Geometry from the spectral point of view

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Spectral geometry

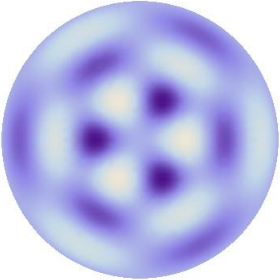
"Can one hear the shape of a drum?" (Kac, 1966)

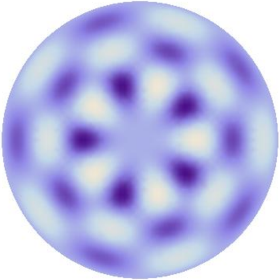
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

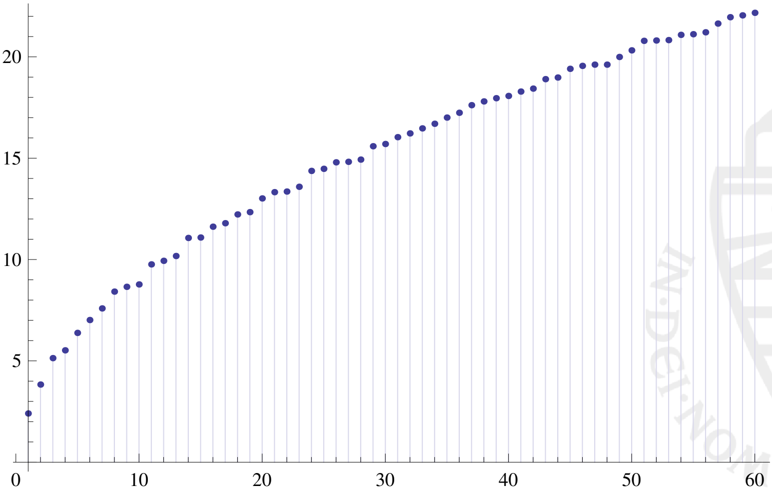
determine the geometry of M?

The disc

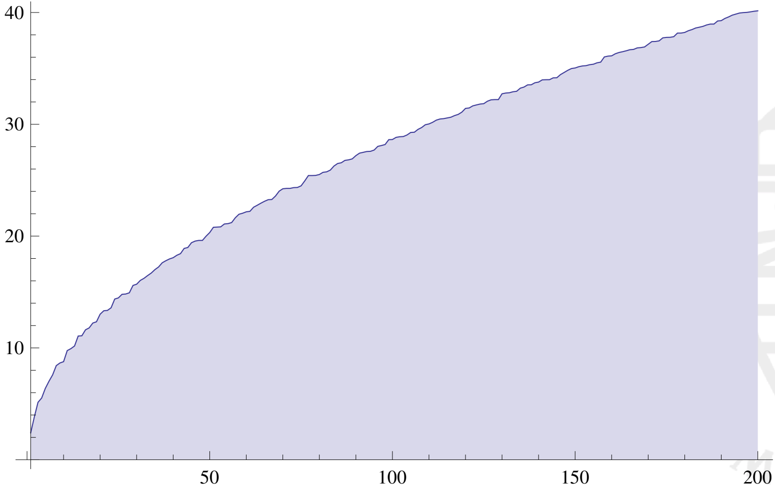




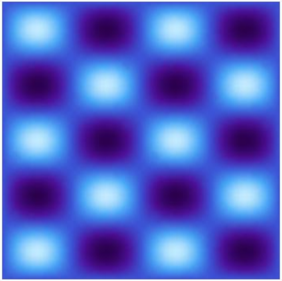
Wave numbers on the disc

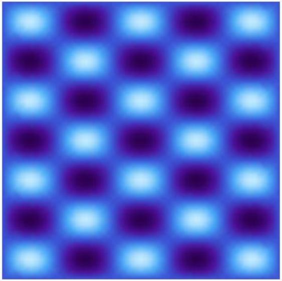


Wave numbers on the disc: high frequencies

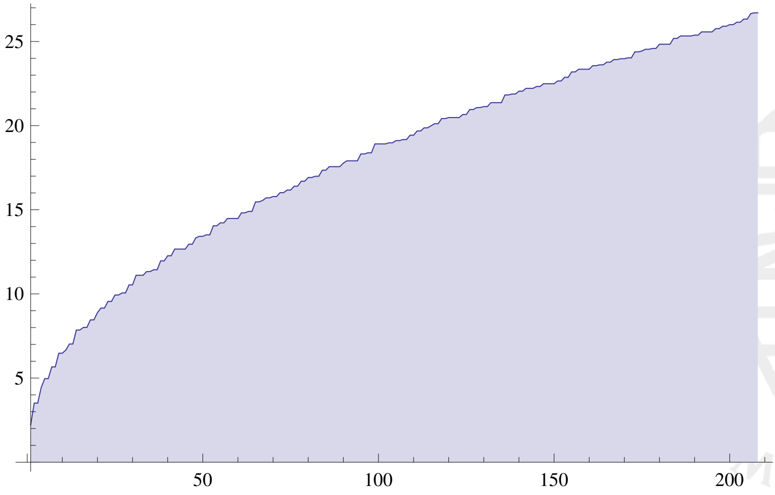


The square



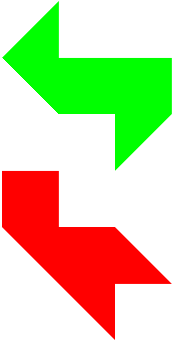


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



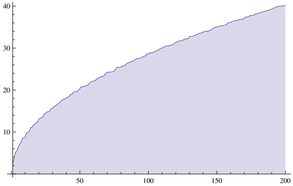
(Gordon, Webb, Wolpert, 1992)

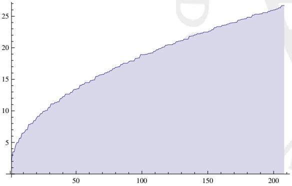
so the answer to Kac's question is no.

Nevertheless, certain information can be extracted from spectrum, such as dimension n of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n} \Lambda^n$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2} \right)^2 = a^2 \frac{\partial^2}{\partial t_1^2} + 2ab \frac{\partial^2}{\partial t_1 \partial t_2} + b^2 \frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then $a^2 = b^2 = -1$ and $ab + ba = 0$

- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:

$$i^2 = j^2 = k^2 = ijk = -1.$$

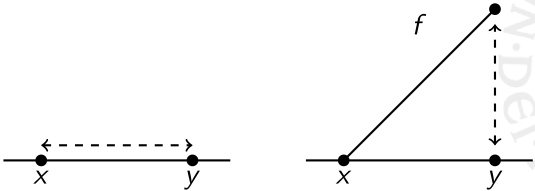
- The Dirac operator on $\mathbb{T}^4$ is

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$

- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

- The geometry of M is not fully determined by spectrum of D_M.
- This is considerably improved by considering besides D_M also the algebra $C^\infty(M)$ of smooth (coordinate) functions on M
- In fact, the Riemannian distance function on M is equal to

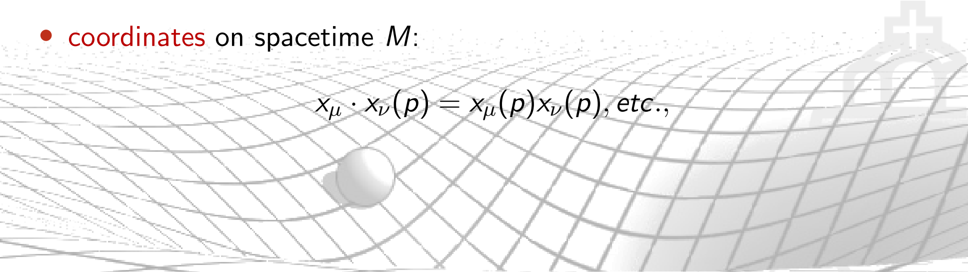
$$d(x,y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$ (*e.g.* $[D_{\mathbb{S}^1}, f] = -i \frac{df}{dt}$)

The combination of coordinates and Dirac operator is of course also central in the description of fermion propagation:

- coordinates on spacetime M:



$$x_\mu \cdot x_\nu(p) = x_\mu(p)x_\nu(p), etc.,$$

- propagation, described by Dirac operator $D_M = i\gamma^\mu\partial_\mu$

$(\mathcal{A}, \mathcal{H}, D)$

Noncommutative fine structure of spacetime

Replace spacetime by
spacetime $\times$ **noncommutative space**: $M \times F$

- F is considered as finite internal space (Kaluza–Klein like)
- F is described by noncommutative matrices, that play the role of coordinates, just as spacetime is described by $x_\mu(p)$.
- 'Propagation' of particles in F is described by a 'Dirac operator' D_F which is actually simply a hermitian matrix.

- Finite space F, discrete topology

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Smooth functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \sup_{f \in C^\infty(F)} \{ |f(p) - f(q)| : \| [D_F, f] \| \leq 1 \}$$

Example: two-point space

$$F = \;_1\bullet \qquad _2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(1,2) = \frac{1}{|c|}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \ldots a_N$ are square matrices of size $n_1, n_2, \ldots, n_N$.

- Hence we will consider the matrix algebra

$$\mathcal{A}_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A finite Dirac operator is still given by a hermitian matrix.

The two-point space can be given a noncommutative structure by considering the algebra $\mathcal{A}_F$ of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extended to real spectral triple:
  - $J : \mathcal{H} \to \mathcal{H}$ real structure (charge conjugation)

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ$$

- Right action of $\mathcal{A}$ on $\mathcal{H}$: $a^{\mathrm{op}} = Ja^* J^{-1}$ so that $(ab)^{\mathrm{op}} = b^{\mathrm{op}} a^{\mathrm{op}}$ and

$$[a^{\mathrm{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy first-order condition if

$$[[D, a], b^{\mathrm{op}}] = 0$$

$$\text{Trace}\, f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = u(u^*)^{\mathrm{op}}$$

- Gauge group: $\mathcal{G}(\mathcal{A}) := \{u(u^*)^{\mathrm{op}} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] \pm Ju[D, u^*]J^{-1}$$

Semigroup of inner perturbations
Chamseddine–Connes-vS (2013)

Extend this to more general perturbations:

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

and this also extends to real spectral triples via the map

$$\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes J\mathcal{A}J^{-1})$$

Perturbation semigroup for matrix algebras

Proposition

Let $\mathcal{A}_F$ *be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of* $\mathcal{A}_F$ *is*

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ *is given by the matrix product in* $\mathcal{A}_F \otimes \mathcal{A}_F$*:*

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

  leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.
- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $\mathcal{A}_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C}))$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

Example: noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Physicists call ϕ_1 and ϕ_2 the Higgs field.
- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Recall, for any involutive algebra $\mathcal{A}$

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- We can consider functions in $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

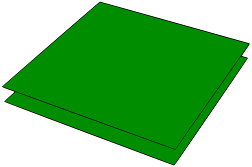
- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\bigg|_{y=x} =: \partial_\mu + A_\mu$$

- Combine (4d) Riemannian spin manifold M with finite noncommutative space F:

$$M \times F$$

- F is internal space at each point of M



- Described by matrix-valued functions on M: algebra $C^\infty(M, \mathcal{A}_F)$

Dirac operator on $M \times F$

- Recall the form of D_M:

$$D_M = \begin{pmatrix} 0 & D_M^+ \\ D_M^- & 0 \end{pmatrix}.$$

- Dirac operator on $M \times F$ is the combination

$$D_{M\times F} = D_M + \gamma_5 D_F = \begin{pmatrix} D_F & D_M^+ \\ D_M^- & -D_F \end{pmatrix}.$$

- The crucial property of this specific form is that it squares to the sum of the two Laplacians on M and F:

$$D_{M\times F}^2 = D_M^2 + D_F^2$$

- Using this, we can expand the heat trace:

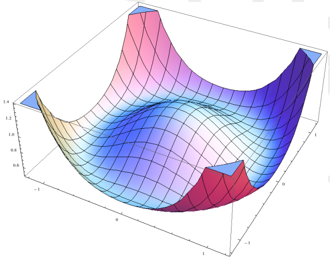
$$\text{Trace}\, e^{-D_{M\times F}^2/\Lambda^2} = \frac{\text{Vol}(M)\Lambda^4}{(4\pi)^2} \text{Trace}\left(1 - \frac{D_F^2}{\Lambda^2} + \frac{D_F^4}{2\Lambda^4}\right) + \mathcal{O}(\Lambda^{-1}).$$

The Higgs mechanism

We apply this to the noncommutative two-point space described before

- Algebra $\mathcal{A}_F = \mathbb{C} \oplus M_2(\mathbb{C})$
- Perturbation of Dirac operator D_M parametrized by gauge bosons for $U(1) \times U(2)$.
- Perturbation of finite Dirac operator D_F parametrized by ϕ_1, ϕ_2.
- Spectral action for the perturbed Dirac operator induces a potential:

$$V(\phi) = -2\Lambda^2(|\phi_1|^2 + |\phi_2|^2) + (|\phi_1|^2 + |\phi_2|^2)^2$$



The spectral Standard Model

Describe $M \times F_{SM}$ by [CCM 2007]

- Finite-dimensional algebra: $C^\infty(M, \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$ (with unimodular unitaries $U(1)_Y \times SU(2)_L \times SU(3)$).
- Dirac operator $D_{M \times F} = D_M + \gamma_5 D_F$ where

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

is a 96×96-dimensional hermitian matrix where 96 is:

$$3 \times 2 \times (\underline{2} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{1} \otimes \underline{1} + \underline{2} \otimes \underline{3} + \underline{1} \otimes \underline{3} + \underline{1} \otimes \underline{3})$$

- 3: families
- 2: anti-particles
- $\underline{2} \otimes \underline{1}$: (ν_L, e_L)
- $\underline{1} \otimes \underline{1}$: ν_R
- $\underline{1} \otimes \underline{1}$: e_R
- $\underline{2} \otimes \underline{3}$: (u_L, d_L)
- $\underline{1} \otimes \underline{3}$: u_R
- $\underline{1} \otimes \underline{3}$: d_R

$$D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The operator S is given by

$$S_l := \begin{pmatrix} 0 & 0 & Y_\nu & 0 \\ 0 & 0 & 0 & Y_e \\ Y_\nu^* & 0 & 0 & 0 \\ 0 & Y_e^* & 0 & 0 \end{pmatrix}, \quad S_q \otimes 1_3 = \begin{pmatrix} 0 & 0 & Y_u & 0 \\ 0 & 0 & 0 & Y_d \\ Y_u^* & 0 & 0 & 0 \\ 0 & Y_d^* & 0 & 0 \end{pmatrix} \otimes 1_3,$$

  where Y_ν, Y_e, Y_u and Y_d are 3×3 mass matrices acting on the three generations.
- The symmetric operator T only acts on the right-handed (anti)neutrinos, $T\nu_R = Y_R\overline{\nu_R}$ for a 3×3 symmetric Majorana mass matrix Y_R, and $Tf = 0$ for all other fermions $f \neq \nu_R$.

Inner perturbations

- Inner perturbations of D_M give a matrix

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ & 0 \\ 0 & W_\mu^- & -W_\mu^3 & 0 \\ 0 & 0 & 0 & (G_\mu^a) \end{pmatrix}$$

corresponding to hypercharge, weak and strong interaction.

- Inner perturbations of D_F give

$$\begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \rightsquigarrow \begin{pmatrix} Y_\nu \phi_1 & -Y_e \overline{\phi}_2 \\ Y_\nu \phi_2 & Y_e \overline{\phi}_1 \end{pmatrix}$$

corresponding to SM-Higgs field. Similarly for Y_u, Y_d.

If we consider the spectral action:

$$\text{Trace}\, f(D_M/\Lambda) \sim c_0 \int F_{\mu\nu} F^{\mu\nu} - c_2' |\phi|^2 + c_0' |\phi|^4 + \cdots$$

we observe [CCM 2007]:

- The couplings of hypercharge, weak and strong interaction are expressed in terms of the single constant c_0 which implies

$$\boxed{g_3^2 = g_2^2 = \frac{5}{3} g_1^2}$$

  In other words, there should be grand unification.

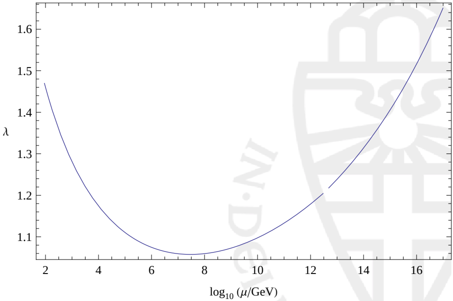
- Moreover, the quartic Higgs coupling λ is related via

$$\lambda \approx 24 \frac{3 + \rho^4}{(3 + \rho^2)^2} g_2^2; \qquad \rho = \frac{m_\nu}{m_{\text{top}}}$$

This can be used to derive predictions as follows:

- Interpret the spectral action as an effective field theory at $\Lambda_{\mathrm{GUT}} \approx 10^{13} - 10^{16}$ GeV.
- Run the quartic coupling λ to SM-energies to predict

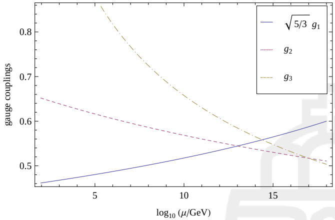
$$m_h^2 = \frac{4\lambda M_W^2}{3g_2^2}$$

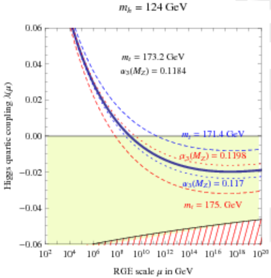


This gives [CCM 2007]

$$\boxed{167 \text{ GeV} \leq m_h \leq 176 \text{ GeV}}$$

1. This prediction is falsified by the now measured value.
2. In the Standard Model there is not the presumed grand unification.
3. There is a problem with the low value of m_h, making the Higgs vacuum un/metastable [Elias-Miro et al. 2011].





- The algebra of the Standard Model arise naturally as a restriction of the following algebra

$$C^\infty(M, \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}))$$

corresponding to a Pati–Salam unification:

$$U(1)_Y \times SU(2)_L \times SU(3) \to SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 fermionic degrees of freedom are structured as

$$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array} \right) \qquad (i = 1, 2, 3)$$

- Again the finite Dirac operator is a 96×96-dimensional matrix (details in [CCS 2013]).

Inner perturbations

- Inner perturbations of D_M now give three gauge bosons:

$$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.

- For the inner perturbations of D_F we distinguish two cases, depending on the initial form of D_F:
  - I The Standard Model $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$
  - II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

Case I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^{b}$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_J^I	1	1	15

Case II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

Action functional

As for the Standard Model, we can compute the spectral action which describes the usual Pati–Salam model with

- unification of the gauge couplings

$$g_R = g_L = g.$$

- A rather involved, fixed scalar potential, still subject to further study

Phenomenology of the spectral Pati–Salam model

However, independently from the spectral action, we can analyze the running at one loop of the gauge couplings [CCS 2015]:

1. We run the Standard Model gauge couplings up to a presumed PS $\to$ SM symmetry breaking scale m_R
2. We take their values as boundary conditions to the Pati–Salam gauge couplings g_R, g_L, g at this scale via

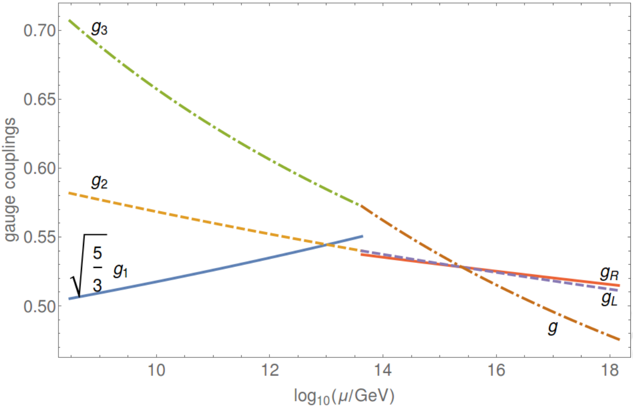
$$\frac{1}{g_1^2} = \frac{2}{3}\frac{1}{g^2} + \frac{1}{g_R^2}, \qquad \frac{1}{g_2^2} = \frac{1}{g_L^2}, \qquad \frac{1}{g_3^2} = \frac{1}{g^2},$$

3. Vary m_R in a search for a unification scale Λ where

$$g_R = g_L = g$$

which is where the spectral action is valid as an effective theory.

For the Standard Model Dirac operator, we have found that with $m_R \approx 4.25 \times 10^{13}$ GeV there is unification at $\Lambda \approx 2.5 \times 10^{15}$ GeV:

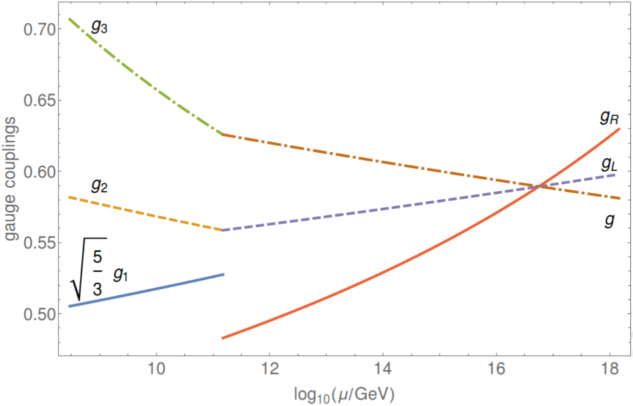


In this case, we can also say something about the scalar particles that remain after SSB:

	$U(1)_Y$	$SU(2)_L$	$SU(3)$
$\begin{pmatrix} \phi_1^0 \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} \phi_{\dot{1}}^{1} \\ \phi_{\dot{1}}^{2} \end{pmatrix}$	1	2	1
$\begin{pmatrix} \phi_2^- \\ \phi_2^0 \end{pmatrix} = \begin{pmatrix} \phi_{\dot{2}}^{1} \\ \phi_{\dot{2}}^{2} \end{pmatrix}$	-1	2	1
σ	0	1	1
η	$-\frac{2}{3}$	1	3

- It turns out that these scalar fields have a little influence on the running of the SM-gauge couplings (at one loop).
- However, this sector contains the real scalar singlet σ that allowed for a realistic Higgs mass and that stabilizes the Higgs vacuum [CC 2012].

For the more general case, we have found that with $m_R \approx 1.5 \times 10^{11}$ GeV there is unification at $\Lambda \approx 6.3 \times 10^{16}$ GeV:



We have arrived at a spectral Pati–Salam model that

- goes beyond the Standard Model
- has a fixed scalar sector once the finite Dirac operator has been fixed (only a few scenarios)
- exhibits grand unification for all of these scenarios (confirmed by [Aydemir–Minic–Sun–Takeuchi 2015])
- the scalar sector has the potential to stabilize the Higgs vacuum and allow for a realistic Higgs mass.

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Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

Grand Unification in the Spectral Pati-Salam Model. *JHEP* 11 (2015) 011. [arXiv:1507.08161]

WvS.

Noncommutative Geometry and Particle Physics. Mathematical Physics Studies, Springer, 2015.

and also: `http://www.noncommutativegeometry.nl`