

Spectral truncations in noncommutative geometry and operator systems

Walter van Suijlekom
(joint with Alain Connes)

A spectral approach to geometry



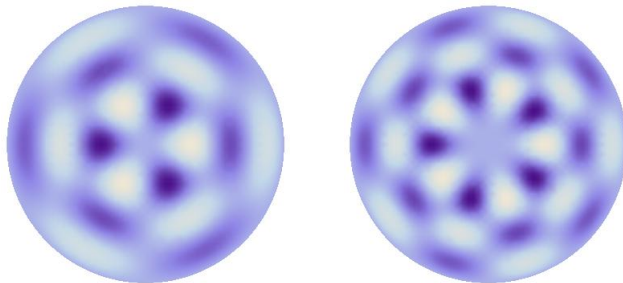
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M , does the **spectrum of wave numbers** k in the **Helmholtz equation**

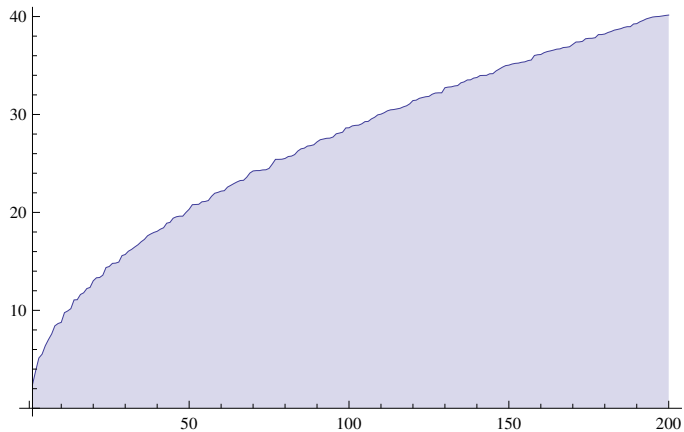
$$\Delta_M u = k^2 u$$

determine the **geometry of M** ?

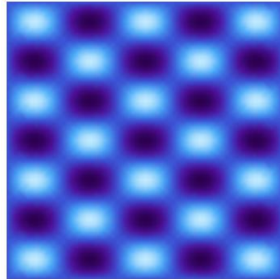
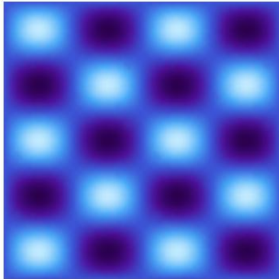
The disc



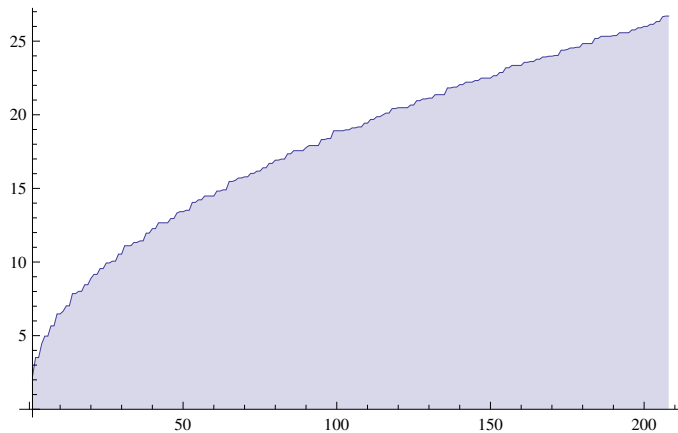
Wave numbers on the disc



The square

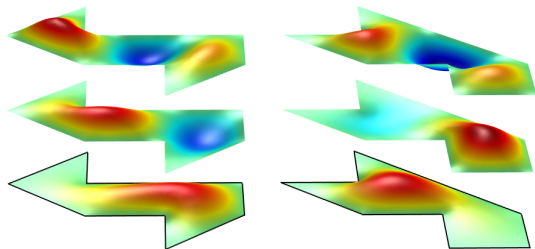


Wave numbers on the square



Isospectral domains

But, there are *isospectral domains* in \mathbb{R}^2 :



(Gordon, Webb, Wolpert, 1992)

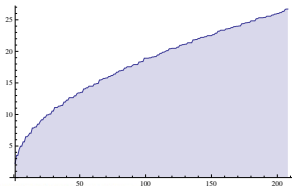
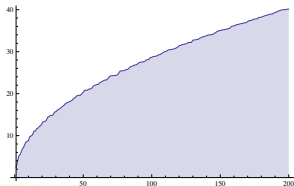
so the answer to Kac's question is **no**

Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension d of M :

$$N(\Lambda) = \#\text{wave numbers} \leq \Lambda \\ \sim \frac{\Omega_d \text{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):



Noncommutative geometry



If combined with the C^ -algebra $C(M)$, then the answer to Kac' question is affirmative.*

Connes' reconstruction theorem [2008]:

$$(C(M), D_M) \longleftrightarrow (M, g)$$

Spectral data

- This mathematical reformulation of geometry in terms of spectral data requires the knowledge of **all eigenvalues** of the Dirac operator.
- From a physical standpoint this is **not very realistic**: detectors have limited energy ranges and resolution.

We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum.

This is in line with earlier work of [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019], [Berendschot 2019] and **based on [arXiv:2004.14115]**

The “usual” story

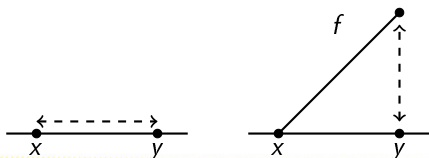
Given cpt Riemannian spin manifold (M, g) with spinor bundle S on M .

- the C^* -algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(M, S)$

\rightsquigarrow spectral triple: $(C(M), L^2(M, S), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples

More generally, we consider a triple (A, \mathcal{H}, D)

- a C^* -algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space \mathcal{H}

Generalized distance function:

- States are positive linear functionals $\phi : A \rightarrow \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A :

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral truncations

Given (A, \mathcal{H}, D) we project onto part of the spectrum of D :

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed **Hilbert subspace**
- $D \mapsto PDP$, still a **self-adjoint operator**
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, **PAP is an operator system**: $(PaP)^* = Pa^*P$.

So, we turn to study $(PAP, P\mathcal{H}, PDP)$.

We expect:

- a **distance formula** on states of PAP .
- a **rich symmetry**: isometries of (A, \mathcal{H}, D) remain isometries of $(PAP, P\mathcal{H}, PDP)$

Operator systems

Definition (Choi-Effros 1977)

An **operator system** is a $*$ -closed vector space E of bounded operators.

For convenience we take E to be finite-dimensional, to contain the identity operator, and act on a fixed Hilbert space \mathcal{H} .

- E is **ordered**: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \quad (\psi \in \mathcal{H}).$$

- in fact, E is **completely ordered**: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on \mathcal{H}^n for any n .

Maps between operator systems E, F are **completely positive maps** in the sense that their extensions $M_n(E) \rightarrow M_n(F)$ are positive for all n .

Isomorphisms are **complete order isomorphisms**

States spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as **positive linear functionals of norm 1**.
- Also, the **dual** E^d of an operator system is an operator system, with

$$E_+^d = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

and similarly for the complete order.

- We have $(E^d)_+^d \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:
pure states on E \longleftrightarrow extreme rays in $(E^d)_+$
and the other way around.

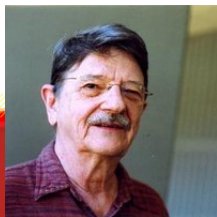
C^* -envelope of an operator system

Arveson introduced the notion of C^* -envelope for operator systems in 1969, Hamana established existence and uniqueness in 1979.

A C^* -extension $\kappa : E \rightarrow A$ of an operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^* -envelope of an operator systems is a C^* -extension $\kappa : E \rightarrow A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \cong \\ & & B \end{array} \quad \exists! \rho$$



Shilov boundaries

There is a useful description of C^* -envelopes using Shilov ideals.

Definition

Let $\kappa : E \rightarrow A$ be a C^* -extension of an operator system. A **boundary ideal** is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \rightarrow A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The **Shilov ideal** is the largest of such boundary ideals.

Proposition

Let $\kappa : E \rightarrow A$ be a C^* -extension. Then there exists a Shilov boundary ideal J and $C_{env}^*(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1 . Accordingly, its C^* -envelope is $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E .

Definition

The *propagation number* $\text{prop}(E)$ of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C_{\text{env}}^*(E)$ is a C^* -algebra.

Returning to harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\mathbb{D})) = 1$.

Proposition

The propagation number is *invariant under complete order isomorphisms*, as well as *under stable equivalence*:

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

Operator system spectral triples

Definition

An *operator system spectral triple* is a triple (E, \mathcal{H}, D) where E is an operator system in $\mathcal{B}(\mathcal{H})$, \mathcal{H} is a Hilbert space and D is a self-adjoint operator in \mathcal{H} with compact resolvent and such that $[D, T]$ is a bounded operator for all $T \in \mathcal{E} \subset E$.

It gives a **distance function** for states ϕ, ψ on E using the same formula:

$$d(\phi, \psi) = \sup_{T \in \mathcal{E}} \{|\phi(T) - \psi(T)| : \|[D, T]\| \leq 1\}$$

We will illustrate this with **spectral truncations of the circle**.

Spectral truncation of the circle

Consider the **circle** $(C(S^1), L^2(S^1), D = -id/dx)$

- Eigenvectors of D are **Fourier modes** $e_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an **operator system**
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a **Toeplitz matrix**

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n) and $(C(S^1)^{(n)}, \mathbb{C}^n, D = \text{diag}\{1, 2, \dots, n\})$ is an operator system spectral triple.

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1 .

Proposition

1. The Shilov boundary of the operator system $C^*(\mathbb{Z})_{(n)}$ is S^1 .
2. The C^* -envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.
3. The propagation number is infinite.

Proposition

1. The **extreme rays** in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1 .
2. The **pure states** of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).



Pure states on the Toeplitz matrices

The **duality** between $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ is given by

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \rightarrow \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. The **extreme rays** in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.
2. The **pure states** of $C(S^1)^{(n+1)}$ are given by functionals $T \mapsto \langle \xi, T\xi \rangle$ where the vector $\xi = (\xi_0, \dots, \xi_n) \in \mathbb{C}^{n+1}$ is such that the polynomial $z \mapsto \sum_k \xi_k z^{n-k}$ has all its zeroes on S^1 .
3. The **pure state space** $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$ is the quotient of the n -torus by the symmetric group on n objects.

Let us illustrate this!

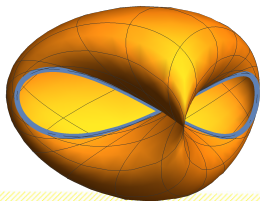
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

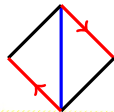
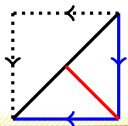
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is \mathbb{T}^2/S_2 , given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!



An old factorization result of Carathéodory

Theorem

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ if and only if T is of the following form:

$$T = V \Delta V^*,$$

where Δ is a positive diagonal matrix and V is a Vandermonde matrix,

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \quad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

Finite Fourier transform and duality

- Fourier transform on the **cyclic group** maps $C(\mathbb{Z}/m\mathbb{Z})$ to $C[\mathbb{Z}/m\mathbb{Z}]$ and vice versa, exchanging **pointwise and convolution** product.
- This can be phrased in terms of a **duality**:

$$\begin{aligned} C[\mathbb{Z}/m\mathbb{Z}] \times C(\mathbb{Z}/m\mathbb{Z}) &\rightarrow \mathbb{C} \\ \langle c, g \rangle &\mapsto \sum_{k,l} c_l g(k) e^{2\pi i k l / m} \end{aligned}$$

compatibly with **positivity**.

- Thus we may consider the above duality for Toeplitz matrices as some sort of generalization of Fourier theory to operator systems.
- However, note that for finite Fourier theory the **symmetries are reduced** from S^1 to $\mathbb{Z}/m\mathbb{Z}$.

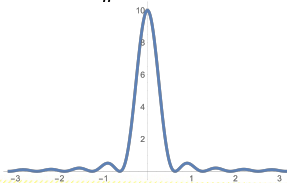
Convergence to the circle

In a recent preprint I analyze the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \rightarrow C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is an **approximate inverse** $S_n : C(S^1)^{(n)} \rightarrow C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \quad S_n(R_n(f)) = F_n * f$$

in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n :

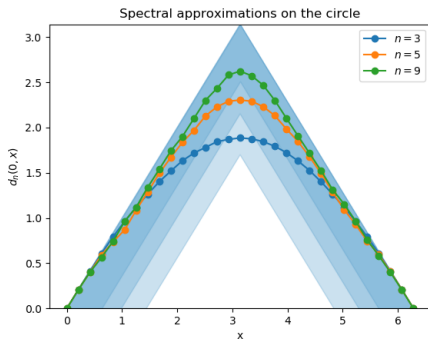


- The fact that S_n is an approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$.

- Some (basic) Python simulations for point evaluation on S^1 :



Gromov–Hausdorff convergence

Recall **Gromov–Hausdorff distance** between two metric spaces:

$$d_{\text{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \rightarrow Z, g : Y \rightarrow Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that **bridges** the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n .

Proposition

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

Outlook

- Established formalism for describing spectral truncations using operator systems, exemplified with truncations on a circle.
- Rich structure: C^* -envelopes, propagation number, stable equivalence, non-unital operator systems etc [Connes–vS 2020]
- Operator systems based on tolerance relations, e.g. metric spaces at finite resolution identifying x, y for which $d(x, y) < \epsilon$.
- Pure state spaces \mathbb{T}^n/S_n for Toeplitz operator systems vs. S^1
- General setup for Gromov–Hausdorff convergence [vS 2020]: applies to Fejér–Riesz operator systems converging to S^1 , matrix algebras converging to a sphere [Rieffel 2004, Barrett–Glaser 2016]

