use those to find the relation between the Pfaffian and the determinant of an antisymmetric matrix.

For a set of anti-commuting Grassmann variables θ_i, we have $\theta_i\theta_j = -\theta_j\theta_i$, and in particular, $\theta_i^2 = 0$. On these Grassmann variables θ_j, we define an integral by

$$\int 1 d\theta_j = 0, \qquad \int \theta_j d\theta_j = 1.$$

If we have a Grassmann vector θ consisting of N components, we define the integral over $D[\theta]$ as the integral over $d\theta_1 \cdots d\theta_N$. Suppose we have two Grassmann vectors η and θ of N components. We then define the integration element as $D[\eta,\theta] = d\eta_1 d\theta_1 \cdots d\eta_N d\theta_N$.

Consider the Grassmann integral over a function of the form $e^{\theta^T\mathfrak{A}\eta}$ for Grassmann vectors θ and η of N components. The $N \times N$-matrix $\mathfrak{A}$ can be considered as a bilinear form on these Grassmann vectors. In the case where θ and η are independent variables, we find

$$\int e^{\theta^T\mathfrak{A}\eta} D[\eta,\theta] = \det\mathfrak{A}, \tag{9.A.1}$$

where the determinant of $\mathfrak{A}$ is given by the formula

$$\det(\mathfrak{A}) = \frac{1}{N!}\sum_{\sigma,\tau\in S_N} (-1)^{|\sigma|+|\tau|} \mathfrak{A}_{\sigma(1)\tau(1)} \cdots \mathfrak{A}_{\sigma(N)\tau(N)},$$

in which S_N denotes the set of all permutations of $\{1,2,\ldots,N\}$. Now let us assume that $\mathfrak{A}$ is an antisymmetric $N \times N$-matrix $\mathfrak{A}$ for $N = 2l$. If we then take $\theta = \eta$, we find

$$\int e^{\frac{1}{2}\eta^T\mathfrak{A}\eta} D[\eta] = \mathrm{Pf}(\mathfrak{A}), \tag{9.A.2}$$

where the *Pfaffian* of $\mathfrak{A}$ is given by

$$\mathrm{Pf}(\mathfrak{A}) = \frac{(-1)^l}{2^l l!}\sum_{\sigma\in S_{2l}} (-1)^{|\sigma|} \mathfrak{A}_{\sigma(1)\sigma(2)} \cdots \mathfrak{A}_{\sigma(2l-1)\sigma(2l)}.$$

Finally, using these Grassmann integrals, one can show that the determinant of a $2l \times 2l$ skew-symmetric matrix $\mathfrak{A}$ is the square of the Pfaffian:

$$\det\mathfrak{A} = \mathrm{Pf}(\mathfrak{A})^2.$$

So, by simply considering one instead of two independent Grassmann variables in the Grassmann integral of $e^{\theta^T \mathfrak{A} \eta}$, we are in effect taking the square root of a determinant.

Notes

Section 9.1. The two-point space

1. The two-point space was first studied in [**59, 71**].

2. The need for KO-dimension 6 for the noncommutative description of the Standard Model has been observed independently by Barrett [**20**] and Connes [**64**].

3. In [**131**, Chapter 9] a proof is given for the claim that the inner fluctuation $\omega + J\omega J^{-1}$ vanishes for commutative algebras. The proof is based on the assumption that the left and right action can be identified, *i.e.* $a = a^0$, for a commutative algebra. Though this holds in the case of the canonical triple describing a spin manifold, it need not be true for arbitrary commutative algebras. Indeed, the almost-commutative manifold $M \times F_X$ provides a counter-example.

What we can say about a commutative algebra, is that there exist no non-trivial inner automorphisms. Thus, it is an important insight that the gauge group $\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$ from Definition 6.4 is larger than the group of inner automorphisms, so that a commutative algebra may still lead to a non-trivial (necessarily abelian) gauge group.

4. It is shown in [**28**] that one can also obtain abelian gauge theories from a one-point space when one works with real algebras (*cf.* Section 3.3).

Section 9.2. Electrodynamics

5. Earlier attempts at a unified description of gravity and electromagnetism originate from the work of Kaluza [**119**] and Klein [**125**] in the 1920's. In their approach, a new (compact) fifth dimension is added to the 4-dimensional spacetime M. The additional components in the 5-dimensional metric tensor are then identified with the electromagnetic gauge potential. Subsequently, it can be shown that the Einstein equations of the 5-dimensional spacetime can be reduced to the Einstein equations plus the Maxwell equations on 4-dimensional spacetime.

6. An interesting question that appears in the context of this Chapter is whether it is possible to describe the *abelian Higgs mechanism* (see *e.g.* [**120**, Section 8.3]) by an almost-commutative manifold. As already noticed, for $M \times F_{ED}$ no scalar fields Φ are generated since A_F commutes with D_F. In terms of the Krajewski diagram for $M \times F_{ED}$,

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

it follows that a component that runs counterdiagonally fails on the first-order condition (*cf.* Lemma 3.10). One is therefore tempted to look at the generalization of inner fluctuations to real spectral triples that do not necessarily satisfy the first-order condition, as was proposed in [**56**]. This generalization is crucial in the applications to Pati–Salam unification (see Note 13 on Page 185), but also in the present case one can show that non-zero off-diagonal components in (9.2.3) then generate a scalar field for which the spectral action yields a spontaneous breaking of the abelian gauge symmetry.

Section 9.A. Grassmann variables, Grassmann integration and Pfaffians

7. For more details we refer the reader to [**26**].

CHAPTER 10

The noncommutative geometry of Yang–Mills fields

In this Chapter we generalize the noncommutative description of Yang–Mills theory to topologically non-trivial gauge configurations.

10.1. Spectral triple obtained from an algebra bundle

Recall from Examples 8.4 and 8.5 that topologically trivial Yang–Mills gauge theory can be described by the almost-commutative manifold

$$M \times F_{YM} = \big(C^\infty(M) \otimes M_N(\mathbb{C}), L^2(S) \otimes M_N(\mathbb{C}), D_M \otimes 1; J_M \otimes (\cdot)^*, \gamma_M \otimes 1\big)\,.$$

In fact, the tensor product of $C^\infty(M)$ with the matrix algebra $M_N(\mathbb{C})$ appearing here is equivalent to restricting the gauge theory to be defined on a *trivial* vector bundle. Indeed, $C^\infty(M) \otimes M_N(\mathbb{C})$ is the algebra of smooth sections of the trivial algebra bundle $M \times M_N(\mathbb{C})$ on M. For the topologically non-trivial case, this suggests considering an arbitrary $*$-algebra bundle with fiber $M_N(\mathbb{C})$. We work in a slightly more general setting more general $*$-algebras are allowed.

Thus, let $\mathfrak{B}$ be some locally trivial $*$-algebra bundle whose fibers are copies of a fixed (finite-dimensional) $*$-algebra A. Furthermore, we require that for each x the fiber $\mathfrak{B}_x$ is endowed with a faithful tracial state τ_x, such that for each $s \in \Gamma^\infty(\mathfrak{B})$ the function $x \mapsto \tau_x s(x)$ is smooth. The corresponding Hilbert–Schmidt inner product in the fiber $\mathfrak{B}_x$ that is induced by τ_x is denoted by $(\cdot,\cdot)_{\mathfrak{B}_x}$. Consequently, the $C^\infty(M)$-valued form

$$\langle\cdot,\cdot\rangle_{\mathfrak{B}} : \Gamma^\infty(\mathfrak{B}) \times \Gamma^\infty(\mathfrak{B}) \to C^\infty(M); \qquad \langle s,t\rangle_{\mathfrak{B}}(x) = (s(x),t(x))_{\mathfrak{B}_x}$$

is a hermitian structure on the $C^\infty(M)$-module $\Gamma^\infty(\mathfrak{B})$, satisfying the conditions of Proposition 6.14.

As in the previous chapters, we assume that M is a compact Riemannian spin manifold on which $S \to M$ is a spinor bundle and $D_M = -ic \circ \nabla^S$ is the Dirac operator. Combining the inner product on spinors with the above hermitian structure naturally induces the following inner product on $\Gamma^\infty(\mathfrak{B} \otimes S)$:

$$(\xi_1,\xi_2) := \int_M (\xi_1(x),\xi_2(x))_{\mathfrak{B}_x \otimes S_x}\,; \quad (\xi_1,\xi_2 \in \Gamma^\infty(\mathfrak{B} \otimes S)), \tag{10.1.1}$$

turning it into a pre-Hilbert space. Its completion with respect to the norm induced by this inner product consists of all square-integrable sections of $\mathfrak{B} \otimes S$, and is denoted by $L^2(\mathfrak{B} \otimes S)$.

Remark 10.1. *Note that we can identify* $\Gamma^\infty(\mathfrak{B}) \otimes_{C^\infty(M)} \Gamma^\infty(S)$ *with* $\Gamma^\infty(\mathfrak{B} \otimes S)$ *as* $C^\infty(M)$*-modules. In what follows, we will use this identification without further notice. The above inner product* (10.1.1) *can then be written as*

$$(s_1 \otimes \psi_1, s_2 \otimes \psi_2) = (\psi_1, \langle s_1, s_2 \rangle_{\mathfrak{B}} \psi_2),$$

where $\langle s_1, s_2 \rangle_{\mathfrak{B}} \in C^\infty(M)$ *acts on* $\Gamma^\infty(S)$ *by pointwise multiplication.*

Theorem 10.2. *In the above notation, let* $\nabla^{\mathfrak{B}}$ *be a hermitian connection (with respect to the Hilbert–Schmidt inner product) on the* $*$*-algebra bundle* $\mathfrak{B}$ *and let* $D_{\mathfrak{B}} = -i\gamma^\mu(\nabla^{\mathfrak{B}}_\mu \otimes 1 + 1 \otimes \nabla^S_\mu)$ *be the twisted Dirac operator on* $\mathfrak{B} \otimes S$*. Then*

$$(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}})$$

is a spectral triple.

Proof. First, it is obvious that fiberwise multiplication of $a \in \Gamma^\infty(\mathfrak{B})$ on $\Gamma^\infty(\mathfrak{B} \otimes S)$ extends to a bounded operator on $L^2(\mathfrak{B} \otimes S)$, since

$$\begin{aligned} \|as \otimes \psi\|^2 &= \int_M \big(\psi(x), (a(x)s(x), a(x)s(x))_{\mathfrak{B}_x} \psi(x)\big)_{S_x} dx \\ &\leq \sup_{x \in M} \{\|a(x)\|_x^2\} \|s \otimes \psi\|^2. \end{aligned}$$

Here $\|\cdot\|_x$ denotes the fiberwise operator C^*-norm. Since M is a compact manifold, the compactness of the resolvent follows from ellipticity of the twisted Dirac operator $D_{\mathfrak{B}}$. Moreover, the commutator $[D_{\mathfrak{B}}, a]$ is bounded for $a \in \Gamma^\infty(\mathfrak{B})$ since $D_{\mathfrak{B}}$ is a first-order differential operator. More precisely, in local coordinates one computes

$$[D_{\mathfrak{B}}, a](s \otimes \psi) = -i\left(\partial_\mu a + [\omega^{\mathfrak{B}}_\mu, a]\right) s \otimes \gamma^\mu \psi,$$

where $\nabla^{\mathfrak{B}}_\mu = \partial_\mu + \omega^{\mathfrak{B}}_\mu$. This operator is bounded on $L^2(\mathfrak{B} \otimes S)$, provided a is differentiable and $\omega^{\mathfrak{B}}_\mu$ is smooth. □

Next, we would like to extend our construction to arrive at a real spectral triple. For this, we introduce an anti-linear operator on $L^2(\mathfrak{B} \otimes S)$ of the form

$$J(s \otimes \psi) = s^* \otimes J_M \psi,$$

with J_M charge conjugation on M as in Definition 4.13. For this operator to be a real structure on our spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}})$, we need some extra conditions on the connection $\nabla^{\mathfrak{B}}$ on $\mathfrak{B}$.

Definition 10.3. *Let* $\mathfrak{B}$ *be a* $*$*-algebra bundle over a manifold* M*. A* $*$-algebra connection ∇ *on* $\mathfrak{B}$ *is a connection on* $\mathfrak{B}$ *that satisfies*

$$\nabla(st) = s\nabla t + (\nabla s)t, \qquad (\nabla s)^* = \nabla s^*; \qquad (s, t \in \Gamma^\infty(\mathfrak{B})).$$

If $\mathfrak{B}$ *is a hermitian* $*$*-algebra bundle and* ∇ *is also a hermitian connection, then* ∇ *is called a* hermitian $*$*-algebra connection.*

Lemma 10.4. *Every locally trivial hermitian* $*$*-algebra bundle* $\mathfrak{B}$ *defined over a compact space* M *admits a hermitian* $*$*-algebra connection.*

Proof. Let $\{U_i\}$ be a finite open covering of M such that $\mathfrak{B}$ is trivialized over U_i for each i. Then on each U_i there exists a hermitian $*$-algebra connection ∇_i, for instance the trivial connection d on U_i. Now, let $\{f_i\}$ be a partition of unity subordinate to the open covering $\{U_i\}$ (note that all f_i are real-valued). Then the linear map ∇ defined by

$$(\nabla s)(x) = \sum_i f_i(x)(\nabla_i s)(x); \qquad (x \in M)$$

is a hermitian $*$-algebra connection on $\Gamma^\infty(\mathfrak{B})$. $\square$

Remark 10.5. *The fact that locally, i.e. on some trivializing neighborhood, the exterior derivative d is a hermitian $*$-algebra connection shows that on such a local chart every hermitian $*$-algebra connection is of the form*

$$d + \omega^{\mathfrak{B}},$$

where $\omega^{\mathfrak{B}}$ is a real connection one-form with values in the real Lie algebra of $*$-derivations of the fiber that are anti-hermitian with respect to the inner product on the fiber. For instance, when the fiber is the $*$-algebra $M_N(\mathbb{C})$ endowed with the Hilbert–Schmidt inner product, this Lie algebra is precisely $\mathrm{ad}(u(N)) \cong su(N)$.

Theorem 10.6. *In addition to the conditions of Theorem 10.2, suppose that $\nabla^{\mathfrak{B}}$ is a hermitian $*$-algebra connection and set $\gamma = 1 \otimes \gamma_M$ as a self-adjoint operator on $L^2(\mathfrak{B} \otimes S)$. Then*

$$(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$$

is a real and even spectral triple whose KO-dimension is equal to the dimension of M.

Proof. First of all, we check that J is anti-unitary:

$$\begin{aligned}(J(s\otimes\psi), J(t\otimes\eta)) &= (J_M\psi, \langle s^*, t^*\rangle J_M\eta) = \Big(J_M\psi, J_M\overline{\langle s^*, t^*\rangle}\eta\Big) \\ &= \Big(\overline{\langle s^*, t^*\rangle}\eta, \psi\Big) = (\langle s, t\rangle\eta, \psi) = (t\otimes\eta, s\otimes\psi)\,,\end{aligned}$$

where we used in the second step that $J_M f = \overline{f} J_M$ for every $f \in C^\infty(M)$, in the third step that J_M is anti-unitary, and in the fourth step that $\langle s, t\rangle = \langle t^*, s^*\rangle$ (by definition of the hermitian structure as a fiberwise trace). Moreover, if $J_M^2 = \epsilon$ it follows that $J^2 = \epsilon$.

We next establish $DJ = \epsilon' JD$ by a local calculation:

$$\begin{aligned}(JD - \epsilon' DJ)(s\otimes\psi) &= J\Big(\nabla^{\mathfrak{B}}_\mu s \otimes (-i\gamma^\mu\psi) + s\otimes D_M\psi\Big) - \epsilon' D_{\mathfrak{B}}(s^*\otimes J_M\psi) \\ &= (\nabla^{\mathfrak{B}}_\mu s)^* \otimes iJ_M\gamma^\mu\psi + s^*\otimes J_M D_M\psi \\ &\quad - \epsilon'\nabla^{\mathfrak{B}}_\mu s^* \otimes (-i\gamma^\mu J_M\psi) - \epsilon' s^*\otimes D_M J_M\psi \\ &= i\Big((\nabla^{\mathfrak{B}}_\mu s)^* - \nabla^{\mathfrak{B}}_\mu s^*\Big)\otimes J_M\gamma^\mu\psi = 0,\end{aligned}$$

since $J_M\gamma^\mu = -\epsilon'\gamma^\mu J_M$, and the last step follows from the definition of a $*$-algebra connection, *i.e.* $(\nabla s)^* = \nabla s^*$ for all $s \in \Gamma^\infty(\mathfrak{B})$.

The commutant property follows easily:

$$\begin{aligned}[a, b^0](s \otimes \psi) &= aJb^*J^{-1}(s \otimes \psi) - Jb^*J^{-1}a(s \otimes \psi) \\ &= aJ(b^*s^* \otimes J_M^*\psi) - Jb^*(s^*a^* \otimes J_M^*\psi) \\ &= asb \otimes \psi - asb \otimes \psi = 0,\end{aligned}$$

where $a, b \in \Gamma^\infty(\mathfrak{B})$ and $s \otimes \psi \in \Gamma^\infty(\mathfrak{B}) \otimes_{C^\infty(M)} \Gamma^\infty(S)$. Since $[a, b^0] = 0$ on $\Gamma^\infty(\mathfrak{B}) \otimes_{C^\infty(M)} \Gamma^\infty(S) \cong \Gamma^\infty(\mathfrak{B} \otimes S)$, it is zero on the entire Hilbert space $L^2(\mathfrak{B} \otimes S)$. It remains to check the order one condition for the Dirac operator. First note that

$$[[D, a], b^0](s \otimes \psi) = -i\gamma^\mu([[\nabla_\mu, a], b^0](s \otimes \psi)); \qquad (a, b, s \in \Gamma^\infty(\mathfrak{B})).$$

This is zero because $[[\nabla, a], b^0](s \otimes \psi)$ is zero:

$$\begin{aligned}&([\nabla_\mu, a]sb) \otimes \psi - Jb^*J^{-1}([\nabla_\mu, a]s \otimes \psi) \\ &\quad = \nabla_\mu(asb) \otimes \psi - a\nabla_\mu(sb) \otimes \psi - \nabla_\mu(as)b \otimes \psi + a(\nabla_\mu s)b \otimes \psi \\ &\quad = ((\nabla_\mu a)sb + a(\nabla_\mu s)b + as(\nabla_\mu b) - a(\nabla_\mu s)b \\ &\qquad - as(\nabla_\mu b) - (\nabla_\mu a)sb - a(\nabla_\mu s)b + a(\nabla_\mu s)b) \otimes \psi, \\ &\quad = 0\end{aligned}$$

using the defining property for $\nabla^{\mathfrak{B}}$ to be a $*$-algebra connection. Thus, J fulfills all of the necessary conditions for a real structure on the spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}})$. The conditions on γ to be a grading operator for this spectral triple are easily checked too. □

10.2. Yang–Mills theory as a noncommutative manifold

The real spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$ that we obtained in Theorem 10.6 will turn out to be the correct triple to describe a topologically non-trivial $PU(N)$-gauge theory on the spin manifold M if the fibers of $\mathfrak{B}$ are taken to be isomorphic to the $*$-algebra $M_N(\mathbb{C})$. Moreover, this triple not only describes a non-trivial $PU(N)$-gauge theory: every $PU(N)$-gauge theory on M is described by such a triple. In this section we prove these claims by first showing how a principal $PU(N)$-bundle can be constructed from this spectral triple. As in the topologically trivial case (*cf.* Remark 8.13) the spectral action applied to this triple will give the Einstein–Yang–Mills action, but now the gauge potential can be interpreted as a connection one-form on the $PU(N)$-bundle P. In fact, the original algebra bundle $\mathfrak{B}$ will turn out to be an associated bundle of the principal bundle P. From now on, then, the fibers of $\mathfrak{B}$ are assumed to be $M_N(\mathbb{C})$.

10.2.1. From algebra bundles to principal bundles. In order to construct a principal $PU(N)$-bundle P out of $\mathfrak{B}$, first of all note that since all $*$-automorphisms of $M_N(\mathbb{C})$ are obtained by conjugation with a unitary element $u \in M_N(\mathbb{C})$ (see Example 6.3), the transition functions of the bundle $\Gamma^\infty(\mathfrak{B})$ take their values in

$$\mathrm{Ad}\, U(N) \cong U(N)/Z(U(N)) \cong PU(N).$$

Thus the bundle $\mathfrak{B}$ provides us with an open covering $\{U_i\}$ of M as well as transition functions $\{g_{ij}\}$ with values in $PU(N)$. Using the reconstruction theorem for principal bundles, we can then construct a principal $PU(N)$-bundle. By construction, the bundle $\mathfrak{B}$ is an associated bundle to P.

Furthermore, for the real spectral triple

$$(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$$

of Theorem 10.6, the hermitian connection $\nabla^{\mathfrak{B}}$ on the bundle $\mathfrak{B}$ can locally be written as $\nabla^{\mathfrak{B}} = d + \omega^{\mathfrak{B}}$, where $\omega^{\mathfrak{B}}$ is a $su(N)$-valued one-form, (*cf.* Remark 10.5). Moreover, the transformation rule for $\omega^{\mathfrak{B}}$ is $\omega_i^{\mathfrak{B}} = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1} \omega_j^{\mathfrak{B}} g_{ij}$, with g_{ij} the $PU(N)$-valued transition function of $\mathfrak{B}$. Comparing this expression with the usual transformation property of a connection one-form, one concludes that the hermitian $*$-algebra connection $\nabla^{\mathfrak{B}}$ on $\mathfrak{B}$ induces a connection one-form on the principal bundle P constructed in the previous paragraph.

Conversely, given a $PU(N)$-gauge theory (P, ω^P) on some compact Riemannian spin manifold, we can construct the locally trivial hermitian $*$-algebra bundle $\mathfrak{B} := P \times_{PU(N)} M_N(\mathbb{C})$, where $PU(N)$ acts on $M_N(\mathbb{C})$ in the usual way. Moreover, the connection ω^P on P induces a hermitian $*$-algebra connection on $\mathfrak{B}$. Following the steps described in the previous paragraph, it is not difficult to see that the principal bundle and connection obtained from the ensuing spectral triple,

$$(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), -i\gamma^\mu(\nabla_\mu^{\mathfrak{B}} \otimes 1 + 1 \otimes \nabla_\mu^S); J, \gamma),$$

coincide with (P, ω^P).

Proposition 10.7. *Let $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$ be as before with M simply connected and $\mathfrak{B}$ a locally trivial $*$-algebra bundle with fiber $M_N(\mathbb{C})$ and a faithful smoothly-varying tracial state. Then:*

(1) there exists a principal $PU(N)$-bundle P such that $\mathfrak{B}$ is an associated bundle of P, as well as a connection one-form ω^P on P corresponding to $\nabla^{\mathfrak{B}}$;

(2) the gauge group $\mathfrak{G}(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S); J)$ of this spectral triple (as in Definition 6.4) is isomorphic to the space of smooth sections of the associated group bundle $\mathrm{Ad}\, P := P \times_{PU(N)} PU(N)$.

Every $PU(N)$-gauge theory (P, ω^P) on M is determined by such a spectral triple.

Proof. The only statement left to prove is (2). If $\mathfrak{B} = P \times_{PU(N)} M_N(\mathbb{C})$, then $\mathcal{U}(\Gamma^\infty(\mathfrak{B})) = \Gamma^\infty(P \times_{PU(N)} U(N))$. As a consequence,

$$\begin{aligned} \mathfrak{G}(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S); J) &\simeq \{uJuJ^{-1} : u \in \Gamma^\infty(P \times_{PU(N)} U(N)\} \\ &\simeq \Gamma^\infty(P \times_{PU(N)} PU(N)), \end{aligned}$$

where we argue as in the proof of Proposition 8.2 (see also Note 4 on Page 125). ☐

10.2.2. Inner fluctuations and spectral action. In this section, we calculate the spectral action for the real spectral triple of Theorem 10.6 in the case that $\dim M = 4$. We show that the spectral action applied to the spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$ produces the Einstein–Yang–Mills action for a connection one-form on the $PU(N)$-bundle P. If $\mathfrak{B}$ is a trivial algebra bundle, this reduces to Example 8.13. In fact, most of these local computations can be adopted in this case as well, since locally the bundle $\mathfrak{B}$ is trivial. Nevertheless, for completeness we include the computations in the case at hand.

First of all, in Remark 10.5 we noticed that locally, *i.e.* on some local trivialization U, the connection $\nabla^{\mathfrak{B}}$ is expressed as $d+\omega^{\mathfrak{B}}$, where $\omega^{\mathfrak{B}}$ is an $su(N)$-valued one-form that acts in the adjoint representation on $\Gamma^\infty(\mathfrak{B})$. Therefore, $\omega^{\mathfrak{B}}$ already induces a connection one-form on P. To get the full gauge potential we need to take the fluctuations of the Dirac operator into account as well.

Recall from Section 6.2 that inner fluctuations of the Dirac operator are given by a perturbation term of the form

$$\omega = \sum_j a_j[D, b_j]; \qquad (a_j, b_j \in \Gamma(\mathfrak{B})), \tag{10.2.1}$$

with the additional condition that $\sum_j a_j[D,b_j]$ is a self-adjoint operator. Explicitly, we have

$$\omega = \sum_j -i\gamma^\mu \circ (a_j[\nabla_\mu, b_j] \otimes 1).$$

Locally, on some trivializing neighborhood U, the expression in (10.2.1) can be written as

$$\omega = \gamma^\mu A_\mu,$$

where A_μ are the components of the one-form $\sum_j a_j[\nabla, b_j]$ with values in $\Gamma^\infty(\mathfrak{B})$. Since ω is self-adjoint, the one-form A_μ can be considered a real one-form taking values in the hermitian elements of $\Gamma^\infty(\mathfrak{B})$.

Similarly, the expression $\omega + J\omega J^{-1}$ is locally written as

$$\gamma^\mu A_\mu - \gamma^\mu J A_\mu J^{-1},$$

since in 4 dimensions γ^μ anti-commutes with J. Writing out the second term gives:

$$(\gamma^\mu J A_\mu J^{-1})(s \otimes \psi) = s A_\mu \otimes \gamma^\mu \psi; \qquad (s \otimes \psi \in \Gamma^\infty(\mathfrak{B} \otimes S)),$$

so that on this local patch, $\omega + J\omega J^{-1}$ can be written as

$$\gamma^\mu \operatorname{ad} A_\mu.$$

Consequently, $\omega + J\omega J^{-1}$ eliminates the $iu(1)$-part of ω, so that ω effectively satisfies the *unimodularity condition*

$$\operatorname{Tr} \omega = 0.$$

Thus, $i \operatorname{ad} A_\mu$ is a one-form on M with values in $\Gamma^\infty(\operatorname{ad} P)$ where $\operatorname{ad} P = P \times_{PU(N)} su(N)$.

The expression for $D + \omega + J\omega J^{-1}$ on a local chart U is then given by

$$D_\omega = -i\gamma^\mu(\nabla^{\mathfrak{B}}_\mu \otimes 1 + 1 \otimes \nabla^S_\mu + i \operatorname{ad} A_\mu \otimes 1),$$

where the connection $\nabla^{\mathfrak{B}}$ can be expressed on U as $d + \omega^{\mathfrak{B}}$ for some unique $su(N)$-valued one-form $\omega^{\mathfrak{B}}$ on U. Thus, on U the fluctuated Dirac operator can be rewritten as

$$D_\omega = -i\gamma^\mu(1 \otimes \nabla^S_\mu + (\partial_\mu + \omega^{\mathfrak{B}}_\mu + i \operatorname{ad} A_\mu) \otimes 1).$$

We interpret $(\omega_\mu^{\mathfrak{B}} + i \operatorname{ad} A_\mu)$ as the full gauge potential on U, acting in the adjoint representation on the spinors. The natural action of an element g in the group $\mathfrak{G}(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S); J) \simeq \Gamma^\infty(\operatorname{Ad} P)$ by conjugation on D_ω then induces the familiar gauge transformation:

$$\omega_\mu^{\mathfrak{B}} + i \operatorname{ad} A_\mu \mapsto (g^{-1}\omega_\mu^{\mathfrak{B}} g + g^{-1}(dg)) + g^{-1}(i \operatorname{ad} A_\mu)g,$$

where the first two terms on the right-hand side are the transformation of $\omega^{\mathfrak{B}}$ under a change of local trivialization, and the last term is the transformation of $i \operatorname{ad} A_\mu$. Therefore, since $\mathfrak{B}$ is an associated bundle of P, it follows that $\omega_\mu^{\mathfrak{B}} + i \operatorname{ad} A_\mu$ induces a $su(N)$-valued connection one-form on the principal $PU(N)$-bundle P that acts on $\Gamma^\infty(\mathfrak{B})$ in the adjoint representation.

Let us summarize what we have obtained so far.

PROPOSITION 10.8. *Let $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B}\otimes S), D_{\mathfrak{B}}; J, \gamma)$ and let P be as before, so that $P \times_{PU(N)} M_N(\mathbb{C}) \simeq \mathfrak{B}$. Then, the inner fluctuations of $D_{\mathfrak{B}}$ are parametrized by sections of $\Gamma^\infty(T^*M \otimes \operatorname{ad} P)$ where $\operatorname{ad} P = P \times_{PU(N)} su(N)$. Moreover, the action of $\mathfrak{G}(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S); J)$ on the inner fluctuations of $D_{\mathfrak{B}}$ by conjugation coincides with the adjoint action of $\Gamma^\infty(\operatorname{Ad} P)$ on $\Gamma^\infty(\operatorname{ad} P)$.*

Let us now proceed to compute the spectral action for these inner fluctuations. We apply the results of Section 8.3, using the following result.

LEMMA 10.9. *For the spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$, the square of the fluctuated Dirac operator is a generalized Laplacian of the form $\Delta^E - F$, with $E = \mathfrak{B} \otimes S$ (notation as in Theorem 8.7), and we have the following local expressions for the corresponding curvature $\Omega^E_{\mu\nu}$ and the bundle endomorphism F:*

$$\begin{aligned} F &= -\frac{1}{4} s \otimes \mathbb{I}_{N^2} + \frac{1}{2} i\gamma^\mu\gamma^\nu \otimes F_{\mu\nu}; \\ \Omega^E_{\mu\nu} &= \Omega^S_{\mu\nu} \otimes \mathbb{I}_{N^2} + i\mathbb{I}_4 \otimes F_{\mu\nu}, \end{aligned}$$

where $F_{\mu\nu}$ is the curvature of the connection $\nabla_\mu^{\mathfrak{B}} + i \operatorname{ad} A_\mu$.

As before, this result allows us to compute the bosonic spectral action for the fluctuated Dirac operator D_ω, essentially reducing the computation in terms of a local trivialization to the trivial case (*cf.* Example 8.13), with the following result.

THEOREM 10.10. *For the spectral triple $(\Gamma^\infty(\mathfrak{B}), L^2(\mathfrak{B} \otimes S), D_{\mathfrak{B}}; J, \gamma)$, the spectral action yields the Yang–Mills action for $\nabla^{\mathfrak{B}} + i \operatorname{ad} A_\mu$ minimally coupled to gravity:*

$$\operatorname{Tr}\left(f(D_\omega/\Lambda)\right) \sim \frac{f(0)}{24\pi^2}\int_M \operatorname{Tr} F_{\mu\nu}F^{\mu\nu}\sqrt{g}dx + N^2 \int_M \mathcal{L}_M(g_{\mu\nu})\sqrt{g}dx,$$

asymptotically as $\Lambda \to \infty$ and up to terms $\propto \Lambda^{-2}$. The Lagrangian $\mathcal{L}_M(g^{\mu\nu})$ is given by (8.4.8).

10.2.3. Topological spectral action. A natural invariant in this topologically non-trivial context is the topological spectral action, given in Equation (7.1.2). With Proposition 7.5 we find that, in general,

$$S_{\text{top}}[\omega] = f(0)\,\text{index}\, D_\omega.$$

Hence, in the setting of Theorem 10.10, using the Atiyah–Singer index theorem (*cf.* Note 16 on Page 82), we find an extra contribution of the form

$$S_{\text{top}}[\omega] = \frac{f(0)}{(2\pi i)^{n/2}} \int_M \hat{A}(M)\,\text{ch}(\mathfrak{B}),$$

in terms of the $\hat{A}$-form of M and the Chern character of the algebra bundle $\mathfrak{B}$.

Notes

1. For an exposition of Yang–Mills theory in terms of principal bundle and connections, we refer to [**7**, Section 2,3] and [**32**].

2. This Chapter extends the noncommutative description of Yang–Mills gauge theory of [**49, 50**] to the topologically non-trivial case; it is based on [**35**]. For a more general treatment of topologically non-trivial almost-commutative geometries we refer to [**40, 41, 34**].

Section 10.1. Spectral triple obtained from an algebra bundle

3. Our approach to locally trivial $*$-algebra bundles gains in substance with the Serre–Swan Theorem, establishing a duality between vector bundles over a topological space X and finite projective modules over $C(X)$ [**169, 182**]. A smooth version was obtained in [**61**] (see also [**131**, Proposition 4.2.1] or [**103**, Section 2.3]). The fiberwise inner product gives rise to the hermitian structure found in Proposition 6.14. A version of the Serre–Swan Theorem for $*$-algebra bundles has been obtained in [**35**].

Section 10.2. Yang–Mills theory as a noncommutative manifold

4. A special case of Proposition 10.7 occurs when $\mathfrak{B}$ is an endomorphism bundle. It follows from a result by Dixmier and Douady in [**77**] (*cf.* [**164**]) that a bundle $\mathfrak{B}$ with continuously varying trace is an endomorphism bundle if and only if the Dixmier–Douady class $\delta(\Gamma(\mathfrak{B})) \in H^3(\mathbb{Z})$ of the C^*-algebra of continuous sections $\Gamma(\mathfrak{B})$ of this bundle is equal to zero. Because the Dixmier–Douady class of the bundle $\mathfrak{B}$ vanishes one can lift the $PU(N)$-valued transition functions g_{ij} to $U(N)$-valued functions μ_{ij} such that $g_{ij} = \text{Ad}\,\mu_{ij}$, and $\mu_{ij}\mu_{jk} = \mu_{ik}$ (see for instance [**164**], Theorem 4.85). One may therefore construct a principal $U(N)$-bundle instead of a $PU(N)$-bundle, to which $\mathfrak{B}$ is associated if and only if $\mathfrak{B}$ is an endomorphism bundle.

CHAPTER 11

The noncommutative geometry of the Standard Model

One of the major applications of noncommutative geometry to physics has been the derivation of the Standard Model of particle physics from a suitable almost-commutative manifold. In this Chapter we present this derivation, using the results of Chapter 8.

11.1. The finite space

Our starting point is the classification of irreducible finite geometries of KO-dimension 6 from Section 3.4, based on the matrix algebra $M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ for $N \geq 1$. We have already seen in Chapter 9 that $N = 1$ is the finite geometry corresponding to electrodynamics. We now proceed and aim for the full Standard Model of particle physics. Let us make the following two additional requirements on the irreducible finite geometry $(A, H_F, D_F; J_F, \gamma_F)$:

(1) The finite-dimensional Hilbert space H_F carries a symplectic structure $I^2 = -1$;
(2) the grading γ_F induces a non-trivial grading on A, by mapping

$$a \mapsto \gamma_F a \gamma_F,$$

and selects an even subalgebra $A^{\mathrm{ev}} \subset A$ consisting of elements that commute with γ_F.

We have already seen in Section 3.4 that the first demand sets $A = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$, represented on the Hilbert space $\mathbb{C}^{2(2k)^2}$. The second requirement sets $k \geq 2$; we will take the simplest $k = 2$ so that $H_F = \mathbb{C}^{32}$. Indeed, this allows for a γ_F such that

$$A^{\mathrm{ev}} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}),$$

where $\mathbb{H}_R$ and $\mathbb{H}_L$ are two copies (referred to as *right* and *left*) of the quaternions; they are the diagonal of $M_2(\mathbb{H}) \subset A$. The Hilbert space can then be decomposed according to the defining representations of A^{ev},

$$H_F = (\mathbb{C}^2_R \oplus \mathbb{C}^2_L) \otimes \mathbb{C}^{4\circ} \oplus \mathbb{C}^4 \otimes (\mathbb{C}^{2\circ}_R \oplus \mathbb{C}^{2\circ}_L). \tag{11.1.1}$$

According to this direct sum decomposition, we write

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix} \tag{11.1.2}$$

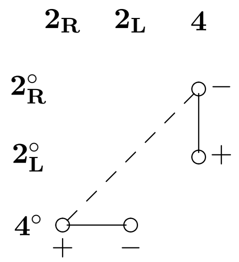


FIGURE 11.1. The Krajewski diagram for the finite real spectral triple $(A^{\mathrm{ev}} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}), H_F, D_F; J_F, \gamma_F)$. The dashed line corresponds to an 'off-diagonal' component of the Dirac operator, thus failing on the first-order condition. The labels + and − represent the value of the grading γ_F on the corresponding summands of H_F.

where

$$S : (\mathbb{C}^2_R \oplus \mathbb{C}^2_L) \otimes \mathbb{C}^{4\circ} \to (\mathbb{C}^2_R \oplus \mathbb{C}^2_L) \otimes \mathbb{C}^{4\circ},$$
$$T : (\mathbb{C}^2_R \oplus \mathbb{C}^2_L) \otimes \mathbb{C}^{4\circ} \to \mathbb{C}^4 \otimes (\mathbb{C}^{2\circ}_R \oplus \mathbb{C}^{2\circ}_L).$$

This gives rise to the Krajewski diagram of Figure 11.1. We now make an additional assumption,

(3) The off-diagonal components T and T^* of the Dirac operator in (11.1.2) are non-zero.

In Figure 11.1 such an off-diagonal component corresponds to the dashed line. As this line runs neither vertically, horizontally, or between the same vertex, it follows from Lemma 3.10 that the corresponding component of D_F breaks the first-order condition.

PROPOSITION 11.1. *Up to $*$-automorphisms of A^{ev}, there is a unique $*$-subalgebra $A_F \subset A^{\mathrm{ev}}$ of maximal dimension that allows $T \neq 0$ in* (11.1.2). *It is given by*

$$A_F = \left\{ \left(q_\lambda, q, \begin{pmatrix} q & 0 \\ 0 & m \end{pmatrix} \right) : \lambda \in \mathbb{C}, q \in \mathbb{H}_L, m \in M_3(\mathbb{C}) \right\} \subset \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}),$$

where $\lambda \mapsto q_\lambda$ is the embedding of $\mathbb{C} \hookrightarrow \mathbb{H}$, with

$$q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}.$$

Consequently, $A_F \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

PROOF. We give a diagrammatic proof. From Figure 11.1, we see that in order to fulfill the first-order condition, we should bring the dashed line to run horizontally or vertically, or to begin and start at the same node on the diagonal. We do so by considering the Krajewski diagrams for subalgebras $A_F \subset A^{\mathrm{ev}}$ which are induced by Figure 11.1. If T is of rank 1, the only possibility is to bring the dashed line to the diagonal. In other words, the subalgebra we are looking for should have a component that is embedded diagonally in $\mathbb{H}_R$ and $M_4(\mathbb{C})$. Such a component can only be $\mathbb{C}$, and the resulting subalgebra is embedded as

$$\begin{aligned}\mathbb{C} \oplus M_3(\mathbb{C}) &\to \mathbb{H}_R \oplus M_4(\mathbb{C});\\ (\lambda, m) &\mapsto \left(\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & m \end{pmatrix} \right).\end{aligned}$$

This breaks the Krajewski diagram to the diagram of Figure 11.2, where the dashed line now connects the two vertices labeled by $(\mathbf{1}, \mathbf{1}^\circ)$. The other edges of Figure 11.1 are now torn apart to the resulting edges in Figure 11.2.

If T has rank greater than 1, then a similar argument shows that one obtains a subalgebra of smaller dimension than A_F. $\square$

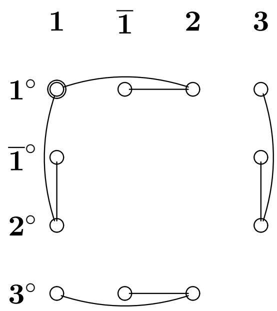


FIGURE 11.2. The Krajewski diagram of the space F_{SM} describing the Standard Model.

In order to connect to the physics of the Standard Model, let us introduce an orthonormal basis for H_F that can be recognized as the fermionic particle content of the Standard Model, and subsequently write the representation of A_F in terms of this basis. Starting with the Krajewski diagram of Figure 11.2, we let the first three nodes in the top row be represented by basis vectors $\{\nu_R, e_R, (\nu_L, e_L)\}$ of the so-called **lepton space** H_l, while the three nodes in the bottom row represent the basis vectors $\{u_R, d_R, (u_L, d_L)\}$ of the **quark space** H_q. Their reflections with respect to the diagonal represent are the **anti-lepton space** $H_{\overline{l}}$ and the **anti-quark space** $H_{\overline{q}}$, spanned by $\{\overline{\nu_R}, \overline{e_R}, (\overline{\nu_L}, \overline{e_L})\}$ and $\{\overline{u_R}, \overline{d_R}, (\overline{u_L}, \overline{d_L})\}$, respectively. The three colors of the quarks are given by a tensor factor $\mathbb{C}^3$ and when we take into account *three generations* of fermions and anti-fermions by tripling the above finite-dimensional Hilbert space we obtain

$$H_F := \left(H_l \oplus H_{\overline{l}} \oplus H_q \oplus H_{\overline{q}} \right)^{\oplus 3}.$$

Note that $H_l = \mathbb{C}^4$, $H_q = \mathbb{C}^4 \otimes \mathbb{C}^3$, $H_{\overline{l}} = \mathbb{C}^4$, and $H_{\overline{q}} = \mathbb{C}^4 \otimes \mathbb{C}^3$. An element $a = (\lambda, q, m) \in A_F$ acts on the space of leptons H_l as $q_\lambda \oplus q$, and acts on the space of quarks H_q as $(q_\lambda \oplus q) \otimes \mathbb{I}_3$. That is,

$$a = (\lambda, q, m) \xrightarrow{H_l} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

$$a = (\lambda, q, m) \xrightarrow{H_q} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix} \otimes \mathbb{I}_3.$$

For the action of a on an anti-lepton $\bar{l} \in H_{\bar{l}}$ we have $a\bar{l} = \lambda \mathbb{I}_4 \bar{l}$, and on an anti-quark $\overline{q} \in H_{\overline{q}}$ we have $a\overline{q} = (\mathbb{I}_4 \otimes m)\overline{q}$.

The $\mathbb{Z}_2$-grading γ_F is such that left-handed particles have eigenvalue $+1$ and right-handed particles have eigenvalue -1. The anti-linear operator J_F interchanges particles with their anti-particles, so $J_F f = \overline{f}$ and $J_F \overline{f} = f$, with f a lepton or quark.

Finally, we write the Dirac operator of (11.1.2) in terms of the decomposition of H_F in particle $(H_l^{\oplus 3} \oplus H_q^{\oplus 3})$ and anti-particles $(H_{\bar{l}}^{\oplus 3} \oplus H_{\bar{q}}^{\oplus 3})$. The operator S will be chosen to be

$$S_l := S|_{H_l^{\oplus 3}} = \begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix},$$

$$S_q \otimes \mathbb{I}_3 := S|_{H_q^{\oplus 3}} = \begin{pmatrix} 0 & 0 & Y_u^* & 0 \\ 0 & 0 & 0 & Y_d^* \\ Y_u & 0 & 0 & 0 \\ 0 & Y_d & 0 & 0 \end{pmatrix} \otimes \mathbb{I}_3,$$

where Y_ν, Y_e, Y_u and Y_d are 3×3 **Yukawa mass matrices** acting on the three generations, and $\mathbb{I}_3$ acting on the three colors of the quarks. The symmetric operator T only acts on the right-handed (anti)neutrinos, so it is given by $T\nu_R = Y_R \overline{\nu_R}$, for a certain 3×3 symmetric **Majorana mass matrix** Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$. Note that ν_R here stands for a vector with 3 components for the number of generations.

Let us summarize what we have obtained so far.

PROPOSITION 11.2. *The data*

$$F_{SM} := (A_F, H_F, D_F; J_F, \gamma_F)$$

as given above define a finite real even spectral triple of KO-dimension 6.

11.2. The gauge theory

11.2.1. The gauge group. We shall now describe the gauge theory corresponding to the almost-commutative manifold $M \times F_{SM}$. In order to determine the gauge group $\mathfrak{G}(F_{SM})$ of Definition 6.4, let us start by examining the subalgebra $(A_F)_{J_F}$ of the algebra A_F of Proposition 11.1, as defined in Section 4.3.1. For an element $a = (\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the relation

$aJ_F = J_F a^*$ now yields $\lambda = \overline{\lambda} = \alpha = \overline{\alpha}$ and $\beta = 0$, as well as $m = \lambda \mathbb{I}_3$. So, $a \in (A_F)_{J_F}$ if and only if $a = (x, x, x)$ for $x \in \mathbb{R}$. Hence we find

$$(A_F)_{J_F} \simeq \mathbb{R}.$$

Next, let us consider the Lie algebra $\mathfrak{h}(F) = \mathfrak{u}\big((A_F)_{J_F}\big)$ of (8.1.1b). Since $\mathfrak{u}(A_F)$ consists of the anti-hermitian elements of A_F, we obtain that the $\mathfrak{h}(F) = \mathfrak{u}\big((A_F)_{J_F}\big)$ is given by the trivial subalgebra $\{0\}$.

Proposition 11.3. *The local gauge group $\mathfrak{G}(F_{SM})$ of the finite space F_{SM} is given by*

$$\mathfrak{G}(F_{SM}) \simeq \big(U(1) \times SU(2) \times U(3)\big)/\{1, -1\},$$

where $\{1, -1\}$ is the diagonal normal subgroup in $U(1) \times SU(2) \times U(3)$.

Proof. The unitary elements of the algebra form the group $\mathcal{U}(A_F) \simeq U(1) \times \mathcal{U}(\mathbb{H}) \times U(3)$. Now, a quaternion $q = q_0\mathbb{I} + iq_1\sigma_1 + iq_2\sigma_2 + iq_3\sigma_3$ is unitary if and only if $|q|^2 = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 = 1$. Using the embedding of $\mathbb{H}$ in $M_2(\mathbb{C})$, we find $|q|^2 = \det(q) = 1$, and this yields the isomorphism $\mathcal{U}(\mathbb{H}) \simeq SU(2)$. Hence, the unitary group $\mathcal{U}(A_F)$ is given by $U(1) \times SU(2) \times U(3)$. By Proposition 8.2, the gauge group is given by the quotient of the unitary group with the subgroup $\mathfrak{H}(F) = \mathcal{U}\big((A_F)_{J_F}\big)$, which is the diagonal normal subgroup

$$\{\pm(1, \mathbb{I}_2, \mathbb{I}_3)\} \subset U(1) \times SU(2) \times U(3). \qquad \square$$

The gauge group that we obtain here is not the gauge group of the Standard Model, because (even ignoring the quotient with the finite group $\{1, -1\}$) we have a factor $U(3)$ instead of $SU(3)$. As mentioned in Proposition 8.3, the unimodularity condition is only satisfied for complex algebras, but in our case, the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is only a real algebra. Therefore, the unimodularity condition is not automatically satisfied. Instead, we shall *require* that the unimodularity condition is satisfied, so for $u = (\lambda, q, m) \in U(1) \times SU(2) \times U(3)$ we impose

$$\det|_{H_F}(u) = 1 \quad \Longrightarrow \quad \big(\lambda \det m\big)^{12} = 1.$$

For $u \in U(1) \times SU(2) \times U(3)$, we denote the corresponding element in $\mathfrak{G}(F_{SM})$ by $U = uJuJ^{-1}$. We shall then consider the subgroup

$$S\mathfrak{G}(F_{SM}) = \Big\{U = uJuJ^{-1} \in \mathfrak{G}(F_{SM}) \mid u = (\lambda, q, m),\ \big(\lambda \det m\big)^{12} = 1\Big\}.$$

The effect of the unimodularity condition is that the determinant of $m \in U(3)$ is identified (modulo the multiplicative group μ_{12} of 12'th roots of unity) with $\overline{\lambda}$. In other words, imposing the unimodularity condition provides us, modulo some finite abelian group, with the gauge group $U(1) \times SU(2) \times SU(3)$. This agrees with the Standard Model, as even the group $U(1) \times SU(2) \times SU(3)$ is actually not the true gauge group of the Standard Model. Indeed, it contains a finite abelian subgroup (isomorphic to) μ_6 which acts trivially on all bosonic and fermionic particles in the Standard Model. The group μ_6 is embedded in $U(1) \times SU(2) \times SU(3)$ by $\lambda \mapsto (\lambda, \lambda^3, \lambda^2)$. The true gauge group of the Standard Model is therefore given by

$$\mathfrak{G}_{SM} := U(1) \times SU(2) \times SU(3)/\mu_6.$$

PROPOSITION 11.4. *The unimodular gauge group $S\mathfrak{G}(F_{SM})$ is isomorphic to*

$$S\mathfrak{G}(F_{SM}) \simeq \mathfrak{G}_{SM} \rtimes \mu_{12}.$$

PROOF. Proposition 11.3 shows that $S\mathfrak{G}(F_{SM}) \simeq S\mathcal{U}(A_F)/\mu_2$, so we determine $S\mathcal{U}(A_F)$. We do so in two steps:

$$\text{(I)} \qquad S\mathcal{U}(A_F) \simeq G \times SU(2) \times SU(3)/\mu_3,$$

where $G = \{(\lambda, \mu) \in U(1) \times U(1) : (\lambda\mu^3)^{12} = 1\}$, containing μ_3 as the subgroup $\{e\} \times \mu_3$, and

$$\text{(II)} \qquad G \simeq \mu_{12} \times U(1).$$

For (I), consider the map

$$(\lambda, \mu, q, m) \in G \times SU(2) \times SU(3) \mapsto (\lambda, q, \mu m) \in SU(A_F).$$

We claim that this map is surjective and has kernel μ_3. If $(\lambda, q, m) \in SU(A_F)$, then there exists $\mu \in U(1)$ such that $\mu^3 = \det m \in U(1)$. Since $(\lambda\mu^3)^{12} = (\lambda \det m)^{12} = 1$, the element (λ, μ, q, m) lies in the pre-image of (λ, q, m). The kernel of the above map consists of pairs $(\lambda, \mu, q, m) \in G \times SU(2) \times SU(3)$ such that $\lambda = 1$, $q = 1$ and $m = \mu^{-1}\mathbb{I}_3$. Since $m \in SU(3)$, this μ satisfies $\mu^3 = 1$. So we have established (I).

For (II) we show that the following sequence is split-exact:

$$1 \to U(1) \to G \to \mu_{12} \to 1,$$

where the group homomorphisms are given by $\lambda \in U(1) \mapsto (\lambda^3, \lambda^{-1}) \in G$ and $(\lambda, \mu) \in G \to \lambda\mu^3 \in \mu_a$. Exactness can be easily checked, and the splitting map is given by $\lambda \in \mu_{12} \to (\lambda, 1) \in G$. In this abelian case, the corresponding action of μ_{12} on $U(1)$ is trivial so that the resulting semi-direct product is

$$G \simeq U(1) \rtimes \mu_{12} \simeq U(1) \times \mu_{12}. \qquad \square$$

A similar argument shows that the gauge algebra of Definition 6.4 is

$$\mathfrak{g}(F_{SM}) \simeq u(1) \oplus su(2) \oplus u(3),$$

and the restriction to traceless matrices gives the gauge algebra of the Standard Model:

$$s\mathfrak{g}(F_{SM}) \simeq u(1) \oplus su(2) \oplus su(3).$$

11.2.2. The gauge and scalar fields. As we have seen in more generality in (8.2.7), the gauge field corresponding to F_{SM} takes values in $\mathfrak{g}(F_{SM})$. We here confirm this result and derive the precise form of the gauge field A_μ of (8.2.1), and also of the scalar field ϕ of (8.2.2).

Take two elements $a = (\lambda, q, m)$ and $b = (\lambda', q', m')$ of the algebra $\mathcal{A} = C^\infty(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$. According to the representation of A_F on H_F, the inner fluctuations $A_\mu = -ia\partial_\mu b$ decompose as

$$\Lambda_\mu := -i\lambda\partial_\mu\lambda'$$

on ν_R,

$$\Lambda'_\mu := -i\overline{\lambda}\partial_\mu\overline{\lambda}'$$

on e_R,

$$Q_\mu := -iq\partial_\mu q'$$

on (ν_l, e_L), and

$$V'_\mu := -im\partial_\mu m'$$

acting on $H_{\overline{q}}$; on all other components of H_F the gauge field A_μ acts as zero. Imposing the hermiticity $\Lambda_\mu = \Lambda_\mu^*$ implies $\Lambda_\mu \in \mathbb{R}$, and also automatically yields $\Lambda'_\mu = -\Lambda_\mu$. Furthermore, $Q_\mu = Q_\mu^*$ implies that Q_μ is a real-linear combination of the Pauli matrices, which span $i\, su(2)$. Finally, the condition that V'_μ be hermitian yields $V'_\mu \in i\, u(3)$, so V'_μ is a $U(3)$ gauge field. As mentioned above, we need to impose the unimodularity condition to obtain an $SU(3)$ gauge field. Hence, we require that the trace of the gauge field A_μ over H_F vanishes, and we obtain

$$\mathrm{Tr}|_{H_{\bar{l}}}\left(\Lambda_\mu \mathbb{I}_4\right) + \mathrm{Tr}|_{H_{\overline{q}}}\left(\mathbb{I}_4 \otimes V'_\mu\right) = 0 \quad \Longrightarrow \quad \mathrm{Tr}(V'_\mu) = -\Lambda_\mu.$$

Therefore, we can define a traceless $SU(3)$ gauge field V_μ by $\overline{V}_\mu := -V'_\mu - \frac{1}{3}\Lambda_\mu$. The gauge field A_μ is given by

$$A_\mu|_{H_l} = \begin{pmatrix} \Lambda_\mu & 0 & \\ 0 & -\Lambda_\mu & \\ & & Q_\mu \end{pmatrix}, \qquad A_\mu|_{H_q} = \begin{pmatrix} \Lambda_\mu & 0 & \\ 0 & -\Lambda_\mu & \\ & & Q_\mu \end{pmatrix} \otimes \mathbb{I}_3,$$

$$A_\mu|_{H_{\bar{l}}} = \Lambda_\mu \mathbb{I}_4, \qquad A_\mu|_{H_{\overline{q}}} = -\mathbb{I}_4 \otimes (\overline{V}_\mu + \frac{1}{3}\Lambda_\mu),$$

for some $U(1)$ gauge field Λ_μ, an $SU(2)$ gauge field Q_μ and an $SU(3)$ gauge field V_μ. The action of the field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ on the fermions is then given by

$$B_\mu|_{H_l} = \begin{pmatrix} 0 & 0 & \\ 0 & -2\Lambda_\mu & \\ & & Q_\mu - \Lambda_\mu \mathbb{I}_2 \end{pmatrix},$$

(11.2.1)

$$B_\mu|_{H_q} = \begin{pmatrix} \frac{4}{3}\Lambda_\mu \mathbb{I}_3 + V_\mu & 0 & \\ 0 & -\frac{2}{3}\Lambda_\mu \mathbb{I}_3 + V_\mu & \\ & & (Q_\mu + \frac{1}{3}\Lambda_\mu \mathbb{I}_2) \otimes \mathbb{I}_3 + \mathbb{I}_2 \otimes V_\mu \end{pmatrix}.$$

Note that the coefficients in front of Λ_μ in the above formulas are precisely the well-known hypercharges of the corresponding particles, as given by the following table:

Particle	ν_R	e_R	ν_L	e_L	u_R	d_R	u_L	d_L
Hypercharge	0	-2	-1	-1	$\frac{4}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Next, let us turn to the scalar field ϕ, which is given by

(11.2.2) $$\phi|_{H_l} = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}, \quad \phi|_{H_q} = \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \otimes \mathbb{I}_3, \quad \phi|_{H_{\bar{l}}} = 0, \quad \phi|_{H_{\overline{q}}} = 0,$$

where we now have, for complex fields ϕ_1, ϕ_2,

$$Y = \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}, \qquad X = \begin{pmatrix} Y_u \phi_1 & -Y_d \overline{\phi}_2 \\ Y_u \phi_2 & Y_d \overline{\phi}_1 \end{pmatrix}.$$

The scalar field Φ is then given by

$$\Phi = D_F + \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} + J_F \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} J_F^* = \begin{pmatrix} S+\phi & T^* \\ T & \overline{(S+\phi)} \end{pmatrix}. \tag{11.2.3}$$

PROPOSITION 11.5. *The action of the gauge group $S\mathfrak{G}(M \times F_{SM})$ on the fluctuated Dirac operator*

$$D_\omega = D_M \otimes \mathbb{I} + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi$$

is implemented by

$$\Lambda_\mu \mapsto \Lambda_\mu - i\lambda\partial_\mu\overline{\lambda}, \quad Q_\mu \mapsto qQ_\mu q^* - iq\partial_\mu q^*, \quad \overline{V}_\mu \mapsto m\overline{V}_\mu m^* - im\partial_\mu m^*,$$
$$\begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda}\, q \begin{pmatrix} \phi_1 + 1 \\ \phi_2 \end{pmatrix},$$

for $\lambda \in C^\infty\big(M, U(1)\big)$, $q \in C^\infty\big(M, SU(2)\big)$ and $m \in C^\infty\big(M, SU(3)\big)$.

PROOF. We simply insert the formulas for the fields obtained in (11.2.1) into the transformations given by (8.2.9). Let us write

$$u = (\lambda, q, m) \in C^\infty\big(M, U(1) \times SU(2) \times SU(3)\big).$$

The term $u\omega u^*$ replaces Q_μ by $qQ_\mu q^*$, and $\overline{V}_\mu$ by $m\overline{V}_\mu m^*$, respectively. We also see that the term $-iu\partial_\mu u^*$ is given by $-i\lambda\partial_\mu\overline{\lambda}$ on ν_R, u_R and $H_{\overline{l}}$, by the expression $-i\overline{\lambda}\partial_\mu\lambda = i\lambda\partial_\mu\overline{\lambda}$ on e_R and d_R, by $-iq\partial_\mu q^*$ on (ν_L, e_L) and (u_L, d_L), and, finally, by $-im\partial_\mu m^*$ on $H_{\overline{q}}$. We thus obtain the desired transformation rules for Λ_μ, Q_μ, and $\overline{V}_\mu$.

For the transformation of ϕ, we separately calculate $u\phi u^*$ and $u[D_F, u^*]$. Since $\phi = 0$ on $H_{\overline{l}}$ and $H_{\overline{q}}$, we may restrict our calculation of $u\phi u^*$ to H_l and H_q. On H_l we find

$$u\phi u^* = \begin{pmatrix} q_\lambda & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} \begin{pmatrix} q_\lambda^* & 0 \\ 0 & q^* \end{pmatrix} = \begin{pmatrix} 0 & q_\lambda Y^* q^* \\ qYq_\lambda^* & 0 \end{pmatrix},$$

which is still hermitian. We then calculate

$$\begin{aligned} qYq_\lambda^* &= \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} Y_\nu\phi_1 & -Y_e\overline{\phi}_2 \\ Y_\nu\phi_2 & Y_e\overline{\phi}_1 \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \overline{\lambda}Y_\nu(\alpha\phi_1 + \beta\phi_2) & \lambda Y_e(\beta\overline{\phi}_1 - \alpha\overline{\phi}_2) \\ \overline{\lambda}Y_\nu(-\overline{\beta}\phi_1 + \overline{\alpha}\phi_2) & \lambda Y_e(\overline{\alpha}\overline{\phi}_1 + \overline{\beta}\overline{\phi}_2) \end{pmatrix}. \end{aligned}$$

A similar computation on H_q gives the same transformation for the ϕ_1 and ϕ_2.

Next, let us calculate the second term $u[D_F, u^*]$. The operator T in D_F only acts on ν_R, and therefore commutes with the algebra. Upon restricting to $H_{\overline{l}}$ and $\mathcal{H}_{\overline{q}}$, the operator $\overline{S}$ commutes with the algebra. Hence, once again we may restrict our calculation to H_l and H_q. The term $u[S, u^*]$ is $uSu^* - S$ and we compute

$$uSu^* = \begin{pmatrix} 0 & q_\lambda Y_0^* q^* \\ qY_0 q_\lambda^* & 0 \end{pmatrix},$$

where $Y_0 = \left(\begin{smallmatrix} Y_\nu & 0 \\ 0 & Y_e \end{smallmatrix}\right)$ on H_l and $Y_0 = \left(\begin{smallmatrix} Y_u & 0 \\ 0 & Y_d \end{smallmatrix}\right)$ on H_q. We find that on H_l,

$$qY_0q_\lambda^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \overline{\lambda}Y_\nu\alpha & \lambda Y_e\beta \\ -\overline{\lambda}Y_\nu\overline{\beta} & \lambda Y_e\overline{\alpha} \end{pmatrix},$$

and a similar expression holds on H_q after replacing Y_ν and Y_e by Y_u and Y_d, respectively.

Combining the two contributions to the transformation, we find that the transformation $u\phi u^* + u[S, u^*]$ maps

$$Y = \begin{pmatrix} Y_\nu\phi_1 & -Y_e\overline{\phi}_2 \\ Y_\nu\phi_2 & Y_e\overline{\phi}_1 \end{pmatrix} \mapsto Y' = \begin{pmatrix} Y_\nu\phi_1' & -Y_e\overline{\phi}_2' \\ Y_\nu\phi_2' & Y_e\overline{\phi}_1' \end{pmatrix},$$

where we defined

$$\phi_1' := \overline{\lambda}(\alpha\phi_1 + \beta\phi_2 + \alpha) - 1, \qquad \phi_2' := \overline{\lambda}(-\overline{\beta}\phi_1 + \overline{\alpha}\phi_2 - \overline{\beta}).$$

Rewriting this in terms of q completes the proof. □

Summarizing, the gauge fields derived from F_{SM} take values in the Lie algebra $u(1) \oplus su(2) \oplus su(3)$ and transform according to the usual Standard Model gauge transformations. The scalar field ϕ transforms as the Standard Model Higgs field in the defining representation of $SU(2)$, with hypercharge -1.

11.3. The spectral action

In this section we calculate the spectral action for the almost-commutative manifold $M \times F_{SM}$ and derive the bosonic part of the Lagrangian of the Standard Model. The general form of this Lagrangian has already been calculated for almost-commutative manifolds in Section 8.12, so we only need to insert the expressions (11.2.1) and (11.2.3) for the fields Φ and B_μ. We start with a few lemmas that capture the rather tedious calculations that are needed to obtain the traces of $F_{\mu\nu}F^{\mu\nu}$, Φ^2, Φ^4 and $(D_\mu\Phi)(D^\mu\Phi)$.

We denote the curvatures of the $U(1), SU(2)$ and $SU(3)$ gauge fields by

$$\begin{aligned} \Lambda_{\mu\nu} &:= \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \\ Q_{\mu\nu} &:= \partial_\mu Q_\nu - \partial_\nu Q_\mu + i[Q_\mu, Q_\nu], \\ V_{\mu\nu} &:= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu]. \end{aligned} \tag{11.3.1}$$

Lemma 11.6. *The trace of the square of the curvature of B_μ is given by*

$$\mathrm{Tr}_{H_F}(F_{\mu\nu}F^{\mu\nu}) = 24\Big(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu})\Big).$$

Proof. Let us first consider the trace over the lepton sector. Using (11.2.1), we find that the curvature $F_{\mu\nu}$ of B_μ can be written as

$$F_{\mu\nu}\Big|_{H_l} = \begin{pmatrix} 0 & 0 & \\ 0 & -2\Lambda_{\mu\nu} & \\ & & Q_{\mu\nu} - \Lambda_{\mu\nu}\mathbb{I}_2 \end{pmatrix},$$

$$F_{\mu\nu}\Big|_{H_{\overline{l}}} = \begin{pmatrix} 0 & 0 & \\ 0 & 2\Lambda_{\mu\nu} & \\ & & \Lambda_{\mu\nu}\mathbb{I}_2 - (\overline{Q})_{\mu\nu} \end{pmatrix},$$

where $(\overline{Q})_{\mu\nu}$ is the curvature of $\overline{Q}_\mu$. The square of the curvature therefore becomes

$$F_{\mu\nu}F^{\mu\nu}\Big|_{H_l} = \begin{pmatrix} 0 & 0 & \\ 0 & 4\Lambda_{\mu\nu}\Lambda^{\mu\nu} & \\ & & Q_{\mu\nu}Q^{\mu\nu} + \Lambda_{\mu\nu}\Lambda^{\mu\nu}\mathbb{I}_2 - 2\Lambda_{\mu\nu}Q^{\mu\nu} \end{pmatrix},$$

$$F_{\mu\nu}F^{\mu\nu}\Big|_{H_{\bar{l}}} = \begin{pmatrix} 0 & 0 & \\ 0 & 4\Lambda_{\mu\nu}\Lambda^{\mu\nu} & \\ & & (\overline{Q})_{\mu\nu}(\overline{Q})^{\mu\nu} + \Lambda_{\mu\nu}\Lambda^{\mu\nu}\mathbb{I}_2 - 2\Lambda_{\mu\nu}(\overline{Q})^{\mu\nu} \end{pmatrix}.$$

Since $Q_{\mu\nu}$ is traceless, the cross-term $-2\Lambda_{\mu\nu}Q^{\mu\nu}$ drops out after taking the trace. Note that since Q_μ is hermitian we have $\overline{Q}_\mu = Q_\mu^T$, and this also holds for $\overline{Q}_{\mu\nu}$. This implies that

$$\operatorname{Tr}\left(\overline{(Q_{\mu\nu})(Q^{\mu\nu})}\right) = \operatorname{Tr}\left((Q_{\mu\nu})^T(Q^{\mu\nu})^T\right) = \operatorname{Tr}\left(Q_{\mu\nu}Q^{\mu\nu}\right).$$

Thus, with three generations we obtain

$$\operatorname{Tr}_{H_l\oplus H_{\bar{l}}}(F_{\mu\nu}F^{\mu\nu}) = 36\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 6\operatorname{Tr}(Q_{\mu\nu}Q^{\mu\nu}).$$

For the quark sector, on H_q, we obtain the curvature

$$F_{\mu\nu}|_{H_q} = \begin{pmatrix} \frac{4}{3}\Lambda_{\mu\nu}\mathbb{I}_3 + V_{\mu\nu} & 0 & \\ 0 & -\frac{2}{3}\Lambda_{\mu\nu}\mathbb{I}_3 + V_{\mu\nu} & \\ & & (Q_{\mu\nu} + \frac{1}{3}\Lambda_{\mu\nu}\mathbb{I}_2)\otimes\mathbb{I}_3 + \mathbb{I}_2\otimes V_{\mu\nu} \end{pmatrix},$$

where we have defined the curvature of the $SU(3)$ gauge field by

$$V_{\mu\nu} := \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu].$$

A similar expression can be derived on $H_{\bar{q}}$.

If we calculate the trace of the square of the curvature $F_{\mu\nu}$, the cross-terms again vanish, so we obtain

$$\begin{aligned}\operatorname{Tr}|_{H_q}(F_{\mu\nu}F^{\mu\nu}) &= \left(\frac{16}{3} + \frac{4}{3} + \frac{1}{3} + \frac{1}{3}\right)\Lambda_{\mu\nu}\Lambda^{\mu\nu} \\ &\quad + 3\operatorname{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + 4\operatorname{Tr}(V_{\mu\nu}V^{\mu\nu}).\end{aligned}$$

We multiply this by a factor of 2 to include the trace over the anti-quarks, and by a factor of 3 for the number of generations. Adding the result to the trace over the lepton sector, we finally obtain

$$\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) = 80\Lambda_{\mu\nu}\Lambda^{\mu\nu} + 24\operatorname{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + 24\operatorname{Tr}(V_{\mu\nu}V^{\mu\nu}). \qquad \square$$

LEMMA 11.7. *The traces of Φ^2 and Φ^4 are given by*

$$\operatorname{Tr}\left(\Phi^2\right) = 4a|H|^2 + 2c,$$
$$\operatorname{Tr}\left(\Phi^4\right) = 4b|H|^4 + 8e|H|^2 + 2d,$$

where H denotes the complex doublet $(\phi_1 + 1, \phi_2)$ and

$$\begin{aligned} a &= \operatorname{Tr}\left(Y_\nu^*Y_\nu + Y_e^*Y_e + 3Y_u^*Y_u + 3Y_d^*Y_d\right), \\ b &= \operatorname{Tr}\left((Y_\nu^*Y_\nu)^2 + (Y_e^*Y_e)^2 + 3(Y_u^*Y_u)^2 + 3(Y_d^*Y_d)^2\right), \\ c &= \operatorname{Tr}\left(Y_R^*Y_R\right), \\ d &= \operatorname{Tr}\left((Y_R^*Y_R)^2\right), \\ e &= \operatorname{Tr}\left(Y_R^*Y_RY_\nu^*Y_\nu\right). \end{aligned} \tag{11.3.2}$$

PROOF. The field Φ is given by (11.2.3), and its square equals

$$\Phi^2 = \begin{pmatrix} (S+\phi)^2 + T^*T & (S+\phi)T^* + T^*\overline{(S+\phi)} \\ T(S+\phi) + \overline{(S+\phi)}T & \overline{(S+\phi)}^2 + TT^* \end{pmatrix}.$$

The square of the off-diagonal part yields $T^*T = TT^* = |Y_R|^2$ on ν_R and $\overline{\nu_R}$, and zero on $l \neq \nu_R, \overline{\nu_R}$. On the lepton sector of the Hilbert space, the component $S+\phi$ is given by

$$S + \phi|_{H_l} = \begin{pmatrix} 0 & Y^* + Y_0^* \\ Y + Y_0 & 0 \end{pmatrix}.$$

We then calculate

$$\mathfrak{X} := (Y+Y_0)^*(Y+Y_0) = |H|^2 \begin{pmatrix} |Y_\nu|^2 & 0 \\ 0 & |Y_e|^2 \end{pmatrix},$$

where we defined the complex doublet $H := (\phi_1 + 1, \phi_2)$. Similarly, we define $\mathfrak{X}' := (Y+Y_0)(Y+Y_0)^*$, and note that $\mathrm{Tr}(\mathfrak{X}) = \mathrm{Tr}(\mathfrak{X}')$ by the cyclic property of the trace. Since $\mathfrak{X} = \mathfrak{X}^*$ and $\mathrm{Tr}(\mathfrak{X}) = \mathrm{Tr}(\mathfrak{X}^T)$, we also have $\mathrm{Tr}(\overline{\mathfrak{X}}) = \mathrm{Tr}(\mathfrak{X})$. Thus, on the lepton sector we obtain

$$\begin{aligned} \mathrm{Tr}_{H_l \oplus H_{\bar{l}}}\left(\Phi^2\right) &= \mathrm{Tr}(\mathfrak{X} + \mathfrak{X}' + \overline{\mathfrak{X}} + \overline{\mathfrak{X}}') + 2|Y_R|^2 \\ &= 4\,\mathrm{Tr}(\mathfrak{X}) + 2|Y_R|^2 = 4(|Y_\nu|^2 + |Y_e|^2)|H|^2 + 2|Y_R|^2. \end{aligned}$$

On the quark sector we similarly find

$$\mathrm{Tr}_{H_q \oplus H_{\bar{q}}}\left(\Phi^2\right) = 4 \cdot 3(|Y_\nu|^2 + |Y_e|^2)|H|^2,$$

leading to the stated formula for $\mathrm{Tr}(\Phi^2)$.

In order to find the trace of Φ^4, we calculate

$$(\mathfrak{X} + T^*T)^2 = |H|^4 \begin{pmatrix} |Y_\nu|^4 & 0 \\ 0 & |Y_e|^4 \end{pmatrix} + 2|H|^2 \begin{pmatrix} |Y_R|^2|Y_\nu|^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} |Y_R|^4 & 0 \\ 0 & 0 \end{pmatrix}.$$

We hence obtain

$$\begin{aligned} \mathrm{Tr}_{H_l \oplus H_{\bar{l}}}\left(\Phi^4\right) &= \mathrm{Tr}\left(4\mathfrak{X}^2 + 4\mathfrak{X}T^*T + 2(T^*T)^2\right) + 4|H|^2|Y_R|^2|Y_\nu|^2 \\ &= 4|H|^4\left(|Y_\nu|^4 + |Y_e|^4\right) + 8|H|^2|Y_R|^2|Y_\nu|^2 + 2|Y_R|^4. \end{aligned}$$

On the quark sector, we obtain a similar result with Y_ν replaced by Y_u and Y_e by Y_d, leaving out the Y_R, and including a factor of 3 for the trace in colour space. □

LEMMA 11.8. *The trace of $(D_\mu\Phi)(D^\mu\Phi)$ is given by*

$$\mathrm{Tr}\left((D_\mu\Phi)(D^\mu\Phi)\right) = 4a|D_\mu H|^2,$$

where H denotes the complex doublet $(\phi_1 + 1, \phi_2)$, and the covariant derivative D_μ on H is defined as

$$D_\mu H = \partial_\mu H + iQ_\mu^a \sigma^a H - i\Lambda_\mu H.$$

PROOF. We need to calculate the commutator $[B_\mu, \Phi]$. We note that B_μ commutes with the off-diagonal part of D_F. It is therefore sufficient to calculate the commutator $[B_\mu, S+\phi]$ on H_l. We shall write $Q_\mu = Q_\mu^1\sigma^1 +$

$Q^2_\mu \sigma^2 + Q^3_\mu \sigma^3$ as a linear combination of Pauli matrices with real coefficients Q^a_μ. By direct calculation on the lepton sector, we then obtain

$$[B_\mu, S+\phi]|_{H_l} = \begin{pmatrix} 0 & 0 & -\overline{Y}_\nu \overline{\chi}_1 & -\overline{Y}_\nu \overline{\chi}_2 \\ 0 & 0 & -\overline{Y}_e \chi_2 & \overline{Y}_e \chi_1 \\ Y_\nu \chi_1 & Y_e \overline{\chi}_2 & 0 & 0 \\ Y_\nu \chi_2 & -Y_e \overline{\chi}_1 & 0 & 0 \end{pmatrix},$$

where we defined the new doublet $\chi = (\chi_1, \chi_2)$ by

$$\chi_1 := (\phi_1 + 1)(Q^3_\mu - \Lambda_\mu) + \phi_2 (Q^1_\mu - iQ^2_\mu),$$
$$\chi_2 := (\phi_1 + 1)(Q^1_\mu + iQ^2_\mu) + \phi_2 (-Q^3_\mu - \Lambda_\mu).$$

We then obtain

$$D_\mu(S+\phi)|_{H_l} = \partial_\mu \phi + i[B_\mu, S+\phi]$$
$$= \begin{pmatrix} 0 & 0 & \overline{Y}_\nu(\partial_\mu \overline{\phi}_1 - i\overline{\chi}_1) & \overline{Y}_\nu(\partial_\mu \overline{\phi}_2 - i\overline{\chi}_2) \\ 0 & 0 & -\overline{Y}_e(\partial_\mu \phi_2 + i\chi_2) & \overline{Y}_e(\partial_\mu \phi_1 + i\chi_1) \\ Y_\nu(\partial_\mu \phi_1 + i\chi_1) & -Y_e(\partial_\mu \overline{\phi}_2 - i\overline{\chi}_2) & 0 & 0 \\ Y_\nu(\partial_\mu \phi_2 + i\chi_2) & Y_e(\partial_\mu \overline{\phi}_1 - i\overline{\chi}_1) & 0 & 0 \end{pmatrix}.$$

As ϕ commutes with the gauge field V_μ, the corresponding formula for $D_\mu(S+\phi)$ on the quark sector is identical (after having tensored with $\mathbb{I}_3$ in colour space).

Since we want to calculate the trace of the square of $D_\mu \Phi$, it is sufficient to determine only the terms on the diagonal of $(D_\mu \Phi)(D^\mu \Phi)$. We find

$$\mathrm{Tr}_{H_l \oplus H_q} \Big((D_\mu(S+\phi))(D^\mu(S+\phi)) \Big) = 2a \Big(|\partial_\mu \phi_1 + i\chi_1|^2 + |\partial_\mu \phi_2 + i\chi_2|^2 \Big),$$

where we have used

$$a = \mathrm{Tr}\left(Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + 3Y_d^* Y_d \right)$$

as in (11.3.2). The column vector H is given by the complex doublet $(\phi_1 + 1, \phi_2)$. We then note that $\partial_\mu \phi + i\chi$ is equal to the covariant derivative $D_\mu H$, so that

$$\mathrm{Tr}_{H_l \oplus H_q} \Big((D_\mu(S+\phi))(D^\mu(S+\phi)) \Big) = 2a |D_\mu H|^2.$$

The trace over $H_{\overline{l}} \oplus H_{\overline{q}}$ yields exactly the same contribution, so we need to multiply this by 2, which gives the desired result. □

PROPOSITION 11.9. *The spectral action of the almost-commutative manifold $M \times F_{SM}$ is given by*

$$\mathrm{Tr}\left(f\Big(\frac{D_\omega}{\Lambda}\Big) \right) \sim \int_M \mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H) \sqrt{g} d^4x + O(\Lambda^{-1}),$$

for the Lagrangian

$$\mathcal{L}(g_{\mu\nu}, \Lambda_\mu, Q_\mu, V_\mu, H) := 96 \mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_A(\Lambda_\mu, Q_\mu, V_\mu) + \mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H),$$

where $\mathcal{L}_M(g_{\mu\nu})$ is defined in Proposition 8.10, $\mathcal{L}_A$ gives the kinetic terms of the gauge fields as

$$\mathcal{L}_A(\Lambda_\mu, Q_\mu, V_\mu) := \frac{f(0)}{\pi^2}\Big(\frac{10}{3} \Lambda_{\mu\nu}\Lambda^{\mu\nu} + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}) \Big),$$

and the Higgs potential $\mathcal{L}_H$ (ignoring the boundary term) equals

$$\mathcal{L}_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H) := \frac{bf(0)}{2\pi^2}|H|^4 + \frac{-2af_2\Lambda^2 + ef(0)}{\pi^2}|H|^2 - \frac{cf_2\Lambda^2}{\pi^2} + \frac{df(0)}{4\pi^2} + \frac{af(0)}{12\pi^2}s|H|^2 + \frac{cf(0)}{24\pi^2}s + \frac{af(0)}{2\pi^2}|D_\mu H|^2.$$

PROOF. We use the general form of the spectral action of an almost-commutative manifold as calculated in Proposition 8.12, and combine it with the previous Lemmas. The gravitational Lagrangian $\mathcal{L}_M$ obtains a factor 96 from the trace over H_F. From Lemma 11.6 we immediately find the term $\mathcal{L}_A$. Combining the formulas of $\operatorname{Tr}\left(\Phi^2\right)$ and $\operatorname{Tr}\left(\Phi^4\right)$ obtained in Lemma 11.7, we find the Higgs potential

$$-\frac{f_2\Lambda^2}{2\pi^2}\operatorname{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2}\operatorname{Tr}(\Phi^4) = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{-2af_2\Lambda^2 + ef(0)}{\pi^2}|H|^2 - \frac{cf_2\Lambda^2}{\pi^2} + \frac{df(0)}{4\pi^2}.$$

The coupling of the Higgs field to the scalar curvature s is given by

$$\frac{f(0)}{48\pi^2}s\operatorname{Tr}(\Phi^2) = \frac{af(0)}{12\pi^2}s|H|^2 + \frac{cf(0)}{24\pi^2}s,$$

where the second term yields a contribution to the Einstein-Hilbert term $-\frac{f_2\Lambda^2}{3\pi^2}s$ of $\mathcal{L}_M$. Finally, the kinetic term of the Higgs field including minimal coupling to the gauge fields is obtained from Lemma 11.8 as

$$\frac{f(0)}{8\pi^2}\operatorname{Tr}\left((D_\mu\Phi)(D^\mu\Phi)\right) = \frac{af(0)}{2\pi^2}|D_\mu H|^2. \qquad \square$$

11.3.1. Coupling constants and unification. In Proposition 11.9 we calculated the bosonic Lagrangian from the spectral action. We now rescale the Higgs and gauge fields $\Lambda_\mu, Q_\mu, V_\mu$ in such a way that their kinetic terms are properly normalized.

We start with the Higgs field, and require that its kinetic term is normalized as usual, *i.e.*,

$$\int_M \frac{1}{2}|D_\mu H|^2\sqrt{g}d^4x.$$

This normalization is evidently achieved by rescaling the Higgs field as

$$H \mapsto \sqrt{\frac{\pi^2}{af(0)}}H. \tag{11.3.3}$$

Next, write the non-abelian gauge fields as $Q_\mu = Q_\mu^a\sigma^a$ and $V_\mu = V_\mu^i\lambda^i$, for the Gell-Mann matrices λ^i and real coefficients V_μ^i. We introduce coupling constants g_1, g_2 and g_3 into the model by rescaling the gauge fields as

$$\Lambda_\mu = \frac{1}{2}g_1Y_\mu, \qquad Q_\mu^a = \frac{1}{2}g_2W_\mu^a, \qquad V_\mu^i = \frac{1}{2}g_3G_\mu^i.$$

Using the relations $\operatorname{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}$ and $\operatorname{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}$, we now find that the Lagrangian $\mathcal{L}_A$ of Proposition 11.9 can be written as

$$\mathcal{L}_A(Y_\mu, W_\mu, G_\mu) = \frac{f(0)}{2\pi^2}\Big(\frac{5}{3}{g_1}^2Y_{\mu\nu}Y^{\mu\nu} + {g_2}^2W_{\mu\nu}W^{\mu\nu} + {g_3}^2G_{\mu\nu}G^{\mu\nu}\Big).$$

It is natural to require that these kinetic terms are properly normalized, and this imposes the relations

$$\frac{f(0)}{2\pi^2}{g_3}^2 = \frac{f(0)}{2\pi^2}{g_2}^2 = \frac{5f(0)}{6\pi^2}{g_1}^2 = \frac{1}{4}. \tag{11.3.4}$$

The coupling constants are then related by

$$ {g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2, $$

which is precisely the relation between the coupling constants at unification, common to grand unified theories (GUT). We shall further discuss this in Section 12.2.

In terms of the rescaled fields, we obtain the following result:

THEOREM 11.10. *The spectral action (ignoring topological and boundary terms) of the almost-commutative manifold $M \times F_{SM}$ is given by*

$$\begin{aligned} S_B = \int_M \Bigg(& \frac{48 f_4 \Lambda^4}{\pi^2} - \frac{c f_2 \Lambda^2}{\pi^2} + \frac{d f(0)}{4\pi^2} + \left(\frac{c f(0)}{24\pi^2} - \frac{4 f_2 \Lambda^2}{\pi^2}\right) s - \frac{3 f(0)}{10\pi^2}(C_{\mu\nu\rho\sigma})^2 \\ & + \frac{1}{4} Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu,a} + \frac{1}{4} G^i_{\mu\nu} G^{\mu\nu,i} + \frac{b\pi^2}{2a^2 f(0)}|H|^4 \\ & - \frac{2a f_2 \Lambda^2 - e f(0)}{a f(0)}|H|^2 + \frac{1}{12} s|H|^2 + \frac{1}{2}|D_\mu H|^2 \Bigg) \sqrt{g} d^4x, \end{aligned}$$

where the covariant derivative $D_\mu H$ is given by

$$D_\mu H = \partial_\mu H + \frac{1}{2} i g_2 W^a_\mu \sigma^a H - \frac{1}{2} i g_1 Y_\mu H. \tag{11.3.5}$$

11.3.2. The Higgs mechanism. Writing down a gauge theory with massive gauge bosons, one encounters the notorious difficulty that the mass terms of these gauge bosons are not gauge invariant. The Higgs field plays a central role in obtaining these mass terms within a gauge theory. The celebrated Higgs mechanism provides a *spontaneous breaking* of the gauge symmetry and thus generates mass terms. In this section we describe how the Higgs mechanism breaks the $U(1) \times SU(2)$ symmetry and introduces mass terms for some of the gauge bosons of the Standard Model.

In Theorem 11.10 we obtained the Higgs Lagrangian $\mathcal{L}_H$. If we drop all the terms that are independent of the Higgs field H, and also ignore the coupling of the Higgs to the gravitational field, we obtain the Lagrangian

$$\mathcal{L}(g_{\mu\nu}, Y_\mu, W^a_\mu, H) := \frac{b\pi^2}{2a^2 f(0)}|H|^4 - \frac{2a f_2 \Lambda^2 - e f(0)}{a f(0)}|H|^2 + \frac{1}{2}|D_\mu H|^2. \tag{11.3.6}$$

We wish to find the value of H for which this Lagrangian obtains its minimum value.

Hence, we consider the Higgs potential

$$\mathcal{L}_{\text{pot}}(H) := \frac{b\pi^2}{2a^2 f(0)}|H|^4 - \frac{2a f_2 \Lambda^2 - e f(0)}{a f(0)}|H|^2. \tag{11.3.7}$$

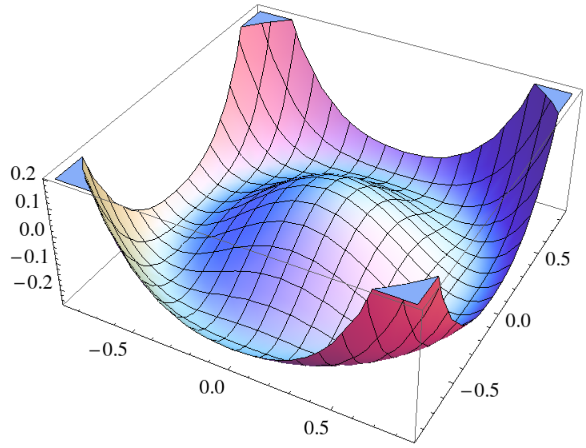


FIGURE 11.3. The potential $\mathcal{L}_{pot}(H)$ of (11.3.7) with $2af_2\Lambda^2 > ef(0)$

If $2af_2\Lambda^2 < ef(0)$, the minimum of this potential is obtained at $H = 0$, and in this case there will be no symmetry breaking. Indeed, the minimum $H = 0$ is symmetric under the full symmetry group $U(1) \times SU(2)$.

We now assume that $2af_2\Lambda^2 > ef(0)$, so that the potential has the form depicted in Figure 11.3. The minimum of the Higgs potential is then reached if the field H satisfies

$$|H|^2 = \frac{2a^2 f_2 \Lambda^2 - aef(0)}{b\pi^2}, \tag{11.3.8}$$

and none such minimum is invariant any more under $U(1) \times SU(2)$. The fields that satisfy this relation are called the *vacuum states* of the Higgs field. We choose a vacuum state $(v, 0)$, where the *vacuum expectation value* v is a real parameter such that v^2 is given by the right-hand side of (11.3.8). From the transformation rule of Proposition 11.5, we see that the vacuum state $(v, 0)$ is still invariant under a subgroup of $U(1) \times SU(2)$. This subgroup is isomorphic to $U(1)$ and is given by

$$\left\{ \left(\lambda, q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \right) : \lambda \in U(1) \right\} \subset U(1) \times SU(2).$$

Let us simplify the expression for the Higgs potential. First, we note that the potential only depends on the absolute value $|H|$. A transformation of the doublet H by an element $(\lambda, q) \in U(1) \times SU(2)$ is written as $H \mapsto uH$ with $u = \overline{\lambda} q$ a unitary matrix. Since a unitary transformation preserves absolute values, we see that $\mathcal{L}_{\text{pot}}(uH) = \mathcal{L}_{\text{pot}}(H)$ for any $u \in U(1) \times SU(2)$. We can use this *gauge freedom* to transform the Higgs field into a simpler form. Consider elements of $SU(2)$ of the form

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

such that $|\alpha|^2 + |\beta|^2 = 1$. The doublet H can in general be written as (h_1, h_2), for some $h_1, h_2 \in \mathbb{C}$. We then see that we may write

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} |H| \\ 0 \end{pmatrix}, \qquad \alpha = \frac{h_1}{|H|}, \quad \beta = \frac{h_2}{|H|},$$

which means that we may always use the gauge freedom to write the doublet H in terms of one real parameter. Let us define a new real-valued field h by setting $h(x) := |H(x)| - v$. We then obtain

$$H = u(x) \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}, \qquad u(x) := \begin{pmatrix} \alpha(x) & -\overline{\beta(x)} \\ \beta(x) & \overline{\alpha(x)} \end{pmatrix}. \tag{11.3.9}$$

Inserting this transformed Higgs field into the Higgs potential, we obtain the following expression in terms of the real parameter v and the real field $h(x)$:

$$\begin{aligned} \mathcal{L}_{\text{pot}}(h) &= \frac{bf(0)}{2\pi^2}(v+h)^4 - \frac{2af_2\Lambda^2 - ef(0)}{\pi^2}(v+h)^2 \\ &= \frac{b\pi^2}{2a^2 f(0)}(h^4 + 4vh^3 + 6v^2h^2 + 4v^3h + v^4) \\ &\quad - \frac{2af_2\Lambda^2 - ef(0)}{af(0)}(h^2 + 2vh + v^2). \end{aligned}$$

Using (11.3.8), the value of v^2 is given by

$$v^2 = \frac{2a^2 f_2 \Lambda^2 - aef(0)}{b\pi^2}.$$

We then see that in $\mathcal{L}_{\text{pot}}$ the terms linear in h cancel out. This is of course no surprise, since the change of variables $|H(x)| \mapsto v + h(x)$ means that at $h(x) = 0$ we are at the minimum of the potential, where the first order derivative of the potential with respect to h must vanish. We thus obtain the simplified expression

$$\mathcal{L}_{\text{pot}}(h) = \frac{b\pi^2}{2a^2 f(0)}\Big(h^4 + 4vh^3 + 4v^2h^2 - v^4\Big). \tag{11.3.10}$$

We now observe that the field $h(x)$ has acquired a mass term and has two self-interactions given by h^3 and h^4. We also have another contribution to the cosmological constant, given by $-v^4$.

11.3.2.1. *Massive gauge bosons.* Next, let us consider what this procedure entails for the remainder of the Higgs Lagrangian $\mathcal{L}_H$. We first consider the kinetic term of H, including its minimal coupling to the gauge fields, given by

$$\mathcal{L}_{\text{min}}(Y_\mu, W_\mu^a, H) := \frac{1}{2}|D_\mu H|^2.$$

The transformation of (11.3.9) is a gauge transformation, and to make sure that $\mathcal{L}_{\text{min}}$ is invariant under this transformation, we also need to transform the gauge fields. The field Y_μ is unaffected by the local $SU(2)$-transformation

$u(x)$. The transformation of $W_\mu = W_\mu^a \sigma^a$ is obtained from Proposition 11.5 and is given by

$$W_\mu \to u W_\mu u^* - \frac{2i}{g_2} u \partial_\mu u^*.$$

One then easily checks that we obtain the transformation $D_\mu H \mapsto u D_\mu H$, so that $|D_\mu H|^2$ is invariant under such transformations. So we can just insert the doublet $(v+h, 0)$ into (11.3.5) and obtain

$$\begin{aligned} D_\mu H &= \partial_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix} + \frac{1}{2} i g_2 W_\mu^a \sigma^a \begin{pmatrix} v+h \\ 0 \end{pmatrix} - \frac{1}{2} i g_1 Y_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix} \\ &= \partial_\mu \begin{pmatrix} h \\ 0 \end{pmatrix} + \frac{1}{2} i g_2 W_\mu^1 \begin{pmatrix} 0 \\ v+h \end{pmatrix} + \frac{1}{2} i g_2 W_\mu^2 \begin{pmatrix} 0 \\ i(v+h) \end{pmatrix} \\ &\quad + \frac{1}{2} i g_2 W_\mu^3 \begin{pmatrix} v+h \\ 0 \end{pmatrix} - \frac{1}{2} i g_1 Y_\mu \begin{pmatrix} v+h \\ 0 \end{pmatrix}. \end{aligned}$$

We can then calculate its square as

$$\begin{aligned} |D_\mu H|^2 &= (D^\mu H)^* (D_\mu H) \\ &= (\partial^\mu h)(\partial_\mu h) + \frac{1}{4} {g_2}^2 (v+h)^2 (W^{\mu,1} W_\mu^1 + W^{\mu,2} W_\mu^2 + W^{\mu,3} W_\mu^3) \\ &\quad + \frac{1}{4} {g_1}^2 (v+h)^2 B'^\mu Y_\mu - \frac{1}{2} g_1 g_2 (v+h)^2 B'^\mu W_\mu^3. \end{aligned}$$

Note that the last term yields a mixing of the gauge fields Y_μ and W_μ^3, parametrized by the electroweak mixing angle θ_w defined by

$$c_w := \cos\theta_w = \frac{g_2}{\sqrt{{g_1}^2 + {g_2}^2}}, \qquad s_w := \sin\theta_w = \frac{g_1}{\sqrt{{g_1}^2 + {g_2}^2}}.$$

Note that the relation ${g_2}^2 = 3{g_1}^2$ for the coupling constants implies that we obtain the values $\cos^2\theta_w = \frac{1}{4}$ and $\sin^2\theta_w = \frac{3}{4}$ at the electroweak unification scale Λ_{EW}. Let us now define new gauge fields by

$$\begin{aligned} W_\mu &:= \frac{1}{\sqrt{2}}(W_\mu^1 + i W_\mu^2), & W_\mu^* &:= \frac{1}{\sqrt{2}}(W_\mu^1 - i W_\mu^2), \\ Z_\mu &:= c_w W_\mu^3 - s_w Y_\mu, & A'_\mu &:= s_w W_\mu^3 + c_w Y_\mu, \end{aligned} \tag{11.3.11}$$

where we have added a prime to A_μ to distinguish the (photon) field from the general form of the inner fluctuations in Equation (8.2.1). We now show that the new fields Z_μ and A'_μ become mass eigenstates. The fields W_μ^1 and W_μ^2 were already mass eigenstates, but the fields W_μ and W_μ^* are chosen so that they obtain a definite charge. We can write

$$\begin{aligned} W_\mu^1 &= \frac{1}{\sqrt{2}}(W_\mu + W_\mu^*), & W_\mu^2 &= \frac{-i}{\sqrt{2}}(W_\mu - W_\mu^*), \\ W_\mu^3 &= s_w A'_\mu + c_w Z_\mu, & Y_\mu &= c_w A'_\mu - s_w Z_\mu, \end{aligned}$$

and inserting this into the expression for $|D_\mu H|^2$ yields

(11.3.12)

$$\frac{1}{2}|D_\mu H|^2 = \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{4}{g_2}^2 (v+h)^2 W^{\mu*} W_\mu + \frac{1}{8}\frac{{g_2}^2}{{c_w}^2}(v+h)^2 Z^\mu Z_\mu.$$

Thus, we see that the fields W_μ, W_μ^* and Z_μ acquire a mass term (where Z_μ has a larger mass than W_μ, W_μ^*) and that the fields A'_μ are massless. The (tree-level) masses of the W-boson and Z-boson are evidently given by

$$M_W = \frac{1}{2} v g_2, \qquad M_Z = \frac{1}{2} v \frac{g_2}{c_w}. \tag{11.3.13}$$

11.4. The fermionic action

In order to obtain the full Lagrangian for the Standard Model, we also need to calculate the fermionic action S_f of Definition 7.3. First, let us have a closer look at the fermionic particle fields and their interactions.

By an abuse of notation, let us write $\nu^\lambda, \overline{\nu}^\lambda, e^\lambda, \overline{e}^\lambda, u^{\lambda c}, \overline{u}^{\lambda c}, d^{\lambda c}, \overline{d}^{\lambda c}$ for a set of independent Dirac spinors. We then write a generic Grassmann vector $\widetilde{\xi} \in \mathcal{H}_{\rm cl}^+$ as follows:

$$\begin{aligned}
\widetilde{\xi} = {} & \nu_L^\lambda \otimes \nu_L^\lambda + \nu_R^\lambda \otimes \nu_R^\lambda + \overline{\nu}_R^\lambda \otimes \overline{\nu_L^\lambda} + \overline{\nu}_L^\lambda \otimes \overline{\nu_R^\lambda} \\
& + e_L^\lambda \otimes e_L^\lambda + e_R^\lambda \otimes e_R^\lambda + \overline{e}_R^\lambda \otimes \overline{e_L^\lambda} + \overline{e}_L^\lambda \otimes \overline{e_R^\lambda} \\
& + u_L^{\lambda c} \otimes u_L^{\lambda c} + u_R^{\lambda c} \otimes u_R^{\lambda c} + \overline{u}_R^{\lambda c} \otimes \overline{u_L^{\lambda c}} + \overline{u}_L^{\lambda c} \otimes \overline{u_R^{\lambda c}} \\
& + d_L^{\lambda c} \otimes d_L^{\lambda c} + d_R^{\lambda c} \otimes d_R^{\lambda c} + \overline{d}_R^{\lambda c} \otimes \overline{d_L^{\lambda c}} + \overline{d}_L^{\lambda c} \otimes \overline{d_R^{\lambda c}},
\end{aligned}$$

where in each tensor product it should be clear that the first component is a Weyl spinor, and the second component is a basis element of H_F. Here $\lambda = 1, 2, 3$ labels the generation of the fermions, and $c = r, g, b$ labels the color index of the quarks.

Let us have a closer look at the gauge fields of the electroweak sector. For the physical gauge fields of (11.3.11) we can write

$$\begin{aligned}
& Q_\mu^1 + iQ_\mu^2 = \frac{1}{\sqrt{2}} g_2 W_\mu, \qquad Q_\mu^1 - iQ_\mu^2 = \frac{1}{\sqrt{2}} g_2 W_\mu^*, \\
& Q_\mu^3 - \Lambda_\mu = \frac{g_2}{2c_w} Z_\mu, \qquad \Lambda_\mu = \frac{1}{2} s_w g_2 A'_\mu - \frac{1}{2} \frac{{s_w}^2 g_2}{c_w} Z_\mu, \\
& -Q_\mu^3 - \Lambda_\mu = -s_w g_2 A'_\mu + \frac{g_2}{2c_w}(1 - 2{c_w}^2) Z_\mu, \\
& Q_\mu^3 + \frac{1}{3}\Lambda_\mu = \frac{2}{3} s_w g_2 A'_\mu - \frac{g_2}{6c_w}(1 - 4{c_w}^2) Z_\mu, \\
& -Q_\mu^3 + \frac{1}{3}\Lambda_\mu = -\frac{1}{3} s_w g_2 A'_\mu - \frac{g_2}{6c_w}(1 + 2{c_w}^2) Z_\mu.
\end{aligned} \tag{11.4.1}$$

Here we have rescaled the Higgs field in (11.3.3), so we can write $H = \frac{\sqrt{af(0)}}{\pi}(\phi_1 + 1, \phi_2)$. We parametrize the Higgs field as

$$H = (v + h + i\phi^0, i\sqrt{2}\phi^-),$$

where ϕ^0 is real and ϕ^- is complex. We write ϕ^+ for the complex conjugate of ϕ^-. Thus, we can write

$$(\phi_1 + 1, \phi_2) = \frac{\pi}{\sqrt{af(0)}}(v + h + i\phi^0, i\sqrt{2}\phi^-). \tag{11.4.2}$$

As in Remark 9.8, we will need to impose a further restriction on the mass matrices in D_F, in order to obtain physical mass terms in the fermionic action. From here on, we will require that the matrices Y_x are anti-hermitian, for $x = \nu, e, u, d$. We then define the hermitian mass matrices m_x by writing

$$Y_x =: -i\frac{\sqrt{af(0)}}{\pi v} m_x. \tag{11.4.3}$$

Similarly, we also take Y_R to be anti-hermitian, and we introduce a hermitian (and symmetric) Majorana mass matrix m_R by writing

$$Y_R = -i\, m_R. \tag{11.4.4}$$

THEOREM 11.11. *The fermionic action of the almost-commutative manifold $M \times F_{SM}$ is given by*

$$S_F = \int_M \big(\mathcal{L}_{kin} + \mathcal{L}_{gf} + \mathcal{L}_{Hf} + \mathcal{L}_R\big)\sqrt{g} d^4x,$$

where, suppressing all generation and color indices, the kinetic terms of the fermions are given by

$$\begin{aligned}\mathcal{L}_{kin} := &-i\langle J_M\overline{\nu}, \gamma^\mu\nabla^S_\mu \nu\rangle - i\langle J_M\overline{e}, \gamma^\mu\nabla^S_\mu e\rangle \\ &- i\langle J_M\overline{u}, \gamma^\mu\nabla^S_\mu u\rangle - i\langle J_M\overline{d}, \gamma^\mu\nabla^S_\mu d\rangle,\end{aligned}$$

the minimal coupling of the gauge fields to the fermions is given by

$$\begin{aligned}\mathcal{L}_{gf} := &\, s_w g_2 A'_\mu\Big(- \langle J_M\overline{e}, \gamma^\mu e\rangle + \tfrac{2}{3}\langle J_M\overline{u}, \gamma^\mu u\rangle - \tfrac{1}{3}\langle J_M\overline{d}, \gamma^\mu d\rangle\Big) \\ &+ \frac{g_2}{4c_w} Z_\mu \Big(\langle J_M\overline{\nu}, \gamma^\mu(1+\gamma_M)\nu\rangle + \langle J_M\overline{e}, \gamma^\mu(4{s_w}^2 - 1 - \gamma_M)e\rangle \\ &\qquad + \langle J_M\overline{u}, \gamma^\mu(-\tfrac{8}{3}{s_w}^2 + 1 + \gamma_M)u\rangle \\ &\qquad + \langle J_M\overline{d}, \gamma^\mu(\tfrac{4}{3}{s_w}^2 - 1 - \gamma_M)d\rangle\Big) \\ &+ \frac{g_2}{2\sqrt{2}} W_\mu\Big(\langle J_M\overline{e}, \gamma^\mu(1+\gamma_M)\nu\rangle + \langle J_M\overline{d}, \gamma^\mu(1+\gamma_M)u\rangle\Big) \\ &+ \frac{g_2}{2\sqrt{2}} W^*_\mu\Big(\langle J_M\overline{\nu}, \gamma^\mu(1+\gamma_M)e\rangle + \langle J_M\overline{u}, \gamma^\mu(1+\gamma_M)d\rangle\Big) \\ &+ \frac{g_3}{2} G^i_\mu\Big(\langle J_M\overline{u}, \gamma^\mu\lambda_i u\rangle + \langle J_M\overline{d}, \gamma^\mu\lambda_i d\rangle\Big),\end{aligned}$$

the Yukawa couplings of the Higgs field to the fermions are given by

$$\begin{aligned}\mathcal{L}_{Hf} := &\, i\left(1 + \frac{h}{v}\right)\Big(\langle J_M\overline{\nu}, m_\nu \nu\rangle + \langle J_M\overline{e}, m_e e\rangle \\ &\qquad + \langle J_M\overline{u}, m_u u\rangle + \langle J_M\overline{d}, m_d d\rangle\Big) \\ &+ \frac{\phi^0}{v}\Big(\langle J_M\overline{\nu}, \gamma_M m_\nu \nu\rangle - \langle J_M\overline{e}, \gamma_M m_e e\rangle \\ &\qquad + \langle J_M\overline{u}, \gamma_M m_u u\rangle - \langle J_M\overline{d}, \gamma_M m_d d\rangle\Big) \\ &+ \frac{\phi^-}{\sqrt{2}v}\Big(\langle J_M\overline{e}, m_e(1+\gamma_M)\nu\rangle - \langle J_M\overline{e}, m_\nu(1-\gamma_M)\nu\rangle\Big) \\ &+ \frac{\phi^+}{\sqrt{2}v}\Big(\langle J_M\overline{\nu}, m_\nu(1+\gamma_M)e\rangle - \langle J_M\overline{\nu}, m_e(1-\gamma_M)e\rangle\Big) \\ &+ \frac{\phi^-}{\sqrt{2}v}\Big(\langle J_M\overline{d}, m_d(1+\gamma_M)u\rangle - \langle J_M\overline{d}, m_u(1-\gamma_M)u\rangle\Big)\end{aligned}$$

$$+\frac{\phi^+}{\sqrt{2}v}\Big(\langle J_M\overline{u}, m_u(1+\gamma_M)d\rangle - \langle J_M\overline{u}, m_d(1-\gamma_M)d\rangle\Big),$$

and, finally, the Majorana masses of the right-handed neutrinos (and left-handed anti-neutrinos) are given by

$$\mathcal{L}_R := i\langle J_M\nu_R, m_R\nu_R\rangle + i\langle J_M\overline{\nu}_L, m_R\overline{\nu}_L\rangle.$$

Proof. The proof is similar to Proposition 9.7, though the calculations are now a little more complicated. From Definition 7.3 we know that the fermionic action is given by $S_F = \frac{1}{2}(J\widetilde{\xi}, D_\omega\widetilde{\xi})$, where the fluctuated Dirac operator is given by

$$D_\omega = D_M \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi.$$

We rewrite the inner product on $\mathcal{H}$ as $(\xi,\psi) = \int_M \langle\xi,\psi\rangle\sqrt{g}d^4x$. As in Proposition 9.7, the expressions for $J\widetilde{\xi} = (J_M\otimes J_F)\widetilde{\xi}$ and $(D_M\otimes 1)\widetilde{\xi}$ are obtained straightforwardly. Using the symmetry of the form $(J_M\widetilde{\chi}, D_M\widetilde{\psi})$, and then we obtain the kinetic terms as

$$\begin{aligned}\frac{1}{2}\langle J\widetilde{\xi}, (D_M\otimes 1)\widetilde{\xi}\rangle &= \langle J_M\overline{\nu}^\lambda, D_M\nu^\lambda\rangle + \langle J_M\overline{e}^\lambda, D_M e^\lambda\rangle \\ &\quad + \langle J_M\overline{u}^{\lambda c}, D_M u^{\lambda c}\rangle + \langle J_M\overline{d}^{\lambda c}, D_M d^{\lambda c}\rangle.\end{aligned}$$

The other two terms in the fluctuated Dirac operator yield more complicated expressions. For the calculation of $(\gamma^\mu\otimes B_\mu)\widetilde{\xi}$, we use (11.2.1) for the gauge field B_μ, and insert the expressions of (11.4). As in Proposition 9.7, we then use the antisymmetry of the form $(J_M\widetilde{\chi}, \gamma^\mu\widetilde{\psi})$. For the coupling of the fermions to the gauge fields, a direct calculation then yields

$$\begin{aligned}&\frac{1}{2}\langle J\widetilde{\xi}, (\gamma^\mu\otimes B_\mu)\widetilde{\xi}\rangle = \\ &\quad s_w g_2 A'_\mu\Big(-\langle J_M\overline{e}^\lambda, \gamma^\mu e^\lambda\rangle + \tfrac{2}{3}\langle J_M\overline{u}^{\lambda c}, \gamma^\mu u^{\lambda c}\rangle - \tfrac{1}{3}\langle J_M\overline{d}^{\lambda c}, \gamma^\mu d^{\lambda c}\rangle\Big) \\ &\quad + \frac{g_2}{4c_w}Z_\mu\Big(\langle J_M\overline{\nu}^\lambda, \gamma^\mu(1+\gamma_M)\nu^\lambda\rangle + \langle J_M\overline{e}^\lambda, \gamma^\mu(4{s_w}^2 - 1 - \gamma_M)e^\lambda\rangle \\ &\qquad + \langle J_M\overline{u}^{\lambda c}, \gamma^\mu(-\tfrac{8}{3}{s_w}^2 + 1 + \gamma_M)u^{\lambda c}\rangle \\ &\qquad + \langle J_M\overline{d}^{\lambda c}, \gamma^\mu(\tfrac{4}{3}{s_w}^2 - 1 - \gamma_M)d^{\lambda c}\rangle\Big) \\ &\quad + \frac{g_2}{2\sqrt{2}}W_\mu\Big(\langle J_M\overline{e}^\lambda, \gamma^\mu(1+\gamma_M)\nu^\lambda\rangle + \langle J_M\overline{d}^{\lambda c}, \gamma^\mu(1+\gamma_M)u^{\lambda c}\rangle\Big) \\ &\quad + \frac{g_2}{2\sqrt{2}}W^*_\mu\Big(\langle J_M\overline{\nu}^\lambda, \gamma^\mu(1+\gamma_M)e^\lambda\rangle + \langle J_M\overline{u}^{\lambda c}, \gamma^\mu(1+\gamma_M)d^{\lambda c}\rangle\Big) \\ &\quad + \frac{g_3}{2}G^i_\mu\lambda_i^{dc}\Big(\langle J_M\overline{u}^{\lambda d}, \gamma^\mu u^{\lambda c}\rangle + \langle J_M\overline{d}^{\lambda d}, \gamma^\mu d^{\lambda c}\rangle\Big),\end{aligned}$$

where in the weak interactions the projection operator $\frac{1}{2}(1+\gamma_M)$ is used to select only the left-handed spinors.

Next, we need to calculate $\frac{1}{2}(J\widetilde{\xi}, (\gamma_M\otimes\Phi)\widetilde{\xi})$. The Higgs field is given by $\Phi = D_F + \phi + J_F\phi J_F^*$, where ϕ is given by (11.2.2). Let us first focus on the four terms involving only the Yukawa couplings for the neutrinos. Using the symmetry of the form $(J_M\widetilde{\chi},\gamma_M\widetilde{\psi})$, we obtain

$$\begin{aligned}\frac{1}{2}&\langle J_M\overline{\nu}_R^\kappa,\gamma_M Y_\nu^{\kappa\lambda}(\phi_1+1)\nu_R^\lambda\rangle+\frac{1}{2}\langle J_M\nu_R^\kappa,\gamma_M Y_\nu^{\lambda\kappa}(\phi_1+1)\overline{\nu}_R^\lambda\rangle\\&+\frac{1}{2}\langle J_M\overline{\nu}_L^\kappa,\gamma_M\overline{Y}_\nu^{\lambda\kappa}(\overline{\phi}_1+1)\nu_L^\lambda\rangle+\frac{1}{2}\langle J_M\nu_L^\kappa,\gamma_M\overline{Y}_\nu^{\kappa\lambda}(\overline{\phi}_1+1)\overline{\nu}_L^\lambda\rangle\\&\qquad=\langle J_M\overline{\nu}_R^\kappa,\gamma_M Y_\nu^{\kappa\lambda}(\phi_1+1)\nu_R^\lambda\rangle+\langle J_M\overline{\nu}_L^\kappa,\gamma_M\overline{Y}_\nu^{\lambda\kappa}(\overline{\phi}_1+1)\nu_L^\lambda\rangle.\end{aligned}$$

Using (11.4.2) and (11.4.3), and dropping the generation labels, we can now rewrite

$$\begin{aligned}\langle J_M\overline{\nu}_R,\gamma_M Y_\nu(\phi_1+1)\nu_R\rangle&+\langle J_M\overline{\nu}_L,\gamma_M\overline{Y}_\nu(\overline{\phi}_1+1)\nu_L\rangle\\&=i\left(1+\frac{h}{v}\right)\langle J_M\overline{\nu},m_\nu\nu\rangle-\frac{\phi^0}{v}\langle J_M\overline{\nu},\gamma_M m_\nu\nu\rangle.\end{aligned}$$

For e,u,d we obtain similar terms, the only difference being that for e and d the sign for ϕ^0 is changed. We also find terms that mix neutrino's and electrons; by the symmetry of the form $(J_M\widetilde{\chi},\gamma_M\widetilde{\psi})$, these are given by the four terms

$$\begin{aligned}\frac{\sqrt{2}}{v}\Big(&\phi^-\langle J_M\overline{e}_L,m_e\nu_L\rangle+\phi^+\langle J_M\overline{\nu}_L,m_\nu e_L\rangle\\&-\phi^-\langle J_M\overline{e}_R,m_\nu\nu_R\rangle-\phi^+\langle J_M\overline{\nu}_R,m_e e_R\rangle\Big).\end{aligned}$$

There are four similar terms with ν and e replaced by u and d, respectively. We can use the projection operators $\frac{1}{2}(1\pm\gamma_M)$ to select left- or right-handed spinors. Lastly, the off-diagonal part T in the finite Dirac operator D_F yields the Majorana mass terms for the right-handed neutrinos (and left-handed anti-neutrinos). Using (11.4.4), these Majorana mass terms are given by

$$\langle J_M\nu_R,\gamma_M Y_R\nu_R\rangle+\langle J_M\overline{\nu}_L,\gamma_M\overline{Y}_R\overline{\nu}_L\rangle=i\langle J_M\nu_R,m_R\nu_R\rangle+i\langle J_M\overline{\nu}_L,m_R\overline{\nu}_L\rangle.$$

Thus, we find that the mass terms of the fermions and their couplings to the Higgs field are given by

$$\begin{aligned}\frac{1}{2}&\langle J\widetilde{\xi},(\gamma_M\otimes\Phi)\widetilde{\xi}\rangle=\\&i\left(1+\frac{h}{v}\right)\Big(\langle J_M\overline{\nu},m_\nu\nu\rangle+\langle J_M\overline{e},m_e e\rangle+\langle J_M\overline{u},m_u u\rangle+\langle J_M\overline{d},m_d d\rangle\Big)\\&+\frac{\phi^0}{v}\Big(\langle J_M\overline{\nu},\gamma_M m_\nu\nu\rangle-\langle J_M\overline{e},\gamma_M m_e e\rangle+\langle J_M\overline{u},\gamma_M m_u u\rangle-\langle J_M\overline{d},\gamma_M m_d d\rangle\Big)\\&+\frac{\phi^-}{\sqrt{2}v}\Big(\langle J_M\overline{e},m_e(1+\gamma_M)\nu\rangle-\langle J_M\overline{e},m_\nu(1-\gamma_M)\nu\rangle\Big)\\&+\frac{\phi^+}{\sqrt{2}v}\Big(\langle J_M\overline{\nu},m_\nu(1+\gamma_M)e\rangle-\langle J_M\overline{\nu},m_e(1-\gamma_M)e\rangle\Big)\\&+\frac{\phi^-}{\sqrt{2}v}\Big(\langle J_M\overline{d},m_d(1+\gamma_M)u\rangle-\langle J_M\overline{d},m_u(1-\gamma_M)u\rangle\Big)\\&+\frac{\phi^+}{\sqrt{2}v}\Big(\langle J_M\overline{u},m_u(1+\gamma_M)d\rangle-\langle J_M\overline{u},m_d(1-\gamma_M)d\rangle\Big)\\&+i\langle J_M\nu_R,m_R\nu_R\rangle+i\langle J_M\overline{\nu}_L,m_R\overline{\nu}_L\rangle,\end{aligned}$$

where we have suppressed all indices. □

In Theorem 11.10 and Theorem 11.11 we have calculated the action functional of Definitions 7.1 and 7.3 for the almost-commutative manifold $M \times F_{SM}$ defined in this Chapter. To summarize, we have geometrically derived:

(1) The full particle contents of the Standard Model, to wit,
  - the W, Z bosons, photons, and gluons, corresponding to the $U(1) \times SU(2) \times SU(3)$ Standard Model gauge group.
  - the Higgs boson.
  - three generations of left and right-handed leptons and quarks.

(2) The dynamics and all interactions of the Standard Model, including
  - self-interactions of the gauge bosons, and coupling to fermions
  - masses for the fermions, including masses for the neutrinos, and coupling to the Higgs field
  - Higgs spontaneous symmetry breaking mechanism, giving masses to the W and Z boson, and also to the Higgs boson itself.

(3) Minimal coupling to gravity.

In addition to the usual Standard Model, there are relations between the coupling constants in the Lagrangian of Theorem 11.10. In the next Chapter, we will analyze this in more detail and derive physical predictions from these relations.

Notes

1. For an exposition of the Standard Model of particle physics, we refer to [**72, 120**].

Section 11.1. The finite space

2. The first description of the finite space F_{SM} yielding the Standard Model (without right-handed neutrinos though) was given by Connes in [**63**], based on [**59, 71**] (see also the review [**146**]). As already mentioned in the Notes to Chapter 7, the spectral action principle was formulated in [**49, 50**] where it was also applied to the Standard Model. Extensive computations on this model can be found in [**168**].

In [**54**] the noncommutative geometric formulation of the Standard Model got in good shape, mainly because of the choice for the finite space to be of KO-dimension 6 [**20, 68**]. This solved the problem of fermion doubling pointed out in [**138**] (see also the discussion in [**65**, Ch. 1, Sect. 16.3]), and at the same time allowed for the introduction of Majorana masses for right-handed neutrinos, along with the seesaw mechanism. Here, we follow [**186**].

The derivation of the Standard Model algebra A_F from the list of finite irreducible geometries of Section 3.4 was first obtained in [**51**], This includes Proposition 11.1 of which we here give an alternative, diagrammatic proof.

The moduli space of Dirac operators D_F of the form (11.1.2) was analyzed in [**54**, Section 2.7] (*cf.* [**65**, Section 1.13.5]) and in [**42**].

Section 11.2. The gauge theory

3. The condition of unimodularity was imposed in the context of the Standard Model in [**54**, Sect. 2.5] (see also [**65**, Ch. 1, Sect. 13.3]). The derivation of the hypercharges from the unimodularity condition is closely related to the equivalence between unimodularity in the almost-commutative Standard Model and anomaly cancellation for the usual Standard Model [**2**].

4. Proposition 11.4 agrees with [**54**, Prop. 2.16] (see also [**65**, Prop. 1.185]). For the derivation of the Standard Model gauge group $\mathfrak{G}_{SM}$, we refer to [**18**].

Section 11.3. The spectral action

5. The coefficients a, b, c, d and e in Lemma 11.7 agree with those appearing in [**54**] (see also [**65**, Ch. 1, Sect. 15.2]).

6. The Higgs mechanism is attributed to Englert, Brout and Higgs [**86, 108**].

7. The form of the Higgs field in (11.3.9) that is obtained after a suitable change of basis is called *unitary gauge* and was introduced by Weinberg in [**193, 194**] (see also [**195**, Chapter 21]).

CHAPTER 12

Phenomenology of the noncommutative Standard Model

In Theorem 11.10 and Theorem 11.11, we have derived the full Lagrangian for the Standard Model from the almost-commutative manifold $M \times F_{SM}$. The coefficients in this Lagrangian are given in terms of:

- the value $f(0)$ and the moments f_2 and f_4 of the function f in the spectral action;
- the cut-off scale Λ in the spectral action;
- the vacuum expectation value v of the Higgs field;
- the coefficients a, b, c, d, e of (11.3.2) that are determined by the mass matrices in the finite Dirac operator D_F.

One can find several relations among these coefficients in the Lagrangian, which we shall derive in the following section. Inspired by the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ obtained from (11.3.4), we will assume that these relations hold at the unification scale. Subsequently, we use the renormalization group equations to obtain predictions for the Standard Model at 'lower' (*i.e.* particle accelerator) energies.

12.1. Mass relations

12.1.1. Fermion masses. Recall from (11.4.3) that we defined the mass matrices m_x of the fermions by rewriting the matrices Y_x in the finite Dirac operator D_F. Inserting the formula (11.4.3) for Y_x into the expression for a given by (11.3.2), we obtain

$$a = \frac{af(0)}{\pi^2 v^2} \operatorname{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big),$$

which yields

$$\operatorname{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big) = \frac{\pi^2 v^2}{f(0)}.$$

From (11.3.13) we know that the mass of the W-boson is given by $M_W = \frac{1}{2} v g_2$. Using the normalization (11.3.4), expressing g_2 in terms of $f(0)$, we can then write

$$f(0) = \frac{\pi^2 v^2}{8{M_W}^2}. \tag{12.1.1}$$

Inserting this into the expression above, we obtain a relation between the fermion mass matrices m_x and the W-boson mass M_W, *viz.*

$$\operatorname{Tr}\big(m_\nu^* m_\nu + m_e^* m_e + 3m_u^* m_u + 3m_d^* m_d\big) = 2{g_2}^2 v^2 = 8{M_W}^2. \tag{12.1.2}$$

If we assume that the mass of the top quark is much larger than all other fermion masses, we may neglect the other fermion masses. In that case, the above relation would yield the constraint

$$m_{\text{top}} \lesssim \sqrt{\frac{8}{3}} M_W. \tag{12.1.3}$$

12.1.2. The Higgs mass. We obtain a mass m_h for the Higgs boson h by writing the term proportional to h^2 in (11.3.10) in the form

$$\frac{b\pi^2}{2a^2 f(0)} 4v^2 h^2 = \frac{1}{2} {m_h}^2 h^2.$$

Thus, the Higgs mass is given by

$$m_h = \frac{2\pi\sqrt{b}v}{a\sqrt{f(0)}}. \tag{12.1.4}$$

Inserting (12.1.1) into this expression for the Higgs mass, we see that M_W and m_h are related by

$${m_h}^2 = 32 \frac{b}{a^2} {M_W}^2.$$

Next, we introduce the quartic Higgs coupling constant λ by writing

$$\frac{b\pi^2}{2a^2 f(0)} h^4 =: \frac{1}{24} \lambda h^4.$$

From (11.3.4) we then find

$$\lambda = 24 \frac{b}{a^2} {g_2}^2, \tag{12.1.5}$$

so that the (tree-level) Higgs mass can be expressed in terms of the mass M_W of the W-boson, the coupling constant g_2 and the quartic Higgs coupling λ as

$${m_h}^2 = \frac{4\lambda {M_W}^2}{3{g_2}^2}. \tag{12.1.6}$$

12.1.3. The seesaw mechanism. Let us consider the mass terms for the neutrinos. The matrix D_F described in Section 11.1 provides the Dirac masses as well as the Majorana masses of the fermions. After a rescaling as in (11.4.3), the mass matrix restricted to the subspace of H_F with basis $\{\nu_L, \nu_R, \overline{\nu_L}, \overline{\nu_R}\}$ is given by

$$\begin{pmatrix} 0 & m_\nu^* & m_R^* & 0 \\ m_\nu & 0 & 0 & 0 \\ m_R & 0 & 0 & \overline{m}_\nu^* \\ 0 & 0 & \overline{m}_\nu & 0 \end{pmatrix}.$$

Suppose we consider only one generation, so that m_ν and m_R are just scalars. The eigenvalues of the above mass matrix are then given by

$$\pm\frac{1}{2} m_R \pm \frac{1}{2}\sqrt{{m_R}^2 + 4{m_\nu}^2}.$$

If we assume that $m_\nu \ll m_R$, then these eigenvalues are approximated by $\pm m_R$ and $\pm\frac{{m_\nu}^2}{m_R}$. This means that there is a heavy neutrino, for which the Dirac mass m_ν may be neglected, so that its mass is given by the Majorana mass m_R. However, there is also a light neutrino, for which the Dirac and Majorana terms conspire to yield a mass $\frac{{m_\nu}^2}{m_R}$, which is in fact much smaller than the Dirac mass m_ν. This is called the *seesaw mechanism*. Thus, even though the observed masses for these neutrinos may be very small, they might still have large Dirac masses (or Yukawa couplings).

From (12.1.2) we obtained a relation between the masses of the top quark and the W-boson by neglecting all other fermion masses. However, because of the seesaw mechanism it might be that one of the neutrinos has a Dirac mass of the same order of magnitude as the top quark. In that case, it would not be justified to neglect all other fermion masses, but instead we need to correct for such massive neutrinos.

Let us introduce a new parameter ρ (typically taken to be of order 1) for the ratio between the Dirac mass m_ν for the tau-neutrino and the mass m_{top} of the top quark at unification scale, so we write $m_\nu = \rho m_{\text{top}}$. Instead of (12.1.3), we then obtain the restriction

$$m_{\text{top}} \lesssim \sqrt{\frac{8}{3+\rho^2}} M_W. \tag{12.1.7}$$

12.2. Renormalization group flow

In this section we evaluate the renormalization group equations (RGEs) for the Standard Model from ordinary energies up to the unification scale. For the validity of these RGEs we need to assume the existence of a 'big desert' up to the grand unification scale. This means that one assumes that:

- there exist no new particles (besides the known Standard Model particles) with a mass below the unification scale;
- perturbative quantum field theory remains valid throughout the big desert.

Furthermore, we also ignore any gravitational contributions to the renormalization group flow.

12.2.1. Coupling constants. In (11.3.1) we introduced the coupling constants for the gauge fields, and we obtained the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$. This is precisely the relation between the coupling constants at (grand) unification, common to grand unified theories (GUT). Thus, it would be natural to assume that our model is defined at the scale Λ_{GUT}. However, it turns out that there is no scale at which the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ holds exactly, as we show below.

The renormalization group β-functions of the (minimal) standard model read

$$\frac{dg_i}{dt} = -\frac{1}{16\pi^2} b_i g_i^3; \qquad b = \left(-\frac{41}{6}, \frac{19}{6}, 7\right),$$

where $t = \log \mu$. At first order, these equations are uncoupled from all other parameters of the Standard Model, and the solutions for the running coupling constants $g_i(\mu)$ at the energy scale μ are easily seen to satisfy

$$g_i(\mu)^{-2} = g_i(M_Z)^{-2} + \frac{b_i}{8\pi^2} \log \frac{\mu}{M_Z}, \tag{12.2.1}$$

where M_Z is the experimental mass of the Z-boson:

$$M_Z = 91.1876 \pm 0.0021 \text{ GeV}.$$

For later convenience, we also recall that the experimental mass of the W-boson is

$$M_W = 80.399 \pm 0.023 \text{ GeV}. \tag{12.2.2}$$

The experimental values of the coupling constants at the energy scale M_Z are known too, and are given by

$$\begin{aligned} g_1(M_Z) &= 0.3575 \pm 0.0001, \\ g_2(M_Z) &= 0.6519 \pm 0.0002, \\ g_3(M_Z) &= 1.220 \pm 0.004. \end{aligned}$$

Using these experimental values, we obtain the running of the coupling constants in Figure 12.1. As can be seen in this figure, the running coupling constants do not meet at any single point, and hence they do not determine a unique unification scale Λ_{GUT}. In other words, the relation ${g_3}^2 = {g_2}^2 = \frac{5}{3}{g_1}^2$ cannot hold exactly at any energy scale, unless we drop the big desert hypothesis. Nevertheless, in the remainder of this section we assume that this relation holds at least approximately and we will come back to this point in the next section. We consider the range for Λ_{GUT} determined by the triangle of the running coupling constants in Figure 12.1. The scale Λ_{12} at the intersection of $\sqrt{\frac{5}{3}}g_1$ and g_2 determines the lowest value for Λ_{GUT}, given by

$$\Lambda_{12} = M_Z \exp\left(\frac{8\pi^2(\frac{3}{5}g_1(M_Z)^{-2} - g_2(M_Z)^{-2})}{b_2 - \frac{3}{5}b_1}\right) = 1.03 \times 10^{13} \text{ GeV}. \tag{12.2.3}$$

The highest value Λ_{23} is given by the solution of $g_2 = g_3$, which yields

$$\Lambda_{23} = M_Z \exp\left(\frac{8\pi^2(g_3(M_Z)^{-2} - g_2(M_Z)^{-2})}{b_2 - b_3}\right) = 9.92 \times 10^{16} \text{ GeV}. \tag{12.2.4}$$

We assume that the Lagrangian we have derived from the almost-commutative manifold $M \times F_{SM}$ is valid at some scale Λ_{GUT}, which we take to be between Λ_{12} and Λ_{23}. All relations obtained in Figure 12.1 are assumed to hold approximately at this scale, and all predictions that will follow from these relations are therefore also only approximate.

12.2.2. Renormalization group equations. The running of the neutrino masses has been studied in a general setting for non-degenerate seesaw scales. In what follows we consider the case where only the tau-neutrino has a large Dirac mass m_ν, which cannot be neglected with respect to the mass of the top-quark. In the remainder of this section we calculate the running of the Yukawa couplings for the top-quark and the tau-neutrino, as well as the running of the quartic Higgs coupling. Let us write y_{top} and y_ν for the Yukawa couplings of the top quark and the tau-neutrino, defined by

$$m_{\text{top}} = \frac{1}{2}\sqrt{2}y_{\text{top}}v, \qquad m_\nu = \frac{1}{2}\sqrt{2}y_\nu v, \tag{12.2.5}$$

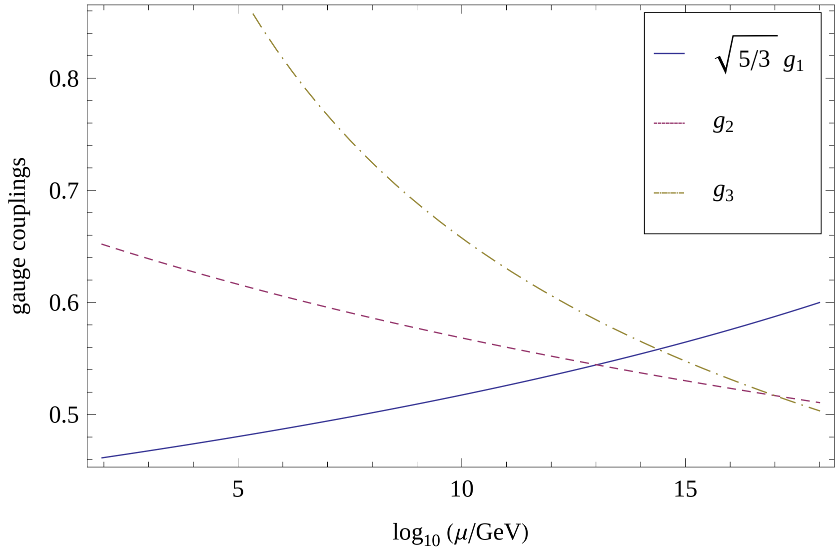


FIGURE 12.1. The running of the gauge coupling constants.

where v is the vacuum expectation value of the Higgs field.

Let m_R be the Majorana mass for the right-handed tau-neutrino. By the Appelquist–Carazzone decoupling theorem (*cf.* Note 5 on Page 184) we can distinguish two energy domains: $E > m_R$ and $E < m_R$. We again neglect all fermion masses except for the top quark and the tau neutrino. For high energies $E > m_R$, the renormalization group equations are given by

$$
\begin{aligned}
\frac{dy_{\text{top}}}{dt} &= \frac{1}{16\pi^2}\left(\frac{9}{2}y_{\text{top}}^2 + y_\nu^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right) y_{\text{top}}, \\
\frac{dy_\nu}{dt} &= \frac{1}{16\pi^2}\left(3y_{\text{top}}^2 + \frac{5}{2}y_\nu^2 - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right) y_\nu, \\
\frac{d\lambda}{dt} &= \frac{1}{16\pi^2}\Bigg(4\lambda^2 - (3{g_1}^2 + 9{g_2}^2)\lambda + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4) \\
&\qquad + 4(3y_{\text{top}}^2 + {y_\nu}^2)\lambda - 12(3y_{\text{top}}^4 + {y_\nu}^4)\Bigg).
\end{aligned}
\tag{12.2.6}
$$

Below the threshold $E = m_R$, the Yukawa coupling of the tau-neutrino drops out of the RG equations and is replaced by an effective coupling

$$\kappa = 2\frac{{y_\nu}^2}{m_R},$$

which provides an effective mass $m_l = \frac{1}{4}\kappa v^2$ for the light tau-neutrino. The renormalization group equations of y_{top} and λ for $E < m_R$ are then given by

$$\frac{dy_{\text{top}}}{dt} = \frac{1}{16\pi^2}\left(\frac{9}{2}y_{\text{top}}^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right)y_{\text{top}},$$

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\Bigg(4\lambda^2 - (3{g_1}^2 + 9{g_2}^2)\lambda + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4) + 12y_{\text{top}}^2\lambda - 36y_{\text{top}}^4\Bigg). \tag{12.2.7}$$

Finally, the equation for y_ν is replaced by an equation for the effective coupling κ given by

$$\frac{d\kappa}{dt} = \frac{1}{16\pi^2}\left(6y_{\text{top}}^2 - 3{g_2}^2 + \frac{\lambda}{6}\right)\kappa. \tag{12.2.8}$$

12.2.3. Running masses. The numerical solutions to the coupled differential equations of (12.2.6) (12.2.7) and (12.2.8) for y_{top}, y_ν and λ depend on the choice of three input parameters:

- the scale Λ_{GUT} at which our model is defined;
- the ratio ρ between the masses m_ν and m_{top};
- the Majorana mass m_R that produces the threshold in the renormalization group flow.

The scale Λ_{GUT} is taken to be either $\Lambda_{12} = 1.03 \times 10^{13}$ GeV or $\Lambda_{23} = 9.92 \times 10^{16}$ GeV, as given by (12.2.3) and (12.2.4), respectively. We now determine the numerical solution to (12.2.6), (12.2.7) and (12.2.8) for a range of values for ρ and m_R. First, we need to start with the initial conditions of the running parameters at the scale Λ_{GUT}. Inserting the top-quark mass $m_{\text{top}} = \frac{1}{2}\sqrt{2}y_{\text{top}}v$, the tau-neutrino mass $m_\nu = \rho m_{\text{top}}$, and the W-boson mass $M_W = \frac{1}{2}g_2 v$ into (12.1.7), we obtain the constraints

$$y_{\text{top}}(\Lambda_{GUT}) \lesssim \frac{2}{\sqrt{3+\rho^2}}g_2(\Lambda_{GUT}), \qquad y_\nu(\Lambda_{GUT}) \lesssim \frac{2\rho}{\sqrt{3+\rho^2}}g_2(\Lambda_{GUT}),$$

where (12.2.1) yields the values $g_2(\Lambda_{12}) = 0.5444$ and $g_2(\Lambda_{23}) = 0.5170$.

Furthermore, from (12.1.5) we obtain an expression for the quartic coupling λ at Λ_{GUT}. Approximating the coefficients a and b from (11.3.2) by $a \approx (3+\rho^2)m_{\text{top}}^2$ and $b \approx (3+\rho^4)m_{\text{top}}^4$, we obtain the boundary condition

$$\lambda(\Lambda_{GUT}) \approx 24\frac{3+\rho^4}{(3+\rho^2)^2}g_2(\Lambda_{GUT})^2.$$

Using these boundary conditions, we can now numerically solve the RG equations of (12.2.6) from Λ_{GUT} down to m_R, which provides us with values for $y_{\text{top}}(m_R)$, $y_\nu(m_R)$ and $\lambda(m_R)$. At this point, the Yukawa coupling y_ν is replaced by the effective coupling κ with boundary condition

$$\kappa(m_R) = 2\frac{y_\nu(m_R)^2}{m_R}.$$

Next, we numerically solve the RG equations of (12.2.7) and (12.2.8) down to M_Z to obtain the values for y_{top}, κ and λ at 'low' energy scales.

The running mass of the top quark at these energies is given by (12.2.5). We find the running Higgs mass by inserting λ into (12.1.6). We shall evaluate these running masses at their own energy scale. For instance, our

Λ_{GUT} (10^{13} GeV)	1.03	1.03	1.03	1.03	1.03	1.03	1.03
ρ	0	0.90	0.90	1.00	1.00	1.10	1.10
m_R (10^{13} GeV)	–	0.25	1.03	0.30	1.03	0.35	1.03
m_{top} (GeV)	183.2	173.9	174.1	171.9	172.1	169.9	170.1
m_l (eV)	0	2.084	0.5037	2.076	0.6030	2.080	0.7058
m_h (GeV)	188.3	175.5	175.7	173.4	173.7	171.5	171.8

Λ_{GUT} (10^{16} GeV)	9.92	9.92	9.92	9.92	9.92
ρ	0	1.10	1.10	1.20	1.20
m_R (10^{13} GeV)	–	0.30	2.0	0.35	9900
m_{top} (GeV)	186.0	173.9	174.2	171.9	173.5
m_l (eV)	0	1.939	0.2917	1.897	6.889×10^{-5}
m_h (GeV)	188.1	171.3	171.6	169.1	171.2

Λ_{GUT} (10^{16} GeV)	9.92	9.92	9.92	9.92
ρ	1.30	1.30	1.35	1.35
m_R (10^{13} GeV)	0.40	9900	100	9900
m_{top} (GeV)	169.9	171.6	169.8	170.6
m_l (eV)	1.866	7.818×10^{-5}	8.056×10^{-3}	8.286×10^{-5}
m_h (GeV)	167.1	169.3	167.4	168.4

TABLE 12.1. Numerical results for the masses m_{top} of the top-quark, m_l of the light tau-neutrino, and m_h of the Higgs boson, as a function of Λ_{GUT}, ρ, and m_R.

predicted mass for the Higgs boson is the solution for μ of the equation $\mu = \sqrt{\lambda(\mu)/3}v$, in which we ignore the running of the vacuum expectation value v.

The effective mass of the light neutrino is determined by the effective coupling κ, and we choose to evaluate this mass at scale M_Z. Thus, we calculate the masses by

$$\begin{aligned} m_{\text{top}}(m_{\text{top}}) &= \frac{1}{2}\sqrt{2}y_{\text{top}}(m_{\text{top}})v, \\ m_l(M_Z) &= \frac{1}{4}\kappa(M_Z)v^2, \\ m_h(m_h) &= \sqrt{\frac{\lambda(m_h)}{3}}v, \end{aligned}$$

where, from the W-boson mass (12.2.2) we can insert the value $v = 246.66 \pm 0.15$. The results of this procedure for m_{top}, m_l and m_h are given in Table 12.1. In this table, we have chosen the range of values for ρ and m_R such that the mass of the top-quark and the light tau-neutrino are in agreement with their experimental values

$$m_{\text{top}} = 172.0 \pm 0.9 \pm 1.3 \text{ GeV}, \qquad m_l \leq 2 \text{ eV}.$$

For comparison, we have also included the simple case where we ignore the Yukawa coupling of the tau-neutrino (by setting $\rho = 0$), in which case there

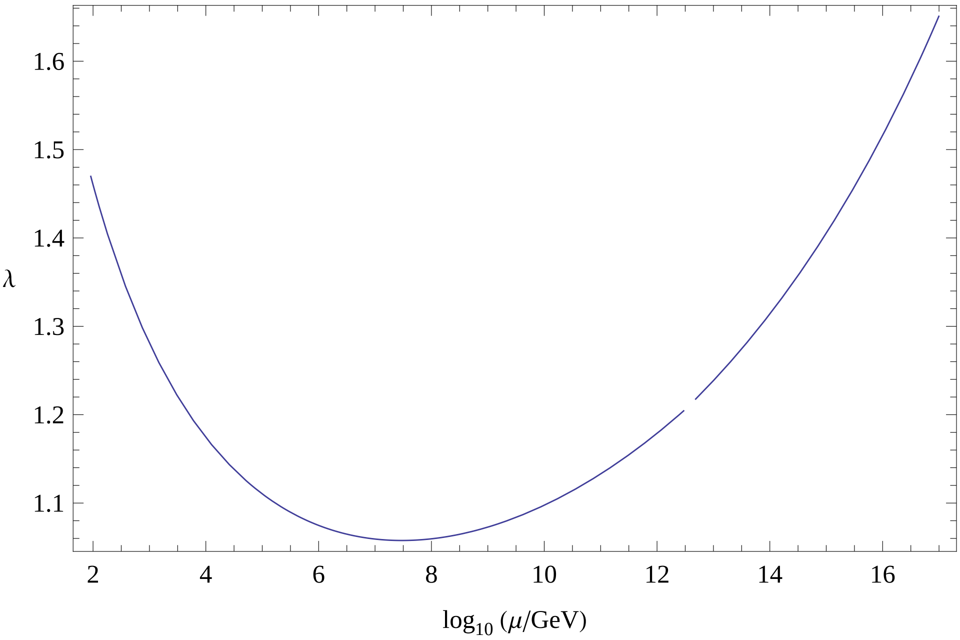


FIGURE 12.2. The running of the quartic Higgs coupling λ for $\Lambda_{GUT} = 9.92\times10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3\times10^{12}$ GeV.

is no threshold at the Majorana mass scale either. As an example, we have plotted the running of λ, y_{top}, y_ν and κ for the values of $\Lambda_{GUT} = \Lambda_{23} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV in Figures 12.2, 12.3, 12.4 and 12.5.

For the allowed range of values for ρ and m_R that yield plausible results for m_{top} and m_l, we see that the mass m_h of the Higgs boson takes its value within the range

$$167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}.$$

The errors in this prediction, which result from the initial conditions (other than m_{top} and m_l) taken from experiment, as well as from ignoring higher-loop corrections to the RGEs, are smaller than this range of possible values for the Higgs mass, and therefore we may ignore these errors.

12.3. Higgs mass: comparison to experimental results

It is time to confront the above predicted range of values with the discovery of a Higgs boson with a mass $m_h \simeq 125.5$GeV at the ATLAS and CMS experiments at the Large Hadron Collider at CERN. At first sight, this experimentally measured value seems to be at odds with the above prediction and seems to falsify the description of the Standard Model as an almost-commutative manifold. However, let us consider more closely the (main) hypotheses on which the above prediction is based, discussing them one-by-one.

The almost-commutative manifold $M \times F_{SM}$: An essential input in the above derivation is the replacement of the background manifold M by a noncommutative space $M \times F_{SM}$. We motivated the structure

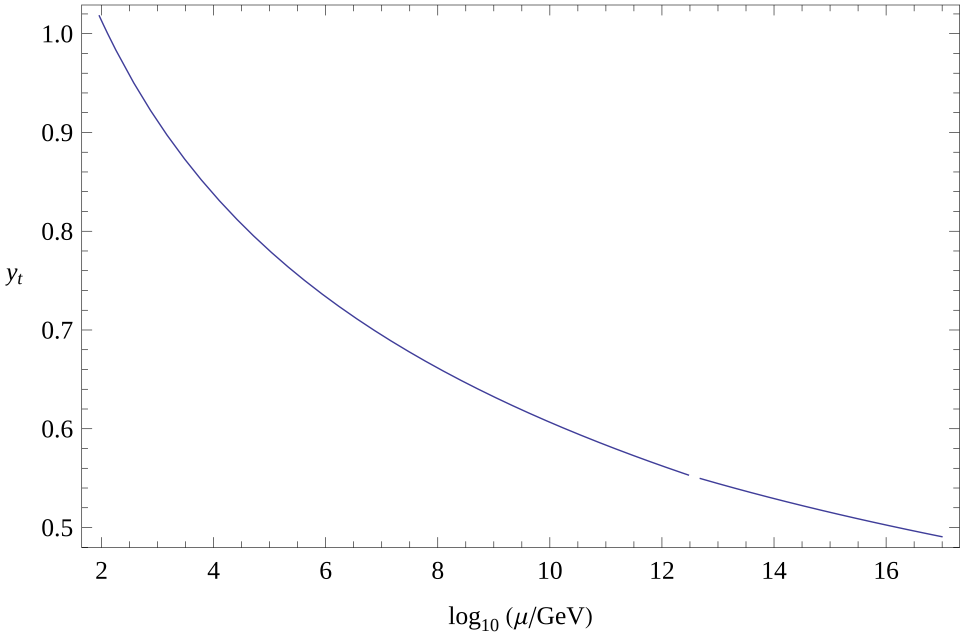


FIGURE 12.3. The running of the top-quark Yukawa coupling y_{top} for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV.

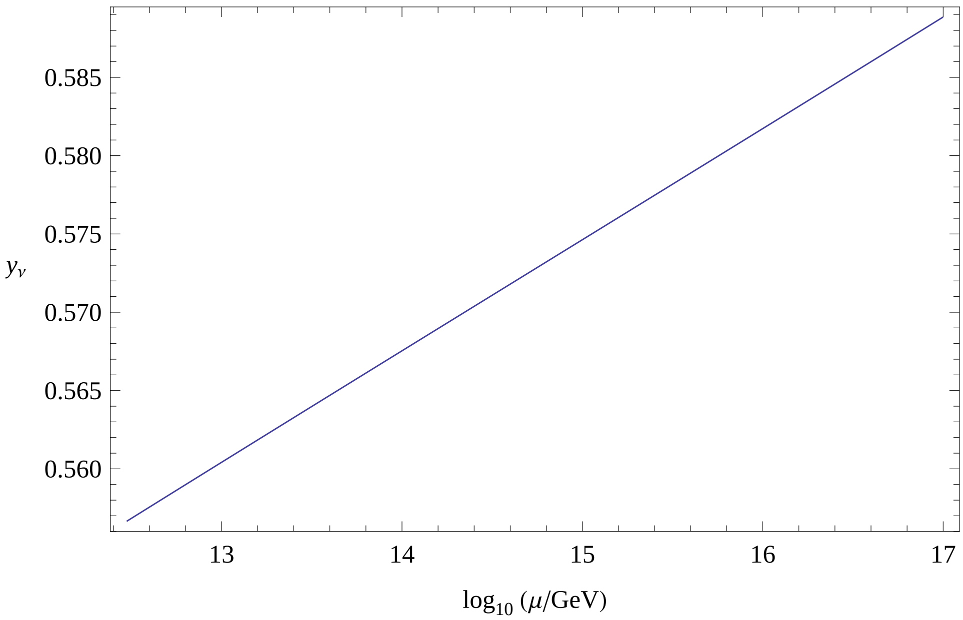


FIGURE 12.4. The running of the tau-neutrino Yukawa coupling y_ν for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV.

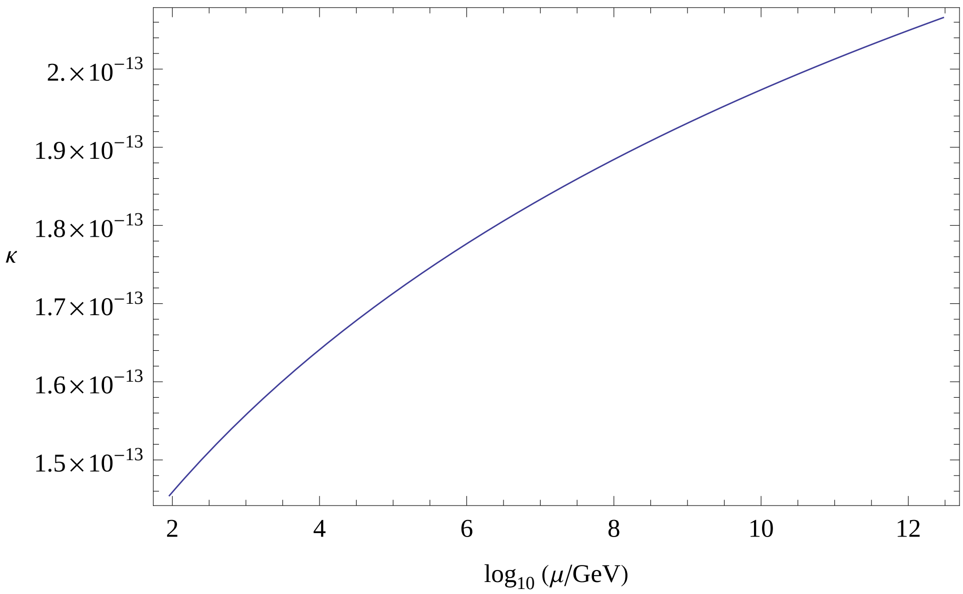


FIGURE 12.5. The running of the effective coupling κ for $\Lambda_{GUT} = 9.92 \times 10^{16}$ GeV, $\rho = 1.2$, and $m_R = 3 \times 10^{12}$ GeV.

of F_{SM} by deriving it from a list of finite irreducible geometries, along the way imposing several mathematical constraints (*cf.* Section 11.1). The strength of this approach was that it allowed for a *derivation* of all the particles and symmetries of the Standard Model from purely geometrical data. Moreover, the spectral action resulted in the Lagrangian of the Standard Model, including Higgs mechanism.

The incompatibility of the prediction of the Higgs mass with experiment might be resolved by considering almost-commutative manifolds that go beyond the Standard Model by dropping some of the aforementioned mathematical constraints; we will discuss a recently proposed possibility in the next Section.

Ultimately, one should also consider noncommutative manifolds that are not the product of M with a finite space F (see Note 7 on Page 185).

The spectral action: The bosonic Lagrangian was derived from the asymptotic expansion of the spectral action $\mathrm{Tr}\, f(D/\Lambda)$.

Adopting Wilson's viewpoint on the renormalization group equation this Lagrangian was considered the bare Lagrangian at the cutoff scale Λ. The renormalization group equations then dictate the running of the renormalized, physical parameters.

Alternatively, one can consider the spectral action for $M \times F_{SM}$ in a perturbative expansion in the fields, as in Section 7.2.2, leading to unexpected and an intriguing behaviour for the propagation of particles at energies larger than the cutoff Λ (see Note 9 on Page 185).

Yet another alternative is to consider Λ as a regularization parameter, allowing for an interpretation of the asymptotic expansion of $\mathrm{Tr}\, f(D/\Lambda)$ as a higher-derivative gauge theory. It turns out that conditions can be formulated on the Krajewski diagram for F that guarantee the (super)renormalizability of the asymptotic expansion of the spectral action for the corresponding almost-commutative manifold $M \times F$ (see Note 9 on Page 185).

Big desert: In our RGE-analysis of the couplings and masses we have assumed the big desert up to the GUT-scale: no more elementary particles than those present in the Standard Model exist at higher energies (and up to the GUT-scale). This is a good working hypothesis, but is unlikely to be true. The main reason for this is the mismatch of the running coupling constants at the GUT-scale (Figure 12.1). This indicates that new physics is expected to appear before this scale. As already suggested, it might very well be that this new physics can be described by considering almost-commutative manifolds that go beyond the Standard Model. We will discuss such a possibility in the next Section.

Renormalization group equations: We exploited renormalization group techniques to run couplings and masses down from the GUT-scale to ordinary energies. The renormalization group equations were derived in a perturbative approach to quantum field theory, which was supposed to be valid at all scales. Moreover, we have adopted the one-loop beta-functions, something which can definitely be improved. Even though this might lead to more accurate predictions, it is not expected to resolve the incompatibility between the predicted range for m_h and the experimentally measured value.

Gravitational effects: In our analysis we have discarded all possible gravitational effects on the running of the couplings constants. It might very well be that gravitational correction terms alter the predicted values to a more realistic value.

12.4. Noncommutative geometry beyond the Standard Model

Let us then drop some of the above hypotheses, and demonstrate how a small correction of the space $M \times F_{SM}$ gives an intriguing possibility to go beyond the Standard Model, solving at the same time a problem with the stability of the Higgs vacuum given the measured low mass m_h.

Namely, in the definition of the finite Dirac operator D_F of Equation 11.1.2, we can replace Y_R by $Y_R\sigma$, where σ is a real scalar field on M. Strictly speaking, this brings us out of the class of almost-commutative manifolds $M \times F$, since part of D_F now varies over M. Nevertheless, it fits perfectly into the more general class of topologically non-trivial almost-commutative geometries. In fact, it is enough to consider the trivial fiber bundle $M \times H_F$, for which an endomorphism $D_F(x) \in \mathrm{End}(H_F)$ is allowed to depend smoothly on $x \in M$.

The scalar field σ can also be seen as the relic of a spontaneous symmetry breaking mechanism, similar to the Higgs field h in the electroweak sector of the Standard Model. Starting point is the almost-commutative manifold

$M \times F_{PS}$ based on the algebra $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ with which we started Chapter 11. The gauge group corresponding to F_{PS} is $SU(2) \times SU(2) \times SU(4)$ and the corresponding model is called **Pati–Salam unification**. It turns out that the spectral action for $M \times F_{PS}$ yields a spontaneous symmetry breaking mechanism that *dynamically* selects the algebra $A_F \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ of Proposition 11.1.

Let us then replace Y_R by $Y_R\sigma$ and analyze the additional terms in the spectral action. In Proposition 11.9 we insert a σ for every Y_R that appears, to arrive at

$$\begin{aligned} \mathcal{L}'_H(g_{\mu\nu}, \Lambda_\mu, Q_\mu, H, \sigma) &:= \frac{bf(0)}{2\pi^2}|H|^4 - \frac{2af_2\Lambda^2}{\pi^2}|H|^2 + \frac{ef(0)}{\pi^2}\sigma^2|H|^2 \\ &-\frac{cf_2\Lambda^2}{\pi^2}\sigma^2 + \frac{df(0)}{4\pi^2}\sigma^4 + \frac{af(0)}{2\pi^2}|D_\mu H|^2 + \frac{1}{4\pi^2}f(0)c(\partial_\mu\sigma)^2, \end{aligned}$$

where we ignored the coupling to the scalar curvature.

As before, we exploit the approximation that m_{top}, m_ν and m_R are the dominant mass terms. Moreover, as before we write $m_\nu = \rho m_{top}$. That is, the expressions for a, b, c, d and e in (11.3.2) now become

$$\begin{aligned} a &\approx m_{top}^2(\rho^2 + 3), \\ b &\approx m_{top}^4(\rho^4 + 3), \\ c &\approx m_R^2, \\ d &\approx m_R^4, \\ e &\approx \rho^2 m_R^2 m_{top}^2. \end{aligned}$$

In a unitary gauge, where $H = \begin{pmatrix} h \\ 0 \end{pmatrix}$, we arrive at the following potential:

$$\mathcal{L}_{pot}(h, \sigma) = \frac{1}{24}\lambda_h h^4 + \frac{1}{2}\lambda_{h\sigma}h^2\sigma^2 + \frac{1}{4}\lambda_\sigma\sigma^4 - \frac{4g_2^2}{\pi^2}f_2\Lambda^2(h^2 + \sigma^2),$$

where we have defined coupling constants

$$\lambda_h = 24\frac{\rho^4 + 3}{(\rho^2 + 3)^2}g_2^2, \qquad \lambda_{h\sigma} = \frac{8\rho^2}{\rho^2 + 3}g_2^2, \qquad \lambda_\sigma = 8g_2^2. \tag{12.4.1}$$

This potential can be minimized, and if we replace h by $v + h$ and σ by $w + \sigma$, respectively, expanding around a minimum for the terms quadratic in the fields, we obtain:

$$\begin{aligned} \mathcal{L}_{pot}(v + h, w + \sigma)|_{\text{quadratic}} &= \frac{1}{6}v^2\lambda_h v^2 + 2vw\lambda_{h\sigma}\sigma h + w^2\lambda_\sigma\sigma^2 \\ &= \frac{1}{2}\begin{pmatrix} h & \sigma \end{pmatrix} M^2 \begin{pmatrix} h \\ \sigma \end{pmatrix}, \end{aligned}$$

where we have defined the mass matrix M by

$$M^2 = 2\begin{pmatrix} \frac{1}{6}\lambda_h v^2 & \lambda_{h\sigma}vw \\ \lambda_{h\sigma}vw & \lambda_\sigma w^2 \end{pmatrix}.$$

This mass matrix can be easily diagonalized, and if we make the natural assumption that w is of the order of m_R, while v is of the order of M_W, so that $v \ll w$, we find that the two eigenvalues are

$$m_+^2 \sim 2\lambda_\sigma w^2 + 2\frac{\lambda_{h\sigma}^2}{\lambda_\sigma}v^2,$$
$$m_-^2 \sim 2\lambda_h v^2 \left(\frac{1}{6} - \frac{\lambda_{h\sigma}^2}{\lambda_h\lambda_\sigma}\right).$$

We can now determine the value of these two masses by running the scalar coupling constants $\lambda_h, \lambda_{h\sigma}$ and λ_σ down to ordinary energy scalar. The renormalization group equations for these couplings are given by

$$\frac{d\lambda_h}{dt} = \frac{1}{16\pi^2}\Bigg(4\lambda_h^2 + 12\lambda_{h\sigma}^2 - (3{g_1}^2 + 9{g_2}^2)\lambda_h + \frac{9}{4}({g_1}^4 + 2{g_1}^2{g_2}^2 + 3{g_2}^4)$$
$$+ 4(3y_{\text{top}}^2 + {y_\nu}^2)\lambda_h - 12(3y_{\text{top}}^4 + {y_\nu}^4)\Bigg),$$
$$\frac{d\lambda_{h\sigma}}{dt} = \frac{1}{16\pi^2}\Bigg(8\lambda_{h\sigma}^2 + 6\lambda_{h\sigma}\lambda_\sigma + 2\lambda_{h\sigma}\lambda_h$$
$$- \frac{3}{2}\left(g_1^2 + 3g_2^2\right)\lambda_{h\sigma} + 2(3y_{\text{top}}^2 + {y_\nu}^2)\lambda_{h\sigma}\Bigg),$$
$$\frac{d\lambda_\sigma}{dt} = \frac{1}{16\pi^2}\left(8\lambda_{h\sigma}^2 + 18\lambda_\sigma^2\right).$$

As before, at lower energy the coupling y_ν drops out of the RG equations and is replaced by an effective coupling.

At one-loop, the other couplings obey the renormalization group equations of the Standard Model, that is, they satisfy (12.2.6) and (12.2.7). As before, we can solve these differential equations, with boundary conditions at Λ_{GUT} given for the scalar couplings by (12.4.1). The result varies with the chosen value for Λ_{GUT} and the parameter ρ. The mass of σ is essentially given by the largest eigenvalue m_+ which is of the order 10^{12} GeV for all values of Λ_{GUT} and the parameter ρ. The allowed mass range for the Higgs, *i.e.* for m_-, is depicted in Figure 12.6. The expected value $m_h = 125.5$ GeV is therefore compatible with the above noncommutative model. Furthermore, this calculation implies that there is a relation (given by the red line in the Figure) between the ratio m_ν/m_{top} and the unification scale Λ_{GUT}.

We conclude that with noncommutative geometry we can proceed beyond the Standard Model, enlarging the field content of the Standard Model by a real scalar field with a mass of the order of 10^{12} GeV. At the time of writing of this book (Spring 2014), this is completely compatible with experiment and also guarantees stability of the Higgs vacuum at higher energy scales. Of course, the final word is to experiment in the years to come. What we can say at this point is that noncommutative geometry provides a fascinating dialogue between abstract mathematics and concrete measurements in experimental high-energy physics.

Notes

1. In the first part of this Chapter, we mainly follow [**54**, Section 5] (see also [**65**, Ch. 1, Section 17]). In Section 12.2 we have also incorporated the running of the neutrino masses as in [**117**] (see also [**186**]).

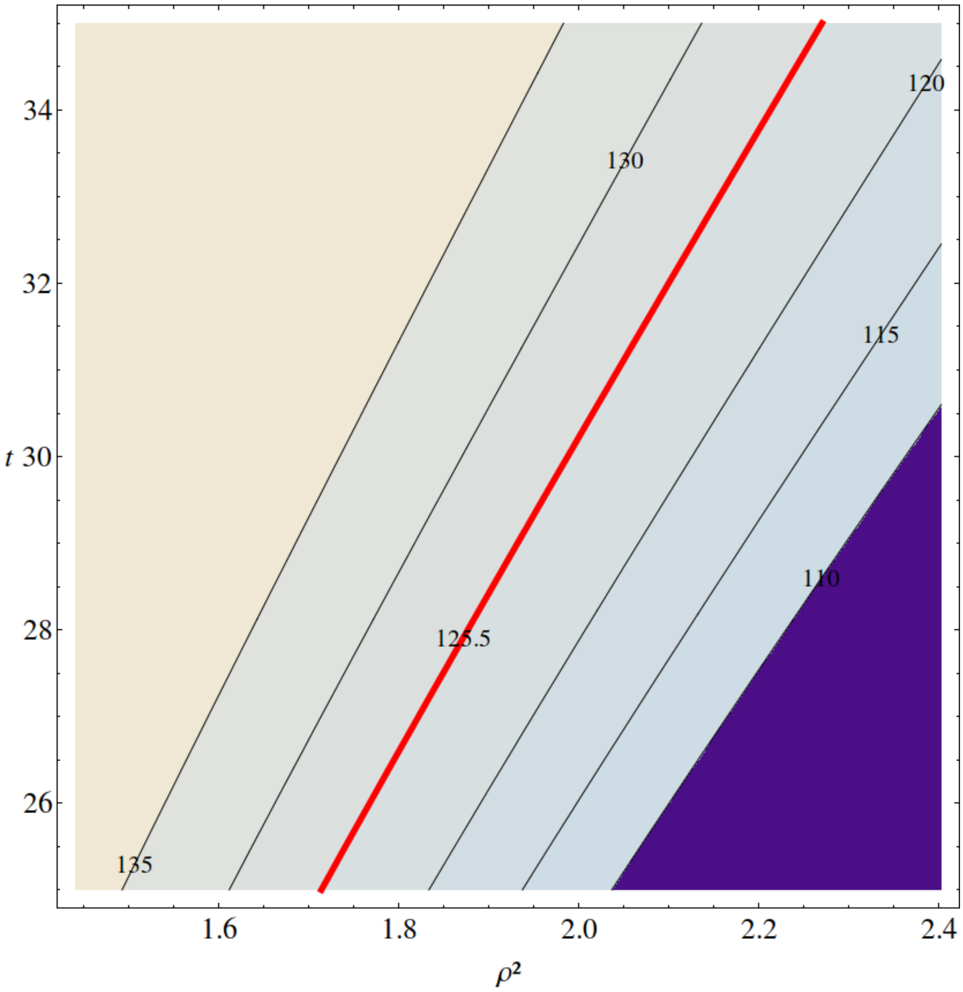


FIGURE 12.6. A contour plot of the Higgs mass m_h as a function of ρ^2 and $t = \log(\Lambda_{GUT}/M_Z)$. The red line corresponds to $m_h = 125.5$ GeV.

Section 12.1. Mass relations

2. Further details on the see-saw mechanism can be found in *e.g.* [**152**].

Section 12.2. Renormalization group flow

3. The renormalization group β-functions of the (minimal) standard model are taken from [**142, 143, 144**] and [**93**]. We simplify the expressions by ignoring the 2-loop contributions, and instead consider only the 1-loop approximation. The renormalization group β-functions are [**142**, Eq. (B.2)] or [**93**, Eq. (A.1)]).

4. The experimental masses of the Z and W-boson and the top quark, as well as the experimental values of the coupling constants at the energy scale M_Z are found in [**156**].

5. In arriving at (12.2.6) we have followed the approach of [**117**] where two energy domains are considered: $E > m_R$ and $E < m_R$. The Appelquist–Carazzone decoupling theorem is found in [**5**]. For the renormalization group equations, we refer to [**143**, Eq. (B.4)], [**4**, Eq. (14) and (15)] and [**144**, Eq. (B.3)].

Section 12.3. Higgs mass: comparison to experimental results

6. The discovery of the Higgs boson at the ATLAS and CMS experiments is published in [**1, 57**].

7. The spectral action has also been computed for spectral triples that are not the product of M with a finite space F, and which are further off the 'commutative shore'. These include the noncommutative torus [**88**], the Moyal plane [**95, 104**], the quantum group $SU_q(2)$ [**110**] and the Podleś sphere S_q^2 [**85**].

8. The generalization of noncommutative geometry to *non-associative geometry* is analyzed in [**91, 36**].

9. The bosonic Lagrangian derived from the spectral action was interpreted in [**49**] à la Wilson [**199**] as the bare Lagrangian at the cutoff scale Λ. A perturbative expansion of the full spectral action was obtained in [**111, 113, 130**], leading to unexpected and an intriguing behaviour for the propagation of particles at energies larger than the cutoff Λ. Alternatively, the interpretation of Λ as a regularization parameter has been worked out in [**179, 181, 177, 180**], including the derivation of renormalizability conditions on the Krajewski diagrams.

10. Other searches beyond the Standard Model with noncommutative geometry include [**172, 173, 175, 174, 176**], adopting a slightly different approach to almost-commutative manifolds as we do (*cf.* Note 3 on Page 97). The intersection between supersymmetry and almost-commutative manifolds is analyzed in [**38, 39, 21, 22, 23**].

11. A possible approach to incorporate gravitational effects in the running of the coupling constants is discussed in [**89**].

Section 12.4. Noncommutative geometry beyond the Standard Model

12. For stability bounds on the Higgs mass, we refer to [**170**].

13. The small correction to the space $M \times F_{SM}$ was realized in [**53**] (and already tacitly present in [**52**]) and we here confirm their conclusions. The class of topologically non-trivial almost-commutative geometries has been worked out in [**40, 41, 34**]. The spontaneous symmetry breaking of the noncommutative description of the Pati–Salam model [**159**] was analyzed in [**56, 55**], after generalizing inner fluctuations to real spectral triples that do not necessarily satisfy the first-order condition (4.3.1).

14. In [**76**] an alternative approach is considered, taking the 'grand' algebra $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ from the list of [**51**], but where now the condition of bounded commutators of D with the algebra is not satisfied.

15. The renormalization group equations for the couplings $\lambda_h, \lambda_{h\sigma}, \lambda_\sigma$ have been derived in [**102**].

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