Fundamental group of non-commutative spaces

Master thesis

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1 Introduction

The Gelfand-Naimark theorem gives a correspondence between locally compact Hausdorff spaces X and commutative C^*-algebras A: for any locally compact Hausdorff space X we can consider the C^*-algebra of continuous functions $C(X)$, and for each commutative C^*-algebra A we can consider the space of characters on A, with the weak$*$-topology. This allows us to think about topological spaces in terms of their corresponding algebra. For example, the space X is compact if and only if the algebra $C(X)$ is unital, and the space X is connected if and only if the algebra $C(X)$ contains no non-trivial projections. In non-commutative geometry we consider C^*-algebras A that are not required to be commutative, and try to find properties of the associated 'non-commutative space' - even though there is not really a topological space on which these properties should hold. This allows us to consider more general spaces: for example, phase space in quantum mechanics [6] or quotients by 'bad' equivalence relations [5]. There are also algebraic analogues for geometries with more structure than just a topology: measure spaces correspond to Von Neumann algebras (see [4] chapter V) and Riemannian spin manifolds correspond to spectral triples (see [4] chapter VI).

In this thesis we are interested in the fundamental group. The fundamental group of a space X can be defined as equivalence classes of continuous maps $\gamma : S^1 \to X$ sending a basepoint $s_0 \in S^1$ to a fixed basepoint $x_0 \in X$, where two such maps are equivalent if there is a base-point preserving continuous homotopy $H : S^1 \times [0, 1] \to X$ between them. To dualise this, we could consider maps from a C^*-algebra A to the C^*-algebra $C([0, 1])$. However this does not seem to be very interesting for non-commutative spaces: since $C([0, 1])$ is commutative, any $*$-homomorphism $\varphi : A \to C([0, 1])$ would send any commutator $ab - ba$ to 0, which for many non-commutative spaces already means that φ is trivial.

Instead, we restrict ourselves to manifolds, and we use that the representations of the fundamental group of a manifold correspond to flat connections on vector bundles over this manifold (this is proven in section 2). From this, the (algebraic hull of) the fundamental group can be reconstructed. To generalise this, we define a notion of flat connections over non-commutative spaces in section 3. In section 4 we apply this on non-commutative tori and toric deformated spaces. In section 5 we try to make the construction functorial.

2 The classical case

In this section we will demonstrate a well-known correspondence between the representations of the fundamental group of a manifold, and the flat connections on vector bundles over the manifold. This can be used to reconstruct the fundamental group (or at least an algebraic hull of the fundamental group) from the category of flat vector bundles. This will give an alternative definition of the fundamental group that is more suitable for generalisation to the non-commutative case, which we will attempt in later sections.

Let M be a connected compact manifold. It is well-known that each connected manifold has a universal cover $p : \tilde{M} \to M$. This is by definition a covering space such that $\tilde{M}$ is simply connected. It is also well-known that p is a $\pi_1(M)$-principal bundle, where $\pi_1(M)$ denotes the fundamental group of M. We show this in the following lemma.

Lemma 2.1. *Let M be a connected compact manifold with universal cover $p : \tilde{M} \to M$. Then p is a $\pi_1(M)$-principal bundle.*

Proof. Pick a basepoint $m_0 \in M$. For each path $\gamma : I \to M$ that starts and ends at m_0, and each point x in the fibre above m_0, there is a unique path $\tilde{\gamma} : I \to \tilde{M}$ above γ that starts at x. It does not have to end at x. This gives an action of the loop space $\Omega_{m_0} M$ on the fibre $p^{-1}(m_0)$, where $x \cdot \gamma$ is the endpoint of the path $\tilde{\gamma}$. Since $\gamma \cdot x$ depends continuously on γ and the fibre is discrete, we see that this action is homotopy invariant, so it factorises through a right action of $\pi_1(M, m_0)$ on the fibre $p^{-1}(m_0)$, which is indeed a group action. This action is transitive because $\tilde{M}$ is connected: if $x, x' \in p^{-1}(m_0)$ there is a path from x to x', this lies above a path $\gamma : I \to M$ that satisfies $x \cdot [\gamma] = x'$ (here $[\gamma]$ denotes the class of a loop γ in $\pi_1(M, m_0)$). The action is free because $\tilde{M}$ is simply connected: if $x \cdot [\gamma] = x$ we know that $\tilde{\gamma}$ is a path from x to x in $\tilde{M}$, so it is homotopic to a constant path, which gives a homotopy between γ and a constant path.

Now we define the action of $\pi_1(M, m_0)$ on the entire manifold $\tilde{M}$. Let $x_0 \in p^{-1}(m_0)$ be a fixed point above m_0. Let $[\gamma] \in \pi_1(M, m_0)$ and $x \in \tilde{M}$. Let $\chi : I \to \tilde{M}$ be a path that starts at x and ends at x_0. Now there is a unique path $\psi : I \to \tilde{M}$ that starts at $y \cdot [\gamma]$ such that $p \circ \psi$ is the reversal of the path $p \circ \chi$. Define $x \cdot [\gamma]$ as the endpoint of ψ. This is a point in the same fibre as x. This depends continuously on the choice of χ, but since the fibre is discrete and $\tilde{M}$ is simply connected, it does not depend on χ at all. This defines the group action of $\pi_1(M, m_0)$ on $\tilde{M}$, and this is again a free transitive action on each fibre. So $p : \tilde{M} \to M$ is a $\pi_1(M, m_0)$-principal bundle. □

Let $\mathrm{Rep}(\pi_1(M))$ denote the category of finite-dimensional representations of $\pi_1(M)$ and let $\mathrm{FlatVec}(M)$ denote the category of finite-dimensional vector bundles with a flat connection. The morphisms in the last category are the morphisms of vector bundles that commute with the connection. We can now show the equivalence between representations of $\pi_1(M)$ and flat vector bundles over M. Since this is a well-known theorem, we only sketch the proof.

Theorem 2.2. *Let M be a manifold. There is an equivalence of categories*

$$\mathrm{FlatVec}(M) \cong \mathrm{Rep}(\pi_1(M)).$$

Proof. Let m_0 be a base point of M. Let E be a vector bundle over M with a flat connection ∇. Let V be the fibre of E at p. The group $\pi_1(M, m_0)$ acts on V in the following way: for $[\gamma] \in \pi_1(M, m_0)$ and $x \in E$, there is a unique path $u : I \to E$ above γ, that starts at x and satisfies $\frac{\nabla u}{dt} = 0$. Define $[\gamma] \cdot x$ as the endpoint of u. This does not depend on the choice of representative $[\gamma]$ because the connection is flat, and it gives a group action of $\pi_1(M, m_0)$ on V. This defines the functor $\mathrm{FlatVec}(M) \to \mathrm{Rep}(\pi_1(M))$.

Conversely, let V be a finite-dimensional representation of $\pi_1(M)$. The group $\pi_1(M)$ acts on the right on $\tilde{M} \times V$ by $g \cdot (x, v) = (x \cdot g, g^{-1} \cdot v)$, using the right action of $\pi_1(M)$ on $\tilde{M}$ and the left action on V. Let $E = \tilde{M} \times_{\pi_1(M)} V$ be the quotient by this action. Its elements are of the form $[x, v]$ with $x \in \tilde{M}, v \in V$, where $[x \cdot g, v] = [x, g \cdot v]$ for $g \in \pi_1(M)$. Since the action of $\pi_1(M)$ on $\tilde{M}$ is fibre-wise, the composition $\tilde{M} \times V \to \tilde{M} \xrightarrow{p} M$ factors through a map $E \to M$. This makes E a vector bundle over M. Now we can define a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ as follows: each section $s \in \Gamma(E)$ can locally be written as $s(m) = [h(m), \tilde{s}(m)]$, with $h : U \subseteq M \to W \subseteq \tilde{M}$ a local section of p and $\tilde{s} : U \to V$. Then for any vector field X define $\nabla_X(s) \in \Gamma(E)$ by $\nabla_X(s)(m) = [h(m), \mathcal{L}_X \tilde{s}(m)]$, where $\mathcal{L}_X$ denotes the Lie derivative. We leave it to the reader to check that this is well-defined, and that ∇ satisfies the Leibniz rule and that it is flat. This gives the functor $\mathrm{Rep}(\pi_1(M)) \to \mathrm{FlatVec}(M)$. It is then easy to check that the two functors defined above are inverse to each other, establishing the equivalence $\mathrm{FlatVec}(M) \cong \mathrm{Rep}(\pi_1(M))$. □

2.1 Tannakian categories

From the category of representations of a group, it is possible to reconstruct (an algebraic hull of) the group. This was done by Saavedra Rivano [14], and Deligne and Milne fixed a gap in the proof [7]. We will state the definitions and the main theorem from [7] without proof.

The notion of a tensor category is modelled on the category of vector spaces with their tensor product. Let $\mathcal{C}$ be a category with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, sending (X, Y) to $X \otimes Y$. In a tensor category there are some given isomorphisms: for any three objects X, Y, Z there is the *associativity constraint* $\varphi_{X,Y,Z}$ which is an isomorphism from $X \otimes (Y \otimes Z)$ to $(X \otimes Y) \otimes Z$. For any two objects X, Y there is also a *commutativity constraint* $\psi_{X,Y}$ which is an isomorphism from $X \otimes Y$ to $Y \otimes X$, which is the inverse of $\psi_{Y,X}$. For all objects X, Y, Z, T the following diagrams have to commute:

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes T)) \arrow[dl] \arrow[dr] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr] & & ((X \otimes Y) \otimes Z) \otimes T,
\end{tikzcd}$$

where all the maps are given by associativity constraints, and

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \longrightarrow & (X \otimes Y) \otimes Z \\
\downarrow & & \downarrow \\
X \otimes (Z \otimes Y) & & Z \otimes (X \otimes Y) \\
\downarrow & & \downarrow \\
(X \otimes Z) \otimes Y & \longrightarrow & (Z \otimes X) \otimes Y,
\end{array}$$

where each map is an associativy constraint or a commutativity constraint. These are called the *pentagon axiom* and the *commutativity axiom* respectively [7]. An object U is called an *identity object* if there is an isomorphism $u : U \to U \otimes U$, and the functor $\mathcal{C} \to \mathcal{C}, X \to U \otimes X$ is an equivalence of categories. Now a tensor category is defined to be a system $(\mathcal{C}, \otimes, \varphi, \psi)$ satisfying the pentagon and hexagon axioms, with an identity object.

Examples of tensor categories include the category vector spaces and the category of R-modules, where R is any commutative ring.

Definition 2.3. Let $\mathcal{C}$ be a tensor category with unit U. A dual of an object X is an object $X^\vee$ satisfying

$$\mathrm{Hom}(T \otimes X, U) = \mathrm{Hom}(T, X^\vee)$$

for all objects T. Then we get $\mathrm{Hom}(X^\vee \otimes X, U) = \mathrm{Hom}(X^\vee, X^\vee)$ so we have a natural morphism $X^\vee \otimes X \to U$. If $X^\vee$ has a dual, we also have $\mathrm{Hom}(X^\vee \otimes X, U) = \mathrm{Hom}(X, (X^\vee)^\vee)$ so we get a natural morphism $X \to (X^\vee)^\vee$. If this is an isomorphism, X is called reflexive. The tensor category $\mathcal{C}$ is called *rigid* if each object has a dual and is reflexive.

For a rigid tensor category there also exist Hom objects: for two objects X and Y we have $\underline{\mathrm{Hom}}(X, Y) = X^\vee \otimes Y$, and this satisfies

$$\mathrm{Hom}(T \otimes X, Y) = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)).$$

The category of vector spaces is not rigid, since an infinite-dimensional vector space is not reflexive. However, the category of finite-dimensional vector spaces is rigid, and so is the category of finitely generated projective modules over a commutative algebra R.

Definition 2.4. A tensor category is called an abelian tensor category if it is abelian and the tensorproduct is bi-additive.

An example of this is the category of finite-dimensional representations of an affine algebraic group scheme. For the definition of an affine algebraic group scheme, see [7] or [15]. If G is an affine algebraic group scheme over $\mathbb{C}$, the category Rep_G of finite-dimensional representations of G is a rigid abelian tensor category. Let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_\mathbb{C}$ be the forgetful functor to the category of finite-dimensional vector spaces. Let $g \in G$. For each object $X \in \mathrm{Rep}_G$ the element g gives an automorphism on $\omega(X)$. These automorphisms are compatible with morphisms in Rep_G and also with the tensor product. So g induces a natural automorphism of the fibre functor ω. We get a map from G to $\mathrm{Aut}^{\otimes \omega}$, which denotes the natural automorphisms of ω that commute with tensor products. The next theorem from [7] shows that this is an isomorphism. It also shows which categories are equivalent to the category of finite-dimensional representations of an affine algebraic group scheme.

Theorem 2.5. *(i) Let G be an affine algebraic group scheme, and let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_{\mathbb{C}}$ be the forgetful functor. Then we have a natural isomorphism $G \to \mathrm{Aut}^{\otimes}(\omega)$.*

(ii) Let $\mathcal{C}$ be an abelian rigid tensor category and let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_{\mathbb{C}}$ be an exact faithful k-linear functor that commutes with tensor products. Assume that $\mathrm{End}(U) = \mathbb{C}$ for a unit $U \in \mathcal{C}$. Then $G = \mathrm{Aut}^{\otimes}(\omega)$ is an affine algebraic group scheme, and $\mathcal{C}$ is equivalent to the category Rep_G.

Proof. See [7], theorem 2.11. □

In the conditions of the second part, the functor ω is called the fibre functor. A category with a fibre functor that satisfies all the conditions of the second part of the theorem is called a *neutral Tannakian category.*

Let Γ be a topological group. The category Rep_Γ of finite-dimensional representations of Γ, with the forgetful functor to $\mathrm{Vec}_{\mathbb{C}}$, is a neutral Tannakian category. Therefore it is equivalent to the category of finite-dimensional representations of an affine group scheme G. In general this affine group scheme is not equal to Γ, unless Γ is a compact group. The affine group scheme G is called the *algebraic hull* of Γ. This concept is called Tannakian duality.

Our plan is now to define a category $\mathcal{C}(X)$ of flat connections on a non-commutative space X with a fibre functor, that satisfies the conditions of the theorem. We will then define the fundamental group of X to be the group scheme of automorphisms of the fibre functor. For a (commutative) manifold M we retrieve the (algebraic hull of) the usual fundamental group of M, by theorem 2.2.

3 Connections over a noncommutative space

In this section we will define connections over a noncommutative space. For our noncommutative spaces we take a differential graded $*$-algebra. We consider the category of all flat connections, and we will prove that under certain conditions this is a neutral Tannakian category. This allows us to define the fundamental group of a space. We also show that the category only depends on the (graded commutative) centre of the algebra.

There are multiple ways to generalise vector bundles to a non-commutative space. Over a commutative space, the vector bundles over a manifold correspond to finitely generated projective modules over the algebra. Over the non-commutative algebra we can look at right modules, but there is no tensor product of right modules. We can look at bimodules, but here we have to remember that over a commutative space, only very special bimodules are allowed: the multiplication on the left is the same as on the right. We can look at bimodules that satisfy this condition for the centre of the algebra. However, we look at a more restrictive class of bimodules, namely those that are a direct summand of a free finite module ([9] defines *diagonal bimodules* as modules that are a summand of a free module).

Connections are then defined as maps $\mathcal{E} \to \mathcal{E} \otimes \Omega^1 \mathcal{A}$ that satisfy the appropriate Leibniz rules. However to satisfy the left Leibniz rule the image should actually be $\Omega^1 A \otimes \mathcal{E}$. This is solved in [8] by using an isomorphism $\sigma : \mathcal{E} \otimes \Omega^1 \mathcal{A} \to \Omega^1 \mathcal{A} \to \mathcal{E}$. However, in their approach the tensor product of two flat connections is not necessarily flat. Therefore we look at graded bimodules $\Omega^\bullet \mathcal{E}$ over the differential algebra $\Omega^\bullet \mathcal{A}$ that are a summand of a free finite module. That way we automatically have $\Omega^1 \mathcal{E} \cong \mathcal{E} \otimes \Omega^1 \mathcal{A} \cong \Omega^1 \mathcal{A} \otimes \mathcal{E}$ (see lemma 3.3), but not all isomorphisms $\mathcal{E} \otimes \Omega^1 \mathcal{A} \to \Omega^1 \mathcal{A} \otimes \mathcal{E}$ are possible. In this way the tensor product of flat modules is always flat (see lemma 3.15).

Definition 3.1. A *graded differential $*$-algebra* or $*$-dga is a graded $*$-algebra $\Omega^\bullet \mathcal{A}$, with $\Omega^0 \mathcal{A} = \mathcal{A}$, together with maps $d : \Omega^n \mathcal{A} \to \Omega^{n+1} \mathcal{A}$ satisfying the graded Leibniz rule $d(\omega\nu) = d(\omega)\nu + (-1)^k \omega d(\nu)$ for $\omega \in \Omega^k, \nu \in \Omega^l$, and $d^2 = 0$.

Note that we do not ask that $\Omega^1 \mathcal{A}$ should be spanned by elements of the form adb with $a, b \in \mathcal{A}$. Therefore $\Omega^\bullet \mathcal{A}$ is in general not a quotient of the universal differential algebra of $\mathcal{A}$.

Notation 3.2. *Let $\Omega^\bullet \mathcal{E}$ be a graded $\Omega^\bullet \mathcal{A}$-bimodule. We define the graded commutator $[\cdot, \cdot] : \Omega^\bullet \mathcal{A} \times \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ as $[\alpha, \varepsilon] = \alpha\varepsilon - (-1)^{kl} \varepsilon\alpha$ for $\alpha \in \Omega^k \mathcal{A}, \varepsilon \in \Omega^l \mathcal{E}$ and $\mathbb{C}$-bilinearly extended. We use the same notation for the graded commutator $[\cdot, \cdot] : \Omega^\bullet \mathcal{A} \times \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{A}$ defined similarly.*

Let $\Omega^\bullet \mathcal{A}$ be any dga. We call a graded $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{E}$ finitely generated projective if there is another graded $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{F}$ satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$ for some integer n, as graded $\Omega^\bullet \mathcal{A}$-bimodules. The following lemma shows why it is convenient to work with these graded fgp $\Omega^\bullet \mathcal{A}$-bimodules instead of just fgp bimodules over $\mathcal{A} = \Omega^0 \mathcal{A}$.

Lemma 3.3. *Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule. Write $\mathcal{A} = \Omega^0\mathcal{A}$ and $\mathcal{E} = \Omega^0\mathcal{E}$. Then for all $k \geq 0$ the multiplication induces isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \xrightarrow{\sim} \Omega^k\mathcal{E}$$
$$\Omega^k\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\sim} \Omega^k\mathcal{E}.$$

Proof. Let $\Omega^\bullet \mathcal{F}$ be another graded $\Omega^\bullet \mathcal{A}$-bimodule satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \cong (\Omega^\bullet \mathcal{A})^n$. Then we have the following commuting diagram:

$$\begin{tikzcd}
\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \oplus \mathcal{F} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[rr] \arrow[d, "\sim"] & & \Omega^k\mathcal{E} \oplus \Omega^k\mathcal{F} \arrow[d, "\sim"] \\
(\mathcal{E} \oplus \mathcal{F}) \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[r, "\sim"] & \mathcal{A}^n \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[r, "\sim"] & (\Omega^k\mathcal{A})^n.
\end{tikzcd}$$

Here the top arrow is the direct sum of the maps $\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \to \Omega^k\mathcal{E}$ and $\mathcal{F} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \to \Omega^k\mathcal{F}$. All the other maps are isomorphisms, so these maps are isomorphisms as well. This shows that the first map in the lemma is an isomorphism and the second one follows analogously. □

So we get isomorhpisms (twists) $\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \xrightarrow{\sim} \Omega^k\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}$. Not all twists are possible: for example, assume that the centre $Z(\mathcal{A})$ of $\mathcal{A}$ commutes with $\Omega^\bullet \mathcal{A}$. We can write $\mathcal{E} = Z(\mathcal{E}) \otimes_{Z(\mathcal{A})} \mathcal{A}$. Since $\Omega^\bullet \mathcal{E}$ is a subbimodule of some $(\Omega^\bullet \mathcal{A})^n$ we know that $Z(\mathcal{E})$ commutes with $\Omega^k\mathcal{A}$. We have $\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} = Z(\mathcal{E}) \otimes_{Z(\mathcal{A})} \Omega^k\mathcal{A}$ and $\Omega^k\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E} = \Omega^k\mathcal{A} \otimes_{Z(\mathcal{A})} Z(\mathcal{E})$. These are naturally isomorphic by switching the factors and this is the only twist we can have. The graded fgp bimodule $\Omega^\bullet \mathcal{E}$ is then completely determined by $\mathcal{E}$, which can be any fgp bimodule over $\mathcal{A}$. This is not the case when $Z(\mathcal{A})$ does not commute with $\Omega^\bullet \mathcal{A}$, see example 3.13. But we do not have to bother with these twists at all: we simply view a connection on $\Omega^\bullet \mathcal{E}$ as a map $\Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ of degree $+1$, satisfying the proper Leibniz rules.

Definition 3.4. Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule. A *connection* on $\Omega^\bullet \mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$$

of degree $+1$ satisfying, for $\varepsilon \in \Omega^k\mathcal{E}$ and $\alpha \in \Omega^l\mathcal{A}$:

$$\nabla(\varepsilon\alpha) = \nabla(\varepsilon)\alpha + (-1)^k \varepsilon d\alpha$$
$$\nabla(\alpha\varepsilon) = d\alpha \cdot \varepsilon + (-1)^l \alpha\nabla(\varepsilon).$$

Definition 3.5. Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule with a connection ∇. The curvature of ∇ is the map $\nabla^2 : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ of degree $+2$.

Remark 3.6. The curvature is an $\Omega^\bullet \mathcal{A}$-bilinear map: for $\varepsilon \in \Omega^k\mathcal{E}$ and $\alpha \in \Omega^l\mathcal{A}$ we have

$$\nabla^2(\varepsilon\alpha) = \nabla(\nabla(\varepsilon)\alpha + (-1)^k \varepsilon d\alpha)$$

$$
\begin{aligned}
&= \nabla^2(\varepsilon)\alpha + (-1)^{k+1}\nabla(\varepsilon)d\alpha + (-1)^k\nabla(\varepsilon)d\alpha + \varepsilon d^2\alpha \\
&= \nabla^2(\varepsilon)\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla^2(\alpha\varepsilon) &= \nabla(d\alpha\cdot\varepsilon + (-1)^l\alpha\nabla(\varepsilon)) \\
&= d^2\alpha\cdot\varepsilon + (-1)^{l+1}d\alpha\nabla(\varepsilon) + (-1)^l d\alpha\nabla(\varepsilon) + \alpha\nabla^2(\varepsilon) \\
&= \alpha\nabla^2(\varepsilon).
\end{aligned}
$$

Definition 3.7. A connection ∇ is called *flat* if the curvature ∇^2 is zero.

Now we can define the category of flat connections.

Definition 3.8. Let $\mathcal{C}(\Omega^\bullet\mathcal{A})$ be the category whose objects are graded fgp $\Omega^\bullet\mathcal{A}$-bimodules with a connection, and whose morphisms are graded fgp $\Omega^\bullet\mathcal{A}$-bimodule morphisms which commute with the connections. Let $\mathcal{C}_{\mathsf{flat}}(\Omega^\bullet\mathcal{A})$ be the full subcategory where the connections are required to be flat.

Example 3.9. Let M be a manifold. The corresponding dga is the differential algebra of the manifold $\Omega^\bullet\mathcal{A} = \Omega^\bullet M$. Any fgp $\Omega^\bullet\mathcal{A}$-bimodule $\Omega^\bullet\mathcal{E}$ is determined by the fgp $\mathcal{A}$-bimodule $\mathcal{E}$. This corresponds to a vector bundle over M by the Serre-Swan theorem. A flat connection on $\Omega^\bullet\mathcal{E}$ corresponds to a (usual) flat connection on this vector bundle. So $\mathcal{C}(\Omega^\bullet\mathcal{A})$ is equivalent to the category of vector bundles over M with a flat connection, which is equivalent to the category of representations of $\pi_1 M$ by theorem 2.2.

Each graded fgp $\Omega^\bullet\mathcal{A}$-bimodule is determined by a graded fgp bimodule over a graded commutative subalgebra. For this we need the following definitions:

Definition 3.10. Let $\Omega^\bullet\mathcal{A}$ be a $*$-dga. We define the graded commutative centre $Z_g(\Omega^\bullet\mathcal{A})$ as

$$Z_g(\Omega^\bullet\mathcal{A}) = \{\alpha\in\Omega^\bullet\mathcal{A} \mid [\alpha,\nu]=0 \text{ for all } \nu\in\Omega^\bullet\mathcal{A}\}.$$

If $\Omega^\bullet\mathcal{E}$ is an fgp $\Omega^\bullet\mathcal{A}$-bimodule we define $Z_g(\Omega^\bullet\mathcal{E})$ as

$$Z_g(\Omega^\bullet\mathcal{E}) = \{\varepsilon\in\Omega^\bullet\mathcal{A} \mid [\varepsilon,\nu]=0 \text{ for all } \nu\in\Omega^l\mathcal{A}\}.$$

Lemma 3.11. *With notations as in the previous definition we have the following:*

(i) The graded commutative centre of the algebra $Z_g(\Omega^\bullet\mathcal{A})$ is a $*$-dga.

(ii) The graded commutative centre of the module $Z_g(\Omega^\bullet\mathcal{E})$ is a graded fgp bimodule over $Z_g(\Omega^\bullet\mathcal{A})$.

(iii) The multiplication induces an isomorphism $Z_g(\Omega^\bullet\mathcal{E})\otimes_{Z_g(\Omega^\bullet\mathcal{A})}\Omega^\bullet\mathcal{A}\xrightarrow{\sim}\Omega^\bullet\mathcal{E}$.

Proof. (i) An easy calculation shows that $Z_g(\Omega^\bullet\mathcal{A})$ is a subalgebra of $\Omega^\bullet\mathcal{A}$, using that $[\alpha\beta,\nu] = \alpha[\beta,\nu] + (-1)^{kl}[\alpha,\nu]\beta$ for $\alpha\in\Omega^\bullet\mathcal{A}, \beta\in\Omega^k\mathcal{A}, \nu\in\Omega^l\mathcal{A}$. It is star-closed because $[\alpha^*,\nu] = (-1)^{kl}[\alpha,\nu^*]^*$ for $\alpha\in\Omega^k\mathcal{A}, \nu\in\Omega^l\mathcal{A}$. It is closed under d because $[d\alpha,\nu] = d[\alpha,\nu] - (-1)^k[\alpha,d\nu]$ for $\alpha\in\Omega^k\mathcal{A}, \nu\in\Omega^\bullet\mathcal{A}$.

(ii) An easy calculation shows that $Z_g(\Omega^\bullet \mathcal{E})$ is a graded bimodule over $Z_g(\Omega^\bullet \mathcal{A})$, using $[\alpha\varepsilon,\nu] = \alpha[\varepsilon,\nu] + (-1)^{kl}[\alpha,\nu]\varepsilon$ for $\alpha \in \Omega^\bullet \mathcal{A}, \varepsilon \in \Omega^k\mathcal{E}, \nu \in \Omega^l\mathcal{A}$. If $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$ it follows directly that $Z_g(\Omega^\bullet \mathcal{E}) \oplus Z_g(\Omega^\bullet \mathcal{F}) = (Z_g(\Omega^\bullet \mathcal{A}))^n$, so $Z_g(\Omega^\bullet \mathcal{E})$ is a graded fgp $Z_g(\Omega^\bullet \mathcal{A})$-bimodule.

(iii) Let $\Omega^\bullet \mathcal{F}$ be another graded fgp $\Omega^\bullet \mathcal{A}$-bimodule satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$. Then we have the following commuting diagram:

$$\begin{tikzcd}
Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \oplus Z_g(\Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[r] \arrow[d, "\sim"] & \Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \arrow[d, "\sim"] \\
(Z_g(\Omega^\bullet \mathcal{E}) \oplus Z_g(\Omega^\bullet \mathcal{F})) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[d, "\sim"] & (\Omega^\bullet \mathcal{A})^n \\
Z_g(\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[r, "\sim"] & Z_g((\Omega^\bullet \mathcal{A})^n) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}. \arrow[u, "\sim"']
\end{tikzcd}$$

The top arrow is the direct sum of the maps $Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{E}$ and $Z_g(\Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{F}$ induced by multiplication. All other arrows in the diagram are isomorphisms, so these maps are isomorphisms as well.

□

So all graded fgp $\Omega^\bullet \mathcal{A}$-bimodules are determined by a graded fgp $Z_g(\Omega^\bullet \mathcal{A})$-bimodule. This is in turn determined by an fgp $Z_g(\Omega^0\mathcal{A})$-bimodule. Note that $Z_g(\Omega^0\mathcal{A})$ may be smaller than the centre of the algebra $\mathcal{A}$.

Theorem 3.12. *We have a natural isomorphism*

$$\mathcal{C}(\Omega^\bullet \mathcal{A}) \xrightarrow{\sim} \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A})).$$

Proof. Let $(\Omega^\bullet \mathcal{E}, \nabla)$ be an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. For $\varepsilon \in Z_g(\Omega^k\mathcal{E})$ and $\alpha \in \Omega^l\mathcal{A}$ we have

$$\nabla(\varepsilon\alpha) = \nabla(\varepsilon)\alpha + (-1)^k\varepsilon d\alpha$$

but also

$$\nabla(\varepsilon\alpha) = (-1)^{kl}\nabla(\alpha\varepsilon) = (-1)^{(k+1)l}\alpha\nabla(\varepsilon) + (-1)^{kl}d\alpha\cdot\varepsilon = (-1)^{(k+1)l}\alpha\nabla(\varepsilon) + (-1)^k\varepsilon d\alpha.$$

So we get

$$\nabla(\varepsilon)\alpha = (-1)^{(k+1)l}\alpha\nabla(\varepsilon).$$

Since this holds for all $\alpha \in \Omega^l\mathcal{A}$ we conclude that $\nabla(\varepsilon) \in Z_g(\Omega^{k+1}\mathcal{E})$. So ∇ restricts to a function $Z_g(\Omega^\bullet \mathcal{E}) \to Z_g(\Omega^\bullet \mathcal{E})$. Then $(Z_g(\Omega^\bullet \mathcal{E}), \nabla_{Z_g(\Omega^\bullet \mathcal{E})})$ is an object of $\mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$. It is easy to see that this is a functorial construction.

Conversely, let $(\Omega^\bullet \mathcal{F}, \nabla)$ be an object of $\mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$. Then we can define the graded fgp $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{F} \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}$, and the connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}(\zeta \otimes \alpha) = \nabla(\zeta) \otimes \alpha + (-1)^k\zeta \otimes d(\alpha)$$

for $\zeta \in \Omega^k \mathcal{F}, \alpha \in \Omega^l \mathcal{A}$. This gives an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. It is then easy to show that this construction is also functorial, and that the two functors thus defined are inverse to each other. $\square$

The above isomorphism also restricts to an isomorphism $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \xrightarrow{\sim} \mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$ as the functor and its inverse clearly preserve flatness.

For a graded commutative dga $\Omega^\bullet \mathcal{A}$ we have a different way to describe the category. Remember that for a graded fgp bimodule $\Omega^\bullet \mathcal{E}$ we have the isomorphisms $\Omega^k \mathcal{E} = \mathcal{E} \otimes_{\mathcal{A}} \Omega^k \mathcal{A}$. The restriction of ∇ to $\mathcal{E}$ is then a map $\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A}$, satisfying $\nabla_0(ea) = \nabla_0(e)a + e \otimes da$. Conversely each such ∇_0 may be extended to $\nabla : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ by setting $\nabla(e \otimes \omega) = \nabla_0(e)\omega + e \otimes d\omega$. So we can describe $\mathcal{C}(\Omega^\bullet \mathcal{A})$ as the category with objects fgp $\mathcal{A}$-modules $\mathcal{E}$ with a connection $\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \mathcal{A}$.

Example 3.13. Let $\mathcal{A}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in M_2(\mathbb{C}) \right\}$ and $\mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \in M_2(\mathbb{C}) \right\}$. Let $\Omega^\bullet \mathcal{A}$ be defined by $\Omega^k \mathcal{A} = \mathcal{A}_0$ for even k and $\Omega^k \mathcal{A} = \mathcal{A}_1$ for odd k, with the differential $d : \mathcal{A}_0 \to \mathcal{A}_1, \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \to \begin{pmatrix} 0 & \beta - \alpha \\ \alpha - \beta & 0 \end{pmatrix}$, and $d : \mathcal{A}_1 \to \mathcal{A}_0, \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \to \begin{pmatrix} \delta + \gamma & 0 \\ 0 & \gamma + \delta \end{pmatrix}$. This is the dga corresponding to the non-commutative space of two points with finite distance.

Now we have $Z_g(\Omega^k \mathcal{A}) = \mathbb{C}$ in even degrees and $Z_g(\Omega^k \mathcal{A}) = 0$ in odd degrees. The fgp bimodules over $Z_g(\Omega^k \mathcal{A})$ are simply determined by a vector space over $\mathbb{C}$, so we get that $\mathcal{C}(\Omega^\bullet \mathcal{A}) \cong \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ is equivalent to the category of vector spaces.

Now we will construct tensor products and duals on the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$.

3.0.1 Tensor products

Lemma 3.14. *Let $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ be objects of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Then $\Omega^\bullet \mathcal{G} = \Omega^\bullet \mathcal{E} \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{F}$ has the structure of an fgp graded bimodule over $\Omega^\bullet \mathcal{A}$, and we can construct a connection $\nabla^{\mathcal{G}}$ satisfying*

$$\nabla^{\mathcal{G}}(\varepsilon \otimes \zeta) = \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^k \varepsilon \otimes \nabla^{\mathcal{F}}(\zeta)$$

for $\varepsilon \in \Omega^k \mathcal{E}, \zeta \in \Omega^\bullet \mathcal{F}$.

Proof. The left action of $\Omega^\bullet \mathcal{A}$ on $\Omega^\bullet \mathcal{E}$ and the right action of $\Omega^\bullet \mathcal{A}$ on $\Omega^\bullet \mathcal{F}$ make $\Omega^\bullet \mathcal{G}$ into a $\Omega^\bullet \mathcal{A}$-bimodule. A grading on the tensor product is given as follows: for any $n \geq 0$ the degree n subspace $\Omega^n \mathcal{G}$ is the linear span of elements $\varepsilon \otimes \zeta$, with $\varepsilon \in \Omega^k \mathcal{E}, \zeta \in \Omega^l \mathcal{F}, k+l = n$. To show that $\Omega^\bullet \mathcal{G}$ is fgp, suppose that $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{E}' \cong \Omega^\bullet \mathcal{A}^k$ and $\Omega^\bullet \mathcal{F} \oplus \Omega^\bullet \mathcal{F}' \cong \Omega^\bullet \mathcal{A}^l$. Then

$$\Omega^\bullet \mathcal{G} \oplus \Omega^\bullet \mathcal{E}' \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{F} \oplus (\Omega^\bullet \mathcal{F}')^k \cong (\Omega^\bullet \mathcal{F})^k \oplus (\Omega^\bullet \mathcal{F}')^k \cong (\Omega^\bullet \mathcal{A})^{kl}.$$

So $\Omega^\bullet \mathcal{G}$ is a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule.

We can then define the connection $\nabla^{\mathcal{G}}$ by

$$\nabla^{\mathcal{G}}(\varepsilon \otimes \zeta) = \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^k \varepsilon \otimes \nabla^{\mathcal{F}}(\zeta)$$

for $\varepsilon \in \Omega^k \mathcal{E}, \zeta \in \Omega^\bullet \mathcal{F}$. This defines $\nabla^{\mathcal{G}}$ on the pure tensors, and it is extended $\mathbb{C}$-linearly. To show that it is well-defined, let $\alpha \in \Omega^l \mathcal{A}$. Then by the above definition we have

$$\begin{aligned}\nabla^{\mathcal{G}}(\varepsilon\alpha \otimes \zeta) &= \nabla^{\mathcal{E}}(\varepsilon\alpha) \otimes \zeta + (-1)^{k+l} \varepsilon\alpha \otimes \nabla^{\mathcal{F}}(\zeta) \\ &= \nabla^{\mathcal{E}}(\varepsilon)\alpha \otimes \zeta + (-1)^k \varepsilon d\alpha \otimes \zeta + (-1)^{k+l} \varepsilon\alpha \otimes \nabla^{\mathcal{F}}(\zeta)\end{aligned}$$

while

$$\begin{aligned}\nabla^{\mathcal{G}}(\varepsilon \otimes \alpha\zeta) &= \nabla^{\mathcal{E}}(\varepsilon) \otimes \alpha\zeta + (-1)^k \varepsilon \otimes \nabla^{\mathcal{F}}(\alpha\zeta) \\ &= \nabla^{\mathcal{E}}(\varepsilon) \otimes \alpha\zeta + (-1)^k \varepsilon \otimes d\alpha \cdot \zeta + (-1)^{k+l} \varepsilon \otimes \alpha \nabla^{\mathcal{F}}(\zeta).\end{aligned}$$

Since these are the same, $\nabla^{\mathcal{G}}$ is well-defined.

Lastly, $\nabla^{\mathcal{G}}$ satisfies the Leibniz rules: for $\varepsilon \in \Omega^k \mathcal{E}, \zeta \in \Omega^l \mathcal{F}, \alpha \in \Omega^n \mathcal{A}$ we have

$$\begin{aligned}\nabla^{\mathcal{G}}(\alpha\varepsilon \otimes \zeta) &= \nabla^{\mathcal{E}}(\alpha\varepsilon) \otimes \zeta + (-1)^{k+n} \alpha\varepsilon \otimes \nabla^{\mathcal{F}}(\zeta) \\ &= d\alpha \cdot \varepsilon \otimes \zeta + (-1)^n \alpha \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^{k+n} \alpha\varepsilon \otimes \nabla^{\mathcal{F}}(\zeta) \\ &= d\alpha \cdot \varepsilon \otimes \zeta + (-1)^n \alpha \nabla^{\mathcal{G}}(\varepsilon \otimes \zeta)\end{aligned}$$

and

$$\begin{aligned}\nabla^{\mathcal{G}}(\varepsilon \otimes \zeta\alpha) &= \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta\alpha + (-1)^k \varepsilon \otimes \nabla^{\mathcal{G}}(\zeta\alpha) \\ &= \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta\alpha + (-1)^k \varepsilon \otimes \nabla^{\mathcal{G}}(\zeta)\alpha + (-1)^{k+l} \varepsilon \otimes \zeta d\alpha \\ &= \nabla^{\mathcal{G}}(\varepsilon \otimes \zeta)\alpha + (-1)^{k+l} \varepsilon \otimes \zeta d\alpha.\end{aligned}$$

□

The curvature on the tensor product is easily calculated:

Lemma 3.15. *In the notation of the previous lemma, we have*

$$(\nabla^{\mathcal{G}})^2 = (\nabla^{\mathcal{E}})^2 \otimes \Omega^\bullet \mathcal{F} \oplus \Omega^\bullet \mathcal{E} \otimes (\nabla^{\mathcal{F}})^2.$$

In particular, the tensor product of flat connections is again flat.

Proof. For $\varepsilon \in \Omega^\bullet \mathcal{E}, \zeta \in \Omega^k \mathcal{F}$ we have

$$\begin{aligned}(\nabla^{\mathcal{G}})^2(\varepsilon \otimes \zeta) &= \nabla^{\mathcal{G}}(\nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^k \varepsilon \otimes \nabla^{\mathcal{F}}(\zeta)) \\ &= (\nabla^{\mathcal{E}})^2(\varepsilon) \otimes \zeta + (-1)^{k+1} \nabla^{\mathcal{E}}(\varepsilon) \otimes \nabla^{\mathcal{F}}(\zeta) \\ &+ (-1)^k \nabla^{\mathcal{E}}(\varepsilon) \otimes \nabla^{\mathcal{F}}(\zeta) + \varepsilon \otimes (\nabla^{\mathcal{F}})^2(\zeta) \\ &= (\nabla^{\mathcal{E}})^2(\varepsilon) \otimes \zeta + \varepsilon \otimes (\nabla^{\mathcal{F}})^2(\zeta).\end{aligned}$$

□

It is easy to see that this tensor product is associative. The tensor product commutes with the equivalence of categories $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ from theorem 3.12. In the commutative case it is easy to see that the tensor product is also commutative; so we have a commutativity constraint in the general case as well.

There is a unit in $\mathcal{C}(\Omega^\bullet \mathcal{A})$: it is the bimodule $\Omega^\bullet \mathcal{A}$ with the connection d. It is easy to see that the isomorphism $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}$ commutes with the connection $\nabla^{\mathcal{E}}$ and the tensor product connection on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}$.

This makes $(\mathcal{C}(\Omega^\bullet \mathcal{A}), \otimes)$ into a tensor category, as well as the subcategory $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$. Now we go on to construct duals.

3.0.2 Duals

Lemma 3.16. *Let $(\Omega^\bullet \mathcal{E}, \nabla)$ be an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Then the dual $\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$ of right-$\Omega^\bullet \mathcal{A}$-linear maps from $\Omega^\bullet \mathcal{E}$ to $\Omega^\bullet \mathcal{A}$ is a fgp $\Omega^\bullet \mathcal{A}$-bimodule. We can construct a connection on $\Omega^\bullet \mathcal{E}^\vee$ satisfying*

$$\langle \nabla^\vee(\theta), \varepsilon \rangle = d\langle \theta, \varepsilon \rangle - (-1)^k \langle \theta, \nabla\varepsilon \rangle$$

for $\theta \in \Omega^k \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}$. Here the angled brackets denote the pairing between $\Omega^\bullet \mathcal{E}^\vee$ and $\Omega^\bullet \mathcal{E}$.

Proof. The bimodule structure is given by $\langle \theta\alpha, \varepsilon \rangle = \langle \theta, \alpha\varepsilon \rangle$ and $\langle \alpha\theta, \varepsilon \rangle = \alpha\langle \theta, \varepsilon \rangle$ for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \alpha \in \Omega^\bullet \mathcal{A}, \varepsilon \in \Omega^\bullet \mathcal{E}$. There is a natural grading where $\Omega^{\bullet k} \mathcal{E}^\vee$ consists of the maps of degree k. If $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \cong \Omega^\bullet \mathcal{A}^n$ we get $\Omega^\bullet \mathcal{E}^\vee \oplus \Omega^\bullet \mathcal{F}^\vee = (\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F})^\vee \cong \Omega^\bullet \mathcal{A}^n$. So $\Omega^\bullet \mathcal{E}^\vee$ is a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule.

We can define the connection $\nabla^\vee$ on $\Omega^\bullet \mathcal{E}^\vee$ by

$$\langle \nabla^\vee(\theta), \varepsilon \rangle = d\langle \theta, \varepsilon \rangle - (-1)^k \langle \theta, \nabla(\varepsilon) \rangle$$

for $\theta \in \Omega^k \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$. Then $\nabla^\vee(\theta)$ is a right-linear map because we get, for $\theta \in \Omega^k \mathcal{E}^\vee, \varepsilon \in \Omega^l \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$:

$$\begin{aligned}
\langle \nabla^\vee(\theta), \varepsilon\alpha \rangle &= d\langle \theta, \varepsilon\alpha \rangle - (-1)^k \langle \theta, \nabla(\varepsilon\alpha) \rangle \\
&= d(\langle (\theta, \varepsilon \rangle \alpha) - (-1)^k \langle \theta, \nabla(\varepsilon)\alpha + (-1)^k \varepsilon d\alpha \rangle \\
&= d\langle \theta, \varepsilon \rangle \alpha + (-1)^{k+l} \langle \theta, \varepsilon \rangle d\alpha - (-1)^k \langle \theta, \nabla(\varepsilon) \rangle \alpha - (-1)^{k+l} \langle \theta, \varepsilon \rangle d\alpha \\
&= \langle \nabla^\vee(\theta), \varepsilon \rangle \alpha.
\end{aligned}$$

This satisfies the Leibniz rules: for $\theta \in \Omega^k \mathcal{E}^\vee, \varepsilon \in \Omega^l \mathcal{E}, \alpha \in \Omega^n \mathcal{A}$ we have

$$\begin{aligned}
\langle \nabla^\vee(\theta\alpha), \varepsilon \rangle &= d\langle \theta\alpha, \varepsilon \rangle - (-1)^{k+n} \langle \theta\alpha, \nabla(\varepsilon) \rangle \\
&= d\langle \theta, \alpha\varepsilon \rangle - (-1)^{k+n} \langle \theta, \alpha\nabla(\varepsilon) \rangle \\
&= d\langle \theta, \alpha\varepsilon \rangle - (-1)^{k+n} \langle \theta, \nabla(\alpha\varepsilon) \rangle + (-1)^k \langle \theta, d\alpha \cdot \varepsilon \rangle \\
&= \langle \nabla^\vee(\theta), \alpha\varepsilon \rangle + (-1)^k \langle \theta d\alpha, \varepsilon \rangle
\end{aligned}$$

$$= \langle \nabla^\vee(\theta)\alpha + (-1)^k \theta d\alpha, \varepsilon\rangle$$

and

$$\begin{aligned}
\langle \nabla^\vee(\alpha\theta), \varepsilon\rangle &= d\langle \alpha\theta, \varepsilon\rangle - (-1)^{k+n}\langle \alpha\theta, \nabla(\varepsilon)\rangle \\
&= d\alpha\langle\theta, \varepsilon\rangle + (-1)^n \alpha d\langle\theta,\varepsilon\rangle - (-1)^{k+n}\alpha\langle\theta, \nabla(\varepsilon)\rangle \\
&= \langle d\alpha\cdot\theta + (-1)^n\alpha\nabla^\vee(\theta), \varepsilon\rangle.
\end{aligned}$$

□

We can compute the curvature of the dual connection:

Lemma 3.17. *In the notation of the previous lemma, the curvature of $\nabla^\vee$ is minus the dual of the curvature of ∇, that is, for $\theta \in \Omega^\bullet \mathcal{E}, \varepsilon \in \Omega^\bullet \mathcal{E}$ we have*

$$\langle (\nabla^\vee)^2(\theta), \varepsilon\rangle = -\langle \theta, \nabla^2(\varepsilon)\rangle.$$

In particular, the dual of a flat connection is again flat.

Proof. We have for $\theta \in \Omega^k\mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}$:

$$\begin{aligned}
\langle ((\nabla^\vee)^2(\theta), \varepsilon\rangle &= d\langle \nabla^\vee(\theta), \varepsilon\rangle - (-1)^{k+1}\langle \nabla^\vee(\theta), \nabla(\varepsilon)\rangle \\
&= d(d\langle\theta,\varepsilon\rangle - (-1)^k\langle\theta, \nabla(\varepsilon)\rangle) \\
&- (-1)^{k+1} d\langle\theta, \nabla(\varepsilon)\rangle - \langle\theta, \nabla^2(\varepsilon)\rangle \\
&= -\langle\theta, \nabla^2(\varepsilon)\rangle.
\end{aligned}$$

□

Writing $\Omega^\bullet \mathcal{E} = Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}$ we have

$$\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{A}) = Z_g(\Omega^\bullet \mathcal{E})^\vee \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}.$$

So the equivalence of categories $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ commutes with the taking of duals. In particular this shows that the dual $\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$ is naturally isomorphic to the space of left-linear functions ${}_{\Omega^\bullet \mathcal{A}}\mathrm{Hom}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$.

Since $\Omega^\bullet E$ is a fgp bimodule, we have for each graded fgp bimodule $\Omega^\bullet \mathcal{F}$ an isomorphism $\mathrm{Hom}(\Omega^\bullet \mathcal{F} \otimes \Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A}) = \mathrm{Hom}(\Omega^\bullet \mathcal{F}, \Omega^\bullet \mathcal{E}^\vee \otimes \Omega^\bullet \mathcal{A})$. An easy calculation shows that this isomorphism continues to hold for morphisms that commute with connections, if connections on $\Omega^\bullet \mathcal{E}$ and $\Omega^\bullet \mathcal{F}$ are given. So $(\Omega^\bullet \mathcal{E}^\vee, \nabla^\vee)$ is a dual object to $(\Omega^\bullet \mathcal{E}, \nabla)$. The morphism $\Omega^\bullet \mathcal{E} \to (\Omega^\bullet \mathcal{E}^\vee)^\vee$ is an isomorphism because $\Omega^\bullet \mathcal{E}$ is fgp. So every object is reflexive, and $(\mathcal{C}(\Omega^\bullet \mathcal{A}), \otimes)$ is a rigid tensor category, and the subcategory $\mathcal{C}_{\mathrm{flat}}$ is also a rigid tensor category.

3.1 Abelianness of the category

In this subsection we will show when the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian. Using theorem 3.12 we can always reduce to a graded commutative algebra. Therefore we will only consider graded commutative algebras in this subsection. We will assume that $\mathcal{A}$ is a unital $*$-algebra that is dense in a unital C^*-algebra A. This will be necessary for some of the constructions below. We also assume that the elements in $\mathcal{A}$ that are invertible in A are also invertible in $\mathcal{A}$, so $\mathcal{A} \cap A^\times = \mathcal{A}^\times$. This is in particular the case if $\mathcal{A}$ is stable under holomorphic functional calculus (see [10], page 134).

The category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is always an additive category: for objects $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ the morphisms from $\Omega^\bullet \mathcal{E}$ to $\Omega^\bullet \mathcal{F}$ form an additive group, and there is an object $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F}$ where the connection is simply given by $\nabla^{\mathcal{E}\oplus\mathcal{F}} = \begin{pmatrix} \nabla^{\mathcal{E}} & 0 \\ 0 & \nabla^{\mathcal{F}} \end{pmatrix}$. In general, $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is not an abelian category. For example, if $\Omega^k \mathcal{A} = 0$ for all $k \geq 1$, then $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is simply the category of fgp modules over $\mathcal{A}$, which is generally not an abelian category. In fact we can easily prove a necessary condition on a graded commutative dga $\Omega^\bullet \mathcal{A}$ if $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian.

Lemma 3.18. *Let $\Omega^\bullet \mathcal{A}$ be a connected graded commutative dga and suppose that $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian. Let $a \in \mathcal{A}$ and suppose that $da = a\omega$ for some $\omega \in \Omega^1 \mathcal{A}$. Then a is either 0 or invertible.*

Proof. Consider the two objects $(\mathcal{A}, d + \omega)$ and $(\mathcal{A}, d)$ of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Since $da = a\omega$ we have a commuting diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{a} & \mathcal{A} \\ \Big\downarrow{\scriptstyle d+\omega} & & \Big\downarrow{\scriptstyle d} \\ \Omega^1 \mathcal{A} & \xrightarrow{a} & \Omega^1 \mathcal{A} \end{array}$$

where a denotes the multiplication by a. So multiplication by a is a morphism between these objects. We then get a short exact sequence $0 \to \ker(a) \to \mathcal{A} \xrightarrow{a} \operatorname{im}(a) \to 0$, and since $\operatorname{im}(a)$ is fgp, this is split and we get $\mathcal{A} \cong \ker(a) \oplus \operatorname{im}(a)$. Since $\mathcal{A}$ is connected this means that either $a = 0$ or a is invertible. $\square$

We will now define a slightly stronger condition on $\mathcal{A}$, and we will show later that this is a sufficient condition for the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ to be abelian.

Definition 3.19. Let $\Omega^\bullet \mathcal{A}$ be a $*$-dga. We say that $\Omega^\bullet \mathcal{A}$ satisfies property Q if it satisfies the following condition:

for all $a \in \mathcal{A}$ with $a \geq 0$ and all $a_1, \ldots, a_s \in \mathcal{A}$ with all $|a_i| \leq a$, and all $\omega_1, \ldots, \omega_s \in \Omega^1 \mathcal{A}$:

if $da = \sum_{i=1}^s a_i \omega_i$, then either $a = 0$ or a is invertible.

This of course implies the condition in lemma 3.18. It also implies that $\mathcal{A}$ is connected, in the sense that there are no non-trivial projections: if $p \in \mathcal{A}$ is a projection, then $dp = d(p^2) = 2pdp$, so $(1 - 2p)dp = 0$ and multiplying by $1 - 2p$ gives $dp = 0$. Then p should be 0 or invertible, so any projection is 0 or 1.

3.1.1 quantum metric differential algebras

We will now show that property Q holds for quantum metric differential algebras. First we introduce the notion of a compact quantum metric space, invented by Rieffel [13]. Let A be a C^*-algebra and let L be a seminorm on A that takes finite values on a dense subalgebra $\mathcal{A}$. We think of L as a Lipschitz norm. This defines a metric on the state space $\mathcal{S}(A)$ by Connes' distance formula: for $\chi, \psi \in S(A)$ we have

$$d_L(\chi, \psi) = \sup\{|\chi(a) - \psi(a)|, a \in \mathcal{A}, L(a) \leq 1\}.$$

This metric then defines a topology on the state space. We already had the weak-$*$ topology, so it is natural to make the following definition:

Definition 3.20. Let A be a unital C^*-algebra and let L be a seminorm on A taking finite values on a dense subalgebra. Then A is called a *compact quantum metric space* if the topology on $S(A)$ induced by the metric d_L coincides with the weak-$*$ topology.

Now we go back to the case that we have a $*$-dga $\Omega^\bullet \mathcal{A}$ and $\mathcal{A}$ is a dense subset of a unital C^*-algebra A. Suppose that a norm $\|\cdot\|$ is given on $\Omega^1\mathcal{A}$, satisfying the inequality $\|a\omega\| \leq \|a\| \cdot \|\omega\|$ for $a \in \mathcal{A}, \omega \in \Omega^1\mathcal{A}$. This defines a seminorm L on $\mathcal{A}$ by $L(a) = \|da\|$. The space $\Omega^\bullet \mathcal{A}$ is called a *quantum metric dga* if A is a compact quantum metric space with this seminorm. If $\Omega^\bullet \mathcal{A}$ is a quantum metric dga, the same holds for $Z_g(\Omega^\bullet \mathcal{A})$, by proposition 2.3 of [13].

Lemma 3.21. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative quantum metric dga and suppose that $\mathcal{A} \cap A^\times = \mathcal{A}^\times$. Then $\Omega^\bullet \mathcal{A}$ satisfies property Q.*

Proof. Let $a \in \mathcal{A}$ with $a \geq 0$, and let $a_1, \ldots, a_s \in \mathcal{A}$ with all $|a_i| \leq a$, and $\omega_1, \ldots, \omega_s \in \Omega^1\mathcal{A}$ satisfying $da = \sum_{i=1}^s a_i\omega_i$. By scaling we may assume that $0 \leq a \leq 1$. Define the polynomial $p_n(x) = \sum_{k=1}^n \frac{1}{k}(1-x)^k$, which is the truncation of the power series of $-\log(x)$. Then we have

$$dp_n(a) = p_n'(a)da = \sum_{i=1}^s \alpha_i\omega_i \sum_{k=1}^n (1-a)^{k-1}.$$

For each $1 \leq i \leq s$ we have

$$\left|\alpha_i \sum_{k=1}^n (1-a)^{k-1}\right| \leq \left|a \sum_{k=1}^n (1-a)^{k-1}\right| = |1 - (1-a)^n| \leq 1.$$

So we get

$$\left\|dp_n(a)\right\| \leq \sum_{i=1}^s \|\omega_i\|,$$

in particular the norm of $dp_n(a)$ is bounded as $n \to \infty$.

If a is neither 0 nor invertible in A, there are points χ, ψ in the Gelfand spectrum of A satisfying $\chi(a) = 0$ and $\psi(a) = t > 0$. Then $\chi(p_n(a)) = \sum_{i=1}^n \frac{1}{k} \to \infty$ as $n \to \infty$, while $\psi(p_n(a)) = \sum_{i=1}^n \frac{1}{k}(1-t)^k \to -\log(t)$ as $n \to \infty$. We get

$$d(\chi, \psi) \geq \frac{|\chi(p_n(a)) - \psi(p_n(a))|}{\left\|dp_n(a)\right\|} \to \infty$$

so $d(\chi, \psi) = \infty$. But the metric d should give the weak-$*$ topology on the spectrum, and the spectrum is connected, so this is a contradiction. So either $a = 0$ or $a \in A^\times$, and in the second case $a \in \mathcal{A} \cap A^\times = \mathcal{A}^\times$. $\square$

We also have the following property for compact quantum metric spaces:

Lemma 3.22. *Let $\Omega^\bullet \mathcal{A}$ be a compact quantum metric space. Then $\{a \in \mathcal{A} \mid da = 0\} = \mathbb{C} \subseteq \mathcal{A}$.*

Proof. Suppose that $da = 0$ but $a \notin \mathbb{C}$. Then there are states χ, ψ with $\chi(a) \neq \psi(a)$. The state space is convex so for $t \in [0,1]$ we have a state $t\chi + (1-t)\psi$, and if $t \to 1$ this converges to χ in the weak-$*$ topology. However for each $t < 1$ we have $[t\chi + (1-t)\psi](a) \neq \chi(a)$. Since $d(Ra) = 0$ for every $R \in \mathbb{R}$ we have

$$d(t\chi + (1-t)\psi, \chi) \geq \left|[t\chi + (1-t)\psi](Ra) - \chi(Ra)\right| = R \cdot \left|[t\chi + (1-t)\psi](a) - \chi(a)\right|.$$

so $d(t\chi + (1-t)\psi, \chi) = \infty$. So the topology on $\mathcal{S}(A)$ induced by the metric does not coincide with the weak-* topology. $\square$

3.1.2 proof of abelianness

In the rest of this section, we will show that if $\Omega^\bullet \mathcal{A}$ satisfies property Q, then $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is an abelian category. Suppose we have a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ in the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$. We have to show that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are also in the category. The most difficult part is to show that these are finitely generated projective modules.

Lemma 3.23. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism between fgp $\mathcal{A}$-modules. Then the following are equivalent:*

- *The $\mathcal{A}$-modules $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are fgp.*
- *There is an $\mathcal{A}$-module homomorphism $\varphi^+ : \mathcal{F} \to \mathcal{E}$ satisfying $\varphi\varphi^+\varphi = \varphi$.*

Proof. Suppose that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are fgp. Then the short exact sequence $0 \to \ker(\varphi) \to \mathcal{E} \to \operatorname{im}(\varphi) \to 0$ is split, so $\mathcal{E} \cong \ker(\varphi) \oplus \operatorname{im}(\varphi)$. The short exact sequence $0 \to \operatorname{im}(\varphi) \to \mathcal{F} \to \operatorname{coker}(\varphi) \to 0$ is also split, so $\mathcal{F} \cong \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$. The map φ then corresponds to the map $\ker(\varphi) \oplus \operatorname{im}(\varphi) \to \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$ sending (a, b) to $(b, 0)$. We can then choose the map $\varphi^+ : \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi) \to \ker(\varphi) \oplus \operatorname{im}(\varphi)$ sending (c, d) to $(0, c)$. It is then easy to check that $\varphi\varphi^+\varphi = \varphi$ (and also $\varphi^+\varphi\varphi^+ = \varphi^+$).

Now suppose there is an $\mathcal{A}$-linear map $\varphi^+ : \mathcal{F} \to \mathcal{E}$ satisfying $\varphi\varphi^+\varphi = \varphi$. The surjection $\mathcal{F} \to \operatorname{coker}(\varphi)$ admits a splitting, sending the equivalence class $[f]$ to $f - \varphi\varphi^+(f)$. This is well-defined because $\varphi\varphi^+\varphi = \varphi$. So the short exact sequence $0 \to \operatorname{im}(\varphi) \to \mathcal{F} \to \operatorname{coker}(\varphi) \to 0$ is split, giving $\mathcal{F} \cong \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$. So $\operatorname{im}(\varphi)$ and $\operatorname{coker}(\varphi)$ are fgp. Then the short exact sequence $0 \to \ker(\varphi) \to \mathcal{E} \to \operatorname{im}(\varphi) \to 0$ is also split, giving $\mathcal{E} \cong \ker(\varphi) \oplus \operatorname{im}(\varphi)$. So $\ker(\varphi)$ is also fgp. □

Remark 3.24. If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi^+ : \mathbb{C}^n \to \mathbb{C}^n$ satisfy $\varphi\varphi^+\varphi = \varphi$ and $\varphi^+\varphi\varphi^+ = \varphi^+$, and $\varphi\varphi^+$ and $\varphi^+\varphi$ are self-adjoint then φ^+ is uniquely determined, and is called the *Moore-Penrose pseudoinverse* [2].

In the case that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are finitely generated projective it is easy to construct connections on these modules.

Lemma 3.25. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{C}(\Omega^\bullet\mathcal{A})$ and suppose that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are finitely generated projective. Then there are natural induced connections on these modules.*

Proof. Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi"] \arrow[d, "\nabla^{\mathcal{E}}"] & \mathcal{F} \arrow[r] \arrow[d, "\nabla^{\mathcal{F}}"] & \operatorname{coker}(\varphi) \\
\mathcal{E} \otimes \Omega^1\mathcal{A} \arrow[r, "\varphi\otimes\Omega^1\mathcal{A}"] & \mathcal{F} \otimes \Omega^1\mathcal{A} \arrow[r] & \operatorname{coker}(\varphi) \otimes \Omega^1\mathcal{A}.
\end{tikzcd}$$

We see that this induces a map $\nabla^{\operatorname{coker}(\varphi)} : \operatorname{coker}(\varphi) \to \operatorname{coker}(\varphi) \otimes \Omega^1\mathcal{A}$ and it is easy to check that it satisfies the Leibniz rules.

We have the isomorphisms $\mathcal{E} \cong \ker(\varphi) \oplus \operatorname{im}(\varphi)$ and $\mathcal{F} \cong \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$, as in the proof of lemma 3.23. Under these isomorphisms, φ corresponds to the map $\ker(\varphi) \oplus \operatorname{im}(\varphi) \to \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$ sending (a, b) to $(b, 0)$. We then get the commuting diagram

$$\begin{tikzcd}
\ker(\varphi) \oplus \operatorname{im}(\varphi) \arrow[r] \arrow[d, "\nabla^{\mathcal{E}}"] & \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi) \arrow[d, "\nabla^{\mathcal{F}}"] \\
\ker(\varphi) \otimes \Omega^1\mathcal{A} \oplus \operatorname{im}(\varphi) \otimes \Omega^1\mathcal{A} \arrow[r] & \operatorname{im}(\varphi) \otimes \Omega^1\mathcal{A} \oplus \operatorname{coker}(\varphi) \otimes \Omega^1\mathcal{A}
\end{tikzcd}$$

From the diagram it follows that $\nabla^{\mathcal{E}}$ restricts to $\ker(\varphi) \to \ker(\varphi) \otimes \Omega^1\mathcal{A}$, and $\nabla^{\mathcal{F}}$ restricts to $\operatorname{im}(\varphi) \to \operatorname{im}(\varphi) \otimes \Omega^1\mathcal{A}$. These are the connections we want. They satisfy the Leibniz rule because they are restrictions of $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. They are also independent of our choice of isomorphisms $\mathcal{E} \cong \ker(\varphi) \oplus \operatorname{im}(\varphi)$ and $\mathcal{F} \cong \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$. □

Definition 3.26. Let $M \in M_n(\mathcal{A})$ be a matrix with coefficients in $\mathcal{A}$. Let $\chi_M \in \mathcal{A}[x]$ be the characteristic polynomial, with coefficients $(-1)^m D_m(M)$, so

$$\chi_M(x) = x^n - D_{n-1}(M)x^{n-1} + D_{n-2}(M)x^{n-2} - \ldots + (-1)^n D_0(M).$$

Note that $D_0(M) = (-1)^n \det(M)$ and $D_{n-1} = \mathrm{Tr}(M)$. In general, $D_m(M)$ is the sum of the determinants of $m \times m$ square submatrices.

We need the inequality below involving D_m. Its proof is an easy calculation after diagonalising M^*M, and not very interesting. The term $2\,\mathrm{Re}(M^*[M,K])$ will appear in the proof of theorem 3.29.

Lemma 3.27. *For $M, K \in M_n(\mathbb{C})$ we have*

$$\left| \frac{d}{dt}_{|t=0} D_m(M^*M + t \cdot 2\,\mathrm{Re}(M^*[M,K])) \right| \leq 4n \|K\|_{\mathrm{HS}}\, D_m(M^*M).$$

Here $2\,\mathrm{Re}(M^*[M,K]) = M^*[M,K] + (M^*[M,K])^*$ and $\|K\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of K.

Proof. The inequality is invariant under a unitary change of basis of M, and M^*M is self-adjoint so we may choose a basis in which M^*M is diagonal, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Now

$$D_m(M^*M) = \sum_{|S|=m} \prod_{i \in S} \lambda_i,$$

where S runs over the m-element subsets of $\{1, 2, \ldots, n\}$. Let

$$M(t) = M^*M + t \cdot M^*[M,K].$$

The t-coefficient in the polynomial $P(t) = D_m(M(t)) \in \mathbb{C}[t]$ is $\frac{d}{dt}_{|t=0} D_m(M(t))$. The matrix $M(t)$ only has multiples of t outside the diagonal. The determinant of an $m \times m$ submatrix is then modulo t^2 equal to the product of the values on its diagonal. So

$$P_m(t) = \sum_{S=|m|} \prod_{i \in S} M(t)_{ii} \mod t^2.$$

We have

$$M(t)_{ii} = \lambda_i + t \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right).$$

The t-coefficient in $P_m(t)$ is then

$$\frac{d}{dt}_{|t=0} D_m(M(t)) = \sum_{|S|=m} \sum_{i \in S} \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right) \prod_{s \in S \setminus \{i\}} \lambda_s.$$

For all i, j, l we have $(M^*)_{ij} M_{li} \leq \frac{1}{2}(|M_{ji}|^2 + |M_{li}|^2) \leq \sum_{k=1}^{n} |M_{ki}|^2 = \lambda_i$. So we get

$$\left| \frac{d}{dt}_{|t=0} D_m(M(t)) \right| = \left| \sum_{|S|=m} \sum_{i \in S} \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right) \prod_{s \in S \setminus \{i\}} \lambda_s \right|$$

$$
\begin{aligned}
&\leq \sum_{|S|=m} \sum_{i \in S} \left| \lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right| \prod_{s \in S \setminus \{i\}} \lambda_s \\
&\leq \sum_{|S|=m} \sum_{i \in S} \left(\lambda_i |K_{ii}| + \sum_{j,l=1}^{n} \lambda_i |K_{jl}| \right) \prod_{s \in S \setminus \{i\}} \lambda_s \\
&\leq \left(\sum_{i=1}^{n} |K_{ii}| + \sum_{j,l=1}^{n} |K_{jl}| \right) \sum_{|S|=m} \prod_{s \in S} \lambda_s \\
&\leq 2n \|K\|_{\mathrm{HS}} \, D_m(M^*M).
\end{aligned}
$$

Now we conclude

$$
\left| \frac{d}{dt}_{|t=0} D_m(M^*M + t \cdot 2\,\mathrm{Re}(M^*[M,K])) \right| = \left| 2\,\mathrm{Re} \left(\frac{d}{dt}_{|t=0} D_m(M(t)) \right) \right| \leq 4n \|K\|_{\mathrm{HS}} \, D_m(M^*M).
$$

□

Remark 3.28. Both sides of this inequality are continuous functions of the entries of M and K. The inequality then still holds for $M, K \in M_n(\mathcal{A})$, since it can be checked at any point in the spectrum (the Hilbert-Schmidt norm is then $\|K\|_{\mathrm{HS}} = \left\| \mathrm{Tr}(K^*K) \right\|^{\frac{1}{2}}$).

We are now ready to prove that $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is an abelian category. We need to show that any $\varphi : \mathcal{E} \to \mathcal{F}$ has a finitely generated projective kernel, image and cokernel. In the first part of the proof we reduce to the case $\varphi : \mathcal{A}^n \to \mathcal{A}^n$. In the second part we prove that each term in the characteristic polynomial of $\varphi^*\varphi$ is either zero or invertible. Lastly we use this to prove that φ has a pseudo-inverse (as in lemma 3.23).

Theorem 3.29. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga satisfying property Q. Then $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is an abelian category.*

Proof. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{C}(\Omega^\bullet \mathcal{A})$. We will show that $\ker(\varphi), \mathrm{im}(\varphi), \mathrm{coker}(\varphi)$ are finitely generated projective $\mathcal{A}$-modules, and then we are done by lemma 3.25.

There is a projective module $\mathcal{G}$ with $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G} \cong \mathcal{A}^n$. We can write $\mathcal{G} \cong p\mathcal{A}^n$ for a projection $p \in \mathrm{End}_{\mathcal{A}}(\mathcal{A}^n)$. Then we can define a connection $\nabla^{\mathcal{G}} : p\mathcal{A}^n \to p\Omega^1\mathcal{A}^n$ by $\nabla^{\mathcal{G}}(g) = pdg$. It is easy to check that this defines a connection on $\mathcal{G}$. This makes $(\mathcal{G}, \nabla^{\mathcal{G}})$ an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$ and it also defines a connection on the direct sum module $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$.

Now the map

$$
\begin{pmatrix} 0 & 0 & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G} \to \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}
$$

is a morphism in $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Its kernel is $\ker(\varphi) \oplus \mathcal{F} \oplus \mathcal{G}$, its image is $0 \oplus \mathrm{im}(\varphi) \oplus 0$ and its cokernel is $\mathcal{E} \oplus \mathrm{coker}(\varphi) \oplus \mathcal{G}$. So it is enough to show that these are finitely generated projective. Therefore it is enough to prove: for a connection $\nabla : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ and a morphism $\varphi : \mathcal{A}^n \to \mathcal{A}^n$, the kernel, image and cokernel of φ are fgp modules.

The connection $\nabla : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ can be written as $\nabla = d + \kappa$, where $\kappa : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ is an $\mathcal{A}$-linear function. We can view κ as an $n \times n$ matrix with coefficients in $\Omega^1\mathcal{A}$. The induced connection on $\mathrm{Hom}_\mathcal{A}(\mathcal{A}^n, \mathcal{A}^n)$, which we still call ∇, satisfies

$$\nabla(\langle f, e\rangle) = \langle \nabla(f), e\rangle + \langle f, \nabla(e)\rangle$$

for $f \in \mathrm{Hom}_\mathcal{A}(\mathcal{A}^n, \mathcal{A}^n)$ and $e \in \mathcal{A}^n$. So

$$d(\langle f, e\rangle) + \kappa\langle f, e\rangle = \langle \nabla(f), e\rangle + \langle f, de\rangle + \langle f, \kappa e\rangle$$

and this gives

$$\nabla(f) = df + [\kappa, f].$$

Since $\varphi : \mathcal{A}^n \to \mathcal{A}^n$ commutes with the connection, we know that $\nabla(\varphi) = 0$ so we conclude that

$$d\varphi = [\varphi, \kappa].$$

Here φ is viewed as an element of $M_n(\mathcal{A})$. We get

$$d(\varphi^*\varphi) = \varphi^* d(\varphi) + d(\varphi)^*\varphi = 2\,\mathrm{Re}(\varphi^*[\varphi, \kappa]).$$

Now let $a_m = D_m(\varphi^*\varphi) \in \mathcal{A}$ be the m-th term of the characteristic polynomial of $\varphi^*\varphi$ (up to sign). Write $\kappa = \sum_{i=1}^s K_i\omega_i$ with $K_i \in M_n(\mathcal{A})$ and $\omega_i \in \Omega^1\mathcal{A}$. We get

$$\begin{aligned} da_m &= dD_m(\varphi^*\varphi) \\ &= \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + td(\varphi^*\varphi)) \\ &= \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t \cdot 2\,\mathrm{Re}(\varphi^*[\varphi, \kappa])) \\ &= \sum_{i=1}^s \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t \cdot 2\,\mathrm{Re}(\varphi^*[\varphi, K_i]))\omega_i \\ &= \sum_{i=1}^s a_i\omega_i \end{aligned}$$

where

$$a_i = \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t \cdot 2\,\mathrm{Re}(\varphi^*[\varphi, K_i])).$$

By lemma 3.27 and remark 3.28 we get $|a_i| \leq 4\|K_i\|_{\mathrm{HS}}\, a_m$. We can now apply property Q (if we put the factor $4\|K_i\|_{\mathrm{HS}}$ in the ω_i) to conclude that a_m is either 0 or invertible.

Now consider the smallest m for which $a_m \neq 0$ (note $a_n = 1$ so this m exists). Then a_m is invertible. The characteristic polynomial of $\varphi^*\varphi$ is now $\chi_{\varphi^*\varphi}(x) = x^n - a_{n-1}x^{n-1} +$

$\ldots + (-1)^{n-m} a_m x^m$. Let $p(x) = x^{-m}\chi_{\varphi^*\varphi} = x^{n-m} - a_{n-1}x^{n-m-1} + \ldots + (-1)^{n-m}a_m$. By Cayley-Hamilton we know $\chi_{\varphi^*\varphi}(\varphi^*\varphi) = 0$, so $\varphi^*\varphi p(\varphi^*\varphi)$ is nilpotent, and also self-adjoint, so $\varphi^*\varphi p(\varphi^*\varphi) = 0$. Then $(\varphi p(\varphi^*\varphi)) \cdot (\varphi p(\varphi^*\varphi))^* = 0$, so in fact already $\varphi p(\varphi^*\varphi) = 0$. Let $q(x) = \frac{1+(-1)^{n-m-1}a_m^{-1}p(x)}{x} \in \mathcal{A}[x]$. Let $\varphi^+ = q(\varphi^*\varphi)\varphi^*$. Then we have

$$\begin{aligned}
\varphi\varphi^+\varphi &= \varphi q(\varphi^*\varphi)\varphi^*\varphi \\
&= \varphi(q \cdot x)(\varphi^*\varphi) \\
&= \varphi(1 + (-1)^{n-m-1}a_m^{-1}p(\varphi^*\varphi)) \\
&= \varphi.
\end{aligned}$$

By lemma 3.23 it follows that the kernel, image and cokernel of φ are finitely generated projective. This concludes the proof of the theorem. □

Corollary 3.30. *With the same conditions as in the theorem, the category $\mathcal{C}_{flat}(\Omega^\bullet \mathcal{A})$ is also abelian.*

Proof. The category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is a full subcategory of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. The kernel, image and cokernel of a morphism in $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is again in $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ because the connections constructed in lemma 3.25 are flat if $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ are flat. □

3.2 Definition of the fundamental group

In this section we will define the fundamental group of a space. We will use theorem 2.5 for this. Since the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is equivalent to the category $\mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$, we will also attach the same fundamental group to $\Omega^\bullet \mathcal{A}$ as to $Z_g(\Omega^\bullet \mathcal{A})$. Therefore we assume in this subsection that $\Omega^\bullet \mathcal{A}$ is graded commutative. We have proven that $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is a rigid tensor category, and under some conditions on $\Omega^\bullet \mathcal{A}$, that it is abelian. What is left is showing that $\text{End}(\Omega^\bullet \mathcal{A}) = \mathbb{C}$ and constructing a fiber functor $\omega : \mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \to \text{Vec}_{\mathbb{C}}$. The first thing can be done easily.

Lemma 3.31. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga and assume that $\{a \in \mathcal{A} \mid da = 0\} = \mathbb{C} \subseteq \mathcal{A}$. Then $\text{End}(\Omega^\bullet \mathcal{A}) = \mathbb{C}$.*

Proof. Let $\theta : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{A}$ be an isomorphism. Since θ is bilinear, for all $\alpha \in \Omega^\bullet \mathcal{A}$ we have $\theta(\alpha) = \alpha\theta(1)$. So θ is determined by $a = \theta(1)$. Since θ has to commute with the connection we get $da = d(\theta(1)) = \theta(d(1)) = 0$. By the assumption, $a \in \mathbb{C}$. □

Note that the condition automatically holds for compact quantum metric spaces by lemma 3.22.

For the fibre functor, pick a point p in the Gelfand spectrum $\widehat{A}$. Then our fibre functor is given by sending a bimodule $\mathcal{E}$ to the localization of its center at p. This is defined as $\mathcal{E} \otimes_{\mathcal{A}} \mathbb{C}$, where the $\mathcal{A}$-module structure on $\mathbb{C}$ is given by p. Note that this depends on a choice of a point in the Gelfand spectrum. This point plays a similar rôle as the base point of the usual fundamental group.

Lemma 3.32. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga that satisfies property Q and let $p \in \widehat{A}$. There is a faithful exact fibre functor $\omega : \mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$ sending $\Omega^\bullet \mathcal{E}$ to $\mathcal{E}_p$.*

Proof. Let $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ be objects of $\mathcal{C}(\Omega^\bullet \mathcal{A})$ and let $\varphi : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{F}$ be a morphism commuting with the connections. Since φ is $\mathcal{A}$-linear, this induces a map $\Omega^\bullet \mathcal{E}_p \to \Omega^\bullet \mathcal{F}_p$, showing that ω is functorial.

To show that ω is faithful, suppose that $\varphi_p = 0$. Since $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian we know that $\mathrm{im}(\varphi)$ is an fgp module. Now look at $\mathrm{im}(\varphi) \otimes_{\mathcal{A}} A$. This is a fgp module over the C*-algebra A, which corresponds to a vector bundle on $\widehat{A}$. It is zero at p, and the rank is locally constant, and $\widehat{A}$ is connected, so $\mathrm{im}(\varphi) \otimes_{\mathcal{A}} A = 0$. Since $\mathrm{im}(\varphi)$ is projective it is flat, and $\mathrm{im}(\varphi) \hookrightarrow \mathrm{im}(\varphi) \otimes_{\mathcal{A}} A$ is an injection, so also $\mathrm{im}(\varphi) = 0$. We conclude that $\varphi = 0$.

The fibre functor is exact because a localisation is always exact. $\square$

Definition 3.33. Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga such that $\Omega^\bullet \mathcal{A}$ satisfies property Q and $\{a \in \mathcal{A} \mid \ker(d) = 0\} = \mathbb{C}$. Let $p \in \widehat{A}$. Then we define $\pi^1(\Omega^\bullet \mathcal{A}, p)$ to be the group scheme of automorphisms of the fibre functor $\omega : \mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$ at p.

Definition 3.34. Let $\Omega^\bullet \mathcal{A}$ be a dga, suppose that $Z_g(\Omega^\bullet \mathcal{A})$ satisfies the condition of definition 3.33 and let $p \in \widehat{Z_g(\mathcal{A})}$. Then we define $\pi^1(\Omega^\bullet \mathcal{A}, p) = \pi^1(Z_g(\Omega^\bullet \mathcal{A}, p))$.

By theorem 2.5 we then have an equivalence of categories $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \cong \mathrm{Rep}(\pi_1(\Omega^\bullet \mathcal{A}, p))$. In examples, we will simply recognise the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ as being equivalent to the category of representations of some (topological) group, which the fundamental group is then the algebraic hull of.

Example 3.35. Let $\Omega^k\mathcal{A} = M_2(\mathbb{C})$ for all k. Let d be given by taken the graded commutator with the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Explicitly, d is given in even degrees by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix}$ and in odd degrees by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 2a & 0 \\ 0 & -2d \end{pmatrix}$. This is the non-commutative space corresponding to the set of two points that are identified.

The graded center of this dga is just $Z_g(\Omega^\bullet \mathcal{A}) = \mathbb{C}, 0, \mathbb{C}, 0, \ldots$ where $\mathbb{C}$ is embedded diagonally in $M_2(\mathbb{C})$. Then $\mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$ is just equivalent to the category of vector spaces. The fundamental group is trivial.

Example 3.36. Consider the following graded commutative dga: let $\mathcal{A} = \mathbb{C}$, $\Omega^1\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, $\Omega^2\mathcal{A} = \mathbb{C}$ and $\Omega^n\mathcal{A} = 0$ for $n \geq 3$, with $d = 0$. The multiplication $\Omega^1\mathcal{A} \otimes \Omega^1\mathcal{A} \to \Omega^2\mathcal{A}$ is given by $(a_1, a_2) \otimes (b_1, b_2) \to a_1 b_2 - a_2 b_1$.

An $\mathcal{A}$-bimodule is now a vector space V. A connection is a $\mathbb{C}$-linear map $\nabla = (\alpha, \beta) : V \to V \oplus V$. The map $\nabla_1 : V \oplus V \to V$ is then given by

$$\nabla_1(v, w) = \nabla_1(v \cdot (1, 0)) + \nabla_1(w \cdot (0, 1)) = \nabla(v) \cdot (1, 0) + \nabla(w) \cdot (0, 1) = -\beta(v) + \alpha(w).$$

So the curvature $\nabla^2 : V \to V$ is $\nabla^2 = [\alpha, \beta]$. The connection is flat if and only if α and β commute, so the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet\mathcal{A})$ is equivalent to the category sdaof vecftor spaces with two commuting isomorphisms. This is in turn equivalent to the category of continuous representations of $\mathbb{R}^2$: the vector space V with the commuting isomorphisms α, β corresponds to the representation $\pi : \mathbb{R}^2 \to \mathrm{End}(V)$, $\pi(t_1, t_2) = \exp(t_1\alpha + t_2\beta)$. The fundamental group of $\Omega^\bullet \mathcal{A}$ is then (the algebraic hull of) $\mathbb{R}^2$.

For graded commutative spaces there is a good notion of functoriality for the fundamental group. For non-commutative spaces it is more difficult, we will revisit this in section 5.

Lemma 3.37. *Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be graded commutative $*$-dga's that satisfy the conditions of definition 3.33. Let $\varphi : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{B}$ be a degree 0 algebra morphism satisfying $\varphi(d\alpha) = d(\varphi(\alpha))$ for all $\alpha \in \Omega^\bullet \mathcal{A}$. Let $q \in \widehat{B}$ and $p = \varphi^*(q) \in \widehat{A}$. Then φ induces a map $\varphi^* : \pi^1(\Omega^\bullet \mathcal{B}, q) \to \pi^1(\Omega^\bullet \mathcal{A}, p)$.*

Proof. If $\mathcal{E}$ is an fgp $\mathcal{A}$-module then $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$ is an fgp $\mathcal{B}$-module. A flat connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ gives a flat connection $\tilde{\nabla} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{B}$, given by $\tilde{\nabla}(e \otimes b) = \nabla(e)b + e \otimes db$. So we get a map $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}(\Omega^\bullet \mathcal{B})$. It is easy to see that it is functorial and also that it commutes with the fibre functors. Then every automorphism of the fibre functor $\mathcal{C}(\Omega^\bullet \mathcal{B}) \to \mathrm{Vec}_{\mathbb{C}}$ can be pulled back to an automorphism of the fibre functor $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$. So we get a map $\varphi^* : \pi^1(\Omega^\bullet \mathcal{B}, q) \to \pi^1(\Omega^\bullet \mathcal{A}, p)$. □

4 Toric noncommutative manifolds

An important class of noncommutative spaces consists of the toric noncommutative manifolds, as described in [1]. We briefly explain the construction that is used there. Let M be a compact manifold with a smooth isometric action of the N-dimensional torus $\mathbb{T}^N$. Let $\theta : \mathbb{R}^N \to \mathbb{R}^N$ be a skew-symmetric linear form. The action of $\mathbb{T}^N$ on the manifold M gives an action α of $C^\infty(\mathbb{T}^N)$ on the algebra $C^\infty(M)$. Each element in $C^\infty(M)$ may be written as

$$a = \sum_{r \in \mathbb{Z}^N} a_r$$

where each a_r is homogeneous of degree $r \in \mathbb{Z}^N$, that is to say

$$\alpha(t)(a_r) = \exp(2\pi i t \cdot r)$$

for all $t \in \mathbb{T}^N$. Now define A to be the algebra that is $C^\infty(M)$ as a vector space, but with the noncommutative multiplication given by

$$a_r \cdot a_s = \chi(r,s) a_s \cdot a_r$$

where $\chi : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{C}$ is the bi-character given by $\chi(r,s) = \exp(2\pi i r \cdot \theta s)$.

The differential algebra can also be deformed and then satisfies

$$\Omega^n A = \sum_{s \in \mathbb{Z}^N} (\Omega^n A)_s$$

and

$$\omega_s \cdot \omega_t = (-1)^{nk} \chi(s,t) \omega_t \cdot \omega_s$$

for $\omega_s \in (\Omega^n A)_s$ and $\omega_t \in (\Omega^k A)_t$. Moreover $\omega_s \cdot \omega_t \in (\Omega^{n+k} A)_{s+t}$ and $d\omega_s \in (\Omega^{n+1} A)_s$.

Instead of $\mathbb{Z}^N$ we can consider the subgroup Γ generated by the s with $A_s \neq 0$. We can also quotient out the subgroup $\{s \in \Gamma \mid \chi(s,t) = 0 \text{ for all } t \in \Gamma\}$. Therefore we can assume that this subgroup is 0. It follows that $Z_g(\Omega^n A) = (\Omega^n A)_0$ for all n. We will consider the category $\mathcal{C}(\Omega^\bullet A)$. By theorem 3.12 this category is equivalent to $\mathcal{C}((\Omega^\bullet A)_0)$. The inclusion $(\Omega^\bullet A)_0 \hookrightarrow \Omega^\bullet A$ of dga's gives a map $\pi_1 M \to \pi^1(\Omega^\bullet A_\theta)$.

We will consider some examples.

Example 4.1. Let A_θ be the rotation algebra, as studied by Rieffel [12] and Connes [3]. Here θ is a real number and the elements of A_θ are formal linear combinations of $u^m v^n$ where the coefficients go to zero faster than any polynomial. The multiplication is given by $uv = \lambda vu$ where $\lambda = e^{2\pi i \theta}$. The one-forms are of the form $adu + bdv$ with $a, b \in A_\theta$ and they satisfy $udu = du \cdot u, udv = \lambda dv \cdot u, vdu = \overline{\lambda} du \cdot v, vdv = dv \cdot v$ and $du \cdot dv = -\lambda dv \cdot du$. The two-forms are of the form $adudv$ with $a \in A_\theta$ ([10], section 12.2).

Assume that θ is irrational. Then the algebra A_θ fits the framework above, where $\Gamma = \mathbb{Z}^2$: we get

$$A_\theta = \sum_{(m,n) \in \mathbb{Z}^2} (A_\theta)_{m,n}$$

where

$$(A_\theta)_{m,n} = u^m v^n \mathbb{C},$$

and the bi-character is given by $\chi((m,n),(m',n')) = \lambda^{mn'+nm'}$. We find

$$(\Omega^1 A_\theta)_{m,n} = u^{m-1} v^n du\mathbb{C} + u^m v^{n-1} dv\mathbb{C}$$

and

$$(\Omega^2 A_\theta)_{m,n} = u^{m-1} v^{n-1} dudv\mathbb{C}.$$

So we get

$$\mathcal{C}(\Omega^\bullet A_\theta) = \mathcal{C}(\mathbb{C}, \mathbb{C}^2, \mathbb{C}).$$

Restricting to the flat ebc's gives

$$\mathcal{C}_{\text{flat}}(\Omega^\bullet A_\theta) = \mathcal{C}_{\text{flat}}(\mathbb{C}, \mathbb{C}^2, \mathbb{C}).$$

By example 3.36 this is equivalent to the category of continuous representations of $\mathbb{R}^2$. So the fundamental group of $\Omega^\bullet A_\theta$ is the algebraic hull of $\mathbb{R}^2$.

Consider a representation $\pi : \mathbb{R}^2 \to \mathrm{GL}(W)$ given by $\pi(t_1, t_2) = \exp(t_1\alpha + t_2\beta)$ with $\alpha, \beta \in \mathrm{End}(W)$ commuting endomorphisms of W. The flat connection corresponding to this representation on the bimodule $W \otimes A_\theta$ is given explicitly by

$$\begin{aligned} \nabla : W \otimes A_\theta &\to W \otimes \Omega^1 A_\theta \\ \nabla(w \otimes a) &= w \otimes da + \alpha(w) \otimes au^{-1}du + \beta(w) \otimes av^{-1}dv. \end{aligned}$$

For the regular torus (with $\theta = 0$) the formula above defines a flat connection as well (but there are more flat connections). From the classical case we know that the flat connections correspond to representations of $\mathbb{Z}^2$. The above flat connection corresponds to the representation $\mathbb{Z}^2 \to \mathrm{GL}(W)$ given by $(n_1, n_2) \to \exp(n_1\alpha + n_2\beta)$. So we have a function $\{$flat connections on A_θ-bimodules $\to$ flat connections on A_0-bimodules$\}$, corresponding to the inclusion $\mathbb{Z}^2 \to \mathbb{R}^2$.

Example 4.2. Again consider A_θ but now assume that $\theta = \frac{p}{q}$ is rational, where p, q are coprime integers. Now we have $\Gamma = (\mathbb{Z}/q\mathbb{Z})^2$ and the bi-character is still given by $\chi((m,n),(m',n')) = \lambda^{mn'+nm'}$. We get

$$Z(A) = (A_\theta)_0 = \sum_{(m,n)\in\mathbb{Z}^2} u^{qm} v^{qn} \mathbb{C}.$$

Note that $Z(A)$ is generated by the commutative unitaries u^q and v^q. As such it is isomorphic to the algebra $\Omega^0\mathbb{T}^2$ of the torus. We also get

$$(\Omega^1 A_\theta)_0 = \sum_{(m,n)\in\mathbb{Z}^2} u^{qm-1} v^{qn} du\mathbb{C} \oplus u^{qm} v^{qn-1} dv\mathbb{C} = Z(A)u^{-1}du + Z(A)v^{-1}dv$$

which is isomorphic to $\Omega^1\mathbb{T}^2$, and

$$(\Omega^2 A_\theta)_0 = \sum_{(m,n)\in\mathbb{Z}^2} u^{qm-1}v^{qn-1}dudv\mathbb{C} = Z(A)u^{-1}duv^{-1}dv$$

which is isomorphic to $\Omega^2\mathbb{T}^2$. We conclude that

$$\mathcal{C}(\Omega^\bullet A_\theta) = \mathcal{C}(\Omega^\bullet\mathbb{T}^2).$$

Restricting to the flat connections gives

$$\mathcal{C}_{\text{flat}}(\Omega^\bullet A_\theta) = \mathcal{C}_{\text{flat}}(\Omega^\bullet\mathbb{T}^2).$$

The classical case (theorem 2.2) tells us that this is equivalent to the category of representations of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$. So the fundamental group of $\Omega^\bullet A_\theta$ is the algebraic hull of $\mathbb{Z}^2$ if θ is rational. The inclusion $\Omega^\bullet A_0 \hookrightarrow \Omega^\bullet A$ corresponds to the morphisms of groups, $\mathbb{Z}^2 \to \mathbb{Z}^2$ that multiplies each entry by q.

Remark 4.3. The continuous representations of $\mathbb{R}^2$ are in bijection to the continuous representations of the dense subgroup $\mathbb{Z}^2 + \Theta\mathbb{Z}^2$, where $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, if θ is irrational. If $\theta = \frac{p}{q}$ then this subgroup is $\frac{1}{q}\mathbb{Z}^2$, which is isomorphic to $\mathbb{Z}^2$. Therefore we can say that for all θ, the fundamental group of $\Omega^\bullet A_\theta$ is $\mathbb{Z}^2 + \Theta\mathbb{Z}^2$, and the inclusion of the fundamental group of the torus $\mathbb{Z}^2$ into this group is the natural one.

Example 4.4. We now consider the non-commutative n-torus A_θ. The n-dimensional torus acts on itself: if we write $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ then this action is simply addition. Let θ be a skew-symmetric $n \times n$ matrix with coefficients in $\mathbb{R}$. Let A_θ be the toric deformation of $A = C^\infty(\mathbb{T}^n)$. Let Λ be the lattice

$$\Lambda = \{r \in \mathbb{Z}^n \mid \theta r \in \mathbb{Z}^n\}.$$

Note that $\chi(r, s) = 1$ for all $s \in \mathbb{Z}^n$ if and only if $r \in \Lambda$. So we get $\Gamma = \mathbb{Z}^n/\Lambda$. Now A_0 is generated by all the $a_r, r \in \Lambda$. It is isomorphic to $C^\infty(\mathbb{T}^m)$ where m is the rank of Λ.

5 Functoriality

A smooth map $f : M \to N$ induces a group homomorphism $\pi_1 f : \pi_1 M \to \pi_1 N$. On algebras, this means that a morphism $\varphi : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{B}$ should induce a group homomorphism $\pi^1\varphi : \pi^1(\Omega^\bullet \mathcal{B}) \to \pi^1(\Omega^\bullet \mathcal{A})$. Here a morphism φ is a degree 0 algebra morphism that commutes with the derivative, so $d \circ \varphi = \varphi \circ d$. If $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ are graded commutative this is easy to show. We now show that it is not possible to extend this to non-commutative spaces. We do this by constructing a non-commutative space $\mathcal{A}$ and morphisms $\Omega^\bullet(S^1) \xrightarrow{\varphi} \Omega^\bullet \mathcal{A} \xrightarrow{\psi} \Omega^\bullet(S^1)$, such that $\psi \circ \varphi$ is the identity on $\Omega^\bullet(S^1)$ but $\pi^1(\Omega^\bullet \mathcal{A})$ is trivial. Then we know that the construction is not functorial, since it is not possible that $\pi^1\varphi \circ \pi^1\psi$ is the identity on $\mathbb{Z}$ if $\pi^1(\Omega^\bullet \mathcal{A})$ is trivial.

First we give an informal explanation on the algebra we will construct. We will make an algebra of functions $\mathbb{R}^2 \to B(L^2(S^1))$, that send the circle S^1 to a commutative subspace of $B(L^2(S^1))$. Since $B(L^2(S^1))$ has a trivial centre, the centre of this algebra will be $C^\infty(\mathbb{R}^2)$, making the fundamental group trivial. The fact that the values of f on S^1 are in a commutative subspace does not change this because S^1 has empty interior. Because the values of f on S^1 are in a commutative subspace it is easy to construct a map $\mathcal{A} \to C^\infty(S^1)$, similar to how one would construct the map $C^\infty(\mathbb{R}^2) \to C^\infty(S^1)$ using the embedding $S^1 \hookrightarrow \mathbb{R}^2$. The map $C^\infty(S^1) \to \mathcal{A}$ is more difficult to construct: one starts on the circle by using multiplication operators. The map can then be extended to $\mathbb{R}^2$ because the space is non-commutative enough.

The non-commutative space is defined as follows: let

$$\mathcal{A} = \{f : \mathbb{R}^2 \to B(L^2(S^1)) \text{ smooth, bounded }, f(U) \subseteq C^\infty(S^1) \subseteq B(L^2(S^1)) \text{ for an open } S^1 \subseteq U \subseteq \mathbb{R}^2\}.$$

Here $L^2(S^1)$ is the Hilbert space of squarely integrable functions on S^1 and $B(L^2(S^1))$ is the C*-algebra of bounded linear operators $L^2(S^1) \to L^2(S^1)$. The functions f have to be smooth in the sense that for all $g \in L^2(S^1)$ and $a \in S^1$, the function $p \to f(p)(g)(a)$ is smooth. The set U is an open neighbourhood of S^1, and for $p \in U \subseteq \mathbb{R}^2$, the operator $f(p) \in B(L^2(S^1))$ has to be a multiplication operator with a smooth map in $C^\infty(S^1)$. For $p \in U$, write $f(p) = M_{h_f(p)}$, where M_h denotes the multiplication operator by h. Then $h_f : U \to C^\infty(S^1)$ is a smooth function. It is easy to see that $\mathcal{A}$ is a dense subalgebra of the C^*-algebra $C_b(\mathbb{R}^2, B(L^2(S^1)))$.

Note that $\mathcal{A}$ has a natural action of $C^\infty(\mathbb{R}^2)$ by multiplication. Now define $\Omega^1\mathcal{A} = \mathcal{A} \otimes_{C^\infty(\mathbb{R}^2)} \Omega^1\mathbb{R}^2$. This has a natural structure of $\mathcal{A}$-bimodule. Since $\Omega^1\mathbb{R}^2 \cong C^\infty(\mathbb{R}^2)dx \oplus C^\infty(\mathbb{R}^2)dy$ we get $\Omega^1\mathcal{A} \cong \mathcal{A}dx + \mathcal{A}dy$. For $f \in \mathcal{A}$ define $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, where $\frac{\partial f}{\partial x} \in \mathcal{A}$ is defined by

$$\frac{\partial f}{\partial x}(x, y)(g)(a) = \frac{\partial f(x, y)(g)(a)}{\partial x}$$

for $(x, y) \in \mathbb{R}^2$, $g \in L^2(S^1)$ and $a \in S^1$. For $(x, y) \in U$ we get $\frac{\partial f}{\partial x}(x, y) = M_{\frac{\partial h_f(x,y)}{\partial x}}$ which is indeed a multiplication operator. The definition for $\frac{\partial f}{\partial y}$ is similar. We can define $\Omega^2\mathcal{A} = \mathcal{A} \otimes_{C^\infty(\mathbb{R})} \Omega^2(\mathbb{R})$ and this works in a similar way.

The centre of $B(L^2(S^1))$ consists only of the scalar multiplications. Let $f \in Z_g(\mathcal{A})$. Then for any $p \in \mathbb{R}^2 \setminus S^1$ it follows that $f(p)$ is a scalar multiplication. Since f should be smooth and $\mathbb{R}^2 \setminus S^1$ we get $Z_g(\mathcal{A}) = C^\infty(\mathbb{R}^2)$. We also get $Z_g(\Omega^1\mathcal{A}) = \Omega^1(\mathbb{R}^2)$ and $Z_g(\Omega^2\mathcal{A}) = \Omega^2(\mathbb{R}^2)$. Hence

$$\pi^1(\Omega^\bullet \mathcal{A}) = \pi^1(Z_g(\Omega^\bullet \mathcal{A})) = \pi_1(\Omega^\bullet \mathbb{R}^2) = \{0\}.$$

Now we will define the map $\varphi : C^\infty(S^1) \to \mathcal{A}$. Let $s_0 \in S^1$ be a base point and for $a \in S^1$, denote by r_a the rotation of S^1 that sends s_0 to a. Let $\mathrm{GL}(L^2(S^1))$ denote the linear automorphisms of the Hilbert space $L^2(S^1)$. This is a contractible manifold by Kuiper's theorem [11]. We have a smooth function $S^1 \to \mathrm{GL}(L^2(S^1))$ sending a to the automorphism $r_a^* : g \to g \circ r_a$. Let $s : \mathbb{R}^2 \to \mathrm{GL}(L^2(S^1))$ be a smooth extension of this map, such that $s(Ra) = s(a)$ for $R > 1 - \varepsilon$ and $a \in S^1$ (for some fixed $0 < \varepsilon < 1$). Now define the map φ by, for $f \in C^\infty(S^1)$ and $p \in \mathbb{R}^2$:

$$\varphi(f)(p) = s_p M_f s_p^{-1} \in B(L^2(S^1)).$$

For $p \in S^1$ and $R > 1 - \varepsilon$ we see that $\varphi(f)(Rp) = M_{f \circ r_p}$, since for $g \in B(L^2(S^1)), a \in S^1$ we have

$$\begin{aligned}\varphi(f)(p)(g)(a) &= (r_p^* M_f r_p^{*-1})(g)(a) = (M_f r_p^{*-1})(g)(r_p(a)) \\ &= f(r_p(a)) r_p^{*-1}(g)(r_p(a)) = f(r_p(a))g(a).\end{aligned}$$

So $\varphi(f)(p)$ is indeed a multiplication operator for p in a neighbourhood of the circle S^1, and $\varphi(f) \in \mathcal{A}$. We have $h_{\varphi(f)}(p) = f \circ r_p$ for $p \in S^1$. At each point $p \in \mathbb{R}^2$ the operator $\varphi(f)(p)$ is just a conjugation of the multiplication operator M_f, therefore φ is an algebra morphism $C^\infty(S^1) \to \mathcal{A}$. The operator $d\varphi : \Omega^1 S^1 \to \Omega^1\mathcal{A}$ can then be defined by the rule $d\varphi(f_1 df_2) = \varphi(f_1)d(\varphi(f_2))$.

Finally we define the map $\psi : \mathcal{A} \to C^\infty(S^1)$. It is given by

$$\psi(f)(p) = h_f(p)(s_0)$$

for $f \in \mathcal{A}, p \in S^1$. This is an algebra morphism. We can then define $d\psi : \Omega^1\mathcal{A} \to \Omega^1 S^1$ by the rule $d\psi(f_1 df_2) = \psi(f_1)d(\psi(f_2))$. Now for $f \in C^\infty(S^1)$ and $p \in S^1$ we have $h_{\varphi(f)}(p) = f \circ r_p$, so $\psi \circ \varphi(f)(p) = f(r_p(s_0)) = f(p)$. So $\psi \circ \varphi$ is the identity on $C^\infty(S^1)$. We also see that $d\psi \circ d\varphi$ is the identity on $\Omega^1 S^1$.

We conclude that π^1 cannot be functorial: if it were we would have $\pi^1(\psi \circ \varphi) : \mathbb{Z} \to \mathbb{Z}$ the identity, while $\pi^1(\varphi)$ is the trivial map $\{0\} \to \mathbb{Z}$ and $\pi^1(\psi)$ is the trivial map $\mathbb{Z} \to \{0\}$.

Of course the functoriality is a very important quality of the fundamental group, together with the fact that it is an invariant for homotopy equivalent functions. It is unfortunate that our definition is not functorial. Perhaps there is a better definition for the fundamental group, such that it is functorial. The above example may help in the search for a better definition of fundamental group.

5.1 Products

For manifolds (and topological spaces in general) it is known that the fundamental group of the product of two spaces is the product of the two fundamental groups: $\pi_1(M \times N) = \pi_1(M) \times \pi_1(N)$. It makes sense to expect that a similar property holds for the fundamental group of a non-commutative space. We have not been able to show this yet but we have some work in this direction.

First, let us define the product of two non-commutative spaces. Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be unital dga's. The product is then the graded tensor product $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ over $\mathbb{C}$. This has a natural grading: the elements of degree n are sums of $\alpha \otimes \beta$ with $\alpha \in \Omega^k \mathcal{A}, \beta \in \Omega^l \mathcal{B}, k + l = n$. For $\alpha \in \Omega^k \mathcal{A}, \alpha' \in \Omega^{k'} \mathcal{A}, \beta \in \Omega^l \mathcal{B}, \beta' \in \Omega^{l'} \mathcal{B}$, the multiplication is given by

$$\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{k'l} \alpha\alpha' \otimes \beta\beta'.$$

The derivative is given by

$$d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^k \alpha \otimes d\beta.$$

From the multiplication the graded commutator is calculated as

$$[\alpha \otimes \beta, \alpha' \otimes \beta'] = (-1)^{k'l}[\alpha, \alpha'] \otimes \beta\beta' + (-1)^{(k+l)k'} \alpha'\alpha \otimes [\beta, \beta'].$$

Recall that the graded centre of $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ is defined as all $\gamma \in \Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ such that $[\gamma, \gamma'] = 0$ for all $\gamma' \in \Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$. We can now proof that this is simply the tensor product of the graded centres of $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$.

Lemma 5.1. *Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be unital dga's. Then $Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}) = Z_g(\Omega^\bullet \mathcal{A}) \otimes Z_g(\Omega^\bullet \mathcal{B})$.*

Proof. It is clear from the formula $[\alpha \otimes \beta, \alpha' \otimes \beta'] = (-1)^{k'l}[\alpha, \alpha'] \otimes \beta\beta' + (-1)^{(k+l)k'} \alpha'\alpha \otimes [\beta, \beta']$ that $Z_g(\Omega^\bullet \mathcal{A}) \otimes Z_g(\Omega^\bullet \mathcal{B}) \subseteq Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$. Now let $\gamma \in Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$. Write $\gamma = \sum_{s=1}^m \alpha_s \otimes \beta_s$ such that all α_s, β_s are homogeneous with m minimal. Since γ is in the graded centre we get for any homogeneous $\alpha' \in \Omega^\bullet \mathcal{A}$ that $[\gamma, \alpha' \otimes 1] = 0$, so $\sum_{s=1}^m \pm[\alpha_s, \alpha'] \otimes \beta_s = 0$. Now suppose that $[\alpha_{s_0}, \alpha'] \neq 0$ for some s_0. Now let $S \subseteq \{1, \ldots, m\}$ be the subset of indices s where α_s has the same degree as α_{s_0} and β_s has the same degree as β_{s_0}. Then $\sum_{s \in S}[\alpha_s, \alpha'] \otimes \beta_s = 0$. Then the β_s cannot be linearly independent, so we can write $\beta_{s_0} = \sum_{s \in S \setminus \{s_0\}} \lambda_s \beta_s$. Writing $\lambda_s = 0$ for $s \notin S$ we get $\gamma = \sum_{s \in \{1,\ldots,m\} \setminus \{s_0\}} (\alpha_s + \lambda_s \alpha_{s_0}) \otimes \beta_s$, which is a contradiction with the minimality of m. So we get $[\alpha_s, \alpha'] = 0$ for all s, and hence $\alpha_s \in Z_g(\Omega^\bullet \mathcal{A})$. Similarly $\beta_s \in Z_g(\Omega^\bullet \mathcal{A})$ for all s, so $\gamma \in Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$. □

So to study the product of non-commutative spaces we can restrict ourselves to graded commutative dga's. We can prove that $\pi^1(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}) = \pi^1(\Omega^\bullet \mathcal{A}) \times \pi^1(\Omega^\bullet \mathcal{B})$ if we assume the following conjecture.

Conjecture 5.2. *Let $\Omega^\bullet \mathcal{A}$ be a unital graded commutative $*$-dga satisfying property Q and $\ker(d) \cap \mathcal{A} = \mathbb{C}$ and let $p \in \widehat{\mathcal{A}}$. Let $\Omega^\bullet B$ be a graded commutative dga. Then the category*

$$\mathcal{C}_{flat}(\Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{B}) = \{\textit{fgp } \mathcal{A} \otimes \Omega^\bullet \mathcal{B}\textit{-modules } \mathcal{E} \textit{ with a flat connection } \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1 \mathcal{A}\}$$

is equivalent to the category

$$\mathrm{Rep}_{\pi^1(\mathcal{A})}(\mathrm{Mod}_{\Omega^\bullet \mathcal{B}}) = \{\textit{representations of } \pi^1(\Omega^\bullet \mathcal{A}) \textit{ on fgp } \Omega^\bullet \mathcal{B}\textit{-modules}\}.$$

Each module $\mathcal{E}$ with a flat connection corresponds to a representation of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ on the localisation $\mathcal{E}_p$.

We can view an fgp $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$-module as a vector bundle over $\Omega^\bullet \mathcal{A}$ where the fibres are vector bundles over $\Omega^\bullet \mathcal{B}$. Suppose that $\mathcal{B} = C(Y)$ for a compact Hausdorff space Y. Then for each $y \in Y$ we get the $\Omega^\bullet \mathcal{A}$-module $\mathcal{E}_y$ with a flat connection, corresponding to a representation of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ on $\mathcal{E}_{py}$. Combining these representations gives a representation no $\mathcal{E}_p$, but we do not know why this should be continuous. We will write a bit more about the conjecture later, first we show how we use it to compute the fundamental group of a product.

Theorem 5.3. *Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be unital graded commutative $*$-dga's satisfying property Q,* $\ker(d) \cap \mathcal{A} = \mathbb{C}$ *and* $\ker(d) \cap \mathcal{B} = \mathbb{C}$. *Let $p \in \widehat{\mathcal{A}}, q \in \widehat{\mathcal{B}}$. If conjecture 5.2 hold for $\Omega^\bullet A$ and $\Omega^\bullet B$ then $\pi^1(\Omega^\bullet A \otimes \Omega^\bullet B, p \otimes q) = \pi^1(\Omega^\bullet A, p) \times \pi^1(\Omega^\bullet B, q)$.*

Proof. Let $p \in \widehat{\mathcal{A}}$ and $q \in \widehat{\mathcal{B}}$. The first three terms of the dga $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ are

$$\mathcal{A} \otimes \mathcal{B}, \Omega^1\mathcal{A} \otimes \mathcal{B} \oplus \mathcal{A} \otimes \Omega^1\mathcal{B}, \Omega^2\mathcal{A} \otimes \mathcal{B} \oplus \Omega^1\mathcal{A} \otimes \Omega^1\mathcal{B} \oplus \mathcal{A} \otimes \Omega^2\mathcal{B}.$$

A connection on a module $\mathcal{E}$ is then given by $\nabla_1 \oplus \nabla_2$ with $\nabla_1 : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ and $\nabla_2 : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{B}$. The curvature of this connection can be computed to be $\nabla_1^2 \oplus [\nabla_1, \nabla_2] \oplus \nabla_2^2$. So ∇ is flat iff ∇_1 and ∇_2 are flat and commute. In this case $(\mathcal{E}, \nabla_1)$ is an object in the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{B})$. This corresponds to a representation γ of $\pi^1(\Omega^\bullet \mathcal{A})$ on $\mathcal{E}_p$. Since ∇_2 commutes with ∇_1 it is an endomorphism of $(\mathcal{E}, \nabla_1)$ in the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{B})$. The localisation $(\nabla_2)_p : \mathcal{E}_p \to \mathcal{E}_p \otimes \Omega^1\mathcal{B}$ is then an endomorphism of $(\mathcal{E}_p, \gamma)$ in the category $\mathrm{Rep}_{\pi^1(\Omega^\bullet \mathcal{A}, p)}(\mathrm{Mod}_{\Omega^\bullet B})$, so $(\nabla_2)_p$ intertwines the action of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ on $\mathcal{E}_p$. Now $(\mathcal{E}_p, (\nabla_2)_p)$ is an object in the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet B)$, so it gives a representation of $\pi^1(\Omega^\bullet \mathcal{B}, q)$ on the vector space $\mathcal{E}_{pq}$. Each group element of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ gives an endomorphism of $(\mathcal{E}_p, (\nabla_2)_p)$ since it commutes with the connection. The localisation of this action to $\mathcal{E}_{pq}$ then commutes with the action of $\pi^1(\Omega^\bullet \mathcal{B}, q)$ on $\mathcal{E}_{pq}$. The two actions of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ and $\pi^1(\Omega^\bullet \mathcal{B}, q)$ on $\mathcal{E}_{pq}$ then combine to an action of $\pi^1(\Omega^\bullet \mathcal{A}, p) \times \pi^1(\Omega^\bullet \mathcal{B}, q)$ on $\mathcal{E}_{pq}$. Each step in this construction can be reversed so we get the equivalence

$$\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}) \cong \mathrm{Rep}(\pi^1(\Omega^\bullet A, p) \times \pi^1(\Omega^\bullet B, q)).$$

So we get

$$\pi^1(\Omega^\bullet A \otimes \Omega^\bullet B, p \otimes q) = \pi^1(\Omega^\bullet A, p) \times \pi^1(\Omega^\bullet B, q).$$

$\square$

6 References

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