



Noncommutative geometry and the BV formalism: Application to a matrix model



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ABSTRACT

We analyze a $U(2)$ -matrix model derived from a finite spectral triple. By applying the BV formalism, we find a general solution to the classical master equation. To describe the BV formalism in the context of noncommutative geometry, we define two finite spectral triples: the BV spectral triple and the BV auxiliary spectral triple. These are constructed from the gauge fields, ghost fields and anti-fields that enter the BV construction. We show that their fermionic actions add up precisely to the BV action. This approach allows for a geometric description of the ghost fields and their properties in terms of the BV spectral triple.

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1. Introduction

Since the early days of noncommutative geometry [1] it has been clear that this mathematical theory is strongly related to gauge theories in physics. Indeed, gauge theories are naturally induced by spectral triples, where the noncommutativity of the pertinent algebra naturally gives rise to non-abelian gauge groups. This has successfully been applied to Yang–Mills gauge theories [2] and to the celebrated Standard Model of particle physics [3]. It is also clear that in the finite-dimensional case, when the algebras are matrix algebras, one obtains hermitian matrix models.

A powerful method to analyze the nature of the gauge symmetries in gauge theories – with the eventual purpose of understanding their rigorous quantization – is the BRST formalism [4–6] and its far-reaching extension, the BV formalism [7,8] (cf. [9–11] for review articles). A first key ingredient in both of these formalisms is Faddeev–Popov *ghost fields* [12], which are introduced to cancel the physically irrelevant gauge symmetries. The BV formalism then proceeds by introducing also so-called *anti-fields* for all previously defined gauge and ghost fields. Moreover, an extended action functional is defined as a solution to the so-called ‘classical master equation’ (cf. Definition 10).

We start this paper by recalling (cf. [13]) the result obtained by applying the BV formalism to a $U(2)$ -matrix model, which is derived from a finite spectral triple on the algebra $M_2(\mathbb{C})$. We find that the gauge structure of this model is richer than expected, requiring also the introduction of ghost-for-ghost fields. After having added the necessary anti-fields, we state the general form of the extended action that solves the classical master equation. Then, the construction is finished by determining the BV auxiliary pairs, which are essential in order to perform a gauge-fixing procedure. As such, our construction fits nicely with previous studies of the BV formalism applied to gauge models derived from noncommutative geometry, such as [14–17].

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As a next step, we here define two spectral triples – the BV spectral triple and the BV auxiliary spectral triple – for which the fermionic action functionals sum up precisely to the so-called BV action functional, which is defined to be the difference between the extended action functional and the initial action. Thus we obtain a noncommutative geometric description of the BV formalism for this particular model, which by itself was derived from a spectral triple.

With this model we give the first description of the BV formalism completely in terms of noncommutative geometric data, that is to say, spectral triples. It serves as a guiding example for higher-rank, $U(n)$ -matrix models and eventually for physically realistic gauge theories defined on a manifold. However, an analysis of these models goes beyond the scope of this paper and is left for future research.

The paper is organized as follows. In Section 2.1 we quickly review the notion of a spectral triple and explain how gauge theories derive from it. Section 2.2 contains a concise overview of the BV formalism, geared towards our finite-dimensional case and essentially following [18] (see also [13]).

In Section 3 we recall what we obtained by applying the BV formalism to a $U(2)$ -matrix model, understood as a gauge theory that is obtained from a spectral triple.

Section 4 is the heart of this paper: we construct a so-called BV spectral triple and BV auxiliary spectral triple and show that the sum of the corresponding fermionic actions coincides with the BV action functional.

2. Preliminaries

2.1. The noncommutative geometry setting

We recall the notion of a *spectral triple* and the construction of the canonically induced gauge theory (cf. [19], [20, Sect. 1.10] and [21, Ch. 6]). This method will be later applied to a finite spectral triple that yields a $U(2)$ -gauge theory, which we want to analyze using the BV formalism.

Definition 1. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of an involutive unital algebra \mathcal{A} , faithfully represented as operators on a Hilbert space \mathcal{H} , together with a self-adjoint operator D on \mathcal{H} , with a compact resolvent, such that the commutators $[D, a]$ are bounded operators for each $a \in \mathcal{A}$.

Remark 2. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be *finite* if \mathcal{H} is finite dimensional. By a classical result the algebra \mathcal{A} in this case has to be a direct sum of matrix algebras, i.e.

$$\mathcal{A} \simeq \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$$

for positive integers n_1, \dots, n_k . Moreover, the required conditions on the self-adjoint operator D are automatically satisfied in this finite-dimensional setting.

Definition 3. An *even spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is one in which the Hilbert space \mathcal{H} is endowed with a $\mathbb{Z}/2$ -grading γ , given by a linear map $\gamma : \mathcal{H} \rightarrow \mathcal{H}$, such that

$$D\gamma = -\gamma D \quad \text{and} \quad \gamma a = a\gamma$$

for all $a \in \mathcal{A}$.

Definition 4. A *real structure of KO-dimension $n \pmod{8}$* on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an anti-linear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ that satisfies

$$J^2 = \epsilon \quad \text{and} \quad JD = \epsilon' DJ$$

together with the condition

$$J\gamma = \epsilon'' \gamma J$$

in the even case. The constants ϵ, ϵ' and ϵ'' depend on the KO-dimension $n \pmod{8}$ as follows:

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Moreover, we require for all $a, b \in \mathcal{A}$ that:

- the action of \mathcal{A} satisfies the *commutation rule*: $[a, Jb^*J^{-1}] = 0$;
- the operator D fulfills the *first-order condition*: $[[D, a], Jb^*J^{-1}] = 0$.

When a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is endowed with such a real structure J , it is said to be a *real spectral triple* and is denoted by $(\mathcal{A}, \mathcal{H}, D, J)$.

Given a possibly real spectral triple, there are two notions of action functionals related to it: the *spectral action* and the *fermionic action* (cf. [2,3,22]).

Definition 5. For a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a suitable real-valued function f , the *spectral action* S_0 is given by

$$S_0[D + M] := \text{Tr}(f(D + M))$$

with, as domain, the set of self-adjoint operators of the form $M = \sum_j a_j [D, b_j]$, for $a_j, b_j \in \mathcal{A}$.

Remark 6. In the finite-dimensional setting, a family of suitable functions f is given by the polynomials in $\text{Pol}_{\mathbb{R}}(x)$.

Definition 7. For a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (finite real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$) the *fermionic action* on \mathcal{H} is given by

$$S_{\text{ferm}}[\varphi] = \frac{1}{2} \langle \varphi, D\varphi \rangle \quad \left(S_{\text{ferm}}[\varphi] = \frac{1}{2} \langle J\varphi, D\varphi \rangle \right); \quad (\varphi \in \mathcal{H}).$$

2.1.1. Gauge theories from spectral triples

We recall the construction of the gauge theory naturally induced by a spectral triple, restricting to the finite-dimensional case. In this context, the appropriate notion of a gauge theory is as follows.

Definition 8. For a real vector space X_0 and a real-valued functional S_0 on X_0 , let $F : \mathcal{G} \times X_0 \rightarrow X_0$ be a group action on X_0 for a given group \mathcal{G} . Then the pair (X_0, S_0) is called a *gauge theory with gauge group* \mathcal{G} if

$$S_0(F(g, M)) = S_0(M)$$

for all $M \in X_0$ and $g \in \mathcal{G}$. The space X_0 is referred to as the *configuration space*, an element $M \in X_0$ is called a *gauge field* and S_0 is the *action functional*.

Given this definition, the derived gauge theory for a finite spectral triple is obtained by the following standard result.

Proposition 9. For a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$, let

$$X_0 = \left\{ M = \sum_j a_j [D, b_j] : M^* = M, a_j, b_j \in \mathcal{A} \right\}$$

be the space of inner fluctuations, the group \mathcal{G} be the unitary elements $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} : uu^* = u^*u = 1\}$ of \mathcal{A} acting on X_0 via the map $(u, M) \mapsto uMu^* + u[D, u^*]$, and S_0 be the spectral action

$$S_0[M] := \text{Tr}(f(D + M)),$$

for any $M \in X_0$ and some $f \in \text{Pol}_{\mathbb{R}}(x)$. Then the pair (X_0, S_0) is a gauge theory with gauge group \mathcal{G} .

2.2. The BV approach to gauge theories

As already mentioned, starting with a gauge theory (X_0, S_0) , the BV construction is a procedure to determine a corresponding *extended theory* (X, \tilde{S}) via the introduction of *ghost/anti-ghost fields*. Here we outline the main aspects of the BV formalism, referring to [10,13,18] and references therein for a more exhaustive presentation.

For notational purposes, it is convenient to fix a basis for X_0 so that a gauge field $M \in X_0$ can be written as a vector $M = (M_a)$, with $a = 1, \dots, n = \dim X_0$ and

$$X_0 \simeq \langle M_1, \dots, M_n \rangle_{\mathbb{R}}.$$

The presence of gauge symmetries in the action demands for the introduction of *ghost fields*. In order to determine the number of required ghost fields, one considers the relations R_i^a ($i = 1, \dots, m_0$) between the partial derivatives $\partial_a S_0$ of the action functional with respect to M_a , i.e.

$$(\partial_a S_0) R_i^a = 0.$$

These relations R_i^a are considered in \mathcal{O}_{X_0} , which is the ring of regular functions on X_0 . Given each relation R_i^a we introduce a *ghost field* C_i for $i = 1, \dots, m_0$. It is useful to assign, for good book-keeping, a *ghost degree* $\text{deg}(\varphi) \in \mathbb{Z}$ and parity $\epsilon(\varphi) \in \{0, 1\}$ to the fields φ obtained so far, with $\epsilon(\varphi) := \text{deg}(\varphi) \pmod{2}$. The parity indicates whether the field is a real variable ($\epsilon = 0$) or a Grassmannian, namely anti-commuting, variable ($\epsilon = 1$). Naturally, we assign

$$\text{deg}(M_a) = 0, \quad \text{deg}(C_i) = 1.$$

However, it might happen that there are additional relations between the R_i^q themselves. If this happens, the gauge theory is called *reducible* and one has to add ghost-for-ghost fields, denoted by E_j , for each such relation-between-relations that appears. The ghost degree of E_j is now 2. This might continue to ghosts-for-ghosts-for-ghosts all the way up to the ‘level of reducibility’ L , which is the highest appearing ghost degree minus 1. We refer e.g. to [10] for full details. We denote the resulting configuration space as follows:

$$\mathcal{E} := \langle M_1, \dots, M_n \rangle_0 \oplus \langle C_1, \dots, C_{m_0} \rangle_1 \oplus \langle E_1, \dots, E_{m_1} \rangle_2 \oplus \dots$$

The key point to the BV formalism is the introduction of anti-fields for all previously introduced gauge fields, ghost fields, ghost-for-ghost fields, *et cetera*. For $\varphi \in \mathcal{E}$ we denote the corresponding anti-field by φ^* and assign ghost degree:

$$\text{deg}(\varphi^*) = -\text{deg}(\varphi) - 1.$$

This results in the vector space

$$\mathcal{E}^*[1] := \dots \oplus \langle E_1^*, \dots, E_{m_1}^* \rangle_{-3} \oplus \langle C_1^*, \dots, C_{m_0}^* \rangle_{-2} \oplus \langle M_1^*, \dots, M_n^* \rangle_{-1}$$

which is modeled on the dual space \mathcal{E}^* , where the notation [1] indicates the shift of degree by one, that is to say,

$$\mathcal{E}^*[1] = \bigoplus_{i \in \mathbb{Z}} [\mathcal{E}^*[1]]^i \quad \text{with} \quad [\mathcal{E}^*[1]]^i = [\mathcal{E}^*]^{i+1}.$$

The fields and anti-fields are combined into an *extended configuration space*

$$\tilde{X} := \mathcal{E} \oplus \mathcal{E}^*[1], \tag{2.1}$$

which has the structure of a super \mathbb{Z} -graded vector space. In view of this construction, the space of functionals on \tilde{X} is described by the *algebra* $\mathcal{O}_{\tilde{X}}$ of *regular functions on* \tilde{X} , which is the symmetric algebra generated by the \mathbb{Z} -graded \mathcal{O}_{X_0} -module \tilde{X} over the ring \mathcal{O}_{X_0} :

$$\mathcal{O}_{\tilde{X}} = \text{Sym}_{\mathcal{O}_{X_0}}(\tilde{X}).$$

Due to the presence of a graded structure on \tilde{X} , $\mathcal{O}_{\tilde{X}}$ is naturally given a graded algebra structure. Moreover, the pairing between \mathcal{E} and $\mathcal{E}^*[1]$ gives rise to a Poisson bracket structure $\{-, -\}$ of degree 1. Explicitly, the Poisson bracket is determined on generators as

$$\{\varphi_i, \varphi_j\} = 0, \quad \{\varphi_i, \varphi_j^*\} = \delta_{ij}, \quad \{\varphi_i^*, \varphi_j^*\} = 0.$$

As a final ingredient for the BV formalism, we come to the extension of the action functional S_0 to \tilde{X} .

Definition 10. Let (X_0, S_0) be a gauge theory. Then an *extended theory* associated to (X_0, S_0) is a pair (\tilde{X}, \tilde{S}) , where \tilde{X} is a super \mathbb{Z} -graded vector space as in (2.1), for \mathcal{E} a $\mathbb{Z}_{\geq 0}$ -graded locally free \mathcal{O}_{X_0} -module with homogeneous components of finite rank such that $[\mathcal{E}]^0 = X_0$, and \tilde{S} is a 0-degree element in $\mathcal{O}_{\tilde{X}}$ such that $\tilde{S}|_{X_0} = S_0$, with $\tilde{S} \neq S_0$, and that solves the ‘classical master equation’ $\{\tilde{S}, \tilde{S}\} = 0$.

We refer to the difference $S_{\text{BV}} = \tilde{S} - S_0$ as the *BV action* of the extended theory (\tilde{X}, \tilde{S}) . Note also that, even though each homogeneous component of the graded vector space \tilde{X} is taken to be finite-dimensional, there is no hypothesis on the number of non-trivial homogeneous components in \tilde{X} which may be infinite.

Definition 11. Given an extended theory (\tilde{X}, \tilde{S}) , the induced *classical BRST cohomology complex* is $(c^*(\tilde{X}), d_{\tilde{S}})$, where

$$c^i(\tilde{X}) = [\text{Sym}_{\mathcal{O}_{X_0}}(\tilde{X})]^i \quad (i \in \mathbb{Z})$$

and $d_{\tilde{S}} := \{\tilde{S}, -\}$ is the coboundary operator.

The fact that the map $d_{\tilde{S}}$ defines a linear and graded-derivative operator of degree 1 over $\mathcal{O}_{\tilde{X}}$ is a consequence of the properties of the Poisson bracket, whereas $(d_{\tilde{S}})^2 = 0$ follows from the (graded) Jacobi identity and the fact that \tilde{S} solves the classical master equation.

We now describe the gauge-fixing of our gauge theory in the context of the BV formalism. This essentially comes down to removing the anti-fields in the action \tilde{S} ; a key role in this construction is played by the choice of a *gauge-fixing fermion*.

Definition 12. Let $\tilde{X} = \mathcal{E} \oplus \mathcal{E}^*[1]$ be the above extended configuration space. A *gauge-fixing fermion* Ψ is defined to be a Grassmannian function $\Psi \in [\mathcal{O}_{\mathcal{E}}]^{-1}$.

From this, given an extended theory (\tilde{X}, \tilde{S}) together with a gauge-fixing fermion Ψ , the corresponding *gauge-fixed theory* is a pair $(\tilde{X}_{\Psi}, \tilde{S}_{\Psi})$ such that $\tilde{X}_{\Psi} = \mathcal{E}$, where \mathcal{E} is the subspace generated by fields and ghost fields, and

$$\tilde{S}_{\Psi} = \tilde{S}(\varphi_i, \varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i})$$

so that $\tilde{S}_\psi \in [\mathcal{O}_\varepsilon]^0$. Given an extended theory (\tilde{X}, \tilde{S}) , the gauge-fixing procedure a priori is not directly applicable, because all the fields/ghost fields in \tilde{X} have non-negative ghost degree, which impedes the definition of a gauge-fixing fermion for the theory. A solution to this problem was first discovered by Batalin and Vilkovisky [7,8] who suggested the introduction of *auxiliary fields* of negative ghost degree. This is done using so-called *trivial pairs*, consisting of fields B, h whose ghost degrees satisfy

$$\text{deg}(h) = \text{deg}(B) + 1.$$

Given a trivial pair (B, h) , the ghost degrees of the corresponding anti-fields (B^*, h^*) are then related by

$$\text{deg}(h^*) = \text{deg}(B^*) - 1.$$

Definition 13. For an extended theory (\tilde{X}, \tilde{S}) and a trivial pair (B, h) , the corresponding *total theory* is a pair $(X_{\text{tot}}, S_{\text{tot}})$, where the total configuration space X_{tot} is the \mathbb{Z} -graded vector space generated by \tilde{X}, B, h together with the corresponding anti-fields B^*, h^* , and the total action is $S_{\text{tot}} = \tilde{S} + S_{\text{aux}}$ with $S_{\text{aux}} = hB^*$.

In other words, the functional S_{tot} is in the algebra of functionals $\mathcal{O}_{X_{\text{tot}}}$ on X_{tot} that is obtained along the same lines as $\mathcal{O}_{\tilde{X}}$. Moreover, this algebra carries a graded Poisson structure, determined by the bracket on $\mathcal{O}_{\tilde{X}}$ and

$$\{B, B^*\} = \{h, h^*\} = 1,$$

with all other combinations of the B, h, B^*, h^* among themselves and with other fields being zero. The fact that S_{tot} does not depend on h^* or B implies that also S_{tot} satisfies the classical master equation and, furthermore, that

$$\{\tilde{S}, -\} = \{S_{\text{tot}}, -\}$$

when we consider $\mathcal{O}_{\tilde{X}}$ as a subalgebra of $\mathcal{O}_{X_{\text{tot}}}$. In fact, it follows that the classical BRST cohomology complex $(\mathcal{C}^\bullet(\tilde{X}), d_{\tilde{S}})$ is quasi-isomorphic to the complex $(\mathcal{C}^\bullet(X_{\text{tot}}), d_{S_{\text{tot}}})$, where

$$\mathcal{C}^\bullet(X_{\text{tot}}) = [\text{Sym}_{\mathcal{O}_{X_0}}(X_{\text{tot}})]^i \quad (i \in \mathbb{Z})$$

and $d_{S_{\text{tot}}} := \{S_{\text{tot}}, -\}$. This is the reason for the terminology *trivial pairs*. Batalin and Vilkovisky showed in [7,8] that the number of trivial pairs that need to be introduced is determined by the aforementioned level of reducibility L of the gauge theory.

Theorem 14. Let (\tilde{X}, \tilde{S}) be an extended theory for a gauge theory with level of reducibility L . Then \tilde{X} is enlarged to give X_{tot} by introducing a collection of trivial pairs $\{(B_i^j, h_i^j)\}$ for $i = 0, \dots, L$ and $j = 1, \dots, i + 1$ such that $\text{deg}(B_i^j) = j - i - 2$ if j is odd, or $\text{deg}(B_i^j) = i - j + 1$ if j is even.

After implementing the gauge-fixing, the pair $(X_{\text{tot}}, S_{\text{tot}})|_\psi$ may still induce a cohomology complex, which is called *gauge-fixed BRST cohomology complex*. In fact, while this always happens if the theory is considered on shell, the existence of this cohomology complex in the off-shell case depends on the explicit form of the action \tilde{S} . For completeness, we give its definition.

Definition 15. For a gauge-fixed theory $(X_{\text{tot}}, S_{\text{tot}})|_\psi$ with $X_{\text{tot}} = Y \oplus Y^*[1]$, the corresponding *gauge-fixed BRST cohomology complex* is $(\mathcal{C}^\bullet(Y), d_{S_{\text{tot}}}|_\psi)$, where

$$\mathcal{C}^k(Y) = [\text{Sym}_{\mathcal{O}_{X_0}}(Y)]^k \quad (k \in \mathbb{Z})$$

and the coboundary operator is given by $d_{S_{\text{tot}}}|_\psi := \{S_{\text{tot}}, -\}|_{\Sigma_\psi}$ for the submanifold Σ_ψ of X_{tot} defined by the gauge-fixing conditions $\varphi_i^* = \frac{\partial \psi}{\partial \varphi_i}$.

3. The BV construction applied to a U(2)-model

The BV construction, reviewed in the previous section, will now be applied to a gauge theory naturally induced by a finite spectral triple on the algebra $M_n(\mathbb{C})$. Indeed, by the construction of Proposition 9, we have that the finite spectral triple

$$(M_n(\mathbb{C}), \mathbb{C}^n, D),$$

for a hermitian $n \times n$ -matrix D , yields a gauge theory (X_0, S_0) with gauge group \mathcal{G} such that

$$X_0 = \{M \in M_n(\mathbb{C}) : M^* = M\}, \quad S_0[M] = \text{Tr}f(M) \quad \text{and} \quad \mathcal{G} = U(n),$$

with f a polynomial in $\text{Pol}_{\mathbb{R}}(x)$ and the adjoint action of \mathcal{G} on X_0 . For simplicity, we will analyze the result of applying the BV construction on this model for $n = 2$. To proceed with the construction, first fix a basis for X_0 given by Pauli matrices (together with the identity matrix):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Denoting by $\{M_a\}_{a=1}^4$ the dual basis of $\{\sigma_a\}_{a=1}^4$, X_0 is isomorphic to a 4-dimensional real vector space generated by four independent initial fields:

$$X_0 \simeq \langle M_1, M_2, M_3, M_4 \rangle_{\mathbb{R}}.$$

Hence the ring of regular functions on X_0 is the ring of polynomials in the variables M_a , $\mathcal{O}_{X_0} = \text{Pol}_{\mathbb{R}}(M_a)$. In terms of the coordinates M_a the spectral action S_0 , defined by a polynomial $f = \sum_{i=0}^r \mu_i x^i$, takes the following explicit form:

$$S_0 = 2 \left[\sum_{a=0}^{\lfloor r/2 \rfloor} \mu_{2a} \left(\sum_{s=0}^a \binom{2a}{2s} (M_1^2 + M_2^2 + M_3^2)^{a-s} M_4^{2s} \right) + \sum_{a=0}^{\lfloor r/2 \rfloor - 1} \mu_{2a+1} \left(\sum_{s=0}^a \binom{2a+1}{2s+1} (M_1^2 + M_2^2 + M_3^2)^{a-s} M_4^{2s+1} \right) \right].$$

However, an action S_0 of this type only represents a family of $U(2)$ -invariant functionals on X_0 . In fact, the most general form for a functional S_0 on X_0 that is invariant under the adjoint action of the gauge group $U(2)$ is as symmetric polynomial in the eigenvalues λ_1, λ_2 of the variable $M \in X_0$, or, equivalently, as polynomial in the symmetric elementary polynomials $a_1 = \lambda_1 + \lambda_2$ and $a_2 = \lambda_1 \lambda_2$. In terms of the coordinates M_a we have:

$$\lambda_i = M_4 \pm \sqrt{M_1^2 + M_2^2 + M_3^2}, \quad a_1 = 2M_4, \quad a_2 = M_4^2 - (M_1^2 + M_2^2 + M_3^2).$$

Hence the generic form for a $U(2)$ -invariant action $S_0 \in \text{Pol}_{\mathbb{R}}(M_a)$ is

$$S_0 = \sum_{k=0}^r (M_1^2 + M_2^2 + M_3^2)^k g_k(M_4), \tag{3.3}$$

where $g_k(M_4) \in \text{Pol}_{\mathbb{R}}(M_4)$. Because the introduction of extra (non-physical) fields is motivated by the necessity of eliminating the symmetries in the action functional S_0 , the BV construction may give rise to different extended configuration spaces \tilde{X} , depending on the explicit form of S_0 . For our $U(2)$ -matrix model we have three different cases:

- 1. If $S_0 \in \text{Pol}_{\mathbb{R}}(M_4)$, there are no symmetries that need to be removed by adding ghost fields. Hence, the construction of \tilde{X} stops at the first stage, after including the anti-fields corresponding to the initial fields in X_0 :

$$\tilde{X} = X_0 \oplus \langle M_1^*, M_2^*, M_3^*, M_4^* \rangle_{-1}.$$

- 2. If $\text{GCD}(\partial_1 S_0, \partial_2 S_0, \partial_3 S_0, \partial_4 S_0) = 1$, three independent ghost fields C_1, C_2, C_3 are inserted to compensate for the three independent relations existing over \mathcal{O}_{X_0} between pairs of partial derivatives of S_0 :

$$M_1(\partial_2 S_0) = M_2(\partial_1 S_0), M_1(\partial_3 S_0) = M_3(\partial_1 S_0), M_2(\partial_3 S_0) = M_3(\partial_2 S_0).$$

After having eliminated these three symmetries, there is still one relation that involves all three terms $\partial_1 S_0, \partial_2 S_0$, and $\partial_3 S_0$. Hence, we have to add a ghost field E of ghost degree 2.

- 3. If $\text{GCD}(\partial_1 S_0, \partial_2 S_0, \partial_3 S_0, \partial_4 S_0) = D \notin \mathbb{R}$, the action S_0 presents additional symmetries to cancel and so the extended configuration space \tilde{X} has to be further enlarged, obtaining that

$$\tilde{X} = \langle K^* \rangle_{-4} \oplus \langle E^*, \dots, E_4^* \rangle_{-3} \oplus \langle C_1^*, \dots, C_6^* \rangle_{-2} \oplus \langle M_1^*, \dots, M_4^* \rangle_{-1} \oplus X_0 \oplus \langle C_1, \dots, C_6 \rangle_1 \oplus \langle E_1, \dots, E_4 \rangle_2 \oplus \langle K \rangle_3.$$

Here we focus on the generic situation (2), for which we have the following result (cf. [13]).

Theorem 16. *Let (X_0, S_0) be a gauge theory with, as configuration space, $X_0 \simeq \langle M_a \rangle_{\mathbb{R}}$ for $a = 1, \dots, 4$, and, as action functional, $S_0 \in \mathcal{O}_{X_0}$ of the form (3.3). If $\text{GCD}(\partial_1 S_0, \partial_2 S_0, \partial_3 S_0, \partial_4 S_0) = 1$, then the minimally extended configuration space \tilde{X} is the following \mathbb{Z} -supergraded real vector space:*

$$\tilde{X} = \langle E^* \rangle_{-3} \oplus \langle C_1^*, C_2^*, C_3^* \rangle_{-2} \oplus \langle M_1^*, \dots, M_4^* \rangle_{-1} \oplus X_0 \oplus \langle C_1, C_2, C_3 \rangle_1 \oplus \langle E \rangle_2.$$

Moreover, the general solution of the classical master equation on \tilde{X} that is linear in the anti-fields, of at most degree 2 in the ghost fields and with coefficients in \mathcal{O}_{X_0} is given by $\tilde{S} = S_0 + S_{\text{BV}}$, for

$$S_{\text{BV}} = \sum_{i,j,k} \epsilon_{ijk} \alpha_k M_i^* M_j C_k + \sum_{i,j,k} C_i^* \left[\frac{\alpha_j \alpha_k}{2\alpha_i} (\beta \alpha_i M_i E + \epsilon_{ijk} C_j C_k) + M_i T \left(\sum_{a,b,c} \epsilon_{abc} \frac{\alpha_b \alpha_c}{2\alpha_i} M_a C_b C_c \right) \right] \tag{3.4}$$

where $\alpha_i, \beta \in \mathbb{R} \setminus \{0\}$, $T \in \text{Pol}_{\mathbb{R}}(M_a)$, and ϵ_{ijk} (ϵ_{abc}) is the totally anti-symmetric tensor in three indices $i, j, k \in \{1, 2, 3\}$ ($a, b, c \in \{1, 2, 3\}$) with $\epsilon_{123} = 1$.

Once the extended theory (\tilde{X}, \tilde{S}) has been constructed, another step is needed to be able to implement the gauge-fixing procedure, namely, we have to introduce the auxiliary fields. Because \tilde{X} contains ghost fields of at most ghost degree 2, the pair (\tilde{X}, \tilde{S}) describes a reducible theory with level of reducibility $L = 1$. Hence, according to [Theorem 14](#), the extended configuration space \tilde{X} has to be enlarged by adding three trivial pairs

$$\{(B_i, h_i)\}_{i=1,2,3} \quad \text{with} \quad \text{deg}(B_i) = -1, \quad \text{and} \quad \text{deg}(h_i) = 0,$$

which correspond to the three ghost fields C_i , together with the two trivial pairs (A_1, k_1) and (A_2, k_2) , corresponding to the ghost field E and satisfying

$$\text{deg}(A_1) = -2, \quad \text{deg}(k_1) = -1, \quad \text{deg}(A_2) = 0, \quad \text{deg}(k_2) = 1.$$

The total theory $(X_{\text{tot}}, S_{\text{tot}})$ now also includes the above auxiliary fields and is given by a \mathbb{Z} -graded vector space $X_{\text{tot}} = Y \oplus Y^*[1]$, with

$$Y = \langle M_a, B_i^*, h_i, k_1^*, A_2 \rangle_0 \oplus \langle C_i, A_1^*, k_2 \rangle_1 \oplus \langle E \rangle_2,$$

for $a = 1, \dots, 4, i = 1, 2, 3$, together with an $S_{\text{tot}} = \tilde{S} + S_{\text{aux}}$, where

$$S_{\text{aux}} := \sum_{i=1}^3 B_i^* h_i + \sum_{j=1}^2 A_j^* k_j. \tag{3.5}$$

4. The BV approach in the framework of NCG

4.1. The BV spectral triple

We now formulate the BV construction for the above $U(2)$ -matrix model in terms of noncommutative geometry. That is, we describe the extended theory (\tilde{X}, \tilde{S}) by means of a spectral triple for which the fermionic action yields \tilde{S} . In order to simplify the computation, we consider the pair (\tilde{X}, \tilde{S}) as described in [Theorem 16](#), but where in formula (3.4) we take the polynomial $T = 0$ and set the real coefficients $\alpha_i = \beta = 1$. Hence, we analyze the case when the action S_{BV} has the following form:

$$S_{\text{BV}} := M_1^*(-M_3C_2 + M_2C_3) + M_2^*(M_3C_1 - M_1C_3) + M_3^*(-M_2C_1 + M_1C_2) + C_1^*(M_1E + C_2C_3) + C_2^*(M_2E - C_1C_3) + C_3^*(M_3E + C_1C_2). \tag{4.6}$$

The construction of the so-called *BV spectral triple*

$$(\mathcal{A}_{\text{BV}}, \mathcal{H}_{\text{BV}}, D_{\text{BV}}, J_{\text{BV}})$$

proceeds in steps, where the form of the algebra \mathcal{A}_{BV} is determined as the last ingredient.

The Hilbert space \mathcal{H}_{BV}

We let \mathcal{H}_{BV} be the following Hilbert space:

$$\mathcal{H}_{\text{BV}} = \mathcal{H}_M \oplus \mathcal{H}_C := M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),$$

where the subscripts M and C refer to the gauge fields and ghost fields; this will be justified below. The inner product structure is given as usual by the Hilbert–Schmidt inner product on each summand $M_2(\mathbb{C})$, that is to say, $\langle -, - \rangle : \mathcal{H}_{\text{BV}} \times \mathcal{H}_{\text{BV}} \rightarrow \mathbb{C}$, with

$$\langle (\varphi_M, \varphi_C), (\varphi'_M, \varphi'_C) \rangle = \text{Tr}(\varphi_M(\varphi'_M)^*) + \text{Tr}(\varphi_C(\varphi'_C)^*),$$

for $\varphi_M, \varphi'_M \in \mathcal{H}_M, \varphi_C, \varphi'_C \in \mathcal{H}_C$. Taking the orthonormal basis of $M_2(\mathbb{C})$ given in (3.2) we can of course identify

$$\mathcal{H}_{\text{BV}} \cong \langle m_1, m_2, m_3, e \rangle \oplus \langle c_1, c_2, c_3, c_4 \rangle \cong \mathbb{C}^8,$$

in terms of which the inner product reads

$$\langle \varphi, \psi \rangle = \sum_{a=1}^3 \bar{m}_{a,\varphi} m_{a,\psi} + \bar{e}_\varphi e_\psi + \sum_{j=1}^4 \bar{c}_{j,\varphi} c_{j,\psi}.$$

Remark 17. The Hilbert space \mathcal{H}_{BV} has also another possible decomposition as direct sum of two vector spaces: $\mathcal{H}_{\text{BV}} = \mathcal{H}_{\text{BV},f} \oplus i \cdot \mathcal{H}_{\text{BV},f}$, with

$$\begin{aligned} \mathcal{H}_{\text{BV},f} &= [i \cdot su(2) \oplus u(1)] \oplus i \cdot u(2) \\ &\simeq \langle M_1, M_2, M_3, iE \rangle_{\mathbb{R}} \oplus \langle C_1, C_2, C_3, C_4 \rangle_{\mathbb{R}}. \end{aligned} \tag{4.7}$$

In (4.7) we denote the real part of the complex variables m_a and c_j by M_a and C_j , respectively, while E is the imaginary part of the complex variable e . This choice of notation is motivated by the fact that these variables coincide with the gauge fields and ghost fields that generate the positively graded part of the extended configuration space \tilde{X} , as we will see in Theorem 22 below. The fourth ghost field C_4 will not enter the fermionic action S_{ferm} as it decouples, being consistent with \tilde{X} having only three ghost fields C_i in our model.

The real structure J_{BV}

Up to this point – with the algebra \mathcal{A}_{BV} and self-adjoint operator D_{BV} yet to be determined – a real structure is simply given by an anti-linear isometry $J_{\text{BV}} : \mathcal{H}_{\text{BV}} \rightarrow \mathcal{H}_{\text{BV}}$, which we take to be

$$J_{\text{BV}}(\varphi_M, \varphi_C) := i \cdot (\varphi_M^*, \varphi_C^*)$$

for $\varphi_M \in \mathcal{H}_M$, $\varphi_C \in \mathcal{H}_C$. In terms of the basis (3.2) we have for $\varphi \in \mathcal{H}_{\text{BV}}$

$$J_{\text{BV}}(\varphi) := i \cdot [\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{e}, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4]^T.$$

The linear operator D_{BV}

The self-adjoint linear operator D_{BV} acting on the Hilbert space \mathcal{H}_{BV} is given by the following expression

$$D_{\text{BV}} := \begin{pmatrix} T & R \\ R^* & S \end{pmatrix}$$

in terms of the decomposition $\mathcal{H}_{\text{BV}} = \mathcal{H}_M \oplus \mathcal{H}_C$. The linear operators R, S, T are defined by

$$\begin{aligned} R : \mathcal{H}_C &\rightarrow \mathcal{H}_M; & S : \mathcal{H}_C &\rightarrow \mathcal{H}_C; & T : \mathcal{H}_M &\rightarrow \mathcal{H}_M; \\ \varphi_C &\mapsto [\beta, \varphi_C], & \varphi_C &\mapsto [\alpha, \varphi_C], & \varphi_C &\mapsto [\alpha, \varphi_C]_+, \end{aligned}$$

where α and β are hermitian, traceless 2×2 -matrices. We stress that thus R and S are derivations of $M_2(\mathbb{C})$, but that T is an odd derivation given in terms of the anti-commutator.

We can write α and β in terms of the Pauli matrices as follows

$$\begin{aligned} \alpha &= \frac{1}{2} [(-C_1^*)\sigma_1 + (-C_2^*)\sigma_2 + (-C_3^*)\sigma_3] \\ \beta &= \frac{1}{2} [(-M_1^*)\sigma_1 + (-M_2^*)\sigma_2 + (-M_3^*)\sigma_3], \end{aligned}$$

where C_i^* and M_i^* are real variables. Then, in terms of the orthonormal basis (3.2) for \mathcal{H}_M and \mathcal{H}_C , we find the following 4×4 -matrices for R, S, T :

$$\begin{aligned} R &:= \begin{pmatrix} 0 & +iM_3^* & -iM_2^* & 0 \\ -iM_3^* & 0 & +iM_1^* & 0 \\ +iM_2^* & -iM_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & S &:= \begin{pmatrix} 0 & +iC_3^* & -iC_2^* & 0 \\ -iC_3^* & 0 & +iC_1^* & 0 \\ +iC_2^* & -iC_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ T &:= \begin{pmatrix} 0 & 0 & 0 & C_1^* \\ 0 & 0 & 0 & C_2^* \\ 0 & 0 & 0 & C_3^* \\ C_1^* & C_2^* & C_3^* & 0 \end{pmatrix}. \end{aligned}$$

Of course, the notation used for the components of α and β has been chosen with purpose: indeed, we will prove that upon inserting all anti-fields in the linear operator D_{BV} , the corresponding fermionic action yields the BV action S_{BV} .

It is not true that the above D_{BV} commutes or anti-commutes with J_{BV} . Instead, we may decompose D_{BV} as

$$D_{\text{BV}} = D_1 + D_2 \quad \text{with} \quad D_1 = \begin{pmatrix} 0 & R \\ R^* & S \end{pmatrix}, \quad D_2 = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$$

for which we find that

$$\begin{aligned} J_{\text{BV}}D_1 &= -D_1J_{\text{BV}}, \\ J_{\text{BV}}D_2 &= +D_2J_{\text{BV}}. \end{aligned}$$

In anticipation of what is to come, this suggests that a real spectral triple of mixed KO -dimension will appear.

The algebra \mathcal{A}_{BV}

We now come to the final ingredient of the BV spectral triple which is the algebra \mathcal{A}_{BV} . We take it to be largest unital subalgebra of the algebra of all linear operator $\mathcal{L}(\mathcal{H}_{\text{BV}})$ that satisfies the commutation rule and first-order condition of Definition 4.

Lemma 18. Let \mathcal{H}_{BV}, J_{BV} and D_{BV} be as defined above. Then the maximal unital subalgebra $\tilde{\mathcal{A}}$ of $\mathcal{L}(\mathcal{H}_{BV})$ that satisfies

$$[a, J_{BV}b^*J_{BV}^{-1}] = 0, \quad [[D_{BV}, a], J_{BV}b^*J_{BV}^{-1}] = 0; \quad (a, b \in \tilde{\mathcal{A}})$$

is given by $\tilde{\mathcal{A}} = M_2(\mathbb{C})$ acting diagonally on \mathcal{H}_{BV} .

Proof. The commutation rule $[a, J_{BV}b^*J_{BV}^{-1}] = 0$ for all $a, b \in \tilde{\mathcal{A}}$ implies that \mathcal{H}_{BV} carries an $\tilde{\mathcal{A}}$ -bimodule structure. This already restricts $\tilde{\mathcal{A}}$ to be a subalgebra of $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, acting diagonally on $\mathcal{H}_M \oplus \mathcal{H}_C$. Then, by a straightforward computation of the double commutator $[[D, (a_1, a_2)], J_{BV}(b_1, b_2)J_{BV}^{-1}]$, it follows that the first-order condition implies $a_1 = a_2$ and $b_1 = b_2$. This selects the subalgebra $M_2(\mathbb{C})$ in $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ as the maximal subalgebra for which both of the above conditions are satisfied. \square

We will denote this maximal subalgebra by \mathcal{A}_{BV} . We now make the encountered phenomenon of mixed KO-dimension more precise by the following result.

Proposition 19. With the above notation,

- (i) $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_1, J_{BV})$ is a real spectral triple of KO-dimension 1.
- (ii) $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_2, J_{BV})$ is a real spectral triple of KO-dimension 7.

Before continuing with our BV spectral triple, we develop some new theory on real spectral triple with mixed KO-dimension.

Definition 20. For $(\mathcal{A}, \mathcal{H}, D)$ a finite spectral triple and J an anti-linear isometry on \mathcal{H} , we say that $(\mathcal{A}, \mathcal{H}, D, J)$ defines a real spectral triple with mixed KO-dimension if J satisfies

$$J^2 = \pm Id \quad \text{and} \quad [a, Jb^*J^{-1}] = 0$$

for $a, b \in \mathcal{A}$, the operator D can be seen as a sum $D = D_1 + D_2$ of two self-adjoint operators D_1, D_2 , which anti-commutes and commutes, respectively, with J :

$$D_1 = -D_1J \quad \text{and} \quad JD_2 = D_2J,$$

and, finally, the first-order condition holds:

$$[[D, a], Jb^*J^{-1}] = 0, \quad (a, b \in \mathcal{A}).$$

The notion of mixed KO-dimension generalizes the usual notion of KO-dimension for real spectral triples allowing the operator D not to fully commute or anti-commute with the isometry J . We notice that, if we are considering a genuinely mixed KO-dimension, that is, if both $D_1, D_2 \neq 0$, then the even case is not allowed.

Proposition 21. Let $(\mathcal{A}, \mathcal{H}, D, J)$ be a real spectral triple of mixed KO-dimension. If $J^2 = +1$ then for all $\varphi, \psi \in \mathcal{H}$:

- (1) the expression $\mathfrak{A}_{D_1}(\varphi, \psi) := \langle J\varphi, D_1\psi \rangle$ defines an anti-symmetric bilinear form on \mathcal{H} ;
- (2) the expression $\mathfrak{A}_{D_2}(\varphi, \psi) := \langle J\varphi, D_2\psi \rangle$ defines a symmetric bilinear form on \mathcal{H} .

On the contrary, if $J^2 = -1$, then \mathfrak{A}_{D_1} is symmetric and \mathfrak{A}_{D_2} is anti-symmetric.

Proof. (1) Bilinearity of \mathfrak{A}_{D_1} is a consequence of J being an anti-linear map, D_1 being a linear operator and the inner product being anti-linear in its first component and linear in the second. For the anti-symmetry, under the assumption that $J^2 = \epsilon$ we compute

$$\langle J\varphi, D_1\psi \rangle = \epsilon \langle J\varphi, J^2D_1\psi \rangle = \epsilon \langle JD_1\psi, \varphi \rangle = -\epsilon \langle D_1J\psi, \varphi \rangle = -\epsilon \langle J\psi, D_1\varphi \rangle$$

using the anti-commutation of D_1 with J and D_1 being a self-adjoint operator.

- (2) This follows *mutatis mutandis* from (1), assuming D_2 to commute with J . \square

We now return to the BV spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ that we constructed for our $U(2)$ -matrix model.

Theorem 22. The data $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$ defined above is a real spectral triple with mixed KO-dimension. Moreover, the fermionic action corresponding to the operator D_{BV} coincides with the BV action in (4.6), i.e.,

$$S_{BV} = \frac{1}{2} \langle J_{BV}(\varphi), D_{BV}\varphi \rangle, \quad \text{with} \quad \varphi \in \mathcal{H}_{BV,f}.$$

Here we interpret the variables that parametrize D and the vector $\varphi \in \mathcal{H}_{BV}$ as follows:

- M_a, E and C_j^* are real variables;
- M_a^* and C_j are Grassmannian variables.

Proof. The first claim is an immediate consequence of Proposition 19. The last statement follows by a straightforward computation. \square

The mixed KO-dimension in the BV spectral triple arises from the particular behavior of the real structure with the operator D_{BV} , which partially commutes and partially *anti-commutes* with J_{BV} . However, this is not due to the fact that the BV spectral triple is a direct sum of real spectral triples of different KO-dimensions. Indeed, then the structure of the real spectral triple would be

$$(\mathcal{A}_1 \oplus \mathcal{A}_2, \mathcal{H}_1 \oplus \mathcal{H}_2, D_1 \oplus D_2, J_1 \oplus J_2).$$

The appearance of a direct sum of the two algebras is in contrast with the structure of \mathcal{A}_{BV} as a simple algebra. As a matter of fact, the mixed KO-dimension has a different significance, allowing us to detect the difference in *parity* of the components of the BV spectral triple, as we will now explain.

Namely, the fields that parametrize the Dirac operator and/or represent vectors in Hilbert space are seen to be structured as follows:

- The anti-fields/anti-ghost fields M_a^* and C_j^* appear as entries of the operator D_{BV} while the fields/ghost fields M_a, C_j , and E are the components of the vectors in the subspace $\mathcal{H}_{BV,f}$.
- The parities of the fields/ghost fields and anti-fields/anti-ghost fields in the BV spectral triple are a consequence of the structure of the real spectral triple. Indeed, the parities chosen in Theorem 22 are precisely those for which both D_1 and D_2 give a non-trivial contribution to the fermionic action S_{ferm} .

4.2. The BV auxiliary spectral triple

In addition to the BV construction, also the technical procedure of introducing auxiliary fields in our $U(2)$ -matrix model can be expressed in terms of a spectral triple: in this section, we construct the so-called *BV auxiliary spectral triple*

$$(\mathcal{A}_{\text{aux}}, \mathcal{H}_{\text{aux}}, D_{\text{aux}}, J_{\text{aux}}),$$

for which the fermionic action coincides with the auxiliary action S_{aux} . We follow the same strategy as for the BV spectral triple: the anti-fields $\{B_j^*\}$, for $j = 1, 2, 3$, and $\{A_l^*\}$, with $l = 1, 2$, parametrize the operator D_{aux} while the auxiliary fields $\{h_j\}$ and $\{k_l\}$ are the components of the vectors in the Hilbert space \mathcal{H}_{aux} . Moreover, we keep in mind the possibility of encountering a real spectral triple with mixed KO-dimension.

Remark 23. Since the action S_{aux} is not bilinear in the fields, it cannot be expected to directly agree with a usual fermionic action. For this reason, we will slightly adapt the definition of a fermion action associated to a real spectral triple.

The Hilbert space \mathcal{H}_{aux}

The Hilbert space describes the field content of the action S_{aux} . So we have

$$\mathcal{H}_{\text{aux}} = \mathcal{H}_h \oplus \mathcal{H}_k := M_2(\mathbb{C}) \oplus \mathbb{C}^2.$$

Again, we take the Pauli matrices (3.2) as an orthonormal basis for $M_2(\mathbb{C})$, so that $\mathcal{H}_{\text{aux}} \cong \mathbb{C}^6$. We also identify the following subspace:

$$\mathcal{H}_{\text{aux},f} = u(2) \oplus i[u(1) \oplus u(1)],$$

and write elements $\chi \in \mathcal{H}_{\text{aux},f}$ suggestively as

$$\chi = [ih_1, ih_2, ih_3, ih_4, k_1, k_2]^T,$$

where h_j and k_l , ($j = 1, \dots, 4, l = 1, 2$) are real variables.

The real structure J_{aux}

The anti-linear isometry J_{aux} is defined similarly as in the BV spectral triple: indeed, $J_{\text{aux}} : \mathcal{H}_h \oplus \mathcal{H}_k \rightarrow \mathcal{H}_h \oplus \mathcal{H}_k$, with

$$J_{\text{aux}}(V, v) := (i \cdot V^*, i \cdot \bar{v}).$$

The operator D_{aux}

In the basis of $M_2(\mathbb{C})$ given by the Pauli matrices we define the operator D_{aux} as

$$D_{\text{aux}} = D_{\text{diag}} + D_{\text{off}} \quad \text{with} \quad D_{\text{diag}} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad D_{\text{off}} = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}$$

where, again in evocative notation in terms of the anti-fields A_i^* and B_j^* , we define

$$P = \frac{1}{2} \begin{pmatrix} +B_1^* + B_2^* + B_3^* & 0 & 0 & +B_1^* - B_2^* - B_3^* \\ 0 & +B_1^* + B_2^* + B_3^* & 0 & -B_1^* + B_2^* - B_3^* \\ 0 & 0 & +B_1^* + B_2^* + B_3^* & -B_1^* - B_2^* + B_3^* \\ +B_1^* - B_2^* - B_3^* & -B_1^* + B_2^* - B_3^* & -B_1^* - B_2^* + B_3^* & +B_1^* + B_2^* + B_3^* \end{pmatrix}$$

and

$$Q = -\frac{i}{3} \begin{pmatrix} A_1^* & A_1^* & A_1^* & 0 \\ A_2^* & A_2^* & A_2^* & 0 \end{pmatrix}.$$

The algebra \mathcal{A}_{aux}

Also in this case, we take the algebra \mathcal{A}_{aux} to be the largest unital subalgebra of $\mathcal{L}(\mathcal{H}_{\text{aux}})$ that completes the triple \mathcal{H}_{aux} , D_{aux} , and J_{aux} to a real spectral triple (with mixed KO-dimension).

Lemma 24. *Let $\mathcal{H}_{\text{aux}}, J_{\text{aux}}$ and D_{aux} be as defined above. Then the maximal unital subalgebra $\tilde{\mathcal{A}}$ of $\mathcal{L}(\mathcal{H}_{\text{aux}})$ on which the commutation rule and first-order condition are fulfilled, that is, which satisfies*

$$[a, J_{\text{aux}} b^* J_{\text{aux}}^{-1}] = 0, \quad [[D_{\text{aux}}, a], J_{\text{aux}} b^* J_{\text{aux}}^{-1}] = 0; \quad (a, b \in \tilde{\mathcal{A}}),$$

is $\tilde{\mathcal{A}} = \mathbb{C}$.

Proof. The fact that \mathcal{H}_{aux} should be a $\tilde{\mathcal{A}}$ -bimodule already restricts $\tilde{\mathcal{A}}$ to be a subalgebra of

$$M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$$

acting by (block diagonal) matrix multiplication on $\mathcal{H}_{\text{aux}} = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. A straightforward computation of the double commutator entering in the first-order condition then selects the diagonal subalgebra $\tilde{\mathcal{A}} = \mathbb{C}$. \square

We will write $\mathcal{A}_{\text{aux}} = \mathbb{C}$ and notice the intriguing agreement between the triviality of the algebra with the triviality of the trivial pairs of auxiliary fields.

Proposition 25. *For $\mathcal{A}_{\text{aux}}, \mathcal{H}_{\text{aux}}, D_{\text{aux}}$ and J_{aux} as previously defined, it holds that*

- (i) $(\mathcal{A}_{\text{aux}}, \mathcal{H}_{\text{aux}}, D_{\text{diag}}, J_{\text{aux}})$ is a real spectral triple of KO-dimension 7;
- (ii) $(\mathcal{A}_{\text{aux}}, \mathcal{H}_{\text{aux}}, D_{\text{off}}, J_{\text{aux}})$ is a real spectral triple of KO-dimension 1.

Proof. Because $\mathcal{A}_{\text{aux}} = \mathbb{C}$, this follows at once from noticing that

$$J_{\text{aux}} D_{\text{diag}} = +D_{\text{diag}} J_{\text{aux}} \quad \text{and} \quad J_{\text{aux}} D_{\text{off}} = -D_{\text{off}} J_{\text{aux}},$$

which can be readily checked. \square

The last ingredient to analyze is the fermionic action. As already noticed in Remark 23, we need to introduce a linear notion of fermionic action. More precisely, the fermionic action corresponding to the operators D_{diag} and D_{off} will be defined using two linear forms $\mathcal{L}_{D_{\text{diag}}}, \mathcal{L}_{D_{\text{off}}}$ instead of a bilinear form \mathfrak{A} , as was done for the BV action. This is a consequence of the fact that the auxiliary action S_{aux} is only linear (rather than quadratic) in the fields. We state the following general, but straightforward result without proof.

Proposition 26. *Let $(\mathcal{A}, \mathcal{H}, D, J)$ be a real spectral triple (possibly with mixed KO-dimension) and fix a vector $v \in \mathcal{H}$. Then the expression*

$$\mathcal{L}_D(\chi) = \frac{1}{2} (\langle Jv, D\chi \rangle + \langle J\chi, Dv \rangle), \quad (\chi \in \mathcal{H})$$

defines a linear form on \mathcal{H} .

In our case of interest, we fix the vector v to be

$$v = \underline{1} = (1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)^T \in \mathbb{C}^6 \cong \mathcal{H}_{\text{aux}}.$$

Theorem 27. $(\mathcal{A}_{\text{aux}}, \mathcal{H}_{\text{aux}}, D_{\text{aux}}, J_{\text{aux}})$ defines a real spectral triple with mixed KO-dimension. Moreover, the fermionic action defined by the linear form $\mathcal{L}_{D_{\text{aux}}}$ coincides with the auxiliary action S_{aux} :

$$S_{\text{aux}} = \frac{1}{2} (\langle J_{\text{aux}}(\underline{1}), D_{\text{aux}}(\chi) \rangle + \langle J_{\text{aux}}(\chi), D_{\text{aux}}(\underline{1}) \rangle), \quad \text{with } \chi \in \mathcal{H}_{\text{aux},f}.$$

Here we interpret the variables that parametrize D_{aux} and the vector $\chi \in \mathcal{H}_{\text{aux}}$ as follows:

- B_j^* , h_l are real variables;
- A_l^* , k_l are Grassmannian variables.

Proof. For a generic vector χ in $\mathcal{H}_{\text{aux},f}$, $\chi = [ih_1, ih_2, ih_3, ih_4, k_1, k_2]^T$, one computes that

$$\langle J_{\text{aux}}(\underline{1}), D_{\text{aux}}(\chi) \rangle = \sum_{j,l} B_j^* h_j + A_l^* k_l - \frac{i}{3} A_l^* h_j$$

$$\langle J_{\text{aux}}(\chi), D_{\text{aux}}(\underline{1}) \rangle = \sum_{j,l} h_j B_j^* - k_l A_l^* + \frac{i}{3} h_j A_l^*$$

with $j = 1, 2, 3, l = 1, 2$. It then follows that

$$\frac{1}{2} [\langle J_{\text{aux}}(\underline{1}), D_{\text{aux}}(\chi) \rangle + \langle J_{\text{aux}}(\chi), D_{\text{aux}}(\underline{1}) \rangle] = \sum_{j=1}^3 B_j^* h_j + \sum_{l=1}^2 A_l^* k_l,$$

whose right-hand side coincides with S_{aux} as defined in (3.5). \square

As expected, the BV auxiliary spectral triple has a similar structure to the one already found for the BV spectral triple:

- The anti-fields B_j^* and A_l^* appear as entries of the operator D_{aux} while the fields h_j and k_l are the components of the vectors in subspace $\mathcal{H}_{\text{aux},f}$.
- The parities of the fields and anti-fields are a consequence of the structure of the real spectral triple, except perhaps for the parity of the anti-fields B_j^* .

5. A possible approach to BV spectral triples

The procedure presented in this paper allows to describe the BV construction of a given gauge theory in the setting of noncommutative geometry. Even though we have restricted ourselves to the case of a $U(2)$ -gauge invariant matrix model, our results suggest a possible way on how to proceed in a more general setting. Indeed, let (X_0, S_0) be the gauge theory derived from a finite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ along the lines of Section 2.1. Then the BV formalism gives rise to an extended theory (X, S) which one tries to capture by a BV spectral triple

$$(\mathcal{A}_{\text{BV}}, \mathcal{H}_{\text{BV}}, D_{\text{BV}}, J_{\text{BV}}).$$

The properties that this spectral triple should satisfy are

1. The algebra \mathcal{A}_{BV} coincides with \mathcal{A} ;
2. The Hilbert space \mathcal{H}_{BV} is spanned by the gauge fields and all ghost fields;
3. The real structure selects the hermitian variables in \mathcal{H}_{BV} .

Of course, the main challenge is now to find the form of the operator D_{BV} in terms of the anti-fields for which the fermionic action coincides with the BV action functional. One of the problems to overcome here is that the BV action might have terms of order higher than 2 in the ghost fields, requiring the introduction of some sort of *multilinear fermionic action*. A first analysis of this is in progress for $U(n)$ -matrix models with $n > 2$.

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