Radboud University Nijmegen

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Truncated Geometry

Spectral approximation of the torus

Thesis MSc Mathematics

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1 Introduction

Noncommutative geometry is a generalization of geometry that is characterized by its algebraic, or spectral description of geometry. A few weeks ago I attended a talk given by Ali Chamseddine, one of the pioneers in using noncommutative geometry to explain the standard model of particle physics. He started his talk by saying that 'algebraic geometry' would be a more appropriate name for the field, but that this name was already taken. He concluded this is probably the reason we use the name 'noncommutative geometry' instead.

The spectral description of geometry is given by *spectral triples* $(\mathcal{A}, \mathcal{H}, D)$, consisting of a Hilbert space $\mathcal{H}$, a dense subalgebra $\mathcal{A}$ of a C^*-algebra A, which is represented faithfully on $\mathcal{H}$, and an (essentially) self-adjoint operator D, acting on $\mathcal{H}$ (Definition 4.1).

To every compact Riemannian spin^c-manifold M, there corresponds a canonical spectral triple

$$(C^\infty(M), L^2(S), D_M). \tag{1.1}$$

Here D_M denotes the *Dirac operator* associated to the manifold M. It acts on the square integrable sections of the *spinor bundle* S. The algebra $C^\infty(M)$ acts on $L^2(S)$ as multiplication operators. It was Alain Connes who showed that from the spectral data (1.1) we can recover the smooth structure, and in particular the topology of M ([18]).

We might also encounter spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where the algebra $\mathcal{A}$ is not commutative. This is why we need to take the word *geometry* with a grain of salt. Not every spectral triple arises from a geometric object. Instead we pretend the object $(\mathcal{A}, \mathcal{H}, D)$ corresponds to some 'noncommutative' space. Therefore, spectral triples are really a generalization of geometry.

In Section 3 we focus on the metrical aspect of the reconstruction of M from the spectral data $(C^\infty(M), L^2(S), D_M)$. Let us briefly describe how this works. First of all, recall that to every Riemannian manifold (M, g) there is associated a distance function on M, making M into a metric space. We will denote this distance function by d_g. If the manifold is in addition compact spin^c, then this distance function can be recovered from the spectral data (1.1) by means of *Connes' distance formula*

$$d_g(p, g) = \sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|[D_M, f]\| \leq 1\}. \tag{1.2}$$

The expression (1.2) can be rewritten as

$$d_g(p, g) = \sup_{f \in C^\infty(M)} \{|\delta_p(f) - \delta_q(f)| : \|[D_M, f]\| \leq 1\}, \tag{1.3}$$

where δ_p denotes the pure state corresponding to evaluation in the point $p \in M$. This motivates us to define a distance on the state space $\mathcal{S}(C^\infty(M))$ given by the formula

$$d(\phi, \psi) = \sup_{f \in C^\infty(M)} \{|\phi(f) - \psi(f)| : \|[D_M, f]\| \leq 1\}, \qquad \phi, \psi \in \mathcal{S}(C^\infty(M)). \tag{1.4}$$

The great feature of formula (1.4) is that it makes sense for any spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}, \qquad \phi, \psi \in \mathcal{S}(\mathcal{A}). \tag{1.5}$$

We are interested in *truncations* of the spectral triple (1.1), where only part of the spectrum of D_M is available. This can be formulated more precisely by choosing an increasing sequence $\{\mathcal{Q}_N\}_N \subseteq B(L^2(S))$ of spectral projections of D_M and considering the triple

$$(\mathcal{Q}_N C^\infty(M)\mathcal{Q}_N, \mathcal{Q}_N L^2(S), \mathcal{Q}_N D_M \mathcal{Q}_N). \tag{1.6}$$

The triple described in (1.6) is not a spectral triple anymore, as $\mathcal{Q}_N C^\infty(M)\mathcal{Q}_N$ is generally no algebra, but an *operator system spectral triple* instead (Definition 4.8). For this generalization of spectral triples the distance formula (1.5) is still well-defined. Henceforth it defines a distance on the state space $\mathcal{S}(\mathcal{Q}_N C^\infty(M)\mathcal{Q}_N)$.

The question we ask ourselves is:

Q: Can we approximate M, as a metric space, using the triple (1.6)?

In order to give an appropriate answer we develop the notion of *Gromov–Hausdorff* distance between operator system spectral triples in Section 4. This is a variation of the notion of quantum Gromov–Hausdorff distance as defined by Rieffel in [7].

In Section 5 we answer the question Q affirmatively for $M = \mathbb{T}^d$ and a suitable sequence of spectral projections $\{\mathcal{Q}_N\}_N$. More precisely, we show that the operator system spectral triple (1.6) converges in Gromov–Hausdorff distance to the canonical spectral triple corresponding to $\mathbb{T}^d$. The proof strongly relies a great deal on the ideas of work in progress by Walter van Suijlekom and Alain Connes ([22]). We extend their results from the circle to the d-dimensional torus using a specific sequence of spectral projections.

Our work strongly compares to the results obtained by F. Latrémolière in [20]. He uses the setting of Rieffel ([7]) to show convergence of the *fuzzy torus* to the the *quantum torus* in quantum Gromov–Hausdorff distance. The work of Rieffel is further developed for operator systems by D. Kerr in [19].

2 Operator theory

Operator, number, please
It's been so many years

Tom Waits, *Martha*

In this section we will recall some theory about C^*-algebras, and exhibit some basic examples. Also we will give the definition of a (concrete) operator system. This section will mainly serve as a résumé. For more details and proofs about C^*-algebras we refer to [14, Chapter 1,2,3 and 5].

2.1 C^*-algebras and operator systems

Definition 2.1. A unital algebra A together with a norm $\|\cdot\| : A \to \mathbb{R}$ is called a *normed algebra* if

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A,$$
$$\|1_A\| = 1.$$

Whenever $(A, \|\cdot\|)$ is a normed algebra such that A is complete with respect to the norm $\|\cdot\|$, we say A is a *Banach algebra*.

One could also allow for non-unital normed algebras, but for the purpose of the text it suffices to consider unital normed algebras exclusively.

Let us give some basic examples.

Example 2.2. Let X be a compact Hausdorff topological space. Then define

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous}\}.$$

The algebra structure on $C(X)$ is given by pointwise addition and multiplication. We define a norm on $C(X)$ by

$$\|f\|_\infty = \sup_{x \in X}\{|f(x)|\}.$$

Then $\|\cdot\|_\infty$ makes $C(X)$ into a Banach algebra.

The key example of a noncommutative normed algebra is given by the set of all bounded linear operators on some normed vector space.

Example 2.3. Let $(V, \|\cdot\|)$ be a normed vector space (complex or real). We define the norm of a linear operator $T : V \to T$ by

$$\|T\| = \sup_{v \in V}\{\|Tv\| : \|v\| \leq 1\}. \tag{2.1}$$

We say that T is a *bounded operator* if $\|T\| < \infty$ and we denote the set of all bounded operators by $B(V)$. The norm given in (2.1) turns $B(V)$ into a Banach algebra.

If V is a finite-dimensional complex vector space, then all linear operators $T : V \to V$ are bounded and $B(V)$ is isomorphic to $M_n(\mathbb{C})$, for $n = \dim V$. This isomorphism boils down to choosing a basis for V. Of course, if V is a finite-dimensional real vector space, then $B(V) \cong M_n(\mathbb{R})$.

Definition 2.4. Let $(A, \|\cdot\|)$ be a (unital) normed algebra. We say an element $a \in A$ is *invertible* if there exists a $b \in A$ such that

$$ab = ba = 1_A.$$

This b is then uniquely determined and we denote it a^{-1}. *The spectrum* of an element $a \in A$ is given by the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible}\}.$$

Theorem 2.5 (Gelfand)**.** *Let $(A, \|\cdot\|)$ be a complex Banach algebra. Then for each $a \in A$, the spectrum $\sigma(a)$ is a non-empty, compact subset of the complex numbers.*

Proof. See [14, Theorem 1.2.5]. □

Example 2.6. If $f \in C(X)$, as in Example 2.2, then $\sigma(f) = \overline{f(X)}$, i.e. the closure of the set $f(X)$.

When V is a finite-dimensional normed vector space, the spectrum of an operator $T \in B(V)$ is just the set of eigenvalues of the matrix corresponding to T.

Definition 2.7. An *involution* $*$ on an algebra A is an antilinear operator $*\colon A \to A$, such that

1. $a^{**} = a$
2. $(ab)^* = b^* a^*$

for all $a, b \in A$. We say the pair $(A, *)$ is a $*$-algebra. We say an element $a \in A$ is *self-adjoint* if $a^* = a$ and we denote the set of all self-adjoint elements in A by A_{sa}. If $E \subseteq A$ is a subspace of A, we say E is $*$-closed if $a^* \in E$, whenever $a \in E$.

A $*$-*homomorphism* ϕ of $*$-algebras $(A, *)$ and $(B, *)$ is a homomorphism of algebras $\phi\colon A \to B$, such that $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Definition 2.8. A C^**-algebra* is a complex Banach algebra $(A, \|\cdot\|)$, which is also a $*$-algebra and satisfies the C^*-identity: $\|a^* a\| = \|a\|^2$ for every $a \in A$.

If $(A, *)$ admits a norm $\|\cdot\|$, which makes it into a C^*-algebra, this norm is the unique norm doing so. Also, any $*$-homomorphism $\phi\colon A \to B$ between C^*-algebras A and B is necessarily norm-decreasing and $\phi(A)$ is a C^*-subalgebra of B. It follows that if ϕ is injective, then $\|\phi(a)\| = \|a\|$.

Example 2.9. If X is a compact Hausdorff space, then $C(X)$ as in Example 2.2 is in fact a C^*-algebra. The involution is given by complex conjugation: $f^*(x) = \overline{f(x)}$.

It turns out that every (unital) commutative C^*-algebra A is isomorphic to $C(X)$ for some compact Hausdorff space X. This is known as *Gelfand duality*, and it allows us to think of C^*-algebras as noncommutative topological spaces ([14, Theorem 2.1.10]).

Example 2.10. If $\mathcal{H}$ is a Hilbert space, then each bounded operator $T\colon \mathcal{H} \to \mathcal{H}$ has an adjoint $T^*\colon \mathcal{H} \to \mathcal{H}$. This map is uniquely defined by the property

$$\langle T\mu, \xi\rangle = \langle \mu, T^*\xi\rangle,$$

for every $\mu, \xi \in H$. With this involution, and the norm from Example 2.3, the Banach algebra $B(\mathcal{H})$ becomes a C^*-algebra.

Definition 2.11. A *representation* of a $*$-algebra A is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and

$$\pi \colon A \to B(\mathcal{H})$$

is a $*$-homomorphism. We say that π is *faithful*, if π is injective.

If $\pi \colon A \to B(H)$ is a faithful representation, then it defines a norm on A, given by

$$\|a\| := \|\pi(a)\|.$$

This norm makes A into a C^*-algebra.

Theorem 2.12 (Gelfand–Naimark)**.** *If A is a C^*-algebra then there exists a Hilbert space $\mathcal{H}$ and a faithful representation $\pi \colon A \to B(H)$.*

Proof. See [14, Theorem 3.4.1]. □

Definition 2.13. An element $a \in A$ is called *positive* if one of the following equivalent conditions is satisfied

1. a is self-adjoint and $\sigma(a) \subseteq [0, \infty)$.
2. There exists a $b \in A$, such that $a = b^*b$.

We denote the set of positive elements by A^+. It is a closed set in the topology induced by $\| \cdot \|$.

For operators on a Hilbert space, there is a particularly nice characterisation of positivity. Namely, if $\mathcal{H}$ is a Hilbert space, then an element $T \in B(\mathcal{H})$ is positive if and only if

$$\langle T\xi, \xi \rangle \geq 0, \text{ for every } \xi \in \mathcal{H}. \tag{2.2}$$

We say that a map

$$\phi \colon A \to B$$

between C^*-algebras A and B is positive if it maps positive elements to positive elements: $\phi(A^+) \subseteq B^+$.

Definition 2.14. Let A be a unital C^*-algebra. A *state* on A is a positive linear functional $\mu \colon A \to \mathbb{C}$ of norm one. We denote $\mathcal{S}(A)$ the set of all states on A. The set $\mathcal{S}(A)$ is called the *state space* of A.

The state space $\mathcal{S}(A)$ is a compact subset of the dual space of A, when equipped with the weak-$*$ topology. If $\mu \in \mathcal{S}(A)$, then the positivity of μ implies that $\mu(A^+) \subseteq [0, \infty)$.

Proposition 2.15. Let A be a C^*-algebra and let $\mu : A \to \mathbb{C}$ be a bounded linear functional. Then the following are equivalent.

- μ is a positive map.
- $\mu(1) = \|\mu\|$.

Proof. See [14, Theorem 3.3.2]. □

The state space $\mathcal{S}(A)$ is thus given by the linear functionals $\mu : A \to \mathbb{C}$ such that $\|\mu\| = \mu(1) = 1$. In the rest of this text we will mostly use this alternative characterization.

Definition 2.16. Given a unital C^*-algebra A, an operator system $E \subseteq A$ is a $*$-closed subspace, containing the identity element.

Our definition is that of a concrete operator system. That is, we define an operator system as a subspace of a given C^*-algebra. In particular E inherits the norm from A and elements $e \in E$ are positive whenever they are positive in A.

The characterization provided by Proposition 2.15 allows us to extend the definition of a state to operator systems.

Definition 2.17. Let $E \subseteq A$ be an operator system. A *state* on E is a linear functional $\mu : E \to \mathbb{C}$ such that

$$\|\mu\| = \mu(1) = 1.$$

We denote $\mathcal{S}(E)$ the set of all the states in E. This set is again called the *state space* of E.

2.2 Matrix C^*-algebras

Given a C^*-algebra A we can form the matrix C^*-algebra $M_n(A) \cong A \otimes M_n(\mathbb{C})$. It consists of matrices of which the entries are elements of A. Multiplication is induced by matrix multiplication combined with multiplication in A. The involution is given by $(a_{ij})^* = (a_{ji}^*)$. Finding a norm on this algebra, which turns it into a C^*-algebra, is not straightforward. That we are able to find such a norm is a very elegant corollary of Theorem 2.12. Indeed if we represent A faithfully on some Hilbert space $\mathcal{H}$:

$$\pi \colon A \to B(\mathcal{H}),$$

then $M_n(A)$ is represented faithfully on $\bigoplus_{j=1}^n \mathcal{H}$ in the following way

$$\begin{aligned} \pi_n \colon M_n(A) &\to M_n(B(\mathcal{H})) \cong B\Big(\bigoplus_{j=1}^n \mathcal{H}\Big) \\ (a_{ij}) &\mapsto (\pi(a_{ij})). \end{aligned}$$

This induces the (unique) C^*-norm on $M_n(A)$ given by $\|A\| = \|\pi_n(A)\|$ (one needs to check $M_n(A)$ is complete in this norm, for details we refer to [14, Page 95]).

Lemma 2.18. Let $\mathcal{H}$ be a Hilbert space and let $T \in B(\mathcal{H})$. Then $T \otimes 1_n \in B(\mathcal{H}) \otimes M_n(\mathbb{C}) \cong B(\mathcal{H} \otimes \mathbb{C}^n)$ has norm

$$\|T \otimes 1_n\| = \|T\|.$$

Proof. Suppose that $(\xi_1, \ldots, \xi_n) \in \mathcal{H} \otimes \mathbb{C}^n$, then

$$\begin{aligned} \|(T \otimes 1_n)(\xi_1, \ldots, \xi_n)\|^2 &= \|(T\xi_1, \ldots, T\xi_n)\|^2 \\ &= \|T\xi_1\|^2 + \cdots + \|T\xi_n\|^2 \\ &\leq \|T\|^2(\|xi_1\|^2 + \ldots \|\xi_n\|^2) \\ &= \|T\|^2 \|(\xi_1, \ldots, \xi_n)\|^2. \end{aligned}$$

For the converse inequality, choose a unit vector $\xi \in \mathcal{H}$ such that $\|T\xi\| > \|A\| - \epsilon$. Then $\widetilde{\xi} = \frac{1}{n}(\xi, \xi, \ldots, \xi) \in \mathcal{H} \otimes \mathbb{C}^n$ is a unit vector and

$$\|(T \otimes 1_n)\widetilde{\xi}\| = \frac{1}{\sqrt{n}}\sqrt{n}\|T\xi\| > \|T\| - \epsilon.$$

□

Given a map $\phi : A \to B$, between C^*-algebras A and B, we construct the induced map

$$\begin{aligned} \phi_n \colon M_n(A) &\to M_n(B) \\ (a_{ij}) &\mapsto (\phi(a_{ij})). \end{aligned} \tag{2.3}$$

In the identification $M_n(A) \cong A \otimes M_n(\mathbb{C})$, ϕ_n corresponds to $\phi \otimes 1_n$.

We should remark something about the notation $\phi \otimes 1_n$ and $T \otimes 1_n$ above, for they are maps of different types of spaces. The map $\phi \otimes 1_n$ is a map between C^*-algebras, whereas the map $T \otimes 1_n$ is a map of Hilbert spaces. Although it easy to compute the norm of $T \otimes 1_n$, like in Lemma 2.18, it requires rather some theory to compute the norm of $\phi \otimes 1_n$ ([6, Chapter 2 and 3]).

3 Connes' distance formula

Distance came in our lives
It always happens
When you're trying to get next to someone
When you want to reach her heart

David Crosby, *Distances*

3.1 Riemannian manifolds as metric spaces

Every Riemannian structure g on a manifold M gives rise to a distance function d_g on M, which turns it into a metric space. This distance function is directly related to the metric g on M. The reason to call d_g a distance function, instead of a metric, is to distinguish it from the Riemannian metric g. The distance between two points $p, q \in M$ is given by the length of the shortest curve between the points p and q. We make this more precise in the definitions below.

Definition 3.1. Given a Riemannian manifold (M, g) and a piecewise smooth curve $\gamma\colon [0,1] \to M$, define the *length* of γ by

$$l(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt.$$

Definition 3.2. Let (M, g) be a Riemannian manifold. For points $p, q \in M$ define the *distance function* on (M, g) by

$$d_g(p, q) = \inf\{l(\gamma) \mid \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\}.$$

A consequence of this definition is that the distance between two points $p, q \in M$ might be infinite if M is not connected. A way to resolve this is to only consider connected Riemannian manifolds (M, g). However we find this too restrictive and we choose instead to allow for infinite distances between points. Hence d_g becomes a *generalized distance function* which is in line with the sort of distance functions we will consider in Section 4.

Proposition 3.3. The distance function defined on (M, g) is a metric on M giving back the topology of M.

Proof. See [1, Definition-Proposition 2.91]. □

Definition 3.4. Given a Riemannian manifold (M, g) the *musical isomorphism* between TM and T^*M is given by the relations

$$\begin{aligned} X^\flat(Y) &= g(X, Y), \qquad \text{for } X, Y \in \Gamma^\infty(TM), \\ \omega(Y) &= g(\omega^\sharp, Y), \qquad \text{for } Y \in \Gamma^\infty(TM), \omega \in \Gamma^\infty(T^*M). \end{aligned} \tag{3.1}$$

For a smooth function f we define the vector field $\operatorname{grad} f = (df)^\sharp$.

That the musical isomorphism is indeed an isomorphism of vector bundles, we can see by noting that $^\sharp$ and $^\flat$ are each others inverses and that they are smooth, because g is a smooth metric. The musical isomorphism will be useful for us to switch between the bundles TM and T^*M. Whether we choose the bundle TM or T^*M to perform a certain construction is then only a matter of convention. For example, in Section 3.4 we follow the approach of [4] and use the cotangent bundle to construct the so-called *Clifford bundle* $\mathbb{C}l(M) \to M$. On the contrary, in [9] this construction is done using the tangent bundle.

3.2 Smoothening Lipschitz functions on $\mathbb{T}^d$

In this section, we attempt to approximate Lipschitz functions on the d-dimensional torus $\mathbb{T}^d$ using smooth functions. The motivation to find such approximations, is to be able to approximate the function $x \mapsto d_g(p_0, x)$, for some fixed $p_0 \in M$. This is a Lipschitz function with Lipschitz constant 1 (see Definition 3.5 below). The supremum in (3.22) is attained with this function, apart from the fact that $x \mapsto d_g(p_0, x)$ may not be smooth. It is necessary therefore, to perform some approximation argument.

The approximation procedure can be extended to arbitrary compact Riemannian manifolds (M, g), as we will explain at the end of the section. However, as we will be mainly concerned with the torus in Section 5, we will only give a sketch of the (more involved) general case.

Definition 3.5. On any metric space (X, d), we say a function $f\colon X \to \mathbb{C}$ is *Lipschitz* whenever

$$\|f\|_{Lip} := \sup_{\substack{x,y\in X \\ x\neq y}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

It turns out, that on a Riemannian manifold (M, g) the real valued Lipschitz functions are precisely those with bounded gradient.

Proposition 3.6. Let (M, g) be a Riemannian manifold and let $f \in C^\infty(M)$, then $\|\operatorname{grad} f\|_\infty = \|f\|_{Lip}$.

Proof. If p and q are points in M, then for any piecewise smooth path γ from p to q we have

$$\begin{aligned} f(q) - f(p) &= f(\gamma(1)) - f(\gamma(0)) \\ &= \int_0^1 \frac{d}{dt} f(\gamma(t))dt \\ &= \int_0^1 df(\dot\gamma(t))dt \\ &= \int_0^1 g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \dot\gamma(t))dt. \end{aligned}$$

So

$$\begin{aligned} |f(p) - f(q)| &\leq \int_0^1 |g_{\gamma(t)}(\operatorname{grad}_{\gamma(t)} f, \dot\gamma(t))|dt \\ &\leq \int_0^1 |\operatorname{grad}_{\gamma(t)} f||\dot\gamma(t))|dt \\ &\leq \|\operatorname{grad} f\|_\infty l(\gamma). \end{aligned}$$

Taking the infimum over all such paths yields $|f(p) - f(q)| \leq \|\operatorname{grad} f\|_\infty d_g(p, q)$, so

$$\frac{|f(p) - f(q)|}{d_g(p, q)} \leq \|\operatorname{grad} f\|_\infty$$

for all $p, q \in M$, which implies $\|f\|_{Lip} \leq \|\operatorname{grad} f\|_\infty$.

For the converse inequality, let $t \mapsto \phi^t(x)$ denote the flow of $\operatorname{grad} f$ at time t, starting at the point x. Let $x_0 \in M$ such that $|\operatorname{grad}_{x_0} f| = \|\operatorname{grad} f\|_\infty$. This point exists as

$x \mapsto |\operatorname{grad}_{x_0} f|$ is a continuous function and M is compact. Choose $\epsilon > 0$ and let U be an open neighbourhood of x_0 such that $x \in U \implies |\operatorname{grad}_x f| > \|\operatorname{grad} f\|_\infty - \epsilon$. Find $\alpha > 0$ such that $\phi^t(x_0) \in U$ for all $0 \leq t \leq \alpha$. Now set $q = \phi^\alpha(x_0)$ and define the smooth curve γ form x_0 to q by $\gamma(t) = \phi^{\alpha t}(x_0)$. Then

$$\begin{aligned}
f(q) - f(x_0) &= \int_0^1 \frac{d}{dt} f(\phi^{\alpha t}(x_0))dt \\
&= \alpha \int_0^1 df(\operatorname{grad}_{\phi^{\alpha t}(x_0)} f)dt \\
&= \alpha \int_0^1 g_{\phi^{\alpha t}(x_0)}(\operatorname{grad}_{\phi^{\alpha t}(x_0)} f, \operatorname{grad}_{\phi^{\alpha t}(x_0)} f)dt \\
&= \alpha \int_0^1 |\operatorname{grad}_{\phi^{\alpha t}(x_0)} f|^2 dt \geq \alpha(\|\operatorname{grad} f\|_\infty - \epsilon)^2.
\end{aligned}$$

Also

$$\begin{aligned}
l(\gamma) &= \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt \\
&= \int_0^1 \sqrt{g_{\phi^{\alpha t}(x_0)}(\alpha \operatorname{grad}_{\phi^{\alpha t}(x_0)} f, \alpha \operatorname{grad}_{\phi^{\alpha t}(x_0)} f)}dt \\
&= \alpha \int_0^1 |\operatorname{grad}_{\phi^{\alpha t}(x_0)} f|dt \\
&\leq \alpha \|\operatorname{grad} f\|_\infty.
\end{aligned}$$

So

$$\begin{aligned}
\frac{f(q) - f(x_0)}{d_g(q, x_0)} &\geq \frac{\alpha(\|\operatorname{grad} f\|_\infty - \epsilon)^2}{\alpha \|\operatorname{grad} f\|_\infty} \\
&= \|\operatorname{grad} f\|_\infty - 2\epsilon + \frac{\epsilon^2}{\|\operatorname{grad} f\|_\infty} \\
&\geq \|\operatorname{grad} f\|_\infty - 2\epsilon.
\end{aligned}$$

As ϵ was arbitrary, we conclude that

$$\sup_{p \neq q} \frac{|f(p) - f(q)|}{d_g(p, q)} \geq \sup_{q \neq x_0} \frac{|f(q) - f(x_0)|}{d_g(q, x_0)} \geq \|\operatorname{grad} f\|_\infty.$$

This shows $\|f\|_{Lip} \geq \|\operatorname{grad} f\|_\infty$, so we have proven the proposition. $\square$

Proposition 3.6 allows us to switch between a metric, or topological characterisation of the 'steepness' of a function, and an analytical one.

The key to approximating continuous functions, and Lipschitz functions in particular, is to introduce suitable mollifier functions. We realize $\mathbb{T}^d$ as $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. Let us define the family of functions κ_ϵ that will play the role of a Dirac net in the convolution algebra $L^1(\mathbb{T}^d)$.

Definition 3.7. Let $\kappa\colon \mathbb{T}^d \to \mathbb{R}$ be the function defined by

$$\kappa(x) = \begin{cases} C \exp \frac{1}{\|x\|^2 - 1} \text{ if } \|x\| < 1 \\ 0 \text{ if } \|x\| \geq 1. \end{cases}$$

Here we have chosen the constant C such that $\int_{\mathbb{T}^d} \kappa = 1$. The function κ is smooth. Next we define

$$\kappa_\epsilon(x) = \frac{1}{\epsilon^d}\kappa\Big(\frac{x}{\epsilon}\Big),$$

for $0 < \epsilon < 1$. The scaled function k_ϵ is smooth, spherically symmetric, positive, supported in an ϵ-ball around the origin and

$$\int_{\mathbb{T}^d} \kappa_\epsilon dx = 1.$$

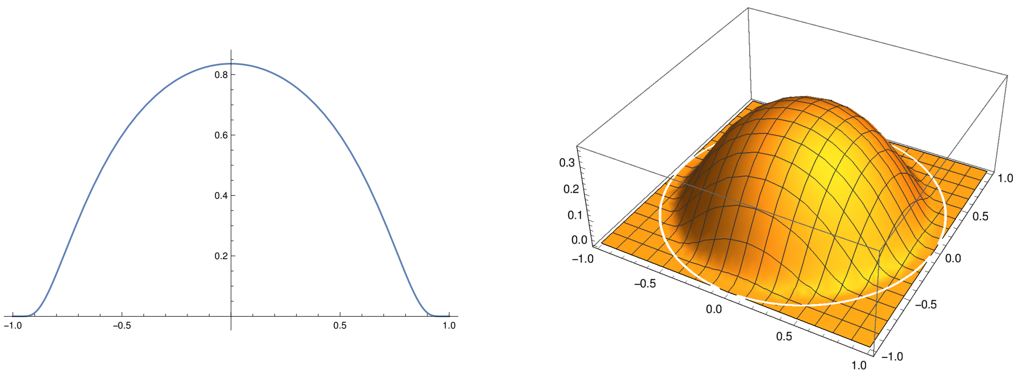

Figure 1: The smooth function κ for $d = 1$ (left) and $d = 2$ (right).

Proposition 3.8. Let $f\colon \mathbb{T}^d \to \mathbb{C}$ be a Lipschitz function, with Lipschitz constant K. Then for every $\epsilon > 0$ there exists a function f_ϵ, such that f_ϵ is smooth, $\|f_\epsilon - f\|_\infty < \epsilon$ and $\|f_\epsilon\|_{Lip} \leq K$.

Proof. Let $\epsilon > 0$ be given. We consider the family of functions

$$\widetilde{f}_r(x) = \int_{\mathbb{R}^n} \kappa_r(\xi) f(x - \xi) d\xi, \tag{3.2}$$

where $0 < r < 1$. The function $\widetilde{f}_r$ is smooth for every r. This follows from a differentiation in the integral argument, see [5, Appendix C]. We claim that $f_\epsilon = \widetilde{f}_{\frac{\epsilon}{K}}$ satisfies

$\|f_\epsilon - f\|_\infty < \epsilon$ and $\|f_\epsilon\|_{Lip} \leq K$. Indeed choosing $r = \frac{\epsilon}{K}$ yields

$$\begin{aligned}
|f(x) - \widetilde{f}_r(x)| &= \Big| \int_{\mathbb{T}^d} \kappa_r(y) f(x-y) dy - f(x) \Big| \\
&= \Big| \int_{\mathbb{T}^d} \kappa_r(y)(f(x-y) - f(x)) dy \Big| \\
&\leq \int_{\mathbb{T}^d} \kappa_r(y) |f(x-y) - f(x)| dy \\
&\leq \int_{\mathbb{T}^d} \kappa_r(y) K d(x-y, x) dy \\
&\leq \int_{\mathbb{T}^d} \kappa_r(y) K d(-y, 0) dy \\
&= \int_{\mathbb{T}^d} \kappa_r(y) K \|y\| dy \\
&\leq \int_{\mathbb{T}^d} \kappa_r(y) \epsilon dy = \epsilon,
\end{aligned}$$

which shows that $\|f_\epsilon - f\|_\infty < \epsilon$. Also we have $\|f_\epsilon\|_{Lip} \leq K$ since

$$\begin{aligned}
|f_\epsilon(x_1) - f_\epsilon(x_2)| &= \Big| \int_{\mathbb{T}^d} \kappa_r(f(x_1 - y) - f(x_2 - y)) dy \Big| \\
&\leq | \int_{\mathbb{T}^d} \kappa_r |f(x_1 - y) - f(x_2 - y)| dy \\
&\leq \int_{\mathbb{T}^d} \kappa_r K d(x_1 - y, x_2 - y) dy \\
&= \int_{\mathbb{T}^d} \kappa_r K d(x_1, x_2) d\xi = K d(x_1, x_2).
\end{aligned}$$

□

As announced, there is a more general statement of Proposition 3.8. We now need to perform the convolution process using normal coordinates on M.

Proposition 3.9. Suppose that (M, g) is a compact, Riemannian manifold, and that $f\colon M \to \mathbb{C}$ is a Lipschitz function with Lipschitz constant K. Then, for all $\epsilon > 0$, we can find a smooth function f_ϵ, such that $\|f - f_\epsilon\|_\infty < \epsilon$ and $\|\operatorname{grad} f_\epsilon\| < K + \epsilon$.

Sketch of proof. Just as in the case of the torus, we attempt to approximate our Lipschitz function by convoluting with some kernel. The appropriate convolution formula is described by Greene and Wu in [3]. We will give a sketch of their approach here.

Let κ_r be a family of smooth, positive functions, such that κ_r has support in $[-r, r]$ and such that each κ_r is constant in a neighbourhood of 0. Furthermore we require that $\int_{v \in \mathbb{R}^n} \kappa_r(\|v\|) d\mu = 1$. Here $n = \dim M$ and μ is Lebesgue meaure on $\mathbb{R}^n$.

We now consider functions $\widetilde{f}_r$, given by

$$\widetilde{f}_r(p) = \int_{v \in T_pM} f(\exp_p(v))\kappa_r(\|v\|)d\Omega_p.$$

Here $d\Omega_p$ is the measure on T_pM obtained from the Riemannian metric of M. The use of the exponential map encodes some of the translation-invariance we made good use of when proving Proposition 3.8.

The claim is that there exists some $r > 0$, such that is we set $f_\epsilon = \widetilde{f}_r$, we have $\|f - f_\epsilon\|_\infty < \epsilon$ and $\|\operatorname{grad} f_\epsilon\| < K + \epsilon$. For the proof of this claim we refer to [2, Lemma 1 and 2], and [3, Lemma 8]. $\square$

Combining Proposition 3.6 and 3.9 we see that for any Lipschitz function $f : M \to \mathbb{R}$ and any $\epsilon > 0$, we can find a smooth function $g : M \to \mathbb{R}$ such that $\|f - g\|_\infty < \epsilon$ and $\|g\|_{Lip} \leq \|f\|_{Lip} + \epsilon$.

3.3 The Clifford algebra

For a detailed treatment of Clifford algebras we refer to [9, Chapter 4] and [4, Chapter 9]. We use [9] as a main guideline in this section.

Definition 3.10. Let V be a vector space over $\mathbb{C}$. A *quadratic form* on V is a map $Q \colon V \to \mathbb{C}$, such that

$$\begin{aligned} Q(\lambda v) &= \lambda^2 Q(v), && \text{for all } \lambda \in \mathbb{C}, v \in V \\ Q(v+w) + Q(v-w) &= 2Q(v) + 2Q(w), && \text{for all } v, w \in V. \end{aligned}$$

Definition 3.11. Let (V, Q) be a vector space over $\mathbb{C}$. Then we define the *Clifford algebra* of (V, Q) by

$$Cl(V,Q) = TV / \langle v \otimes v - Q(v)1 \rangle_{v \in V}.$$

Here TV denotes the tensor algebra of V (see appendix C), and $\langle v \otimes v - Q(v)1 \rangle_{v \in V}$ denotes the ideal in TV generated by the expressions $v \otimes v - Q(v)1$, where $v \in V$.

In other words, $Cl(V, Q)$ is the algebra generated by the vector space V, where the elements are subject to the relation

$$v^2 = Q(v)1, \qquad \text{for every } v \in V. \tag{3.3}$$

To every quadratic form Q on some vector space V, there is associated a pairing $g_Q \colon V \times V \to \mathbb{C}$, given by

$$g_Q(v,w) = \frac{1}{2}\big(Q(v+w) - Q(v) - Q(w)\big).$$

The relations (3.3) are then equivalent to the defining relations

$$vw + wv = 2g_Q(v,w) \qquad \text{for every } v, w \in V. \tag{3.4}$$

We can retrieve Q from this pairing by $Q(v) = g_Q(v, v)$.

One can easily check that if $\{e_j\}_{j=1}^n$ is a basis for the vector space V, then

$$\{e_{i_1} e_{i_2} \dots e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}_{k=0}^n$$

is a basis for $Cl(V, Q)$. So if V is an n-dimensional vector space, then $Cl(V, Q)$ is 2^n-dimensional.

There is a $\mathbb{Z}_2$ grading on the Clifford algebra $Cl(V, Q)$, which is given by

$$\chi(v_1 v_2 \dots v_k) = (-1)^k v_1 v_2 \dots v_k.$$

So we can decompose $Cl(V, Q)$ as

$$Cl(V, Q) = \big(Cl(V, Q)\big)^0 \oplus \big(Cl(V, Q)\big)^1,$$

where

$$\begin{aligned} \big(Cl(V, Q)\big)^0 &= \{v \in Cl(V) \mid \chi v = v\chi\} \\ \big(Cl(V, Q)\big)^1 &= \{v \in Cl(V) \mid \chi v = -v\chi\}, \end{aligned}$$

which are called the even and the odd part of the $Cl(V, Q)$ respectively.

Example 3.12. The vector space $\mathbb{C}^n$ has a standard quadratic form Q_n given by

$$Q_n(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} (x_j)^2.$$

We denote

$$\mathbb{C}l_n = Cl(\mathbb{C}^n, Q_n).$$

If we denote $\{e_j\}$ the standard basis for $\mathbb{C}^n$, then $\mathbb{C}l_n$ is the complex algebra generated by the vectors $e_j, 1 \le j \le n$. The defining relations (3.4) now become

$$e_i e_j + e_j e_i = \pm\delta_{ij}. \tag{3.5}$$

The Clifford algebras $\mathbb{C}l_n, n \ge 1$, are completely classified and they are subject to so-called *Bott–periodicity*:

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{3.6}$$

Another feature is that we have the following relations

$$(\mathbb{C}l_{n+1})^0 \cong \mathbb{C}l_n. \tag{3.7}$$

Bott-periodicity (3.6) implies that the algebras $\mathbb{C}l_{n+2}$ and $\mathbb{C}l_n$ are *Morita equivalent.* For a more detailed treatment of Bott–periodicity, Morita equivalence and the relations (3.7) we refer to [9].

We will classify the algebras $\mathbb{C}l_n$. By Bott–periodicity it is enough to compute $\mathbb{C}l_1$ and $\mathbb{C}l_2$.

Lemma 3.13. $\mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C}l_2 \cong M_2(\mathbb{C})$.

Proof. $\mathbb{C}l_1$ is the complex algebra generated by the elements $1, e_1$, subject to the relation $e_1^2 = 1$ and $\mathbb{C}l_2$ is the algebra generated by $1, e_1, e_2$, subject to the relations $e_1^2 = e_2^2 = 1$. One can check that the following are maps on these generating vectors inducing algebra isomorphisms:

$$\begin{array}{ccc} \mathbb{C}l_1 \to \mathbb{C} \oplus \mathbb{C} & \qquad & \mathbb{C}l_2 \to M_2(\mathbb{C}) \\ 1 \mapsto (1, 1) & & 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ e_1 \mapsto (1, -1) & & e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & & e_2 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{array}$$

□

Combining Lemma 3.13 and $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} M_2(\mathbb{C})$, we see that

$$\begin{aligned} \mathbb{C}l_n &\cong M_{2^m}(\mathbb{C}), && n = 2m. \\ \mathbb{C}l_n &\cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}), && n = 2m+1. \end{aligned} \tag{3.8}$$

Using (3.7) and (3.8) we see that

$$\left(\mathbb{C}l_{2m+1}\right)^0 \cong M_{2^m}(\mathbb{C}). \tag{3.9}$$

3.4 Connes' distance formula

After having done the preliminary work on approximating Lipschitz functions on Riemannian manifolds, we are now ready to prove Connes' distance formula. An excellent reference for the material covered in this section is provided by [4, Chapter 9].

Throughout this section (M, g) will always denote a compact Riemannian manifold.

Using the musical isomorphism (3.1) we can equip T^*M with the metric g^{-1}, that is defined by

$$g^{-1}(\omega_1, \omega_2) = g(\omega_1^\sharp, \omega_2^\sharp). \tag{3.10}$$

We want to construct the Clifford bundle over a manifold M form the cotangent bundle T^*M. As we will always work with complex Clifford algebras and the bundle T^*M is real, we must complexify the cotangent bundle. That is, we should consider the bundle $T^*M \otimes_{\mathbb{R}} \mathbb{C}$. The pairing g^{-1} on T^*M extends to $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ by

$$g^{-1}(\omega_1 + i\xi_1, \omega_2 + i\xi_2) = g^{-1}(\omega_1, \omega_2) - g^{-1}(\xi_1, \xi_2) + ig^{-1}(\omega_1, \xi_2) + ig^{-1}(\xi_1, \omega_2).$$

Definition 3.14. Let (M, g) be a Riemannian manifold. The *Clifford bundle* $\mathbb{C}l(M)$ over M is the complex algebra bundle with fibres $(\mathbb{C}l(M))_p = Cl(T_p^*M \otimes_{\mathbb{R}} \mathbb{C})$. Here $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ is equipped with the quadratic form corresponding to g_p^{-1}. So

$$Q_{g_p^{-1}}(\omega) = g_p^{-1}(\omega, \omega) \qquad \omega \in T^*M \otimes_{\mathbb{R}} \mathbb{C}. \tag{3.11}$$

The transition functions are given by $h^\alpha_\beta(v_1 v_2 \dots v_k) = h^\alpha_\beta(v_1) h^\alpha_\beta(v_2) \dots h^\alpha_\beta(v_k)$, where h^α_β denote the transition functions of $T^*M \otimes_{\mathbb{R}} \mathbb{C}$. The transition functions are skew-symmetric and satisfy the cocycle condition, so indeed we obtain a complex algebra bundle $\mathbb{C}l(M)$ over M.

If (U, ϕ) is a local trivializing chart for T^*M, then we can find a local orthonormal basis $\{dx^\mu\}_{\mu=1}^n$ with respect to the metric g^{-1}. Then $\mathbb{C}l(U)$ is the complex algebra generated by the elements dx^μ, subject to the defining relations

$$dx^\mu dx^\nu + dx^\nu dx^\mu = 2\delta^{\mu\nu}. \tag{3.12}$$

Definition 3.15. A Riemannian manifold (M, g) is called spin^c if there exists a complex bundle $S \to M$ and an algebra bundle isomorphism

$$\begin{aligned} \mathbb{C}l(M) &\cong \mathrm{End}(S), && \text{when } M \text{ is even-dimensional,} \\ \left(\mathbb{C}l(M)\right)^0 &\cong \mathrm{End}(S), && \text{when } M \text{ is odd-dimensional.} \end{aligned} \tag{3.13}$$

We call S the spinor bundle, and the smooth sections $\Gamma^\infty(S)$ we call the spinors.

Note that locally we can always find such a bundle S. Indeed, if (U, ϕ) is a trivializing chart, then we saw that

$$\begin{aligned} \mathbb{C}l(U) &\cong U \times M_{2^m}(\mathbb{C}), \quad \text{if } n = 2m \\ \big(\mathbb{C}l(U)\big)^0 &\cong U \times M_{2^m}(\mathbb{C}), \quad \text{if } n = 2m+1. \end{aligned} \tag{3.14}$$

So if we choose S_U the trivial bundle $S_U = \mathbb{C}^{2^m} \times U$, then we have the desired isomorphism (3.13). Consequently, whether a Riemannian manifold is spin^c, depends if we can patch these trivializations together to form a global bundle S. This corresponds to the vanishing of the *Dixmier-Douady class* of the vector bundle $\mathbb{C}l(M)$ ([4, Section 9.2]).

Definition 3.16. Let (M, g) be a spin^c-manifold, with corresponding spinor bundle $S \to M$. The isomorphism of bundles (3.13) induces an isomorphism of C^∞-modules

$$\begin{aligned} c\colon \Gamma^\infty(\mathbb{C}l(M)) &\xrightarrow{\sim} \Gamma^\infty(\mathrm{End}(S)), \quad \text{when } M \text{ is even-dimensional}, \\ c\colon \Gamma^\infty\Big(\big(\mathbb{C}l(M)\big)^0\Big) &\xrightarrow{\sim} \Gamma^\infty(\mathrm{End}(S)), \quad \text{when } M \text{ is odd-dimensional}. \end{aligned} \tag{3.15}$$

This ismorphism c is called the Clifford action.

Using (3.12) and (3.14) we can compute the Clifford action locally. First we inductively define the matrices $\gamma_j^{(n)} \in M_{2^m}(\mathbb{C})$. Here again n and m are related by $n = 2m$ if n is even, or $n = 2m+1$ if n is odd. Set $\gamma_1^{(1)} = 1$, and for $n > 1$ odd we define

$$\gamma_j^{(n)} = \begin{pmatrix} 0 & \gamma_j^{(n-2)} \\ \gamma_j^{(n-2)} & 0 \end{pmatrix}, \qquad \gamma_{n-1}^{(n)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_n^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad 1 \le j \le n-2. \tag{3.16}$$

For n even, we define $\gamma_j^{(n)} = \gamma_j^{(n+1)}$. For $n = 3$, this just yields the well known Pauli matrices. One easily checks that the matrices satisfy

$$\gamma_j^{(n)}\gamma_k^{(n)} + \gamma_k^{(n)}\gamma_j^{(n)} = 2\delta_{jk}, \qquad \Big(\gamma_j^{(n)}\Big)^* = \Big(\gamma_j^{(n)}\Big)^2 = 1, \tag{3.17}$$

for each j, k, n. Now if $\{dx^\mu\}_{\mu=1}^n$ is a local orthonormal frame for the metric g^{-1} on T^*M, then setting $c(dx^\mu) = \gamma^\mu \equiv \gamma_\mu^{(n)} \in M_{2^m}(\mathbb{C})$, defines the desired isomorphism (3.15), as the matrices γ^μ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}. \tag{3.18}$$

Apart from the Clifford action, we also want a connection on the Clifford bundle. The Levi-Civita connection ∇^g on TM defines a connection on T^*M, via the musical isomorphism (3.1). This connection on T^*M we will also denote by ∇^g. The Clifford bundle $\mathbb{C}l(M)$ is generated by the bundle T^*M. The Levi-Civita connection on T^*M then extends (after complexifying) to a connection on $\mathbb{C}l(M)$, which we will also call the Levi-Civita connection and which we will denote by ∇. It is recursively defined by $\nabla|_{\Omega^1(M)} = \nabla^g$ and

$$\nabla(\mu\lambda) = \nabla(\mu)\lambda + \mu\nabla(\lambda) \qquad \text{for } \mu, \lambda \in \Gamma^\infty(\mathbb{C}l(M)).$$

Proposition 3.17. Let (M, g) be a spin^c–manifold, with spinor bundle $S \to M$, Clifford action c and Levi–Civita connection ∇ on $\mathbb{C}l(M)$. Then there exists a connection ∇^S on S satisfying the Leibniz rule

$$\nabla^S(c(v)s) = c(\nabla v)s + c(v)\nabla^S(s) \text{ for all } v \in \Gamma^\infty(\mathbb{C}l(M)), s \in \Gamma^\infty(S). \tag{3.19}$$

Proof. See [4, Theorem 9.8]. □

Definition 3.18. Let (M, g) be a spin^c manifold and let c denote the Clifford action (3.15). We define $\hat{c} : \Gamma^\infty(\mathbb{C}l(M)) \otimes \Gamma^\infty(S) \to \Gamma^\infty(S)$ by

$$\hat{c}(v \otimes s) = c(v)s.$$

Now if ∇^S is a connection on the spinor bundle S, satisfying the Leibniz rule (3.19), then the *Dirac operator* associated to the connection ∇^S and the Clifford action c is defined by

$$D = -i(\hat{c} \circ \nabla^S). \tag{3.20}$$

Proposition 3.19. Suppose D is the Dirac operator on $\Gamma^\infty(S)$ and $f \in C^\infty(M)$ acts on $\Gamma^\infty(S)$ by multiplication. Then we have

$$[D, f] = -ic(df).$$

Proof. For any $s \in \Gamma^\infty(S)$

$$\begin{aligned}
[D, f]s &= D(fs) - f(Ds) \\
&= -i\hat{c}(\nabla(fs)) + if\hat{c}(\nabla(s)) \\
&= -i\hat{c}(df \otimes s + f\nabla(s)) + if\hat{c}(\nabla(s)) \\
&= -ic(df)s.
\end{aligned}$$

□

Suppose M is a manifold and $E \to M$ is some complex vector bundle, then there exists a smooth Hermitian structure

$$h\colon \Gamma^\infty(E) \times \Gamma^\infty(E) \to C^\infty(M),$$

which is linear in the first entry and conjugate linear in the second entry.

Definition 3.20. Let (M, g) be a Riemannian spin^c-manifold and let $S \to M$ be its spinor bundle. Let $h : \Gamma^\infty(S) \times \Gamma^\infty(S) \to C^\infty(M)$ be a smooth Hermitian structure, then we define an inner product on $\Gamma^\infty(S)$ by

$$\langle s_1, s_2 \rangle = \int_M h(s_1, s_2)\sqrt{g}dx. \tag{3.21}$$

We define the space of square integrable spinors, denoted $L^2(S)$, to be the Hilbert space completion of $\Gamma^\infty(S)$ with respect to the inner product (3.21). The construction of $L^2(S)$ is independent of the chosen metric on the spinor bundle S.

The inner product (3.21) defines a norm on $\Gamma^\infty(S)$. Therefore, if an operator A is acting on $\Gamma^\infty(S)$, then we can also define its norm

$$\|A\| = \sup_{s \in \Gamma^\infty(S)} \{\|As\| : \|s\| \leq 1\}.$$

For example, a function $f \in C^\infty(M)$ acts on $\Gamma^\infty(S)$ by pointwise multiplication ($\Gamma^\infty(S)$ is a $C^\infty(M)$-module) and $\|f\| = \|f\|_\infty$. More generally, if $B \in \Gamma^\infty(\mathrm{End}(S))$, then $\|B\| = \sup_{p \in M} \|B(p)\|$.

Theorem 3.21 (Connes' distance formula)**.** *Let (M, g) be a Riemannian spin^c-manifold with spinor bundle $S \to M$ and Dirac operator associated to some Clifford connection ∇^S. Let d_g be the metric associated to (M, g), then we can recover d_g with the formula*

$$d_g(p, q) = \sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|[D, f]\| \leq 1\}. \tag{3.22}$$

Proof. By Proposition 3.19 we know $\|[D, f]\| = \|c(df)\|$. We claim that we have the equality $\|c(df)\| = \|\operatorname{grad} f\|_\infty$. As $\|c(df)\| = \sup_{p \in M} \|c(df)(p)\|$, it is enough to show $\|c(df)(p)\| = \|\operatorname{grad}_p f\|$ for every $p \in M$. Choose $p \in M$ and let $\{dx^\mu\}_{\mu=1}^n$ be an orthonormal frame on a neighbourhood U around p, with respect to the metric g^{-1} on $T^*M\Big|_U$. Then we know that the Clifford action is given (locally) by

$$c(dx^\mu) = \gamma^\mu,$$

with the gamma matrices γ^μ as in (3.16). Now we compute

$$\begin{aligned}
\|c(df)(p)\|^2 &= \|c(df)^*(p)c(df)(p)\| \\
&= \left\| c\left(\sum_\mu \partial_\mu f dx^\mu\right)^* (p) c\left(\sum_\mu \partial_\nu f dx^\nu\right)(p) \right\| \\
&= \left\| \sum_{\mu,\nu} \overline{\partial_\mu f(p)} \partial_\nu f(p) \left(\gamma^\mu\right)^* \gamma^\nu \right\| \\
&= \left\| \sum_\mu \overline{\partial_\mu f(p)} \partial_\mu f(p) \otimes 1_{2^m} \right\| \\
&= \left\| \sum_\mu \overline{\partial_\mu f(p)} \partial_\mu f(p) \right\| \\
&= \|\operatorname{grad}_p f\|^2,
\end{aligned}$$

where we use the relations (3.17) and (3.18) for the fourth equality and Lemma 2.18 for the fifth equality. Therefore Connes' distance formula is equivalent to

$$d_g(p, q) = \sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|\operatorname{grad} f\|_\infty \leq 1\}.$$

We prove the two inequalities. For the first inequality, we know that

$$|f(p) - f(q)| \leq \|\operatorname{grad} f\|_\infty d_g(p, q),$$

which we saw in the proof of Proposition 3.6. This yields the inequality

$$\sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|\operatorname{grad} f\|_\infty \leq 1\} \leq d_g(p, q).$$

For the converse inequality we need the results of Section 3.2. The function f_p, defined by $f_p(q) = d_g(p, q)$ is Lipschitz with Lipschitz constant 1. Indeed, for $x, y \in M$,

$$|f_p(x) - f_p(y)| = |d_g(p, x) - d_g(p, y)| \leq d_g(x, y),$$

by the converse triangle inequality. If we let $\epsilon > 0$ arbitrary, then according to Proposition 3.9 we can find a smooth function f_ϵ such that $\|f_p - f_\epsilon\|_\infty < \epsilon$ and $\|\operatorname{grad} f_\epsilon\| < 1 + \epsilon$, which implies $\|\operatorname{grad} \frac{f_\epsilon}{1+\epsilon}\| \leq 1$. Also we have that

$$|f_\epsilon(x) - f_\epsilon(y)| \geq |f_p(x) - f_p(y)| - 2\epsilon,$$

for all $x, y \in M$. Therefore

$$\begin{aligned}\sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|\operatorname{grad} f\| \leq 1\} &\geq \frac{|f_\epsilon(p) - f_\epsilon(q)|}{1+\epsilon} \\ &\geq \frac{|f_p(p) - f_p(q)| - 2\epsilon}{1+\epsilon} \\ &= \frac{d_g(p,q) - 2\epsilon}{1+\epsilon}.\end{aligned}$$

This last expression tends to $d_g(p,q)$ as ϵ tends to 0, so we have proven the other inequality

$$\sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|\operatorname{grad} f\|_\infty \leq 1\} \geq d_g(p,q),$$

completing the proof of the theorem. $\square$

4 Noncommutative geometry

> "Sometimes, if you stand on the bottom rail of a bridge and lean over to watch the river slipping slowly away beneath you, you will suddenly know everything there is to be known."
>
> A.A. Milne

We can extend Connes' distance formula (3.22) to noncommutative spaces, as we will explain in this section. The analytical tools that we use in this section may sometimes be quite technical. For the theory on compact operators and self-adjoint operators we refer to [21, Chapters 4, 13].

4.1 Spectral triples

Definition 4.1. A *spectral triple* is given by a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a dense, $*$-closed subalgebra of a unital C^*-algebra A, that acts faithfully on $\mathcal{H}$ and D is an essentially self-adjoint operator on H, with compact resolvent and such that $[D, a] \in B(\mathcal{H})$ for each $a \in \mathcal{A}$.

Example 4.2. To every compact, Riemannian spin^c–manifold there corresponds a canonical spectral triple

$$(C^\infty(M), L^2(S), D_M),$$

where $L^2(S)$ denotes the space of square integrable spinors, as in Definition 3.20, and D_M denotes the Dirac operator of M (with domain $C^\infty(M)$). The algebra $C^\infty(M)$ acts on $L^2(S)$ by multiplication. For the technical proofs of the fact that D_M is essentially self-adjoint and has compact resolvent we refer to [4, Chapter 10].

In Section 3 we showed that we can recover the distance function on M from the data $(C^\infty(M), L^2(S), D_M)$, making use of Connes' distance formula (3.22). This gives us back the topology of M. It is a deep theorem by Alain Connes that we can also recover the smooth structure from the same data ([18],[4, Theorem 11.2]). Therefore, spectral triples $(\mathcal{A}, \mathcal{H}, D)$ are really a generalization of geometry.

We can rewrite Connes' distance formula (3.22) as

$$\sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \|[D, f]\| \leq 1\} = \sup_{f \in C^\infty(M)} \{|\delta_p(f) - \delta_q(f)| : \|[D, f]\| \leq 1\},$$

where $\delta_p \in \mathcal{S}(C^\infty(M))$ denotes the pure state $f \mapsto f(p)$. This motivates us to define a (generalized) distance function on $\mathcal{S}(C^\infty(M))$ given by

$$d(\phi, \psi) = \sup_{f \in C^\infty(M)} \{|\phi(f) - \psi(f)| : \|[D, f]\| \leq 1\}, \qquad \phi, \psi \in \mathcal{S}(C^\infty(M)). \tag{4.1}$$

We use the word 'generalized', for the supremum in (4.1) could be infinite a priori. The rest of the axioms of a metric are all satisfied. The great feature about (4.1) is that it also makes sense for spectral triples.

Proposition 4.3. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, then the formula

$$d_{\mathcal{A}}(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}, \qquad \phi, \psi \in \mathcal{S}(\mathcal{A}) \tag{4.2}$$

defines a (generalized) distance function on $\mathcal{S}(\mathcal{A})$.

Proof. It is clear that $d_{\mathcal{A}}(\phi, \phi) = 0$. Suppose $\phi \neq \psi$, then $\phi(a) \neq \psi(a)$ for some $a \in \mathcal{A}$. Now consider $a' = \frac{a}{\|[D,a]\|}$. We see that $\phi(a') \neq \psi(a')$ and that $\|[D, a']\| \leq 1$, therefore $d_{\mathcal{A}}(\phi, \psi) > 0$. For the triangle inequality consider $\phi, \psi, \xi \in \mathcal{S}(\mathcal{A})$, then

$$\begin{aligned}
d_{\mathcal{A}}(\phi, \xi) &= \sup_{a \in \mathcal{A}}\{|\phi(a) - \xi(a)| : \|[D, a]\| \leq 1\} \\
&\leq \sup_{a \in \mathcal{A}}\{|\phi(a) - \psi(a)| + |\psi(a) - \xi(a)| : \|[D, a]\| \leq 1\} \\
&\leq \sup_{a \in \mathcal{A}}\{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\} + \sup_{a \in \mathcal{A}}\{|\psi(a) - \xi(a)| : \|[D, a]\| \leq 1\} \\
&= d_{\mathcal{A}}(\phi, \psi) + d_{\mathcal{A}}(\psi, \xi).
\end{aligned}$$

□

It will be useful to notice the following ([7]):

Remark 4.4. In the formula (4.2), defining the distance on the state space, it suffices to take the supremum over all self-adjoint elements. That is

$$d_{\mathcal{A}}(\phi, \psi) = \sup_{a \in \mathcal{A}}\{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\} = \sup_{a \in \mathcal{A}_{sa}}\{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}.$$

We can see this as follows. Let $\phi, \psi \in \mathcal{S}(\mathcal{A})$ and $\epsilon > 0$ be given. Then there is an $a \in \mathcal{A}$ such that $\|[D, a]\| \leq 1$ and

$$|\phi(a) - \psi(a)| > d_{\mathcal{A}}(\phi, \psi) - \epsilon.$$

So there exists $\alpha \in \mathbb{C}, |\alpha| = 1$ such that

$$\phi(\alpha a) - \psi(\alpha a) > d_{\mathcal{A}}(\phi, \psi) - \epsilon.$$

If we now set $b = \frac{\alpha a + (\alpha a)^*}{2}$, then b is self-adjoint and

$$\begin{aligned}
\phi(b) - \psi(b) &= \frac{\phi(\alpha a) - \psi(\alpha a)}{2} + \frac{\phi\big((\alpha a)^*\big) - \psi\big((\alpha a)^*\big)}{2} \\
&= \frac{\phi(\alpha a) - \psi(\alpha a)}{2} + \frac{\overline{\phi(\alpha a)} - \overline{\psi(\alpha a)}}{2} \\
&= \frac{\phi(\alpha a) - \psi(\alpha a)}{2} + \frac{\phi(\alpha a) - \psi(\alpha a)}{2} > d_{\mathcal{A}}(\phi, \psi) - \epsilon.
\end{aligned}$$

Here we used that $\phi(a^*) = \overline{\phi(a)}$ for positive linear functionals ϕ, and that $\phi(\alpha a) - \psi(\alpha a)$ is real. As $[D, a^*] = -[D, a]^*$, we have that $\|[D, a^*]\| = \|[D, a]\|$, and so $\|[D, b]\| \leq 1$. This proves the equality.

4.1.1 Some examples

We compute the distance formula formula (4.2) induces on $\mathcal{S}(\mathcal{A})$ for several examples.

Example 4.5. Consider the spectral triple on a two point space

$$\left(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\right), \tag{4.3}$$

for some $t \in \mathbb{R}, t \neq 0$. The algebra $\mathbb{C}\oplus\mathbb{C}$ acts on $\mathbb{C}^2$ by

$$(x, y) \cdot v = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} v, \qquad v \in \mathbb{C}^2.$$

The state space $\mathcal{S}(\mathbb{C}\oplus\mathbb{C})$ is given by $\{\phi_\lambda | \lambda \in [0,1]\}$. Here ϕ_λ denotes the linear functional

$$(x, y) \mapsto \lambda x + (1-\lambda) y. \tag{4.4}$$

We compute the distance $d(\phi_{\lambda_1}, \phi_{\lambda_2}), \lambda_1 \neq \lambda_2$ as given by (4.2)

$$\left\| \left[D, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right] \right\| = \left\| \begin{pmatrix} 0 & t(y-x) \\ t(x-y) & 0 \end{pmatrix} \right\| = |t||x-y|.$$

So $\|[D,(x,y)]\| \leq 1 \implies |x-y| \leq \frac{1}{|t|}$, which implies

$$\begin{aligned} d(\phi_{\lambda_1}, \phi_{\lambda_2}) &= \sup_{(x,y)\in\mathbb{C}\oplus\mathbb{C}} \{|\phi_{\lambda_1}(x,y) - \phi_{\lambda_2}(x,y)|, \|[D,(x,y)]\| \leq 1\} \\ &= \sup_{(x,y)\in\mathbb{C}\oplus\mathbb{C}} \{|\lambda_1 - \lambda_2||x-y|, \|[D,(x,y)]\| \leq 1\} \\ &\leq \frac{|\lambda_1 - \lambda_2|}{|t|}. \end{aligned}$$

We may consider the element $\left(0, \frac{1}{|t|}\right)$, for which $\left\| \left[D, \left(0, \frac{1}{|t|}\right) \right] \right\| \leq 1$, so that

$$\left| \phi_{\lambda_1}\left(0, \frac{1}{|t|}\right) - \phi_{\lambda_2}\left(0, \frac{1}{|t|}\right) \right| = \frac{|\lambda_1 - \lambda_2|}{|t|}.$$

We conclude that $d(\phi_{\lambda_1}, \phi_{\lambda_2}) = \frac{|\lambda_1 - \lambda_2|}{|t|}$. If $t = 0$, then $d(\phi_{\lambda_1}, \phi_{\lambda_2}) = \infty$, whenever $\lambda_1 \neq \lambda_2$.

Example 4.6. The first noncommutative example is given by the spectral triple

$$\left(M_2(\mathbb{C}), \mathbb{C}^2, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right), \tag{4.5}$$

for some $x, y \in \mathbb{R}, x \neq y$. The action of $M_2(\mathbb{C})$ on $\mathbb{C}^2$ is just given by matrix multiplication. Again we compute the distance induced by Connes' distance formula (4.2). This time we restrict the distance to the pure state space $\mathcal{P}(M_2(\mathbb{C}))$, which is isomorphic to $\mathbb{CP}^1$ ([23, Proposition 2.9]). An element $[z : w] \in \mathbb{CP}^1$ corresponds to the pure state $\phi_{[z:w]}$, given by

$$M \mapsto \frac{1}{|z|^2 + |w|^2} \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, M \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle, \qquad M \in M_2(\mathbb{C}). \tag{4.6}$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ we have

$$\|[D, M]\| = \left\| \begin{pmatrix} 0 & b(x-y) \\ c(y-x) & 0 \end{pmatrix} \right\| = \max\{|b|, |c|\}|x-y|. \tag{4.7}$$

Write $\phi_z = \phi_{[z:1]}$. Then we compute

$$\phi_{z_1}(M) - \phi_{z_2}(M) = \frac{1}{1+|z_1|^2}\big(a|z_1|^2 + b\overline{z_1} + cz_1 + d\big) - \frac{1}{1+|z_2|^2}\big(a|z_2|^2 + b\overline{z_2} + cz_2 + d\big). \tag{4.8}$$

If we choose $M_N = \begin{pmatrix} 0 & |x-y|^{-1} \\ |x-y|^{-1} & N \end{pmatrix}$, then from (4.7) we see that $\|[D, M_N]\| \leq 1$ and form (4.8) it is clear that $\lim_{N\to\infty} |\phi_{z_1}(M_N) - \phi_{z_2}(M_N)| = \infty$, whenever $|z_1| \neq |z_2|$. It follows that $d(\phi_{z_1}, \phi_{z_2}) = \infty$, whenever $|z_1| \neq |z_2|$. If instead $|z_1| = |z_2|$, we can use (4.8) to calculate

$$\begin{aligned} |\phi_{z_1}(M) - \phi_{z_2}(M)| &= \frac{1}{1+|z_1|^2}\big|\big(b(z_1 - \overline{z_2}) + c(z_1 - \overline{z_2})\big)\big| \\ &\leq 2\frac{|z_1 - z_2|}{1+|z_1|^2}\max\{|b|, |c|\}. \end{aligned} \tag{4.9}$$

This implies

$$d(\phi_{z_1}, \phi_{z_2}) \leq 2\frac{|z_1 - z_2|}{1+|z_1|^2}|x-y|^{-1}.$$

For the converse inequality, we choose an element

$$M = \begin{pmatrix} 0 & \frac{z_1 - z_2}{|z_1 - z_2||x-y|} \\ \frac{z_1 - z_2}{|z_1 - z_2||x-y|} & 0 \end{pmatrix}.$$

Then indeed $\|[D, M]\| \leq 1$ and $|\phi_{z_1}(M) - \phi_{z_2}(M)| = 2\frac{|z_1 - z_2|}{1+|z_1|^2}|x-y|^{-1}$. We conclude

$$d(\phi_{z_1}, \phi_{z_2}) = 2\frac{|z_1 - z_2|}{1+|z_1|^2}|x-y|^{-1}.$$

Let us now compute the distance between the state ϕ_1 and the state 'at infinity': $\phi_{[1:0]}$. If $M'_N \in M_2(\mathbb{C})$ is the matrix

$$M'_N = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix},$$

then according to (4.7), $\|[D, M'_N]\| = 0$, for every N. Also

$$|\phi_{[1:0]}(M'_N) - \phi_{[0:1]}(M'_N)| = \left|\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, M\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle - \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, M\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\rangle\right| = 2N \xrightarrow{N\to\infty} \infty,$$

so that $d(\phi_{[1:0]}, \phi_{[0:1]}) = \infty$. We have now completely determined the distance the spectral triple (4.5) defines on $\mathbb{CP}^1$. The map

$$\begin{aligned} &\mathbb{CP}^1 \xrightarrow{\sim} S^2 \subseteq \mathbb{C} \oplus \mathbb{R} \\ &[z:1] \mapsto \Big(2\frac{z}{1+|z|^2}, \frac{|z|^2 - 1}{|z|^2 + 1}\Big) \\ &[1:0] \mapsto (0, 1) \end{aligned} \tag{4.10}$$

is a diffeomorphism. We see that the distance the spectral triple (4.5) induces on S^2 via the map (4.10) is infinite between different latitude lines of S^2 and on latitude lines it is, up to the factor $|x-y|^{-1}$, given by the chord distance between the two points.

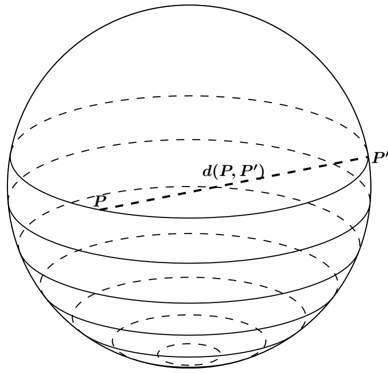

Figure 2: An illustration of the distance function that the formula (4.2) and the spectral triple (4.5) induce on the sphere. On each lattitude line the distance between points is given by the chord distance. The distance between lattitude lines is infinite.

In the special case of the canonical spectral triple for some compact Riemannian spin^c-manifold, we can guarantee the distance function (4.2) only takes finite values.

Proposition 4.7. Given a compact Riemannian spin^c-manifold (M, g), with corresponding Dirac operator D_M, the distance formula

$$d(\phi, \psi) = \sup_{f \in C^\infty(M)} \{|\phi(f) - \psi(f)| : \|[D_M, f]\| \leq 1\}$$

induces the weak-$*$ topology on $\mathcal{S}(C^\infty(M))$.

Proof. Let $\omega_n, \omega \in \mathcal{S}(C^\infty(M))$. We need to prove that

$$\lim_{n\to\infty} d(\omega_n, \omega) = 0 \iff \lim_{n\to\infty} \omega_n(f) - \omega(f) = 0 \text{ for all } f \in C^\infty(M).$$

Suppose we are given that $\lim_{n\to\infty} d(\omega_n, \omega) = 0$. Take $f \in C^\infty(M)$, which we may assume to be real-valued according to Remark 4.4. If $\|[D_M, f]\| = \|\operatorname{grad} f\|_\infty = 0$, then using Proposition 3.6, we conclude that $\|f\|_{Lip} = 0$, which implies that f is constant. So f is a multiple of the identity element in $C^\infty(M)$: $f = \lambda 1_{C^\infty(M)}, \lambda \in \mathbb{R}$. Now

$$\begin{aligned}|\omega_n(f) - \omega(f)| &= |\omega_n(\lambda 1_{C^\infty(M)}) - \omega(\lambda 1_{C^\infty(M)})| \\ &= |\lambda\omega_n(1_{C^\infty(M)}) - \lambda\omega(1_{C^\infty(M)})| \\ &= |\lambda 1 - \lambda 1| = 0.\end{aligned}$$

If $\|[D_M, f]\| \neq 0$, then $\|[D_M, \frac{f}{\|[D_M,f]\|}]\| = 1$, so

$$\begin{aligned}|\omega_n(f) - \omega(f)| &= \|[D_M, f]\| \Big|\omega_n\Big(\frac{f}{\|[D_M, f]\|}\Big) - \omega\Big(\frac{f}{\|[D_M, f]\|}\Big)\Big| \\ &\leq \|[D_M, f]\| d(\omega_n, \omega) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Conversely, suppose we know that $\lim_{n\to\infty} \omega_n(f) - \omega(f) = 0$ for all $f \in C^\infty(M)$. Since $C^\infty(M)$ is dense in $C(M)$, we can extend ω_n and ω to states on $C(M)$. We then still have that $\lim_{n\to\infty} \omega_n(f) - \omega(f) = 0$ for every $f \in C(M)$. Indeed, if $f \in C(M)$, and $g_n \in C^\infty(M)$ is a sequence of functions converging to f and $\epsilon > 0$ is arbitrary, then we can find N and M natural numbers such that $n \geq N \implies \|f - g_n\| < \frac{\epsilon}{3}$ and $m \geq M \implies |\omega_m(g_N) - \omega(g_N)| < \frac{\epsilon}{3}$. Then we see that for $m \geq M$ we have

$$\begin{aligned}|\omega_m(f) - \omega(f)| &\leq |\omega_m(f) - \omega_m(g_N)| + |\omega_m(g_N) - \omega(g_N)| + |\omega(g_N) - \omega(f)| \\ &\leq 2\|g_N - f\| + |\omega_m(g_N) - \omega(g_N)| < \epsilon.\end{aligned}$$

We now argue by contradiction. Suppose that $\lim_{n\to\infty} d(\omega_n, \omega) \neq 0$. Then again using Remark 4.4, we can find an $\epsilon > 0$ and a sequence of real-valued functions $f_n \in C^\infty(M)$, $\|[D_M, f_n]\| \leq 1$, such that

$$|\omega_n(f_n) - \omega(f_n)| > \epsilon.$$

Since adding a multiple of the identity function to f_n does not change the above expression, or the value $\|[D_M, f_n]\|$, we may pick a point $p_0 \in M$ and assume $f_n(p_0) = 0$ for all n. As all the f_n are real-valued and smooth, we can apply Proposition 3.6 to conclude $\|f_n\|_{Lip} = \|[D_M, f_n]\| \leq 1$. This implies the family $\{f_n\}_n$ is an equicontinuous family. Because M is compact, the conditions $f_n(p_0) = 0$ and $\|f\|_{Lip} \leq 1$ imply that the family $\{f_n\}_n$ is uniformly bounded. So we can apply the Arzelá–Ascoli Theorem, which states that $\{f_n\}_n$ has a convergent subsequence (in $C(M)$). We may therefore switch to a convergent subsequence $\{f_n\}_n$, converging to some $f \in C(M)$. Now choose N large enough so that $n \geq N \implies \|f_n - f\| < \frac{\epsilon}{4}$. Then for $n \geq N$ we have

$$\begin{aligned}
|\omega_n(f) - \omega(f)| &= |\omega_n(f) - \omega_n(f_n) + \omega_n(f_n) - \omega(f_n) + \omega(f_n) - \omega(f)| \\
&\geq \Big| |\omega_n(f_n) - \omega(f_n)| - |\omega_n(f) - \omega_n(f_n) + \omega(f_n) - \omega(f)| \Big| \\
&\geq \frac{\epsilon}{2}
\end{aligned}$$

contradicting the assumption that $|\omega_n(f) - \omega(f)| \xrightarrow{n\to\infty} 0$. ☐

As the dual space of any normed vector space is compact in the weak-* topology by the Banach-Alaoglu Theorem, $d(\cdot, \cdot)$, can only take finite values on $\mathcal{S}(C^\infty(M))$.

4.2 Operator system spectral triples

For the purpose of this text we extend the definition of a spectral triple.

Definition 4.8. An *operator system spectral triple* is a triple $(\mathcal{E}, \mathcal{H}, D)$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{E}$ is a $*$-closed dense subspace of an operator system $E \subseteq B(\mathcal{H})$, such that $1_{B(\mathcal{H})} \in \mathcal{E}$ and D is an essentially selfadjoint operator on $\mathcal{H}$ with compact resolvent and such that $[D, a] \in B(\mathcal{H})$ for each $a \in \mathcal{E}$.

Proposition 4.9. Suppose we are given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an orthogonal projection $\mathcal{Q} \in B(\mathcal{H})$ that commutes with D. Then $(\mathcal{Q}\mathcal{A}\mathcal{Q}, \mathcal{Q}\mathcal{H}, \mathcal{Q}D\mathcal{Q})$ is an operator system spectral triple.

Proof. Since $\mathcal{A}$ is a dense subspace of a C^*-algebra A, the linear space $\mathcal{Q}\mathcal{A}\mathcal{Q}$ is a dense subspace of $\mathcal{Q}A\mathcal{Q}$, which is an operator system. Furthermore $\mathcal{Q}$ is the identity operator on $\mathcal{Q}\mathcal{H}$ and $(\mathcal{Q}a\mathcal{Q})^* = \mathcal{Q}a^*\mathcal{Q}$, so $\mathcal{Q}\mathcal{A}\mathcal{Q}$ is $*$-closed. To show the operator $\mathcal{Q}D\mathcal{Q}$ has compact resolvent, we need to show $(i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1}$ is bounded as an operator on $\mathcal{Q}\mathcal{H}$. To show this, we notice first of all that we have the equality of operators on $\mathcal{Q}\mathcal{H}$ ([9, Page 113])

$$\begin{aligned}
(i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})\mathcal{Q}(i + D)^{-1}\mathcal{Q} &= \mathcal{Q}(i + D)\mathcal{Q}(i + D)^{-1}\mathcal{Q} - \mathcal{Q}(i + D)(i + D)^{-1}\mathcal{Q} + \mathcal{Q} \\
&= \mathcal{Q}[i + D, \mathcal{Q}](i + D)^{-1}\mathcal{Q}.
\end{aligned}$$

If we now multiply with the term $(i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1}$ on the left we obtain the equality

$$\mathcal{Q}(i + D)^{-1}\mathcal{Q} = (i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1}\mathcal{Q}[i + D, \mathcal{Q}](i + D)^{-1}\mathcal{Q} + (i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1},$$

again as operators on $\mathcal{QH}$. So we obtain the expression

$$(i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1} = \mathcal{Q}(i+D)^{-1}\mathcal{Q} - (i\mathcal{Q} + \mathcal{Q}D\mathcal{Q})^{-1}\mathcal{Q}[i+D,\mathcal{Q}](i+D)^{-1}\mathcal{Q}.$$

The left hand side is compact, as we know that $(i+D)^{-1}$ is compact, showing that $\mathcal{Q}D\mathcal{Q}$ has compact resolvent.

Let $\mathcal{D}(D)$ denote the domain of D. In order to show that $\mathcal{Q}D\mathcal{Q}$ is essentially self-adjoint, we need to check it is densily defined, symmetric and that $(\mathcal{Q}D\mathcal{Q})^{**}$ is self-adjoint, (with domain $\mathcal{Q}\mathcal{D}(D)$). As $\mathcal{D}(D)$ is dense in $\mathcal{H}$, it is clear that $\mathcal{Q}\mathcal{D}(D) \subseteq \mathcal{QH}$ is dense. That $\mathcal{Q}D\mathcal{Q}$ is symmetric follows from the fact that D is symmetric and that $\mathcal{Q}$ is an orthogonal projection. Furthermore we have that

$$(\mathcal{Q}D\mathcal{Q})^{**} = (\mathcal{Q}D^*\mathcal{Q})^* = \mathcal{Q}D^{**}\mathcal{Q},$$

which follows from [21, Theorem 13.2]. This shows $(\mathcal{Q}D\mathcal{Q})^{**}$ is self-adjoint, as $\mathcal{Q}$ is an orthogonal projection and D^{**} is self-adjoint by assumption. Lastly, since $\mathcal{Q}$ commutes with D, the commutator $[\mathcal{Q}D\mathcal{Q}, \mathcal{Q}a\mathcal{Q}] = \mathcal{Q}[D,a]\mathcal{Q}$ is bounded. □

In particular, if we choose $\mathcal{Q} = 1_{B(\mathcal{H})}$, we see that every spectral triple is an example of an operator system spectral triple. We say the operator system spectral triple $(\mathcal{QAQ}, \mathcal{QH}, \mathcal{Q}D\mathcal{Q})$ is a *truncation* of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Even now, formula (4.2) makes sense, and in this way we obtain a distance function on the state space $\mathcal{S}(\mathcal{E})$:

$$d_{\mathcal{E}}(\phi,\psi) = \sup_{a\in\mathcal{E}}\{|\phi(a)-\psi(a)| : \|[D,a]\| \leq 1\}, \qquad \phi,\psi \in \mathcal{S}(\mathcal{E}). \tag{4.11}$$

If (4.11) induces the weak-$*$ topology on $\mathcal{S}(\mathcal{E})$, then the operator system spectral triple $(\mathcal{E}, \mathcal{H}, D)$ is a quantum metric space as defined by Rieffel in [7]. The Lip-norm is then given by $L(a) = \|[D,a]\|$. Requiring that (4.11) induces the weak-$*$ topology on $\mathcal{S}(\mathcal{E})$ would exclude examples like 4.6. Proposition 4.10 and 4.16 below state examples of operator system spectral triples that are also quantum metric spaces. The operator system spectral triples we consider in the final section are of this form.

Proposition 4.10. Suppose $(\mathcal{E}, \mathcal{H}, D)$ is an operator system spectral triple with finite dimensional operator system $\mathcal{E}$. Then the distance formula (4.11) induces the weak-$*$ topology on $\mathcal{S}(\mathcal{E})$ if and only if

$$[D,a] = 0 \text{ if and only if } a \in \mathbb{C}\cdot 1_{\mathcal{E}}. \tag{4.12}$$

Proof. See [15, Proposition 3.1], and [16, Proposition 4.2]. □

4.3 Gromov–Hausdorff distance

Now that we have some idea of what the distance induced by Connes' distance formula looks like, let us try to compare two operator system spectral triples. The appropriate notion of the distance between operator system spectral triples relies on the notion of Gromov–Hausdorff distance between metric spaces.

Let (X,d) be a metric space and let $C \subseteq X$. For $\epsilon > 0$ we define the ϵ-neighbourhood of C by

$$\mathcal{N}_\epsilon(C) = \{x \in X \mid \text{ there is } y \in C \text{ such that } d(x,y) < \epsilon\}.$$

Definition 4.11. Let (X, d) be a metric space and let $C, D \subseteq X$ be closed subspaces. The *Hausdorff distance* between C and D is defined by

$$\mathrm{dist}_H(C, D) = \inf\{\epsilon > 0 \mid C \subseteq \mathcal{N}_\epsilon(D) \text{ and } D \subseteq \mathcal{N}_\epsilon(C)\}.$$

If we moreover require X to be compact, then $\mathrm{dist}_H(C, D)$ is guaranteed to be finite.

Definition 4.12. Given two metric spaces (X, d_X) and (Y, d_Y), we define the *Gromov–Hausdorff distance* between these spaces to be

$$\mathrm{dist}_{GH} = \inf\left\{\mathrm{dist}_H(f(X), g(Y)) \mid {}^{f\colon X\to Z,\, g\colon Y\to Z}_{\text{isometric imbeddings for some metric space } (Z, d_Z)}\right\}.$$

We are now ready to define Gromov–Hausdorff distance between operator system spectral triples. The definition is inspired by the notion of quantum Gromov–Hausdorff distance between quantum metric spaces as defined by Rieffel in [7].

Definition 4.13. Suppose $\mathcal{O}_1 = (\mathcal{E}_1, \mathcal{H}_1, D_1)$ and $\mathcal{O}_2 = (\mathcal{E}_2, \mathcal{H}_2, D_2)$ are operator system spectral triples, then we define the *Gromov–Hausdorff* distance between them by

$$\mathrm{dist}^o_{GH}(\mathcal{O}_1, \mathcal{O}_2) = \mathrm{dist}_{GH}\left((\mathcal{S}(\mathcal{E}_1), d_{\mathcal{E}_1}), (\mathcal{S}(\mathcal{E}_2), d_{\mathcal{E}_2})\right).$$

Here $d_{\mathcal{E}_1}$ and $d_{\mathcal{E}_2}$ are as in (4.11). We use the notation dist^o_{GH} to distinguish between Gromov–Hausdorff distance between operator system spectral triples and the usual notion of Gromov–Hausdorff distance between metric spaces.

Example 4.14. If we denote

$$\mathcal{O}_t = \left(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\right)$$

the spectral triple from Example 4.5, and d_t the distance induced by Connes' distance formula on $\mathcal{S}(\mathbb{C} \oplus \mathbb{C})$, then we saw that

$$(\mathcal{S}(\mathbb{C} \oplus \mathbb{C}), d_t) \cong \left[0, \frac{1}{|t|}\right] \subseteq \mathbb{R}$$

as metric spaces. Suppose that $t_1 \neq t_2$, then we can embed both $\mathcal{O}_{t_1}$ and $\mathcal{O}_{t_2}$ isometrically into the interval $\left[0, \max\left\{\frac{1}{|t_1|}, \frac{1}{|t_2|}\right\}\right]$, using the map $t \mapsto t$. We then see that $\mathrm{dist}^o_{GH}(\mathcal{O}_{t_1}, \mathcal{O}_{t_2}) \leq \left|\frac{1}{|t_1|} - \frac{1}{|t_2|}\right|$. In particular, if t_1 converges to t_2, then $\mathcal{O}_{t_1}$ converges to $\mathcal{O}_{t_2}$ in Gromov–Hausdorff distance.

There is a more generic way to compute the distance between operator system spectral triples ([7]).

Definition 4.15. Let $\mathcal{O}_1 = (\mathcal{E}_1, \mathcal{H}_1, D_1)$ and $\mathcal{O}_2 = (\mathcal{E}_2, \mathcal{H}_2, D_2)$ be operator system spectral triples. A *weak bridge* between $\mathcal{O}_1$ and $\mathcal{O}_2$ is a seminorm $\mathcal{B}$ on $\mathcal{E}_1 \oplus \mathcal{E}_2$ that satisfies the following properties:

1. For any $a_1 \in \mathcal{E}_1$, there exists an $a_1^\dagger \in \mathcal{E}_2$ such that

$$\left\|\left[D_2, a_1^\dagger\right]\right\|, \mathcal{B}(a_1, a_1^\dagger) \leq \|[D_1, a_1]\|.$$

2. For any $a_2 \in \mathcal{E}_2$ there exists an $a_2^\dagger \in \mathcal{E}_1$ such that

$$\left\|\left[D_1, a_2^\dagger\right]\right\|, \mathcal{B}(a_2^\dagger, a_2) \leq \|[D_2, a_2]\|.$$

In the above definition we view $\mathcal{E}_1 \oplus \mathcal{E}_2$ as a subspace of $\mathcal{A}_1 \oplus \mathcal{A}_2$, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are subspaces of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively.

Once again, our definition of a weak bridge between operator system spectral triples is modelled on the definition of a bridge between quantum metric spaces ([7, Section 5]).

We have the natural projections

$$\begin{tikzcd} & \mathcal{E}_1 \oplus \mathcal{E}_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{E}_1 & & \mathcal{E}_2 \end{tikzcd} .$$

These projections induce maps

$$\begin{tikzcd} & \mathcal{S}(\mathcal{E}_1 \oplus \mathcal{E}_2) & \\ \mathcal{S}(\mathcal{E}_1) \arrow[ur, "\mathcal{S}(\pi_1)"] & & \mathcal{S}(\mathcal{E}_2) \arrow[ul, "\mathcal{S}(\pi_2)"'] \end{tikzcd} ,$$

defined by $\mathcal{S}(\pi_1)(\phi)(a_1, a_2) = \phi(a_1)$ and $\mathcal{S}(\pi_2)(\psi)(a_1, a_2) = \psi(a_2)$. Clearly $\mathcal{S}(\pi_1)$ and $\mathcal{S}(\pi_2)$ are injections.

Proposition 4.16. Let $\mathcal{B}$ be a weak bridge between operator system spectral triples $\mathcal{O}_1 = (\mathcal{E}_1, \mathcal{H}_1, D_1)$ and $\mathcal{O}_2 = (\mathcal{E}_2, \mathcal{H}_2, D_2)$. Then, if we equip $\mathcal{S}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ with the metric

$$d_{\mathcal{B}}(\phi, \psi) = \sup_{(a_1,a_2)\in\mathcal{E}_1\oplus\mathcal{E}_2} \{|\phi(a_1, a_2) - \psi(a_1, a_2)| : |[D_1, a_1]\|, \|[D_2, a_2]\|, \mathcal{B}(a_1, a_2) \leq 1\}, \tag{4.13}$$

and $\mathcal{S}(\mathcal{E}_1)$ and $\mathcal{S}(\mathcal{E}_2)$ with the metric given by Connes' distance formula, then the maps

$$\begin{tikzcd} & \mathcal{S}(\mathcal{E}_1 \oplus \mathcal{E}_2) & \\ \mathcal{S}(\mathcal{E}_1) \arrow[ur, "\mathcal{S}(\pi_1)"] & & \mathcal{S}(\mathcal{E}_2) \arrow[ul, "\mathcal{S}(\pi_2)"'] \end{tikzcd}$$

are isometric embeddings.

Proof. Let $\phi, \psi \in \mathcal{S}(\mathcal{E}_1)$. We need to show that

$$d_{\mathcal{B}}\big(\mathcal{S}(\pi_1)\phi, \mathcal{S}(\pi_1)\psi\big) = d_{\mathcal{E}_1}(\phi, \psi).$$

Let us prove the two inequalities. First of all

$$\begin{aligned} d_{\mathcal{E}_1}(\phi, \psi) &= \sup_{a_1\in\mathcal{E}_1} \{|\phi(a_1) - \psi(a_1)| : \|[D_1, a_1]\| \leq 1\} \\ &\geq \sup_{(a_1,a_2)\in\mathcal{E}_1\oplus\mathcal{E}_2} \{|\phi(a_1) - \psi(a_1)| : \|[D_1, a_1]\|, \|[D_2, a_2]\|, \mathcal{B}(a_1, a_2) \leq 1\} \\ &= \sup_{(a_1,a_2)\in\mathcal{E}_1\oplus\mathcal{E}_2} \{|\mathcal{S}(\pi_1)\phi(a_1, a_2) - \mathcal{S}(\pi_1)\psi(a_1, a_2)| : \|[D_1, a_1]\|, \|[D_2, a_2]\|, \mathcal{B}(a_1, a_2) \leq 1\} \\ &= d_{\mathcal{B}}\big(\mathcal{S}(\pi_1)\phi, \mathcal{S}(\pi_1)\psi\big). \end{aligned}$$

For the converse inequality we use property 1 of Definition 4.15. For each $a_1 \in \mathcal{E}_1$ we can find an $a_1^\dagger \in \mathcal{E}_2$ such that

$$\left\|\left[D_2, a_1^\dagger\right]\right\|, \mathcal{B}(a_1, a_1^\dagger) \leq \|[D_1, a_1]\|.$$

Now we see that

$$\begin{aligned}
d_{\mathcal{E}_1}(\phi,\psi) &= \sup_{a_1\in\mathcal{E}_1}\{|\phi(a_1)-\psi(a_1)| : \|[D_1,a_1]\|\le 1\}\\
&= \sup_{a_1\in\mathcal{E}_1}\{|\phi(a_1)-\psi(a_1)| : \|[D_1,a_1]\|, \|[D_2,a_1^\dagger]\|, \mathcal{B}(a_1,a_1^\dagger)\le 1\}\\
&\le \sup_{(a_1,a_2)\in\mathcal{E}_1\oplus\mathcal{E}_2}\{|\phi(a_1)-\psi(a_1)| : \|[D_1,a_1]\|, \|[D_2,a_2]\|, \mathcal{B}(a_1,a_2)\le 1\}\\
&= \sup_{(a_1,a_2)\in\mathcal{E}_1\oplus\mathcal{E}_2}\{|\mathcal{S}(\pi_1)\phi(a_1,a_2)-\mathcal{S}(\pi_1)\psi(a_1,a_2)| : \|[D_1,a_1]\|, \|[D_2,a_2]\|, \mathcal{B}(a_1,a_2)\le 1\}\\
&= d_{\mathcal{B}}\big(\mathcal{S}(\pi_1)\phi, \mathcal{S}(\pi_1)\psi\big).
\end{aligned}$$

That $\mathcal{S}(\pi_2)$ is an isometry can be proven in a completely analogous way, this time using property 2. of Definition 4.15. □

Thus every weak bridge $\mathcal{B}$ between operator system spectral triples provides us with isometric embeddings

$$\begin{tikzcd}
& \mathcal{S}(\mathcal{E}_1\oplus\mathcal{E}_2) & \\
\mathcal{S}(\mathcal{E}_1) \arrow[ur, "\mathcal{S}(\pi_1)"] & & \mathcal{S}(\mathcal{E}_2) \arrow[ul, "\mathcal{S}(\pi_2)"']
\end{tikzcd}.$$

The next thing we need to do, in order to obtain an upper bound for $\mathrm{dist}^o_{GH}(\mathcal{O}_1,\mathcal{O}_2)$, is to compute the distance

$$\mathrm{dist}_H\left(\mathcal{S}(\pi_1)\big(\mathcal{S}(\mathcal{E}_1)\big), \mathcal{S}(\pi_2)\big(\mathcal{S}(\mathcal{E}_2)\big)\right).$$

Upon choosing the weak bridge $\mathcal{B}$ appropriately, we hope to arise at a good estimate for this distance, and accordingly for $\mathrm{dist}^o_{GH}(\mathcal{O}_1,\mathcal{O}_2)$ as well.

5 Truncated geometry

Those little quarrels that tore us apart
Oh, gee, I can see they were wrong from the start
But now that you've come back
My dream of life is here to stay

Billie Holiday, *Dream of life*

In the previous section we have seen the definition of an operator system spectral triple. In particular we saw that if we are given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an orthogonal projection $\mathcal{Q}$ on $\mathcal{H}$, which commutes with D, then $(\mathcal{QAQ}, \mathcal{QH}, \mathcal{Q}D\mathcal{Q})$ is an operator system spectral triple (Proposition 4.9). Also we developed the notion of Gromov–Hausdorff distance between operator system spectral triples. We could therefore ask ourselves what the Gromov–Hausdorff distance is between $(\mathcal{A}, \mathcal{H}, D)$ and the truncated spectral triple $(\mathcal{QAQ}, \mathcal{QH}, \mathcal{Q}D\mathcal{Q})$. Pushing this further, if $\{\mathcal{Q}_N\}_N$ is some sequence of spectral projections associated to D, that converges to $1_{\mathcal{H}}$ in the strong operator topology, the natural question to ask is:

Does $(\mathcal{Q}_N\mathcal{A}\mathcal{Q}_N, \mathcal{Q}_N\mathcal{H}, \mathcal{Q}_N D\mathcal{Q}_N)$ converge to $(\mathcal{A}, \mathcal{H}, D)$ in Gromov–Hausdorff distance?

In this section we answer this question affirmatively in the case the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the canonical spectral triple associated to the d-dimensional torus $\mathbb{T}^d$ and $\{\mathcal{Q}_N\}_N$ is the sequence of *rectangular* spectral projections associated to $D_{\mathbb{T}^d}$ (defined in Definition 5.1 below).

In Section 5.3 we define maps between the operator system spectral triples $(\mathcal{Q}_N\mathcal{A}\mathcal{Q}_N, \mathcal{Q}_N\mathcal{H}, \mathcal{Q}_N D\mathcal{Q}_N)$ and $(\mathcal{A}, \mathcal{H}, D)$. In Section 5.4 we use these maps to build a weak bridge between the two spaces. Both constructions rely heavily on ideas from work in progress by Walter van Suijlekom and Alain Connes ([22]).

5.1 The torus

Let us commence by determining the canonical spectral triple associated to $\mathbb{T}^d$ explicitly.

The algebra: The algebra is given by $\mathcal{A} = C^\infty(\mathbb{T}^d)$. We have an inclusion

$$\bigotimes_{j=1}^{d} C^\infty(S^1) \subseteq C^\infty(\mathbb{T}^d),$$

given by $(f_1 \otimes \cdots \otimes f_d)(\theta_1, \ldots, \theta_d) = f_1(\theta_1)f_2(\theta_2)\ldots f_d(\theta_d)$. In fact, $\bigotimes_{j=1}^{d} C^\infty(S^1)$ is a dense subset of $C^\infty(\mathbb{T}^d)$, with respect to the supremum norm on $C^\infty(\mathbb{T}^d)$. This can be seen in the following way. Using Fourier theory we know we can write every function $f \in C^\infty(\mathbb{T}^d)$ as a series

$$f(\theta) = \sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot\theta}, \qquad e^{in\cdot\theta} = e^{i(n_1\theta_1+\cdots+n_d\theta_d)},$$

such that the coefficients a_n fall off quicker than any polynomial. In particular, for any

$\epsilon > 0$, we can find an $N \in \mathbb{N}$ such that

$$\left\| f - \sum_{\substack{n\in\mathbb{Z}^d \\ |n_1|,\dots,|n_d|\le N}} a_n e^{in\cdot\theta} \right\| < \epsilon,$$

proving the assertion.

The Hilbert space: The manifold $\mathbb{T}^d$ has global coordinates $(\theta^1, \dots, \theta^d) \in [-\pi, \pi]^d$ and has trivializable tangent space $T\mathbb{T}^d$ with global frame $\{\frac{\partial}{\partial\theta^\mu}\}_{\mu=1}^d$. We equip the tangent space with the flat metric:

$$g\Big(\frac{\partial}{\partial\theta^\mu}, \frac{\partial}{\partial\theta^\nu}\Big) = \delta^{\mu\nu}. \tag{5.1}$$

Therefore also the cotangent space $T^*\mathbb{T}^d$ is trivializable and has global, orthonormal frame $\{d\theta^\mu\}_{\mu=1}^d$, with respect to the metric g^{-1}:

$$g^{-1}(d\theta^\mu, d\theta^\nu) = \delta^{\mu\nu}. \tag{5.2}$$

Using (3.14) we see that $\mathbb{C}l(\mathbb{T}^d) \cong \mathbb{T}^d \times M_{2^m}(\mathbb{C})$, where $d = 2m$ or $d = 2m+1$. Thus the spinor bundle S is given by the trivial bundle of dimension 2^m:

$$S \cong \mathbb{T}^d \times \mathbb{C}^{2^m}. \tag{5.3}$$

It follows that the spinor module is given by $\Gamma^\infty(S) = C^\infty(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$ and so the Hilbert space of square integrable spinors is given by

$$L^2(\mathbb{T}^d) \cong L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}. \tag{5.4}$$

In this section we will frequently identify

$$L^2(\mathbb{T}^d) \cong \bigotimes_{\mu=1}^d L^2(\mathbb{T}^1). \tag{5.5}$$

This isomorphism of Hilbert spaces follows just because $\mathbb{T}^d = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$ (d times).

The Dirac operator: As $T\mathbb{T}^d$ is trivializable, the Levi-Civita connection ∇^g is just given by the exterior derivative d. A connection ∇^S on S that satisfies the Leibniz rule (3.18) is given by

$$\nabla^S(f \otimes s) = df \otimes s. \tag{5.6}$$

Now using (3.20) and $c(d\theta^\mu) = \gamma^\mu$ as in (3.16) we can compute the Dirac operator:

$$\begin{aligned}
D_{\mathbb{T}^d}(f \otimes s) &= -i\big(\hat{c} \circ \nabla^S\big)(f \otimes s) \\
&= -i\big(\hat{c}(df \otimes s)\big) \\
&= -i\left(\hat{c}\left(\sum_{\mu=1}^d \partial_\mu f d\theta^\mu \otimes s\right)\right) \\
&= -i\left(\sum_{\mu=1}^d \partial_\mu f c(d\theta^\mu) s\right) \\
&= \sum_{\mu=1}^d -i\partial_\mu f \gamma^\mu(s).
\end{aligned}$$

Here ∂_μ denotes the operator $\frac{\partial}{\partial\theta^\mu}$. The Dirac operator $D_{\mathbb{T}^d}$ acting on $L^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^m}$ is thus given by

$$D_{\mathbb{T}^d} = \sum_{\mu=1}^{d} -i\partial_\mu \otimes \gamma^\mu. \tag{5.7}$$

With respect to the identification (5.5), the Dirac operator becomes

$$D_{\mathbb{T}^d} = \sum_{\mu=1}^{d} \Big(1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ \mu}}{-i\frac{d}{dx}} \otimes \cdots \otimes 1\Big) \otimes \gamma^\mu. \tag{5.8}$$

Here $\underset{\mu}{\uparrow}$ means that the term is on position μ.

We have thus established that the canonical spectral triple corresponding to $\mathbb{T}^d$ is given by

$$\left(C^\infty(\mathbb{T}^d), L^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^m}, \sum_{\mu=1}^{d} -i\partial_\mu \otimes \gamma^\mu\right).$$

5.2 The rectangularly truncated torus

We now introduce an increasing sequence of projections $\{\mathcal{Q}_N\}_N$ on $L^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^m}$. In Appendix D we compute the spectrum of $D_{\mathbb{T}^d}$. It is given by the set

$$\Big\{ \pm\sqrt{n_1^2+n_2^2+\cdots+n_d^2} \mid n \in \mathbb{Z}^d \Big\}.$$

Finding an increasing sequence of spectral projections is then equivalent to choosing an increasing sequence of finite subsets $K_N \subseteq \mathbb{Z}^d$, such that $\bigcup_N K_N = \mathbb{Z}^d$. Then we can define $\mathcal{Q}_N$ to be the orthogonal projection onto the eigenspaces corresponding to the eigenvalues

$$\Big\{ \pm\sqrt{n_1^2+n_2^2+\cdots+n_d^2} \mid n \in K_N \Big\}. \tag{5.9}$$

Given any spectral triple $(\mathcal{A},\mathcal{H},D)$, there is a canonical sequence of orthogonal projections given by $\chi_{[-N,N]}(D_{\mathbb{T}^d})$. Here $\chi_{[-N,N]}$ is the characteristic function on the interval $[-N,N]\subseteq\mathbb{R}$. In the case of the torus, this corresponds to the sequence $K_N^{\bigcirc} = \{n\in\mathbb{Z}^d : \|n\|_2 \le N\}$. However, choosing K_N in this way makes it very hard to reduce the higher-dimensional case to the 1-dimensional case. Therefore we consider the sequence of projections induced by (5.9) for the sets

$$K_N^{\square} = \{n\in\mathbb{Z}^d : |n_1|,\ldots,|n_d| \le N\}.$$

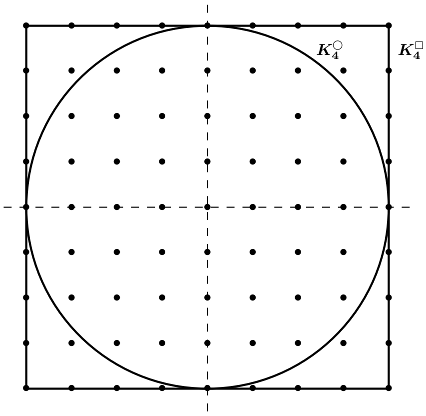

Figure 3: An illustration of the sets $K_N^{\bigcirc}, K_N^{\square} \subseteq \mathbb{Z}^2$. The points inside the circle correspond to the set $K_4^{\bigcirc}$ and all the points inside the square correspond to the set $K_4^{\square}$.

It is then clear that $\|D_N\| \leq \sqrt{d}N$. We can also give a more concrete definition of the spectral projections we obtain in this way.

Definition 5.1. Define the increasing sequence of projections $\{\mathcal{Q}_N\}_N$ on $L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$ by

$$\mathcal{Q}_N(f \otimes s) = Q_N(f) \otimes s, \tag{5.10}$$

where Q_N is the orthogonal projection in $B(L^2(\mathbb{T}^d))$ given by

$$Q_N\left(\sum_{n \in \mathbb{Z}^d} a_n e^{in\cdot\theta}\right) = \sum_{\substack{n \in \mathbb{Z}^d \\ |n_1|,\dots,|n_d| \leq N}} a_n e^{in\cdot\theta}. \tag{5.11}$$

In other words, $\mathcal{Q}_N = Q_N \otimes 1_{2^m}$ as acting on $L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$. We can decompose Q_N as well, using the identification (5.5). Indeed, if $P_N \in B(L^2(\mathbb{T}^1))$ denotes the orthogonal projection given by

$$P_N\Big(\sum_{n \in \mathbb{Z}} a_n e^{in\theta}\Big) = \sum_{n=-N}^{N} a_n e^{in\theta}, \tag{5.12}$$

then $Q_N = P_N^{\otimes d} = P_N \otimes \dots P_N$, as in appendix C.

From now on we will write

$$\mathcal{A}^d = C^\infty(\mathbb{T}^d), \qquad \mathcal{A}_N^d = \mathcal{Q}_N C^\infty(\mathbb{T}^d)\mathcal{Q}_N.$$

Definition 5.2. Define the map $R^d \colon \mathcal{A}^d \to \mathcal{A}_N^d$ by

$$R^d(f \otimes 1_{2^m}) = \mathcal{Q}_N(f \otimes 1_{2^m})\mathcal{Q}_N, \tag{5.13}$$

the canonical projection onto the truncated algebra.

The map R^d depends on N. However, we omit to stress this, as it would lead to very heavy notation. When we mention the map R^d it is always understood that we have fixed some $N \in \mathbb{N}_{\geq 1}$ beforehand.

Definition 5.3. Let $D_{\mathbb{T}^d}$ be the Dirac operator acting on $L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$, as in (5.7). We write $D_{\mathbb{T}^d}^N = \mathcal{Q}_N D_{\mathbb{T}^d} \mathcal{Q}_N$. By the *rectangularly truncated torus* we mean the operator system spectral triple (Proposition 4.9)

$$\big(\mathcal{A}_N^d, \mathcal{Q}_N\big(L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}\big), D_{\mathbb{T}^d}^N\big). \tag{5.14}$$

Notice that, because $\mathcal{Q}_N = Q_N \otimes 1_{2^m}$, we have the equality $\mathcal{Q}_N\big(L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}\big) = \big(Q_N L^2(\mathbb{T}^d)\big) \otimes \mathbb{C}^{2^m}$. Furthermore we have that

$$\begin{aligned} B(L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}) &\cong B(L^2(\mathbb{T}^d)) \otimes M_{2^m}(\mathbb{C}), \\ B(Q_N L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}) &\cong B(Q_N L^2(\mathbb{T}^d)) \otimes M_{2^m}(\mathbb{C}). \end{aligned}$$

The algebra $\mathcal{A}^d$ is represented on $L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$ by

$$\begin{aligned} \pi \colon \mathcal{A}^d &\to B(L^2(\mathbb{T}^d)) \otimes M_{2^m}(\mathbb{C}) \\ f &\mapsto f \otimes 1_{2^m}, \end{aligned} \tag{5.15}$$

where $f \in B(L^2(\mathbb{T}^d))$ denotes pointwise multiplication with the function f. Therefore we can view

$$\begin{aligned} \mathcal{A}^d &\subseteq B(L^2(\mathbb{T}^d)) \otimes M_{2^m}(\mathbb{C}), \\ \mathcal{A}^d_N &\subseteq B(Q_N L^2(\mathbb{T}^d)) \otimes M_{2^m}(\mathbb{C}), \end{aligned}$$

given by

$$\begin{aligned} \mathcal{A}^d &= \{f \otimes 1_{2^m} \mid f \in C^\infty(\mathbb{T}^d)\}, \\ \mathcal{A}^d_N &= \{T \otimes 1_{2^m} \mid T \in Q_N C^\infty(\mathbb{T}^d) Q_N\}, \end{aligned} \tag{5.16}$$

where this time $C^\infty(\mathbb{T}^d)$ and $Q_N C^\infty(\mathbb{T}^d) Q_N$ act on $L^2(\mathbb{T}^d)$ and $Q_N L^2(\mathbb{T}^d)$ respectively. We will identify the elements $f \in C^\infty(\mathbb{T}^d)$ and $f \otimes 1_{2^m} \in \mathcal{A}^d$, and the elements $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$ and $T \otimes 1_{2^m} \in \mathcal{A}^d_N$. It follows form Lemma 2.18 that

$$\begin{aligned} \|f \otimes 1_{2^m}\| &= \|f\|, \\ \|T \otimes 1_{2^m}\| &= \|T\|, \end{aligned}$$

for $f \in C^\infty(\mathbb{T}^d)$ and $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$.

5.3 Maps of operator systems

We already have a map $R^d \colon \mathcal{A}^d \to \mathcal{A}^d_N$, defined in Definition 5.2. We also want a map $\breve{R}^d \colon \mathcal{A}^d_N \to \mathcal{A}^d$ in the converse direction. Before we attempt to construct such a map in the general case of the d-dimensional torus $\mathbb{T}^d$, we first stay a little more down to Earth and we investigate the case $d = 1$ in more detail. The exposition of the one-dimensional case follows the lines of [22]. The general results will rely heavily on the the reduction to one dimension.

5.3.1 The circle

Recall that $P_N \in B(L^2(\mathbb{T}^1))$ denotes the orthogonal projection given by

$$P_N\Big(\sum_{n \in \mathbb{Z}} a_n e^{in\theta}\Big) = \sum_{n=-N}^{N} a_n e^{in\theta}.$$

An orthonormal basis for the Hilbert space $P_N L^2(\mathbb{T}^1)$ is given by the set $\{e_{-N}, e_{-N+1}, \ldots, e_N\}$. With respect to this basis, elements of $P_N C^\infty(\mathbb{T}^1) P_N$ are just matrices. They have a very specific form.

Proposition 5.4. For an element $f = \sum_{n\in\mathbb{Z}} a_n e_n$ in $C^\infty(\mathbb{T}^1)$ the corresponding element $T = P_N f P_N \in P_N C^\infty(\mathbb{T}^1) P_N$ can be written as the matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots & a_{-2N-1} \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots & a_{-2N} \\ a_2 & a_1 & a_0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_0 & a_{-1} \\ a_{2N+1} & a_{2N} & a_{2N-1} & \dots & a_1 & a_0 \end{pmatrix}. \tag{5.17}$$

Equivalently, $T_{mn} = a_{m-n}$. Furthermore $\left[-iP_N \frac{d}{dx} P_N, T\right]_{mn} = (m-n)a_{m-n}$.

Proof.

$$\begin{aligned} P_N f P_N e_n &= P_N f e_n \\ &= P_N \sum_{m\in\mathbb{Z}} a_m e^{im\theta} e_n \\ &= P_N \sum_{m\in\mathbb{Z}} a_m e^{i(m+n)\theta} \\ &= P_N \sum_{m\in\mathbb{Z}} a_{m-n} e^{in\theta} \\ &= \sum_{|m|\le N} a_{m-n} e^{in\theta}. \end{aligned}$$

So indeed $T_{mn} = a_{m-n}$. Furthermore

$$\left[-iP_N \frac{d}{dx} P_N, P_N f P_N\right] = P_N \left[-i\frac{d}{dx}, f\right] P_N = -iP_N f' P_N,$$

and f' corresponds to the Fourier series $\sum i n a_n e^{in\theta}$. So using the first result $T_{mn} = a_{m-n}$ we see that $\left[-iP_N \frac{d}{dx} P_N, T\right]_{mn} = (m-n)a_{m-n}$. ☐

Similarly as in Definition 5.2 we define the projection map $R\colon C^\infty(\mathbb{T}^1) \to P_N C^\infty(\mathbb{T}^1) P_N$ by

$$R(f) = P_N f P_N.$$

The natural action of $\mathbb{T}^1$ on $C^\infty(\mathbb{T}^1)$ is given by $\alpha_x f(\theta) = f(\theta - x)$. So

$$\alpha_x\big(\sum a_n e^{in\theta}\big) = \sum a_n e^{in(\theta-x)} = \sum a_n e^{-inx} e^{in\theta}.$$

Moreover we see that $\mathbb{T}^1$ acts on $P_N C^\infty(\mathbb{T}^1) P_N$ by

$$\alpha_x(T)_{mn} = e^{-i(m-n)x} T_{mn}. \tag{5.18}$$

From Proposition 5.4 it then follows that R commutes with α_x.

Definition 5.5. Define the vector $\psi \in P_N L^2(\mathbb{T}^1)$ by

$$\psi = \frac{1}{\sqrt{2N+1}}\big(e_{-N} + e_{-N+1} + \dots + e_N\big).$$

Then we define $\breve{R}\colon P_N C^\infty(\mathbb{T}^1) P_N \to C^\infty(\mathbb{T}^1)$ by

$$\breve{R}(T)(x) = \mathrm{Tr}(|\psi\rangle\langle\psi|\alpha_x(T)).$$

Again we omit the dependence $\breve{R}$ of N in the notation.

Proposition 5.6. We have the following equalities

$$\breve{R}(R(f))(x) = \sum_{n=-2N}^{2N} \Big(1 - \frac{|n|}{2N+1}\Big) a_n e^{inx} = (F_{2N+1} * f)(x), \tag{5.19}$$

$$R(\breve{R}(T)) = T - \frac{1}{2N+1} S(B(T)), \tag{5.20}$$

where F_{2N+1} denotes the Fejér kernel (see Appendix A), a_n denote the Fourier coefficients of f, and where $S, B \in L(P_N C^\infty(\mathbb{T}^1) P_N)$ are given by $B(T) = [-iP_N \frac{d}{dx}, T]$ and $S(T) = (T_{2N+1} - T^*_{2N+1}) \circ T$, i.e Schur multiplication with the matrix $T_{2N+1} - T^*_{2N+1}$, where T_{2N+1} is given by (B.1):

$$T_{mn} = \begin{cases} 1 \text{ if } n \leq m \\ 0 \text{ if } n > m \end{cases}, T_{2N+1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Proof. The operator $|\psi\rangle\langle\psi|$ is given by the matrix $\psi\psi^t$, which is the $(2N+1)\times(2N+1)$ matrix with every entry equal to $\frac{1}{2N+1}$. Therefore we have that

$$\begin{aligned} \breve{R}(R(f))(x) &= \mathrm{Tr}(\psi\psi^t \alpha_x(R(f))) \\ &= \sum_{n=-N}^{N} \big(\psi\psi^t \alpha_x(R(f))\big)_{nn} \\ &= \sum_{n=-N}^{N} \sum_{m=-N}^{N} \big(\psi\psi^t\big)_{nm} \alpha_x(R(f))_{mn} \\ &= \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{m=-N}^{N} \alpha_x(R(f))_{mn} \\ &= \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{m=-N}^{N} e^{-i(m-n)x} a_{m-n} \\ &= \frac{1}{2N+1} \sum_{n=-2N}^{2N} \big(2N+1-|n|\big) a_n e^{inx} \\ &= \sum_{n=-2N}^{2N} \Big(1 - \frac{|n|}{2N+1}\Big) a_n e^{inx}. \end{aligned}$$

Using Lemma A.3 we see that this shows that $\breve{R}(R(f))(x) = \big(F_{2N+1} * f\big)(x)$, proving (5.19).

For the proof of (5.20), suppose $T = P_N g P_N$, for some $g \in C^\infty(\mathbb{T}^1)$, given by the Fourier series $\sum b_n e^{in\theta}$. Then, using (5.19) and Proposition 5.4, we see

$$\begin{aligned}
\Big(R(\breve{R}(T))\Big)_{mn} &= \Big(1 - \frac{|m-n|}{2N+1}\Big) b_{m-n} \\
&= b_{m-n} - \frac{|m-n|}{2N+1} b_{m-n} \\
&= T_{mn} - \Big((T_{2N+1} - T^*_{2N+1}) \circ \Big(\frac{m-n}{2N+1} b_{m-n} \Big)_{mn} \Big)_{mn} \\
&= T_{mn} - \frac{1}{2N+1} \Big((T_{2N+1} - T^*_{2N+1}) \circ \Big[-iP_N \frac{d}{dx}, T \Big] \Big)_{mn},
\end{aligned}$$

proving that $R(\breve{R}(T)) = T - \frac{1}{2N+1} S(B(T))$. $\square$

Lemma 5.7. The maps R and $\breve{R}$ satisfy

$$\Big[-i\frac{d}{dx}, \breve{R}(T) \Big] = \breve{R}\Big(\Big[-iP_N \frac{d}{dx} P_N, T \Big] \Big), \tag{5.21}$$

$$\Big[-iP_N \frac{d}{dx} P_N, R(f) \Big] = R\Big(\Big[-i\frac{d}{dx}, f \Big] \Big). \tag{5.22}$$

Proof. Suppose that $T = P_N f P_N$, where f is given by the Fourier series $\sum a_n e^{in\theta}$. Then combining Proposition 5.4 and Proposition 5.6 and using that $[\frac{d}{dx}, \breve{R}(T)](x) = i\frac{d}{dx}\breve{R}(T)(x)$, we see

$$\Big[-i\frac{d}{dx}, \breve{R}(T) \Big](x) = \sum_{n=-2N}^{2N} \Big(1 - \frac{|n|}{2N+1}\Big) n a_n e^{inx} = \breve{R}\Big(\Big[-iP_N \frac{d}{dx} P_N, T \Big] \Big)(x),$$

which proves (5.21). For (5.22), notice that the operator $[-i\frac{d}{dx}, f]$ equals the multiplication operator $-if'$. The n'th Fourier coefficient of $-if'$ is given by na_n. Then, according to Proposition 5.6

$$\begin{aligned}
\Big[-iP_N \frac{d}{dx} P_N, R(f) \Big]_{mn} &= (m-n) R(f)_{mn} \\
&= (m-n) a_{m-n} \\
&= R\Big(\Big[-i\frac{d}{dx}, f \Big] \Big)_{mn},
\end{aligned}$$

proving that $\big[-iP_N \frac{d}{dx} P_N, R(f)\big] = R\big(\big[-i\frac{d}{dx}, f\big]\big)$. $\square$

5.3.2 The Torus

In this section we relate the map

$$R \colon C^\infty(\mathbb{T}^1) \to P_N C^\infty(\mathbb{T}^1) P_N$$

to the map R^d, as in Definition 5.2. Also we use the map

$$\breve{R} \colon P_N C^\infty(\mathbb{T}^1) P_N \to C^\infty(\mathbb{T}^1)$$

to construct a map

$$\breve{R}^d \colon \mathcal{A}^d_N \to \mathcal{A}^d,$$

so that eventually we have maps in both directions:

$$\mathcal{A}^d \underset{\breve{R}^d}{\overset{R^d}{\rightleftarrows}} \mathcal{A}_N^d \ . \tag{5.23}$$

Let us commence with relating the maps R and R^d. From the map $R\colon C^\infty(\mathbb{T}^1) \to P_N C^\infty(\mathbb{T}^1) P_N$, we construct the map

$$\begin{gathered} R^{\otimes d}\colon \bigotimes_{j=1}^{d} C^\infty(\mathbb{T}^1) \to Q_N \bigotimes_{j=1}^{d} \big(C^\infty(\mathbb{T}^1)\big)\, Q_N. \\ f_1 \otimes \cdots \otimes f_d \mapsto R(f_1) \otimes \cdots \otimes R(f_d), \end{gathered} \tag{5.24}$$

where we have identified

$$\bigotimes_{j=1}^{d} \big(P_N C^\infty(\mathbb{T}^1) P_N\big) \cong Q_N \bigotimes_{j=1}^{d} \big(C^\infty(\mathbb{T}^1)\big)\, Q_N.$$

More concretely $R^{\otimes d}(f) = Q_N f Q_N$, from which we deduce that $\|R^{\otimes d}\| \leq 1$. As $\bigotimes_{j=1}^{d} C^\infty(\mathbb{T}^1)$ is a dense subset of $C^\infty(\mathbb{T}^d)$ on which $R^{\otimes d}$ is bounded, it extends to the map

$$\begin{gathered} R^{\otimes d}\colon C^\infty(\mathbb{T}^d) \to Q_N C^\infty(\mathbb{T}^d) Q_N \\ f \mapsto Q_N f Q_N. \end{gathered} \tag{5.25}$$

Using (5.11) it is then clear that $R^d = R^{\otimes d} \otimes 1_{2^m}$. So

$$R^d(f \otimes 1_{2^m}) = R^{\otimes d}(f) \otimes 1_{2^m}.$$

From the map $\breve{R}\colon P_N C^\infty(\mathbb{T}^1) P_N \to C^\infty(\mathbb{T}^1)$, we construct the map

$$\begin{gathered} \breve{R}^{\otimes d}\colon \bigotimes_{j=1}^{d} \big(P_N C^\infty(\mathbb{T}^1) P_N\big) \to \bigotimes_{j=1}^{d} C^\infty(\mathbb{T}^1) \\ T_1 \otimes \cdots \otimes T_d \mapsto \breve{R}(T_1) \otimes \cdots \otimes \breve{R}(T_d), \end{gathered} \tag{5.26}$$

which we view as a map

$$\breve{R}^{\otimes d}\colon Q_N C^\infty(\mathbb{T}^d) Q_N \to C^\infty(\mathbb{T}^d). \tag{5.27}$$

Here we used that

$$\bigotimes_{j=1}^{d} \big(P_N C^\infty(\mathbb{T}^1) P_N\big) \cong Q_N \left(\bigotimes_{j=1}^{d} C^\infty(\mathbb{T}^1)\right) Q_N = Q_N C^\infty(\mathbb{T}^d) Q_N.$$

Definition 5.8. Using the map

$$\breve{R}^{\otimes d}\colon \ : Q_N C^\infty(\mathbb{T}^d) Q_N \to C^\infty(\mathbb{T}^d)$$

and using the identification (5.16), we define the map

$$\breve{R}^d\colon \mathcal{A}_N^d \to \mathcal{A}^d \tag{5.28}$$

by $\breve{R}^d = \breve{R}^{\otimes d} \otimes 1_{2^m}$, as in (2.3). So

$$\breve{R}^d(T \otimes 1_{2^m}) = \breve{R}^{\otimes d}(T) \otimes 1_{2^m}. \tag{5.29}$$

Lemma 5.9. The maps R^d and $\breve{R}^d$ as defined in Definitions 5.2 and 5.8 are contractions. That is

$$\begin{aligned} \|R^d\| &\leq 1 \\ \|\breve{R}^d\| &\leq 1. \end{aligned} \tag{5.30}$$

Proof. The strategy of the proof is to show first that the maps

$$C^\infty(\mathbb{T}^d) \underset{\breve{R}^{\otimes d}}{\overset{R^{\otimes d}}{\rightleftarrows}} Q_N C^\infty(\mathbb{T}^d) Q_N,$$

are contractions. We will then use this to show that also R^d and $\breve{R}^d$ contractive.

For now, let us view $R^{\otimes d}$ as a map $R^{\otimes d} \colon C^\infty(\mathbb{T}^d) \to B(L^2(\mathbb{T}^d))$. As we already remarked, $R^{\otimes d}$ is just given by

$$R^{\otimes d}(f) = Q_N f Q_N.$$

Since Q_N is an orthogonal projection, we know $\|Q_N\| \leq 1$ and so we have

$$\|R^{\otimes d}(f)\| = \|Q_N f Q_N\| \leq \|f\|, \tag{5.31}$$

for any $f \in C^\infty(\mathbb{T}^d)$. This shows $\|R^{\otimes d}\| \leq 1$. If $f \otimes 1_{2^m} \in \mathcal{A}^d$, then we apply Lemma 2.18 twice to see that

$$\begin{aligned} \|R^d(f \otimes 1_{2^m})\| &= \|R^{\otimes d}(f) \otimes 1_{2^m}\| \\ &= \|R^{\otimes d}(f)\| \\ &\leq \|f\| = \|f \otimes 1_{2^m}\|, \end{aligned}$$

proving that $\|R^d\|$ is a contraction as well.

To show that $\|\breve{R}^{\otimes d}\| \leq 1$, we first show that we have the identity

$$\breve{R}^{\otimes d}(T)(\theta) = \mathrm{Tr}\left(|\psi_d\rangle\langle\psi_d|\alpha_\theta(T)\right), \tag{5.32}$$

where $\psi_d = \psi \otimes \cdots \otimes \psi$ for ψ as in Definition 5.1. $\alpha \equiv \alpha^{\otimes d}$ denotes the action of $\mathbb{T}^d$ on $\mathcal{A}_N^d$. On pure tensors it is given by

$$\alpha_\theta(T_1 \otimes \cdots \otimes T_d) = \alpha_{\theta_1}(T_1) \otimes \cdots \otimes \alpha_{\theta_d}(T_d).$$

Now for pure tensors we have that

$$\begin{aligned} \breve{R}^{\otimes d}(T_1 \otimes \cdots \otimes T_d)(\theta) &= (\breve{R}T_1) \otimes \cdots \otimes (\breve{R}T_d)(\theta) \\ &= (\breve{R}T_1)(\theta_1) \ldots (\breve{R}T_d)(\theta_d) \\ &= \mathrm{Tr}\left(|\psi\rangle\langle\psi|\alpha_{\theta_1}(T_1)\right) \ldots \mathrm{Tr}\left(|\psi\rangle\langle\psi|\alpha_{\theta_d}(T_d)\right) \\ &= \mathrm{Tr}\left(|\psi\rangle\langle\psi|\alpha_{\theta_1}(T_1) \otimes \cdots \otimes |\psi\rangle\langle\psi|\alpha_{\theta_d}(T_d)\right) \\ &= \mathrm{Tr}\left(|\psi^d\rangle\langle\psi^d|\alpha_{\theta_1}(T_1) \otimes \cdots \otimes \alpha_{\theta_d}(T_d)\right) \\ &= \mathrm{Tr}\left(|\psi^d\rangle\langle\psi^d|\alpha_\theta(T_1 \otimes \cdots \otimes T_d)\right), \end{aligned}$$

so that by extending this equality linearly, we see that we have $\breve{R}^{\otimes d}(T)(\theta) = \mathrm{Tr}\left(|\psi_d\rangle\langle\psi_d|\alpha_\theta(T)\right)$ for all $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$. Then

$$\begin{aligned} \left|\breve{R}^{\otimes d}(T)(\theta)\right| &= \left|\mathrm{Tr}\left(|\psi_d\rangle\langle\psi_d|\alpha_\theta(T)\right)\right| \\ &= \left\||\psi_d\rangle\langle\psi_d|\alpha_\theta(T)\right\|_1 \\ &\leq \||\psi_d\rangle\langle\psi_d|\|_1 \|\alpha_\theta(T)\| \leq \|T\|. \end{aligned}$$

Here we used the Hölder inequality for Schatten operators: $\|AB\|_1 \leq \|A\|_1 \|B\|_\infty$(see [17, Chapter 3, Section 7]). If now $T \otimes 1_{2^m} \in \mathcal{A}_N^d$, then again it follows from applying Lemma 2.18 that

$$\begin{aligned}\|\breve{R}^d(T \otimes 1_{2^m})\| &= \|\breve{R}^{\otimes d}(T) \otimes 1_{2^m}\| \\ &= \|\breve{R}^{\otimes d}(T)\| \\ &\leq \|T\| = \|T \otimes 1_{2^m}\|,\end{aligned}$$

which shows that $\|\breve{R}^d\| \leq 1$. □

Proposition 5.10. The maps $\mathcal{A}^d \underset{\breve{R}^d}{\overset{R^d}{\rightleftarrows}} \mathcal{A}_N^d$ induce maps $\mathcal{S}(\mathcal{A}_N^d) \underset{\mathcal{S}(\breve{R}^d)}{\overset{\mathcal{S}(R^d)}{\rightleftarrows}} \mathcal{S}(\mathcal{A}^d)$, which are given by

$$\begin{aligned}\mathcal{S}(R^d)(\phi)(f) &= \phi(R^d(f)) \\ \mathcal{S}(\breve{R}^d)(\psi)(T) &= \psi(\breve{R}^d(T)).\end{aligned} \tag{5.33}$$

Proof. According to the definition of a state we need to check that

$$\begin{aligned}\|\mathcal{S}(R^d)(\phi)\| &= \mathcal{S}(R^d)(\phi)(1_{\mathcal{A}^d}) = 1 \\ \|\mathcal{S}(\breve{R}^d)(\psi)\| &= \mathcal{S}(\breve{R}^d)(\psi)(1_{\mathcal{A}_N^d}) = 1,\end{aligned}$$

for any $\phi \in \mathcal{S}(\mathcal{A}_N^d), \psi \in \mathcal{S}(\mathcal{A}^d)$. Note that if we can prove

$$\begin{aligned}\mathcal{S}(R^d)(\phi)(1_{\mathcal{A}^d}) &= 1 \\ \mathcal{S}(\breve{R}^d)(\psi)(1_{\mathcal{A}_N^d}) &= 1,\end{aligned} \tag{5.34}$$

we automatically have the inequalities $\|\mathcal{S}(R^d)(\phi)\|, \|\mathcal{S}(\breve{R}^d)(\psi)\| \geq 1$. Lemma 5.9 provides us with the reverse inequality. Therefore it remains to prove (5.34). Because ϕ and ψ are states, they satisfy $\phi(1_{\mathcal{A}_N^d}) = 1$ and $\psi(1_{\mathcal{A}^d}) = 1$. Since also $R^d = R^{\otimes d} \otimes 1_{2^m}$ and $\breve{R}^d = \breve{R}^{\otimes d} \otimes 1_{2^m}$, the proof of (5.34) reduces to showing that $R(1_{C^\infty(\mathbb{T}^1)}) = 1_{P_N C^\infty(\mathbb{T}^1) P_N}$ and $\breve{R}(1_{P_N C^\infty(\mathbb{T}^1) P_N}) = 1_{C^\infty(\mathbb{T}^1)}$. Clearly the unit of $C^\infty(\mathbb{T}^d)$ equals $\mathbb{1}$, the function with value 1 in every point, and $1_{P_N C^\infty(\mathbb{T}^1) P_N} = I_{2N+1}$, the $2N+1 \times 2N+1$ identity matrix. We simply calculate

$$\begin{aligned}\breve{R}(I_{2N+1})(x) &= \mathrm{Tr}(\psi\psi^* \alpha_x(I_{2N+1})) \\ &= \sum_{n=-N}^{N} \big(\psi\psi^* \alpha_x(I_{2N+1})\big)_{nn} \\ &= \sum_{n,m=-N}^{N} \big(\psi\psi^*\big)_{nm} \big(\alpha_x(I_{2N+1})\big)_{mn} \\ &= \sum_{n,m=-N}^{N} \frac{1}{2N+1} e^{-i(m-n)x} (I_{2N+1})_{mn} = 1,\end{aligned}$$

where we use that $\psi\psi^*$ is the $2N+1 \times 2N+1$ matrix with value $\frac{1}{2N+1}$ at each entry.

To show that $R(\mathbb{1}) = I_{2N+1}$, notice that the function $\mathbb{1}$ is given by the trivial Fourier series $\sum_n a_n e^{in\theta}$, with $a_0 = 1$ and $a_n = 0$ if $n \neq 0$. We use Proposition 5.4 to compute

$$R(\mathbb{1})_{mn} = a_{m-n} = \delta_{mn} = \big(I_{2N+1}\big)_{mn},$$

which shows $R(\mathbb{1}) = I_{2N+1}$. □

5.4 Building the bridge

We want to construct a weak bridge between $\left(\mathcal{A}^d_N, \mathcal{Q}_N\left(L^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^m}\right), D^N_{\mathbb{T}^d}\right)$ and $\left(\mathcal{A}^d, L^2(\mathbb{T}^d)\otimes\mathbb{C}^{2^m}, D_{\mathbb{T}^d}\right)$. We build this weak bridge using the maps R^d and $\breve{R}^d$, that we defined in the previous section. The construction is very similar to the one Rieffel performs in [7, Section 9]. The maps R^d and $\breve{R}^d$ should be nice in the sense they ought to satisfy the four lemmas that we will prove in this section ([22]). The lemmas guarantee that the bridge we construct satisfies the requirements 1 and 2 from Definition 4.15, so that we can apply Proposition 4.16. Also, we use the lemmas to show that we can bound

$$\mathrm{dist}_H\left(\mathcal{S}(\pi_1)\big(\mathcal{S}(\mathcal{A}^d)\big), \mathcal{S}(\pi_2)\big(\mathcal{S}(\mathcal{A}^d_N)\big)\right),$$

as explained below Proposition 4.16.

5.4.1 Four lemmas

Lemma 5.11. Suppose $f\in\mathcal{A}^d$, then

$$\left[D^N_{\mathbb{T}^d}, R^d(f)\right] = R^d\left([D_{\mathbb{T}^d}, f]\right).$$

Proof. As $\mathcal{Q}_N$ commutes with $D_{\mathbb{T}^d}$ we see that

$$\begin{aligned}\left[D^N_{\mathbb{T}^d}, (R^d(f))\right] &= [\mathcal{Q}_N D_{\mathbb{T}^d}\mathcal{Q}_N, \mathcal{Q}_N f \mathcal{Q}_N]\\ &= \mathcal{Q}_N\,[D_{\mathbb{T}^d}, f]\,\mathcal{Q}_N\\ &= R^d\left([D_{\mathbb{T}^d}, f]\right).\end{aligned}$$

□

Lemma 5.12. Suppose $T\in\mathcal{A}^d_N$, then

$$\left[D_{\mathbb{T}^d}, \breve{R}^d(T)\right] = \breve{R}^d\left(\left[D^N_{\mathbb{T}^d}, T\right]\right).$$

Proof. We prove the lemma in two steps. First of all we will show that for any $T\in Q_N C^\infty(\mathbb{T}^d)Q_N$ we have

$$\left[\partial_\mu, \breve{R}^{\otimes d}(T)\right] = \breve{R}^{\otimes d}\left[Q_N\partial_\mu Q_N, T\right]. \tag{5.35}$$

Then we will show the lemma follows from (5.35). If $T = T_1\otimes\cdots\otimes T_d$ a pure tensor

($T_\mu \in Q_N C^\infty(\mathbb{T}^1) Q_N$), then using $Q_N = P_N^{\otimes d}$ and Lemma 5.7 (5.22), we see

$$\begin{aligned}
\left[\partial_\mu, \breve{R}^{\otimes d}(T)\right] &= \left[1 \otimes \cdots \otimes \overset{\mu}{\overset{\downarrow}{\frac{d}{dx}}} \otimes \cdots \otimes 1, \breve{R}(T_1) \otimes \cdots \otimes \breve{R}(T_d)\right] \\
&= \breve{R}(T_1) \otimes \cdots \otimes \left[\frac{d}{dx}, \breve{R}(T_\mu)\right] \otimes \cdots \otimes \breve{R}(T_d) \\
&= \breve{R}(T_1) \otimes \cdots \otimes \breve{R}\left(\left[P_N \frac{d}{dx} P_N, T_\mu\right]\right) \otimes \cdots \otimes \breve{R}(T_d) \\
&= \breve{R}^{\otimes d}\left(T_1 \otimes \cdots \otimes \left[P_N \frac{d}{dx} P_N, T_\mu\right] \otimes \cdots \otimes T_d\right) \\
&= \breve{R}^{\otimes d}\left(\left[Q_N \left(1 \otimes \cdots \otimes \overset{\mu}{\overset{\downarrow}{\frac{d}{dx}}} \otimes \cdots \otimes 1\right) Q_N, Q_N \left(T_1 \otimes \cdots \otimes T_d\right) Q_N\right]\right) \\
&= \breve{R}^{\otimes d}\left(\left[Q_N \partial_\mu Q_N, T_1 \otimes \cdots \otimes T_d\right]\right) = \breve{R}^{\otimes d}\left(\left[Q_N \partial_\mu Q_N, T\right]\right).
\end{aligned}$$

Here we also used that $Q_N T Q_N = T$, as $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$ already. Taking linear combinations of pure tensors yields the statement for all $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$. We now make use of the equality (5.35). If $T \otimes 1_{2^m} \in \mathcal{A}_N^d$, then

$$\begin{aligned}
\left[D_{\mathbb{T}^d}, \breve{R}^d(T \otimes 1_{2^m})\right] &= \left[\sum_{\mu=1}^d -i\partial_\mu \otimes \gamma^\mu, \breve{R}^{\otimes d}(T) \otimes 1_{2^m}\right] \\
&= -i \sum_{\mu=1}^d \left[\partial_\mu, \breve{R}^{\otimes d}(T)\right] \otimes \gamma^\mu \\
&= -i \sum_{\mu=1}^d \breve{R}^{\otimes d}\left(\left[Q_n \partial_\mu Q_N, T\right]\right) \otimes \gamma^\mu \\
&= \breve{R}^d\left(-i \sum_{\mu=1}^d \left[Q_n \partial_\mu Q_N, T\right] \otimes \gamma^\mu\right) \\
&= \breve{R}^d\left(\left[-i \sum_{\mu=1}^d Q_n \partial_\mu Q_N \otimes \gamma^\mu, T \otimes 1_{2^m}\right]\right) \\
&= \breve{R}^d\left(\left[\mathcal{Q}_N \left(\sum_{\mu=1}^d -i\partial_\mu \otimes \gamma^\mu\right) \mathcal{Q}_N, T \otimes 1_{2^m}\right]\right) \\
&= \breve{R}^d\left(\left[\mathcal{Q}_N D_{\mathbb{T}^d} \mathcal{Q}_N, T \otimes 1_{2^m}\right]\right) \\
&= \breve{R}^d\left(\left[D_{\mathbb{T}^d}^N, T \otimes 1_{2^m}\right]\right),
\end{aligned}$$

proving the lemma. $\square$

Lemma 5.13. For each $d \geq 1$ there exists a sequence $\{\gamma_N\}_N$ of real numbers, converging to zero, such that

$$\left\| f - \breve{R}^d R^d(f) \right\| \leq \gamma_N \left\| [D_{\mathbb{T}^d}, f] \right\|,$$

for every $f \in \mathcal{A}^d_{sa}$.

Proof. Let F^d_N be the Féjer kernel as in Appendix A. Then if $f = f_1 \otimes \cdots \otimes f_d \in \bigotimes_{j=1}^d C^\infty(S^1) \subseteq C^\infty(\mathbb{T}^d)$ we have that $\breve{R}^{\otimes d} R^{\otimes d}(f) = F^d_{2N+1} * f$. Indeed it follows from Proposition 5.6 (5.19) that

$$\begin{aligned}\breve{R}^{\otimes d} R^{\otimes d}(f) &= \breve{R}R(f_1) \otimes \cdots \otimes \breve{R}R(f_d)\\ &= (F_{2N+1} * f_1) \otimes \cdots \otimes (F_{2N+1} * f_d)\\ &= F^d_N * f.\end{aligned}$$

As $\left\| R^{\otimes d} \right\|, \left\| \breve{R}^{\otimes d} \right\| \leq 1$ and $\left\| F^d_{2N+1} * (f - g) \right\|_\infty \leq \| f - g \|_\infty$, we have that

$$\breve{R}^{\otimes d} R^{\otimes d}(\lim_n f_n) = \lim_n \breve{R}^{\otimes d} R^{\otimes d}(f_n) = \lim_n F^d_{2N+1} * (f_n) = F^d_{2N+1} * (\lim_n f_n),$$

whenever $\{f_n\}$ is a sequence of pure tensors converging to some $f \in C^\infty(\mathbb{T}^d)$. So we see that we have $\breve{R}^{\otimes d} R^{\otimes d}(f) = F^d_{2N+1} * f$ for all $f \in C^\infty(\mathbb{T}^d)$. Now if $f \otimes 1_{2^m} \in \mathcal{A}^d_{sa}$, then $f \in C^\infty(\mathbb{T}^d)$ is real valued, and therefore Proposition 3.6 applies. Furthermore, if we denote $\|y\|_\infty = \max\{|y_1|, \ldots, |y_d|\}$, then

$$\begin{aligned}\left| f(\theta) - \breve{R}^{\otimes d} R^{\otimes d}(f)(\theta) \right| &= \left| f(\theta) - (F^d_{2N+1} * f)(\theta) \right|\\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F^d_{2N+1}(y) \left| f(\theta) - f(\theta - y) \right| dy\\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_2 \|f\|_{Lip} dy\\ &\leq \frac{\sqrt{d}}{(2\pi)^d} \left(\int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_\infty dy \right) \|\mathrm{grad}\, f\|_\infty\\ &= \frac{\sqrt{d}}{(2\pi)^d} \left(\int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_\infty dy \right) \|[D_{\mathbb{T}^d}, f \otimes 1_{2^m}]\|.\end{aligned} \tag{5.36}$$

Here we use that $\|[D_{\mathbb{T}^d}, f \otimes 1_{2^m}]\| = \| \mathrm{grad}\, f\|_\infty$, which we saw in the proof of Theorem 3.21. From the estimate (5.36) we obtain the bound

$$\|f - \breve{R}^{\otimes d} R^{\otimes d}(f)\| \leq \frac{\sqrt{d}}{(2\pi)^d} \left(\int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_\infty dy \right) \|[D_{\mathbb{T}^d}, f \otimes 1_{2^m}]\|,$$

which we use to conclude that

$$\begin{aligned}\|f \otimes 1_{2^m} - \breve{R}^d R^d(f \otimes 1_{2^m})\| &= \left\| \left(f - \breve{R}^{\otimes d} R^{\otimes d}(f) \right) \otimes 1_{2^m} \right\|\\ &= \|f - \breve{R}^{\otimes d} R^{\otimes d}(f)\| \leq \frac{\sqrt{d}}{(2\pi)^d} \left(\int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_\infty dy \right) \|[D_{\mathbb{T}^d}, f \otimes 1_{2^m}]\|.\end{aligned}$$

We set $\gamma_N = \frac{\sqrt{d}}{(2\pi)^d} \int_{\mathbb{T}^d} F^d_{2N+1}(y) \|y\|_\infty dy$, so that

$$\|f \otimes 1_{2^m} - \breve{R}^d R^d(f)\| \leq \gamma_N \|[D_{\mathbb{T}}, f \otimes 1_{2^m}]\|.$$

We are left to show that γ_N converges to 0. Choose $\epsilon > 0$ and choose M such that $N \geq M$ implies

$$\int_{\epsilon \leq |x| \leq \pi} F_{2N+1}(x)dx < \epsilon.$$

Then

$$\begin{aligned}
\gamma_N &= \frac{\sqrt{d}}{(2\pi)^d} \int_{\mathbb{T}^d} F^d_{2N+1}(y)\|y\|_\infty dy \\
&= \frac{\sqrt{d}}{(2\pi)^d} \left(\int_{\mathbb{T}^d \setminus [-\epsilon,\epsilon]^d} F^d_{2N+1}(y)\|y\|_\infty dy + \int_{[-\epsilon,\epsilon]^d} F^d_{2N+1}(y)\|y\|_\infty dy \right) \\
&\leq \frac{\sqrt{d}}{(2\pi)^d} \left(\pi \int_{\mathbb{T}^d \setminus [-\epsilon,\epsilon]^d} F^d_{2N+1}(y)dy + \epsilon \int_{[-\epsilon,\epsilon]^d} F^d_{2N+1}(y)dy \right) \\
&\leq \frac{\sqrt{d}}{(2\pi)^d} \Big(\pi\epsilon + \epsilon \Big).
\end{aligned}$$

As $\epsilon > 0$ was arbitrary we can conclude that $\gamma_N \to 0$. □

Lemma 5.14. For each $d \geq 1$ there exists a sequence $\{\gamma'_N\}_N$ of real numbers, converging to zero, such that

$$\left\| T - R^d \check{R}^d(T) \right\| \leq \gamma'_N \left\| [D^N_{\mathbb{T}^d}, T] \right\|,$$

for every $T \in \mathcal{A}^d_N$.

Proof. Let the maps $B, S \in L(P_N C^\infty(\mathbb{T}^1) P_N)$ be as in Proposition 5.6. So $B(T) = [-iP_N \frac{d}{dx} P_N, T]$ and $S(T) = (T_{2N+1} - T^*_{2N+1}) \circ T$, for the matrix T_{2N+1} given by (B.1):

$$T_{mn} = \begin{cases} 1 \text{ if } n \leq m \\ 0 \text{ if } n > m \end{cases} , T_{2N+1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Then we claim to have the following identity

$$\begin{aligned}
1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - R^{\otimes d} \check{R}^{\otimes d} &= \frac{1}{2N+1} \sum_{j=1}^{d} 1 \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 - \dots \\
&\quad \dots \left(\frac{1}{2N+1} \right)^2 \sum_{i<j\leq d} 1 \otimes \dots \otimes \overset{\overset{i}{\downarrow}}{SB} \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 + \dots \\
&\quad \dots (-1)^{d+1} \left(\frac{1}{2N+1} \right)^d SB \otimes SB \otimes \dots \otimes SB.
\end{aligned} \tag{5.37}$$

Here we denote $1 = 1_{P_N C^\infty(\mathbb{T}^1) P_N}$. The equality (5.37) follows directly from applying the result (5.20) from Proposition 5.6 and Proposition C.2. Indeed

$$
\begin{aligned}
1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - R^{\otimes d}\breve{R}^{\otimes d} &= 1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - \left(R\breve{R}\right)^{\otimes d} \\
&= 1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - \left(1 - \frac{1}{2N+1} SB\right)^{\otimes d} \\
&= 1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - \Bigg(1_{Q_N C^\infty(\mathbb{T}^d) Q_N} - \frac{1}{2N+1}\sum_{j=1}^{d} 1 \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 + \dots \\
&\qquad \dots \left(\frac{1}{2N+1}\right)^2 \sum_{i<j\leq d} 1 \otimes \dots \otimes \overset{\overset{i}{\downarrow}}{SB} \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 - \dots \\
&\qquad\qquad \dots (-1)^d \left(\frac{1}{2N+1}\right)^d SB \otimes SB \otimes \dots \otimes SB \Bigg) \\
&= \frac{1}{2N+1}\sum_{j=1}^{d} 1 \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 - \dots \\
&\qquad \dots \left(\frac{1}{2N+1}\right)^2 \sum_{i<j\leq d} 1 \otimes \dots \otimes \overset{\overset{i}{\downarrow}}{SB} \otimes \dots \otimes \overset{\overset{j}{\downarrow}}{SB} \otimes \dots \otimes 1 + \dots \\
&\qquad\qquad \dots (-1)^{d+1} \left(\frac{1}{2N+1}\right)^d SB \otimes SB \otimes \dots \otimes SB.
\end{aligned}
$$

Note that if $T = T_1 \otimes \dots \otimes T_d \in Q_N C^\infty(\mathbb{T}^d) Q_N$ is a pure tensor, then

$$
(1 \otimes \dots \otimes \overset{\overset{\nu}{\downarrow}}{B} \otimes \dots 1)(T) = T_1 \otimes \dots \otimes [-iP_N \frac{d}{dx} P_N, T_\nu] \otimes \dots \otimes T_d = [Q_N \partial_\nu Q_N, T]. \tag{5.38}
$$

Taking finite linear combinations of such pure tensors and applying the above equality yields the statement for all $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$. In Lemma 5.11 we saw that

$$
[D^N_{\mathbb{T}^d}, T \otimes 1_{2^m}] = -i \sum_{\mu=1}^{d} [Q_N \partial_\mu Q_N, T] \otimes \gamma^\mu,
$$

for each $T \otimes 1_{2^m} \in \mathcal{A}^d_N$. Hence we have the following equality

$$
\begin{aligned}
\frac{i}{2}\{1 \otimes \gamma^\nu, [D^N_{\mathbb{T}^d}, T \otimes 1_{2^m}]\} &= \frac{i}{2}\{1 \otimes \gamma^\nu, -i \sum_{\mu=1}^{d} [Q_N \partial_\mu Q_N, T] \otimes \gamma^\mu\} \\
&= \frac{1}{2}\sum_{\mu=1}^{d} [Q_N \partial_\mu Q_N, T] \otimes \{\gamma^\nu, \gamma^\mu\} \\
&= [Q_N \partial_\nu Q_N, T] \otimes 1_{2^m} \\
&= (1 \otimes \dots \otimes \overset{\overset{\nu}{\downarrow}}{B} \otimes \dots 1)(T) \otimes 1_{2^m} \\
&= \left((1 \otimes \dots \otimes \overset{\overset{\nu}{\downarrow}}{B} \otimes \dots 1) \otimes 1_{2^m} \right) (T \otimes 1_{2^m}),
\end{aligned} \tag{5.39}
$$

for all $T \otimes 1_{2^m} \in \mathcal{A}_N^d$. For the third equality we used (3.17). The term $1 \otimes \gamma^\mu \equiv id_{L^2(\mathbb{T}^d)} \otimes \gamma^\mu$ acts on $L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$ and its norm is given by

$$\begin{aligned} \|1 \otimes \gamma^\mu\|^2 &= \|(1 \otimes \gamma^\mu)^*(1 \otimes \gamma^\mu)\| \\ &= \|1 \otimes (\gamma^\mu)^*(\gamma^\mu)\| \\ &= \|1 \otimes 1_{2^m}\| = 1. \end{aligned} \tag{5.40}$$

Combining (5.39) and (5.40) yields

$$\left\| \left((1 \otimes \cdots \otimes \overset{\nu}{\overset{\downarrow}{B}} \otimes \dots 1) \otimes 1_{2^m} \right) (T \otimes 1_{2^m}) \right\| = \left\| \frac{i}{2} \left\{ 1 \otimes \gamma^\nu, [D_{\mathbb{T}^d}^N, T \otimes 1_{2^m}] \right\} \right\| \leq \|[D_{\mathbb{T}^d}^N, T \otimes 1_{2^m}]\|. \tag{5.41}$$

In particular, as $\|D_{\mathbb{T}^d}^N\| \leq \sqrt{d}N$, the inequality (5.41) implies

$$\left\| \left((1 \otimes \cdots \otimes \overset{\nu}{\overset{\downarrow}{B}} \otimes \dots 1) \otimes 1_{2^m} \right) (T \otimes 1_{2^m}) \right\| \leq 2\sqrt{d}N \|T \otimes 1_{2^m}\|. \tag{5.42}$$

Applying the inequality (5.42) $k-1$ times and the estimate (5.41) once, we see that if the tensor $1 \otimes \cdots \otimes B \otimes \cdots \otimes B \otimes \cdots \otimes 1$ contains B at k entries, then

$$\|((1 \otimes \cdots \otimes B \otimes \cdots \otimes B \otimes \cdots \otimes 1) \otimes 1_{2^m}) (T \otimes 1_{2^m})\| \leq \left(2N\sqrt{d}\right)^{k-1} \|[D_{\mathbb{T}^d}^N, T \otimes 1_{2^m}]\|. \tag{5.43}$$

We now turn our attention to the operator

$$1 \otimes \cdots \otimes S \otimes \cdots \otimes 1 = 1 \otimes \cdots \otimes T_{2N+1} \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes T_{2N+1}^* \otimes \cdots \otimes 1.$$

First of all, notice that applying the operator 1 is the same as to take the Schur product with the matrix consisting only of 1's. Let us denote this matrix with 1 in each entry as J. So for $T \in Q_N C^\infty(\mathbb{T}^d) Q_N$

$$\Big(1 \otimes \cdots \otimes S \otimes \cdots \otimes 1\Big)(T) = \Big(J \otimes \cdots \otimes T_{2N+1} \otimes \cdots \otimes J\Big) \circ T - \Big(J \otimes \cdots \otimes T_{2N+1}^* \otimes \cdots \otimes J\Big) \circ T.$$

We want to find a matrix squaring to $J \otimes \cdots \otimes T_{2N+1} \otimes \cdots \otimes J$, so that we can apply Proposition B.2 to estimate the norm of $1 \otimes \cdots \otimes S \otimes \cdots \otimes 1$. In appendix B we found a matrix A_{2N+1} squaring to T_{2N+1} and estimated the norm of its columns. It remains, therefore, to find a matrix squaring to J. It is easily seen that $\frac{1}{\sqrt{2N+1}} J$ is an excellent choice. Hence

$$\left(\frac{1}{(2N+1)^{\frac{d-1}{2}}} \Big(1 \otimes \cdots \otimes A_{2N+1} \otimes \cdots \otimes 1 \Big) \right)^2 = J \otimes \cdots \otimes T_{2N+1} \otimes \cdots \otimes J. \tag{5.44}$$

Let us briefly argue that the norm of the columns of $\frac{1}{(2N+1)^{\frac{d-1}{2}}} \Big(J \otimes \cdots \otimes A_{2N+1} \otimes \cdots \otimes J \Big)$ is the same as the norm of the columns of A_{2N+1}. It suffices to show that the norm of the columns of $\frac{1}{\sqrt{2N+1}} J \otimes A_{2N+1}$ and $\frac{1}{\sqrt{2N+1}} A_{2N+1} \otimes J$ is the same as the norm of the columns of A_{2N+1}, as we can then apply this equality repeatedly. The Kronecker product of the matrices J and A_{2N+1} is given by

$$(J \otimes A_{2N+1})_{(2N+1)(n-1)+k, (2N+1)(m-1)+l} = J_{nm} a_{kl} = a_{kl}.$$

Here a_{kl} denotes the (k,l)-entry of the matrix A_{2N+1}. The norm of column $(2N+1)(m-1)+l$ of the matrix $J \otimes A_{2N+1}$ is therefore given by

$$\begin{aligned}
\sqrt{\sum_{n,k=1}^{2N+1} \left|(J \otimes A_{2N+1})_{(2N+1)(n-1)+k,(2N+1)(m-1)+l}\right|^2} &= \sqrt{\sum_{n,k=1}^{2N+1} |a_{kl}|^2} \\
&= \sqrt{\sum_{k=1}^{2N+1} 2N+1|a_{kl}|^2} \\
&= \sqrt{2N+1}\sqrt{\sum_{k=1}^{2N+1} |a_{kl}|^2},
\end{aligned}$$

which is just $\sqrt{2N+1}$ times the norm of column l of the matrix A_{2N+1}. Therefore the norm of the columns of $\frac{1}{\sqrt{2N+1}} J \otimes A_{2N+1}$ is the same as the norm of the columns of A_{2N+1}. Analagously one can show the same is true for the matrix $\frac{1}{\sqrt{2N+1}} A_{2N+1} \otimes J$. For the norm of the columns of A_{2N+1} we found the upper bound (B.4), and thus we can use the same upperbound for the norm of the columns of the matrix

$$\frac{1}{(2N+1)^{\frac{d-1}{2}}} \Big(J \otimes \cdots \otimes A_{2N+1} \otimes \cdots \otimes J\Big).$$

Using the same reasoning we see that

$$\left(\frac{1}{(2N+1)^{\frac{d-1}{2}}} \Big(J \otimes \cdots \otimes A^*_{2N+1} \otimes \cdots \otimes J\Big)\right)^2 = J \otimes \cdots \otimes T^*_{2N+1} \otimes \cdots \otimes J_{2N+1} \quad (5.45)$$

where again we obtain the same estimate for the norm of the columns as in (B.4). Altogether we see that

$$\|1 \otimes \cdots \otimes S \otimes \cdots \otimes 1\| \leq 2 + \frac{2}{\pi}\big(1 + \log(2N)\big). \quad (5.46)$$

Hence, if the tensor $1 \otimes \cdots \otimes S \otimes \cdots \otimes S \otimes \cdots \otimes 1$ contains S at k entries, we have the estimate

$$\|1 \otimes \cdots \otimes S \otimes \cdots \otimes S \otimes \cdots \otimes 1\| \leq \left(2\Big(1 + \frac{1}{\pi}\big(1 + \log(2N)\big)\Big)\right)^k.$$

Using Lemma 2.18, we see that we then have

$$\begin{aligned}
&\left\|\Big(\Big(1 \otimes \cdots \otimes S \otimes \cdots \otimes S \otimes \cdots \otimes 1\Big) \otimes 1_{2^m}\Big)(T \otimes 1_{2^m})\right\| \\
&\qquad = \left\|\Big(1 \otimes \cdots \otimes S \otimes \cdots \otimes S \otimes \cdots \otimes 1\Big)\big(T\big) \otimes 1_{2^m}\right\| \\
&\qquad = \left\|\Big(1 \otimes \cdots \otimes S \otimes \cdots \otimes S \otimes \cdots \otimes 1\Big)T\right\| \\
&\qquad \leq \left(2\Big(1 + \frac{1}{\pi}\big(1 + \log(2N)\big)\Big)\right)^k \|T\| \\
&\qquad = \left(2\Big(1 + \frac{1}{\pi}\big(1 + \log(2N)\big)\Big)\right)^k \|T \otimes 1_{2^m}\|.
\end{aligned} \quad (5.47)$$

Combining (5.37), (5.43) and (5.47) yields

$$
\begin{aligned}
\Big\|T\otimes 1_{2^m} - R^d\breve{R}^d(T\otimes 1_{2^m})\Big\| &= \Bigg\|\Bigg(\frac{1}{2N+1}\sum_{j=1}^{d}\Big(1\otimes\cdots\otimes \overset{\overset{j}{\downarrow}}{SB}\otimes\cdots\otimes 1\Big)\otimes 1_{2^m} - \ldots \\
&\quad \ldots\left(\frac{1}{2N+1}\right)^2\sum_{i<j\le d}\Big(1\otimes\cdots\otimes \overset{\overset{i}{\downarrow}}{SB}\otimes\cdots\otimes \overset{\overset{j}{\downarrow}}{SB}\otimes\cdots\otimes 1\Big)\otimes 1_{2^m} + \ldots \\
&\quad \ldots(-1)^{d+1}\left(\frac{1}{2N+1}\right)^d\Big(SB\otimes SB\otimes\cdots\otimes SB\Big)\otimes 1_{2^m}\Bigg)(T\otimes 1_{2^m})\Bigg\| \\
&\le \sum_{j=1}^{d}\frac{2}{2N+1}\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\|[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}]\| + \ldots \\
&\quad \cdots\sum_{i<j\le d}\left(\frac{2}{2N+1}\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^2 2\sqrt{d}N\|[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}]\| + \ldots \\
&\quad \cdots + \left(\frac{2}{2N+1}\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^d\big(2\sqrt{d}N\big)^{d-1}\|[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}]\| \\
&= \frac{1}{2N+1}\Bigg(\binom{d}{1}2\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big) + \ldots \\
&\quad \cdots\binom{d}{2}\left(2\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^2\frac{2\sqrt{d}N}{2N+1} + \ldots \\
&\quad \cdots + \binom{d}{d}\left(2\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^d\Big(\frac{2\sqrt{d}N}{2N+1}\Big)^{d-1}\Bigg)\|[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}]\| \\
&= \frac{1}{2N+1}\left(\sum_{j=1}^{d}\binom{d}{j}\left(2\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^j\Big(\frac{2\sqrt{d}N}{2N+1}\Big)^{j-1}\right)\|[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}]\|.
\end{aligned}
$$

The term

$$
\gamma'_N = \frac{1}{2N+1}\sum_{j=1}^{d}\binom{d}{j}\left(2\Big(1+\frac{1}{\pi}\big(1+\log(2N)\big)\Big)\right)^j\Big(\frac{2\sqrt{d}N}{2N+1}\Big)^{j-1}
$$

converges to zero since all the terms converge to zero. Also we have

$$
\big\|T\otimes 1_{2^m} - R^d\breve{R}^d(T\otimes 1_{2^m})\big\| \le \gamma'_N\big\|\big[D^N_{\mathbb{T}^d}, T\otimes 1_{2^m}\big]\big\|.
$$

□

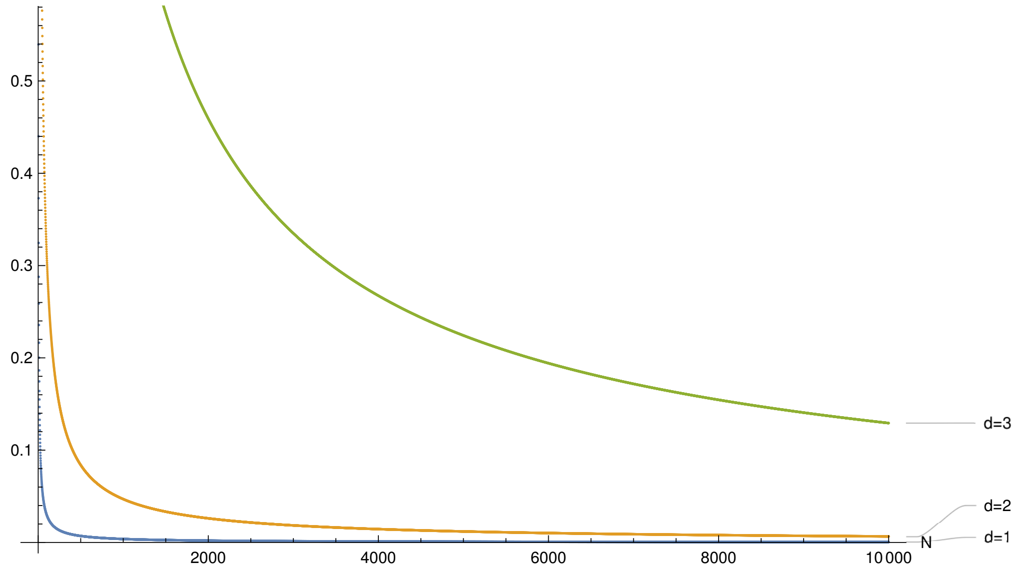

Figure 4: The sequence γ'_N, for $N = 1, \ldots 10000$, for the torus in dimension 1 (blue), 2 (orange) and 3 (green).

5.4.2 Gromov–Hausdorff convergence

We are now ready to prove the main theorem of this thesis. Let $\mathcal{O}$ and $\mathcal{O}_N$ denote the operator system spectral triples

$$\mathcal{O} = \left(\mathcal{A}^d, L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}, D_{\mathbb{T}^d}\right), \qquad \mathcal{O}_N = \left(\mathcal{A}_N^d, \mathcal{Q}_N\left(L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}\right), D_{\mathbb{T}^d}^N\right),$$

the spectral triple associated to the d-dimensional torus and the rectangularly truncated torus as defined in Definition 5.3 respectively.

Theorem 5.15. *The rectangularly truncated torus $\mathcal{O}_N$, converges to the canonical spectral triple $\mathcal{O}$ in Gromov–Hausdorff distance.*

Proof. Let $\epsilon > 0$. Then choose N large enough such that $\gamma_N, \gamma'_N \leq \epsilon$, where γ_N and γ'_N are as in Lemmas 5.13 and 5.14 respectively. We build a weak bridge $\mathcal{B}$, in the sense of Definition 4.15, between $\mathcal{O}$ and $\mathcal{O}_N$. We define

$$\mathcal{B}(f, T) = \max\left\{\frac{1}{\epsilon}\|f - \breve{R}^d(T)\|, \frac{1}{\epsilon}\|T - R^d(f)\|\right\}, \tag{5.48}$$

for $f \in \mathcal{A}^d, T \in \mathcal{A}_N^d$. We need to check the requirements 1 and 2 from Definition 4.15. If $f \in \mathcal{A}^d$, then set $f^\dagger = R^d(f) \in \mathcal{A}_N^d$. By a similar argument as in Remark 4.4, it suffices to consider $f \equiv f \otimes 1_{2^m} \in \mathcal{A}_{sa}^d$, so that f is real-valued. Then according to Lemma 5.13

$$\begin{aligned}\mathcal{B}(f, f^\dagger) &= \max\left\{\frac{1}{\epsilon}\|f - \breve{R}^d(R^d(f))\|, \frac{1}{\epsilon}\|R^d(f) - R^d(f)\|\right\} \\ &= \frac{1}{\epsilon}\|f - \breve{R}^d(R^d(f))\| \leq \|[D_{\mathbb{T}^d}, f]\|.\end{aligned}$$

Also by Lemma 5.11 and Lemma 5.9

$$\begin{aligned}\left\|[D_{\mathbb{T}^d}^N, f^\dagger]\right\| &= \left\|[D_{\mathbb{T}^d}^N, R^d(f)]\right\| \\ &= \left\|R^d\left([D_{\mathbb{T}^d}, f]\right)\right\| \\ &\leq \|[D_{\mathbb{T}^d}, f]\|.\end{aligned}$$

So $\mathcal{B}$ satisfies requirement 1 from Definition 4.15. Choosing $T^\dagger = \breve{R}(T) \in \mathcal{A}^d$ for $T \in \mathcal{A}^d_N$ and using Lemma 5.9, 5.12 and 5.14 shows that $\mathcal{B}$ also satisfies requirement 2 from Definition 4.15. Proposition 4.16 tells us that if we equip the space $\mathcal{S}(\mathcal{A}^d \oplus \mathcal{A}^d_N)$ with the metric

$$d_{\mathcal{B}}(\phi, \psi) = \sup_{f \in \mathcal{A}^d, T \in \mathcal{A}^d_N} \{|\phi(f,T) - \psi(f,T)| : \|[D_{\mathbb{T}^d}, f]\|, \left\|\left[D^N_{\mathbb{T}^d}, T\right]\right\|, \mathcal{B}(f,T) \leq 1\}, \tag{5.49}$$

the maps

$$\begin{tikzcd} & \mathcal{S}(\mathcal{A}^d \oplus \mathcal{A}^d_N) & \\ \mathcal{S}(\mathcal{A}^d) \arrow[ur, "\mathcal{S}(\pi_1)"] & & \mathcal{S}(\mathcal{A}^d_N) \arrow[ul, "\mathcal{S}(\pi_2)"'] \end{tikzcd}$$

are isometric embeddings. We now show that

$$\mathrm{dist}_H \left(\mathcal{S}(\pi_1)\big(\mathcal{S}(\mathcal{A}^d)\big), \mathcal{S}(\pi_2)\big(\mathcal{S}(\mathcal{A}^d_N)\big)\right) \leq \epsilon, \tag{5.50}$$

from which we can then conclude that $\mathrm{dist}^q_{GH}(\mathcal{O}, \mathcal{O}_N) \leq \epsilon$. Take $\phi \in \mathcal{S}(\mathcal{A}^d)$. We want to find $\psi \in \mathcal{S}(\mathcal{A}^d_N)$, such that $d_{\mathcal{B}}(\mathcal{S}(\pi_1)\phi, \mathcal{S}(\pi_2)\psi) \leq \epsilon$. We choose $\psi = \mathcal{S}(\breve{R}^d)\phi$, which is a state on $\mathcal{A}^d_N$ according to Proposition 5.10. Then indeed

$$\begin{aligned} d_{\mathcal{B}}\left(\mathcal{S}(\pi_1)\phi, \mathcal{S}(\pi_2)\psi\right) &= \sup_{f \in \mathcal{A}^d, T \in \mathcal{A}^d_N} \left\{ |\phi(f) - \phi(\breve{R}^d(T))| : \substack{\|[D_{\mathbb{T}^d}, f]\|, \|[D^N_{\mathbb{T}^d}, T]\| \leq 1 \\ \mathcal{B}(f,T) \leq 1} \right\} \\ &\leq \sup_{f \in \mathcal{A}^d, T \in \mathcal{A}^d_N} \left\{ \|f - \breve{R}^d(T)\| : \substack{\|[D_{\mathbb{T}^d}, f]\|, \|[D^N_{\mathbb{T}^d}, T]\| \leq 1 \\ \|f - \breve{R}^d(T)\|, \|T - R^d(f)\| \leq \epsilon} \right\} \\ &\leq \epsilon, \end{aligned}$$

which shows that $\mathcal{S}(\pi_1)\big(\mathcal{S}(\mathcal{A}^d)\big) \subseteq \mathcal{N}_\epsilon\Big(\mathcal{S}(\pi_2)\big(\mathcal{S}(\mathcal{A}^d_N)\big)\Big)$. A very similar computation shows that

$$d_{\mathcal{B}}\left(\mathcal{S}(\pi_1)\mathcal{S}(R^d)(\psi), \mathcal{S}(\pi_2)\psi\right) \leq \epsilon,$$

for $\psi \in \mathcal{S}(\mathcal{A}^d_N)$, implying that $\mathcal{S}(\pi_2)\big(\mathcal{S}(\mathcal{A}^d_N)\big) \subseteq \mathcal{N}_\epsilon\Big(\mathcal{S}(\pi_1)\big(\mathcal{S}(\mathcal{A}^d)\big)\Big)$. Thus we have established (5.50) so we conclude that $\mathcal{O}_N$ converges to $\mathcal{O}$ in Gromov–Hausdorff distance. $\square$

According to Proposition 4.7 and 4.10 the operator system spectral triples $\mathcal{O}$ and $\mathcal{O}_N$ are also quantum metric spaces ([7]). That is, the formula (4.11) induces the weak-$*$ topology on the state space. Therefore we can also speak of the quantum Gromov–Hausdorff distance between $\mathcal{O}$ and $\mathcal{O}_N$ as quantum metric spaces. The weak bridge $\mathcal{B}$, defined in (5.48) satisfies some additional requirements and is in fact a bridge of quantum metric spaces between $\mathcal{O}$ and $\mathcal{O}_N$ ([7, Definition 5.1]). This means $\mathcal{B}$ induces the weak-$*$ topology on $\mathcal{S}(\mathcal{A}^d \oplus \mathcal{A}^d_N)$ through (5.49). One can then show that $\mathcal{O}_N$ converges to $\mathcal{O}$ in quantum Gromov–Hausdorff distance for quantum metric spaces, using the exact same proof strategy as for the proof of Theorem 5.15.

A The Fejér kernel

The Féjer kernel is an important tool in Fourier theory on the circle. For more detail on the Fourier theory and the Féjer kernel we refer to [13, Chapter 2]. In this appendix we will recall some important properties.

Definition A.1. The *N'th Fejér kernel* $F_N : \mathbb{T}^1 \to \mathbb{C}$ is defined by

$$F_N(\theta) = \sum_{n=-2N}^{2} N\Big(1 - \frac{|n|}{N}\Big)e^{in\theta}.$$

The Fejér kernel is a so-called *good kernel*.

Lemma A.2. Let F_N denote the N'th Fejér kernel. Then we have the identity

$$F_N(\theta) = \frac{1}{N}\frac{\sin^2\left(\frac{N\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}. \tag{A.1}$$

Furthermore F_N satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(\theta)d\theta = 1$$

$$\text{For every } \delta > 0, \int_{\delta\le|\theta|\le\pi} F_N(\theta)d\theta \to 0 \text{ as } N \to \infty.$$

Proof. Denoting $\omega = e^{i\theta}$ we see

$$\begin{aligned}
F_N(\theta) &= \sum_{n=-N}^{N}\Big(1 - \frac{|n|}{N}\Big)\omega^n \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=-m}^{m}\omega^n \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\frac{\omega^{-m} - \omega^{m+1}}{1-\omega} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\frac{\omega^{-m-\frac{1}{2}} - \omega^{m+\frac{1}{2}}}{\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\frac{\sin((m+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})} \\
&= \frac{1}{2N\sin^2\left(\frac{\theta}{2}\right)}\sum_{m=0}^{N-1} 2\sin\big((m+\frac{1}{2})\theta\big)\sin\big(\frac{\theta}{2}\big) \\
&= \frac{1}{2N\sin^2\left(\frac{\theta}{2}\right)}\sum_{m=0}^{N-1}\Big(\cos(m\theta) - \cos\big((m+1)\theta\big)\Big) \\
&= \frac{1-\cos(N\theta)}{2N\sin^2\left(\frac{\theta}{2}\right)} = \frac{\sin^2\left(\frac{N\theta}{2}\right)}{N\sin^2\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

In the above computation we used the trigonometric identities $\cos(2\phi) - \cos(2\psi) = -2\sin(\phi+\psi)\sin(\phi-\psi)$ and $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$. The closed formula (A.1) for the Fejér kernel shows that F_N is positive for every N. We easily compute

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(\theta)d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-2N}^{2N}\Big(1-\frac{|n|}{2N+1}\Big)e^{in\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1d\theta = 1.\end{aligned}$$

Finally, suppose $\delta > 0$ is given, then there exists some constant $c_\delta > 0$ such that $\sin^2(\frac{\theta}{2}) \geq c_\delta$, whenever $\delta \leq |\theta| \leq \pi$. Therefore, if $\delta \leq |\theta| \leq \pi$, we have

$$F_N(\theta) = \frac{1}{N}\frac{\sin^2\left(\frac{N\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)} \leq \frac{1}{Nc_\delta}.$$

So indeed we have that

$$\int_{\delta\leq|\theta|\leq\pi} F_N(\theta)d\theta \to 0 \text{ as } N \to \infty.$$

$\square$

Lemma A.3. Let F_N denote the N'th Fejér kernel and let f be given by the absolute convergent Fourier series $f = \sum a_n e^{in\theta}$, then

$$\big(F_N * f\big)(\theta) = \sum_{n=-N}^{N}\Big(1-\frac{|n|}{N}\Big)a_n e^{in\theta}$$

Proof. We simply calculate

$$\begin{aligned}\big(F_N * f\big)(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)f(\theta-x)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-N}^{N}\Big(1-\frac{|n|}{N}\Big)e^{inx}\sum_{m\in\mathbb{Z}} a_m e^{im(\theta-x)}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-N}^{N}\sum_{m\in\mathbb{Z}}\Big(1-\frac{|n|}{N}\Big)e^{i(n-m)x}a_m e^{im\theta}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-N}^{N}\Big(1-\frac{|n|}{N}\Big)a_n e^{in\theta}dx = \sum_{n=-N}^{N}\Big(1-\frac{|n|}{N}\Big)a_n e^{in\theta}.\end{aligned}$$

We used that we may interchange summation and integration, as the Fourier series of f converges absolutely. $\square$

Definition A.4. We define the *multidimensional Fejér kernel* F_N^d as

$$F_N^d = F_N \otimes \cdots \otimes F_N,$$

where F_N denotes the Fejér kernel on the circle. So for $\theta \in \mathbb{T}^d$, we have $F_N^d(\theta) = F_N(\theta_1)F_N(\theta_2)\ldots F_N(\theta_d)$.

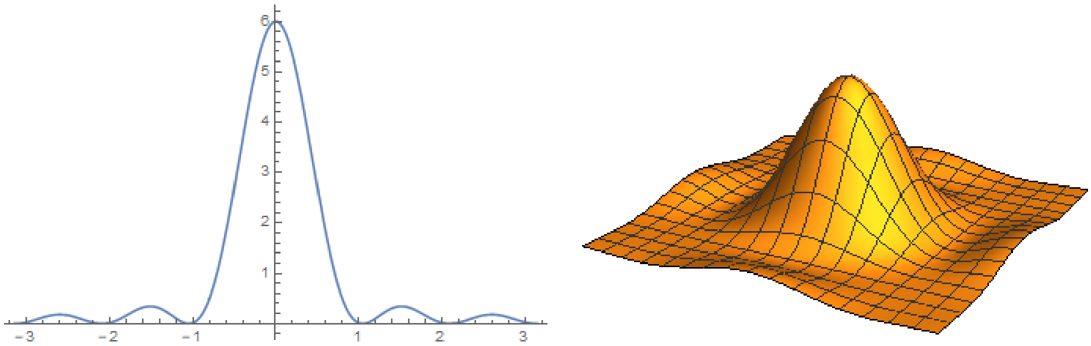

Figure 5: The one-dimensional Fejér kernel F_6 (left) and the two-dimensional Fejér kernel F_3^2 (right).

B The Schur product

We define a matrix multiplication different from the usual one. It is called the Schur product, also known as the Hadamard product.

Definition B.1. Define the *Schur product* of $A, B \in M_n(\mathbb{C})$ by

$$(A \circ B)_{mn} = A_{mn} B_{mn}.$$

We define $s_B \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$s_B(A) = B \circ A.$$

Proposition B.2. Suppose we can write $B \in M_n(\mathbb{C})$ as a usual matrix product $B = S^* R$ for some $S, R \in M_n(\mathbb{C})$, then $\|s_B\| \leq c(S)c(R)$, where $c(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}$, the maximum of the norms of the columns of $A \in M_n(\mathbb{C})$.

Proof. Write w_j, v_j for the j'th column of S and R respectively. Then

$$b_{ij} = \sum_k S^*_{ik} R_{kj} = \sum_k \overline{S_{ki}} R_{kj} = \langle w_i, v_j \rangle.$$

Also, clearly $\|w_j\| \leq c(S)$ and $\|v_j\| \leq c(R)$. Let $(\cdot, \cdot)$ denote the Hilbert–Schmidt inner product on $M_n(\mathbb{C})$ given by $(A, B) = \mathrm{Tr}(A^* B) = \sum_{i,j} \overline{a_{ij}} b_{ij}$. Suppose now we are given two unit vectors $\lambda, \mu \in \mathbb{C}^n$. Then define the matrices $\tilde{w}$ and $\tilde{v}$ by

$$\tilde{v}_{ij} = \lambda_i (v_i)_j, \tilde{w}_{ij} = \mu_i (w_i)_j.$$

We now compute

$$\begin{aligned}
\langle s_B(A)\lambda, \mu \rangle &= \sum_i \overline{(A \circ B\lambda)_i} \mu_i \\
&= \sum_{i,j} \overline{(B \circ A)_{ij} \lambda_j} \mu_i \\
&= \sum_{i,j} \overline{b_{ij} a_{ij} \lambda_j} \mu_i \\
&= \sum_{i,j} \overline{\langle w_i, v_j \rangle a_{ij} \lambda_j} \mu_i \\
&= \sum_{i,j,k} \overline{a_{ij} \lambda_j (v_j)_k} \mu_i (w_i)_k.
\end{aligned}$$

And on the other hand we also have

$$\begin{aligned}
(A\tilde{v}, \tilde{w}) &= \sum_{i,k} \overline{A\tilde{v}_{ik}} \tilde{w}_{ik} \\
&= \sum_{i,j,k} \overline{a_{ij}\tilde{v}_{jk}} \mu_i (w_i)_k \\
&= \sum_{i,j,k} \overline{a_{ij}\lambda_j (v_j)_k} \mu_i (w_i)_k .
\end{aligned}$$

So we obtain the equality $\langle s_B(A)\lambda, \mu\rangle = (A\tilde{v}, \tilde{w})$. Therefore we can estimate

$$\begin{aligned}
|\langle s_B(A)\lambda, \mu\rangle|^2 &= |(A\tilde{v}, \tilde{w})|^2 \\
&\leq (A\tilde{v}, A\tilde{v})(\tilde{w}, \tilde{w}) \\
&= \operatorname{Tr}(\tilde{v}^* A^* A\tilde{v})(\tilde{w}, \tilde{w}) \\
&= \operatorname{Tr}(A^* A\tilde{v}\tilde{v}^*)(\tilde{w}, \tilde{w}) \\
&\leq \|A^* A\| \|\tilde{v}\tilde{v}^*\|_1 (\tilde{w}, \tilde{w}) \\
&= \|A\|^2 \operatorname{Tr}(\tilde{v}\tilde{v}^*)(\tilde{w}, \tilde{w}) \\
&= \|A\|^2 (\tilde{v}, \tilde{v})(\tilde{w}, \tilde{w}).
\end{aligned}$$

We now claim that we have the inequalities $(\tilde{v}, \tilde{v}) \leq c(R)^2, (\tilde{w}, \tilde{w}) \leq c(S)^2$, which would complete our proof. Indeed we have

$$\begin{aligned}
(\tilde{w}, \tilde{w}) &= \sum_{i,j} \overline{\tilde{w}_{ij}} \tilde{w}_{ij} \\
&= \sum_{i,j} \overline{\mu_i (w_i)_j} \mu_i (w_i)_j \\
&= \sum_{i,j} |\mu_i|^2 |(w_i)_j|^2 \\
&= \sum_{i} |\mu_i|^2 |\sum_{j} |(w_i)_j|^2 \\
&\leq \sum_{i} |\mu_i|^2 c(S)^2 = c(S)^2,
\end{aligned}$$

as μ is a unit vector. A similar calculation shows $(\tilde{v}, \tilde{v}) \leq c(R)^2$. ☐

Proposition B.3. Let $T_N \in M_N(\mathbb{C})$ be the matrix given by

$$T_{mn} = \begin{cases} 1 \text{ if } n \leq m \\ 0 \text{ if } n > m \end{cases}, T_N = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}. \tag{B.1}$$

Then $\|s_{T_N}\| \leq 1 + \frac{1}{\pi}\big(1 + \log(N-1)\big)$.

Proof. We claim that the matrix $A_N = (a_{jk})$, given by

$$a_{jk} = \begin{cases} 0 \text{ if } k > j \\ 1 \text{ if } k = j \\ \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2(j-k)-1}{2(j-k)} = \prod_{m=1}^{j-k} \frac{2m-1}{2m} \text{ if } j < k \end{cases} \tag{B.2}$$

squares to T_N. We can then use Proposition B.2 to estimate the s_{T_N}. We prove this claim by induction. First of all the claim is trivially true for $N = 1$. Suppose now that we have proven $A_M^2 = T_M$ for all $M \leq N$. We prove the statement for $N+1$. We simply calculate

$$\left(A_{N+1}\right)^2_{jk} = \sum_{i=1}^{N+1} a_{ji}a_{ik} = \begin{cases} 0 \text{ if } k > i \\ \sum_{i=k}^{j} a_{ji}a_{ik} \text{ if } k \leq j. \end{cases} \tag{B.3}$$

From expression (B.1) it is clear that $\left(A_{N+1}\right)^2_{jj} = 1$. We proceed for the case where $j < k$. If $j \leq N$ it follows from the induction hypothesis that

$$\left(A_{N+1}\right)^2_{jk} = \sum_{i=k}^{j} a_{ji}a_{ik} = \left(A_N\right)^2_{jk} = 1.$$

Note that if $k \geq 2$, then $a_{jk} = a_{j-1,k-1}$. So if $k \geq 2$

$$\begin{aligned} \left(A_{N+1}\right)^2_{jk} &= \sum_{i=k}^{j} a_{ji}a_{ik} \\ &= \sum_{i=k}^{j} a_{j-1,i-1}a_{i-1,k-1} \\ &= \sum_{i=k-1}^{j-1} a_{j-1,i}a_{i,k-1} \\ &= \left(A_N\right)^2_{j-1,k-1} = \left(T_N\right)_{j-1,k-1} = \left(T_N\right)_{jk} = \left(T_{N+1}\right)_{jk}. \end{aligned}$$

Here we used the induction hypothesis in the last line. The only case that remains is the case where $j = N+1$ and $k = 1$. So we must show that

$$\sum_{i=1}^{N+1} a_{N+1,i}a_{i,1} = 1.$$

To prove this, we consider the power series corresponding to the function $\frac{1}{1-x}$ and $\frac{1}{\sqrt{1-x}}$, both with positive radius of convergence 1. It is well known the power series corresponding to $\frac{1}{1-x}$ is given by

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

The power series corresponding to $\frac{1}{\sqrt{1-x}}$ is given by

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \left(\prod_{j=1}^{n} \frac{2l-1}{2l} \right) x^n, \qquad |x| < 1.$$

We show this. Again, we argue by induction. The induction hypothesis is that

$$\left(\frac{d}{dx}\right)^n (1-x)^{-\frac{1}{2}} = \left(\prod_{l=1}^{n} \frac{2l-1}{2} \right) (1-x)^{-\frac{2n+1}{2}}$$

This is clear for $n = 0$. For the inductive step we compute

$$\begin{aligned}
\left(\frac{d}{dx}\right)^{n+1}(1-x)^{-\frac{1}{2}} &= \frac{d}{dx}\left(\frac{d}{dx}\right)^{n}(1-x)^{-\frac{1}{2}} \\
&= \left(\prod_{l=1}^{n}\frac{2l-1}{2}\right)\frac{d}{dx}(1-x)^{-\frac{2n+1}{2}} \\
&= \left(\prod_{l=1}^{n}\frac{2l-1}{2}\right)\frac{2n+1}{2}(1-x)^{-\frac{2(n+1)+1}{2}} \\
&= \left(\prod_{l=1}^{n+1}\frac{2l-1}{2}\right)(1-x)^{-\frac{2(n+1)+1}{2}},
\end{aligned}$$

Then it follows that $\frac{1}{\sqrt{1-x}}$ is given by the power series

$$\begin{aligned}
\frac{1}{\sqrt{1-x}} &= \sum_{n=0}^{\infty}\left(\frac{1}{n!}\left(\prod_{l=1}^{n}\frac{2l-1}{2}\right)(1-x)^{-\frac{2n+1}{2}}\bigg|_{x=0}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{n!}\prod_{l=1}^{n}\frac{2l-1}{2}\right)x^n \\
&= \sum_{n=0}^{\infty}\left(\prod_{l=1}^{n}\frac{2l-1}{2l}\right)x^n = \sum_{n=0}^{\infty}a_{n+1,1}x^n.
\end{aligned}$$

As $\frac{1}{\sqrt{1-x}}$ squares to $\frac{1}{1-x}$, we know that

$$\begin{aligned}
\sum_{n=0}^{\infty}x^n &= \Big(\sum_{n=0}^{\infty}a_{n+1,1}x^n\Big)^2 \\
&= \sum_{n=0}^{\infty}\Big(\sum_{m+i=n}a_{m+1,1}a_{i+1,1}\Big)x^n.
\end{aligned}$$

So we conclude that $\sum_{m+i=n}a_{m+1,1}a_{i+1,1} = 1$, for each n. In particular, choosing $n = N$ we see that

$$\begin{aligned}
1 &= \sum_{m+i=N}a_{m+1,1}a_{i+1,1} \\
&= \sum_{i=0}^{N}a_{N+1-i,1}a_{i+1,1} \\
&= \sum_{i=0}^{N}a_{N+1,i+1}a_{i+1,1} \\
&= \sum_{i=1}^{N+1}a_{N+1,i}a_{i,1},
\end{aligned}$$

which is exactly what we wanted to show! So we have proven the claim $A_N^2 = T_N$. Note that $A_N^* = A_N$, so that

$$c(A_N^*)c(A_N) = c(A_N)^2 = 1 + \sum_{m=1}^{N-1}\Big(\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2m-1}{2m}\Big)^2.$$

By Wallis' formula ([11, Page 697]),

$$\Big(\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2m-1}{2m}\Big)^2 < \frac{2}{\pi}\frac{1}{2m} = \frac{1}{\pi}\frac{1}{m}.$$

So we conclude that

$$c(A_N^*)c(A_N) \leq 1 + \frac{1}{\pi}\sum_{m=1}^{N-1}\frac{1}{m} \leq 1 + \frac{1}{\pi}\big(1+\log(N-1)\big). \tag{B.4}$$

Proposition B.2 now yields the result directly. □

C The tensor algebra

Given two vector spaces V, W over the same field k, we can form their tensor product $V \otimes W$, generated by the elements

$$v \otimes w, v \in V, w \in W,$$

which we call pure tensors. The elements of the space $V \otimes W$ are subject to the relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \tag{C.1}$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \tag{C.2}$$

$$\lambda v \otimes w = v \otimes \lambda w = \lambda(v \otimes w), \tag{C.3}$$

where $\lambda \in k, v, v_1, v_2 \in V, w, w_1, w_2 \in W$. $V \otimes W$ is again a vectorspace over the field k. One can make this construction more rigorous by defining the tensor product of $V \otimes W$ to be the quotient space of the vector space over k, generated by the formal symbols $v \otimes w$, modulo some ideal capturing precisely the relations (C.1), (C.2), (C.3). For more detail we refer to [12].

One can check that taking the tensor product is associative, in the sense that

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

So it makes sense to define $V^{\otimes n} = V \otimes \cdots \otimes V$ (n times).

Definition C.1. Given a vector space V over some field k, we define the *tensor algebra* of V, denoted TV, by

$$TV = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

Here $V^{\otimes 0}$ denotes the field k itself, viewed as a vector space over k.

Given some element $x \in V$, we define $x^{\otimes n} \in V^{\otimes n}$ by

$$x^{\otimes n} = x \otimes x \otimes \cdots \otimes x \ (n \text{ times}).$$

Proposition C.2. Let V be a vector space and let x be an element in V. Then

$$\begin{aligned}(1-x)^{\otimes n} = 1 \otimes \cdots \otimes 1 - \sum_{j=1}^{d} 1 \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 + \ldots \\ \cdots \sum_{i<j\leq d} 1 \otimes \cdots \otimes \overset{\overset{i}{\downarrow}}{x} \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 - \ldots \\ \ldots (-1)^n x \otimes x \otimes \cdots \otimes x.\end{aligned}$$

Proof. We argue by induction. For $n = 1$, the statement is clear. Suppose now that we have already obtained the result for some $n \geq 1$, then

$$\begin{aligned}
(1-x)^{\otimes n+1} &= (1-x)^{\otimes n} \otimes (1-x) \\
&= \Bigg(1 \otimes \cdots \otimes 1 - \sum_{j=1}^{n} 1 \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 + \ldots \\
&\qquad\qquad \cdots \sum_{i<j\leq n} 1 \otimes \cdots \otimes \overset{\overset{i}{\downarrow}}{x} \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 - \ldots \\
&\qquad\qquad\qquad\qquad \ldots (-1)^n x \otimes x \otimes \cdots \otimes x \Bigg) \otimes \big(1-x\big) \\
&= 1 \otimes \cdots \otimes 1 - \sum_{j=1}^{n+1} 1 \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 + \ldots \\
&\qquad\qquad \cdots \sum_{i<j\leq n+1} 1 \otimes \cdots \otimes \overset{\overset{i}{\downarrow}}{x} \otimes \cdots \otimes \overset{\overset{j}{\downarrow}}{x} \otimes \cdots \otimes 1 - \ldots \\
&\qquad\qquad\qquad\qquad \ldots (-1)^{n+1} SB \otimes x \otimes \cdots \otimes x.
\end{aligned}$$

□

D The spectrum of $D_{\mathbb{T}^d}$

In Section 5.1 we determined that the canonical spectral triple corresponding to the d-dimensional torus is given by

$$\left(C^\infty(\mathbb{T}^d), L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}, \sum_{\mu=1}^{d} -i\partial_\mu \otimes \gamma^\mu \right),$$

where $d = 2m$ if d is even and $d = 2m+1$ if d is odd. In this appendix we show that the spectrum $\sigma(D_{\mathbb{T}^d})$ of $D_{\mathbb{T}^d} = \sum_{\mu=1}^{d} -i\partial_\mu \otimes \gamma^\mu$ is given by

$$\sigma(D_{\mathbb{T}^d}) = \left\{ \pm\sqrt{n_1^2 + \ldots n_d^2} \mid n \in \mathbb{Z}^d \right\}.$$

As $D_{\mathbb{T}^d}$ has compact resolvent ([4, Section 10]), we know that all the spectrum of $D_{\mathbb{T}^d}$ is point spectrum. We are thus looking for values $\lambda \in \mathbb{R}$ for which there exists a spinor $\psi \in L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^m}$ such that

$$D_{\mathbb{T}^d}\psi = \lambda\psi. \tag{D.1}$$

We can decompose ψ into pure tensors

$$\psi = \sum_j \psi_j \otimes v_j, \qquad \psi_j \in L^2(\mathbb{T}^d), v_j \in \mathbb{C}^{2^m}. \tag{D.2}$$

Then applying the operator $D_{\mathbb{T}^d}$ once more to Equation (D.1) yields

$$D_{\mathbb{T}^d}^2\psi = \lambda^2\psi. \tag{D.3}$$

We can compute $D_{\mathbb{T}^d}^2$ more explicitly. Indeed, using the relations (3.17) and (3.18), we see that

$$\begin{aligned} D_{\mathbb{T}^d}^2 &= \left(\sum_{\mu=1}^{d} -i\partial_\mu \otimes \gamma^\mu \right)^2 \\ &= \sum_{\mu,\nu=1}^{d} (-i\partial_\mu)(-i\partial_\nu) \otimes \gamma^\mu\gamma^\nu \\ &= \sum_{\mu=1}^{d} -\partial_\mu^2 \otimes 1_{2^m} \\ &= \Delta \otimes 1_{2^m}. \end{aligned}$$

Here Δ denotes the operator $\sum_{\mu=1}^{d} -\partial_\mu^2$ on $L^2(\mathbb{T}^d)$. Then Equation (D.3) is equivalent to

$$\nabla\psi_j = \lambda^2\psi_j, \qquad \text{for all } j. \tag{D.4}$$

This has solutions $\psi_j = e^{in\cdot\theta} = e^{i(n_1\theta_1+\cdots+n_d\theta_d)}$, with $n \in \mathbb{Z}^d$ such that

$$n_1^2 + \cdots + n_d^2 = \lambda^2.$$

It follows that

$$\sigma(D_{\mathbb{T}^d}) \subseteq \left\{ \pm\sqrt{n_1^2 + \ldots n_d^2} \mid n \in \mathbb{Z}^d \right\}.$$

To show equality, it suffices to show that $\sigma(D_{\mathbb{T}^d})$ is a symmetric set. So we need to show $\lambda \in \sigma(D_{\mathbb{T}^d}) \implies -\lambda \in \sigma(D_{\mathbb{T}^d})$. We do this by introducing the operator $\tau : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ defined by

$$(\tau f)(\theta_1, \ldots, \theta_d) = f(-\theta_1, \ldots, -\theta_d).$$

The operator $\tau \otimes 1_n$ anti-commutes with $D_{\mathbb{T}^d}$. Then, if ψ is an eigenvector with eigenvalue λ, the spinor $(\tau \otimes 1_{2^m})\psi$ is an eigenvector with eigenvalue $-\lambda$. This shows the spectrum of $D_{\mathbb{T}^d}$ is a symmetric set and we conclude that

$$\sigma(D_{\mathbb{T}^d}) = \left\{ \pm\sqrt{n_1^2 + \ldots n_d^2} \mid n \in \mathbb{Z}^d \right\}.$$

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