Factorization of Dirac operators in unbounded KK-theory

Walter van Suijlekom

(with Jens Kaad, Bram Mesland and Adam Rennie, Luuk Verhoeven)

11 June 2019

Radboud University Nijmegen

Motivation for unbounded KK-theory

- Kasparov's bivariant K-theory: the backbone of Connes' noncommutative differential geometry
- Many constructions in differential geometry can be captured in *unbounded* KK-theory [BJ 1983]
- With the added value that geometry becomes visible:
  - gauge degrees of freedom (Particle Physics [CC,CCvS])
  - curvature
- A key role played by the internal Kasparov product

- We investigate a notion of curvature that emerges from the unbounded refinement of the underlying (and guiding) construction in KK-theory.
- We work in three classes of examples:
  - Morita equivalences (applications to particle physics) [Chamseddine–Connes–Marcolli, vS]
  - Riemannian submersions Kaad–vS, Mesland–Rennie–vS, ...]
  - Riemannian immersions: $S^1 \hookrightarrow \mathbb{R}^2$ [vS–Verhoeven]

Spectral triples and Morita equivalence

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a (fgp) Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$:

- Hilbert space: $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$
- Choose (hermitian) connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$
- Define an operator $D' := 1 \otimes_\nabla D$ by

$$(1 \otimes_\nabla D)(\xi \otimes h) = (1 \otimes \pi_D)(\nabla(\xi))h + \xi \otimes (Dh).$$

where $\pi_D(a\delta(b)) = a[D, b]$.

Then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, 1 \otimes_\nabla D)$ is a spectral triple [C96] representing the internal Kasparov product of $[(\mathcal{E}, 0)]$ and $[(H, F_D)]$.

- Even though this is completely trivial from a KK-theory point of view (due to Morita equivalence), in the applications there is much interest in the specific form of the connection ∇
- In fact, for this class of examples we can show that

$$R_\nabla := 1 \otimes_\nabla D^2 - (1 \otimes_\nabla D)^2 = \pi_D(\nabla^2)$$

  as an operator from $\mathcal{E} \otimes_\mathcal{A} \mathrm{Dom}(D^2) \to \mathcal{E} \otimes_\mathcal{A} H$.
- In this context curvature can be obtained as the difference of two symmetric operators.

Morita self-equivalences

- Now take $\mathcal{A} = \mathcal{B}$ and $\mathcal{E} = \mathcal{A}$ as well, then $\mathcal{H}' \cong \mathcal{H}$
- But we may still choose a connection $\nabla : \mathcal{A} \to \Omega^1(\mathcal{A})$ so that

$$1 \otimes_\nabla D \equiv D + \pi_D(\omega)$$

- The connection one-form is $\pi_D(\omega)^* = \pi_D(\omega) = \pi_D(\nabla(1))$, or,

$$\pi_D(\omega) = \sum_j a_j[D, b_j]; \qquad (a_j, b_j \in \mathcal{A})$$

- Semigroup $\mathrm{Pert}(\mathcal{A})$ of inner perturbations [CCvS 2013]:

$$D \mapsto 1 \otimes_\nabla D \mapsto 1 \otimes_{\nabla'} (1 \otimes_\nabla D) \mapsto \cdots$$

- Curvature for Morita self-equivalences becomes

$$R_\nabla = \pi_D(\nabla^2) = -\pi_D(\delta\omega) - \pi_D(\omega)^2$$

- In physical applications $\mathcal{A} = C^\infty(M, A_F)$ for some finite-dimensional matrix algebra A_F and ω is parametrized by gauge fields, scalar (Higgs) fields [CCvS 2017, vS 2014].

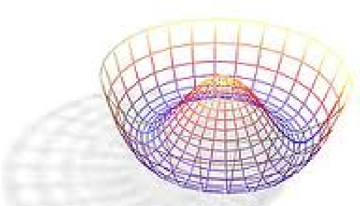
Two-point space
Connes–Lott, 1990

- Consider the spectral triple

$$\left(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}\oplus\mathbb{C}, D = \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}\right)$$

- Inner perturbations: $\pi_D(\omega) = \begin{pmatrix} 0 & c\phi \\ \overline{c}\overline{\phi} & 0 \end{pmatrix}$
- Curvature:

$$\begin{aligned} R_\nabla &= (1 \otimes_\nabla D^2) - (1 \otimes_\nabla D)^2 \\ &\equiv D^2 - (D + \pi_D(\omega))^2 = |c|^2(|\phi + 1|^2 - 1)1_2 \end{aligned}$$



$$R_\nabla^2 = (|c|^2 - |\widetilde{\phi}|^2)^2$$

Standard Model as an almost-commutative spacetime

Describe $M \times F_{SM}$ by [CCM 2007]

- Matrix-valued function $A = C^\infty(M, \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).
- Dirac operator $D_{M\times F} = D_M + \gamma_M \otimes D_F$ where

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

is a 96×96-dimensional hermitian matrix where 96 is:

$$\underset{\text{families}}{3} \times \underset{\text{anti-particles}}{2} \times \big(\underset{(\nu_L, e_L)}{\underline{2} \otimes \underline{1}} + \underset{\nu_R}{\underline{1} \otimes \underline{1}} + \underset{e_R}{\underline{1} \otimes \underline{1}} + \underset{(u_L, d_L)}{\underline{2} \otimes \underline{3}} + \underset{u_R}{\underline{1} \otimes \underline{3}} + \underset{d_R}{\underline{1} \otimes \underline{3}} \big)$$

The Dirac operator on F_{SM}

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.

- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Inner perturbations

We find

- Inner perturbations of D_M give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

  corresponding to hypercharge, weak and strong interaction.

- Inner perturbations of D_F give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

  corresponding to SM-Higgs field. Similarly for Y_u, Y_d.

- and Beyond the Standard Model...

Riemannian submersions
joint work with Jens Kaad, 2016

Consider the following setup:

- A Riemannian submersion

$$\pi : M \to B$$

  of compact spin^c manifolds M and B
- It is well-known from wrong-way functoriality [CS, 1984] that

$$[M] = \pi! \widehat{\otimes}_{C(B)} [B] \qquad (*)$$

- Can we write the Dirac operator on M as a tensor sum:

$$D_M = D_V \otimes \gamma + 1 \otimes_\nabla D_B + \widetilde{c}(\Omega)?$$

  for some unbounded KK-cycle $(C^\infty(M), X, D_V)$, representing the internal KK-product $(*)$, while allowing for curvature Ω of $\pi : M \to B$?

Riemannian submersions

- Let M and B be closed Riemannian manifolds and let

$$\pi : M \to B$$

be a smooth and surjective map

- This is a Riemannian submersion when $d\pi$ is surjective and

$$d\pi : (\ker d\pi)^{\perp} \to \mathscr{X}(B) \otimes_{C^\infty(B)} C^\infty(M)$$

is an isometric isomorphism

- This gives rise to vertical and horizontal vector fields

$$\mathscr{X}(M) \cong \mathscr{X}_V(M) \oplus \mathscr{X}_H(M).$$

- On $\mathscr{X}(M)$ one can introduce the direct sum connection

$$\nabla^{\oplus} = P_V \nabla^M P_V \oplus \pi^* \nabla^B,$$

- the second fundamental form:

$$S(X, Y, Z) := \langle \nabla^V_{(1-P)Z}(PX) - [(1-P)Z, PX], PY \rangle_M,$$

- the curvature of $\pi : M \to B$:

$$\Omega(X, Y, Z) := -\langle [(1-P)X, (1-P)Y], PZ \rangle_M$$

which combined yield a tensor $\omega \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^2(M)$

Proposition (Bismut, 1986)

The Levi-Civita connection ∇^M is related to the direct sum connection $\nabla^{\oplus}$ by the following formula

$$\langle \nabla^M_X Y, Z \rangle_M = \langle \nabla^{\oplus}_X Y, Z \rangle_M + \omega(X)(Y, Z).$$

Spin geometry and Clifford modules

Suppose that M and B are (even-dim) spin^c manifolds, so we have

$$\mathrm{Cl}(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_M), \qquad \mathrm{Cl}(B) \cong \mathrm{End}_{C^\infty(B)}(\mathscr{E}_B)$$

and hermitian Clifford connections $\nabla^{\mathscr{E}_M}$ and $\nabla^{\mathscr{E}_B}$.

- We define the horizontal spinor module:

$$\mathscr{E}_H := \mathscr{E}_B \otimes_{C^\infty(B)} C^\infty(M); \qquad \nabla^{\mathscr{E}_H} := \pi^* \nabla^{\mathscr{E}_B}$$

  and $\mathrm{Cl}_H(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_H)$

- We define the vertical spinor module:

$$\mathscr{E}_V := \mathscr{E}_H^* \otimes_{\mathrm{Cl}_H(M)} \mathscr{E}_M,$$

$$\nabla_X^{\mathscr{E}_V}(\phi \otimes s) = \nabla_X^{\mathscr{E}_H^*}(\phi) \otimes s + \phi \otimes \nabla_X^{\mathscr{E}_M}(s) + \phi \otimes \frac{1}{4} c(\omega(X))(s),$$

  and $\mathrm{Cl}_V(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_V)$

- Finally,

$$\mathscr{E}_H \otimes_{C^\infty(M)} \mathscr{E}_V \cong \mathscr{E}_M,$$

  compatibly with $\mathrm{Cl}_H(M) \widehat{\otimes}_{C^\infty(M)} \mathrm{Cl}_V(M) \cong \mathrm{Cl}(M)$.

The vertical operator

- We define a C^*-correspondence X from $C(M)$ to $C(B)$ by completing $\mathscr{E}_V$ with respect to

$$\langle \phi_1, \phi_2 \rangle_X(b) := \int_{F_b} \langle \phi_1, \phi_2 \rangle_{\mathscr{E}_V} \, d\mu_{F_b}$$

- The following local expression defines an odd symmetric unbounded operator

$$(D_V)_0(\xi) = i \sum_{j=1}^{\dim(F)} c_V(e_j) \nabla_{e_j}^{\mathscr{E}_V}(\xi)$$

where $\{e_j\}$ is a local orthonormal frame for $\mathscr{X}_V(M)$

Proposition

The triple $(C^\infty(M), X, D_V)$ is an even unbounded Kasparov module from $C(M)$ to $C(B)$ with grading operator $\gamma_X : X \to X$.

The horizontal operator and the connection

- The Dirac operator $D_B : \mathrm{Dom}(D_B) \to L^2(\mathscr{E}_B)$ is locally

$$D_B = i \sum_{\alpha=1}^{\dim(B)} c(f_\alpha) \nabla^{\mathscr{E}_B}_{f_\alpha} : \mathscr{E}_B \to L^2(\mathscr{E}_B)$$

  Clearly $(C^\infty(B), L^2(\mathscr{E}_B), D_B; \gamma_B)$ is an (even) spectral triple

- The following defines a hermitian connection on X

$$\nabla^X_Z(\xi) = \nabla^{\mathscr{E}_V}_{Z_H}(\xi) + \frac{1}{2} k(Z_H) \cdot \xi$$

  with $k = (\mathrm{Tr} \otimes 1)(D_V) \in \Omega^1(M)$ the mean curvature

Lemma

The following local expression defines an odd symmetric unbounded operator in $X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B)$:

$$(1 \otimes_\nabla D_B)(\xi \otimes r) := \xi \otimes D_B r + i \sum_\alpha \nabla^X_{f_\alpha}(\xi) \otimes c(f_\alpha) r.$$

The tensor sum

- The tensor sum we are after is given by

$$(D_V \times_\nabla D_B)_0 := D_V \otimes \gamma_B + 1 \otimes_\nabla D_B : \\ \mathrm{Dom}(D_V \times_\nabla D_B)_0 \to X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B)$$

- The closure of this symmetric operator is denoted $D_V \times_\nabla D_B$.

Theorem (KvS, 2016)

Under the unitary isomorphism $V : X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B) \to L^2(\mathscr{E}_M)$ we have the identity

$$V(D_V \times_\nabla D_B)V^* = D_M - \frac{i}{8}\widetilde{c}(\Omega).$$

Factorization of Dirac operators

Theorem (KvS,2016)

The even spectral triple $(C^\infty(M), L^2(\mathscr{E}_M), D_M)$ *is the unbounded Kasparov product of the even unbounded KK-cycle* $(C^\infty(M), X, D_V)$ *with the even spectral triple* $(C^\infty(B), L^2(\mathscr{E}_B), D_B)$ *up to the curvature term* $-\frac{i}{8}\widetilde{c}(\Omega)$.

Sketch of proof

1. The classes $[D_M]$, $[S]$, $[D_B]$ in bounded KK-theory given by bounded transforms coincide with the classes $[M]$, $\pi!$, $[B]$.
2. Wrong-way functoriality [CS, 1984]: the map $f_M : M \to \{\text{pt}\}$ can be factorized as $f_M = f_B \circ \pi$ with $f_B : B \to \{\text{pt}\}$ and gives rise to

$$f_M! = \pi! \widehat{\otimes}_{C(B)} f_B!$$

3. Identify $f_M! = [M]$ and $f_B! = [B]$.

- Let us reconsider curvature in this context:

$$R_\nabla = 1 \otimes_\nabla D_B^2 - (1 \otimes_\nabla D_B)^2 = \pi_{D_B}(\nabla^2)$$

as an (unbounded) operator on $\mathscr{E}_V \otimes \mathrm{Dom}\, D_B^2$

- This is a symmetric operator that is D_V-bounded, at leading order it is:

$$[(f_i)_H, (f_j)_H] - [f_i, f_j]_H = \sum_k \Omega((f_i)_H, (f_j)_H), e_k) \cdot e_k$$

in terms of the frame of vertical vector fields $\{e_k\}$.

- In fact, one can show that

$$R_\nabla = c_H \circ \left((\nabla^{\mathscr{E}_V})^2 + dk \right)$$

in terms of horizontal Clifford multiplication c_H, the curvature of $\nabla^{\mathscr{E}_V}$ and the mean curvature k

- Consider the embedding $\imath : S^n \hookrightarrow \mathbb{R}^{n+1}$.
- By [CS 1984] we know that

$$[S^n] = \imath_! \otimes [R^{n+1}]$$

as classes in KK-theory.

- Let us consider the unbounded version, which starts by writing

$$D_{\mathbb{R}^{n+1}} = \gamma^1 \frac{1}{r} D_{S^n} + \gamma^2 \left(i \frac{\partial}{\partial r} + \frac{n}{2r} \right)$$

- This is (a local expression for) an essentially self-adjoint operator with volume form $r^n dr \wedge d\sigma_{S^n}$

The immersion KK-cycle

- We introduce the immersion module between $C(S^n)$ and $C_0(\mathbb{R}^{n+1})$:
$$X = C_0(S^n \times (-\epsilon, \epsilon))$$
equipped with $C_0(\mathbb{R}^{n+1})$-valued inner product given by
$$\langle \psi_1, \psi_2 \rangle_X (r, \theta) = \frac{1}{r^n} \overline{\psi_1(r, \theta)} \psi_2(r, \theta)$$
- Symmetric operator $S_0 : \mathrm{Dom}(S_0) \to X$ defined as multiplication operator with some suitable $f \equiv f(r)$, such as
$$f(r) = \frac{\pi}{2\epsilon} \tan \frac{\pi(r-1)}{2\epsilon}$$
Closure is denoted $S = \overline{S_0}$.

Proposition

The triple $(C^\infty(S^n), X, S)$ is an unbounded KK-cycle from $C(S^n)$ to $C_0(\mathbb{R}^{n+1})$.

The internal Kasparov product

- We introduce a connection ∇ on X by

$$\nabla(\psi) = [D, \psi] - \frac{n}{2r} ic(dr)$$

- The tensor sum is given by

$$\begin{aligned} D_\times &= S \otimes 1 + 1 \otimes_\nabla D_{\mathbb{R}^{n+1}} \\ &= \gamma^1 \frac{1}{r} D_{S^n} + \gamma^2 \left(i \frac{\partial}{\partial r} \right) + \gamma^3 f(r) \end{aligned}$$

- As an operator on $L^2((-\epsilon, \epsilon)) \otimes \mathbb{C}^2$ we have

$$T := \gamma^2 \left(i \frac{\partial}{\partial r} \right) + \gamma^3 f(r) = \begin{pmatrix} 0 & i \frac{\partial}{\partial r} + if(r) \\ i \frac{\partial}{\partial r} - if(r) & 0 \end{pmatrix}$$

for which we compute that index $T = 1$

We now arrive at the final result

Theorem

The triple $C^\infty(S^n), X \otimes_{C_0(\mathbb{R}^{n+1})} L^2(\mathscr{E}_{\mathbb{R}^{n+1}}, D_\times$ *is an unbounded representative both of the internal Kasparov product of* $\imath_!$ *and* $[\mathbb{R}^{n+1}]$ *as well as of* $[S^n]$ *and* $[T] = 1 \in KK(\mathbb{C}, \mathbb{C})$

The proof is a check of Kucerovsky's conditions

Interestingly, the curvature can also be computed. Since

$$\nabla = [D, \cdot] - \frac{n}{2r} ic(dr)$$

we find

$$R_\nabla = \pi_{D^2}(\omega) - [D, \pi_D(\omega)]_+ - \pi_D(\omega)^2 = -\frac{n^2}{2r^2}$$

which is (proportional to) the sectional curvature of the n-sphere...

Outlook

- We have started to investigate a new notion of curvature that arises in unbounded KK-theory, based on the difference of two natural symmetric operators that appear in the (unbounded version of the) internal Kasparov product
- This notion of curvature is in concordance with the applications to particle physics, Riemannian submersions, immersions of spheres into Euclidean planes
- and much more to be explored!