II. Spectral action, expansions.

Consider $(\mathcal{A}, \mathcal{H}, D)$, fin. summ. $\mathrm{Pert}(\mathcal{A})$ acts to give $\sum a_j D b_j = D + A$.
$A \in \Omega^1_D(\mathcal{A})$.

<u>Spectral action functional</u>

$$S_\Lambda[A] = \mathrm{tr}\, f\left(\frac{D+A}{\Lambda}\right) \qquad \Lambda > 0.$$

Manifest invariance under $\mathcal{U}(\mathcal{A}) \hookrightarrow \mathrm{Pert}(\mathcal{A})$.

Also, there is a topological spectral action

$$S_{top}[A] = \mathrm{tr}\, \gamma\, f\left(\frac{D+A}{\Lambda}\right)$$

We will assume throughout that f is a Laplace transf.:

$$f(x) = \int_{t>0} g(t)\, e^{-t x^2}\, dt \qquad (*)$$

Prop. $S_{top}[A] = f(0)\ \mathrm{index}\,(D+A)$

Proof McKean-Singer: $\mathrm{tr}\, \gamma\, e^{-t(D+A)^2/\Lambda^2} = \mathrm{index}(D+A)$

while $f(0) = \int g(t)\, dt$ $\square$

At least two interesting expansions for $S_\Lambda[A]$:

- (Asymptotic) expansion in Λ; tells us effective behav. for energies $< \Lambda$
- Taylor expansion in A; tells about propagators, interactions

<u>and their combination</u> in physics applications

Asymptotic expansion in $\Lambda \to \infty$.

Suppose $\mathrm{tr}\, e^{-t(D+A)^2} \sim \sum_{\alpha} c_\alpha t^\alpha \quad (t \to 0)$

Prop [CCM] $S_\Lambda[A] \sim \sum_{\beta \in Sd} f_\beta \Lambda^\beta \operatorname{res}_{z=\beta} \mathrm{tr}\, |D+A|^{-z}$

$$+ f(0)\, \underbrace{\mathrm{tr}\, |D+A|^{-z}\Big|_{z=0}}_{\zeta_{D+A}(0)} + O(\Lambda^{-1})$$

where $f_\beta = \int f(v) v^{\beta-1} dv$.

Prop [CC, 2006] If $(A, \mathcal{H}, D)$ th. summable then

$$\zeta_{D+A}(0) - \zeta_D(0) = -⨍ \log(1 + AD^{-1})$$

$$= \sum_{n>0} \frac{(-1)^n}{n} ⨍ (AD^{-1})^n$$

where $⨍(\cdot) = \operatorname{res}_{z=0} \mathrm{tr}((\cdot)\, |D|^{-z})$.

Suppose $\displaystyle\int AD^{-1} = 0$ $\to$ least action principle: EOM for D. $\forall A \in \Omega^1_D(A)$.

Then the quadratic form $Q(A,A) = \displaystyle\int (AD^{-1})^2$ is degenerate: $Q(A, [D,X]) = 0$. $\forall X \in \mathcal{A}$.

Proof: Consider

$$\int AD^{-1}[D,X]D^{-1} = -\int A[D^{-1},X]$$

$$= \int [X,A]D^{-1} = 0$$

by assumption, since $[X,A] \in \Omega^1_D(A)$. $\square$

Open problem: Can we write

$$\int AD^{-1}AD^{-1} = \langle A, T(A) \rangle_{\Omega^1_D(A)}$$

in terms of some (self-adj.?) operator T where $\langle A_1, A_2 \rangle = \displaystyle\int A_1^* A_2 |D|^{-n}$?

Taylor expansion in A [vS, JFA 2011, Skripka, JFA 2014, JOT 2018]

Prop. [vS, 2011, Skripka, 2014 ...]

Suppose (A, H, D) finitely-summable and f a Laplace transform (as in ⊗) Then there is a Taylor expansion:

$$S_\Lambda[A] = \sum_{n=0}^{N} \frac{1}{n} f'[\lambda_{i_1}, \dots, \lambda_{i_n}] A_{i_1 i_2} \cdots A_{i_n i_1} + R$$

where $A_{ij} = \langle \psi_i, A \psi_j \rangle$; $D\psi_i = \lambda_i \psi_i$ eigenbasis

and

$$\| R \| = O(\|A\|^{N+1})$$

Cor.

$$S_\Lambda[A] = \sum_{n=0}^{N} \frac{1}{2\pi i n} \operatorname{tr} \oint f'(\lambda)\, \underbrace{A(\lambda - D)^{-1} \cdots A(\lambda - D)^{-1}}_{n \text{ times}} + R$$

Again, there is a degeneracy!

if $\sum A_{ii} f'[\lambda_i] = tr(A f'(D)) = 0 \quad \forall A \in \Omega'_D(A)$ $(*')$

$\hookrightarrow$ EOM for D.

then $Q(A,A) = \sum A_{ij} A_{ji} f'[\lambda_i, \lambda_j]$ is degenerate

in the sense that $Q(A, [D,X]) = 0 \quad \forall X \in A$.

Proof: Note that $[D,X]_{ij} = (\lambda_i - \lambda_j) X_{ij}$.

Then

$$Q(A,[D,X]) = \sum A_{ij} X_{ji} \, f'[\lambda_i, \lambda_j] (\lambda_i - \lambda_j)$$

$$= \sum A_{ij} X_{ji} \left(f'(\lambda_i) - f'(\lambda_j) \right)$$

$$= tr[A, X] f'(D) = 0 \text{ by } (*')$$

Open problem: Can we write

$$Q(A_1, A_2) = \langle A_1, T(A_2) \rangle_{\Omega'_D(A)}$$

for some operator T and inner product on $\Omega'_D(A)$?

Examples of the spectral action:

- M Riem. spin^c mfd $\quad \dim M = 4$

$$S_\Lambda \sim \int \mathcal{L}_M \sqrt{g}\, d^4x + O(\Lambda^{-1})$$

$$\mathcal{L}_M(g_{\mu\nu}) = \frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} s_M + f(0)(\ldots)$$

- $\underline{\underline{M \times F}} \qquad F := (A_F, H_F, D_F)$

$$S_\Lambda[A] \sim \int \mathcal{L} \sqrt{g}\, d^4x + O(\Lambda^{-1})$$

Here $A = \gamma^\mu B_\mu + \gamma_M \phi$ and

$$\mathcal{L} = N\mathcal{L}_M + \mathcal{L}_B[B_\mu] + \mathcal{L}_\phi$$

$$\mathcal{L}_B(B_\mu) = \frac{f(0)}{24\pi^2} tr(F_{\mu\nu} F^{\mu\nu})$$

$$\mathcal{L}_\phi[B_\mu, \phi, g_{\mu\nu}] = -\frac{f_2}{2\pi^2}\Lambda^2 tr\,\phi^2 + \frac{f(0)}{8\pi^2} tr\,\phi^4$$

$$+ \frac{f(0)}{48\pi^2} s_M tr\,\phi^2 + \frac{f(0)}{8\pi^2} tr\, D_\mu \phi D^\mu \phi$$

• NC torus ($n=4$) [Gayral-Iochum-Vassilevich '06]
[Essouabri-Iochum-Levy-Sitarz '08]

$$S_\Lambda[A] \sim 8\pi^2 \Lambda^4 f_4 - \frac{4\pi^2}{3} f(0)\, \tau(F_{\mu\nu} F^{\mu\nu}) + O(\Lambda^{-2})$$

• Product geometry: $D = D_M \otimes 1 + \gamma \otimes D_{S^1}$
[CC08, uncanny]

$$\operatorname{tr} h\left(\frac{D^2}{\Lambda^2}\right) \sim 2\Lambda \operatorname{tr} k\left(\frac{D_M^2}{\Lambda^2}\right) \quad \text{as } \Lambda \to \infty$$

where $$k(x) = \int_x^\infty (u-x)^{-1/2} h(u)\, du$$

Proof:

Take $h(x) = e^{-bx}$.

$$\rightsquigarrow \operatorname{tr} e^{-b\frac{D^2}{\Lambda^2}} = 2 \operatorname{tr} e^{-b\frac{D_M^2}{\Lambda^2}} \operatorname{tr} e^{-b\frac{D_1^2}{\Lambda^2}}$$

But $$\operatorname{tr} e^{-b\frac{D_1^2}{\Lambda^2}} \sim \sqrt{\pi}\, \Lambda b^{-1/2} + O(\Lambda^{-k}) \quad \forall k$$

$$\Rightarrow \operatorname{tr} e^{-b\frac{D^2}{\Lambda^2}} \sim 2\Lambda \operatorname{tr}\left(\sqrt{\pi}\, b^{-1/2} e^{-b\frac{D_M^2}{\Lambda^2}}\right)$$

Now note that
$$\int_x^\infty (u-x)^{-1/2} e^{-bu}\, du = \sqrt{\pi}\, b^{-1/2} e^{-bx}$$

• Expansion in powers of A [ILV 11]: $D_\mu + \gamma A^\mu$

$$S_\Lambda[A] = \frac{\Lambda^{d-4}}{(4\pi)^{d/2}} \int d^d p \; \mathrm{tr}\, \tilde{F}^{\mu\nu}(p)\, w_\Lambda(p^2)\, \hat{F}_{\mu\nu}(p) + O(F^3)$$

$$w_\Lambda(p^2) \underset{p^2\to\infty}{\sim} -2^{\frac{d}{2}+1} c_f \Lambda^4 p^{-4} + o(p^{-4})$$

for f in a suitable class