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Geometry and the quantum: basics. (English summary)
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This paper proposes a generalization of the Heisenberg commutation relation $[p,q]=i\hbar$ to arrive at a quantization of spin manifolds. The momentum p is naturally replaced by the Dirac operator, but the generalization of position q to dimension n is more difficult to capture in a single object. Noncommutative geometry suggests looking for an algebra $\mathcal{A}$ of coordinates that together with a Hilbert space $\mathcal{H}$ and a Dirac operator D forms a so-called spectral triple. In the present paper, the authors consider an element $Y\in\mathcal{A}\otimes C$ with coefficients in the Clifford algebra C in dimension $n+1$ (where n is even), which is normalized as

$$Y^2=1,\qquad Y^*=Y.$$

The (one-sided) higher analogue of the Heisenberg commutation relations that the authors propose is then

$$\frac{1}{n!}\langle Y[D,Y]\cdots[D,Y]\rangle=\gamma,$$

with n terms $[D,Y]$, where $\langle\cdot\rangle$ is a trace in the Clifford algebra (not to be confused with the matrix structure of D). This relation already appeared in [A. Connes, J. Math. Phys. **41** (2000), no. 6, 3832–3866; MR1768641; A. Connes and G. Landi, Comm. Math. Phys. **221** (2001), no. 1, 141–159; MR1846904] in the construction of spherical manifolds. This is confirmed by the authors in Theorem 1, showing that a solution to the one-sided equation exists for the spectral triple related to a compact Riemannian spin manifold if and only if the manifold M breaks as the disjoint sum of spheres of unit volume, so-called quanta of geometry. One way to see this is by using the index formula [A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994 (Chapter IV.2.β); MR1303779]:

$$⨍\gamma\langle Y[D,Y]\cdots[D,Y]\rangle D^{-n}=2^{n/2+1}\mathrm{degree}(Y),$$

where the normalization of Y allows one to consider it as a map from M to S^n. The one-sided equation implies that the map Y is a covering map so that necessarily the connected components of M are spheres. The volume of M is then proportional to the degree of Y appearing on the right-hand side of the index formula.

A refinement is possible when there is a real structure J (closely related to Tomita's anti-isomorphism operator). This leads to the two-sided equation

$$\frac{1}{n!}\langle Z[D,Z]\cdots[D,Z]\rangle=\gamma,\qquad Z=2EJEJ^{-1}-1$$

where $E=\frac{1}{2}(1+Y_+)\oplus\frac{1}{2}(1+Y_-)$ with respect to the decomposition $Y=Y_+\oplus Y_-$ in even and odd degree of the Clifford algebra.

In Theorem 6 the authors show that the two-sided equation still implies that the volume of M is quantized but that M no longer breaks into small disjoint connected components. In fact, for $n=4$ solutions are given by smooth connected compact spin 4-manifolds, precisely the class of relevant geometries to be considered in the context of spectral triples. We give a brief sketch of the proof. In analogy with the approach taken for the one-sided equation, one considers the set $D(M)$ of pairs of smooth maps $\phi_\pm\colon M \to S^n$ (corresponding to $Y_\pm$) such that the differential form

$$\phi_+^*(\alpha) + \phi_-^*(\alpha)$$

does not vanish anywhere on M, where α is the volume form on the sphere S^n. One then considers the following invariant:

$$q(M) := \{\text{degree}(\phi_+) + \text{degree}(\phi_-) : (\phi_+, \phi_-) \in D(M)\} \subset \mathbb{Z}$$

and shows that a solution to the two-sided equation exists if and only if the volume of M belongs to $q(M) \subset \mathbb{Z}$. In addition, the authors show in Theorem 12 that the set $q(M)$ contains all integers $m \geq 5$ for any smooth connected compact spin 4-manifold.

As a final intriguing remark, note that for $n = 4$ the Clifford algebra is

$$C_+ \oplus C_- = M_2(\mathbb{H}) \oplus M_4(\mathbb{C}),$$

which are precisely the algebraic constituents of the Standard Model of particle physics that appear in the work of the first two authors and the reviewer.

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