Connections and the fundamental group of non-commutative spaces

Master thesis

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1 Introduction

Many geometric spaces can be analysed by looking at a corresponding algebra. Properties of the geometric space then give analogous properties of the algebra. This algebra will always be commutative. In non-commutative geometry, one studies similar algebras without the commutativity condition and tries to still find geometric properties of these algebras.

For example, the Gelfand-Naimark theorem gives a correspondence between locally compact Hausdorff spaces X and commutative C^*-algebras A: for any locally compact Hausdorff space X we can consider the C^*-algebra of continuous functions $C(X)$, and for each commutative C^*-algebra A we can consider the space of characters on A, with the weak$*$-topology. This allows us to think about topological spaces in terms of their corresponding algebra. For example, the space X is compact if and only if the algebra $C(X)$ is unital, and the space X is connected if and only if the algebra $C(X)$ contains no non-trivial projections. In non-commutative geometry we consider C^*-algebras A that are not required to be commutative, and try to find properties of the associated 'non-commutative space' - even though there is not really a topological space on which these properties should hold. This allows us to consider more general spaces: for example, phase space in quantum mechanics [7] or quotients by 'bad' equivalence relations [6]. There are also algebraic analogues for geometries with more structure than just a topology: measure spaces correspond to Von Neumann algebras (see [5] chapter V) and Riemannian spin manifolds correspond to spectral triples (see [5] chapter VI).

In this thesis we are interested in the fundamental group. The fundamental group of a space X can be defined as equivalence classes of continuous maps $\gamma : S^1 \to X$ sending a basepoint $s_0 \in S^1$ to a fixed basepoint $x_0 \in X$, where two such maps are equivalent if there is a base-point preserving continuous homotopy $H : S^1 \times [0, 1] \to X$ between them. To dualise this, we could consider maps from a C^*-algebra A to the C^*-algebra $C([0, 1])$. However this does not seem to be very interesting for non-commutative spaces: since $C([0, 1])$ is commutative, any $*$-homomorphism $\varphi : A \to C([0, 1])$ would send any commutator $ab - ba$ to 0, which for many non-commutative spaces already means that φ is trivial.

Another way to define the fundamental group of a manifold is as the automorphism group of the universal cover. To generalise this we could look at non-commutative generalisations of covering spaces, as in [3] or [13]. We use a slightly different approach: we use that the representations of the fundamental group of a manifold correspond to flat connections on vector bundles over this manifold (this is proven in section 2, using the universal covering space of the manifold). From this, the (algebraic hull of) the fundamental group can be reconstructed. To generalise this, we define a notion of flat connections over non-commutative spaces in section 3. We show that (under some extra conditions) it is a neutral Tannakian category. This is then used to define the fundamental group of the non-commutative space, similar to the approach in [16]. In section 4 we apply this on non-commutative tori. In section 5 we try to make the construction functorial and consider the product of spaces. In section 6 we consider a broader definition of connections.

For simplicity, we assume everywhere in this thesis that our algebras are unital (corresponding to compact manifolds).

2 The classical case

In this section we will demonstrate a well-known correspondence between the representations of the fundamental group of a manifold, and the flat connections on vector bundles over the manifold. This can be used to reconstruct the fundamental group (or at least an algebraic hull of the fundamental group) from the category of flat vector bundles. This will give an alternative definition of the fundamental group that is more suitable for generalisation to the non-commutative case, which we will attempt in later sections.

Let M be a connected compact manifold. It is well-known that each connected manifold has a universal cover $p : \tilde{M} \to M$. This is by definition a covering space such that $\tilde{M}$ is simply connected. It is also well-known that p is a $\pi_1(M)$-principal bundle, where $\pi_1(M)$ denotes the fundamental group of M (see for instance [12], section 1.3) . We show this in the following lemma.

Lemma 2.1. *Let M be a connected compact manifold with universal cover $p : \tilde{M} \to M$. Then p is a $\pi_1(M)$-principal bundle.*

Proof. Pick a basepoint $m_0 \in M$. For each path $\gamma : I \to M$ that starts and ends at m_0, and each point x in the fibre above m_0, there is a unique path $\tilde{\gamma} : I \to \tilde{M}$ above γ that starts at x. It does not have to end at x. This gives an action of the loop space $\Omega_{m_0}M$ on the fibre $p^{-1}(m_0)$, where $x \cdot \gamma$ is the endpoint of the path $\tilde{\gamma}$. Since $\gamma \cdot x$ depends continuously on γ and the fibre is discrete, we see that this action is homotopy invariant, so it factorises through a right action of $\pi_1(M, m_0)$ on the fibre $p^{-1}(m_0)$, which is indeed a group action. This action is transitive because $\tilde{M}$ is connected: if $x, x' \in p^{-1}(m_0)$ there is a path from x to x', this lies above a path $\gamma : I \to M$ that satisfies $x \cdot [\gamma] = x'$ (here $[\gamma]$ denotes the class of a loop γ in $\pi_1(M, m_0)$). The action is free because $\tilde{M}$ is simply connected: if $x \cdot [\gamma] = x$ we know that $\tilde{\gamma}$ is a path from x to x in $\tilde{M}$, so it is homotopic to a constant path, which gives a homotopy between γ and a constant path.

Now we define the action of $\pi_1(M, m_0)$ on the entire manifold $\tilde{M}$. Let $x_0 \in p^{-1}(m_0)$ be a fixed point above m_0. Let $[\gamma] \in \pi_1(M, m_0)$ and $x \in \tilde{M}$. Let $\chi : I \to \tilde{M}$ be a path that starts at x and ends at x_0. Now there is a unique path $\psi : I \to \tilde{M}$ that starts at $x_0 \cdot [\gamma]$ such that $p \circ \psi$ is the reversal of the path $p \circ \chi$. Define $x \cdot [\gamma]$ as the endpoint of ψ. This is a point in the same fibre as x. This depends continuously on the choice of χ, but since the fibre is discrete and $\tilde{M}$ is simply connected, it does not depend on χ at all. This defines the group action of $\pi_1(M, m_0)$ on $\tilde{M}$, and this is again a free transitive action on each fibre. So $p : \tilde{M} \to M$ is a $\pi_1(M, m_0)$-principal bundle. □

Let $\mathrm{Rep}(\pi_1(M))$ denote the category of finite-dimensional representations of $\pi_1(M)$ and let $\mathrm{FlatVec}(M)$ denote the category of finite-dimensional vector bundles with a flat connection. The morphisms in the last category are the morphisms of vector bundles that commute with the connection. We can now show the equivalence between representations of $\pi_1(M)$ and flat vector bundles over M. Since this is a well-known theorem (again see [12], section 1.3), we only sketch the proof.

Theorem 2.2. *Let M be a manifold. There is an equivalence of categories*

$$\mathrm{FlatVec}(M) \cong \mathrm{Rep}(\pi_1(M)).$$

Proof. Let m_0 be a base point of M. Let E be a vector bundle over M with a flat connection ∇. Let V be the fibre of E at p. The group $\pi_1(M, m_0)$ acts on V in the following way: for $[\gamma] \in \pi_1(M, m_0)$ and $x \in E$, there is a unique path $u : I \to E$ above γ, that starts at x and satisfies $\frac{\nabla u}{dt} = 0$. Define $[\gamma] \cdot x$ as the endpoint of u. It can be shown that this does not depend on the choice of representative $[\gamma]$ because the connection is flat, and it gives a group action of $\pi_1(M, m_0)$ on V. This defines the functor $\mathrm{FlatVec}(M) \to \mathrm{Rep}(\pi_1(M))$.

Conversely, let V be a finite-dimensional representation of $\pi_1(M)$. The group $\pi_1(M)$ acts on the right on $\tilde{M} \times V$ by $g \cdot (x, v) = (x \cdot g, g^{-1} \cdot v)$, using the right action of $\pi_1(M)$ on $\tilde{M}$ and the left action on V. Let $E = \tilde{M} \times_{\pi_1(M)} V$ be the quotient by this action. Its elements are of the form $[x, v]$ with $x \in \tilde{M}, v \in V$, where $[x \cdot g, v] = [x, g \cdot v]$ for $g \in \pi_1(M)$. Since the action of $\pi_1(M)$ on $\tilde{M}$ is fibre-wise, the composition $\tilde{M} \times V \to \tilde{M} \xrightarrow{p} M$ factors through a map $E \to M$. This makes E a vector bundle over M. Now we can define a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ as follows: each section $s \in \Gamma(E)$ can locally be written as $s(m) = [h(m), \tilde{s}(m)]$, with $h : U \subseteq M \to W \subseteq \tilde{M}$ a local section of p and $\tilde{s} : U \to V$. Then for any vector field X define $\nabla_X(s) \in \Gamma(E)$ by $\nabla_X(s)(m) = [h(m), \mathcal{L}_X \tilde{s}(m)]$, where $\mathcal{L}_X$ denotes the Lie derivative. We leave it to the reader to check that this is well-defined, and that ∇ satisfies the Leibniz rule and that it is flat (or see [12], section 1.3). This gives the functor $\mathrm{Rep}(\pi_1(M)) \to \mathrm{FlatVec}(M)$. It is then easy to check that the two functors defined above are inverse to each other, establishing the equivalence $\mathrm{FlatVec}(M) \cong \mathrm{Rep}(\pi_1(M))$. $\square$

2.1 Tannakian categories

From the category of representations of a group, it is possible to reconstruct (an algebraic hull of) the group. This was done by Saavedra Rivano [20] as a student of Grothendieck, and Deligne and Milne fixed a gap in the proof [8]. We will state the definitions and the main theorem from [8] without proof.

The notion of a tensor category is modelled on the category of vector spaces with their tensor product. Let $\mathcal{C}$ be a category with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, sending (X, Y) to $X \otimes Y$. In a tensor category there are some given isomorphisms: for any three objects X, Y, Z there is the *associativity constraint* $\varphi_{X,Y,Z}$ which is an isomorphism from $X \otimes (Y \otimes Z)$ to $(X \otimes Y) \otimes Z$. For any two objects X, Y there is also a *commutativity constraint* $\psi_{X,Y}$ which is an isomorphism from $X \otimes Y$ to $Y \otimes X$, which is the inverse of $\psi_{Y,X}$. For all objects X, Y, Z, T the following diagrams have to commute:

$$\begin{tikzcd}[column sep=small]
 & X \otimes (Y \otimes (Z \otimes T))) \arrow[dl] \arrow[dr] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr] & & ((X \otimes Y) \otimes Z) \otimes T,
\end{tikzcd}$$

where all the maps are given by associativity constraints, and

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r] \arrow[d] & (X \otimes Y) \otimes Z \arrow[d] \\
X \otimes (Z \otimes Y) \arrow[d] & Z \otimes (X \otimes Y) \arrow[d] \\
(X \otimes Z) \otimes Y \arrow[r] & (Z \otimes X) \otimes Y,
\end{tikzcd}$$

where each map is an associativy constraint or a commutativity constraint. These are called the *pentagon axiom* and the *commutativity axiom* respectively [8]. An object U is called an *identity object* if there is an isomorphism $u : U \to U \otimes U$, and the functor $\mathcal{C} \to \mathcal{C}, X \to U \otimes X$ is an equivalence of categories. Now a tensor category is defined to be a system $(\mathcal{C}, \otimes, \varphi, \psi)$ satisfying the pentagon and hexagon axioms, with an identity object.

Examples of tensor categories include the category vector spaces and the category of R-modules, where R is any commutative ring.

Definition 2.3. Let $\mathcal{C}$ be a tensor category with unit U. A dual of an object X is an object $X^\vee$ satisfying

$$\mathrm{Hom}(T \otimes X, U) = \mathrm{Hom}(T, X^\vee)$$

for all objects T. Then we get $\mathrm{Hom}(X^\vee \otimes X, U) = \mathrm{Hom}(X^\vee, X^\vee)$ so we have a natural morphism $X^\vee \otimes X \to U$. If $X^\vee$ has a dual, we also have $\mathrm{Hom}(X^\vee \otimes X, U) = \mathrm{Hom}(X, (X^\vee)^\vee)$ so we get a natural morphism $X \to (X^\vee)^\vee$. If this is an isomorphism, X is called reflexive. The tensor category $\mathcal{C}$ is called *rigid* if each object has a dual and is reflexive.

For a rigid tensor category there also exist Hom objects: for two objects X and Y we have $\underline{\mathrm{Hom}}(X, Y) = X^\vee \otimes Y$, and this satisfies

$$\mathrm{Hom}(T \otimes X, Y) = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y)).$$

The category of vector spaces is not rigid, since an infinite-dimensional vector space is not reflexive. However, the category of finite-dimensional vector spaces is rigid, and so is the category of finitely generated projective modules over a commutative algebra R.

Definition 2.4. A tensor category is called an abelian tensor category if it is abelian and the tensorproduct is bi-additive.

Remark 2.5. Recall that a category $\mathcal{C}$ is called abelian if $\mathrm{Hom}(X, Y)$ is an abelian group for all objects X and Y, composition is linear, finite direct sums exist and every morphism has a kernel, cokernel and image

An example of this is the category of finite-dimensional representations of an affine algebraic group scheme. For the definition of an affine algebraic group scheme, see [8] or [22]. If G is an affine algebraic group scheme over $\mathbb{C}$, the category Rep_G of finite-dimensional representations of G is a rigid abelian tensor category. Let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_{\mathbb{C}}$ be the forgetful functor to the category of finite-dimensional vector spaces. Let $g \in G$. For each object $X \in \mathrm{Rep}_G$ the element g gives an automorphism on $\omega(X)$. These automorphisms are compatible with morphisms in Rep_G and also with the tensor product. So g induces a natural automorphism of the fibre functor ω. We get a map from G to $\mathrm{Aut}^{\otimes \omega}$, which denotes the natural automorphisms of ω that commute with tensor products. The next theorem from [8] shows that this is an isomorphism. It also shows which categories are equivalent to the category of finite-dimensional representations of an affine algebraic group scheme.

Theorem 2.6. *(i) Let G be an affine algebraic group scheme, and let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_{\mathbb{C}}$ be the forgetful functor. Then we have a natural isomorphism $G \to \mathrm{Aut}^{\otimes}(\omega)$.*

(ii) Let $\mathcal{C}$ be an abelian rigid tensor category and let $\omega : \mathrm{Rep}_G \to \mathrm{Vec}_{\mathbb{C}}$ be an exact faithful k-linear functor that commutes with tensor products. Assume that $\mathrm{End}(U) = \mathbb{C}$ for a unit $U \in \mathcal{C}$. Then $G = \mathrm{Aut}^{\otimes}(\omega)$ is an affine algebraic group scheme, and $\mathcal{C}$ is equivalent to the category Rep_G.

Proof. See [8], theorem 2.11. □

In the conditions of the second part, the functor ω is called the fibre functor. A category with a fibre functor that satisfies all the conditions of the second part of the theorem is called a *neutral Tannakian category.*

Let Γ be a topological group. The category Rep_Γ of finite-dimensional representations of Γ, with the forgetful functor to $\mathrm{Vec}_{\mathbb{C}}$, is a neutral Tannakian category. Therefore it is equivalent to the category of finite-dimensional representations of an affine group scheme G. In general this affine group scheme is not equal to Γ, unless Γ is a compact group. The affine group scheme G is called the *algebraic hull* of Γ. This concept is called Tannakian duality.

Our plan is now to define a category $\mathcal{C}(X)$ of flat connections on a non-commutative space X with a fibre functor, that satisfies the conditions of the theorem. We will then define the fundamental group of X to be the group scheme of automorphisms of the fibre functor. For a (commutative) manifold M we retrieve the (algebraic hull of the) usual fundamental group of M, by theorem 2.2. A disadvantage of this approach is that different groups may have the same category of representations, hence the same algebraic hull, and our method cannot distinguish between these groups. For example, the topological group $\mathbb{R}$ and dense subgroups of it have the same representations (see lemma 8.2).

3 Connections over a non-commutative space

In this section we will define connections over a non-commutative space. For our non-commutative spaces we take a differential graded algebra. We consider the category of all flat connections for this algebra, and we will prove that under certain conditions this is a neutral Tannakian category. This allows us to define the fundamental group of a space. We also show that the category only depends on the (graded commutative) centre of the algebra.

There are multiple ways to generalise vector bundles to a non-commutative space. Over a commutative space, the vector bundles over a manifold correspond to finitely generated projective modules over the algebra. Over the non-commutative algebra we can look at right modules, but there is no tensor product of right modules. We can look at bimodules, but here we have to remember that over a commutative space, only very special bimodules are allowed: the multiplication on the left is the same as on the right. We can look at bimodules that satisfy this condition for the centre of the algebra. Instead we look at a more restrictive class of bimodules, namely those that are a direct summand of a free finite module ([10] defines *diagonal bimodules* as modules that are a summand of a free module).

Connections are then defined as maps $\mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ that satisfy the appropriate Leibniz rules. However to satisfy the left Leibniz rule the image should actually be $\Omega^1 A \otimes \mathcal{E}$. This is solved in [9] by using an isomorphism $\sigma : \mathcal{E} \otimes \Omega^1\mathcal{A} \to \Omega^1\mathcal{A} \to \mathcal{E}$. In their approach the tensor product of two flat connections is not necessarily flat. Instead we will look at graded bimodules $\Omega^\bullet \mathcal{E}$ over the differential algebra $\Omega^\bullet \mathcal{A}$ that are a summand of a free finite module. That way we automatically get isomorphisms $\Omega^1\mathcal{E} \cong \mathcal{E} \otimes \Omega^1\mathcal{A} \cong \Omega^1\mathcal{A} \otimes \mathcal{E}$ (see lemma 3.5). When defined in this way we will see that the tensor product of flat connections is always flat (see lemma 3.18).

Notation 3.1. *If α is a homogeneous element of a graded algebra or module, we denote the degree by $|\alpha|$. If we use this notation we always assume that α is homogeneous.*

Definition 3.2. A *graded differential algebra* or dga is a graded algebra $\Omega^\bullet\mathcal{A}$, with $\Omega^0\mathcal{A} = \mathcal{A}$, together with a $\mathbb{C}$-linear map $d : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{A}$ of degree $+1$ satisfying the Leibniz rule

$$d(\omega\nu) = d\omega \cdot \nu + (-1)^{|\omega|}\omega d\nu$$

for $\omega, \nu \in \Omega^\bullet \mathcal{A}$, and $d^2 = 0$.

Note that we do not ask that $\Omega^1\mathcal{A}$ should be spanned by elements of the form adb with $a, b \in \mathcal{A}$. Therefore $\Omega^\bullet \mathcal{A}$ is in general not a quotient of the universal differential algebra of $\mathcal{A}$.

Notation 3.3. *Let $\Omega^\bullet \mathcal{E}$ be a graded $\Omega^\bullet \mathcal{A}$-bimodule. We define the graded commutator $[\cdot,\cdot] : \Omega^\bullet \mathcal{A} \times \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ as $[\alpha, \varepsilon] = \alpha\varepsilon - (-1)^{|\alpha|\cdot|\varepsilon|}\varepsilon\alpha$ for $\alpha \in \Omega^\bullet \mathcal{A}, \varepsilon \in \Omega^\bullet \mathcal{E}$ and $\mathbb{C}$-bilinearly extended to non-homogeneous elements. We use the same notation for the graded commutator $[\cdot,\cdot] : \Omega^\bullet \mathcal{A} \times \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{A}$ defined similarly.*

Definition 3.4. Let $\Omega^\bullet \mathcal{A}$ be any dga. We call a graded $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{E}$ *finitely generated projective* (fgp) if there is another graded $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{F}$ satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$ for some integer n, as graded $\Omega^\bullet \mathcal{A}$-bimodules.

The following lemma shows why it is convenient to work with these graded fgp $\Omega^\bullet \mathcal{A}$-bimodules instead of just fgp bimodules over $\mathcal{A} = \Omega^0\mathcal{A}$.

Lemma 3.5. *Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule. Write $\mathcal{A} = \Omega^0\mathcal{A}$ and $\mathcal{E} = \Omega^0\mathcal{E}$. Then for all $k \geq 0$ the multiplication induces isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \xrightarrow{\sim} \Omega^k\mathcal{E}$$
$$\Omega^k\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\sim} \Omega^k\mathcal{E}.$$

Proof. Let $\Omega^\bullet \mathcal{F}$ be another graded $\Omega^\bullet \mathcal{A}$-bimodule satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \cong (\Omega^\bullet \mathcal{A})^n$. Then we have the following commuting diagram:

$$\begin{tikzcd}
\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \oplus \mathcal{F} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[rr] \arrow[d, "\sim"] & & \Omega^k\mathcal{E} \oplus \Omega^k\mathcal{F} \arrow[d, "\sim"] \\
(\mathcal{E} \oplus \mathcal{F}) \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[r, "\sim"] & \mathcal{A}^n \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \arrow[r, "\sim"] & (\Omega^k\mathcal{A})^n.
\end{tikzcd}$$

Here the top arrow is the direct sum of the maps $\mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \to \Omega^k\mathcal{E}$ and $\mathcal{F} \otimes_{\mathcal{A}} \Omega^k\mathcal{A} \to \Omega^k\mathcal{F}$. All the other maps are isomorphisms, so these maps are isomorphisms as well. This shows that the first map in the lemma is an isomorphism and the second one follows analogously. $\square$

Definition 3.6. Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule. A *connection* on $\Omega^\bullet \mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$$

of degree $+1$ satisfying the following equations for $\varepsilon \in \Omega^\bullet \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$:

$$\nabla(\varepsilon\alpha) = \nabla(\varepsilon)\alpha + (-1)^{|\varepsilon|}\varepsilon d\alpha,$$
$$\nabla(\alpha\varepsilon) = d\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha\nabla(\varepsilon).$$

Remark 3.7. The equations in this definition are the right Leibniz rule and the left Leibniz rule respectively. They are well-defined because the elements of $\Omega^\bullet \mathcal{E}$ can be multiplied with elements of $\Omega^\bullet \mathcal{A}$ both on the left and the right.

Definition 3.8. Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule with a connection ∇. The curvature of ∇ is the map $\nabla^2 : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ of degree $+2$.

Remark 3.9. The curvature is an $\Omega^\bullet \mathcal{A}$-bilinear map: for $\varepsilon \in \Omega^\bullet \mathcal{E}$ and $\alpha \in \Omega^\bullet \mathcal{A}$ we have

$$\nabla^2(\varepsilon\alpha) = \nabla(\nabla(\varepsilon)\alpha + (-1)^{|\varepsilon|}\varepsilon d\alpha)$$

$$
\begin{aligned}
&= \nabla^2(\varepsilon)\alpha + (-1)^{|\varepsilon|+1}\nabla(\varepsilon)d\alpha + (-1)^{|\varepsilon|}\nabla(\varepsilon)d\alpha + \varepsilon d^2\alpha \\
&= \nabla^2(\varepsilon)\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla^2(\alpha\varepsilon) &= \nabla(d\alpha \cdot \varepsilon + (-1)^{|\alpha|}\alpha\nabla(\varepsilon)) \\
&= d^2\alpha \cdot \varepsilon + (-1)^{|\alpha|+1}d\alpha\nabla(\varepsilon) + (-1)^{|\alpha|}d\alpha\nabla(\varepsilon) + \alpha\nabla^2(\varepsilon) \\
&= \alpha\nabla^2(\varepsilon).
\end{aligned}
$$

Definition 3.10. A connection ∇ is called *flat* if the curvature ∇^2 is zero.

Now we can define the category of flat connections.

Definition 3.11. We define $\mathcal{C}(\Omega^\bullet \mathcal{A})$ to be the category whose objects are graded fgp $\Omega^\bullet \mathcal{A}$-bimodules with a connection, and whose morphisms are graded fgp $\Omega^\bullet \mathcal{A}$-bimodule morphisms which commute with the connections. Let $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ be the full subcategory where the connections are required to be flat.

Example 3.12. Let M be a manifold. The corresponding dga is the de Rham differential algebra of the manifold $\Omega^\bullet \mathcal{A} = \Omega^\bullet{}_{\text{deRham}} M$. Any fgp $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{E}$ is determined by the fgp $\mathcal{A}$-bimodule $\mathcal{E}$. This corresponds to a vector bundle over M by the Serre-Swan theorem. A flat connection on $\Omega^\bullet \mathcal{E}$ corresponds to a (usual) flat connection on this vector bundle. So $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is equivalent to the category of vector bundles over M with a flat connection, which is equivalent to the category of representations of $\pi_1 M$ by theorem 2.2.

3.1 connections over the graded centre

Each graded fgp $\Omega^\bullet \mathcal{A}$-bimodule is determined by a graded fgp bimodule over a graded commutative subalgebra. For this we need the following definitions:

Definition 3.13. Let $\Omega^\bullet \mathcal{A}$ be a dga. We define the graded commutative centre $Z_g(\Omega^\bullet \mathcal{A})$ as

$$Z_g(\Omega^\bullet \mathcal{A}) = \{\alpha \in \Omega^\bullet \mathcal{A} \mid [\alpha, \nu] = 0 \text{ for all } \nu \in \Omega^\bullet \mathcal{A}\}.$$

If $\Omega^\bullet \mathcal{E}$ is an fgp $\Omega^\bullet \mathcal{A}$-bimodule we define $Z_g(\Omega^\bullet \mathcal{E})$ as

$$Z_g(\Omega^\bullet \mathcal{E}) = \{\varepsilon \in \Omega^\bullet \mathcal{A} \mid [\varepsilon, \nu] = 0 \text{ for all } \nu \in \Omega^\bullet \mathcal{A}\}.$$

Lemma 3.14. *With notations as in the previous definition we have the following, part of which is shown in [9], page 12:*

(i) The graded commutative centre of the algebra $Z_g(\Omega^\bullet \mathcal{A})$ is a dga.

(ii) The graded commutative centre of the module $Z_g(\Omega^\bullet \mathcal{E})$ is a graded fgp bimodule over $Z_g(\Omega^\bullet \mathcal{A})$.

(iii) The multiplication induces an isomorphism $Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \xrightarrow{\sim} \Omega^\bullet \mathcal{E}$.

Proof. (i) An easy calculation shows that $Z_g(\Omega^\bullet \mathcal{A})$ is a subalgebra of $\Omega^\bullet \mathcal{A}$, using that $[\alpha\beta, \nu] = \alpha[\beta, \nu] + (-1)^{|\beta|\cdot|\nu|}[\alpha, \nu]\beta$ for $\alpha, \beta, \nu \in \Omega^\bullet \mathcal{A}$. It is closed under d because $[d\alpha, \nu] = d[\alpha, \nu] - (-1)^{|\alpha|}[\alpha, d\nu]$ for $\alpha, \nu \in \Omega^\bullet \mathcal{A}$.

(ii) An easy calculation shows that $Z_g(\Omega^\bullet \mathcal{E})$ is a graded bimodule over $Z_g(\Omega^\bullet \mathcal{A})$, using $[\alpha\varepsilon, \nu] = \alpha[\varepsilon, \nu] + (-1)^{|\varepsilon|\cdot|\nu|}[\alpha, \nu]\varepsilon$ for $\alpha, \nu \in \Omega^\bullet \mathcal{A}, \varepsilon \in \Omega^\bullet \mathcal{E}$. If $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$ it follows directly that $Z_g(\Omega^\bullet \mathcal{E}) \oplus Z_g(\Omega^\bullet \mathcal{F}) = (Z_g(\Omega^\bullet \mathcal{A}))^n$, so $Z_g(\Omega^\bullet \mathcal{E})$ is a graded fgp $Z_g(\Omega^\bullet \mathcal{A})$-bimodule.

(iii) Let $\Omega^\bullet \mathcal{F}$ be another graded fgp $\Omega^\bullet \mathcal{A}$-bimodule satisfying $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} = (\Omega^\bullet \mathcal{A})^n$. Then we have the following commuting diagram:

$$\begin{tikzcd}
Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \oplus Z_g(\Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[r] \arrow[d, "\sim"] & \Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \arrow[d, "\sim"] \\
(Z_g(\Omega^\bullet \mathcal{E}) \oplus Z_g(\Omega^\bullet \mathcal{F})) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[d, "\sim"] & (\Omega^\bullet \mathcal{A})^n \\
Z_g(\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \arrow[r, "\sim"] & Z_g((\Omega^\bullet \mathcal{A})^n) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}. \arrow[u, "\sim"']
\end{tikzcd}$$

The top arrow is the direct sum of the maps $Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{E}$ and $Z_g(\Omega^\bullet \mathcal{F}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{F}$ induced by multiplication. All other arrows in the diagram are isomorphisms, so these maps are isomorphisms as well.

□

So all graded fgp $\Omega^\bullet \mathcal{A}$-bimodules are determined by a graded fgp $Z_g(\Omega^\bullet \mathcal{A})$-bimodule. This is in turn determined by an fgp $Z_g(\Omega^0 \mathcal{A})$-bimodule. Note that $Z_g(\Omega^0 \mathcal{A})$ may be smaller than the centre of the algebra $\mathcal{A}$.

Theorem 3.15. *We have a natural isomorphism*

$$\mathcal{C}(\Omega^\bullet \mathcal{A}) \xrightarrow{\sim} \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A})).$$

Proof. Let $(\Omega^\bullet \mathcal{E}, \nabla)$ be an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. For $\varepsilon \in Z_g(\Omega^\bullet \mathcal{E})$ and $\alpha \in \Omega^\bullet \mathcal{A}$ we have

$$\nabla(\varepsilon\alpha) = \nabla(\varepsilon)\alpha + (-1)^{|\varepsilon|}\varepsilon d\alpha$$

but also

$$\begin{aligned}
\nabla(\varepsilon\alpha) &= (-1)^{|\varepsilon|\cdot|\alpha|}\nabla(\alpha\varepsilon) \\
&= (-1)^{(|\varepsilon|+1)|\alpha|}\alpha\nabla(\varepsilon) + (-1)^{|\varepsilon|\cdot|\alpha|}d\alpha \cdot \varepsilon \\
&= (-1)^{(|\varepsilon|+1)|\alpha|}\alpha\nabla(\varepsilon) + (-1)^{|\varepsilon|}\varepsilon d\alpha.
\end{aligned}$$

So we get

$$\nabla(\varepsilon)\alpha = (-1)^{(|\varepsilon|+1)|\alpha|}\alpha\nabla(\varepsilon).$$

Since this holds for all $\alpha \in \Omega^\bullet \mathcal{A}$ we conclude that $\nabla(\varepsilon) \in Z_g(\Omega^\bullet \mathcal{E})$. So ∇ restricts to a function $Z_g(\Omega^\bullet \mathcal{E}) \to Z_g(\Omega^\bullet \mathcal{E})$. Then $(Z_g(\Omega^\bullet \mathcal{E}), \nabla_{Z_g(\Omega^\bullet \mathcal{E})})$ is an object of $\mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$. It is easy to see that this is a functorial construction.

Conversely, let $(\Omega^\bullet \mathcal{F}, \nabla)$ be an object of $\mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$. Then we can define the graded fgp $\Omega^\bullet \mathcal{A}$-bimodule $\Omega^\bullet \mathcal{F} \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}$, and the connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}(\zeta \otimes \alpha) = \nabla(\zeta) \otimes \alpha + (-1)^{|\zeta|}\zeta \otimes d(\alpha)$$

for $\zeta \in \Omega^\bullet \mathcal{F}, \alpha \in \Omega^\bullet \mathcal{A}$. This gives an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. It is then easy to show that this construction is also functorial, and that the two functors thus defined are inverse to each other. □

The above isomorphism also restricts to an isomorphism $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \xrightarrow{\sim} \mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$ as the functor and its inverse clearly preserve flatness.

For a graded commutative dga $\Omega^\bullet \mathcal{A}$ we have a different way to describe the category. Remember that for a graded fgp bimodule $\Omega^\bullet \mathcal{E}$ we have the isomorphisms $\Omega^k\mathcal{E} = \mathcal{E} \otimes_{\mathcal{A}} \Omega^k\mathcal{A}$. The restriction of ∇ to $\mathcal{E}$ is then a map $\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$, satisfying $\nabla_0(ea) = \nabla_0(e)a + e \otimes da$. Conversely each such ∇_0 may be extended to $\nabla : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{E}$ by setting $\nabla(e \otimes \omega) = \nabla_0(e)\omega + e \otimes d\omega$. So we can describe $\mathcal{C}(\Omega^\bullet \mathcal{A})$ as the category with objects fgp $\mathcal{A}$-modules $\mathcal{E}$ with a connection $\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$.

Example 3.16. Let $\mathcal{A}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in M_2(\mathbb{C}) \right\}$ and $\mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \in M_2(\mathbb{C}) \right\}$. Let $\Omega^\bullet \mathcal{A}$ be defined by $\Omega^k\mathcal{A} = \mathcal{A}_0$ for even k and $\Omega^k\mathcal{A} = \mathcal{A}_1$ for odd k, with the differential $d : \mathcal{A}_0 \to \mathcal{A}_1, \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \to \begin{pmatrix} 0 & \beta - \alpha \\ \alpha - \beta & 0 \end{pmatrix}$, and $d : \mathcal{A}_1 \to \mathcal{A}_0, \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \to \begin{pmatrix} \delta + \gamma & 0 \\ 0 & \gamma + \delta \end{pmatrix}$. This is the dga corresponding to the non-commutative space of two points with finite distance (see for example [15], pages 116-8).

Now we have $Z_g(\Omega^k\mathcal{A}) = \mathbb{C}$ in even degrees and $Z_g(\Omega^k\mathcal{A}) = 0$ in odd degrees. The fgp bimodules over $Z_g(\Omega^k\mathcal{A})$ are simply determined by a vector space over $\mathbb{C}$, so we get that $\mathcal{C}(\Omega^\bullet \mathcal{A}) \cong \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ is equivalent to the category of vector spaces.

3.2 Tensor products

In this subsection we will construct the tensor product for the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$.

Proposition 3.17. *Let $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ be objects of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Then $\Omega^\bullet \mathcal{G} = \Omega^\bullet \mathcal{E} \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{F}$ has the structure of an fgp graded bimodule over $\Omega^\bullet \mathcal{A}$, and we can construct a connection $\nabla^{\mathcal{G}}$ satisfying*

$$\nabla^{\mathcal{G}}(\varepsilon \otimes \zeta) = \nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^{|\varepsilon|}\varepsilon \otimes \nabla^{\mathcal{F}}(\zeta)$$

for $\varepsilon \in \Omega^\bullet \mathcal{E}, \zeta \in \Omega^\bullet \mathcal{F}$.

Proof. The left action of $\Omega^\bullet \mathcal{A}$ on $\Omega^\bullet \mathcal{E}$ and the right action of $\Omega^\bullet \mathcal{A}$ on $\Omega^\bullet \mathcal{F}$ make $\Omega^\bullet \mathcal{G}$ into a $\Omega^\bullet \mathcal{A}$-bimodule. A grading on the tensor product is given as follows: for any $n \geq 0$ the degree n subspace $\Omega^n\mathcal{G}$ is the linear span of elements $\varepsilon \otimes \zeta$, with $\varepsilon \in \Omega^k\mathcal{E}, \zeta \in \Omega^l\mathcal{F}, k+l = n$. To show that $\Omega^\bullet \mathcal{G}$ is fgp, suppose that $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{E}' \cong \Omega^\bullet \mathcal{A}^k$ and $\Omega^\bullet \mathcal{F} \oplus \Omega^\bullet \mathcal{F}' \cong \Omega^\bullet \mathcal{A}^l$. Then

$$\Omega^\bullet \mathcal{G} \oplus \Omega^\bullet \mathcal{E}' \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{F} \oplus (\Omega^\bullet \mathcal{F}')^k \cong (\Omega^\bullet \mathcal{F})^k \oplus (\Omega^\bullet \mathcal{F}')^k \cong (\Omega^\bullet \mathcal{A})^{kl}.$$

So $\Omega^\bullet \mathcal{G}$ is a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule.

We can then define the connection $\nabla^\mathcal{G}$ by

$$\nabla^\mathcal{G}(\varepsilon \otimes \zeta) = \nabla^\mathcal{E}(\varepsilon) \otimes \zeta + (-1)^{|\varepsilon|}\varepsilon \otimes \nabla^\mathcal{F}(\zeta)$$

for $\varepsilon \in \Omega^\bullet \mathcal{E}, \zeta \in \Omega^\bullet \mathcal{F}$. This defines $\nabla^\mathcal{G}$ on the pure tensors, and it is extended $\mathbb{C}$-linearly. To show that it is well-defined, let $\alpha \in \Omega^\bullet \mathcal{A}$. Then by the above definition we have

$$\begin{aligned}\nabla^\mathcal{G}(\varepsilon\alpha \otimes \zeta) &= \nabla^\mathcal{E}(\varepsilon\alpha) \otimes \zeta + (-1)^{|\varepsilon|+|\alpha|}\varepsilon\alpha \otimes \nabla^\mathcal{F}(\zeta)\\ &= \nabla^\mathcal{E}(\varepsilon)\alpha \otimes \zeta + (-1)^{|\varepsilon|}\varepsilon d\alpha \otimes \zeta + (-1)^{|\varepsilon|+|\alpha|}\varepsilon\alpha \otimes \nabla^\mathcal{F}(\zeta)\end{aligned}$$

while

$$\begin{aligned}\nabla^\mathcal{G}(\varepsilon \otimes \alpha\zeta) &= \nabla^\mathcal{E}(\varepsilon) \otimes \alpha\zeta + (-1)^{|\varepsilon|}\varepsilon \otimes \nabla^\mathcal{F}(\alpha\zeta)\\ &= \nabla^\mathcal{E}(\varepsilon) \otimes \alpha\zeta + (-1)^{|\varepsilon|}\varepsilon \otimes d\alpha \cdot \zeta + (-1)^{|\varepsilon|+|\alpha|}\varepsilon \otimes \alpha\nabla^\mathcal{F}(\zeta).\end{aligned}$$

Since these are the same, $\nabla^\mathcal{G}$ is well-defined.

Lastly, $\nabla^\mathcal{G}$ satisfies the Leibniz rules: for $\varepsilon \in \Omega^\bullet \mathcal{E}, \zeta \in \Omega^{\bullet l} \mathcal{F}, \alpha \in \Omega^\bullet \mathcal{A}$ we have

$$\begin{aligned}\nabla^\mathcal{G}(\alpha\varepsilon \otimes \zeta) &= \nabla^\mathcal{E}(\alpha\varepsilon) \otimes \zeta + (-1)^{|\varepsilon|+|\alpha|}\alpha\varepsilon \otimes \nabla^\mathcal{F}(\zeta)\\ &= d\alpha \cdot \varepsilon \otimes \zeta + (-1)^{|\alpha|}\alpha\nabla^\mathcal{E}(\varepsilon) \otimes \zeta + (-1)^{|\varepsilon|+|\alpha|}\alpha\varepsilon \otimes \nabla^\mathcal{F}(\zeta)\\ &= d\alpha \cdot \varepsilon \otimes \zeta + (-1)^{|\alpha|}\alpha\nabla^\mathcal{G}(\varepsilon \otimes \zeta)\end{aligned}$$

and

$$\begin{aligned}\nabla^\mathcal{G}(\varepsilon \otimes \zeta\alpha) &= \nabla^\mathcal{E}(\varepsilon) \otimes \zeta\alpha + (-1)^{|\varepsilon|}\varepsilon \otimes \nabla^\mathcal{G}(\zeta\alpha)\\ &= \nabla^\mathcal{E}(\varepsilon) \otimes \zeta\alpha + (-1)^{|\varepsilon|}\varepsilon \otimes \nabla^\mathcal{G}(\zeta)\alpha + (-1)^{|\varepsilon|+|\zeta|}\varepsilon \otimes \zeta d\alpha\\ &= \nabla^\mathcal{G}(\varepsilon \otimes \zeta)\alpha + (-1)^{|\varepsilon|+|\zeta|}\varepsilon \otimes \zeta d\alpha.\end{aligned}$$

□

The curvature on the tensor product is easily calculated:

Lemma 3.18. *In the notation of the previous lemma, we have*

$$(\nabla^\mathcal{G})^2 = (\nabla^\mathcal{E})^2 \otimes \Omega^\bullet \mathcal{F} \oplus \Omega^\bullet \mathcal{E} \otimes (\nabla^\mathcal{F})^2.$$

In particular, the tensor product of flat connections is again flat.

Proof. For $\varepsilon \in \Omega^\bullet \mathcal{E}, \zeta \in \Omega^\bullet \mathcal{F}$ we have

$$\begin{aligned}(\nabla^{\mathcal{G}})^2(\varepsilon \otimes \zeta) &= \nabla^{\mathcal{G}}(\nabla^{\mathcal{E}}(\varepsilon) \otimes \zeta + (-1)^{|\zeta|}\varepsilon \otimes \nabla^{\mathcal{F}}(\zeta)) \\ &= (\nabla^{\mathcal{E}})^2(\varepsilon) \otimes \zeta + (-1)^{|\zeta|+1}\nabla^{\mathcal{E}}(\varepsilon) \otimes \nabla^{\mathcal{F}}(\zeta) \\ &+ (-1)^{|\zeta|}\nabla^{\mathcal{E}}(\varepsilon) \otimes \nabla^{\mathcal{F}}(\zeta) + \varepsilon \otimes (\nabla^{\mathcal{F}})^2(\zeta) \\ &= (\nabla^{\mathcal{E}})^2(\varepsilon) \otimes \zeta + \varepsilon \otimes (\nabla^{\mathcal{F}})^2(\zeta).\end{aligned}$$

□

Remark 3.19. The above lemma does not apply for some other definitions of connections on bimodules, see for example [21], example 2.13.

It is easy to see that this tensor product is associative. The tensor product commutes with the equivalence of categories $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ from theorem 3.15. In the commutative case it is easy to see that the tensor product is also commutative; so we have a commutativity constraint in the general case as well.

There is a unit in $\mathcal{C}(\Omega^\bullet \mathcal{A})$: it is the bimodule $\Omega^\bullet \mathcal{A}$ with the connection d. It is easy to see that the isomorphism $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}$ commutes with the connection $\nabla^{\mathcal{E}}$ and the tensor product connection on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{A}$.

This makes $(\mathcal{C}(\Omega^\bullet \mathcal{A}), \otimes)$ into a tensor category, as well as the subcategory $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$.

3.3 Duals

We will now construct dual objects in the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$.

Proposition 3.20. *Let $(\Omega^\bullet \mathcal{E}, \nabla)$ be an object of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. The dual module $\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$ of right-$\Omega^\bullet \mathcal{A}$-linear maps from $\Omega^\bullet \mathcal{E}$ to $\Omega^\bullet \mathcal{A}$ is an fgp $\Omega^\bullet \mathcal{A}$-bimodule. We can construct a connection on $\Omega^\bullet \mathcal{E}^\vee$ satisfying*

$$\langle \nabla^\vee(\theta), \varepsilon\rangle = d\langle \theta, \varepsilon\rangle - (-1)^{|\theta|}\langle \theta, \nabla\varepsilon\rangle$$

for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}$. Here the angled brackets denote the pairing between $\Omega^\bullet \mathcal{E}^\vee$ and $\Omega^\bullet \mathcal{E}$.

Proof. The bimodule structure is given by $\langle \theta\alpha, \varepsilon\rangle = \langle \theta, \alpha\varepsilon\rangle$ and $\langle \alpha\theta, \varepsilon\rangle = \alpha\langle \theta, \varepsilon\rangle$ for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \alpha \in \Omega^\bullet \mathcal{A}, \varepsilon \in \Omega^\bullet \mathcal{E}$. There is a natural grading where $\Omega^{\bullet k} \mathcal{E}^\vee$ consists of the homogeneous maps of degree k. If $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F} \cong \Omega^\bullet \mathcal{A}^n$ we get $\Omega^\bullet \mathcal{E}^\vee \oplus \Omega^\bullet \mathcal{F}^\vee = (\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F})^\vee \cong \Omega^\bullet \mathcal{A}^n$. So $\Omega^\bullet \mathcal{E}^\vee$ is a graded fgp $\Omega^\bullet \mathcal{A}$-bimodule.

We can define the connection $\nabla^\vee$ on $\Omega^\bullet \mathcal{E}^\vee$ by

$$\langle \nabla^\vee(\theta), \varepsilon\rangle = d\langle \theta, \varepsilon\rangle - (-1)^{|\theta|}\langle \theta, \nabla(\varepsilon)\rangle$$

for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$. This is well-defined because $\langle \nabla^\vee(\theta), -\rangle$ is indeed a right-linear map with this definition: for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$:

$$\langle \nabla^\vee(\theta), \varepsilon\alpha\rangle = d\langle \theta, \varepsilon\alpha\rangle - (-1)^{|\theta|}\langle \theta, \nabla(\varepsilon\alpha)\rangle$$

$$= d(\langle(\theta,\varepsilon\rangle\alpha) - (-1)^{|\theta|}\langle\theta, \nabla(\varepsilon)\alpha + (-1)^{|\theta|}\varepsilon d\alpha\rangle$$
$$= d\langle\theta,\varepsilon\rangle\alpha + (-1)^{|\theta|+|\varepsilon|}\langle\theta,\varepsilon\rangle d\alpha - (-1)^{|\theta|}\langle\theta,\nabla(\varepsilon)\rangle\alpha - (-1)^{|\theta|+|\varepsilon|}\langle\theta,\varepsilon\rangle d\alpha$$
$$= \langle\nabla^\vee(\theta),\varepsilon\rangle\alpha.$$

This satisfies the Leibniz rules: for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}, \alpha \in \Omega^\bullet \mathcal{A}$ we have

$$\begin{aligned}\langle\nabla^\vee(\theta\alpha),\varepsilon\rangle &= d\langle\theta\alpha,\varepsilon\rangle - (-1)^{|\theta|+|\alpha|}\langle\theta\alpha,\nabla(\varepsilon)\rangle \\ &= d\langle\theta,\alpha\varepsilon\rangle - (-1)^{|\theta|+|\alpha|}\langle\theta,\alpha\nabla(\varepsilon)\rangle \\ &= d\langle\theta,\alpha\varepsilon\rangle - (-1)^{|\theta|+|\alpha|}\langle\theta,\nabla(\alpha\varepsilon)\rangle + (-1)^{|\theta|}\langle\theta, d\alpha\cdot\varepsilon\rangle \\ &= \langle\nabla^\vee(\theta),\alpha\varepsilon\rangle + (-1)^{|\theta|}\langle\theta d\alpha,\varepsilon\rangle \\ &= \langle\nabla^\vee(\theta)\alpha + (-1)^{|\theta|}\theta d\alpha,\varepsilon\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\nabla^\vee(\alpha\theta),\varepsilon\rangle &= d\langle\alpha\theta,\varepsilon\rangle - (-1)^{|\theta|+|\alpha|}\langle\alpha\theta,\nabla(\varepsilon)\rangle \\ &= d\alpha\langle\theta,\varepsilon\rangle + (-1)^{|\alpha|}\alpha d\langle\theta,\varepsilon\rangle - (-1)^{|\theta|+|\alpha|}\alpha\langle\theta,\nabla(\varepsilon)\rangle \\ &= \langle d\alpha\cdot\theta + (-1)^{|\alpha|}\alpha\nabla^\vee(\theta),\varepsilon\rangle.\end{aligned}$$

□

We can compute the curvature of the dual connection:

Lemma 3.21. *In the notation of the previous lemma, the curvature of $\nabla^\vee$ is minus the dual of the curvature of ∇, that is, for $\theta \in \Omega^\bullet \mathcal{E}, \varepsilon \in \Omega^\bullet \mathcal{E}$ we have*

$$\langle(\nabla^\vee)^2(\theta),\varepsilon\rangle = -\langle\theta,\nabla^2(\varepsilon)\rangle.$$

In particular, the dual of a flat connection is again flat.

Proof. We have for $\theta \in \Omega^\bullet \mathcal{E}^\vee, \varepsilon \in \Omega^\bullet \mathcal{E}$:

$$\begin{aligned}\langle((\nabla^\vee)^2(\theta),\varepsilon\rangle &= d\langle\nabla^\vee(\theta),\varepsilon\rangle - (-1)^{|\theta|+1}\langle\nabla^\vee(\theta),\nabla(\varepsilon)\rangle \\ &= d(d\langle\theta,\varepsilon\rangle - (-1)^{|\theta|}\langle\theta,\nabla(\varepsilon)\rangle) \\ &\quad - (-1)^{|\theta|+1}d\langle\theta,\nabla(\varepsilon)\rangle - \langle\theta,\nabla^2(\varepsilon)\rangle \\ &= -\langle\theta,\nabla^2(\varepsilon)\rangle.\end{aligned}$$

□

Writing $\Omega^\bullet \mathcal{E} = Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}$ we have

$$\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(Z_g(\Omega^\bullet \mathcal{E}) \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{A}) = Z_g(\Omega^\bullet \mathcal{E})^\vee \otimes_{Z_g(\Omega^\bullet \mathcal{A})} \Omega^\bullet \mathcal{A}.$$

So the equivalence of categories $\mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}(Z_g(\Omega^\bullet \mathcal{A}))$ commutes with the taking of duals. In particular this shows that the dual $\Omega^\bullet \mathcal{E}^\vee = \mathrm{Hom}_{\Omega^\bullet \mathcal{A}}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$ is naturally isomorphic to the space of left-linear functions ${}_{\Omega^\bullet \mathcal{A}}\mathrm{Hom}(\Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A})$.

Since $\Omega^\bullet E$ is a fgp bimodule, we have for each graded fgp bimodule $\Omega^\bullet \mathcal{F}$ an isomorphism $\mathrm{Hom}(\Omega^\bullet \mathcal{F} \otimes \Omega^\bullet \mathcal{E}, \Omega^\bullet \mathcal{A}) = \mathrm{Hom}(\Omega^\bullet \mathcal{F}, \Omega^\bullet \mathcal{E}^\vee \otimes \Omega^\bullet \mathcal{A})$. An easy calculation shows that this isomorphism continues to hold for morphisms that commute with connections, if connections on $\Omega^\bullet \mathcal{E}$ and $\Omega^\bullet \mathcal{F}$ are given. So $(\Omega^\bullet \mathcal{E}^\vee, \nabla^\vee)$ is a dual object to $(\Omega^\bullet \mathcal{E}, \nabla)$. The morphism $\Omega^\bullet \mathcal{E} \to (\Omega^\bullet \mathcal{E}^\vee)^\vee$ is an isomorphism because $\Omega^\bullet \mathcal{E}$ is fgp. So every object is reflexive, and $(\mathcal{C}(\Omega^\bullet \mathcal{A}), \otimes)$ is a rigid tensor category, and the subcategory $\mathcal{C}_{\text{flat}}$ is also a rigid tensor category.

3.4 Abelianness of the category

In this subsection we will show when the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian. Using theorem 3.15 we can always reduce to a graded commutative algebra. Therefore we will only consider graded commutative algebras in this subsection. We will assume that $\mathcal{A}$ is a unital $*$-algebra that is dense in a unital C^*-algebra A. This will be necessary for some of the constructions below. Moreover we also need the star operation on $\Omega^\bullet \mathcal{A}$.

Definition 3.22. A $*$-dga is a dga $\Omega^\bullet \mathcal{A}$ with a linear involution $*$, satisfying $(\alpha\beta)^* = \beta^*\alpha^*$ and $d(\alpha^*) = d(\alpha)^*$.

We also assume that the elements in $\mathcal{A}$ that are invertible in A are also invertible in $\mathcal{A}$, so $\mathcal{A} \cap A^\times = \mathcal{A}^\times$. This is in particular the case if $\mathcal{A}$ is stable under holomorphic functional calculus (see [11], page 134).

The category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is always an additive category: given two objects $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ the morphisms from $\Omega^\bullet \mathcal{E}$ to $\Omega^\bullet \mathcal{F}$ form an additive group, and there is an object $\Omega^\bullet \mathcal{E} \oplus \Omega^\bullet \mathcal{F}$ where the connection is simply given by $\nabla^{\mathcal{E}\oplus\mathcal{F}} = \begin{pmatrix} \nabla^{\mathcal{E}} & 0 \\ 0 & \nabla^{\mathcal{F}} \end{pmatrix}$. In general, $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is not an abelian category. For example, if $\Omega^k \mathcal{A} = 0$ for all $k \geq 1$, then $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is simply the category of fgp modules over $\mathcal{A}$, which is generally not an abelian category. In fact we can easily prove a necessary condition on a graded commutative dga $\Omega^\bullet \mathcal{A}$ if $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian.

Lemma 3.23. *Let $\Omega^\bullet \mathcal{A}$ be a connected graded commutative dga and suppose that $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian. Let $a \in \mathcal{A}$ and suppose that $da = a\omega$ for some $\omega \in \Omega^1\mathcal{A}$. Then a is either 0 or invertible.*

Proof. Consider the two objects $(\mathcal{A}, d + \omega)$ and $(\mathcal{A}, d)$ of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. Since $da = a\omega$ we have a commuting diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{a} & \mathcal{A} \\ \big\downarrow{\scriptstyle d+\omega} & & \big\downarrow{\scriptstyle d} \\ \Omega^1\mathcal{A} & \xrightarrow{a} & \Omega^1\mathcal{A} \end{array}$$

where a denotes the multiplication by a. So multiplication by a is a morphism between these objects. We then get a short exact sequence $0 \to \ker(a) \to \mathcal{A} \xrightarrow{a} \mathrm{im}(a) \to 0$. This is a short exact sequence of $\mathcal{A}$-modules, and since $\mathrm{im}(a)$ is an fgp $\mathcal{A}$-module, it is split, and we get $\mathcal{A} \cong \ker(a) \oplus \mathrm{im}(a)$. Since $\mathcal{A}$ is connected this means that either $\mathrm{im}(a) = 0$, which means that $a = 0$, or $\mathrm{im}(a) = \mathcal{A}$, which means that a is invertible. □

We will now define a slightly stronger condition on $\mathcal{A}$, and we will show later that this is a sufficient condition for the category $\mathcal{C}(\Omega^\bullet \mathcal{A})$ to be abelian.

Definition 3.24. Let $\Omega^\bullet \mathcal{A}$ be a $*$-dga. We say that $\Omega^\bullet \mathcal{A}$ satisfies property Q if it satisfies the following condition:

for all $a \in \mathcal{A}$ with $a \geq 0$ and all $a_1, \ldots, a_s \in \mathcal{A}$ with all $|a_i| \leq a$, and all $\omega_1, \ldots, \omega_s \in \Omega^1 \mathcal{A}$:

if $da = \sum_{i=1}^s a_i \omega_i$, then either $a = 0$ or a is invertible.

If $\Omega^\bullet \mathcal{A}$ is graded commutative, this easily implies the conclusion in lemma 3.23: if $da = a\omega$, we have $a^*a \geq 0$ and $d(a^*a) = d(a)^*a + a^*da = aa^*(\omega + \omega^*)$, so aa^* is 0 or invertible, hence a is 0 or invertible. It also implies that $\mathcal{A}$ is connected, in the sense that there are no non-trivial projections: if $p \in \mathcal{A}$ is a projection, then $dp = d(p^2) = 2pdp$, so $(1 - 2p)dp = 0$ and multiplying by $1 - 2p$ gives $dp = 0$. Then p should be 0 or invertible, so any projection is 0 or 1.

3.4.1 Quantum metric differential algebras

We will now show that property Q holds for quantum metric differential algebras. First we introduce the notion of a compact quantum metric space, invented by Rieffel [19]. Let A be a C^*-algebra and let L be a seminorm on A that takes finite values on a dense subalgebra $\mathcal{A}$. We think of L as a Lipschitz norm. This defines a metric on the state space $\mathcal{S}(A)$ by Connes' distance formula: for $\chi, \psi \in \mathcal{S}(A)$ we have

$$d_L(\chi, \psi) = \sup\{|\chi(a) - \psi(a)|, a \in \mathcal{A}, L(a) \leq 1\}.$$

This metric then defines a topology on the state space. We already had the weak-$*$ topology, so it is natural to make the following definition:

Definition 3.25. Let A be a unital C^*-algebra and let L be a seminorm on A taking finite values on a dense subalgebra. Then A is called a *compact quantum metric space* if the topology on $\mathcal{S}(A)$ induced by the metric d_L coincides with the weak-$*$ topology.

Now we go back to the case that we have a $*$-dga $\Omega^\bullet \mathcal{A}$ and $\mathcal{A}$ is a dense subset of a unital C^*-algebra A. Suppose that a norm $\|\cdot\|$ is given on $\Omega^1\mathcal{A}$, satisfying the inequality $\|a\omega\| \leq \|a\| \cdot \|\omega\|$ for $a \in \mathcal{A}, \omega \in \Omega^1\mathcal{A}$. This defines a seminorm L on $\mathcal{A}$ by $L(a) = \|da\|$. The space $\Omega^\bullet \mathcal{A}$ is called a *quantum metric dga* if A is a compact quantum metric space with this seminorm. If $\Omega^\bullet \mathcal{A}$ is a quantum metric dga, the same holds for $Z_g(\Omega^\bullet \mathcal{A})$, by proposition 2.3 of [19].

Lemma 3.26. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative quantum metric dga and suppose that $\mathcal{A} \cap A^\times = \mathcal{A}^\times$. Then $\Omega^\bullet \mathcal{A}$ satisfies property Q.*

Proof. Let $a \in \mathcal{A}$ with $a \geq 0$, and let $a_1, \ldots, a_s \in \mathcal{A}$ with all $|a_i| \leq a$, and $\omega_1, \ldots, \omega_s \in \Omega^1\mathcal{A}$ satisfying $da = \sum_{i=1}^{s} a_i\omega_i$. By scaling we may assume that $0 \leq a \leq 1$. Define the polynomial $p_n(x) = \sum_{k=1}^{n} \frac{1}{k}(1-x)^k$, which is the truncation of the power series of $-\log(x)$. Then we have

$$dp_n(a) = p_n'(a)da = \sum_{i=1}^{s} \alpha_i\omega_i \sum_{k=1}^{n}(1-a)^{k-1}.$$

For each $1 \leq i \leq s$ we have

$$\left|\alpha_i \sum_{k=1}^{n}(1-a)^{k-1}\right| \leq \left|a \sum_{k=1}^{n}(1-a)^{k-1}\right| = |1-(1-a)^n| \leq 1.$$

So we get

$$\left\|dp_n(a)\right\| \leq \sum_{i=1}^{s}\|\omega_i\|\,,$$

in particular the norm of $dp_n(a)$ is bounded as $n \to \infty$.

If a is neither 0 nor invertible in A, there are points χ, ψ in the Gelfand spectrum of A satisfying $\chi(a) = 0$ and $\psi(a) = t > 0$. Then $\chi(p_n(a)) = \sum_{i=1}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$, while $\psi(p_n(a)) = \sum_{i=1}^{n} \frac{1}{k}(1-t)^k \to -\log(t)$ as $n \to \infty$. We get

$$d(\chi, \psi) \geq \frac{|\chi(p_n(a)) - \psi(p_n(a))|}{\left\|dp_n(a)\right\|} \to \infty$$

so $d(\chi, \psi) = \infty$. But the metric d should give the weak-$*$ topology on the spectrum, and the spectrum is connected, so this is a contradiction. So either $a = 0$ or $a \in A^\times$, and in the second case $a \in \mathcal{A} \cap A^\times = \mathcal{A}^\times$. □

3.4.2 Proof of abelianness

In the rest of this section, we will show that if $\Omega^\bullet\mathcal{A}$ satisfies property Q, then $\mathcal{C}(\Omega^\bullet\mathcal{A})$ is an abelian category. Suppose we have a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ in the category $\mathcal{C}(\Omega^\bullet\mathcal{A})$. We have to show that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are also in the category. The most difficult part is to show that these are finitely generated projective modules.

Lemma 3.27. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism between fgp $\mathcal{A}$-modules. Then the following are equivalent:*

- *The $\mathcal{A}$-modules $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are fgp.*
- *There is an $\mathcal{A}$-module homomorphism $\varphi^+ : \mathcal{F} \to \mathcal{E}$ satisfying $\varphi\varphi^+\varphi = \varphi$.*

Proof. Suppose that $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$ are fgp. Then the short exact sequence $0 \to \ker(\varphi) \to \mathcal{E} \to \operatorname{im}(\varphi) \to 0$ is split, so $\mathcal{E} \cong \ker(\varphi) \oplus \operatorname{im}(\varphi)$. The short exact sequence $0 \to \operatorname{im}(\varphi) \to \mathcal{F} \to \operatorname{coker}(\varphi) \to 0$ is also split, so $\mathcal{F} \cong \operatorname{im}(\varphi) \oplus \operatorname{coker}(\varphi)$. The map φ then corresponds to the map $\ker(\varphi) \oplus \mathrm{im}(\varphi) \to \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi)$ sending (a, b) to $(b, 0)$. We can then choose the map $\varphi^+ : \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi) \to \ker(\varphi) \oplus \mathrm{im}(\varphi)$ sending (c, d) to $(0, c)$. It is then easy to check that $\varphi\varphi^+\varphi = \varphi$ (and also $\varphi^+\varphi\varphi^+ = \varphi^+$).

Now suppose there is an $\mathcal{A}$-linear map $\varphi^+ : \mathcal{F} \to \mathcal{E}$ satisfying $\varphi\varphi^+\varphi = \varphi$. The surjection $\mathcal{F} \to \mathrm{coker}(\varphi)$ admits a splitting, sending the equivalence class $[f]$ to $f - \varphi\varphi^+(f)$. This is well-defined because $\varphi\varphi^+\varphi = \varphi$. So the short exact sequence $0 \to \mathrm{im}(\varphi) \to \mathcal{F} \to \mathrm{coker}(\varphi) \to 0$ is split, giving $\mathcal{F} \cong \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi)$. So $\mathrm{im}(\varphi)$ and $\mathrm{coker}(\varphi)$ are fgp. Then the short exact sequence $0 \to \ker(\varphi) \to \mathcal{E} \to \mathrm{im}(\varphi) \to 0$ is also split, giving $\mathcal{E} \cong \ker(\varphi) \oplus \mathrm{im}(\varphi)$. So $\ker(\varphi)$ is also fgp. $\square$

Remark 3.28. If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ and $\varphi^+ : \mathbb{C}^n \to \mathbb{C}^n$ satisfy $\varphi\varphi^+\varphi = \varphi$ and $\varphi^+\varphi\varphi^+ = \varphi^+$, and $\varphi\varphi^+$ and $\varphi^+\varphi$ are self-adjoint then φ^+ is uniquely determined, and is called the *Moore-Penrose pseudoinverse* [2].

In the case that $\ker(\varphi), \mathrm{im}(\varphi), \mathrm{coker}(\varphi)$ are finitely generated projective it is easy to construct connections on these modules.

Lemma 3.29. *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{C}(\Omega^\bullet\mathcal{A})$ and suppose that* $\ker(\varphi)$, $\mathrm{im}(\varphi)$ *and* $\mathrm{coker}(\varphi)$ *are finitely generated projective. Then there are natural induced connections on these modules.*

Proof. Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi"] \arrow[d, "\nabla^{\mathcal{E}}"] & \mathcal{F} \arrow[r] \arrow[d, "\nabla^{\mathcal{F}}"] & \mathrm{coker}(\varphi) \\
\mathcal{E} \otimes \Omega^1\mathcal{A} \arrow[r, "\varphi\otimes\Omega^1\mathcal{A}"] & \mathcal{F} \otimes \Omega^1\mathcal{A} \arrow[r] & \mathrm{coker}(\varphi) \otimes \Omega^1\mathcal{A}.
\end{tikzcd}$$

We see that this induces a map $\nabla^{\mathrm{coker}(\varphi)} : \mathrm{coker}(\varphi) \to \mathrm{coker}(\varphi) \otimes \Omega^1\mathcal{A}$ and it is easy to check that it satisfies the Leibniz rules.

We have the isomorphisms $\mathcal{E} \cong \ker(\varphi) \oplus \mathrm{im}(\varphi)$ and $\mathcal{F} \cong \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi)$, as in the proof of lemma 3.27. Under these isomorphisms, φ corresponds to the map $\ker(\varphi) \oplus \mathrm{im}(\varphi) \to \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi)$ sending (a, b) to $(b, 0)$. We then get the commuting diagram

$$\begin{tikzcd}
\ker(\varphi) \oplus \mathrm{im}(\varphi) \arrow[r] \arrow[d, "\nabla^{\mathcal{E}}"] & \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi) \arrow[d, "\nabla^{\mathcal{F}}"] \\
\ker(\varphi) \otimes \Omega^1\mathcal{A} \oplus \mathrm{im}(\varphi) \otimes \Omega^1\mathcal{A} \arrow[r] & \mathrm{im}(\varphi) \otimes \Omega^1\mathcal{A} \oplus \mathrm{coker}(\varphi) \otimes \Omega^1\mathcal{A}
\end{tikzcd}$$

From the diagram it follows that $\nabla^{\mathcal{E}}$ restricts to $\ker(\varphi) \to \ker(\varphi) \otimes \Omega^1\mathcal{A}$, and $\nabla^{\mathcal{F}}$ restricts to $\mathrm{im}(\varphi) \to \mathrm{im}(\varphi) \otimes \Omega^1\mathcal{A}$. These are the connections we want. They satisfy the Leibniz rule because they are restrictions of $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. They are also independent of our choice of isomorphisms $\mathcal{E} \cong \ker(\varphi) \oplus \mathrm{im}(\varphi)$ and $\mathcal{F} \cong \mathrm{im}(\varphi) \oplus \mathrm{coker}(\varphi)$. $\square$

Definition 3.30. Let $M \in M_n(\mathcal{A})$ be a matrix with coefficients in $\mathcal{A}$. Let $\chi_M \in \mathcal{A}[x]$ be the characteristic polynomial, with coefficients $(-1)^m D_m(M)$, so

$$\chi_M(x) = x^n - D_{n-1}(M)x^{n-1} + D_{n-2}(M)x^{n-2} - \ldots + (-1)^n D_0(M).$$

Note that $D_0(M) = (-1)^n \det(M)$ and $D_{n-1} = \mathrm{Tr}(M)$. In general, $D_m(M)$ is the sum of the determinants of $m \times m$ square submatrices.

We need the inequality below involving D_m. Its proof is an easy calculation after diagonalising M^*M, and not very interesting. Its proof can be found in the appendix. The term $2\,\mathrm{Re}(M^*[M,K])$ will appear in the proof of theorem 3.33.

Lemma 3.31. *For $M, K \in M_n(\mathbb{C})$ we have*

$$\left| \frac{d}{dt}_{|t=0} D_m(M^*M + t \cdot 2\,\mathrm{Re}(M^*[M,K])) \right| \leq 4n \|K\|_{\mathrm{HS}}\, D_m(M^*M).$$

Here $2\,\mathrm{Re}(M^*[M,K]) = M^*[M,K] + (M^*[M,K])^*$ and $\|K\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of K.

Remark 3.32. Both sides of this inequality are continuous functions of the entries of M and K. The inequality then still holds for $M, K \in M_n(\mathcal{A})$, since it can be checked at any point in the spectrum (the Hilbert-Schmidt norm is then $\|K\|_{\mathrm{HS}} = \left\|\mathrm{Tr}(K^*K)\right\|^{\frac{1}{2}}$).

We are now ready to prove that $\mathcal{C}(\Omega^\bullet\mathcal{A})$ is an abelian category. We need to show that any $\varphi : \mathcal{E} \to \mathcal{F}$ has a finitely generated projective kernel, image and cokernel. In the first part of the proof we reduce to the case $\varphi : \mathcal{A}^n \to \mathcal{A}^n$. In the second part we prove that each term in the characteristic polynomial of $\varphi^*\varphi$ is either zero or invertible. Lastly we use this to prove that φ has a pseudo-inverse (as in lemma 3.27).

Theorem 3.33. *Let $\Omega^\bullet\mathcal{A}$ be a graded commutative dga satisfying property Q. Then the category $\mathcal{C}(\Omega^\bullet\mathcal{A})$ is abelian.*

Proof. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{C}(\Omega^\bullet\mathcal{A})$. We will show that $\ker(\varphi), \mathrm{im}(\varphi), \mathrm{coker}(\varphi)$ are finitely generated projective $\mathcal{A}$-modules, and then we are done by lemma 3.29.

There is a projective module $\mathcal{G}$ with $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G} \cong \mathcal{A}^n$. We can write $\mathcal{G} \cong p\mathcal{A}^n$ for a projection $p \in \mathrm{End}_{\mathcal{A}}(\mathcal{A}^n)$. Then we can define a connection $\nabla^{\mathcal{G}} : p\mathcal{A}^n \to p\Omega^1\mathcal{A}^n$ by $\nabla^{\mathcal{G}}(g) = pdg$. It is easy to check that this defines a connection on $\mathcal{G}$ (it is called the Grassmannian connection). This makes $(\mathcal{G}, \nabla^{\mathcal{G}})$ an object of $\mathcal{C}(\Omega^\bullet\mathcal{A})$ and it also defines a connection on the direct sum module $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$.

Now the map

$$\begin{pmatrix} 0 & 0 & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G} \to \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$$

is a morphism in $\mathcal{C}(\Omega^\bullet\mathcal{A})$. Its kernel is $\ker(\varphi) \oplus \mathcal{F} \oplus \mathcal{G}$, its image is $0 \oplus \mathrm{im}(\varphi) \oplus 0$ and its cokernel is $\mathcal{E} \oplus \mathrm{coker}(\varphi) \oplus \mathcal{G}$. So it is enough to show that these are finitely generated projective. Therefore it is enough to prove: for a connection $\nabla : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ and a morphism $\varphi : \mathcal{A}^n \to \mathcal{A}^n$, the kernel, image and cokernel of φ are fgp modules.

The connection $\nabla : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ can be written as $\nabla = d + \kappa$, where $\kappa : \mathcal{A}^n \to \Omega^1\mathcal{A}^n$ is an $\mathcal{A}$-linear function. We can view κ as an $n \times n$ matrix with coefficients in $\Omega^1\mathcal{A}$. The induced connection on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A}^n)$, which we still call ∇, satisfies

$$\nabla(\langle f, e\rangle) = \langle \nabla(f), e\rangle + \langle f, \nabla(e)\rangle$$

for $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A}^n)$ and $e \in \mathcal{A}^n$. So

$$d(\langle f, e\rangle) + \kappa\langle f, e\rangle = \langle \nabla(f), e\rangle + \langle f, de\rangle + \langle f, \kappa e\rangle$$

and this gives

$$\nabla(f) = df + [\kappa, f].$$

Since $\varphi : \mathcal{A}^n \to \mathcal{A}^n$ commutes with the connection, we know that $\nabla(\varphi) = 0$ so we conclude that

$$d\varphi = [\varphi, \kappa].$$

Here φ is viewed as an element of $M_n(\mathcal{A})$. We get

$$d(\varphi^*\varphi) = \varphi^* d(\varphi) + d(\varphi)^*\varphi = 2\,\mathrm{Re}(\varphi^*[\varphi, \kappa]).$$

Now let $a_m = D_m(\varphi^*\varphi) \in \mathcal{A}$ be the m-th term of the characteristic polynomial of $\varphi^*\varphi$ (up to sign). Write $\kappa = \sum_{i=1}^{s} K_i\omega_i$ with $K_i \in M_n(\mathcal{A})$ and $\omega_i \in \Omega^1\mathcal{A}$. We get

$$\begin{aligned}
da_m &= dD_m(\varphi^*\varphi) \\
&= \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + td(\varphi^*\varphi)) \\
&= \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t\cdot 2\,\mathrm{Re}(\varphi^*[\varphi, \kappa])) \\
&= \sum_{i=1}^{s} \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t\cdot 2\,\mathrm{Re}(\varphi^*[\varphi, K_i]))\omega_i \\
&= \sum_{i=1}^{s} a_i\omega_i
\end{aligned}$$

where

$$a_i = \frac{d}{dt}_{|t=0} D_m(\varphi^*\varphi + t\cdot 2\,\mathrm{Re}(\varphi^*[\varphi, K_i])).$$

By lemma 3.31 and remark 3.32 we get $|a_i| \le 4\|K_i\|_{\mathrm{HS}}\, a_m$. We can now apply property Q (if we put the factor $4\|K_i\|_{\mathrm{HS}}$ in the ω_i) to conclude that a_m is either 0 or invertible.

Now consider the smallest m for which $a_m \neq 0$ (note $a_n = 1$ so this m exists). Then a_m is invertible. The characteristic polynomial of $\varphi^*\varphi$ is now $\chi_{\varphi^*\varphi}(x) = x^n - a_{n-1}x^{n-1} + \ldots + (-1)^{n-m}a_m x^m$. Let $p(x) = x^{-m}\chi_{\varphi^*\varphi} = x^{n-m} - a_{n-1}x^{n-m-1} + \ldots + (-1)^{n-m}a_m$. By

Cayley-Hamilton we know $\chi_{\varphi^*\varphi}(\varphi^*\varphi) = 0$, so $\varphi^*\varphi p(\varphi^*\varphi)$ is nilpotent, and also self-adjoint, so $\varphi^*\varphi p(\varphi^*\varphi) = 0$. Then $(\varphi p(\varphi^*\varphi)) \cdot (\varphi p(\varphi^*\varphi))^* = 0$, so in fact already $\varphi p(\varphi^*\varphi) = 0$. Let $q(x) = \frac{1+(-1)^{n-m-1}a_m^{-1}p(x)}{x} \in \mathcal{A}[x]$. Let $\varphi^+ = q(\varphi^*\varphi)\varphi^*$. Then we have

$$\begin{aligned}
\varphi\varphi^+\varphi &= \varphi q(\varphi^*\varphi)\varphi^*\varphi \\
&= \varphi(q \cdot x)(\varphi^*\varphi) \\
&= \varphi(1 + (-1)^{n-m-1}a_m^{-1}p(\varphi^*\varphi)) \\
&= \varphi.
\end{aligned}$$

By lemma 3.27 it follows that the kernel, image and cokernel of φ are finitely generated projective. This concludes the proof of the theorem. □

Corollary 3.34. *With the same conditions as in the theorem, the category $\mathcal{C}_{flat}(\Omega^\bullet \mathcal{A})$ is also abelian.*

Proof. The category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is a full subcategory of $\mathcal{C}(\Omega^\bullet \mathcal{A})$. The kernel, image and cokernel of a morphism in $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is again in $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ because the connections constructed in lemma 3.29 are flat if $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ are flat. □

3.5 Definition of the fundamental group

In this section we will define the fundamental group of a dga. We will use theorem 2.6 for this. Since the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is equivalent to the category $\mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$, we will also attach the same fundamental group to $\Omega^\bullet \mathcal{A}$ as to $Z_g(\Omega^\bullet \mathcal{A})$. Therefore we assume in this subsection that $\Omega^\bullet \mathcal{A}$ is graded commutative. We have proven that $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is a rigid tensor category, and under some conditions on $\Omega^\bullet \mathcal{A}$, that it is abelian. What is left is showing that $\text{End}(\Omega^\bullet \mathcal{A}) = \mathbb{C}$ and constructing a fiber functor $\omega : \mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \to \text{Vec}_{\mathbb{C}}$. The first thing can be done easily.

Lemma 3.35. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga satisfying property Q. Then the algebra of endomorphisms is* $\text{End}(\Omega^\bullet \mathcal{A}) = \mathbb{C}$.

Proof. Let $\theta : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{A}$ be an isomorphism. Since θ is bilinear, for all $\alpha \in \Omega^\bullet \mathcal{A}$ we have $\theta(\alpha) = \alpha\theta(1)$. So θ is determined by $a = \theta(1)$. Since θ has to commute with the connection we get $da = d(\theta(1)) = \theta(d(1)) = 0$. Let λ be a complex number in the spectrum of a. Then we have $d(a - \lambda) = 0$, but $a - \lambda$ is not invertible. Since $\mathcal{A}$ satisfies property Q it follows that $a = \lambda \in \mathbb{C}$. □

For the fibre functor, pick a point p in the Gelfand spectrum $\widehat{A}$. Then our fibre functor is given by sending a bimodule $\mathcal{E}$ to the localization of its center at p. This is defined as $\mathcal{E} \otimes_{\mathcal{A}} \mathbb{C}$, where the $\mathcal{A}$-module structure on $\mathbb{C}$ is given by p. Note that this depends on a choice of a point in the Gelfand spectrum. This point plays a similar rôle as the base point of the usual fundamental group.

Lemma 3.36. *Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga that satisfies property Q and let $p \in \widehat{A}$. There is a faithful exact fibre functor $\omega : \mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$ sending $\Omega^\bullet \mathcal{E}$ to $\mathcal{E}_p$.*

Proof. Let $(\Omega^\bullet \mathcal{E}, \nabla^{\mathcal{E}})$ and $(\Omega^\bullet \mathcal{F}, \nabla^{\mathcal{F}})$ be objects of $\mathcal{C}(\Omega^\bullet \mathcal{A})$ and let $\varphi : \Omega^\bullet \mathcal{E} \to \Omega^\bullet \mathcal{F}$ be a morphism commuting with the connections. Since φ is $\mathcal{A}$-linear, this induces a map $\Omega^\bullet \mathcal{E}_p \to \Omega^\bullet \mathcal{F}_p$, showing that ω is functorial.

To show that ω is faithful, suppose that $\varphi_p = 0$. Since $\mathcal{C}(\Omega^\bullet \mathcal{A})$ is abelian we know that $\mathrm{im}(\varphi)$ is an fgp module. Now look at $\mathrm{im}(\varphi) \otimes_{\mathcal{A}} A$. This is a fgp module over the C*-algebra A, which corresponds to a vector bundle on $\widehat{A}$. It is zero at p, and the rank is locally constant, and $\widehat{A}$ is connected, so $\mathrm{im}(\varphi) \otimes_{\mathcal{A}} A = 0$. Since $\mathrm{im}(\varphi)$ is projective it is flat, and $\mathrm{im}(\varphi) \hookrightarrow \mathrm{im}(\varphi) \otimes_{\mathcal{A}} A$ is an injection, so also $\mathrm{im}(\varphi) = 0$. We conclude that $\varphi = 0$.

The fibre functor is exact because a localisation is always exact. □

Definition 3.37. Let $\Omega^\bullet \mathcal{A}$ be a graded commutative dga such that $\Omega^\bullet \mathcal{A}$ satisfies property Q. Let $p \in \widehat{A}$. Then we define $\pi^1(\Omega^\bullet \mathcal{A}, p)$ to be the group scheme of automorphisms of the fibre functor $\omega : \mathcal{C}(\Omega^\bullet \mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$ at p.

Definition 3.38. Let $\Omega^\bullet \mathcal{A}$ be a dga, suppose that $Z_g(\Omega^\bullet \mathcal{A})$ satisfies property Q and let $p \in \widehat{Z_g(\mathcal{A})}$. Then we define $\pi^1(\Omega^\bullet \mathcal{A}, p) = \pi^1(Z_g(\Omega^\bullet \mathcal{A}, p))$.

By theorem 2.6 we have an equivalence of categories $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \cong \mathrm{Rep}(\pi_1(\Omega^\bullet \mathcal{A}, p))$. In examples, we will simply recognise the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ as being equivalent to the category of representations of some (topological) group, which the fundamental group is then the algebraic hull of.

Remark 3.39. If $\pi^1(Z_g(\Omega^\bullet \mathcal{A}, p))$ is independent of the point p we will just write it as $\pi^1(Z_g(\Omega^\bullet \mathcal{A}))$. This may be expected if $\mathcal{A}$ is connected.

Example 3.40. Let $\Omega^k \mathcal{A} = M_2(\mathbb{C})$ for all k. Let d be given by taken the graded commutator with the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Explicitly, d is given in even degrees by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix}$ and in odd degrees by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 2a & 0 \\ 0 & -2d \end{pmatrix}$. This is the non-commutative space corresponding to the set of two points that are identified.

The graded centre of this dga is just $Z_g(\Omega^\bullet \mathcal{A}) = \mathbb{C}, 0, \mathbb{C}, 0, \ldots$ where $\mathbb{C}$ is embedded diagonally in $M_2(\mathbb{C})$. Then $\mathcal{C}_{\text{flat}}(Z_g(\Omega^\bullet \mathcal{A}))$ is just equivalent to the category of vector spaces. The fundamental group is trivial.

Example 3.41. Consider the following graded commutative dga: let $\mathcal{A} = \mathbb{C}$, $\Omega^1 \mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, $\Omega^2 \mathcal{A} = \mathbb{C}$ and $\Omega^n \mathcal{A} = 0$ for $n \geq 3$, with $d = 0$. The multiplication $\Omega^1 \mathcal{A} \otimes \Omega^1 \mathcal{A} \to \Omega^2 \mathcal{A}$ is given by $(a_1, a_2) \otimes (b_1, b_2) \to a_1 b_2 - a_2 b_1$.

An fgp $\mathcal{A}$-bimodule is now simply a finite-dimensional vector space V. A connection is a $\mathbb{C}$-linear map $\nabla = (\alpha, \beta) : V \to V \oplus V$. The map $\nabla_1 : V \oplus V \to V$ is then given by

$$\nabla_1(v, w) = \nabla_1(v \cdot (1, 0)) + \nabla_1(w \cdot (0, 1)) = \nabla(v) \cdot (1, 0) + \nabla(w) \cdot (0, 1) = -\beta(v) + \alpha(w).$$

So the curvature $\nabla^2 : V \to V$ is $\nabla^2 = [\alpha, \beta]$. The connection is flat if and only if α and β commute, so the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet\mathcal{A})$ is equivalent to the category of vector spaces with two commuting endomorphisms. This is in turn equivalent to the category of continuous representations of $\mathbb{R}^2$: the vector space V with the commuting endomorphisms α, β corresponds to the representation $\pi : \mathbb{R}^2 \to \mathrm{End}(V)$, $\pi(t_1, t_2) = \exp(t_1\alpha + t_2\beta)$. All representations of $\mathbb{R}$ are of this form by lemma 8.2 in the appendix. The fundamental group of $\Omega^\bullet\mathcal{A}$ is then (the algebraic hull of) $\mathbb{R}^2$.

For graded commutative spaces there is a good notion of functoriality for the fundamental group. For non-commutative spaces it is more difficult, we will revisit this in section 5.

Lemma 3.42. *Let $\Omega^\bullet\mathcal{A}$ and $\Omega^\bullet\mathcal{B}$ be graded commutative $*$-dga's that satisfy property Q. Let $\varphi : \Omega^\bullet\mathcal{A} \to \Omega^\bullet\mathcal{B}$ be a degree 0 algebra morphism satisfying $\varphi(d\alpha) = d(\varphi(\alpha))$ for all $\alpha \in \Omega^\bullet\mathcal{A}$. Let $q \in \widehat{B}$ and $p = \varphi^*(q) \in \widehat{A}$. Then φ induces a map $\varphi^* : \pi^1(\Omega^\bullet\mathcal{B}, q) \to \pi^1(\Omega^\bullet\mathcal{A}, p)$.*

Proof. If $\mathcal{E}$ is an fgp $\mathcal{A}$-module then $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$ is an fgp $\mathcal{B}$-module. A flat connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ gives a flat connection $\tilde{\nabla} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{B}$, given by $\tilde{\nabla}(e \otimes b) = \nabla(e)b + e \otimes db$. So we get a map $\mathcal{C}(\Omega^\bullet\mathcal{A}) \to \mathcal{C}(\Omega^\bullet\mathcal{B})$. It is easy to see that it is functorial and also that it commutes with the fibre functors. Then every automorphism of the fibre functor $\mathcal{C}(\Omega^\bullet\mathcal{B}) \to \mathrm{Vec}_{\mathbb{C}}$ can be pulled back to an automorphism of the fibre functor $\mathcal{C}(\Omega^\bullet\mathcal{A}) \to \mathrm{Vec}_{\mathbb{C}}$. So we get a map $\varphi^* : \pi^1(\Omega^\bullet\mathcal{B}, q) \to \pi^1(\Omega^\bullet\mathcal{A}, p)$. □

4 Non-commutative tori

In this section we will consider the non-commutative torus, also called the rotation algebra. Let $\mathcal{A}_\theta$ be the rotation algebra, as studied by Rieffel [17] and Connes [4], and described in [11], chapter 12. Here θ is a real number and $\mathcal{A}_\theta$ is generated by two unitaries u and v. The algebra consists of formal linear combinations of $u^m v^n$ where the coefficients go to zero faster than any polynomial. The multiplication is given by $uv = \lambda vu$ where $\lambda = e^{2\pi i\theta}$. The elements of $\Omega^1\mathcal{A}_\theta$ are of the form $adu + bdv$ with $a, b \in \mathcal{A}_\theta$ and they satisfy $udu = du \cdot u, udv = \lambda dv \cdot u, vdu = \overline{\lambda} du \cdot v, vdv = dv \cdot v$ and $du \cdot dv = -\lambda dv \cdot du$. The elements of $\Omega^2\mathcal{A}_\theta$ are of the form $adudv$ with $a \in \mathcal{A}_\theta$ (see [11], section 12.2). The algebra $\mathcal{A}_\theta$ has a natural $\mathbb{Z}^2$-grading where $u^m v^n$ has degree (m, n).

Note that an integer value of θ gives back the algebra $C^\infty(\mathbb{T}^2)$ of the usual torus. The non-commutative torus looks rather differently for θ irrational and θ rational. In both cases we will compute the graded centre of $\Omega^\bullet \mathcal{A}_\theta$ and from there the fundamental group.

Example 4.1. Assume that θ is irrational. For all m, n we have $u^m v^n \cdot u = \lambda^{-n} u \cdot u^m v^n$ and $u^m v^n \cdot v = \lambda^m v \cdot u^m v^n$. If $(m, n) \neq (0, 0)$ then λ^{-n} and λ^m cannot both be 0. So if an element $a \in \mathcal{A}_\theta$ has a non-zero term of degree (m, n) then either $[a, u]$ has a non-zero term in degree $(m + 1, n)$ or $[a, v]$ has a non-zero term in degree $(m, n + 1)$. In particular a cannot be in the centre of $\mathcal{A}_\theta$. We conclude that $Z_g(\mathcal{A}_\theta) = \mathbb{C}$ (as is also shown in [11], chapter 12). To compute $Z_g(\Omega^1\mathcal{A}_\theta)$ note that $u^m v^n du \cdot u = \lambda^{-n} u \cdot u^m v^n du$ and $u^m v^n du \cdot v = \lambda^{m+1} v \cdot u^m v^n du$. So if a term $u^m v^n du$ is in the graded centre exactly when $m = -1$ and $n = 0$. Indeed $u^{-1}du$ is in the graded centre. Similarly we see that $v^{-1}dv$ is in the graded centre, and $Z_g(\Omega^1\mathcal{A}_\theta) = u^{-1}du \cdot \mathbb{C} \oplus v^{-1}dv \cdot \mathbb{C}$. In the same way we compute that $Z_g(\Omega^2\mathcal{A}_\theta) = u^{-1}du \cdot v^{-1}dv \cdot \mathbb{C}$. We conclude that the dga $Z_g(\Omega^\bullet \mathcal{A})$ is isomorphic to the dga constructed in example 3.41. Hence, the fundamental group of $\Omega^\bullet \mathcal{A}_\theta$ is isomorphic to (the algebraic hull of) $\mathbb{R}^2$.

Apparently the flat connections on fgp $\Omega^\bullet \mathcal{A}_\theta$-bimodules correspond to continuous representations of $\mathbb{R}^2$. We can give the correspondence explicitly. Any continuous representation of $\mathbb{R}^2$ on a vector space W is given by $(t_1, t_2) \to \exp(t_1\alpha + t_2\beta)$ with $\alpha, \beta \in \mathrm{End}(W)$ commuting endomorphisms of the vector space. The corresponding module is $\Omega^\bullet \mathcal{E} = W \otimes \Omega^\bullet \mathcal{A}_\theta$, and the connection is given by

$$\begin{aligned}
&\nabla : W \otimes \Omega^\bullet \mathcal{A}_\theta \to W \otimes \Omega^\bullet \mathcal{A}_\theta \\
&\nabla(w \otimes a) = w \otimes da + \alpha(w) \otimes u^{-1}du \cdot a + \beta(w) \otimes v^{-1}dv \cdot a.
\end{aligned}$$

Note that all fgp $\Omega^\bullet \mathcal{A}_\theta$-bimodules are free by lemma 3.14, as $Z_g(\Omega^\bullet \mathcal{A}_\theta) = \mathbb{C}$.

Remark 4.2. This can be contrasted with [16], which proposes that the fundamental group of the irrational rotation algebra is (the algebraic hull of) $\mathbb{Z}$.

Example 4.3. Now assume that θ is rational. We write $\theta = \frac{p}{q}$ with p, q coprime integers. Since $\lambda^q = 1$ we see that both u^q and v^q are in the centre of $\mathcal{A}_\theta$. The graded centre is generated by the commuting unitaries u^q and v^q (see also [11], chapter 12). Its elements are power series in u^q and v^q where the coefficients go to zero faster than any polynomial. This is isomorphic to the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on the (commutative) 2-torus. We have $Z_g(\Omega^1\mathcal{A}_\theta) = Z_g(\mathcal{A}_\theta)\cdot u^{-1}du \oplus Z_g(\mathcal{A}_\theta)\cdot v^{-1}dv$. This is also generated by u^q and v^q and their derivations, as $u^{-q}du^q = q\cdot u^{-1}du$ and $v^{-q}dv^q = q\cdot v^{-1}dv$. So $Z_g(\Omega^1\mathcal{A}_\theta)\cong\Omega^1\mathbb{T}^2$. Similarly, $Z_g(\Omega^2\mathcal{A}_\theta)\cong\Omega^2\mathbb{T}^2$. We see that $Z_g(\Omega^\bullet\mathcal{A}_\theta)\cong\Omega^\bullet\mathbb{T}^2$. Hence the fundamental group of $\mathcal{A}_\theta$ is the same as that of the classical manifold $\mathbb{T}^2$, so it is (the algebraic hull of) $\mathbb{Z}^2$.

Remark 4.4. For any θ we have an inclusion map $Z_g(\Omega^\bullet\mathcal{A}_\theta)\hookrightarrow\Omega^\bullet\mathcal{A}_0$. This gives a map $\pi^1(\Omega^\bullet\mathcal{A}_0)\to\pi^1(Z_g(\Omega^\bullet\mathcal{A}_\theta)) = \pi^1(\Omega^\bullet\mathcal{A}_\theta)$. In the case that θ is irrational, this comes from the inclusion $\mathbb{Z}^2\to\mathbb{R}^2$. In the case that $\theta=\frac{p}{q}$ it comes from the multiplication $\mathbb{Z}^2\xrightarrow{\cdot q}\mathbb{Z}^2$. With this map we can distinguish the rational rotation algebras for different values of q.

For any θ the flat connections over $\Omega^\bullet\mathcal{A}_\theta$ correspond to the continuous representations of the group $(\mathbb{Z}+\theta\mathbb{Z})^2$: for irrational θ, this is a dense subgroup of $\mathbb{R}^2$ which has the same representations as $\mathbb{R}^2$ (see lemma 8.3 in the appendix), and for rational θ we have $\mathbb{Z}+\theta\mathbb{Z}\cong\mathbb{Z}$. The map $\pi^1(\Omega^\bullet\mathcal{A}_0)\to\pi^1(\Omega^\bullet\mathcal{A}_\theta)$ is then always given by the inclusion $\mathbb{Z}^2\hookrightarrow(\mathbb{Z}+\theta\mathbb{Z})^2$.

The irrational tori can be generalised to higher dimensions, as in [18].

Example 4.5. Let Θ be a skew-symmetric $n\times n$ matrix with coefficients in $\mathbb{R}$. We use the notation of [11], paragraph 12.2. The algebra $\mathcal{A}_\Theta$ is generated by unitaries $u_1,\dots,u_n$ satisfying $u_iu_j = e^{2\pi i\Theta_{ij}}u_ju_i$. We can also write this as $u_iu_j = \tau(e_i,e_j)^2u_ju_i$ where $\tau: (\mathbb{Z}^n)^2\to\mathbb{C}$ is the two-cocycle defined by $\tau(r,s)=\exp(\pi i r^t\Theta s)$ and $e_i\in\mathbb{Z}^n$ denotes the i-th unit vector. A general term in $\mathcal{A}_\Theta$ is a polynomial in the u_i with coefficients in the Schwarz space $\mathcal{S}(\mathbb{Z}^n)$, that is, the coefficients go to zero faster than any polynomial. This is a generalisation of the two-dimensional non-commutative torus, which we get back by taking $n=2$ and the matrix $\Theta=\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix}$. It is the non-commutative interpretation of the quotient of $\mathbb{R}^n$ by $\mathbb{Z}^n+\Theta\mathbb{Z}^n$, which is simply $\mathbb{R}^n/\mathbb{Z}^n$ if $\Theta=0$.

For any $r\in\mathbb{Z}$ we define the *Weyl element*

$$u^r = \exp\left(\pi i\sum_{j<k} r_j\Theta_{jk}r_k\right)u_1^{r_1}\cdots u_n^{r_n}.$$

These are linearly independent and generate the algebra $\mathcal{A}_\Theta$, and they satisfy

$$u^ru^s = \tau(r,s)u^{r+s}.$$

The one-form module $\Omega^1\mathcal{A}_\Theta$ is free with generators $\{u_i^{-1}du_i\}$. We use these generators because they are in the centre of $\Omega^1\mathcal{A}_\Theta$. The two-form module $\Omega^2\mathcal{A}_\Theta$ is free with generators $\{u_i^{-1}du_i\cdot u_j^{-1}du_j, i<j\}$.

Define the lattice $\Lambda=\{r\in\mathbb{Z}^n\mid\Theta r\in\mathbb{Z}^n\}=\{r\in\mathbb{Z}^n\mid\tau(r,s)=1\text{ for all }s\in\mathbb{Z}^n\}$ (which is reciprocal of the lattice used in [11], paragraph 12.2). Then the centre $Z_g(\mathcal{A})$ is generated by the u^r with $r\in\Lambda$. Let m be the rank of Λ and let $r_1,\dots,r_m$ be a basis, and let $r_{m+1},\ldots,r_n$ be elements of $\mathbb{Z}^n$ such that $r_1,r_2,\ldots,r_n$ are linearly independent. Then $Z_g(\mathcal{A})$ is generated by m independent unitaries u^{r_i}, so it is isomorphic to $C^\infty(\mathbb{T}^m)$, the algebra of smooth functions on the (commutative) m-torus. The one-form module $\Omega^1\mathbb{T}^m$ then corresponds to the submodule of $\Omega^1\mathcal{A}$ generated by the $u^{-r_i}du^{r_i}$, with $1 \le i \le m$. Let V be the $n-m$-dimensional vector space spanned by the $u^{-r_i}du^{r_i}$, with $m < i \le n$. We get

$$\begin{aligned}
Z_g(\Omega^1\mathcal{A}) &= Z_g(\mathcal{A})u_1^{-1}du_1 \oplus Z_g(\mathcal{A})u_2^{-1}du_2 \oplus \ldots \oplus Z_g(\mathcal{A})u_n^{-1}du_n \\
&= Z_g(\mathcal{A})u^{-r_1}du^{r_1} \oplus Z_g(\mathcal{A})u^{-r_2}du^{r_2} \oplus \ldots \oplus Z_g(\mathcal{A})u_{-r_n}du^{r_n} \\
&\cong \Omega^1\mathbb{T}^m \oplus Z_g(\mathcal{A})u^{-r_{m+1}}du^{r_{m+1}} \oplus Z_g(\mathcal{A})u^{-r_{m+2}}du^{r_{m+2}} \oplus \ldots \oplus Z_g(\mathcal{A})u^{-r_n}du^{r_n} \\
&\cong \Omega^1\mathbb{T}^m \oplus \Omega^0\mathbb{T}^m \otimes V
\end{aligned}$$

Similarly

$$\begin{aligned}
Z_g(\Omega^2\mathcal{A}) &= \bigoplus_{i<j} Z_g(\mathcal{A})u^{-r_i}du^{r_i}u^{-r_j}du^{r_j} \\
&= \bigoplus_{i<j\le m} Z_g(\mathcal{A})u^{-r_i}du^{r_i}u^{-r_j}du^{r_j} \\
&\oplus \bigoplus_{i\le m<j} Z_g(\mathcal{A})u^{-r_i}du^{r_i}u^{-r_j}du^{r_j} \\
&\oplus \bigoplus_{m<i\le j} Z_g(\mathcal{A})u^{-r_i}du^{r_i}u^{-r_j}du^{r_j} \\
&= \Omega^2\mathbb{T}^m \oplus \Omega^1\mathbb{T}^m \otimes V \oplus \Omega^0\mathbb{T}^m \otimes V^{\wedge 2},
\end{aligned}$$

where $V^{\wedge 2}$ denotes the wedge product of V with itself.

So $Z_g(\Omega^\bullet\mathcal{A})$ is the differential algebra

$$\Omega^0\mathbb{T}^m, \Omega^1\mathbb{T}^m \oplus \Omega^0\mathbb{T}^m \otimes V, \Omega^2\mathbb{T}^m \oplus \Omega^1\mathbb{T}^m \otimes V \oplus \Omega^0\mathbb{T}^m \otimes V^{\wedge 2}.$$

The zero-degree differential here is given by $da = (da, 0)$ and the first degree differential is given by $d(\omega, a\otimes v) = (d\omega, da\otimes v, 0)$. Let $\mathcal{E}$ be any fgp $\Omega^0\mathbb{T}^m$-module. A connection $\nabla : \mathcal{E} \to \mathcal{E}\otimes\Omega^1\mathbb{T}^m \oplus \mathcal{E}\otimes V$ can be written as $\nabla = \tilde{\nabla} \oplus L$ where $\tilde{\nabla} : \mathcal{E} \to \mathcal{E}\otimes\Omega^1\mathbb{T}^m$ is a connection and $L : \mathcal{E} \to \mathcal{E}\otimes V$ is a $\Omega^0\mathbb{T}^m$-linear map. The curvature of this connection is $\nabla^2 = \tilde{\nabla}^2 \oplus \tilde{\nabla}L \oplus L\tilde{\nabla} \oplus L^{\wedge 2}$. Here $\tilde{\nabla}$ also denotes the induced connection $\mathcal{E}\otimes V \to \mathcal{E}\otimes\Omega^1\mathbb{T}^m\otimes\mathcal{V}$ and L also denotes the induced linear map $\mathcal{E}\otimes\Omega^1\mathbb{T}^m \to \mathcal{E}\otimes V\otimes\Omega^1\mathbb{T}^m$. Choosing an $n-m$-dimensional basis for V we may write $L = (L_1,\ldots,L_{n-m})$ where the L_j are $\Omega^0\mathbb{T}^m$-linear maps $\mathcal{E}\to\mathcal{E}$. Then we see that the connection ∇ is flat if and only if $\tilde{\nabla}$ is flat and commutes with each L_j and the L_j commute with each other.

We can now show that the flat connections on $\mathcal{E}$ correspond to an action of $\mathbb{Z}^m\times\mathbb{R}^{n-m}$ on the fibre W of $\mathcal{E}$. Since the fundamental group of $\mathbb{T}^m$ equals $\mathbb{Z}^m$ we know that the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet\mathbb{T}^m)$ is equivalent to the category of finite-dimensional representations of $\mathbb{Z}^m$. Since $(\mathcal{E},\tilde{\nabla})$ is an object in this category it gives a representation $\tilde{\pi} : \mathbb{Z}^m \to \mathrm{GL}(W)$. The linear maps $L_j : \mathcal{E}\to\mathcal{E}$ commute with $\tilde{\nabla}$ so they are morphisms in $\mathcal{C}_{\text{flat}}(\Omega^\bullet\mathbb{T}^m)$, so they give morphisms $l_j : W \to W$ that intertwine $\tilde{\pi}$. Since the L_j commute with each other, the l_j also commute with each other. Now the representation $\pi : \mathbb{Z}^m \times \mathbb{R}^{n-m} \to \mathrm{GL}(V)$ is given by $\pi(g, t_1, t_2, \ldots, t_{n-m}) = \tilde{\pi}(g) \exp(t_1 l_1 + t_2 l_2 + \ldots + t_{n-m} l_{n-m})$.

This construction can also be reversed. So this shows that the fundamental group of $\Omega^\bullet \mathcal{A}_\Theta$ equals (the algebraic hull of) $\mathbb{Z}^m \times \mathbb{R}^{n-m}$. The subgroup $\mathbb{Z}^n + \Theta\mathbb{Z}^n \subseteq \mathbb{R}^n$ is dense in $\{u \in \mathbb{R}^n \mid r^t u \in \mathbb{Z} \text{ for all } r \in \Lambda\}$, and this is isomorphic to $\mathbb{Z}^m \times \mathbb{R}^{n-m}$. So for any Θ the fundamental group of $\Omega^\bullet \mathcal{A}_\Theta$ equals the algebraic hull of $\mathbb{Z}^n + \Theta\mathbb{Z}^n$. The inclusion map $Z_g(\Omega^\bullet \mathcal{A}_\Theta) \to \Omega^\bullet \mathcal{A}_0$ induces a map $\pi^1(\Omega^\bullet \mathcal{A}_0) \to \pi^1(\Omega^\bullet \mathcal{A}_\Theta)$, corresponding to the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^n + \Theta\mathbb{Z}^n$.

Remark 4.6. If G is a (discrete) group that acts freely and properly on $\mathbb{R}^n$, then $\mathbb{R}^n \to \mathbb{R}^n/G$ is the universal cover of $\mathbb{R}^n/G$ and it is a G-principal bundle. In the example above we show that the fundamental group corresponding to the non-commutative realisation of the quotient of $\mathbb{R}^n$ by $\mathbb{Z}^n + \Theta\mathbb{Z}^n$ can be identified with $\mathbb{Z}^n + \Theta\mathbb{Z}^n$, which is exactly as we would expect. Of course, since we get the fundamental group from its representations we cannot actually distinguish between $\mathbb{Z}^n + \Theta\mathbb{Z}^n$ and $\mathbb{Z}^m \times \mathbb{R}^{n-m}$ as fundamental group of A_Θ.

5 Functoriality: a counterexample

A smooth map $f : M \to N$ induces a group homomorphism $\pi_1 f : \pi_1 M \to \pi_1 N$. On algebras, this means that a morphism $\varphi : \Omega^\bullet \mathcal{A} \to \Omega^\bullet \mathcal{B}$ should induce a group homomorphism $\pi^1\varphi : \pi^1(\Omega^\bullet \mathcal{B}) \to \pi^1(\Omega^\bullet \mathcal{A})$. Here a morphism φ is a degree 0 algebra morphism that commutes with the derivative, so $d \circ \varphi = \varphi \circ d$. If $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ are graded commutative this is easy to show: if $\Omega^\bullet \mathcal{E}$ is a graded fpg $\Omega^\bullet \mathcal{A}$-bimodule, then we can tensor it with $\Omega^\bullet \mathcal{B}$ to get the graded fgp $\Omega^\bullet \mathcal{B}$-bimodule $\Omega^\bullet \mathcal{E} \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{B}$. It is easy to construct a flat connection on $\Omega^\bullet \mathcal{E} \otimes_{\Omega^\bullet \mathcal{A}} \Omega^\bullet \mathcal{B}$, given one on $\Omega^\bullet \mathcal{E}$. This gives a map $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A}) \to \mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{B})$, and this in turn determines the map $\pi^1\varphi : \pi^1(\Omega^\bullet \mathcal{B}) \to \pi^1(\Omega^\bullet \mathcal{A})$ (see lemma 3.42).

However, in the case that the dga's are not graded commutative it is not so easy. Tensoring on the right with $\Omega^\bullet \mathcal{B}$ only gives an $\Omega^\bullet \mathcal{A} - \Omega^\bullet \mathcal{B}$-bimodule, while tensoring with $\Omega^\bullet \mathcal{B}$ on both sides gives a bimodule that is clearly too large. In fact, we will now show that it is impossible to define a notion of functoriality forour notion of the fundamental group that extends the one we have for manifolds. We do this by constructing a non-commutative space $\mathcal{A}$ and morphisms $\Omega^\bullet(S^1) \xrightarrow{\varphi} \Omega^\bullet \mathcal{A} \xrightarrow{\psi} \Omega^\bullet(S^1)$, such that $\psi \circ \varphi$ is the identity on $\Omega^\bullet(S^1)$ but $\pi^1(\Omega^\bullet \mathcal{A})$ is trivial. Then we know that the construction is not functorial, since it is not possible that $\pi^1\varphi \circ \pi^1\psi$ is the identity on $\mathbb{Z}$ if $\pi^1(\Omega^\bullet \mathcal{A})$ is trivial.

First we give an informal explanation on the algebra we will construct. We will make an algebra of functions $\mathbb{R}^2 \to B(L^2(S^1))$, that send the circle S^1 to a commutative subspace of $B(L^2(S^1))$. Since $B(L^2(S^1))$ has a trivial centre, the centre of this algebra will be $C^\infty(\mathbb{R}^2)$, making the fundamental group trivial. The fact that the values of f on S^1 are in a commutative subspace does not change this because S^1 has empty interior. Because the values of f on S^1 are in a commutative subspace it is easy to construct a map $\mathcal{A} \to C^\infty(S^1)$, similar to how one would construct the map $C^\infty(\mathbb{R}^2) \to C^\infty(S^1)$ using the embedding $S^1 \hookrightarrow \mathbb{R}^2$. The map $C^\infty(S^1) \to \mathcal{A}$ is more difficult to construct: one starts on the circle by using multiplication operators. This map can then be extended to $\mathbb{R}^2$.

The non-commutative space is defined as follows: let

$$\mathcal{A} = \{f : \mathbb{R}^2 \to B(L^2(S^1)) \text{ smooth, bounded, } f(U) \subseteq C^\infty(S^1) \text{ for an open } S^1 \subseteq U \subseteq \mathbb{R}^2\}.$$

Here $L^2(S^1)$ is the Hilbert space of squarely integrable functions on S^1 and $B(L^2(S^1))$ is the C*-algebra of bounded linear operators $L^2(S^1) \to L^2(S^1)$. The functions f have to be smooth in the sense that for all $g \in L^2(S^1)$ and $a \in S^1$, the function $p \to f(p)(g)(a)$ is smooth. The set U is an open neighbourhood of S^1, and for $p \in U \subseteq \mathbb{R}^2$, the operator $f(p) \in B(L^2(S^1))$ has to be a multiplication operator with a smooth map in $C^\infty(S^1)$. For $p \in U$, write $f(p) = M_{h_f(p)}$, where M_h denotes the multiplication operator by h. Then $h_f : U \to C^\infty(S^1)$ is a smooth function. It is easy to see that $\mathcal{A}$ is a dense subalgebra of the C^*-algebra $\{f \in C_b(\mathbb{R}^2, B(L^2(S^1))) \mid f(S^1) \subseteq C(S^1) \subseteq B(L^2(S^1))\}$.

Note that $\mathcal{A}$ has a natural action of $C^\infty(\mathbb{R}^2)$ by multiplication. Now define $\Omega^1\mathcal{A} = \mathcal{A} \otimes_{C^\infty(\mathbb{R}^2)} \Omega^1\mathbb{R}^2$. This has a natural structure of $\mathcal{A}$-bimodule. Since $\Omega^1\mathbb{R}^2 \cong C^\infty(\mathbb{R}^2)dx \oplus C^\infty(\mathbb{R}^2)dy$ we get $\Omega^1\mathcal{A} \cong \mathcal{A}dx + \mathcal{A}dy$. For $f \in \mathcal{A}$ define $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, where $\frac{\partial f}{\partial x} \in \mathcal{A}$

is defined by

$$\frac{\partial f}{\partial x}(x,y)(g)(a) = \frac{\partial f(x,y)(g)(a)}{\partial x}$$

for $(x,y) \in \mathbb{R}^2$, $g \in L^2(S^1)$ and $a \in S^1$. For $(x,y) \in U$ we get $\frac{\partial f}{\partial x}(x,y) = M_{\frac{\partial h_f(x,y)}{\partial x}}$ which is indeed a multiplication operator. The definition for $\frac{\partial f}{\partial y}$ is similar. We can define $\Omega^2\mathcal{A} = \mathcal{A} \otimes_{C^\infty(\mathbb{R})} \Omega^2(\mathbb{R})$ and this works in a similar way.

The centre of $B(L^2(S^1))$ consists only of the scalar multiplications. Let $f \in Z_g(\mathcal{A})$. Suppose there is a $p \in \mathbb{R}^2 \setminus S^1$ such that $f(p)$ is not scalar. Then there is $f_2 \in \mathcal{A}$ such that $f_2(p)$ does not commute with $f(p)$, but then f does not commute with f_2, contradiction. So $f(p)$ is scalar for all $p \notin S^1$. Since f should be continuous and $\mathbb{R}^2 \setminus S^1$ is dense in $\mathbb{R}^2$ we get $Z_g(\mathcal{A}) = C^\infty(\mathbb{R}^2)$. We also get $Z_g(\Omega^1\mathcal{A}) = \Omega^1(\mathbb{R}^2)$ and $Z_g(\Omega^2\mathcal{A}) = \Omega^2(\mathbb{R}^2)$. Hence

$$\pi^1(\Omega^\bullet\mathcal{A}) = \pi^1(Z_g(\Omega^\bullet\mathcal{A})) = \pi_1(\Omega^\bullet\mathbb{R}^2) = \{0\}.$$

Now we will define the map $\varphi : C^\infty(S^1) \to \mathcal{A}$. Let $s_0 \in S^1$ be a base point and for $a \in S^1$, denote by r_a the rotation of S^1 that sends s_0 to a. Let $\mathrm{GL}(L^2(S^1))$ denote the linear automorphisms of the Hilbert space $L^2(S^1)$. This is a contractible manifold by Kuiper's theorem [14]. We have a smooth function $S^1 \to \mathrm{GL}(L^2(S^1))$ sending a to the automorphism $r_a^* : g \to g \circ r_a$. Let $s : \mathbb{R}^2 \to \mathrm{GL}(L^2(S^1))$ be a smooth extension of this map, such that $s(Ra) = s(a)$ for $R > 1-\varepsilon$ and $a \in S^1$ (for some fixed $0 < \varepsilon < 1$). Now define the map φ by, for $f \in C^\infty(S^1)$ and $p \in \mathbb{R}^2$:

$$\varphi(f)(p) = s_p M_f s_p^{-1} \in B(L^2(S^1)).$$

For $p \in S^1$ and $R > 1-\varepsilon$ we see that $\varphi(f)(Rp) = M_{f\circ r_p}$, since for $g \in B(L^2(S^1)), a \in S^1$ we have

$$\begin{aligned}\varphi(f)(p)(g)(a) &= (r_p^* M_f r_p^{*-1})(g)(a) = (M_f r_p^{*-1})(g)(r_p(a)) \\ &= f(r_p(a)) r_p^{*-1}(g)(r_p(a)) = f(r_p(a))g(a).\end{aligned}$$

So $\varphi(f)(p)$ is indeed a multiplication operator for p in a neighbourhood of the circle S^1, and $\varphi(f) \in \mathcal{A}$. We have $h_{\varphi(f)}(p) = f \circ r_p$ for $p \in S^1$. At each point $p \in \mathbb{R}^2$ the operator $\varphi(f)(p)$ is just a conjugation of the multiplication operator M_f, therefore φ is an algebra morphism $C^\infty(S^1) \to \mathcal{A}$. The operator $d\varphi : \Omega^1 S^1 \to \Omega^1\mathcal{A}$ can then be defined by the rule $d\varphi(f_1 df_2) = \varphi(f_1)d(\varphi(f_2))$.

Finally we define the map $\psi : \mathcal{A} \to C^\infty(S^1)$. It is given by

$$\psi(f)(p) = h_f(p)(s_0)$$

for $f \in \mathcal{A}, p \in S^1$. This is an algebra morphism. We can then define $d\psi : \Omega^1\mathcal{A} \to \Omega^1 S^1$ by the rule $d\psi(f_1 df_2) = \psi(f_1)d(\psi(f_2))$. Now for $f \in C^\infty(S^1)$ and $p \in S^1$ we have $h_{\varphi(f)}(p) = f \circ r_p$, so $\psi \circ \varphi(f)(p) = f(r_p(s_0)) = f(p)$. So $\psi \circ \varphi$ is the identity on $C^\infty(S^1)$. We also see that $d\psi \circ d\varphi$ is the identity on $\Omega^1 S^1$.

We conclude that π^1 cannot be functorial: if it were we would have $\pi^1(\psi \circ \varphi) : \mathbb{Z} \to \mathbb{Z}$ the identity, while $\pi^1(\varphi)$ is the trivial map $\{0\} \to \mathbb{Z}$ and $\pi^1(\psi)$ is the trivial map $\mathbb{Z} \to \{0\}$.

Of course the functoriality is a very important quality of the fundamental group, together with the fact that it is an invariant for homotopy equivalent functions. It is unfortunate that our definition is not functorial. Perhaps there is a better definition for the fundamental group, such that it is functorial. Alternativately, another definition of morphisms could be used. The above example may help in the search for better definitions of the fundamental group or of morphisms between non-commutative spaces.

5.1 Products

For manifolds (and topological spaces in general) it is known that the fundamental group of the product of two spaces is the product of the two fundamental groups: $\pi_1(M \times N) = \pi_1(M) \times \pi_1(N)$. It makes sense to expect that a similar property holds for the fundamental group of a non-commutative space. We have not been able to show this yet but we have some work in this direction that we include here.

First, let us define the product of two non-commutative spaces. Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be unital dga's. The product is then the graded tensor product $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ over $\mathbb{C}$. This has a natural grading: the elements of degree n are sums of $\alpha \otimes \beta$ with $\alpha \in \Omega^k \mathcal{A}, \beta \in \Omega^l \mathcal{B}, k + l = n$. For $\alpha, \alpha' \in \Omega^\bullet \mathcal{A}, \beta, \beta' \in \Omega^\bullet \mathcal{B}$, the multiplication is given by

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{|\beta| \cdot |\alpha'|} \alpha\alpha' \otimes \beta\beta'.$$

The derivative is given by

$$d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^{|\alpha|} \alpha \otimes d\beta.$$

From the multiplication the graded commutator is calculated as

$$[\alpha \otimes \beta, \alpha' \otimes \beta'] = (-1)^{|\beta| \cdot |\alpha'|} [\alpha, \alpha'] \otimes \beta\beta' + (-1)^{(|\alpha| + |\beta|) \cdot |\alpha'|} \alpha'\alpha \otimes [\beta, \beta'].$$

Recall that the graded centre of $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ is defined as all $\gamma \in \Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ such that $[\gamma, \gamma'] = 0$ for all $\gamma' \in \Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$. We can now prove that this is simply the tensor product of the graded centres of $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$.

Lemma 5.1. *Let $\Omega^\bullet \mathcal{A}$ and $\Omega^\bullet \mathcal{B}$ be unital dga's. Then we have $Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}) = Z_g(\Omega^\bullet \mathcal{A}) \otimes Z_g(\Omega^\bullet \mathcal{B})$.*

Proof. The inclusion $Z_g(\Omega^\bullet \mathcal{A}) \otimes Z_g(\Omega^\bullet \mathcal{B}) \subseteq Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$ is clear from the formula for the graded commutator above. Now let $\gamma \in Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$. Write $\gamma = \sum_{s=1}^m \alpha_s \otimes \beta_s$ such that all α_s, β_s are homogeneous with m minimal. Since γ is in the graded centre we get for any homogeneous $\alpha' \in \Omega^\bullet \mathcal{A}$ that $[\gamma, \alpha' \otimes 1] = 0$, so $\sum_{s=1}^m \pm[\alpha_s, \alpha'] \otimes \beta_s = 0$. Now suppose that $[\alpha_{s_0}, \alpha'] \neq 0$ for some s_0. Now let $S \subseteq \{1, \ldots, m\}$ be the subset of indices s where α_s has the same degree as α_{s_0} and β_s has the same degree as β_{s_0}. Then $\sum_{s \in S} [\alpha_s, \alpha'] \otimes \beta_s = 0$. Then the β_s cannot be linearly independent, so we can write $\beta_{s_0} = \sum_{s \in S \setminus \{s_0\}} \lambda_s \beta_s$. Writing

$\lambda_s = 0$ for $s \notin S$ we get $\gamma = \sum_{s\in\{1,\dots,m\}\setminus\{s_0\}}(\alpha_s + \lambda_s\alpha_{s_0}) \otimes \beta_s$, which is a contradiction with the minimality of m. So we get $[\alpha_s, \alpha'] = 0$ for all s, and hence $\alpha_s \in Z_g(\Omega^\bullet \mathcal{A})$. Similarly $\beta_s \in Z_g(\Omega^\bullet \mathcal{A})$ for all s, so $\gamma \in Z_g(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$. □

So to study the product of non-commutative spaces we can restrict ourselves to graded commutative dga's. Let $\Omega^\bullet \mathcal{A}, \Omega^\bullet \mathcal{B}$ be graded commutative dga's, and $p \in \widehat{\mathcal{A}}, q \in \widehat{\mathcal{B}}$ for which the fundamental group is defined. Then $p \otimes q \in \widehat{\mathcal{A} \otimes \mathcal{B}}$. Let $\Omega^\bullet \mathcal{E}$ be a graded fgp $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$-bimodule. The first three terms of the dga $\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}$ are

$$\mathcal{A} \otimes \mathcal{B}, \quad \Omega^1\mathcal{A} \otimes \mathcal{B} \oplus \mathcal{A} \otimes \Omega^1\mathcal{B}, \quad \Omega^2\mathcal{A} \otimes \mathcal{B} \oplus \Omega^1\mathcal{A} \otimes \Omega^1\mathcal{B} \oplus \mathcal{A} \otimes \Omega^2\mathcal{B}.$$

A connection on $\mathcal{E}$ is then given by $\nabla = \nabla_1 \oplus \nabla_2$ with $\nabla_1 : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ and $\nabla_2 : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{B}$. The Leibniz rules for ∇ give that ∇_1 is a connection on $\mathcal{E}$ as a $\Omega^\bullet \mathcal{A}$-bimodule, and ∇_2 is a connection on $\mathcal{E}$ as a $\Omega^\bullet \mathcal{B}$-bimodule. Moreover, ∇_1 is $\mathcal{B}$-linear and ∇_2 is $\mathcal{A}$-linear. The curvature of ∇ can be computed to be $\nabla_1^2 \oplus [\nabla_1, \nabla_2] \oplus \nabla_2^2$. So ∇ is flat if and only if ∇_1 and ∇_2 are flat and anti-commute. Since ∇_1 is $\mathcal{B}$-linear, it localises to a map $(\nabla_1)_q : \mathcal{E}_q \to \mathcal{E}_q \otimes \Omega^1\mathcal{A}$. Then $(\mathcal{E}_q, (\nabla_1)_q)$ is an object in the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$. It corresponds to a representation of $\pi^1(\Omega^\bullet \mathcal{A}, p)$ on the vector space $\mathcal{E}_{pq}$. Similarly, ∇_2 localises to a connection $(\nabla_2)_p : \mathcal{E}_p \to \mathcal{E}_p \otimes \Omega^1\mathcal{B}$, and this corresponds to a representation of $\pi^1(\Omega^\bullet \mathcal{B}, q)$.

Somehow the fact that ∇_1 and ∇_2 anti-commute should mean that the actions of $\pi^1(\Omega^\bullet \mathcal{A})$ and $\pi^1(\Omega^\bullet \mathcal{B})$ on $\mathcal{E}_{pq}$ should commute. We would then have an action of $\pi^1(\Omega^\bullet \mathcal{A}) \times \pi^1(\Omega^\bullet \mathcal{B})$ on $\mathcal{E}_{pq}$. If this correspondence can then also be reversed, it follows that the category of flat connections over the tensor product $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B})$ is equivalent to the category of representations of $\pi^1(\Omega^\bullet \mathcal{A}, p) \times \pi^1(\Omega^\bullet \mathcal{B}, q)$, and then it follows that

$$\pi^1(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}, p \otimes q) = \pi^1(\Omega^\bullet \mathcal{A}, p) \times \pi^1(\Omega^\bullet \mathcal{B}, q).$$

Example 5.2. In example 4.5, we examined the dga

$$\Omega^0\mathbb{T}^m, \Omega^1\mathbb{T}^m \oplus \Omega^0\mathbb{T}^m \otimes V, \Omega^2\mathbb{T}^m \oplus \Omega^1\mathbb{T}^m \otimes V \oplus \Omega^0\mathbb{T}^m \otimes V^{\wedge 2},$$

where V is an $n-m$-dimensional vector space. This is (up to degree 2) isomorphic to the tensor product of m copies of the dga $\Omega^\bullet \mathbb{T}^1$, and $n-m$ copies of the dga which just has $\mathbb{C}$ in degrees 0 and 1. The fundamental group is indeed the algebraic hull of $\mathbb{Z}^n \times \mathbb{R}^{n-m}$.

6 More general connections

So far we have only considered connections over modules that are finitely generated projective over the differential algebra $\Omega^\bullet \mathcal{A}$. This is quite a restriction, and all connections are reduced to a connection over the graded centre of the module. In this section we define a more general notion of connection, that works on all modules over $\mathcal{A}$. It is similar to the construction of [9], but we require that the connection $\mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ extends to $\mathcal{E} \otimes \Omega^n\mathcal{A} \to \mathcal{E} \otimes \Omega^{n+1}\mathcal{A}$ for all n, still satisfying the appropriate Leibniz rules. The main advantage of this is that the tensor product of two flat connections is always flat.

Let $\mathcal{E}$ be any $\mathcal{A}$-bimodule. We are going to consider connections on $\mathcal{E}$, that satisfy both left and right Leibniz rules. For this we need to make an identification $\mathcal{E} \otimes \Omega^1\mathcal{A} \xrightarrow{\sim} \Omega^1\mathcal{A} \otimes \mathcal{E}$. Since we want to extend the connection we also want isomorphisms $\mathcal{E} \otimes \Omega^n\mathcal{A} \xrightarrow{\sim} \Omega^n\mathcal{A} \otimes \mathcal{E}$ for all n. This leads to the following definition.

Definition 6.1. Let $\mathcal{E}$ be an $\mathcal{A}$-bimodule. An *extended twist* for $\mathcal{E}$ is a series of $\mathcal{A}$-bimodule isomorphisms

$$\sigma_n : \mathcal{E} \otimes \Omega^n\mathcal{A} \to \Omega^n\mathcal{A} \otimes E,$$

with $n \geq 0$, such that σ_0 is the identity and the following diagram commutes for all $n \geq 1$:

$$\begin{tikzcd}
\mathcal{E} \otimes \Omega^1\mathcal{A} \otimes \Omega^{n-1}\mathcal{A} \arrow[r, "\sigma_1 \otimes \Omega^{n-1}\mathcal{A}"] \arrow[d] & \Omega^1\mathcal{A} \otimes \mathcal{E} \otimes \Omega^{n-1}\mathcal{A} \arrow[r, "\Omega^1\mathcal{A} \otimes \sigma_{n-1}"] & \Omega^1\mathcal{A} \otimes \Omega^{n-1}\mathcal{A} \otimes \mathcal{E} \arrow[d] \\
\mathcal{E} \otimes \Omega^n\mathcal{A} \arrow[rr, "\sigma_n"] & & \Omega^n\mathcal{A} \otimes \mathcal{E}.
\end{tikzcd}$$

Now we can give the definition of a connection on the bimodule $\mathcal{E}$.

Definition 6.2. Let $\mathcal{E}$ be an $\mathcal{A}$-bimodule. An *extended bimodule connection* (ebc) on $\mathcal{E}$ consists of an extended twist σ for $\mathcal{E}$ and for each $n \geq 0$ two $\mathbb{C}$-linear functions

$$\nabla_{L,n} : \Omega^n\mathcal{A} \otimes \mathcal{E} \to \Omega^{n+1}\mathcal{A} \otimes \mathcal{E}$$

and

$$\nabla_{R,n} : \mathcal{E} \otimes \Omega^n\mathcal{A} \to \mathcal{E} \otimes \Omega^{n+1}\mathcal{A}$$

satisfying the Leibniz rules:

$$\nabla_{L,n}(\omega \cdot \nu \otimes e) = d\omega \cdot \nu \otimes e + (-1)^k \omega \cdot \nabla_{L,l}(\nu \otimes e)$$

and

$$\nabla_{R,n}(e \otimes \omega \cdot \nu) = \nabla_{R,k}(e \otimes \omega) \cdot \nu + (-1)^k e \otimes \omega \cdot d\nu$$

for $e \in \mathcal{E}$, and $\omega \in \Omega^k\mathcal{A}, \nu \in \Omega^l\mathcal{A}$ with $k+l = n$. Moreover they have to satisfy the relation

$$\sigma_{n+1} \circ \nabla_{R,n} = \nabla_{L,n} \circ \sigma_n$$

for $n \geq 0$.

Remark 6.3. Given any right connection $\nabla_{R,0} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ that satisfies the Leibniz rule $\nabla_{R,0}(ea) = \nabla_{R,0}(e)a + e \otimes da$, there can be at most one ebc that extends it: the formula $\nabla_{R,0}(ae) = a\nabla_{R,0}(e) + \sigma_1^{-1}(da \otimes e)$ determines σ_1, and with that also $\nabla_{L,0}$, and the higher connections must be given by $\nabla_{R,n}(e \otimes \omega) = \nabla_{R,0}(e)\omega + e \otimes d\omega$ and $\nabla_{L,n}(\omega \otimes e) = d\omega \otimes e + (-1)^k \omega \nabla_{L,0}(e)$ for $e \in \mathcal{E}$ and $\omega \in \Omega^k\mathcal{A}$. Conversely, not every right connection extends to an ebc, even if it satisfies the formula $\nabla_{R,0}(ae) = a\nabla_{R,0}(e) + \sigma_1^{-1}(da \otimes e)$ for some isomorphism $\sigma_1 : \mathcal{E} \otimes \Omega^1\mathcal{A} \to \Omega^1\mathcal{A} \otimes \mathcal{E}$, because it does not follow that $\sigma_{n+1} \circ \nabla_{R,n} = \nabla_{L,n} \circ \sigma_n$ in general (see example 6.4). We call the right connection $\nabla_{R,0}$ *extendable* if it extends to an ebc.

Remark 6.4. Suppose that $\Omega^\bullet \mathcal{A}$ is graded commutative and $\mathcal{E}$ is a symmetric $\mathcal{A}$-module. Then any right connection $\nabla_{R,0} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1\mathcal{A}$ that satisfies the right Leibniz rule, automatically satisfies the left Leibniz rule. It can be extended to an ebc where the twist $\sigma_n : \mathcal{E} \otimes \Omega^n\mathcal{A} \to \Omega^n\mathcal{A} \otimes \mathcal{E}$ just switches the two factors. In particular, when $\Omega^\bullet \mathcal{A}$ is the differential graded algebra of a manifold M, the ebc's on finitely generated projective $\mathcal{A}$-bimodules correspond to usual connections on vector bundles over M.

Definition 6.5. For any ebc we can consider the right curvature $\nabla_R^2 = \nabla_{R,1} \circ \nabla_{R,0}$ and the left curvature $\nabla_L^2 = \nabla_{L,1} \circ \nabla_{L,0}$. These are as usual linear functions and they are related by $\sigma_2 \circ \nabla_R^2 = \nabla_L^2$. We say that the ebc is *flat* when one of ∇_R^2, ∇_L^2 is zero (and therefore both are zero).

Example 6.6. Let $\mathcal{E}$ be the trivial bimodule $\mathcal{E} = \mathcal{A}$. Then we have identifications $\mathcal{E} \otimes \Omega^n\mathcal{A} = \Omega^n\mathcal{A} \otimes \mathcal{E} = \Omega^n\mathcal{A}$. We can then take the extended twist to be the identity map, and $\nabla_{R,n} = \nabla_{L,n} = d : \Omega^n\mathcal{A} \to \Omega^{n+1}\mathcal{A}$. This gives a flat ebc on $\mathcal{A}$ which we call the trivial ebc.

6.1 Tensor products

If $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{A}$-bimodules with an ebc, we can construct an ebc on the tensor product $\mathcal{E} \otimes \mathcal{F}$. The construction for the tensor product generalises the formula on page 56 in [9].

Proposition 6.7. *Let $\mathcal{E}, \mathcal{F}$ be $\mathcal{A}$-bimodules, with extended twists $\sigma^{\mathcal{E}}$ and $\sigma^{\mathcal{F}}$ and ebc's $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. We can then define an extended twist $\sigma = \sigma^{\mathcal{E}\otimes\mathcal{F}}$ and ebc $\nabla = \nabla^{\mathcal{E}\otimes\mathcal{F}}$ on $\mathcal{E} \otimes \mathcal{F}$ with the formulas*

$$\sigma_n = (\sigma_n^{\mathcal{E}} \otimes \mathcal{F}) \circ (\mathcal{E} \otimes \sigma_n^{\mathcal{F}}),$$

$$\nabla_{L,n}(\omega \otimes e \otimes f) = d\omega \otimes e \otimes f + (-1)^n \omega \cdot \nabla_{L,0}^{\mathcal{E}}(e) \otimes f + (-1)^n \omega \cdot \left(\sigma_1^{\mathcal{E}} \otimes \mathcal{F}\right)(e \otimes \nabla_{L,0}(f))$$

$$\nabla_{R,n}(e \otimes f \otimes \omega) = \left(\mathcal{E} \otimes (\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla_{R,0}^{\mathcal{E}}(e) \otimes f) \cdot \omega + e \otimes \nabla_{R,0}^{\mathcal{F}}(f) \cdot \omega + e \otimes f \otimes d\omega.$$

for $\omega \in \Omega^n\mathcal{A}$ and $e \in \mathcal{E}, f \in \mathcal{F}$.

If $\mathcal{F} = \mathcal{A}$ and ∇^F is the trivial ebc, and we identify $\mathcal{E} \otimes \mathcal{A} \cong \mathcal{A} \otimes \mathcal{E} \cong \mathcal{E}$ the ebc's $\nabla^{\mathcal{E}\otimes\mathcal{A}}$ and $\nabla^{\mathcal{A}\otimes\mathcal{E}}$ simply become $\nabla^{\mathcal{E}}$.

If $\mathcal{G}$ is another $\mathcal{A}$-bimodule with extended twist σ^G and ebc ∇^G, the induced connections on $(\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{G}$ and $\mathcal{E} \otimes (\mathcal{F} \otimes \mathcal{G})$ are the same.

Proof. To show that σ defines an extended twist, consider the diagram, constructed from the twists $\sigma^{\mathcal{E}}$ and $\sigma^{\mathcal{F}}$:

$$\begin{tikzcd}
\mathcal{E}\otimes\mathcal{F}\otimes\Omega^1\mathcal{A}\otimes\Omega^{n-1}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{E}\otimes\Omega^1\mathcal{A}\otimes\mathcal{F}\otimes\Omega^{n-1}\mathcal{A} \arrow[r] \arrow[d] & \Omega^1\mathcal{A}\otimes\mathcal{E}\otimes\mathcal{F}\otimes\Omega^{n-1}\mathcal{A} \arrow[d] \\
\mathcal{E}\otimes\mathcal{F}\otimes\Omega^n\mathcal{A} \arrow[d] & \mathcal{E}\otimes\Omega^1\mathcal{A}\otimes\Omega^{n-1}\mathcal{A}\otimes\mathcal{F} \arrow[r] \arrow[dl] & \Omega^1\mathcal{A}\otimes\mathcal{E}\otimes\Omega^{n-1}\mathcal{A}\otimes\mathcal{F} \arrow[d] \\
\mathcal{E}\otimes\Omega^n\mathcal{A}\otimes\mathcal{F} \arrow[r] & \Omega^n\mathcal{A}\otimes\mathcal{E}\otimes\mathcal{F} & \Omega^1\mathcal{A}\otimes\Omega^{n-1}\mathcal{A}\otimes\mathcal{E}\otimes\mathcal{F} \arrow[l]
\end{tikzcd}$$

The upper-right square clearly commutes, the left pentagon commutes because $\sigma^{\mathcal{F}}$ is an extended twist, and the lower pentagon commutes because $\sigma^{\mathcal{E}}$ is an extended twist. So the entire diagram commutes, and this shows that σ is an extended twist.

Consider the formula

$$\nabla_{R,0}(e\otimes f) = \left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(e)\otimes f) + e\otimes\nabla^{\mathcal{F}}_{R,0}(f).$$

To show that this is well-defined we have to show that it gives the same result when applied to $ea\otimes f$ or $e\otimes af$. This holds, because

$$\begin{aligned}
&\left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(ea)\otimes f) + ea\otimes\nabla^{\mathcal{F}}_{R,0}(f)\\
&=\left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(e)a\otimes f) + e\otimes(\sigma_1^{\mathcal{F}})^{-1}(da\otimes f) + e\otimes a\nabla^{\mathcal{F}}_{R,0}(f)\\
&=\left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(e)\otimes af) + e\otimes\nabla^{\mathcal{F}}_{R,0}(af).
\end{aligned}$$

In the last line we used that $\nabla^{\mathcal{F}}_{R,0} = (\sigma_1^{\mathcal{F}})^{-1}\circ\nabla^{\mathcal{F}}_{L,0}$ and that $\nabla^{\mathcal{F}}_{L,0}$ satisfies the left Leibniz rule.

Now we show that $\nabla^{\mathcal{F}}_{R,0}$ satisfies the right Leibniz rule:

$$\begin{aligned}
\nabla_{R,0}(e\otimes fa) &= \left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(e) + fa) + e\otimes\nabla^{\mathcal{F}}_{R,0}(fa)\\
&= \left(\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1}\right)(\nabla^{\mathcal{E}}_{R,0}(e)\otimes f)a + e\otimes\nabla^{\mathcal{F}}_{R,0}(f)a + e\otimes f\otimes da\\
&= \nabla_{R,0}(e\otimes f)a + e\otimes f\otimes da.
\end{aligned}$$

Since $\nabla_{R,n}$ is given by $\nabla_{R,n}(e\otimes f\otimes\omega) = \nabla_{R,0}(e\otimes f)\omega + e\otimes f\otimes d\omega$, it follows that $\nabla_{R,n}$ is also well-defined and satisfies the right Leibniz rules. Similarly, it follows that $\nabla_{L,n}$ is well-defined and satisfies the left Leibniz rules.

It remains to show that $\sigma_{n+1}\circ\nabla_{R,n} = \nabla_{L,n}\circ\sigma_n$. First note that for $e\in\mathcal{E}, f\in\mathcal{F}, \omega\in\Omega^n\mathcal{A}$ we have

$$\nabla_{R,n}(e\otimes f\otimes\omega) = (\mathcal{E}\otimes(\sigma_1^{\mathcal{F}})^{-1})(\nabla^{\mathcal{E}}_{R,0}(e)\otimes f)\cdot\omega + e\otimes\nabla^{\mathcal{F}}_{R,n}(f\otimes\omega).$$

Then we get

$$(\mathcal{E}\otimes\sigma^{\mathcal{F}}_{n+1})\circ\nabla_{R,n}(e\otimes f\otimes\omega)$$

$$
\begin{aligned}
&=(\mathcal{E} \otimes \sigma^{\mathcal{F}}_{n+1})((\mathcal{E} \otimes (\sigma^{\mathcal{F}}_1)^{-1})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes f) \cdot \omega) + e \otimes \sigma^{\mathcal{F}}_{n+1} \nabla^{\mathcal{F}}_{R,n}(f \otimes \omega) \\
&=\nabla^{\mathcal{E}}_{R,0}(e) \cdot \sigma^{\mathcal{F}}_n(f \otimes \omega) + e \otimes \nabla^{\mathcal{F}}_{L,n} \sigma^{\mathcal{F}}_n(f \otimes \omega).
\end{aligned}
$$

So

$$
\begin{aligned}
&(\mathcal{E} \otimes \sigma^{\mathcal{F}}_{n+1}) \circ \nabla_{R,n} \circ (\mathcal{E} \otimes (\sigma^{\mathcal{F}}_n)^{-1})(e \otimes \omega \otimes f) \\
=&\nabla^{\mathcal{E}}_{R,0}(e) \cdot \omega \otimes f + e \otimes \nabla^{\mathcal{F}}_{L,n}(\omega \otimes f) \\
=&\nabla^{\mathcal{E}}_{R,0}(e) \cdot \omega \otimes f + e \otimes d\omega \otimes f + (-1)^n e \otimes \omega \cdot \nabla^{\mathcal{F}}_{L,0}(f).
\end{aligned}
$$

Suppose that $\mathcal{F} = \mathcal{A}$ and $\nabla^{\mathcal{F}}$ is the trivial ebc. Then $\nabla_{R,0}(e \otimes a) = \nabla^{\mathcal{E}}_{R,0} \otimes a + e \otimes da$. This is equal to $\nabla^{\mathcal{E}}_{R,0}$ by the Leibniz rule. So we get $\nabla = \nabla^{\mathcal{E}}$. Similarly, when $\mathcal{E} = \mathcal{A}$ and $\nabla^{\mathcal{E}}$ is the trivial ebc, we get $\nabla_{L,0}(a \otimes e) = da \otimes e + \nabla^{\mathcal{F}}_{L,0}(f) = \nabla^{\mathcal{F}}_{L,0}(a \otimes e)$, so $\nabla = \nabla^{\mathcal{F}}$.

Finally we show the last assertion. It is clear that the extended twists on $\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{G}$ are the same. To show that the ebc's are the same, it is enough to consider $\nabla_{R,0}$ and this follows from the computation

$$
\begin{aligned}
&\nabla^{(\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{G}}_{R,0}(e \otimes f \otimes g) \\
=&(\mathcal{E} \otimes \mathcal{F} \otimes (\sigma^{\mathcal{G}}_1)^{-1})(\nabla^{\mathcal{E} \otimes \mathcal{F}}_{R,0}(e \otimes f) \otimes g) + e \otimes f \otimes \nabla^{\mathcal{G}}_{R,0}(g) \\
=&(\mathcal{E} \otimes \mathcal{F} \otimes (\sigma^{\mathcal{G}}_1)^{-1})((\mathcal{E} \otimes (\sigma^{\mathcal{F}}_1)^{-1})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes f) \otimes g) \\
&+ (\mathcal{E} \otimes \mathcal{F} \otimes (\sigma^{\mathcal{G}}_1)^{-1})(e \otimes \nabla^{\mathcal{F}}_{R,0}(f) \otimes g) + e \otimes f \otimes \nabla^{G}_{R,0}(g) \\
=&(\mathcal{E} \otimes (\sigma^{\mathcal{F} \otimes \mathcal{G}}_1)^{-1})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes f \otimes g) + e \otimes \nabla^{\mathcal{F} \otimes \mathcal{G}}_{R,0}(f \otimes g) \\
=&\nabla^{\mathcal{E} \otimes (\mathcal{F} \otimes \mathcal{G})}_{R,0}(e \otimes f \otimes g).
\end{aligned}
$$

□

We can also compute the curvature of this new connection.

Proposition 6.8. *With the notation of the previous proposition, we have*

$$
\nabla^2_R = (\mathcal{E} \otimes (\sigma^{\mathcal{F}}_2)^{-1}) \circ ((\nabla^{\mathcal{E}}_R)^2 \otimes \mathcal{F}) + \mathcal{E} \otimes (\nabla^{\mathcal{F}}_R)^2.
$$

In particular, the tensor product of two flat connections is again flat.

Proof. We have

$$
\begin{aligned}
&(\mathcal{E} \otimes \sigma^{\mathcal{F}}_2) \circ \nabla^2_R(e \otimes f) \\
=&(\mathcal{E} \otimes \sigma^{\mathcal{F}}_2) \nabla_{R,1}((\mathcal{E} \otimes (\sigma^{\mathcal{F}}_1)^{-1})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes f) + e \otimes \nabla^{\mathcal{F}}_{R,0}(f)) \\
=&((\sigma^{\mathcal{E}}_2)^{-1} \otimes \mathcal{F}) \nabla_{L,1}(\sigma^{\mathcal{E}}_1 \otimes \mathcal{F})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes f) + (\mathcal{E} \otimes \sigma^{\mathcal{F}}_2) \nabla_{R,1}(e \otimes \nabla^{\mathcal{F}}_{R,0}(f)) \\
=&((\sigma^{\mathcal{E}}_2)^{-1} \otimes \mathcal{F}) \nabla_{L,1}(\nabla^{\mathcal{E}}_{L,0}(e) \otimes f) + (\mathcal{E} \otimes \sigma^{\mathcal{F}}_2) \nabla_{R,1}(e \otimes \nabla^{\mathcal{F}}_{R,0}(f)) \\
=&((\sigma^{\mathcal{E}}_2)^{-1} \otimes \mathcal{F})((\nabla^{\mathcal{E}}_L)^2(e) \otimes f) - ((\sigma^{\mathcal{E}}_2)^{-1} \otimes \mathcal{F})(\Omega^1 \mathcal{A} \otimes \sigma^{\mathcal{E}}_1 \otimes \mathcal{F})(\nabla^{\mathcal{E}}_{L,0}(e) \otimes \nabla^{\mathcal{F}}_{L,0}(f)) \\
&+ (\mathcal{E} \otimes \sigma^{\mathcal{F}}_2)(\mathcal{E} \otimes (\sigma^{\mathcal{F}}_1)^{-1} \otimes \Omega^1 \mathcal{A})(\nabla^{\mathcal{E}}_{R,0}(e) \otimes \nabla^{\mathcal{F}}_{R,0}(f)) + (\mathcal{E} \otimes \sigma^{\mathcal{F}}_2)(e \otimes (\nabla^{\mathcal{F}}_R)^2(f))
\end{aligned}
$$

$$
\begin{aligned}
=&(\nabla^{\mathcal{E}}_R)^2(e)\otimes f-((\sigma^{\mathcal{E}}_1)^{-1}\otimes\Omega^1\mathcal{A}\otimes\mathcal{F})(\nabla^{\mathcal{E}}_{L,0}(e)\otimes\nabla^{\mathcal{F}}_{L,0}(f))\\
&+(\mathcal{E}\otimes\Omega^1\mathcal{A}\otimes\sigma^{\mathcal{F}}_1)(\nabla^{\mathcal{E}}_{R,0}(e)\otimes\nabla^{\mathcal{F}}_{R,0}(f))+e\otimes(\nabla^{\mathcal{F}}_L)^2(f)\\
=&(\nabla^{\mathcal{E}}_R)^2(e)\otimes f-\nabla^{\mathcal{E}}_{R,0}(e)\cdot\nabla^{\mathcal{F}}_{L,0}(f)\\
&+\nabla^{\mathcal{E}}_{R,0}(e)\cdot\nabla^{\mathcal{F}}_{L,0}(f)+e\otimes(\nabla^{\mathcal{F}}_L)^2(f)\\
=&(\nabla^{\mathcal{E}}_R)^2(e)\otimes f+e\otimes(\nabla^{\mathcal{F}}_L)^2(f).
\end{aligned}
$$

Note that the cross terms are both equal to $\nabla^{\mathcal{E}}_{R,0}(e)\cdot\nabla^{\mathcal{F}}_{L,0}(f)$, and they cancel. □

The tensor product is not commutative: in general, $\mathcal{E}\otimes\mathcal{F}$ is not isomorphic to $\mathcal{F}\otimes\mathcal{E}$. If we restrict to finitely generated projective $\mathcal{A}$-bimodules, then $\mathcal{E}\otimes\mathcal{F}\cong\mathcal{F}\otimes\mathcal{E}$, but the tensor product connections do not have to be the same (see example 6.4).

6.2 Duals

For any finitely generated projective $\mathcal{A}$-bimodule $\mathcal{E}$ consider the left dual $\mathcal{E}^\vee=\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ considering of the right-linear functions $\mathcal{E}\to\mathcal{A}$. Let $\langle,\rangle$ denote the pairing $\mathcal{E}^\vee\otimes\mathcal{E}\to\mathcal{A}$. The dual $\mathcal{E}^\vee$ is again an $\mathcal{A}$-bimodule: for $\theta\in\mathcal{E}^\vee, a\in\mathcal{A}, e\in\mathcal{E}$ we have $\langle a\theta,e\rangle=a\langle\theta,e\rangle$ and $\langle\theta a,e\rangle=\langle\theta,ae\rangle$. We also denote by $\langle,\rangle$ the pairing $(\Omega^k\mathcal{A}\otimes\mathcal{E}^\vee)\times(\mathcal{E}\otimes\Omega^l\mathcal{A})\to\Omega^{k+l}\mathcal{A}$ given by $\langle(\omega\otimes\theta,e\otimes\nu)=\omega\langle\theta,e\rangle\nu$.

If (σ,∇) is an ebc on $\mathcal{E}$ we can define the linear map $\sigma^\vee_n:\mathcal{E}^\vee\otimes\Omega^n\mathcal{A}\to\Omega^n\mathcal{A}\otimes\mathcal{E}^\vee$ by the formula

$$\langle\sigma^\vee_n(\theta\otimes\omega),e\rangle=\langle\theta,\sigma_n^{-1}(\omega\otimes e)\rangle.$$

If $\sigma^\vee_n$ is an isomorphism, it is easy to compute that it is an extended twist (but it might not be the case that $\sigma^\vee_n$ is an isomorphism). If it is, we can construct an ebc on the dual.

Lemma 6.9. *Let $\mathcal{E}$ be an fgp $\mathcal{A}$-bimodule with extended twist σ and ebc ∇. Suppose that $\sigma^\vee$ as defined above is an extended twist. Then there is an ebc $\nabla^\vee$ on $\mathcal{E}^\vee$ satisfying*

$$d\langle\alpha,\beta\rangle=\langle\nabla^\vee_{L,k}(\alpha),\beta\rangle+(-1)^k\langle\alpha,\nabla_{R,l}(\beta)\rangle$$

for $\alpha\in\Omega^k\mathcal{A}\otimes\mathcal{E}^\vee,\beta\in\mathcal{E}\otimes\Omega^l\mathcal{A}$.

Proof. Consider the formula

$$\langle\nabla^\vee_{L,0}(\theta),e\rangle=d\langle\theta,e\rangle-\langle\theta,\nabla_{R,0}(e)\rangle\in\Omega^1\mathcal{A}.$$

This defines $\nabla^\vee_{L,0}(\theta)\in\Omega^1\mathcal{A}\otimes\mathcal{E}^\vee$ because the right-hand side is right-linear in e: for $a\in\mathcal{A}$ we have

$$
\begin{aligned}
&d\langle\theta,ea\rangle-\langle\theta,\nabla_{R,0}(ea)\rangle\\
=&d(\langle\theta,e\rangle a)-\langle\theta,\nabla_{R,0}(e)a\rangle-\langle(\theta,eda\rangle)\\
=&(d\langle\theta,e\rangle a+\langle\theta,e\rangle da-\langle\theta,\nabla_{R,0}(e)\rangle a-\langle\theta,eda\rangle\\
=&(d\langle\theta,e\rangle-\langle\theta,\nabla_{R,0}(e)\rangle)a.
\end{aligned}
$$

Furthermore, the function $\nabla^\vee_{L,0}$ satisfies the left Leibniz rule because for $a \in \mathcal{A}, \theta \in \mathcal{E}^\vee, e \in \mathcal{E}$ we have

$$\begin{aligned}&\langle \nabla^\vee_{L,0}(a\theta), e\rangle\\ =&d\langle a\theta, e\rangle - \langle a\theta, \nabla_{R,0}(e)\rangle\\ =&da \cdot \langle \theta, e\rangle + ad\langle \theta, e\rangle - a\langle \theta, \nabla_{R,0}(e)\rangle\\ =&\langle da \otimes \theta, e\rangle + \langle a\nabla^\vee_{L,0}(\theta), e\rangle.\end{aligned}$$

Now we define $\nabla^\vee_{L,n}$ by $\nabla^\vee_{L,n}(\omega \otimes \theta) = d\omega \otimes \theta + (-1)^n \omega \nabla^\vee_{L,0}(\theta)$ for $\omega \in \Omega^n\mathcal{A}, \theta \in \mathcal{E}^\vee$. This automatically satisfies the left Leibniz rules. We check the formula in the lemma. For $\omega \in \Omega^k\mathcal{A}, \theta \in \mathcal{E}^\vee, e \in \mathcal{E}, \nu \in \Omega^l\mathcal{A}$ we have

$$\begin{aligned}\langle \nabla^\vee_{L,k}(\omega \otimes \theta), e \otimes \nu\rangle &= \langle d\omega \otimes \theta, e \otimes \nu\rangle + (-1)^k \omega \langle \nabla^\vee_{L,0}(\theta), e\rangle \nu\\ &= d\omega \cdot \langle \theta, e\rangle \nu + (-1)^k \omega d\langle \theta, e\rangle \cdot \nu - (-1)^k \omega \langle \theta, \nabla_{R,0}(e)\rangle \nu,\end{aligned}$$

and

$$\langle \omega \otimes \theta, \nabla_{R,l}(e \otimes \nu)\rangle = \omega\langle \theta, \nabla_{R,0}(e)\rangle \nu + \omega \langle \theta, e\rangle d\nu,$$

while

$$d\langle \omega \otimes \theta, e \otimes \nu\rangle = d(\omega\langle \theta, e\rangle \nu) = d\omega \cdot \langle \theta, e\rangle \nu + (-1)^k \omega d\langle \theta, e\rangle \cdot v + (-1)^k \omega \langle \theta, e\rangle d\nu.$$

So we see that indeed

$$d\langle \omega \otimes \theta, e \otimes \nu\rangle = \langle \nabla^\vee_{L,k}(\omega \otimes \theta), e \otimes \nu\rangle + (-1)^k \langle \omega \otimes \theta, \nabla_{R,l}(e \otimes \nu)\rangle.$$

Now define $\nabla^\vee_{R,0}$ by $\nabla^\vee_{R,0} = \sigma_1^{\vee -1} \circ \nabla^\vee_{L,0}$. This satisfies the right Leibniz rule, because for $\theta \in \mathcal{E}^\vee, a \in \mathcal{A}, e \in \mathcal{E}$ we have

$$\langle \sigma_1^\vee \nabla^\vee_{R,0}(\theta), e\rangle = \langle \nabla^\vee_{L,0}(\theta), e\rangle = d\langle \theta, e\rangle - \langle \theta, \nabla_{R,0}(e)\rangle$$

and then

$$\begin{aligned}&\langle \sigma_1^\vee \nabla^\vee_{R,0}(\theta a), e\rangle\\ =&d\langle \theta a, e\rangle - \langle \theta a, \nabla_{R,0}(e)\rangle\\ =&d\langle \theta, ae\rangle - \langle \theta, a\nabla_{R,0}(e)\rangle\\ =&d\langle \theta, ae\rangle - \langle \theta, \nabla_{R,0}(ae)\rangle + \langle \theta, \sigma_1^{-1}(da \otimes e)\rangle\\ =&\langle \sigma_1^\vee \nabla^\vee_{R,0}(\theta), ae\rangle + \langle \sigma_1^\vee(\theta \otimes da), e\rangle\\ =&\langle \sigma_1^\vee(\nabla^\vee_{R,0}(\theta)a + \theta \otimes da), e\rangle.\end{aligned}$$

Now define $\nabla^\vee_{R,n}$ by $\nabla^\vee_{R,n}(\theta \otimes \omega) = \nabla^\vee_{R,0}(\theta)\omega + \theta \otimes d\omega$ for $\theta \in \mathcal{E}^\vee, \omega \in \Omega^n\mathcal{A}$. This automatically satisfies the right Leibniz rules. It remains to check that $\sigma^\vee_{n+1} \circ \nabla^\vee_{R,n} = \nabla^\vee_{L,n} \circ \sigma^\vee_n$. For this, let $\theta \in \mathcal{E}^\vee, \omega \in \Omega^n\mathcal{A}, e \in \mathcal{E}$. We have

$$\langle \sigma^\vee_{n+1}(\nabla^\vee_{R,n}(\theta \otimes \omega)), e\rangle$$

$$
\begin{aligned}
=&\langle \sigma^\vee_{n+1}(\sigma_1^{\vee -1}\nabla^\vee_{L,0}(\theta)\otimes\omega) + \sigma^\vee_{n+1}(\theta\otimes d\omega), e\rangle\\
=&\langle (\Omega^1\mathcal{A}\otimes\sigma^\vee_n)(\nabla^\vee_{L,0}(\theta)\otimes\omega), e\rangle + \langle\theta, \sigma_{n+1}^{-1}(d\omega\otimes e)\rangle\\
=&\langle \nabla^\vee_{L,0}(\theta), \sigma_n^{-1}(\omega\otimes e)\rangle + \langle\theta, \sigma_{n+1}^{-1}(d\omega\otimes e)\rangle.
\end{aligned}
$$

Meanwhile, we have

$$
\begin{aligned}
&\langle \nabla^\vee_{L,n}(\sigma^\vee_n(\theta\otimes\omega)), e\rangle\\
=&d\langle \sigma^\vee_n(\theta\otimes\omega), e\rangle - (-1)^n\langle \sigma^\vee_n(\theta\otimes\omega), \nabla_{R,0}(e)\rangle\\
=&d\langle \theta, \sigma_n^{-1}(\omega\otimes e)\rangle - (-1)^n\langle \theta, (\sigma_n^{-1}\otimes\Omega^1\mathcal{A})(\omega\otimes\nabla_{R,0}(e))\rangle\\
=&\langle \nabla^\vee_{L,0}(\theta), \sigma_n^{-1}(\omega\otimes e)\rangle + \langle\theta, \nabla_{R,n}\sigma_n^{-1}(\omega\otimes e)\rangle - (-1)^n\langle\theta, (\sigma_n^{-1}\otimes\Omega^1\mathcal{A})(\omega\otimes\sigma_1^{-1}\nabla_{L,0}(e))\rangle\\
=&\langle \nabla^\vee_{L,0}(\theta), \sigma_n^{-1}(\omega\otimes e)\rangle + \langle\theta, \sigma_{n+1}^{-1}\nabla_{L,n}(\omega\otimes e)\rangle - (-1)^n\langle\theta, \sigma_{n+1}^{-1}(\omega\nabla_{L,0}(e))\rangle\\
=&\langle \nabla^\vee_{L,0}(\theta), \sigma_n^{-1}(\omega\otimes e)\rangle + \langle\theta, \sigma_{n+1}^{-1}(d\omega\otimes e)\rangle,
\end{aligned}
$$

So indeed $\sigma^\vee_{n+1}\circ\nabla^\vee_{R,n} = \nabla^\vee_{L,n}\circ\sigma^\vee_n$.

□

It is easy to check that $\langle\nabla^2_L(\alpha), \beta\rangle + \langle\alpha, \nabla^2_R(\beta)\rangle = 0$, in particular, the dual of a flat connection is again flat. Similar things can be done for the right dual ${}^\vee\mathcal{E} ={}_\mathcal{A}\operatorname{Hom}(\mathcal{E}, \mathcal{A})$.

6.3 Categories of flat connections

We have defined a more general notion of connection. We can now make different categories of flat connections, where we specify which modules go into the category.

- We can take the category of all $\mathcal{A}$-bimodules with a flat connection. This is a very large category. In the classical case it is much more than just the vector bundles, as vector bundles correspond to finitely generated projective modules. It has tensor products, but no duals.

- We can require that the bimodules in the category are finitely generated right projective, that is, for all bimodules $\mathcal{E}$ in the category there is a bimodule $\mathcal{F}$ such that $\mathcal{E}\oplus\mathcal{F}\cong\mathcal{A}^n$ for some integer n, as right modules. This still does not generalise the classical case, because if $\mathcal{A}$ is commutative, the bimodules do not have to be symmetric. However, if we also require that the bimodules are central, that is, $ae = ea$ for $a\in Z(\mathcal{A}), e\in\mathcal{E}$, it does generalise the classical case. This is similar to what is done in [10].

- Slightly more restrictive than the item above, we can ask that the modules be finitely generated right projective and left projective, but not necessarily by the same module.

- We can require that all modules are finitely generated projective, on the left and right simultaneously (in [10] these are called diagonal bimodules). This seems more natural than the examples above. We will call this category $\mathcal{C}'(\Omega^\bullet\mathcal{A})$. We will give examples below.

- Suppose that $Z(\mathcal{A})$ commutes with $\Omega^n\mathcal{A}$ for all n. Then for each fgp $\mathcal{A}$-bimodule $\mathcal{E}$ we have a natural isomorphism $\mathcal{E} \cong Z(\mathcal{E}) \otimes_{Z(\mathcal{A})} \mathcal{A}$ (confer lemma 3.14). We then have

  $$\mathcal{E} \otimes \Omega^n\mathcal{A} \cong Z(\mathcal{E}) \otimes_{Z(\mathcal{A})} \mathcal{A} \otimes \Omega^n\mathcal{A} \cong Z(\mathcal{E}) \otimes_{Z(\mathcal{A})} \Omega^n\mathcal{A} \cong \Omega^n\mathcal{A} \otimes_{Z(\mathcal{A})} Z(\mathcal{E}) \cong \Omega^n\mathcal{A} \otimes \mathcal{E}.$$

  Call this isomorphism σ_n. Then σ is an extended twist for $\mathcal{E}$, we call it the canonical twist. We can take the category with the fgp $\mathcal{A}$-bimodules and only the canonical twist. In fact, this category is isomorphic to the category $\mathcal{C}(\Omega^\bullet\mathcal{A})$ defined in section 3.

The category $\mathcal{C}'(\Omega^\bullet\mathcal{A})$ has a tensor product, that does not have to be commutative. It sometimes has left and right duals, that is, if $(\mathcal{E}, \nabla^{\mathcal{E}})$ is an object, we can at least sometimes construct duals for the bimodule of right-linear maps $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ as well as for the bimodule of left-linear maps ${}_{\mathcal{A}}\mathrm{Hom}(\mathcal{E}, \mathcal{A})$. The problem why this may not always work is because the dual of the extended twist might not be an isomorphism. We do not have a necessary condition for abelianness. However, there are theorems stating that under some extra conditions on the category $\mathcal{C}'_{\mathrm{flat}}(\Omega^\bullet\mathcal{A})$ it should be equivalent to the category of representations of a Hopf algebra (see [1]). We indeed find this in some examples below. Note that Hopf algebras (or more restrictively quantum groups) are themselves non-commutative generalisations of groups, so it could make sense to expect that the non-commutative generalisation of the fundamental group is actually a Hopf algebra.

6.4 Example: $M_2(\mathbb{C})$

Let $\Omega^k\mathcal{A} = M_2(\mathbb{C})$ for all k. Let d be given by taken the graded commutator with the matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, as in example 3.40. Let $\mathcal{E}$ be an fgp bimodule over $\mathcal{A}$, this is of the form $W \otimes_{\mathbb{C}} \mathcal{A}$ with W some vector space. Then a twist $\sigma_1 : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} \to \Omega^1\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}$ corresponds to an isomorphism $W \otimes_{\mathbb{C}} \mathcal{A} \to W \otimes_{\mathbb{C}} \mathcal{A}$, which is given by a linear isomorphism $\gamma : W \to W$. If we have an extended twist $\sigma_n : \mathcal{E} \otimes_{\mathcal{A}} \Omega^n\mathcal{A} \to \Omega^n\mathcal{A} \otimes_{\mathcal{A}} \mathcal{E}$, this is then given by $\gamma^n \otimes_{\mathbb{C}} \mathcal{A} : W \otimes_{\mathbb{C}} \mathcal{A} \to W \otimes_{\mathbb{C}} \mathcal{A}$.

Proposition 6.10. *Let ∇_L, ∇_R be an extended bimodule connection on $\mathcal{E}$ and also write ∇_L, ∇_R for the compositions with the isomorphism $\mathcal{E} \otimes \Omega^n\mathcal{A} \to \mathcal{E}$. Then any extended bimodule connection for the extended twist given by γ is of the form*

$$\nabla_{L,0}(w \otimes a) = w \otimes Da - \gamma(w) \otimes aD + \delta(w) \otimes a$$
$$\nabla_{L,1}(w \otimes a) = w \otimes Da + \gamma(w) \otimes aD - \delta(w) \otimes a,$$

for some linear function $\delta : W \to W$ with $\gamma\delta + \delta\gamma = 0$. This connection is flat precisely when $\gamma^2 = 1 + \delta^2$.

Proof. It is easy to check that the function $\nabla_{L,0}(w \otimes a) = w \otimes Da - \gamma(w) \otimes aD$ satisfies the left and right Leibniz rules $\nabla_{L,0}(w \otimes ab) = a\nabla_{L,0}(w \otimes b) + w \otimes [D, a]b$ and $\nabla_{L,0}(w \otimes ab) = \nabla_{L,0}(w \otimes a)b + \gamma(w) \otimes a[D, b]$. So for any connection we see that $\nabla_{L,0}(w \otimes a) - w \otimes Da - \gamma(w) \otimes aD$ must be a $\mathcal{A}$-bilinear function, so we can write

$$\nabla_{L,0}(w \otimes a) = w \otimes Da - \gamma(w) \otimes aD + \delta(w) \otimes a.$$

Then we get

$$\nabla_{L,1}(w \otimes a) = \nabla_{L,1}(w \otimes 1 \cdot a) = 2D \cdot (w \otimes a) - \nabla_{0,L}(w \otimes a) = w \otimes Da + \gamma(w) \otimes aD - \delta(w) \otimes a.$$

We also get

$$\nabla_{R,0}(w \otimes a) = \gamma^{-1}(w) \otimes Da - w \otimes aD + \gamma^{-1}\delta(w) \otimes a$$

and

$$\begin{aligned}\nabla_{R,1}(w \otimes a) &= \nabla_{R,1}(w \otimes a \cdot 1) \\ &= \nabla_{R,0}(w \otimes a) + (w \otimes a) \cdot 2D \\ &= \gamma^{-1}(w) \otimes Da + w \otimes aD + \gamma^{-1}\delta(w) \otimes a.\end{aligned}$$

So we see that $(\gamma^2 \otimes_{\mathbb{C}} \mathcal{A}) \circ \nabla_{R,1} = \nabla_{L,1} \circ (\gamma \otimes_{\mathbb{C}} \mathcal{A})$ only holds when $\gamma\delta + \delta\gamma = 0$. When this holds, the connection is actually extendable and $\nabla_{L,n}$ and $\nabla_{R,n}$ only depend on the parity of n.

The curvature of this connection can be directly computed:

$$\begin{aligned}\nabla_L^2(w \otimes a) &= \nabla_{L,1}(w \otimes Da - \gamma(w) \otimes aD + \delta(w) \otimes a) \\ &= w \otimes D^2a - \gamma(w) \otimes DaD + \delta(w) \otimes Da \\ &+ \gamma(w) \otimes DaD - \gamma^2(w) \otimes aD^2 + \gamma\delta(w) \otimes aD \\ &- \delta(w) \otimes Da + \delta\gamma(w) \otimes aD - \delta^2(w) \otimes a \\ &= (-1 + \gamma^2 - \delta^2)(w) \otimes a.\end{aligned}$$

So the curvature is flat precisely when $\gamma^2 = 1 + \delta^2$. $\square$

Corollary 6.11. *Let $X = \mathbb{C}[\gamma, \gamma^{-1}, \delta]/(\gamma\delta + \delta\gamma, \gamma^2 + \delta^2 - 1)$ be a Hopf algebra with comultiplication given by $\Delta\gamma = \gamma \otimes \gamma, \Delta\delta = \delta \otimes 1 + \gamma \otimes \delta$, antipode given by $S\gamma = \gamma^{-1}, S\delta = \delta\gamma^{-1}$ and counit given by $\varepsilon(\gamma) = 1, \varepsilon(\delta) = 0$. Then the category $\mathcal{C}'_{flat}(\Omega^\bullet M_2(\mathbb{C}))$ is equivalent to the category of representations of X, as rigid monoidal categories.*

Proof. By the proposition, the flat connections correspond to representations of the algebra $X = \mathbb{C}[\gamma, \gamma^{-1}, \delta]/(\gamma\delta + \delta\gamma, \gamma^2 + \delta^2 - 1)$.

The trivial ebc on $\mathcal{A}$ corresponds to $W = \mathbb{C}$ with $\gamma = 1$ and $\delta = 0$. This corresponds to the counit on X given by $\varepsilon(\gamma) = 1$ and $\varepsilon(\delta) = 0$.

If we have another bimodule $\mathcal{E}' = W' \otimes \mathcal{A}$ and a flat extended bimodule connection given by γ', δ', then we can compute the tensor product of these connections on $\mathcal{E} \otimes \mathcal{E}' \cong W \otimes_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{A}$. This is given by

$$\nabla_{L,0}(w \otimes w' \otimes a)$$

$$
\begin{aligned}
=&\nabla_{L,0}((w\otimes 1)\otimes(w'\otimes a))\\
=&\nabla^{\mathcal{E}}_{L,0}(w\otimes 1)\otimes(w'\otimes a)+(\sigma^{\mathcal{E}}_1\otimes\mathcal{E}')(w\otimes 1\otimes\nabla^{\mathcal{E}'}_{L,0}(w'\otimes a))\\
=&(w\otimes D-\gamma(w)\otimes D+\delta(w)\otimes 1)\otimes w'\otimes a\\
&+\gamma(w)\otimes 1\otimes(w'\otimes Da-\gamma'(w)\otimes aD+\delta(w')\otimes a)\\
=&w\otimes w'\otimes Da-\gamma(w)\otimes\gamma'(w)\otimes aD+(\delta(w)\otimes w'-\gamma(w)+\delta(w'))\otimes a.
\end{aligned}
$$

This corresponds to a coalgebra structure on X given by $\Delta(\gamma)=\gamma\otimes\gamma$ and $\Delta(\delta)=\delta\otimes 1+\gamma\otimes\delta$.

Consider the left dual $\mathcal{E}^\vee=\mathrm{Hom}_{\mathcal{A}}(W\otimes_{\mathbb{C}}\mathcal{A},\mathcal{A})\cong W^\vee\otimes_{\mathbb{C}}\mathcal{A}$ with $W^\vee$ the dual space to W. The last isomorphism is given by the pairing $\langle\theta\otimes a,w\otimes a'\rangle=\langle\theta,w\rangle\otimes aa'$. Any morphism $\alpha:W\to W$ induces a dual morphism $\alpha^\vee:W^\vee\to W^\vee$. It is easy to see that $\sigma^\vee_n:W^\vee\otimes_{\mathbb{C}}\mathcal{A}\otimes\Omega^n\mathcal{A}\cong W^\vee\otimes_{\mathbb{C}}\mathcal{A}\to\Omega^n\mathcal{A}\otimes(W^\vee\otimes_{\mathbb{C}}\mathcal{A})\cong W^\vee\otimes\mathcal{A}$ is given by $\sigma^\vee_n(\theta\otimes\omega)=(\gamma^\vee)^{-n}(\theta)\otimes\omega$. This is an isomorphism. Now the connection on $\mathcal{E}^\vee$ is given by

$$
\begin{aligned}
&\langle\nabla^\vee_{L,0}(\theta\otimes a),w\otimes 1\rangle\\
=&d\langle\theta\otimes a,w\otimes 1\rangle-\langle\theta\otimes a,\nabla_{R,0}(w\otimes 1)\rangle\\
=&\langle\theta,w\rangle da-\langle\theta\otimes a,(\gamma^{-1}(w)-w)\otimes D+\gamma^{-1}\delta(w)\otimes 1)\rangle\\
=&\langle\theta,w\rangle(Da-aD)+\langle\theta-(\gamma^\vee)^{-1}(\theta),w\rangle aD-\langle(\gamma^{-1}\delta)^\vee\theta,w\rangle a\\
=&\langle\theta,w\rangle Da-\langle(\gamma^\vee)^{-1}(\theta),w\rangle aD-\langle(\gamma^{-1}\delta)^\vee(\theta),w\rangle a\\
=&\langle\theta\otimes Da-(\gamma^{-1})^\vee(\theta)\otimes aD+(\delta\gamma^{-1})^\vee(\theta)\otimes a,w\otimes 1\rangle
\end{aligned}
$$

so

$$\nabla^\vee_{L,0}(\theta\otimes a)=\theta\otimes Da-(\gamma^{-1})^\vee(\theta)\otimes aD+(\delta\gamma^{-1})^\vee(\theta)\otimes a.$$

This corresponds to the antipode S on X given by $S\gamma=\gamma^{-1}$ and $S\delta=\delta\gamma^{-1}$. $\square$

An alternative definition for the fundamental group would use the category $\mathcal{C}'_{\text{flat}}(\Omega^\bullet\mathcal{A})$. It would then in general give a non-commutative, non-cocommutative Hopf algebra, instead of a group. Of course, for this alternative definition a good solution would need to be found for duals, and moreover, for the abelianness of the category. As another example, we show that $\mathcal{C}'(\Omega^\bullet\mathcal{A}_\theta)$ and $\mathcal{C}(\Omega^\bullet\mathcal{A}_\theta)$ are equal for the non-commutative tori $\mathcal{A}_\theta$. For simplicity, we only prove it for the irrational 2-torus.

Example 6.12. Let θ be irrational and let $\Omega^\bullet\mathcal{A}_\theta$ be the irrational 2-torus, as in example 4.1. Now we will show that any ebc on an fga A_θ bimodule automatically has a trivial twist, implying that $\mathcal{C}'(\Omega^\bullet\mathcal{A}_\theta)$ and $\mathcal{C}(\Omega^\bullet\mathcal{A}_\theta)$ are the same. Since the centre of $\mathcal{A}$ is just $\mathbb{C}$, each fgp $\mathcal{A}_\theta$-bimodule $\mathcal{E}$ is isomoprhic to $Z(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{A}_\theta$, so we only have to consider free modules. Let W be a vector space and consider the free module $\mathcal{E}=W\otimes\mathcal{A}_\theta$ and suppose we have an extended twist σ and ebc ∇. The centre $Z(\Omega^1\mathcal{A}_\theta)=\{\omega\in\Omega^1\mathcal{A}_\theta\mid a\omega=\omega a\text{ for all }a\in\mathcal{A}_\theta\}$ is the linear span of $u^{-1}du$ and $v^{-1}dv$. The centre of $\mathcal{E}\otimes\Omega^1\mathcal{A}_\theta$ is then $W\otimes(u^{-1}du\mathbb{C}\oplus v^{-1}dv\mathbb{C})$. So the twist σ_1 has to map $W\otimes(u^{-1}du\mathbb{C}\oplus v^{-1}dv\mathbb{C})$ this to itself, since it is an $\mathcal{A}_\theta$ bimodule homomorphism.

We can write

$$\nabla_{L,0}(w \otimes a) = w \otimes da + L(w \otimes a)$$

where $L : W \otimes \mathcal{A}_\theta \to W \otimes \Omega^1 \mathcal{A}_\theta$ is a left-linear function. We can then write $L(w \otimes a) = \alpha_{ij}(w) \otimes a u^i v^j du + \beta_{ij}(w) \otimes a u^i v^j dv$ with $\alpha_{ij}, \beta_{ij} \in \mathrm{End}(W)$. Here we sum over i and j. Now the formula $\nabla_{L,0}(ea) = \nabla_{L,0}(e)a + \sigma_1(e \otimes da)$ with $e = w \otimes 1$ gives

$$\begin{aligned}(1 - \sigma_1)(w \otimes da) &= L(w \otimes 1)a - L(w \otimes a) \\ &= \alpha_{ij}(w) \otimes [u^i v^j du, a] + \beta_{ij}(w) \otimes [u^i v^j dv, a].\end{aligned}$$

Now apply this for $a = u$. Then the left-hand side is an element of $W \otimes (du\mathbb{C} \oplus uv^{-1}dv\mathbb{C})$. In the right-hand side, the terms $[u^i v^j du, u]$ give a scalar multiple of $u^{i+1} v^j du$ and the terms $[u^i v^j dv, u]$ give a scalar multiple of $u^{i+1} v^j dv$. However, $[u^{-1}du, u] = [v^{-1}dv, u] = 0$, so we get no terms with du or $uv^{-1}dv$ on the right-hand side. So the left-hand side must be zero and we have $\sigma_1(w \otimes du) = w \otimes du$. Since we can do the same thing for $a = v$ we also have $\sigma_1(w \otimes dv) = w \otimes dv$, so σ_1 is the identity on $W \otimes \Omega^1 \mathcal{A}_\theta$. In other words, the twist is the canonical twist. This shows that $C'(\Omega^\bullet \mathcal{A}_\theta) \cong C(\Omega^\bullet \mathcal{A}_\theta)$.

7 Conclusion and outlook

We have defined a notion of connection on finitely generated projective bimodules over a differential graded algebra. We have defined the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ of these bimodules with flat connections, and shown that it is equal to the category of flat connections over the graded centre $Z_g(\Omega^\bullet \mathcal{A})$. We have constructed a tensor product in this category and shown that it admits dual objects. The category is also abelian for a large class of non-commutative spaces. This was used to define an affine algebraic group scheme, which we called the fundamental group of $\Omega^\bullet \mathcal{A}$. We computed the fundamental group in the following examples:

$\Omega^\bullet \mathcal{A}$	$\pi^1(\Omega^\bullet \mathcal{A})$ is algebraic hull of
${\Omega^\bullet}_{\text{deRham}} M$ for a manifold M	$\pi_1(M)$
irrational rotation algebra $\Omega^\bullet \mathcal{A}_\theta, \theta \in \mathbb{Q}$	$\mathbb{R}^2$, or $(\mathbb{Z} + \theta\mathbb{Z})^2$
rational rotation algebra $\Omega^\bullet \mathcal{A}_\theta, \theta \notin \mathbb{Q}$	$\mathbb{Z}^2 \cong (\mathbb{Z} + \theta\mathbb{Z})^2$
higher-dimensional non-commutative torus $\Omega^\bullet \mathcal{A}_\Theta$	$\mathbb{Z}^n + \Theta\mathbb{Z}^n$

We tried to make the construction of the fundamental group functorial, but so far this has not worked. In section 6 we also looked at a more general notion of connection, that retains more of the non-commutativity of the space.

Some open problems remain:

- Is the fundamental group $\pi^1(\Omega^\bullet \mathcal{A}, p)$ independent of the basepoint $p \in \widehat{\mathcal{A}}$ for a connected graded commutative dga $\Omega^\bullet \mathcal{A}$?
- Is there any way to recover some functoriality for π^1?
- Does the fundamental group respect products: is it true that $\pi^1(\Omega^\bullet \mathcal{A} \otimes \Omega^\bullet \mathcal{B}, p \otimes q) = \pi^1(\Omega^\bullet \mathcal{A}, p) \times \pi^1(\Omega^\bullet \mathcal{B}, q)$?
- When are there duals in the category $\mathcal{C}'(\Omega^\bullet \mathcal{A})$ described in section 6? When is it an abelian category?
- If a more direct definition of the fundamental group were found, it would be natural to ask if the category $\mathcal{C}_{\text{flat}}(\Omega^\bullet \mathcal{A})$ is still equivalent to the category of representations of this group.

8 Appendix

We have put here some proofs of lemmas that are a bit technical and would otherwise disrupt the flow of the argument. First we prove lemma 3.31.

Lemma 8.1. *For $M, K \in M_n(\mathbb{C})$ we have*

$$\left| \frac{d}{dt}_{|t=0} D_m(M^*M + t \cdot 2\operatorname{Re}(M^*[M,K])) \right| \leq 4n\|K\|_{\mathrm{HS}}\, D_m(M^*M).$$

Here $2\operatorname{Re}(M^*[M,K]) = M^*[M,K] + (M^*[M,K])^*$ and $\|K\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of K.

Proof. The inequality is invariant under a unitary change of basis of M, and M^*M is self-adjoint so we may choose a basis in which M^*M is diagonal, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Now

$$D_m(M^*M) = \sum_{|S|=m} \prod_{i \in S} \lambda_i,$$

where S runs over the m-element subsets of $\{1, 2, \ldots, n\}$. Let

$$M(t) = M^*M + t \cdot M^*[M,K].$$

The t-coefficient in the polynomial $P(t) = D_m(M(t)) \in \mathbb{C}[t]$ is $\frac{d}{dt}_{|t=0} D_m(M(t))$. The matrix $M(t)$ only has multiples of t outside the diagonal. The determinant of an $m \times m$ submatrix is then modulo t^2 equal to the product of the values on its diagonal. So

$$P_m(t) = \sum_{S=|m|} \prod_{i \in S} M(t)_{ii} \mod t^2.$$

We have

$$M(t)_{ii} = \lambda_i + t \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right).$$

The t-coefficient in $P_m(t)$ is then

$$\frac{d}{dt}_{|t=0} D_m(M(t)) = \sum_{|S|=m} \sum_{i \in S} \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right) \prod_{s \in S \setminus \{i\}} \lambda_s.$$

For all i, j, l we have $(M^*)_{ij} M_{li} \leq \frac{1}{2}(|M_{ji}|^2 + |M_{li}|^2) \leq \sum_{k=1}^{n} |M_{ki}|^2 = \lambda_i$. So we get

$$\left| \frac{d}{dt}_{|t=0} D_m(M(t)) \right| = \left| \sum_{|S|=m} \sum_{i \in S} \left(\lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right) \prod_{s \in S \setminus \{i\}} \lambda_s \right|$$

$$
\begin{aligned}
&\leq \sum_{|S|=m} \sum_{i\in S} \left| \lambda_i K_{ii} - \sum_{j,l=1}^{n} (M^*)_{ij} K_{jl} M_{li} \right| \prod_{s\in S\setminus\{i\}} \lambda_s \\
&\leq \sum_{|S|=m} \sum_{i\in S} \left(\lambda_i |K_{ii}| + \sum_{j,l=1}^{n} \lambda_i |K_{jl}| \right) \prod_{s\in S\setminus\{i\}} \lambda_s \\
&\leq \left(\sum_{i=1}^{n} |K_{ii}| + \sum_{j,l=1}^{n} |K_{jl}| \right) \sum_{|S|=m} \prod_{s\in S} \lambda_s \\
&\leq 2n \|K\|_{\mathrm{HS}}\, D_m(M^*M).
\end{aligned}
$$

Now we conclude

$$
\begin{aligned}
&\left| \frac{d}{dt}_{|t=0} D_m(M^*M + t\cdot 2\operatorname{Re}(M^*[M,K])) \right| = \\
&\qquad\qquad \left| 2\operatorname{Re}\left(\frac{d}{dt}_{|t=0} D_m(M(t)) \right) \right| \leq 4n\|K\|_{\mathrm{HS}}\, D_m(M^*M).
\end{aligned}
$$

□

Now we consider representations of $\mathbb{R}$ and of dense subgroups. Let V be a finite-dimensional vector space and $\alpha \in \operatorname{End}(V)$, and view $\mathbb{R}$ as additive topological group. There is a continuous representation $\pi : \mathbb{R} \to \mathrm{GL}(V)$ given by $\pi(t) = \exp(t\alpha)$. In fact every continuous representation has this form.

Lemma 8.2. *Let V be a finite-dimensional vector space and $\pi : \mathbb{R} \to \mathrm{GL}(V)$ a continuous representation. Then there is a unique $\alpha \in \operatorname{End}(V)$ such that $\pi(t) = \exp(t\alpha)$ for all $t \in \mathbb{R}$.*

Proof. Since π is continuous there is a $\delta > 0$ such that for all $t \in \mathbb{R}$ with $|t| < \delta$ we have $\|\pi(t) - 1\| < 1$. Let $0 < t < \delta$. The formal power series $\log(x+1)$ has radius of convergence 1, so $\alpha = \frac{1}{t}\log(\pi(t))$ is well-defined. Then $\pi(t) = \exp(t\alpha)$. Also $\alpha' = \frac{2}{t}\log(\pi(t/2))$ is well-defined. Since $\log(x^2) = 2\log(x)$ whenever $|x-1| < 1$ and $|x^2 - 1| < 1$, we get $\alpha' = \alpha$. So $\pi(t/2) = \exp(t/2\alpha)$. By induction we get $\pi(t2^{-n}) = \exp(t2^{-n}\alpha)$ for positive integers n. For integers m it also follows that $\pi(tm2^{-n}) = pi(t2^{-n})^m = \exp(tm2^{-n}\alpha)$. Since the set $\{tm2^{-n} \mid m,n \in \mathbb{Z}\}$ is already dense in $\mathbb{R}$ it follows that $\pi(x) = \exp(x\alpha)$ for all $x \in \mathbb{R}$. □

This result also holds for continuous representations of dense subgroups of $\mathbb{R}$. This is used in the computation of the fundamental group for non-commutative tori in section 4.

Lemma 8.3. *Let $G \subseteq \mathbb{R}$ be a dense subgroup of $\mathbb{R}$. Let V be a finite-dimensional vector space. Each representation $\pi : G \to \mathrm{GL}(V)$ extends to a representation of $\mathbb{R}$, and is therefore of the form $\pi(t) = \exp(t\alpha)$ for some $\alpha \in \operatorname{End}(V)$.*

Proof. Since the representation is continuous there is a $\delta > 0$ such that for $t \in [-\delta, \delta] \cap G$ we have $\|\pi(t) - 1\| \leq 1$, so $\|\pi(t)\| \leq 2$. For all positive integers n we have $\pi(nt) = \pi(t)^n$, and it follows that π is bounded on bounded intervals.

Now we show from this that π is uniformly continuous on bounded intervals. Let $\varepsilon > 0$. There is $\delta > 0$ such that for all $t \in G \cap (-\delta, \delta)$ we have $\|\pi(t) - 1\| < \varepsilon$. Let $M > 0$ and let $x, y \in G \cap [-M, M]$ with $|x - y| < \delta$. Then $\|\pi(x) - \pi(y)\| \leq \|\pi(x)\| \cdot \|1 - \pi(y - x)\| \leq \|\pi(x)\| \cdot \varepsilon$. Since π is bounded on $G \cap [-M, M]$ we see that π is uniformly continuous on bounded intervals. Then it can be extended uniquely to a function $\mathbb{R} \to \mathrm{End}(V)$, and it follows easily that this is still a representation. $\square$

9 References

[1] T.K. Bakke. *Hopf algebras and monoidal categories*. PhD thesis, University of Tromsø, 2007.

[2] S.L. Campbell. On continuity of the Moore-Penrose and Drazin generalized inverses. *Linear Algebra and its Applications*, 18(1):53–57, 1977.

[3] C.R. Canlubo. Non-commutative covering spaces, 2016. arXiv:1612.08673.

[4] A. Connes. C^*-algèbres et géométrie différentielle. *C. R. Acad. Sci. Paris. Sér. A*, 290:599–604, 1980.

[5] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, 1994.

[6] A. Connes and M. Marcolli. A walk in the noncommutative garden, 2006. arXiv:math/0601054.

[7] J. Cuntz. Quantum spaces and their noncommutative topology. *Notices of the American Mathematical Society*, 48:793–799, January 2001.

[8] P. Deligne and J.S. Milne. *Hodge Cycles, Motives, and Shimura Varieties*, chapter Tannakian Categories. Springer, Berlin, Heidelberg, 1982.

[9] M. Dubois-Violette. Lectures on graded differential algebras and noncommutative geometry. *Math.Phys.Stud.*, 23:245–306, 2001.

[10] M. Dubois-Violette and W. Michor. Dérivations et calcul différentiel non commutatif ii. *C. R. Acad. Sci. Paris Sér. I Math.*, 319:927–931, 1994.

[11] J. M. Gracia-Bondia, J. C. Varilly, and H. Figueroa. *Elements of noncommutative geometry*. Birkhaüser, 2001.

[12] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[13] P.R. Ivankov. Universal covering space of the noncommutative torus. Unpublished, October 2018.

[14] N. Kuiper. The homotopy type of the unitary group of Hilbert space. *Topology*, 3:19–30, 1965.

[15] G. Landi. *An introduction to noncommutative spaces and their geometries*. Springer, 1997.

[16] S. Mahanta and W.D. van Suijlekom. Noncommutative Tori and the Riemann-Hilbert correspondence. *J. Noncomm. Geom.*, pages 261–287, 2009.

[17] M.A. Rieffel. C*-algebras associated with irrational rotations. *Pacific J. Math.*, 93:415–429, 1981.

[18] M.A. Rieffel. Projective modules over higher-dimensional non-commutative tori. *Can. J. Math.*, XL:257–338, 1988.

[19] M.A. Rieffel. Compact Quantum Metric Spaces. *Contemp. Math.*, 365:315–330, 2004.

[20] N. Saavedra Rivano. Catégories tannakiennes. *Lecture Notes in Mathematics*, 265, 1972.

[21] W.D. van Suijlekom, M. Khalkhali, and G. Landi. Holomorphic structures on the quantum projective line. *Int. Math. Res. Not. IMRN*, 2010.

[22] W. C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.