Spectral truncations in noncommutative geometry

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A spectral approach to geometry

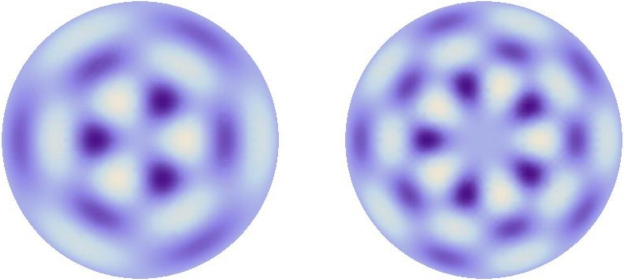
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

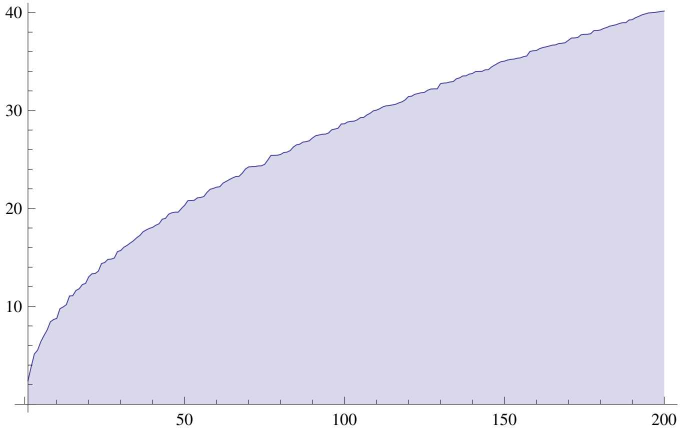
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

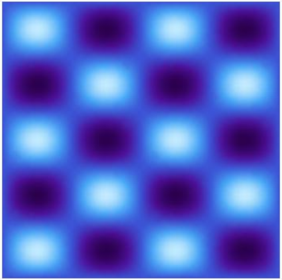
The disc

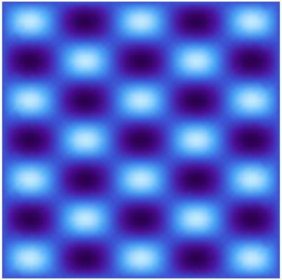


Wave numbers on the disc

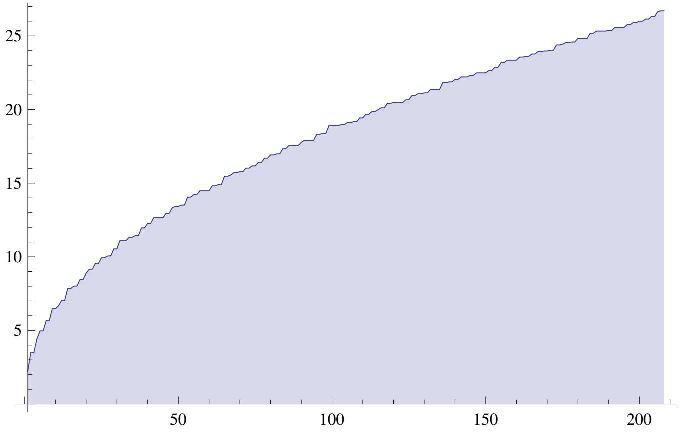


The square



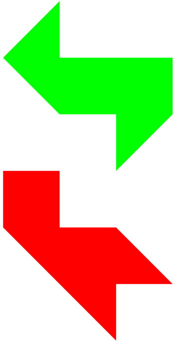


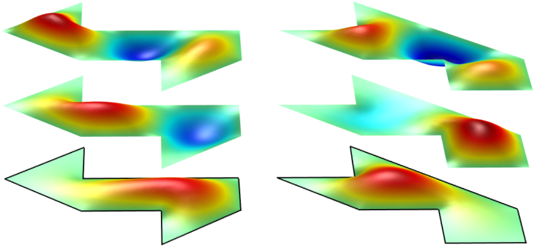
Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:





(Gordon, Webb, Wolpert, 1992)

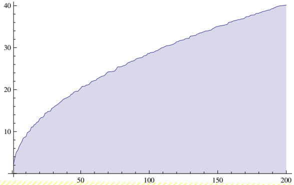
so the answer to Kac's question is no

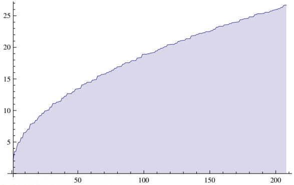
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension d of M:

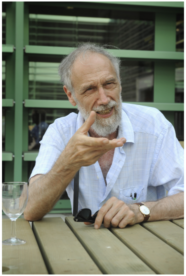
$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda \sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Noncommutative geometry



If combined with an algebra of coordinates on M, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), \not{\partial}_M) \longleftrightarrow (M, g)$$

Spectral data

- This mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum.

This is in line with earlier work of [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019], [Berendschot 2019] and based on [arXiv:2004.14115]

The “usual” story

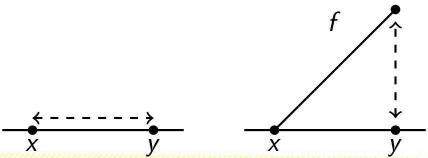
Given Riemannian spin manifold (M, g) with spinor bundle S on M.

- coordinate algebra $C(M)$
- propagation on M: self-adjoint Dirac operator $\not{\partial}_M$
- both acting on Hilbert space $L^2(M, S)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(M, S), \not{\partial}_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \left\{ |f(x) - f(y)| : \|[\not{\partial}_M, f]\| \leq 1 \right\}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a (C^*)-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1 (e.g. vector states)
- Pure states are extreme points of state space (e.g. evaluation at a point)
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Application in physics

- GR (in parallel with reconstruction)
- Gauge theories can be described by a spectral triple, e.g.

$$(C(M) \otimes A_F, L^2(M,S) \otimes \mathcal{H}_F, \partial\!\!\!/_M \otimes 1 + 1 \otimes D_F)$$

- Gauge group: unitaries $\mathcal{U}(A)$
- Group of isometries of $(A, \mathcal{H}, D)$: all unitaries on $\mathcal{H}$ that induce automorphisms on A and commute with D.

Reviewed in [Chamseddine–vS 2019]

Spectral truncations

Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Definition

An operator system is a $*$-closed vector space of bounded operators.

And, indeed, PAP is an operator system: $(PaP)^* = Pa^*P$.

So, we turn to study $(PAP, P\mathcal{H}, PDP)$.

We expect:

- a distance formula on states of PAP.
- a rich symmetry: isometries of $(A, \mathcal{H}, D)$ remain isometries of $(PAP, P\mathcal{H}, PDP)$

Operator systems

Definition (Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.

For convenience we take E to be finite-dimensional, to contain the identity operator, and act on a fixed Hilbert space $\mathcal{H}$.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is completely ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

States spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- Also, the dual E^d of an operator system is an operator system, with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

and similarly for the complete order.
- We have $(E^d)^d_+ \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

and the other way around.

Operator system spectral triples

Definition

An operator system spectral triple is a triple $(E, \mathcal{H}, D)$ where E is an operator system in $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ is a Hilbert space and D is a self-adjoint operator in $\mathcal{H}$ with compact resolvent and such that $[D, T]$ is a bounded operator for all $T \in \mathcal{E} \subset E$.

It gives a distance function for states ϕ, ψ on E using the same formula:

$$d(\phi, \psi) = \sup_{T \in \mathcal{E}} \{ |\phi(T) - \psi(T)| : \|[D, T]\| \leq 1 \}$$

We will illustrate this with spectral truncations of the circle.

Spectral truncation of the circle

Consider the circle $(C(S^1), L^2(S^1), D = -id/dx)$

- Eigenvectors of D are Fourier modes $e_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

$\rightsquigarrow$ operator system spectral triple $\left(C(S^1)^{(n)}, \mathbb{C}^n, D = \mathrm{diag}\{1, 2, \ldots, n\}\right)$

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$.*

Pure states on the Toeplitz matrices

The duality between $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ is given by

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.*
2. *The pure states of $C(S^1)^{(n+1)}$ are given by functionals $T \mapsto \langle \xi, T\xi \rangle$ where the vector $\xi = (\xi_0, \dots, \xi_n) \in \mathbb{C}^{n+1}$ is such that the polynomial $z \mapsto \sum_k \xi_k z^{n-k}$ has all its zeroes on S^1.*
3. *The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$ is the quotient of the n-torus by the symmetric group on n objects.*

Let us illustrate this!

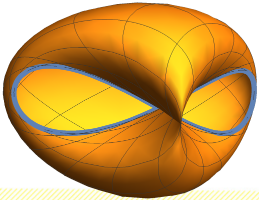
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is $\mathbb{T}^2/S_2$, given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi = \frac{1}{\sqrt{4 + 2\cos(x - y)}} \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!

An old factorization result of Carathéodory

Theorem
Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ if and only if T is of the following form:

$$\boxed{T = V \Delta V^*,}$$

where Δ is a positive diagonal matrix and V is a Vandermonde matrix,

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Finite Fourier transform and duality

- Fourier transform on the cyclic group maps $C(\mathbb{Z}/m\mathbb{Z})$ to $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ and vice versa, exchanging pointwise and convolution product.
- This can be phrased in terms of a duality:

$$\mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \times C(\mathbb{Z}/m\mathbb{Z}) \to \mathbb{C}$$
$$\langle c, g \rangle \mapsto \sum_{k,l} c_l g(k) e^{2\pi i k l/m}$$

  compatibly with positivity.
- Thus we may consider the above duality for Toeplitz matrices as some sort of generalization of Fourier theory to operator systems.
- However, note that for finite Fourier theory the symmetries are reduced from S^1 to $\mathbb{Z}/m\mathbb{Z}$.

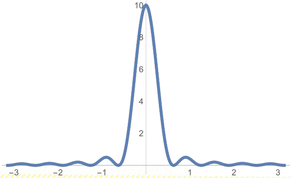
Convergence to the circle

In ongoing work I analyze the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is an approximate inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

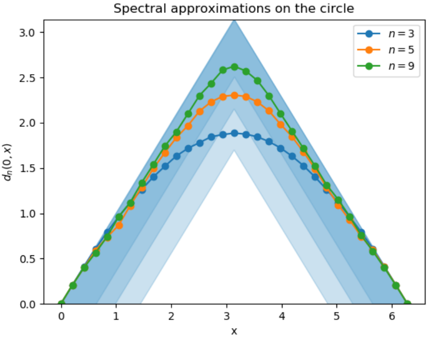
in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n:



- The fact that S_n is an approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi,\psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi,\psi)$$

  where $\gamma_n \to 0$ as $n \to \infty$.
- Some (basic) Python simulations for point evaluation on S^1:



Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that bridges the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within any ϵ for n large enough.

Proposition

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

Outlook

- Established formalism for describing spectral truncations using operator systems, exemplified with truncations on a circle.
- Rich mathematical structure: C^*-envelopes, propagation number, stable equivalence, non-unital operator systems etc [Connes–vS 2020]
- Operator systems based on tolerance relations, e.g. metric spaces at finite resolution identifying x, y for which $d(x, y) < \epsilon$.
- General setup for Gromov–Hausdorff convergence [vS 2020]: applies to Fejér–Riesz operator systems converging to S^1, matrix algebras converging to a sphere [Rieffel 2004, Barrett–Glaser 2016]