I. Geometry from the spectral point of view.

Basic principle:

All information about the structure of spacetime that we gathered in experiments is spectral such as: redshifts, absorption spectra, etc.

Question for today:

How to extract a smooth manifold structure and observables from a set of real numbers (the spectrum)?

Mark Kac (1966) "Can one hear the shape of a drum?"

$$\Delta_M u = k^2 u \qquad (M, g)$$

$\hookrightarrow$ wave numbers. "$\sqrt{\Delta_M}$"

Paul Dirac: (1928) Relativistic version of Schrödinger:

$$i \frac{\partial}{\partial t} \psi(x,t) = \left(\Delta + V(x,t)\right) \psi(x,t)$$

$$\Delta = -\left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$$

"Square root of Laplacian"

S^1: $\quad \Delta_{S^1} = -\left(\frac{\partial}{\partial x}\right)^2 \qquad D_{S^1} = i\frac{\partial}{\partial x}.$

$D_{S^1}^2 = \Delta_{S^1}$ $\qquad$ eigenvalues: $n \in \mathbb{Z}$

$$D_{S^1} e^{inx} = -n\, e^{inx}$$

$\mathbb{T}^2 = S^1 \times S^1$: $\quad \Delta_{\mathbb{T}^2} = -\left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2$

$$D_{\mathbb{T}^2} = a\frac{\partial}{\partial x^1} + b\frac{\partial}{\partial x^2} \rightsquigarrow a^2 = b^2 = -1, \quad ab = 0 \quad ↯$$

But: matrix-valued solutions exist:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rightsquigarrow \mathbb{H}$$

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} \\ -\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & 0 \end{pmatrix} \to D_{\mathbb{T}^2}^2 = \Delta_{\mathbb{T}^2} \cdot \mathbb{I}_2.$$

eigenfunctions:

$$\psi_{\vec{n}}^{(\pm)}(\vec{x}) = \begin{pmatrix} e^{i\vec{n}\cdot\vec{x}} \\ \mp\frac{in_1 + n_2}{\sqrt{n_1^2 + n_2^2}}\, e^{i\vec{n}\cdot\vec{x}} \end{pmatrix}; \quad D_{\mathbb{T}^2}\psi_{\vec{n}}^{(\pm)} = \pm\sqrt{n_1^2 + n_2^2}\,\psi_{\vec{n}}^{(\pm)}$$

$\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$

$$\Delta_{\mathbb{T}^4} = -\left(\frac{\partial}{\partial x^1}\right)^2 - \cdots - \left(\frac{\partial}{\partial x^4}\right)^2$$

$$D_{\mathbb{T}^4} = a\frac{\partial}{\partial x^1} + \cdots + d\frac{\partial}{\partial x^4} \qquad a^2 = \cdots = d^2 = -1$$

$ab + ba = 0$
$ac + ca = 0 \ldots$

$\rightsquigarrow a = 1,\ b = i,\ c = j,\ d = k$ $M_2(\mathbb{H})$

Quaternions

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} + j\frac{\partial}{\partial x^3} + k\frac{\partial}{\partial x^4} \\ -\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} + j\frac{\partial}{\partial x^3} + k\frac{\partial}{\partial x^4} & 0 \end{pmatrix}$$

$\Rightarrow D^2_{\mathbb{T}^4} = \Delta_{\mathbb{T}^4}\, \mathbb{I}_4$ spectrum $\pm\sqrt{n_1^2 + \cdots + n_4^2}$

Ref: Spectral model of PP - WS

In general, a Dirac operator exists whenever (M, g) is a Riemannian spin^c manifold.

Then we have:

- $\Gamma^\infty(S_M)$: spinors, carrying a representation of Clifford algebra generated by $\gamma^\mu = \gamma(dx^\mu)$ satisfying
$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$
- ∇^{S_M}, spin connection, lift of Levi-Civita:
$$\nabla^{S_M}(\gamma(\omega) s) = \gamma(\nabla^{LC}(\omega)) s + \gamma(\omega) \nabla^{S_M}(s)$$

Ref: Lawson-Michelsohn. Spin Geometry

- Unique if M is spin; $J_M : \Gamma(S_M) \to \Gamma(S_M)$

- Dirac operator

$$D_M : \Gamma^\infty(S_M) \longrightarrow \Gamma^\infty(S_M)$$

$$s \longmapsto \sum_\mu \gamma^\mu (\nabla^S_\mu s)$$

- Lichnerowicz:

$$D_M^2 = \Delta_M + \frac{1}{4} \sigma_M .$$

Spectrum: if M compact then D_M is an essentially self-adjoint operator with cpt resolvent:

$$D_M = \sum_{n \in \mathbb{Z}} \lambda_n |\psi_n\rangle\langle\psi_n|$$

If M even-dim, then $\gamma_M : \Gamma^\infty(S) \to \Gamma^\infty(S)$ $\gamma_M D_M = -D_M \gamma_M$

Math question: does D_M capture all geometric information about (M,g)?

Marc Kac (1966) "Can one hear the shape of a drum?"

No... isospectral but non-isometric examples.

Some geometric info: $\mathrm{tr}\, e^{-tD_M^2} \sim t^{-d/2}\mathrm{Vol}(M) + t^{-d/2+1}\int_M \cdots$

Noncommutative geometry adds to D_M more local spectral information about M

"locally hearing the shape of a drum"

$$\left(C^\infty(M), L^2(S_M), D_M ; \gamma_M, J_M \right)$$

↳ smooth functions $f: M \to \mathbb{C}$ to localize, act on $L^2(S_M)$

Reconstruction of (M,g) is possible (C96, C08)

Let us illustrate this for the metric distance.

Distance function

$$d(x,y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \| [D_M, f] \| \leq 1 \}$$

Prop $d(x,y) = d_g(x,y)$.

Proof: First we compute

$$[D_M, f] = -i\, c(dx^\mu) [\nabla^S_\mu, f]$$

$$= -i\, c(dx^\mu)\, \partial_\mu f = -i\, c(df)$$

Hence

$$\| [D_M, f] \|^2 = \| c(d\bar{f})\, c(df) \|$$

grad $f = (df)^\#$

$$= \| g^{-1}(d\bar{f}, df) \| = \sup_{x \in M} \| \mathrm{grad}_x f \|^2$$

$$=: \| \mathrm{grad} f \|_\infty^2$$

We claim that $\| \mathrm{grad}\, f \|_\infty = \| f \|_1$

where

$$\| f \|_1 = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_g(x,y)} \quad (\text{Lipschitz norm})$$

First, consider a smooth path $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$. Then

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f(\gamma(t))\, dt$$

$$= \int_0^1 (d_{\gamma(t)} f)(\dot{\gamma}(t))\, dt$$

$$= \int_0^1 g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \dot{\gamma}(t))\, dt$$

By Cauchy-Schwartz we then have

$$|f(x) - f(y)| \leq \|\operatorname{grad} f\|_\infty \, \ell(\gamma)$$

This also holds for a geodesic γ so that $\ell(\gamma) = d_g(x,y)$ and we find $\|f\|_1 \leq \|\operatorname{grad} f\|_\infty$.

Vice versa, suppose instead that $\exists\, x \in M$ so that $\|\operatorname{grad}_x f\| > \|f\|_1 + \varepsilon$.

Consider again $\gamma : [0,1] \to M$ s.t. $\gamma(0) = x$.

Then $\exists\, \delta > 0$ s.t. $\forall\, 0 < t < \delta$ we have

$$\left| \frac{1}{t}\left(f(\gamma(t)) - f(\gamma(0))\right) - g_x\left(\mathrm{grad}_x f, \dot\gamma(0)\right)\right| < \frac{\varepsilon}{2}$$

$\Rightarrow$

$$\left|\frac{1}{t}\left(f(\gamma(t)) - f(\gamma(0))\right)\right| > \left|g_x\left(\mathrm{grad}_x f, \dot\gamma(0)\right)\right| - \frac{\varepsilon}{2}$$

Now take $\dot\gamma(0) = \dfrac{\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}$ so $\|\dot\gamma(0)\| = 1$

and parametrize γ naturally so that

$l(\gamma(0) \to \gamma(t)) = t$ Then the above inequality

yields

$$\left| f(\gamma(t)) - f(\gamma(0))\right| > \left(\|\mathrm{grad}_x f\| - \frac{\varepsilon}{2}\right) t$$

$$> \left(\|f\|_1 + \frac{\varepsilon}{2}\right) l(\gamma(0) \to \gamma(t))$$

$$> \left(\|f\|_1 + \frac{\varepsilon}{2}\right) d_g(\gamma(0), \gamma(t))$$

But this implies $\|f\|_1 > \|f\|_1 + \frac{\varepsilon}{2}$ $\lightning$.

Now the distance function: under the

condition $\|[D_n, f]\| \leq 1$ the above result implies that

$$|f(x) - f(y)| \leq d_g(x,y).$$

This upper bound is also attained by taking $f(z) = d_g(x, z)$. This indeed satisfies

$$\|d_g(x,\cdot)\|_1 = \sup_{y,z} \frac{|d_g(x,y) - d_g(y,z)|}{d_g(y,z)} \leq 1$$

by reverse triangle inequality.

We now abstract the following notion of a noncommutative manifold.

Defn A spectral triple $(A, \mathcal{H}, D)$ is given by

A: $*$-algebra of bdd operators on

$\mathcal{H}$: Hilbert space

D: ess. self-adj. operator in $\mathcal{H}$

s.t. - $[D, a]$ extends to bdd commutator $\forall a \in A$

- $(1+D^2)^{-1}$ is cpt. operator.

grading: γ. $\quad \gamma D = -D\gamma$

real structure: J $\quad J^2 = \varepsilon, \ JD = \varepsilon' DJ, \ J\gamma = \varepsilon'' \gamma J$

$\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$.

Ex $D_M ; (A_F, H_F, D_F)$

Distance function: $\varphi, \psi \in \mathcal{S}(A)$:

$$d(\varphi, \psi) = \sup_{a \in A} \{ |\varphi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral action: $\operatorname{tr}\left(f\left(\frac{D}{\Lambda}\right)\right)$ $\quad \Lambda \in \mathbb{R}_+$, f cutoff.