Factorization of Dirac operators in unbounded KK-theory

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2 March 2017

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Motivation

Consider the following setup:

- A Riemannian submersion

$$\pi : M \to B$$

  of compact spin^c manifolds M and B

  e.g. a connected compact Lie group G acting freely on M and $B = M/G$.

- It is well-known from wrong-way functoriality [CS, 1984] that

$$[M] = \pi! \widehat{\otimes}_{C(B)} [B] \qquad (*)$$

- Can we write the Dirac operator on M as a tensor sum:

$$D_M = S \otimes \gamma + 1 \otimes_\nabla D_B?$$

  for some unbounded KK-cycle $(C^\infty(M), X, S)$, representing the internal KK-product $(*)$?

Key application: toric noncommutative manifolds

- Suppose $\mathbb{T}^n$ acts on M. Then [CL, 2001]

$$(C^\infty(M_\theta), L^2(\mathscr{E}_M), D_M)$$

is a spectral triple, defining a class in $KK(C(M_\theta), \mathbb{C})$

- At the C^*-algebraic level, one can identify [vS, 2015] a continuous C^*-bundle $\mathfrak{B}$ so that

$$C(M_\theta) = \Gamma(M/\mathbb{T}^n, \mathfrak{B})$$

- The above tensor sum factorization (if $\mathbb{T}^n$ acts *freely*) means that this bundle picture extends to the geometric level:

$$[D_M] = [S] \otimes_{C(M/\mathbb{T}^n)} [D_{M/\mathbb{T}^n}]$$

where S is a family of Dirac operators acting on a Hilbert $C(M_\theta) - C(M/\mathbb{T}^n)$-module X.

For Riemannian submersions $\pi : M \to B$ of compact spin^c manifolds, the factorization can be done [KvS, 2016], by introducing

- a C^*-correspondence X between $C(M)$ and $C(B)$
- A regular self-adjoint operator

$$S : \mathrm{Dom}(S) \to X$$

  with compact resolvent (family of vertical Dirac operators), defining an unbounded KK-cycle from $C(M)$ to $C(B)$.
- A hermitian connection

$$\nabla : \mathrm{Dom}(\nabla) \to X \widehat{\otimes}_{C(B)} \Omega^1(B)$$

Then,

1. $S\widehat{\otimes}\gamma$ and $1\widehat{\otimes}_\nabla D_B$ define symmetric operators on (suitable domains in) $X\widehat{\otimes}_{C(B)}L^2(\mathscr{E}_B)$
2. There is a unitary isomorphism

$$V : X\widehat{\otimes}_{C(B)}L^2(\mathscr{E}_B) \to L^2(\mathscr{E}_M)$$

3. We have the following identity of self-adjoint operators on $\mathrm{Dom}(D_M)$:

$$V(S\widehat{\otimes}\gamma + 1\widehat{\otimes}_\nabla D_B)V^* = D_M + \widetilde{c}(\Omega)$$

   where Ω is the curvature of $\pi : M \to B$.

4. The tensor sum is an unbounded representative of the internal KK-product of the corresponding elements in KK-theory.

Riemannian submersions

- Let M and B be closed Riemannian manifolds and let

$$\pi : M \to B$$

be a smooth and surjective map

- This is a Riemannian submersion when $d\pi$ is surjective and

$$d\pi : (\ker d\pi)^{\perp} \to \mathscr{X}(B) \otimes_{C^\infty(B)} C^\infty(M)$$

is an isometric isomorphism

- This gives rise to vertical and horizontal vector fields

$$\mathscr{X}(M) \cong \mathscr{X}_V(M) \oplus \mathscr{X}_H(M).$$

- On $\mathscr{X}(M)$ one can introduce the direct sum connection

$$\nabla^{\oplus} = P_V \nabla^M P_V \oplus \pi^* \nabla^B,$$

- the second fundamental form:

$$S(X,Y,Z) := \left\langle \nabla^V_{(1-P)Z}(PX) - [(1-P)Z, PX], PY \right\rangle_M,$$

- the curvature of $\pi : M \to B$:

$$\Omega(X,Y,Z) := -\left\langle [(1-P)X, (1-P)Y], PZ \right\rangle_M$$

which combined yield a tensor $\omega \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^2(M)$

Proposition (Bismut, 1986)

The Levi-Civita connection ∇^M is related to the direct sum connection $\nabla^{\oplus}$ by the following formula

$$\langle \nabla^M_X Y, Z \rangle_M = \langle \nabla^{\oplus}_X Y, Z \rangle_M + \omega(X)(Y,Z).$$

Spin geometry and Clifford modules

Suppose that M and B are (even-dim) spin^c manifolds, so we have

$$\mathrm{Cl}(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_M), \qquad \mathrm{Cl}(B) \cong \mathrm{End}_{C^\infty(B)}(\mathscr{E}_B)$$

and hermitian Clifford connections $\nabla^{\mathscr{E}_M}$ and $\nabla^{\mathscr{E}_B}$.

- We define the horizontal spinor module:

  $$\mathscr{E}_H := \mathscr{E}_B \otimes_{C^\infty(B)} C^\infty(M); \qquad \nabla^{\mathscr{E}_H} := \pi^* \nabla^{\mathscr{E}_B}$$

  and $\mathrm{Cl}_H(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_H)$
- We define the vertical spinor module:

  $$\mathscr{E}_V := \mathscr{E}_H^* \otimes_{\mathrm{Cl}_H(M)} \mathscr{E}_M,$$

  $$\nabla_X^{\mathscr{E}_V}(\phi \otimes s) = \nabla_X^{\mathscr{E}_H^*}(\phi) \otimes s + \phi \otimes \nabla_X^{\mathscr{E}_M}(s) + \phi \otimes \frac{1}{4} c(\omega(X))(s),$$

  and $\mathrm{Cl}_V(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_V)$
- Finally,

  $$\mathscr{E}_H \otimes_{C^\infty(M)} \mathscr{E}_V \cong \mathscr{E}_M,$$

  compatibly with $\mathrm{Cl}_H(M) \widehat{\otimes}_{C^\infty(M)} \mathrm{Cl}_V(M) \cong \mathrm{Cl}(M)$.

The vertical operator

- We define a C^*-correspondence X from $C(M)$ to $C(B)$ by completing $\mathscr{E}_V$ with respect to

$$\langle \phi_1, \phi_2 \rangle_X(b) := \int_{F_b} \langle \phi_1, \phi_2 \rangle_{\mathscr{E}_V} \, d\mu_{F_b}$$

- The following local expression defines an odd symmetric unbounded operator

$$S_0(\xi) = i \sum_{j=1}^{\dim(F)} c_V(e_j) \nabla^{\mathscr{E}_V}_{e_j}(\xi)$$

where $\{e_j\}$ is a local orthonormal frame for $\mathscr{X}_V(M)$

Proposition

The triple $(C^\infty(M), X, S)$ is an even unbounded Kasparov module from $C(M)$ to $C(B)$ with grading operator $\gamma_X : X \to X$.

The horizontal operator and the connection

- The Dirac operator $D_B : \mathrm{Dom}(D_B) \to L^2(\mathscr{E}_B)$ is locally

$$D_B = i \sum_{\alpha=1}^{\dim(B)} c(f_\alpha)\nabla^{\mathscr{E}_B}_{f_\alpha} : \mathscr{E}_B \to L^2(\mathscr{E}_B)$$

  Clearly $(C^\infty(B), L^2(\mathscr{E}_B), D_B; \gamma_B)$ is an (even) spectral triple
- The following defines a hermitian connection on X

$$\nabla^X_Z(\xi) = \nabla^{\mathscr{E}_V}_{Z_H}(\xi) + \frac{1}{2}k(Z_H) \cdot \xi$$

  with $k = (\mathrm{Tr} \otimes 1)(S) \in \Omega^1(M)$ the mean curvature

Lemma

The following local expression defines an odd symmetric unbounded operator in $X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B)$:

$$(1 \otimes_\nabla D_B)(\xi \otimes r) := \xi \otimes D_B r + i \sum_\alpha \nabla^X_{f_\alpha}(\xi) \otimes c(f_\alpha) r.$$

The tensor sum

- The tensor sum we are after is given by

$$(S \times_\nabla D_B)_0 := S \otimes \gamma_B + 1 \otimes_\nabla D_B : \mathrm{Dom}(S \times_\nabla D_B)_0 \to X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B)$$

- The closure of this symmetric operator is denoted $S \times_\nabla D_B$.

Theorem (KvS, 2016)

Under the unitary isomorphism $V : X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B) \to L^2(\mathscr{E}_M)$ we have the identity

$$V(S \times_\nabla D_B)V^* = D_M - \frac{i}{8}\widetilde{c}(\Omega).$$

Main result

Theorem

The even spectral triple $(C^\infty(M), L^2(\mathscr{E}_M), D_M)$ is the unbounded Kasparov product of the even unbounded KK-cycle $(C^\infty(M), X, S)$ with the even spectral triple $(C^\infty(B), L^2(\mathscr{E}_B), D_B)$ up to the (bounded) curvature term $-\frac{i}{8}\widetilde{c}(\Omega)$.

Sketch of proof

1. The classes $[D_M]$, $[S]$, $[D_B]$ in bounded KK-theory given by bounded transforms coincide with the classes $[M]$, $\pi!$, $[B]$.
2. Wrong-way functoriality [CS, 1984]: the map $f_M : M \to \{\mathrm{pt}\}$ can be factorized as $f_M = f_B \circ \pi$ with $f_B : B \to \{\mathrm{pt}\}$ and gives rise to

$$f_M! = \pi! \widehat{\otimes}_{C(B)} f_B!$$

3. Identify $f_M! = [M]$ and $f_B! = [B]$.

What happens in cases where $\pi : M \to B$ is not a submersion,

> *e.g. a connected compact Lie group G acting on M and $B = M/G$.*

Prototype: the four-sphere $\mathbb{S}^4$ with a $\mathbb{T}^2$-action (the starting point for the Connes–Landi four-sphere)

The four-sphere

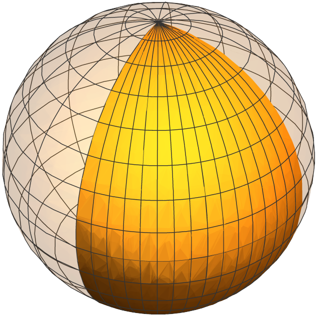
- Toroidal coordinates:
  $0 \leq \theta_1, \theta_2 < 2\pi$,
  $0 \leq \varphi \leq \pi/2$,
  $-\pi/2 \leq \psi \leq \pi/2$, and write

$$z_1 = e^{i\theta_1} \cos\varphi \cos\psi;$$
$$z_2 = e^{i\theta_2} \sin\varphi \cos\psi;$$
$$x = \sin\psi$$

- Action of $\mathbb{T}^2$ is by translating the θ_1, θ_2-coordinates

- The orbit space $\mathbb{S}^4/\mathbb{T}^2 \cong Q^2$ is a closed quadrant in the two-sphere, parametrized by

$$0 \leq \varphi \leq \pi/2, \qquad -\pi/2 \leq \psi \leq \pi/2$$



- Principal stratum $\mathbb{S}^4_0$ is a trivial $\mathbb{T}^2$-principal fiber bundle over the interior Q^2_0 of Q^2
- Moreover, $\pi : \mathbb{S}^4_0 \to Q^2_0$ is a Riemannian submersion for the metric on Q^2_0 induced by the round metric on $\mathbb{S}^2$

Spin geometry of $\mathbb{S}^4$

- The Dirac operator for the round metric is a $\mathbb{T}^2$-invariant self-adjoint operator

$$D_{\mathbb{S}^4} : \mathrm{Dom}(D_{\mathbb{S}^4}) \to L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$$

with local expression:

$$\begin{aligned} D_{\mathbb{S}^4} = {} & i \frac{1}{\cos\varphi\cos\psi}\gamma^1 \frac{\partial}{\partial\theta_1} + i \frac{1}{\sin\varphi\cos\psi}\gamma^2 \frac{\partial}{\partial\theta_2} \\ & + i \frac{1}{\cos\psi}\gamma^3 \left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \right) + i\gamma^4 \left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi \right) \end{aligned}$$

- Since the 'subprincipal' stratum is of codimension two, it follows that $C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^4 \subset L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$ is a core for $D_{\mathbb{S}^4}$

Vertical operator

- We consider a C^*-correspondence X from $C_0(\mathbb{S}^4_0)$ to $C_0(Q^2_0)$, defined as the Hilbert C^*-completion of $C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^2$ wrt

$$\langle s, t\rangle_X = \int_{\mathbb{T}^2} \overline{s(\theta_1, \theta_2, \varphi, \psi)} \cdot t(\theta_1, \theta_2, \varphi, \psi) d\theta_1 d\theta_2 \cdot \sin\varphi \cos\varphi \cos^2\psi,$$

- Then there is a unitary isomorphism

$$u : X \otimes_{C_0(Q^2_0)} L^2(Q^2_0) \otimes \mathbb{C}^2 \to L^2(\mathbb{S}^4_0) \otimes \mathbb{C}^4$$

- We define a symmetric operator $S_0 : C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^2 \to X$:

$$S_0 = i \frac{1}{\cos\varphi \cos\psi} \sigma^1 \frac{\partial}{\partial\theta_1} + i \frac{1}{\sin\varphi \cos\psi} \sigma^2 \frac{\partial}{\partial\theta_2}.$$

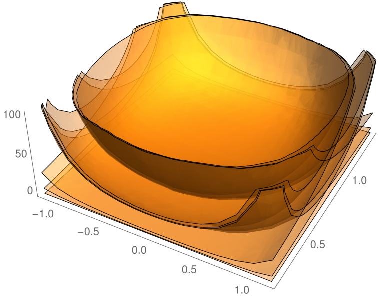
We denote its closure by $S : \mathrm{Dom}(S) \to X$.

- One can find families of eigenvectors $\Psi^{\pm}_{n_1 n_2}$ in $C^\infty(\mathbb{T}^2) \otimes \mathbb{C}^2$.

$$S\left(f\Psi^{\pm}_{n1,n_2}\right) = \pm\lambda_{n_1 n_2} \cdot f\Psi^{\pm}_{n_1 n_2}.$$

for any $f \in C_c^\infty(Q_0^2)$ where

$$\lambda_{n_1 n_2} = \sqrt{\frac{n_1^2}{\cos^2\varphi\cos^2\psi} + \frac{n_2^2}{\sin^2\varphi\cos^2\psi}}$$



- Regularity, self-adjointness and compactness of the resolvent: $\rightsquigarrow$ S defines an unbounded KK-cycle from $C_0(\mathbb{S}^4_0)$ to $C_0(Q^2_0)$

Horizontal operator

We define an operator $T_0 : C_c^\infty(Q_0^2) \otimes \mathbb{C}^2 \to L^2(Q_0^2) \otimes \mathbb{C}^2$ as the restriction of the Dirac operator on $\mathbb{S}^2$ to the open quadrant Q_0^2:

$$T_0 = i \frac{1}{\cos\psi} \sigma^1 \frac{\partial}{\partial\varphi} + i\sigma^2 \left(\frac{\partial}{\partial\psi} - \frac{1}{2} \tan\psi \right).$$

This is a symmetric unbounded operator; denote its closure by $T : \mathrm{Dom}(T) \to L^2(Q_0^2) \otimes \mathbb{C}^2$. Then T defines a half-closed chain [Hil, 2010] from $C_0(Q_0^2)$ to $\mathbb{C}$

Tensor sum of S and T

We lift the operators S and T to $X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2$.

- For S we take $S \widehat{\otimes} \gamma$ and have on $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ that

$$u(S\widehat{\otimes}\gamma)u^* = i\frac{1}{\cos\varphi\cos\psi}\gamma^1\frac{\partial}{\partial\theta_1} + i\frac{1}{\sin\varphi\cos\psi}\gamma^2\frac{\partial}{\partial\theta_2},$$

- For the operator T, we first need a hermitian connection on X

$$\nabla_{\partial/\partial\varphi} = \frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \qquad \nabla_{\partial/\partial\psi} = \frac{\partial}{\partial\psi} - \tan\psi$$

- The operator $1\widehat{\otimes}_\nabla T$ then becomes

$$u(1\widehat{\otimes}_\nabla T)u^* = i\frac{1}{\cos\psi}\gamma^3\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi\right) + i\gamma^4\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi\right)$$

- The tensor sum we are after is given by

$$(S \times_\nabla T)_0 := S \hat{\otimes} \gamma + 1 \hat{\otimes}_\nabla T$$

- This is a symmetric operator and we denote its closure by

$$S \times_\nabla T : \mathrm{Dom}(S \times_\nabla T) \to X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2.$$

- Since $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^2 \otimes_{C_c^\infty(Q_0^2)} C_c^\infty(Q_0^2) \otimes \mathbb{C}^2$ is a core for $S \times_\nabla T$ which is mapped by u to the core $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ of $D_{\mathbb{S}^4}$, it follows that

$$\boxed{u(S \times_\nabla T)u^* = D_{\mathbb{S}^4}}$$

as an equality on $\mathrm{Dom}(D_{\mathbb{S}^4})$

- A connection and positivity property imply that this is an unbounded representative of the internal product

$$KK(C_0(\mathbb{S}^4), C_0(Q_0^2)) \otimes_{C_0(Q_0^2)} KK(C_0(Q_0^2), \mathbb{C}) \to KK(C_0(\mathbb{S}^4), \mathbb{C})$$

Outlook: almost-regular fibrations

work in progress with Jens Kaad

We consider almost-regular fibrations in the following sense:

- Let M be a closed Riemannian manifold, together with an embedded submanifold $P \subseteq M$ without boundary of codimension greater than 1, and suppose that there exists a Riemannian manifold B_0 so that there is a Riemannian submersion $\pi_0 : M \setminus \overline{P} \to B_0$
- Major class of examples given by connected compact Lie groups G acting on closed Riemannian manifolds M
- Tensor sum factorization of D_M corresponds to

$$KK(C_0(M\backslash\overline{P}), C_0(B_0)) \otimes_{C_0(B_0)} KK(C_0(B), \mathbb{C}) \to KK(C_0(M\backslash\overline{P}), \mathbb{C})$$