II. Inner perturbations.

$$(\mathcal{A}, \mathcal{H}, D) \rightsquigarrow \operatorname{tr} f(D/\Lambda)$$

Most natural : unitary equivalence.

<u>Defn.</u> $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are called unitarily equivalent if there exists isomorphism $\alpha : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2$ and unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.

$$\alpha(a) = U a U^*$$

$$D_2 = U D_1 U^*.$$

Special case: $u \in \mathcal{U}(A)$: $\begin{pmatrix} A_1 = A_2 = A \\ \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \end{pmatrix}$

$$\alpha(u) = u a u^* = \alpha_u(a)$$

inner automorphisms $\nearrow$ $\alpha : \mathcal{U}(A) \to \mathrm{Inn}(A)$.

$$u D u^* = D + u[D, u^*]$$

pure gauge fields

extend this to arbitrary gauge fields.

1) Morita equivalence $\quad A \underset{M}{\sim} B$ (unital $\mathbb{C}$-algebras)

${}_B\mathcal{E}_A$, ${}_A\mathcal{F}_B$:

$$\mathcal{E} \otimes_A \mathcal{F} \cong B \qquad \mathcal{F} \otimes_B \mathcal{E} \cong A.$$

A, B commutative $\quad A \underset{M}{\sim} B \iff A \cong B.$

Transfer $(A, \mathcal{H}, D)$ to $(B, \mathcal{H}', D')$

$$\mathcal{H}' = \mathcal{E} \otimes_A \mathcal{H}$$

$$D' = 1 \otimes_\nabla D,$$

$$(1 \otimes_\nabla D)(e \otimes \xi) = \nabla e \cdot \xi + e \otimes D\xi$$

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D(A)$$

$$\left\{ \begin{array}{l} \Omega^1_D(A) = \{ \sum a_j [D, b_j] : a_j, b_j \in A \} \\ \nabla(ea) = \nabla(e)a + e \otimes [D, a]. \end{array} \right.$$

Thm $(A, \mathcal{H}, D)$ spectral triple $\Rightarrow$

$(B, \mathcal{E} \otimes_A \mathcal{H}, 1 \otimes_\nabla D)$ spectral triple.

(extends to real sp. tr.)

Morita self-equivalence. Take $B = A, \mathcal{E} = A$

Then $\nabla : A \to \Omega^1_D(A)$ determined by

$A = \nabla(1) = \sum a_j [D, b_j]$

$1 \otimes_\nabla D = D_A = D + A$

inner perturbations

Special case: $\mathcal{E} = {}_{\alpha_u(A)}A_A \rightsquigarrow$ pure gauge fields

② Semigroup of inner perturbations

[CCS 13]

Recall: $A^{op} = \{ a^{op} : a \in A \}$ vect. space

$$a^{op} b^{op} = (ba)^{op}$$

Defn. Perturbation semigroup:

$$\mathrm{Pert}(A) = \Big\{ \sum a_j \otimes b_j^{op} \in A \otimes A^{op} \;\Big|\; \sum a_j b_j = 1,$$
$$\sum a_j \otimes b_j^{op} = \sum b_j^{*} \otimes (a_j^{*})^{op} \Big\}$$

Semigroup law inherited from $A \otimes A^{op}$.

N.B. $\left(\sum a_j \otimes b_j^{o}\right)\left(\sum a'_k \otimes (b'_k)^{o}\right)$ normalized:

$$\sum a_j a'_k \cdot b'_k b_j = \sum a_j b_j = 1.$$

Observations:

- $\mathcal{U}(A) \to \mathrm{Pert}(A)$

  $u \mapsto u \otimes (u^*)^{op}$

- $\mathrm{Pert}(A)$ acts on self-adj. op. D:

$$D \mapsto \sum_j a_j D b_j = D + \underbrace{a_j [D, b_j]}_{\text{gauge field}}$$

So, semigroup structure on gauge fields

Ex: (manifold)

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) : \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

$$\frac{\partial}{\partial x^\mu} \longmapsto \frac{\partial}{\partial y^\mu} f(x,y)\Big|_{y=x} =: A_\mu \in C^\infty(M)$$

More generally, for $C^\infty(M, M_N(\mathbb{C}))$

$A_\mu \in C^\infty(M, \mathfrak{u}(N))$ $\mathfrak{u}(N)$-gauge field

semigroup structure on gauge fields

Ex two-point space

$$\left(\mathbb{C}^2, \mathbb{C}^2, D = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}\right)$$

$$\mathrm{Pert}(\mathbb{C}^2) = \{ e_{11} \otimes e_{11} + z\, e_{11} \otimes e_{22} + \bar{z}\, e_{22} \otimes e_{11} + e_{22} \otimes e_{22} \} \cong \mathbb{C}$$

$\mathcal{U}(\mathbb{C}^2) \longrightarrow U(1) \subset \mathrm{Pert}(\mathbb{C}^2)$

$$D = \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 0 & zc \\ \bar{z}\bar{c} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \bar{\phi} \\ \phi & 0 \end{pmatrix}$$

moreover ϕ transforms under $U(1)$ as $\phi \mapsto \lambda \phi$

Ex.

$\mathrm{Pert}(\mathbb{C}^N) \cong \mathbb{C}^{N(N-1)/2}$, $\mathrm{Pert}(M_N(\mathbb{C})) = V \times S$ etc.

Ex: Noncommutative two-point space

$$\left(\mathbb{C} \oplus \mathbb{H},\ \mathbb{C} \oplus \mathbb{C}^2,\ D = \begin{pmatrix} 0 & c & 0 \\ \bar{c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$\mathrm{Pert}(\mathbb{C} \oplus \mathbb{H}) \cong \mathbb{H}^{\mathbb{C}} \times \mathrm{Pert}(\mathbb{H})$$

$$D \longmapsto \begin{pmatrix} 0 & c z_1 & c z_2 \\ \bar{c}\bar{z}_1 & 0 & 0 \\ \bar{c}\bar{z}_2 & 0 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \bar{\phi}_1 & \bar{\phi}_2 \\ \phi_1 & 0 & 0 \\ \phi_2 & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ column in $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus \mathbb{H})$

Moreover, $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transforms under $\mathcal{U}(\mathbb{C} \oplus \mathbb{H}) = U(1) \times SU(2)$

as $\Phi \longmapsto \bar{\lambda} u . \Phi \qquad \left((\lambda, u) \in U(1) \times SU(2) \right)$

Ex. AC mfds $\left(C^\infty(M) \otimes A_F,\ L^2(S_M) \otimes H_F,\ D \otimes 1 + \gamma \otimes D_F \right)$

$$D \otimes 1 + \gamma \otimes D_F \longmapsto D \otimes 1 + \gamma \otimes D_F + \gamma^\mu A_\mu + \gamma \otimes \Phi$$

$$\begin{cases} A_\mu \in C^\infty(M, \mathfrak{u}(A_F)) \\ \Phi \in C^\infty(M, L(\mathcal{H}_F)) \end{cases}$$

Ex. NC torus (Tenn v Nuland, BSc 2016)
($\lambda = e^{i\theta}$) (my website)

$$\mathrm{Per}(A_\theta) = \{ \hat{c} \in A_\theta \otimes A_\theta : \hat{c}\,\tau = \tau ;\ \hat{c}\,\Omega = \Omega \hat{c}^{\,t} \}$$

$$\begin{cases} A_\theta \otimes A_\theta \subseteq B(\ell^\infty(\mathbb{Z}^4)) & \hat{c} = \sum c_{hlmn}\, u^h v^l \otimes u^m v^n \\ \hat{c}\,\varphi(p,q,r,s) = \sum c_{hlmn}\, \lambda^{-ph - rm}\, \varphi(p-l, q-h, r-n, s-m) & \end{cases}$$

$$\begin{cases} \tau(p,q,r,s) = \lambda^{-pq}\, \delta_{q+r,\, p+s} \\ \Omega\varphi(p,q,r,s) = \varphi(r,s,p,q) \end{cases}$$