Spectral action and entropy

NCG is built on spectra of operators.

spectral triple $(A, \mathcal{H}, D)$ (C95)

- Riemannian spin manifold M

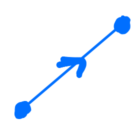
$$\rightsquigarrow \left(C^\infty(M), L^2(S_M), D_M \right)$$

- Particle Physics:

$$\left(C^\infty(M, M_N(\mathbb{C})), L^2(S_M) \otimes V, D_M \otimes 1 + \gamma_M \otimes \phi \right)$$

Both describe one-particle propagation:

$$D \psi = 0.$$



possibly coupled to background gauge field

$D \rightsquigarrow D + W$
driven by $\mathrm{Pert}(A)$.

Interaction and dynamics described by spectral action: $\mathrm{tr}\, f(D/\Lambda)$ (CC96)

Technique: heat expansions

$$\mathrm{tr}\, f(D/\Lambda) \sim \Lambda^{-d} a_d(D) + \Lambda^{-d+2} a_{d-2}(D) + \Lambda^{-d+4} a_{d-4}(D)$$

e.g. $\dim M = 4$:

$$a_4(D) \sim \mathrm{vol}(M)$$

$$a_2(D) \sim \int R$$

$$a_0(D) \sim \int F \wedge * F$$

Second quantization

Recent paper w/ Chamseddine & Connes on "spectral action and entropy" is a first step. (Dong-Khalkhali) CMP.

We will build a dictionary:

one-particle	"second-quantized"
A $*$-algebra	action of $\mathrm{Pert}(A)$ on $\{\sigma_t\}$
$\mathcal{H}$ Hilbert space	$\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ (Clifford algebra)
D unbdd self-adj. op.	$\{\sigma_t\}$ 1-p. group of autom. on $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H})$ corresp. to $e^{itD} \in \mathrm{Aut}(\mathcal{H}_{\mathbb{R}})$
Spectral action	entropy of KMS_β-state for σ_t

We start by replacing $\mathcal{H}$ by the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$.

$\mathcal{H}_{\mathbb{R}}$: real vector space underlying $\mathcal{H}$
so $\mathcal{H}_{\mathbb{R}}$ Eucl. vector space.

$$g : \mathcal{H}_{\mathbb{R}} \times \mathcal{H}_{\mathbb{R}} \to \mathbb{R}$$
$$(v, w) \mapsto \langle v, w \rangle.$$

$\mathrm{Cliff}(\mathcal{H}_{\mathbb{R}})$: $\gamma(v)\gamma(w) + \gamma(w)\gamma(v)$

$$= 2g(v,w)$$

Complexification $\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) = \mathrm{Cliff}(\mathcal{H}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

Next, consider operator D.
It gives rise to one-parameter family of orthogonal transfo. $\{e^{itD}\}$ on $\mathcal{H}_{\mathbb{R}}$.

This induces a one-param. family of automorphisms of $Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ via

$$\sigma_t(\gamma(v)) = \gamma(\sigma_t(v))$$

This is how D manifests itself at 2nd quantized level.

Inner perturbations map $D \mapsto D_A$ and accordingly $\sigma_t^D \to \sigma_t^{D_A}$.

A	$\sigma_t^D \mapsto \sigma_t^{D_A}$.
$\mathcal{H}$	$Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$
D	$\{\sigma_t^D\}$ arising from e^{itD}

C^*-dynamical system $\left(\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}), \sigma_t^D\right)$

$\exists!$ KMS_β-state : $\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) \to \mathbb{C}$

$$\varphi(a\,\sigma_t(b))\Big|_{t=i\beta} = \varphi(ba)$$

a, b norm-dense subalgebra (of σ-analytic vectors)

Example: $M_2(\mathbb{C}) = \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2)$

$\hookrightarrow$ σ^1, σ^2 Pauli.

$\varphi : M_2(\mathbb{C}) \to \mathbb{C}$ $\quad \sigma_t = e^{i\beta H}(\cdot)e^{-i\beta H}$.

$\varphi(a) = tr(\rho a)$ $\quad KMS_\beta \Rightarrow \rho = e^{-\beta H}$

More generally, for $M_N(\mathbb{C})$. $\qquad H \geq 0$.

Representations of $Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$. [GVF'01]

$V = \mathcal{H}_{\mathbb{R}}$. + complex structure I

$\rightsquigarrow$ complexification V_I $\quad I^2 = -1$

$$\gamma_I : Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}}) \to \mathcal{L}(\Lambda(V_I))$$

$$v \longmapsto a_I^*(v) + a_I(v)$$

$a_I^*(v) = v \wedge \dots$ $\quad a_I$ adj..

$\Omega_I \in \Lambda^0(V_I)$

Prop. γ_I is irreducible

Physical Hilbert space: instead of given complex structure on $\mathcal{H}$ we consider

$$I := i(E_+ - E_-)$$

$E_\pm$ spectral proj. of D.

This amounts to e^{itD} acting as $e^{it|D|}$

Prop. (i) $\gamma_I(\sigma_t(a)) = \Lambda(e^{it|D|})\,\gamma_I(a)\,\Lambda(e^{-it|D|})$

(ii) KMS_β-state $\varphi_\beta(a) = \frac{1}{Z} tr\left(\Lambda(e^{-\beta|D|})\gamma_I(a)\right)$

$a \in Cliff_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$

Density matrix $\rho = \Lambda(e^{-\beta|D|})$

We consider von Neumann entropy

$$S(\phi) = -tr\,\rho\log\rho$$

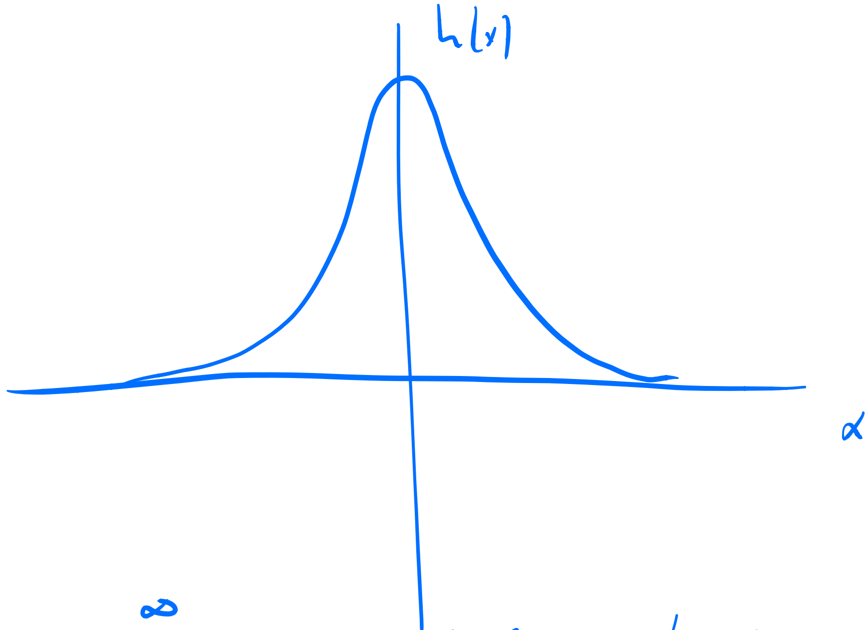
Thm Entropy of KMS_β-state φ_β is given by spectral action for the function $h(x) = \Sigma(e^{-x})$ where $\Sigma(x)$ is entropy of partition of unit interval in two intervals with size of ratio x.

$$\mathcal{E}(x) = \log x+1 - \frac{x\log x}{x+1} \qquad \rho = \frac{1}{1+x}(1\ x)$$

$$= -\rho\log\rho = -\frac{1}{1+x}\log\frac{1}{1+x} - \frac{x}{1+x}\log\frac{x}{1+x} = \frac{1+x}{1+x}\log 1+x - \frac{x}{1+x}\log x$$

Consider $h(x)$:

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^{x}} + \log(1+e^{-x})$$



$$\begin{cases} h(x) = \int_0^\infty g(t)\, e^{-tx^2} & \text{Laplace transf.} \\ g(t) = \dfrac{-1}{8\sqrt{\pi}\, t^{5/2}} \displaystyle\sum_{n\in\mathbb{Z}} \underbrace{(-1)n^2 q^{n^2}}_{q\frac{\partial}{\partial q}\theta_4(0;q)}\ ; & q = e^{-1/4t} \end{cases}$$

Thm

Heat expansion: $\operatorname{tr} e^{-(\beta D)^2} \sim \sum_k \beta^{2k} b_k \Rightarrow$

$$\operatorname{tr} h(\beta D) \sim \sum \beta^{2k} \gamma(k)\, b_k$$

$$\gamma(k) = \frac{1-2^{-2k}}{k} \pi^{-k} \zeta(2k)$$

Riemann ζ-function:

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$\vdots$

$$\gamma(-1) = \frac{9\zeta(3)}{2}$$

$$\gamma\left(-\tfrac{1}{2}\right) = \frac{\pi^{3/2}}{3}$$

$$\gamma(0) = \log 2$$

$$\gamma\left(\tfrac{1}{2}\right) = \frac{1}{2\sqrt{\pi}}$$

$$\gamma(1) = \frac{1}{8}$$

$$\gamma\left(\tfrac{3}{2}\right) = \frac{7\zeta(3)}{8\pi^{5/2}}$$

$\vdots$