Factorization of Dirac operators in unbounded KK-theory

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27 March 2017

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Motivation

Consider the following setup:

- A Riemannian submersion

$$\pi : M \to B$$

of spin^c manifolds M and B

- It is well-known from wrong-way functoriality [CS, 1984] that

$$[M] = \pi! \widehat{\otimes}_{C_0(B)} [B] \qquad (*)$$

- Moreover, the Dirac operator on M can be written as a tensor sum:

$$D_M = S \widehat{\otimes} \gamma + 1 \widehat{\otimes}_\nabla D_B$$

representing the internal KK-product $(*)$ [KvS, 2016].

This talk: can we obtain similar factorization for singular fibrations, that is, if the fiber dimension of $\pi : M \to B$ is allowed to vary?

Key application: toric noncommutative manifolds

- Let $\mathbb{T}^n$ act on N by isometries with principal stratum $M \subseteq N$.
- Deformation quantization both at the topological [Rie, 1993] and geometric level [CL, 2001]:

$$(C^\infty(N_\theta), L^2(\mathscr{E}_N), D_N)$$

- At the C^*-algebraic level, using a real structure one can identify [vS, 2015] a continuous C^*-bundle $\mathfrak{B}$ so that

$$C(N_\theta) = \Gamma(N/\mathbb{T}^n, \mathfrak{B})$$

- Gauge theory: $\mathrm{Inn}(\mathcal{A})$ acts fiberwise on $\mathfrak{B}$.
- A tensor sum factorization of the above type means that this bundle picture extends to the geometric level:

$$D_N = S \widehat{\otimes} \gamma + 1 \widehat{\otimes}_\nabla D_{M/\mathbb{T}^n}$$

where S is a suitably defined family of Dirac operators.

Setup: almost-regular fibrations

We consider singular fibrations in the following sense:

Definition

Let N be a closed Riemannian manifold, together with an embedded submanifold $P \subset N$ without boundary of codimension greater than 1. If there exists a proper Riemannian submersion $\pi : M := N \setminus \overline{P} \to B$ we call the data (N, P, B, π) an almost-regular fibration.

- We say that (N, P, B, π) is an almost-regular fibration of spin^c manifolds if both N and B are spin^c manifolds.
- Major class of examples given by connected compact Lie groups G acting on closed Riemannian manifolds N

For such almost-regular fibrations (N, P, B, π) of spin^c manifolds, the factorization can be done [KvS] by introducing

- a C^*-correspondence X between $C_0(M)$ and $C_0(B)$
- A regular selfadjoint operator

$$S : \mathrm{Dom}(S) \to X$$

  with compact resolvent (family of vertical Dirac operators), defining an unbounded KK-cycle from $C_0(M)$ to $C_0(B)$.

- A hermitian connection

$$\nabla : \mathrm{Dom}(\nabla) \to X \widehat{\otimes}_{C_0(B)} \Omega^1(B)$$

Then

- $S\widehat{\otimes}\gamma$ and $1\widehat{\otimes}_\nabla D_B$ define symmetric unbounded operators on $X\widehat{\otimes}_{C_0(B)}L^2(\mathscr{E}_B^c)$
- Both $\mathscr{E}_M^c := \Gamma_c^\infty(M, \mathcal{S}_N)$ and $\mathscr{E}_N := \Gamma^\infty(N, \mathcal{S}_N)$ form a core for the selfadjoint operator D_N on $\mathrm{Dom}(D_N) \subset L^2(\mathscr{E}_N)$.

Theorem (KvS)

Up to unitary isomorphism we have the following identity of selfadjoint operators

$$S\widehat{\otimes}\gamma + 1\widehat{\otimes}_\nabla D_B = D_N + \widetilde{c}(\Omega).$$

where Ω is the curvature of $\pi : M \to B$.

Suppose that

- $\pi : M \to B$ is a proper and surjective submersion
- M is equipped with a Riemannian metric in the fiber direction.
- E is a smooth hermitian vector bundle on M and write $\mathscr{E} = \Gamma^\infty(M, E)$ and $\mathscr{E}^c = \Gamma_c^\infty(M, E)$.
- $\mathscr{D}$ is a first-order vertical differential operator on $\mathscr{E}$
- $\mathscr{D}$ is vertically elliptic, in the sense that

  the map $[\mathscr{D}, f](x) : E_x \to E_x$ *is invertible whenever* $(d_V f)(x)$ *is non-trivial.*

We define a C^*-correspondence X from $C_0(M)$ to $C_0(B)$ using the hermitian structure on $\mathscr{E}^c$ and integration along the fibers:

$$\langle s, t\rangle_X(b) := \int_{F_b} \langle s, t\rangle_{\mathscr{E}^c}$$

Moreover, $\mathscr{D}$ gives rise to an unbounded operator $D_0 : \mathscr{E}^c \to X$ by setting $D_0(s) = \mathscr{D}(s)$.

Theorem (KvS)

If the above operator D_0 is symmetric then the closure $D = \overline{D_0}$ is selfadjoint and regular and $m(f)(1 + D^2)^{-1/2}$ is a compact operator for all $f \in C_c^\infty(M)$. In other words, (X, D) is an unbounded Kasparov module from $C_0(M)$ to $C_0(B)$.

Riemannian submersions

- Let M and B be Riemannian manifolds and let

$$\pi : M \to B$$

be a smooth and surjective map
- This is a Riemannian submersion when $d\pi$ is surjective and

$$d\pi : (\ker d\pi)^{\perp} \to \mathscr{X}(B) \otimes_{C^\infty(B)} C^\infty(M)$$

is an isometric isomorphism
- This gives rise to vertical and horizontal vector fields

$$\mathscr{X}(M) \cong \mathscr{X}_V(M) \oplus \mathscr{X}_H(M).$$

On $\mathscr{X}(M)$ one can introduce the direct sum connection

$$\nabla^{\oplus} = \nabla^M_V \oplus \pi^* \nabla^B,$$

and associate to it the second fundamental form and the curvature Ω of the fibration which yield a tensor $\omega \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^2(M)$

Proposition (Bismut, 1986)

The Levi-Civita connection ∇^M is related to the direct sum connection $\nabla^{\oplus}$ by the following formula

$$\langle \nabla^M_X Y, Z \rangle_M = \langle \nabla^{\oplus}_X Y, Z \rangle_M + \omega(X)(Y, Z).$$

Suppose that M and B are (even-dim) spin^c manifolds, so we have

$$\mathrm{Cl}(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_M), \qquad \mathrm{Cl}(B) \cong \mathrm{End}_{C^\infty(B)}(\mathscr{E}_B)$$

and hermitian Clifford connections $\nabla^{\mathscr{E}_M}$ and $\nabla^{\mathscr{E}_B}$.

- We define the horizontal spinor module:

$$\mathscr{E}_H := \mathscr{E}_B \otimes_{C^\infty(B)} C^\infty(M); \qquad \nabla^{\mathscr{E}_H} := \pi^* \nabla^{\mathscr{E}_B}$$

  and $\mathrm{Cl}_H(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_H)$
- We define the vertical spinor module:

$$\mathscr{E}_V := \mathscr{E}_H^* \otimes_{\mathrm{Cl}_H(M)} \mathscr{E}_M; \qquad \nabla_X^{\mathscr{E}_V} = \nabla_X^{\mathscr{E}_H^*} \otimes 1 + 1 \otimes \nabla_X^{\mathscr{E}_M} + \frac{1}{4} c(\omega(X))$$

  and $\mathrm{Cl}_V(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_V)$
- Finally,

$$\mathscr{E}_H \otimes_{C^\infty(M)} \mathscr{E}_V \cong \mathscr{E}_M,$$

  compatibly with $\mathrm{Cl}_H(M) \widehat{\otimes}_{C^\infty(M)} \mathrm{Cl}_V(M) \cong \mathrm{Cl}(M)$.

The vertical operator

- We thus have a C^*-correspondence X from $C_0(M)$ to $C_0(B)$ by completing $\mathscr{E}_V^c := \mathscr{E}_V \otimes_{C^\infty(M)} C_c^\infty(M)$ with respect to

$$\langle \phi_1, \phi_2 \rangle_X(b) := \int_{F_b} \langle \phi_1, \phi_2 \rangle_{\mathscr{E}_V^c}$$

- The following local expression defines a symmetric operator

$$S_0(\xi) = i \sum_{j=1}^{\dim(F)} c(e_j) \nabla_{e_j}^{\mathscr{E}_V}(\xi); \qquad (\xi \in \mathscr{E}_V^c)$$

Proposition

If $\pi : M \to B$ is a proper Riemannian submersion of spin^c manifolds, then (X, S) is an even unbounded Kasparov module from $C_0(M)$ to $C_0(B)$.

The horizontal operator and the connection

- The Dirac operator $D_B : \mathrm{Dom}(D_B) \to L^2(\mathscr{E}_B^c)$ is locally

$$D_B = i \sum_{\alpha=1}^{\dim(B)} c(f_\alpha) \nabla_{f_\alpha}^{\mathscr{E}_B^c} : \mathscr{E}_B^c \to L^2(\mathscr{E}_B^c)$$

and $(C_c^\infty(B), L^2(\mathscr{E}_B^2), D_B)$ is a half-closed chain [Hil, 2010]

- The following defines a hermitian connection on X

$$\nabla_Z^X(\xi) = \nabla_{Z_H}^{\mathscr{E}_V}(\xi) + \frac{1}{2} k(Z_H) \cdot \xi$$

with $k \in \Omega^1(M)$ the mean curvature.

Lemma

The following local expression defines an odd symmetric unbounded operator in $X \widehat{\otimes}_{C_0(B)} L^2(\mathscr{E}_B^c)$:

$$(1 \otimes_\nabla D_B)(\xi \otimes r) := \xi \otimes D_B r + i \sum_\alpha \nabla_{f_\alpha}^X(\xi) \otimes c(f_\alpha) r.$$

The tensor sum

- The tensor sum we are after is given by

$$(S \times_\nabla D_B)_0 := S \otimes \gamma_B + 1 \otimes_\nabla D_B : \mathrm{Dom}(S \times_\nabla D_B)_0 \to X \widehat{\otimes}_{C_0(B)} L^2(\mathscr{E}_B^c)$$

- The closure of this symmetric operator is denoted $S \times_\nabla D_B$.
- Since $P \subseteq N$ has codimension > 1, it follows that the closures of the following two operators coincide:
  1. the first-order differential operator D_N on $\mathscr{E}_N := \Gamma^\infty(N, \mathscr{E}_N)$
  2. the restriction of D_N to $\mathscr{E}_M^c := \Gamma_c^\infty(M, \mathscr{E}_N)$ where $M = N \setminus \overline{P}$

Theorem (KvS)

Under the unitary isomorphism $V : X \widehat{\otimes}_{C_0(B)} L^2(\mathscr{E}_B^c) \to L^2(\mathscr{E}_N)$ we have the identity of selfadjoint operators

$$V(S \times_\nabla D_B)V^* = D_N + \widetilde{c}(\Omega).$$

Application: toric noncommutative manifolds

- Consider the class of examples coming from actions of a connected compact Lie group G acting on a spin^c manifold N with principal stratum M and with $B = M/G$ a spin^c manifold.
- Prototype: the four-sphere $\mathbb{S}^4$ with a $\mathbb{T}^2$-action (the starting point for the Connes–Landi four-sphere)

The four-sphere

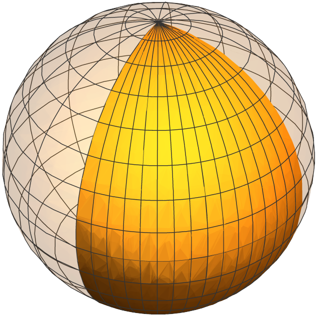
- Toroidal coordinates: $0 \leq \theta_1, \theta_2 < 2\pi$, $0 \leq \varphi \leq \pi/2$, $-\pi/2 \leq \psi \leq \pi/2$, and write

$$z_1 = e^{i\theta_1} \cos\varphi \cos\psi;$$
$$z_2 = e^{i\theta_2} \sin\varphi \cos\psi;$$
$$x = \sin\psi$$

- Action of $\mathbb{T}^2$ is by translating the θ_1, θ_2-coordinates

- The orbit space $\mathbb{S}^4/\mathbb{T}^2 \cong Q^2$ is a closed quadrant in the two-sphere, parametrized by

$$0 \leq \varphi \leq \pi/2, \qquad -\pi/2 \leq \psi \leq \pi/2$$



- Principal stratum $\mathbb{S}^4_0$ is a trivial $\mathbb{T}^2$-principal fiber bundle over the interior Q^2_0 of Q^2
- Moreover, $\pi : \mathbb{S}^4_0 \to Q^2_0$ is a Riemannian submersion for the metric on Q^2_0 induced by the round metric on $\mathbb{S}^2$

Spin geometry of $\mathbb{S}^4$

- The Dirac operator for the round metric is a $\mathbb{T}^2$-invariant selfadjoint operator

$$D_{\mathbb{S}^4} : \mathrm{Dom}(D_{\mathbb{S}^4}) \to L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$$

with local expression:

$$D_{\mathbb{S}^4} = i \frac{1}{\cos\varphi \cos\psi} \gamma^1 \frac{\partial}{\partial\theta_1} + i \frac{1}{\sin\varphi\cos\psi}\gamma^2 \frac{\partial}{\partial\theta_2} + i\frac{1}{\cos\psi}\gamma^3 \left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \right) + i\gamma^4 \left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi \right)$$

- Since the 'subprincipal' stratum is of codimension two, it follows that $C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^4 \subset L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$ is a core for $D_{\mathbb{S}^4}$

Vertical operator

- We consider a C^*-correspondence X from $C_0(\mathbb{S}^4_0)$ to $C_0(Q^2_0)$, defined as the Hilbert C^*-completion of $C^\infty_c(\mathbb{S}^4_0) \otimes \mathbb{C}^2$ wrt

$$\langle s, t\rangle_X = \int_{\mathbb{T}^2} \overline{s(\theta_1, \theta_2, \varphi, \psi)} \cdot t(\theta_1, \theta_2, \varphi, \psi) d\theta_1 d\theta_2 \cdot \sin\varphi \cos\varphi \cos^2\psi,$$

- There is a unitary isomorphism (induced by pointw. multipl.)

$$u : X \otimes_{C_0(Q^2_0)} L^2(Q^2_0) \otimes \mathbb{C}^2 \to L^2(\mathbb{S}^4_0) \otimes \mathbb{C}^4$$

- We define a symmetric operator $S_0 : C^\infty_c(\mathbb{S}^4_0) \otimes \mathbb{C}^2 \to X$:

$$S_0 = i \frac{1}{\cos\varphi \cos\psi} \sigma^1 \frac{\partial}{\partial\theta_1} + i \frac{1}{\sin\varphi \cos\psi} \sigma^2 \frac{\partial}{\partial\theta_2}.$$

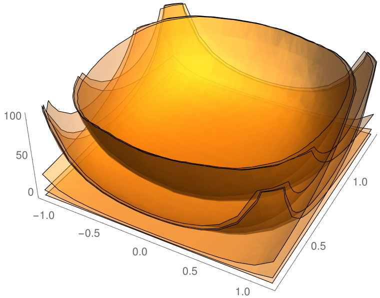
We denote its closure by $S : \mathrm{Dom}(S) \to X$.

- One can find families of eigenvectors $\Psi^{\pm}_{n_1 n_2}$ in $C^\infty(\mathbb{T}^2) \otimes \mathbb{C}^2$.

$$S\left(f\Psi^{\pm}_{n1,n_2}\right) = \pm\lambda_{n_1 n_2} \cdot f\Psi^{\pm}_{n_1 n_2}.$$

for any $f \in C_c^\infty(Q_0^2)$ where

$$\lambda_{n_1 n_2} = \sqrt{\frac{n_1^2}{\cos^2\varphi\cos^2\psi} + \frac{n_2^2}{\sin^2\varphi\cos^2\psi}}$$

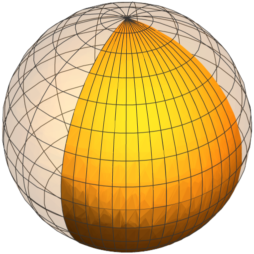


- Regularity, selfadjointness and compactness of the resolvent: S defines an unbdd Kasparov module from $C_0(\mathbb{S}_0^4)$ to $C_0(Q_0^2)$

Horizontal operator

We define an operator $T_0 : C_c^\infty(Q_0^2) \otimes \mathbb{C}^2 \to L^2(Q_0^2) \otimes \mathbb{C}^2$ as the restriction of the Dirac operator on $\mathbb{S}^2$ to the open quadrant Q_0^2:

$$T_0 = i\frac{1}{\cos\psi}\sigma^1\frac{\partial}{\partial\varphi} + i\sigma^2\left(\frac{\partial}{\partial\psi} - \frac{1}{2}\tan\psi\right).$$



This is a symmetric operator. Its closure T defines a half-closed chain [Hil, 2010] from $C_0(Q_0^2)$ to $\mathbb{C}$

Tensor sum of S and T

We lift the operators S and T to $X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2$.

- For S we take $S \widehat{\otimes} \gamma$ and have on $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ that

$$u(S \widehat{\otimes} \gamma) u^* = i \frac{1}{\cos\varphi \cos\psi} \gamma^1 \frac{\partial}{\partial \theta_1} + i \frac{1}{\sin\varphi \cos\psi} \gamma^2 \frac{\partial}{\partial \theta_2},$$

- For the operator T, we first need a hermitian connection on X

$$\nabla_{\partial/\partial\varphi} = \frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \qquad \nabla_{\partial/\partial\psi} = \frac{\partial}{\partial\psi} - \tan\psi$$

- The operator $1 \widehat{\otimes}_\nabla T$ then becomes

$$u(1 \widehat{\otimes}_\nabla T) u^* = i \frac{1}{\cos\psi} \gamma^3 \left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \right) + i\gamma^4 \left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi \right)$$

- The tensor sum we are after is given by

$$(S \times_\nabla T)_0 := S \hat{\otimes} \gamma + 1 \hat{\otimes}_\nabla T$$

- This is a symmetric operator and we denote its closure by

$$S \times_\nabla T : \mathrm{Dom}(S \times_\nabla T) \to X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2.$$

- Since $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^2 \otimes_{C_c^\infty(Q_0^2)} C_c^\infty(Q_0^2) \otimes \mathbb{C}^2$ is a core for $S \times_\nabla T$ which is mapped by u to the core $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ of $D_{\mathbb{S}^4}$, it follows that

$$\boxed{u(S \times_\nabla T)u^* = D_{\mathbb{S}^4}}$$

as an equality on $\mathrm{Dom}(D_{\mathbb{S}^4})$

- A connection and positivity property imply that this is an unbounded representative of the internal product

$$KK(C_0(\mathbb{S}^4), C_0(Q_0^2)) \otimes_{C_0(Q_0^2)} KK(C_0(Q_0^2), \mathbb{C}) \to KK(C_0(\mathbb{S}^4), \mathbb{C})$$

Summary

For almost-regular fibrations (N, P, B, π) of spin^c manifolds we have obtained with $M = N \setminus \overline{P}$:

- an unbounded Kasparov module (X, S) from $C_0(M)$ to $C_0(B)$ describing the vertical data
- a connection ∇ on X that allows to lift the Dirac operator D_B to a symmetric operator $1 \widehat{\otimes}_\nabla D_B$ on $X \widehat{\otimes}_{C_0(B)} L^2(\mathscr{E}_B^c)$
- an identity of selfadjoint operators on $\mathrm{Dom}(D_N)$

$$V \left(S \widehat{\otimes} \gamma + 1 \widehat{\otimes}_\nabla D_B \right) V^* = D_N + \widetilde{c}(\Omega).$$

- Applications: connected compact Lie groups acting on compact Riemannian spin^c manifolds

References

- J. Kaad and WvS. Riemannian submersions and factorization of Dirac operators. arXiv:1610.02873.
- WvS. Localizing gauge theories from noncommutative geometry. *Adv. Math.* 290 (2016) 682-708. arXiv:1411.6482
- S. Brain, B. Mesland, and WvS. Gauge theory for spectral triples and the unbounded Kasparov product. *J. Noncommut. Geom.* 10 (2016), 135 206. arXiv:1306.1951