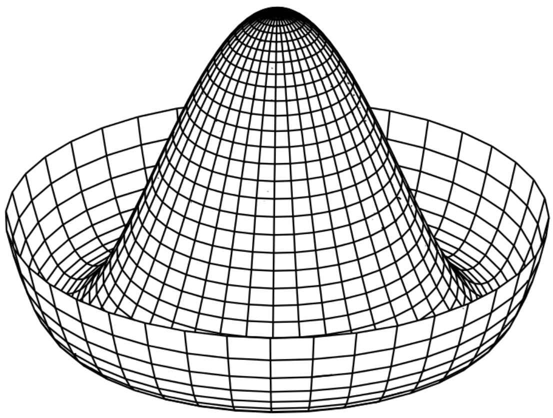
Group actions, critical points and critical fields

Oscar de Wit

Supervisor: Dr. Walter van Suijlekom



Bachelor's thesis Mathematics

Extended as part of the Radboud University Faculty of Science Bachelor's Honours programme

Date: August 2020

Acknowledgements

I would like to thank the people that helped me during this Bachelor's project. First of all, I would like to express my gratitude to my supervisor Walter van Suijlekom. After having explained my interests for doing a Bachelor's project with him, Walter suggested the first topics for me to focus on and learn about. During the year, he was very accessible, did numerous valuable suggestions, gave valuable feedback on my writing and made the suggestion to visit Kasia Rejzner at the University of York, as part of the Faculty of Science Honours Programme in which I participated. I would like to thank Kasia Rejzner for the meetings we had and the opportunity to speak at the Mathematical Physics seminar at the Mathematics Department at the University of York. Great thanks also go to Sam Crawford, who aided me in settling at the Mathematics PhD office at the University of York and with whom I had a few encouraging discussions. Furthermore, I would like to thank Andrew Stacey for answering my questions about smooth structures and topological considerations. Also, I would like to thank Teun van Nuland, for helping me during a lunch break at the beginning of the project, and Gert Heckman, for an insightful discussion about Lie algebras. Furthermore, I would like to thank Tobias Diez for his response on my questions about his paper on the Slice Theorem in the Fréchet setting. Lastly, I would like to thank the Honours Programme of the Faculty of Science at the Radboud University for providing the resources to travel abroad for my Bachelor's project.

Contents

Chapter 1

Introduction and background from symmetry breaking

1.1 General introduction

Symmetries are ubiquitous in both theoretical physics and mathematics. Sometimes the symmetries of theoretical physics are an object of study for mathematicians and sometimes the symmetries of mathematicians are found to be applicable to theoretical physics. This Bachelor's thesis will be concerned with symmetry, but not only that.

Inspired by the idea of how symmetries seem to underlie (parts of) the reality we live in – by breaking these symmetries –, this thesis takes the role symmetry plays in the symmetry breaking process of the electroweak theory as a starting point. In section 1.2, we will explain this background on symmetry breaking from classical field theory in theoretical physics. Then, in section 1.2.4 we explain how some parts of this symmetry breaking process can be generalized in a mathematical context.

Secondly, in chapter 2, a mathematical generalization of the symmetric potentails, as commonly used in symmetry breaking in physics, will be studied. More specifically, from a purely mathematical perspective, we ask how the set of critical points of such potentials is related to the critical points of the same potential after symmetry reduction. That is, by taking quotients with respect to the group action associated to a symmetry on the domain manifold of the potential. This will be done by considering the orbit spaces of certain submanifolds of the domain manifold of the potential. Now, this is mostly a question of pure mathematics, but at the end of this chapter we will provide some examples of how this can describe the critical points of potentials such as the Higgs potential.

Chapter 3 will be more of an analytic nature. The motivation for this chapter comes from the breaking of so-called local symmetries in classical field theories and the ground states that are used in that approach, as explained in 1.2. More formally, we will investigate what the relation is between finding critical functions of what we will call the composition map and the integral map. The composition map maps fields on some spacetime domain to real-valued functions which are those fields composed with a fixed real-valued potential defined on the codomain of those fields. The domain of the potential and the codomain of the field can be seen as some general configuration space. The integral map maps the same fields to the real number value of the integral of this composed function over the spacetime domain of the fields. This map can be seen as an analogue of only integrating the potential energy density in the action functional for classical field theories. To find the critical functions (or just critical points) of these maps, we will have to construct a smooth structure on the set of smooth functions and define the differential, similar as in the theory of finite-dimensional manifolds. This will lead us to investigate the theory of infinite-dimensional manifolds. Pursuing this idea, we will be lead to a view on finding a globally critical field, which we can apply to the situation where there is a symmetry involved as in chapter 2.

Lastly, in chapter 4, we will give a conclusion and an outlook.

1.2 An introduction to symmetry breaking in classical field theories of physics

This section serves as a brief and informal introduction to the ideas that inspired the topics of this Bachelor's thesis. First, we take a look at the frameworks of Newtonian mechanics and classical field theory.

For many theories of physics the Lagrangian or the Lagrangian density is central. The first serves as an powerful analytical tool for Newtonian mechanics, and the second for field theories such as the electromagnetic Maxwell equations. The usual expression for the Lagrangian is $\mathcal{L} = T - V$ where T stands for the kinetic energy and V for the potential energy. For Newtonian mechanics, we use coordinates, $x(t)$, which are functions of time t, where $x(t) \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Usually, in this case, the kinetic energy T at a certain time can be written as a function of $\dot{x}(t) = \frac{dx}{dt}(t)$ and the potential energy depends on $x(t)$. Now, the dynamical evolution of the coordinates over time is prescribed by a set of differential equations – the Euler-Lagrange equations – that depend on the form of T and V. More specifically, these differential equations minimize the so-called action functional $S = \int_{t_1}^{t_2} \mathcal{L}$ over all possible paths $x(t)$ that satisfy some constraints at time t_1 and t_2. Of particular interest are the critical points of the potential, as it is common that these give constant states[1], i.e. $x(t)$ is constant for all time. This is because the differential equations usually depend on the first derivative of V, and if x is a critical point of V with zero velocity at some starting time, there will be no variance of x over time. Moreover, these constant states can represent minimal energy states if x is a global minimum of V. That is, the sum $T + V$ is a minimal value of V for all t, where we note that $T = 0$ if $x(t)$ is constant. For more details on the Lagrangian setting for Newtonian mechanics we refer to Chapter 7 in [34].

For field theories, instead of coordinates, we consider fields φ which are functions defined over spacetime mapping into some configuration space. For the moment, we can think of spacetime as the space $\mathbb{R}^4$. A common configuration space is $\mathbb{C}^n$ for some positive integer n. As fields are not only functions of time, in this case, the fields can be thought of as smeared out over spacetime. Now, the Lagrangian *density* reflects this smearing and it is assumed that this Lagrangian density can be written as a function of φ and space and time derivatives $\partial_\mu \varphi$. The kinetic energy density T is usually a function of the space and time derivatives $\partial_\mu \varphi$ and the potential energy density a function that only depends on φ. Now, the action functional for fields $S = \int \mathcal{L}$ is an integral of the Lagrangian density over the entirety of spacetime (with an appropriate Lorentzian invariant volume element) and a specific functional minimization of this action[2] gives the Euler-Lagrange field equations that, together with appropriate initial conditions, determine the evolution of the fields over all of spacetime. For more details on the Lagrangian density for classical field theories we refer to Chapter 2 in [9].

[1] This should not be thought of as a quantum mechanical state.

[2] When taking the functional derivative, perturbation functions with *compact support* could be used, such that the integrals are bounded.

Similar to Newtonian mechanics, fields that assume a constant value over all of spacetime whose value is also a minimal and critical value of the potential are of interest. Again, these states can reflect a global minimal energy condition, but a direct relation with the integral of the total energy density may not exist, as this integral may be unbounded.

Now, we turn to our first example of *global* symmetry breaking for classical field theories. Later on, the distinction between *local* and *global* symmetry breaking will become clear.

1.2.1 Global symmetry breaking and gauge invariance

Consider a one-dimensional complex scalar field φ ($\varphi(x) \in \mathbb{C}$ for each $x \in R^4$) and a Lagrangian density

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - V(\varphi), \text{ where } V(\varphi) = -a|\varphi|^2 + b|\varphi|^4,$$

such that $a, b > 0$ are real-valued constants and $T = \partial_\mu \varphi^\dagger \partial^\mu \varphi$ is the kinetic energy density and V the potential energy density. (We use the Einstein summation convention, the metric $(+,-,-,-)$, $\varphi^\dagger$ for the complex conjugate and $|\varphi| = \sqrt{\varphi\varphi^\dagger}$.) Now, this Lagrangian is invariant under a *global* transformation (group action) $\varphi \mapsto e^{i\theta}\varphi$, where θ is a real-valued constant.

Now, there (at least) two ways to view the breaking of this global symmetry. (See for example [5] and [33], where the following is largely drawn from.) Firstly, we can pick a ground state such that $\varphi(x) = \varphi_0$ where φ_0 is a global minimum of the potential V. In this case, this a complex number of the form $\sqrt{\frac{a}{2b}}e^{i\theta}$, such that it is any element of the *orbit under the group action* – in this case the group $U(1)$ acting on $\mathbb{C}$ by multiplication (see also the plot on the titlepage). Then, if we write the field as $\varphi = \varphi_0 + \sqrt{\frac{1}{2}}(\chi + i\psi)$, where χ and ψ are two new real-valued fields, we get a new Lagrangian in terms of these two fields. In particular, the χ field is approximately the field of a bosonic particle with mass a and ψ corresponds approximately to a massless bosonic particle. More specifically, the Euler-Lagrange equations are the Klein-Gordon equations with and without mass, in a certain approximation. This approximation means that we neglect products of χ and ψ with more than three product factors. The act of picking a preferred ground state, or angle θ in this case, can be seen as the point where the symmetry is broken.

The other approach for finding these same Klein-Gordon equations uses a field transformation that does not violate symmetry. For this, we consider – without picking a preferred ground state – two consecutive field reparametrizations. First, we consider

$$\varphi = \sqrt{\frac{1}{2}}\rho e^{i\theta}, \;\; \rho = \sqrt{2\varphi\varphi^\dagger}, \quad \theta = \frac{1}{2i}\ln\left(\frac{\varphi}{\varphi^\dagger}\right),$$

where we assume $\varphi \neq 0$ and 'ln' is the principal value logarithm and we note that now θ can vary over spacetime. Secondly, we reparametrize ρ and θ by

$$\rho = \chi + \sqrt{\frac{a}{b}}, \text{ such that } \chi > -\sqrt{\frac{a}{b}}, \quad \theta = \sqrt{\frac{b}{a}}\psi.$$

Again, substituting this back into the Lagrangian density, we now assume that $\partial_\mu\psi$ can be neglected to some higher power, which is an assumption that does *not* violate the global symmetry. This is because ψ maps to $\psi + \alpha$ under a global transformation, for some fixed real-valued α, so the first order derivative stays the same. Therefore, in this case, the symmetry is *not* broken. Again, neglecting products of χ and $\partial_\mu\psi$ with three or more factors, we get the same Klein-Gordon equations as in the first approach.

This field transformation is related to so-called *gauge invariant* variables. In the above situation, χ and $\partial_\mu\psi$ are such gauge invariant variables. We note that this is also related to *fixing a gauge*, as done with, for example, the Coulomb gauge in electromagnetism. For more details on this notion, see for example [33].

1.2.2 Local symmetry breaking and gauge invariance

We can reconsider the same Lagrangian density as above, but now we want to able to apply the *local gauge transformation* $\varphi(x) \mapsto e^{i\theta(x)}\varphi(x)$ to the field, where θ is some differentiable function over spacetime. We observe that under this transformation $\varphi(x) \mapsto e^{i\theta(x)}\varphi(x) = \varphi'(x)$ the Lagrangian densities are not the same, i.e. $\mathcal{L}(\varphi) \neq \mathcal{L}(\varphi')$. Yet, we can introduce a new vector field, A_μ, and a gauge transformation for that field, defined as

$$\varphi(x) \mapsto e^{i\theta(x)}\varphi(x), \quad A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x),$$

such that the Lagrangian *is* invariant under this gauge transformation. An example of introducing such a new field is the introduction of the electromagnetic field in the Klein-Gordon equation, allowing for electromagnetic interactions. Now, the new Lagrangian becomes

$$\mathcal{L} = D_\mu\varphi^\dagger D^\mu\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\varphi), \quad D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where the positive real constant e can be associated with the value of the charge. The new derivative, D_μ, is called a covariant derivative. In this case, the minimal energy field solutions are those of the form $\varphi = \sqrt{\frac{a}{2b}}e^{i\theta}$, where θ may vary freely, and $A_\mu = \frac{1}{e}\partial_\mu\theta$.[3] Similar to the first method of breaking a global symmetry, we can pick a constant ground state $\varphi = \varphi_0$ and introduce the same fields χ and ψ as before. This way, expanding around the ground state of φ_0, we again get a Klein-Gordon equation with mass for the χ field and, after introducing $B_\mu = A_\mu - \frac{1}{e}\sqrt{\frac{2b}{a}}\partial_\mu\psi$ and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, we note that B_μ describes a vector boson with mass.

Alternatively, we could have begun with the field transformations

$$\varphi = \sqrt{\frac{1}{2}}\rho e^{i\theta}, \ \rho = \sqrt{2\varphi\varphi^\dagger}, \quad \theta = \frac{1}{2i}\ln\left(\frac{\varphi}{\varphi^\dagger}\right),$$

with in addition $A_\mu = B_\mu + \frac{1}{e}\partial_\mu\theta$. Rewriting the Lagrangian, we see that it becomes independent of θ. Now, we again expand around the ground state $\rho = \sqrt{\frac{a}{2b}}$, which does not break symmetry, such that $\rho = \chi + \sqrt{\frac{a}{b}}$, and we get the same equations as in the above approach. Lastly, we note that χ and B_μ are now the gauge invariant variables.

1.2.3 The bosonic electroweak model

The existence of the electromagnetic field as well as the $W^\pm$ and Z vector boson fields can be seen as result of the breaking of the local symmetry of the group $U(1) \times SU(2)$. For this, we consider a 2-dimensional complex vector field instead, $\varphi = (\varphi_A, \varphi_B)$, but we start with the same Lagrangian, $\mathcal{L} = \partial_\mu\varphi_A^\dagger\partial^\mu\varphi_A + \partial_\mu\varphi_B^\dagger\partial^\mu\varphi_B - V(\varphi)$, where $V(\varphi) =$

[3] We note that the minimal energy condition is not specified deliberately. Constant fields that assume a critical value of V could also serve as a minimal energy field, as is done in [5], but we must note that the energy is invariant under a local gauge transformation which give the non-constant fields.

$-a(|\varphi_A|^2+|\varphi_B|^2)+b(|\varphi_A|^2+|\varphi_B|^2)^2$. This Lagrangian naturally has a global symmetry by the group action of $U(1)\times SU(2)$ on $\mathbb{C}^2$ by matrix multiplication. Promoting this global symmetry to a local one, we introduce the vector field B_μ for the local $U(1)$ symmetry and the three vector fields W^k_μ with $k=1,2,3$ for the local $SU(2)$ symmetry. The three fields W^k_μ each are associated to a generator of the group $SU(2)$, which are the Pauli matrices. Again, we can reformulate the transformations such that for a new (covariant derivative) D_μ the Lagrangian becomes invariant under local gauge transformations.

The symmetry may be broken by a combination of gauge fixing and perturbations around some ground state. We start by choosing a ground state $\varphi=(0,\varphi_0)$, such that it is a critical point of the potential. Next, we may insert the perturbed field $\varphi=(0,\varphi_0+h(x)/\sqrt{2})$, where h is a real-valued scalar field, and use that any other $\mathbb{C}^2$-field sufficiently close to $(0,\varphi_0)$ can be brought to the form of φ by a local $U(1)\times SU(2)$ gauge transformation (see for example section 6.2 in [29]). Then, after rewriting the Lagrangian with new reparametrizations of B_μ and W^k_μ into A_μ, $W^\pm_\mu$ and Z_μ, we get a massive scalar boson field for h, a massless vector boson field for A_μ, two charged massive vector boson fields for $W^\pm_\mu$ and a neutral massive vector boson field for Z_μ. (For more details on the formulas of the parametrizations and the fields, see for example [5].) These four fields are related to, respectively, the electromagnetic photon and to the $W^\pm$ and Z subatomic particles.

1.2.4 Generalizing mathematical parts of the symmetry breaking process

Having learned about the process of symmetry breaking, we can generalize some parts of the mathematical details of this process. Notably, we can consider spacetime and the configuration space to be arbitrary manifolds, say M and N, and the symmetry as a smooth group action of some Lie group G on the configuration space N. The potential would be a smooth G-invariant function V from N to $\mathbb{R}$.

Next, we will put the orbits of this group action central. It happens so that, under certain conditions, orbits are diffeomorphic to the quotient manifold of the left coset space G/G_x where x is any point along the orbit and G_x is its isotropy group. Therefore, we can take this local knowledge of the isotropy group as a starting point to see the form of the orbits. Moreover, for global symmetry breaking, the dimension of the isotropy group of the chosen absolute minimum of the potential says something about the number of massless particles and the number of particles with mass after symmetry breaking. (This is related to the so-called Goldstone theorem and the massless particles are often called Goldstone bosons and the massive particles Higgs bosons. For more details on this, see for example [15] and [19].)

Inspired by this, we will be interested in how the critical points of V can be described by a reduced potential that has some quotient space as its domain, with the symmetry of G quotiented out.

Furthermore, as the field varies over spacetime and therefore has infinite degrees of freedom, we would like to look at the potential energy density $V(\varphi)$ and find globally critical fields in some way, i.e. not only constant fields that assume one critical value of V. More specifically, this will be done by looking only at the integral of $V(\varphi)$ over M, with some additional conditions on M, such that the integral exists, and the critical fields of the map $\varphi\mapsto\int V(\varphi)$, similar to finding critical fields for the action functional.

In other words, the first study can be seen as elucidating the form of the potential and its relation with the group action G, as also seen in global symmetry breaking, and the second as an alternative viewing on the ground states in local symmetry breaking.

1.3 Mathematical outline

The main content of this thesis is divided in two chapters: chapter 2 and chapter 3. Chapter 2 is concerned with smooth group actions of Lie groups, G, on smooth finite-dimensional manifolds with or without boundary, M. From hereon, if we say manifolds, it will mean smooth finite-dimensional, second countable manifolds with or without boundary. With these manifolds, M, we study general smooth functions $f : M \to \mathbb{R}$ which are associated to the potentials as discussed in section 1.2. Specifically, we study the critical points of these functions, f, with the assumption that they are invariant under a smooth group action of a Lie group G on the domain manifold M. This means that, if we write $g \cdot x$ for the application of the group element $g \in G$ on $x \in M$, we have $f(g \cdot x) = f(x)$ for each x in M. Then, we are interested in the potential on a domain but with the symmetry 'turned off', or in this case with the group action quotiented out. For this, we cannot use the orbit space M/G, as this space is not a manifold in general. However, the domain can be partitioned by certain submanifolds called strata. These strata $M_{(H)}$ are defined by a subgroup H of G and are the collection of points in M for which the isotropy group is conjugated to H. That is, if we write G_x for the isotropy group of $x \in M$, G_x is conjugated to H if there is some $g \in G$ such that $gG_xg^{-1} = H$. In section 2.3, these strata will be shown to be embedded submanifolds of M and the orbit spaces $M_{(H)}/G$ will be shown to be natural manifolds. This will be done by the use of proper group actions and the so-called Slice Theorem, which will be discussed in sections 2.1 and 2.2, respectively. In section 2.4 we show how, under certain conditions, the critical points of $f : M \to \mathbb{R}$ are related to the critical points of each restricted and reduced function $\tilde{f}_{(H)} : M_{(H)}/G \to \mathbb{R}$ for each strata of M. That is, restricting f to $M_{(H)}$ gives $f_{(H)} : M_{(H)} \to \mathbb{R}$ and going to the orbit space gives $\tilde{f}_{(H)} : M_{(H)}/G \to \mathbb{R}$, by the G-invariance of f.

In chapter 3, inspired by the ground states as covered in section 1.2.2, we study if we can 'globally' differentiate the potential of a field to find critical fields similar to these ground states. That is, if M and N are manifolds and $C^\infty(M, N)$ the set of smooth functions from M to N, we want to study the composition map and the integral map. The composition map maps f to $V \circ f$ from $C^\infty(M, N)$ to $C^\infty(M, \mathbb{R})$, if $V : N \to \mathbb{R}$ is a smooth potential. The integral map maps f to the real-value of the integral $\int_M V \circ f$ over M, assuming M to be a compact and orientable manifold, such that the integral exists using the pullback Lebesgue measure via charts of M. To speak of critical fields, we need a smooth structure on $C^\infty(M, N)$. This will be done by locally modelling this space on open subsets of Fréchet spaces, which result in Fréchet manifolds. The topology and calculus of Fréchet spaces are discussed in section 3.1 and the smooth structure of $C^\infty(M, N)$ for M compact is discussed in section 3.2. In section 3.3 we will show that the critical fields of the composition map and the integral map are equal, for an appropriate definition of critical fields for the composition map. Lastly, at the end of this section we will show how the critical functions are related by local gauge transformations of the infinite-dimensional Fréchet-Lie group $C^\infty(M, G)$, i.e. an action, under certain conditions for the connectedness of M.

Chapter 2

Orbits, strata and critical points

2.1 Orbits, isotropy groups and proper actions

In this section we are interested in characterising the orbits of a smooth group action by a Lie group G on a manifold M. We will show that, if the action is proper, that the orbits $G \cdot x = \{g \cdot x : g \in G\}$ are embedded submanifolds of M. In this case, we also have a diffeomorphism between $G \cdot x$ and the left coset space G/G_x of G modulo the isotropy group of x. Lastly, we introduce the strata $M_{(H)}$, the sets with points that have conjugated isotropy groups.

In the theory of manifolds there is a distinction between the definitions of *immersed* and *embedded* submanifolds of M. Therefore, we first recall some definitions. (For more details, see for example Chapters 4 and 5 in [21].)

Definition 2.1.1. An *immersed submanifold* of M is a subset $S \subseteq M$ with some topology (not necessarily the subspace topology), such that it is a topological manifold, and with a smooth structure such that the inclusion map $S \hookrightarrow M$ is a smooth immersion. A *smooth immersion* is a smooth function $F : M \to N$ with rank equal to the dimension of M, so dF_p is injective for all $p \in M$.

An *embedded submanifold* of M is a subset $S \subseteq M$ *with* the subspace topology and a smooth structure such that the inclusion map is a smooth embedding. A *smooth embedding* is a smooth immersion which is also a homeomorphism onto its image, with the subspace topology for this image, inherited from the topology of the codomain.

A *smooth submersion* is defined as a smooth map between two manifolds that has a surjective differential at each point.

Group actions can be compatible with the topological (smooth structure) of both the group itself and the space on which it is acting in the following way.

Definition 2.1.2. Let G be a topological group (Lie group) and let X be a topological space (manifold). Then, a group action of G on X is continuous (smooth) if the group action map

$$\varphi : G \times X \to X, \ (g, x) \mapsto g \cdot x,$$

is continuous (smooth), where the product $G \times X$ is equipped with the product topology (product smooth structure).

Now, the *orbit space* is defined as the quotient space defined by the following equivalence relation: $x \sim y$ if and only if there is an element g in G such that $g \cdot x = y$, i.e. $x \sim y$ if and only if they lie in the same orbit. This equivalence relation gives the set of equivalence classes, denoted by $X/\sim$. Now, if X is a topological space, for any equivalence relation $\sim$, the set of equivalence classes $X/\sim$ can be equipped with a natural topology by declaring the *quotient map* $\pi : X \to X/\sim$, that sends x to its equivalence class, to be an open map. This topology is called the *quotient topology*. To make the association between this orbit space and the group of the group action clear, we will mainly write X/G for the orbit space of a G-action on X. For more details on topological quotient spaces, see for example section 22 in [25].

The subsequent Theorem 2.1.6 will allow us to view the left coset space G/G_x as a manifold. A proof of this theorem, as for Theorem 21.17 in [21], depends on two important theorems, namely the Quotient Manifold Theorem and the Closed Sub-Group Theorem.

First we need the following definitions concerning group actions.

Definition 2.1.3. A group action on a set X for a group G is *free* if for all x in X the statement $gx = x$ for any g in G implies $g = e$, the identity element of G. This is equivalent to saying that $G_x = \{e\}$ for all x in X.

A group action is *transitive* if for every pair x, y in X there is a g in G such that $g \cdot x = y$. This is equivalent to saying that this group action possesses one orbit.

A continuous group action on a topological space X by a topological group G is *proper* if the map $G \times X \to X \times X$ by $(g, x) \mapsto (g \cdot x, x)$ is proper, i.e. inverses of compact sets are compact.

A map $f : X \to Y$, between two sets, X and Y, on which the group G both acts is called *G-equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all x in X.

Theorem 2.1.4 (Quotient Manifold Theorem)**.** Let G be a Lie group and M a manifold. If G acts freely and properly on M, then, the orbit space M/G can be endowed with a unique smooth structure such that the quotient map $\pi : M \to M/G$ is a smooth submersion.

For a proof, see for example Theorem 21.10 in [21].

Theorem 2.1.5 (Closed Subgroup Theorem)**.** Suppose G is a Lie group and H a subgroup that is also a closed subset of G. Then H is an embedded submanifold of G.

For a proof, see for example Theorem 20.12 in [21].

Let us recall that the *left coset space* of G modulo a subgroup H is defined by the following equivalence relation: for g and g' in G, $g \sim g'$ if and only if there is an element h in H such that $g = g'h$. Equivalently, $g \sim g'$ if and only if g lies in $g'H = \{g'h : h \in H\}$.

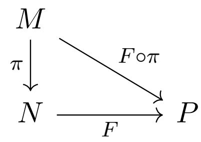
Theorem 2.1.6. Let G be a Lie group and H a closed subgroup of G. Then the left coset space G/H has a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion. Moreover, the action of G on G/H, by $g_1(g_2H) = g_1g_2H$ for g_1 and g_2 in G, is smooth and transitive.

This statement is adapted from Theorem 21.17 in [21], where a proof can also be found.

Let $\theta^{(x)} : G \to M$ be the map that sends g to $g \cdot x$. It is clear that this map is smooth and therefore continuous. We see that G_x is a closed subgroup of G, as G_x is the inverse image of $\{x\}$ under $\theta^{(x)}$. Note that $\{x\}$ is closed in M, because M is Hausdorff. Therefore, by Theorem 2.1.6, for each point x in M, G/G_x is a topological manifold with a unique smooth structure, such that the quotient map $\pi : G \to G/G_x$ is a smooth submersion.

The next theorem is very useful for many proposition that will follow.

Theorem 2.1.7. Suppose M, N and P are manifolds and $\pi : M \to N$ is a surjective smooth submersion. A map $F : N \to P$ is smooth if and only if $F \circ \pi$ is smooth.



For a proof, see for example Theorem 4.29 in [21].

Now, we will prove that $G \cdot x$ is an immersed submanifold of M, if G is a Lie group that acts smoothly on M and x lies in M. In the proof of this proposition we will use the Equivariant Rank Theorem and the Global Rank Theorem for smooth injections.

Theorem 2.1.8 (Equivariant Rank Theorem)**.** Let M and N be manifolds and suppose G is a Lie group such that it acts on both M and N. Suppose further that G acts transitively on M. If $F : M \to N$ is an equivariant smooth map, then F has constant rank.

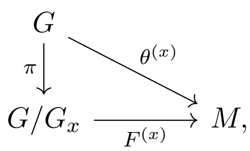
For a proof, see for example Theorem 7.25 in [21].

Theorem 2.1.9 (Global Rank Theorem for smooth injections)**.** Suppose M and N are manifolds and suppose $F : M \to N$ is a smooth injection of constant rank. Then F is a smooth immersion.

For a proof, see for example Theorem 4.14 in [21].

Proposition 2.1.10. Let G be a Lie group and M a manifold on which G acts smoothly. For each x in M, the orbit $G \cdot x$ is a topological manifold with a unique smooth structure such that it is an immersed submanifold of M.

Proof. We consider the diagram:



where $\theta^{(x)}$ is defined as above and π is the quotient map, which is smooth (see the comment after Theorem 2.1.6). Let $[g]$ denote the left coset of g in G/G_x. Define $F^{(x)} : G/G_x \to M$ by $[g] \mapsto g \cdot x$. This map is well-defined, because if $g' = gh_x$ for some $h_x \in G_x$, then $F^{(x)}(g'G_x) = gh_x \cdot x = g \cdot x$. Now, as π is a surjective smooth submersion and $\theta^{(x)} = F^{(x)} \circ \pi$, we have that $F^{(x)}$ is smooth, by Theorem 2.1.7. It is easy to see that $F^{(x)}$ is G-equivariant. Indeed, if $g, g' \in G$, we have that $F^{(x)}(g(g'G_x)) = F^{(x)}((gg')G_x) = (gg') \cdot x = g \cdot (g' \cdot x) = g \cdot F^{(x)}(g'G_x)$. Also, G acts smoothly on both M and G/G_x and G acts transitively on G/G_x by Theorem 2.1.6. Now, by the Equivariant Rank Theorem we get that $F^{(x)}$ is of constant rank. Moreover, $F^{(x)}$ is injective, because for $g_1, g_2 \in G$ we have that $g_1 \cdot x = g_2 \cdot x$ implies $g_1^{-1}g_2 \in G_x$ and so $g_2 \in g_1 G_x$. Therefore, $F^{(x)}$ is a smooth immersion by Theorem 2.1.9. The orbit

$G \cdot x$ can now be equipped with a unique smooth structure (e.g. using Proposition 5.18 in [21]) such that it is an immersed submanifold of M and $F^{(x)} : G/G_x \to G \cdot x$ is a diffeomorphism. □

In the course of this proof, we encountered the following corollary.

Corollary 2.1.11. Let M be a manifold and G a Lie group that acts smoothly on M. Then, the orbit $G \cdot x$ is diffeomorphic to G/G_x, with the smooth structures of the previous proposition.

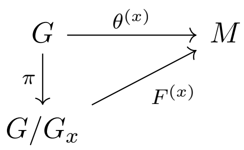
Still, it could be that $G \cdot x$ does not necessarily carry the subspace topology of the parent space M, but only that the inclusion $G \cdot x \hookrightarrow M$ is a smooth immersion. In Proposition 2.1.13 we will show that this is the case if the smooth action of G on M is proper.

Proposition 2.1.12. Suppose M and N are manifolds and $F : M \to N$ an injective smooth immersion. If F is a proper map (i.e. inverse images of compact sets are compact), then F is a smooth embedding.

For a proof, see for example Proposition 4.22 in [21].

Proposition 2.1.13. If G is a Lie group acting smoothly on a manifold M, such that the action is proper, then the orbits are embedded submanifolds of M.

Proof. Consider the following commutative diagram.



We show that $F^{(x)}$ is a proper map and then apply Proposition 2.1.12. If K is a compact subset of M, then $(F^{(x)})^{-1}(K) = \pi((\theta^{(x)})^{-1}(K))$. As $\theta^{(x)}$ is a proper map and π is continuous we see that the inverse image of any compact subset K of M under $F^{(x)}$ is compact. So, $F^{(x)}$ is a proper map. Since $F^{(x)}$ is also a smooth injective immersion (see the proof of Proposition 2.1.10), $F^{(x)}$ is a smooth embedding by Proposition 2.1.12. Therefore, the image of $F^{(x)}$, the orbit $G \cdot x$, is an embedded submanifold of M. (For the proof of the claim that images of smooth embeddings are embedded submanifolds, see for example Proposition 5.2 in [21].) □

With this result in mind, it is useful to know some examples or conditions such that the action is proper. In general, we have the following condition that gives a proper action. First, we discuss a condition which is equivalent to the action being proper.

Proposition 2.1.14. Let G be a topological group that acts continuously on a topological space X. The action of G on X is proper if and only if for any sequences (p_i) and (g_i) in X and G, respectively, such that both (p_i) and $(g_i \cdot p_i)$ converge, we have that (g_i) has a converging subsequence.

For a proof, see for example Proposition 21.5 in [21].

Proposition 2.1.15. Suppose M is a manifold and G a compact Lie group that acts smoothly on M. Then, the action of G on M is proper. Therefore, the orbits of this action are embedded submanifolds of M.

Proof. Each manifold is first countable. Therefore, G being compact implies G being sequentially compact (for a proof, see for example Proposition 15.2.8 and Corollary 15.2.11 in [11]). So, each sequence (g_i) in G always has a converging subsequence. Hence, the action of G on M is proper. □

Note that the Lie groups $U(n)$ and $SU(n)$ for positive integers n are compact (for a proof, see for example section 1.3.1 in [14]) and these are the type of groups that arise in applications in theoretical physics and are of interest in the 'Applications' section 2.5.

It is useful to see some examples of group actions which are not proper and to see what orbit spaces might arise for certain group actions.

Proposition 2.1.16. If a topological group acts continuously and properly on a topological space, then the orbit space is Hausdorff.

For a proof, see for example Proposition 21.4 in [21].

Example 2.1.17. Let Γ be the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and let $O(1,1)$ be the set of 2×2 invertible matrices g that satisfy $g^T\Gamma g = \Gamma$. We claim that this is a Lie group (see for example section 1.2.3 in [14]). This Lie group is not compact. Indeed, if we consider the sequence $(n)_{n\in\mathbb{N}}$, the matrix sequence

$$\begin{pmatrix} \cosh(n) & \sinh(n) \\ \sinh(n) & \cosh(n) \end{pmatrix}$$

lies in G but does not have a converging subsequence.

Let $O(1,1)$ act smoothly on $\mathbb{R}^2$ by matrix multiplication. This action is not proper: the inverse image of the compact set $\{((0,0),(0,0))\}$ under the map $\varphi : G\times\mathbb{R}^2 \to \mathbb{R}^2\times\mathbb{R}^2$ by $(A,x) \mapsto (Ax,x)$ is not compact. Indeed, this inverse image is equal to $G\times\{0\}$, which is not compact since G is not compact.

Example 2.1.18. Let the set $S^1 \subseteq \mathbb{C}$ of complex numbers with modulus one act on the complex plane as $S^1 \times \mathbb{C} \ni (z,w) \mapsto zw$ by multiplication. This group action is smooth and as S^1 is compact it is also proper. It can be proven that the topological orbit space $\mathbb{C}/S^1$ is homeomorphic to $[0,\infty)$, which is still a manifold, but now with boundary. (This is Example 21.2 (c) in [21].)

Example 2.1.19. Let $\mathbb{R}$ act on the two-torus $T^2 = S^1\times S^1$ by $(t,(w,z)) \mapsto (e^{2\pi it}w, e^{2\pi i\alpha t}z)$, where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is irrational. This a smooth group action. Each orbit can be proven to lie dense in T^2. Therefore, it follows that $T^2/\mathbb{R}$ as a quotient space of this action has the trivial topology where the only opens are the whole space and the empty set. Therefore, it is not even Hausdorff. Hence, it cannot be a manifold. We also see that this action could not have been proper by Proposition 2.1.16. (For more details, see Example 21.3 in [21].)

In example 2.1.17, the action is proper if we exclude the origin of $\mathbb{R}^2$, because the action is free on that space and so a sequence (g_i) in $O(1,1)$ has to be bounded if the sequence $(g_i \cdot p_i)$ is to be convergent. Also, if we exclude the origin in example 2.1.18, the orbit space would be the manifold without boundary $(0,\infty)$.

With these 'pathologies' in mind, we will find the concept of strata very useful. One way to think about these strata is that they partition the manifold on which the Lie group acts into separate submanifolds – the strata – such that the action is of a similar 'type'. For example, the type of the action on the origin in examples 2.1.17 and 2.1.18 could be said to be very different from the action outside of the origin. The idea of strata will allow us to take the orbit space for each separate strata and this orbit space will be a natural manifold.

Definition 2.1.20. Let G be a group and H and K subgroups of G. Then, we say H and K are *conjugate* if there is a $g \in G$, such that $H = gKg^{-1}$. We denote (H) for the set of all subgroups of G conjugate with H. We define the set $M_{(H)} = \{x \in M : (G_x) = (H)\} = \{x \in M \mid G_x \text{ is conjugate to } H\}$ as the (H)-*stratum*. We also define $M^H = \{h \in H : h \cdot m = m\}$ to be the *fixed point set* of H.

Note that being conjugate is an equivalence relation on the subgroups of a group, such that each (H) is an equivalence class. If H and K are subgroups of a group G and they are conjugate, we write $H \sim K$. Also, note that the sets $M_{(H)}$ partition M into strata.

Next, we derive some elementary results related to the isotropy groups and orbits. The first proposition shows that each point of an orbit lies in the same stratum.

Proposition 2.1.21. Let G be an action on a set X. Then $G_{g \cdot x} = gG_xg^{-1}$. Therefore $G_{g \cdot x} \sim G_x$.

Proof. Suppose $h \in G_{g \cdot x}$, then $hg \cdot x = g \cdot x \iff g^{-1}hg \in G_x \iff h \in gG_xg^{-1}$. □

The following proposition allows us to see strata as a set of points which have diffeomorphic orbits.

Proposition 2.1.22. Suppose G is a Lie group acting on a manifold M and $x, y \in M$. If $G_x \sim G_y$, then $G \cdot x \cong G \cdot y$ are diffeomorphic.

Proof. The idea is to use the diffeomorphisms $G \cdot x \cong G/G_x$ of Corollary 2.1.11. If $G_x \sim G_y$, there is a $g \in G$ such that $G_y = gG_xg^{-1} = G_{g \cdot x}$ (see the proof of the previous proposition). Therefore, we have the following diagram.

$$\begin{array}{ccc} G \cdot x & \longrightarrow & G \cdot y \\ {\scriptstyle \mathrm{id}}\downarrow & & \uparrow{\scriptstyle 2} \\ G \cdot (g \cdot x) & \xrightarrow[1]{} & G/G_{g \cdot x} \end{array}$$

We know that we have diffeomorphism at the arrows 1 and 2, by Corollary 2.1.11, so we can compose them with id to get a diffeomorphism between $G \cdot x$ and $G \cdot y$. □

Notice that we have a partial converse of this statement: if $G \cdot x$ and $G \cdot y$ are G-equivariantly diffeomorphic with respect to the G-action on M, then $G_x \sim G_y$.

2.2 The Slice Theorem

Now we know that a manifold M can be partitioned into its strata, our goal now is to show that each (H)-stratum for any Lie subgroup H of G is an embedded submanifold of M. We also want to show that the orbit spaces $M_{(H)}/G$ are manifolds.

For this, we will make use of the so-called *Slice Theorem*. This theorem will allow us to locally model open neighbourhoods of orbits as embedded submanifolds of M.

First, we will define certain submanifolds of M which are the so-called *slices* associated to the group action of a Lie group G that acts smoothly on M. A good picture to have in mind is that the slices are 'cross sections of tubes' such that the tube consists of a 'bundle' of orbits.

Definition 2.2.1. Let a Lie group G act smoothly on a manifold M. A *slice* for the group action of G at a point x_0 in M is an embedded submanifold S of M such that

1. $T_{x_0}M = T_{x_0}(G \cdot x_0) \oplus T_{x_0}S$ as a direct sum and where $G \cdot x_0$ is viewed as an (immersed) submanifold of M.

2. For each point x in S we have $T_xM = T_x(G \cdot x) + T_xS$ (not necessarily as a direct sum).

3. S is G_{x_0}-invariant: $G_{x_0} \cdot S \subseteq S$.

4. If $x \in S$, $g \in G$ and $g \cdot x \in S$, then $g \in G_{x_0}$.

(This definition is taken from Definition 2.3.1 in [8].)

The next definition and construction will be used in the formulation of the Slice Theorem.

Definition 2.2.2. Let G be a Lie group and H a Lie subgroup, such that H acts smoothly on a manifold M. In this case, the product manifold $G \times M$ carries a certain H-action: the action map $H \times (G \times M) \to G \times M$ given by $(h, (g, x)) \mapsto (gh^{-1}, hx)$. The orbit space associated to this group action is denoted by $G \times_H M = (G \times M)/H$.

We claim that the action of H on the product manifold $G \times M$ is free. Indeed, if $(g, x) \in G \times M$ and $h \in H$ is such that $(gh^{-1}, hx) = (g, x)$, then $gh^{-1} = g$ implies $h = e$. If the action of H on M is assumed to be proper, we may use this, in combination with the fact that the H-action is free, to apply the Quotient Manifold Theorem (Theorem 2.1.4). Therefore, the orbit space $G \times_H M$ is a manifold. This orbit space can also be equipped with a G-action by $(g', [g, x]) \mapsto [g'g, x]$. It is easy to see that this action is well-defined. Also, this action is smooth by the diagram:

$$\begin{array}{ccc} G \times (G \times M) & \xrightarrow{\quad 1 \quad} & G \times M \\ {\scriptstyle \mathrm{id}_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times (G \times_H M) & \xrightarrow[\quad 2 \quad]{} & G \times_H M, \end{array}$$

where the map 1 is defined as $(g', (g, x)) \mapsto (g'g, x)$ and the map 2 as $(g', [g, x]) \mapsto [g'g, x]$, which is the group action map for this action. By Theorem 2.1.7, we have that map 2 is smooth, so the group action is indeed smooth.

We proceed with a discussion of the proof of the Slice Theorem. Now, one could see the Slice Theorem as showing the existence of a slice at each point of M for a G-action when the group G satisfies some conditions or as showing the existence of a so-called tubular neighbourhood of each point, i.e. a G-invariant open neighbourhood containing the orbit of that point. Yet the second statement is largely a corollary of the existence of a slice. Therefore, if we say the Slice Theorem we will mean the existence of a slice for a G-action.

The steps in the proof of the existence of a slice that we will follow are as follows. These steps are drawn from sections 2.1 to 2.4 in [8].

Let M be a manifold and G a Lie group that acts smoothly and properly on M. Let x_0 be any point of M. First we find an open, G_{x_0}-invariant neighbourhood of x_0 in M and a diffeomorphism mapping this open set onto an open neighbourhood of 0 in the tangent space $T_{x_0}M$. This can be done via Bochner's so-called Linearization

Theorem (Theorem 1 in [2]). Secondly, we also have an induced action of G_{x_0} on $T_{x_0}M$ via $(g, v) \mapsto d(L_g)_{x_0}v$, the so-called *isotropy action*, where $L_g : M \to M$ by $m \mapsto g \cdot m$ and $d(L_g)_{x_0}$ is the differential of this map at x_0. The diffeomorphism from Bochner's Linearization Theorem will also be G_{x_0}-equivariant. Thirdly, using the inverse of this diffeomorphism we can find a slice S at x_0, proving the existence of a slice. Finally, as a corollary, we can construct a G-equivariant diffeomorphism between the orbit space $G \times_{G_{x_0}} V$ as discussed above, for some vector space V on which G_{x_0} acts identically as on the slice S, and a G-invariant open neighbourhood of x_0 (an open neighbourhood of the orbit of x_0), where the G-action on the orbit space $G \times_{G_{x_0}} V$ is as above.

For the proof of Bochner's Linearization Theorem, the Haar integral on a Lie group is needed and as this piece of theory lays beyond the scope of this thesis, the details concerning the use of the Haar integral will not be covered. For more details, in general, on the proof of the Slice Theorem, we refer to [8] and [13].

Theorem 2.2.3 (Bochner's Linearization Theorem)**.** Let a compact Lie group H act smoothly on a manifold M and let $x_0 \in M^H$, i.e. $h \cdot x_0 = x_0$ for each $h \in H$. Then, there exists a H-equivariant diffeomorphism from a neighbourhood of the origin in $T_{x_0}M$ onto an H-invariant neighbourhood of x_0 in M.

Proof. First, we construct a H-invariant open neighbourhood of $x_0 \in M$. We follow the proof of Theorem 2.2.1 in [8]. As we will see, we can also find a H-invariant open neighbourhood contained in every open neighbourhood of x_0.

Indeed, because for every $h \in H$ the functions $H \to M$ by $h \mapsto h \cdot x$ and $M \to M$ by $x \mapsto h \cdot x$ are continuous, for every $x \in M$ and for every open neighbourhood U' of x_0, there exist open neighbourhoods in H for each $h \in H$, say $W(h)$ and $U(h)$ in U', such that the image of $W(h) \times U(h)$ under the H-action is in U'. This can be done by taking the inverse image of U' for both maps. Since H is compact, there is a finite subset of $\{W(h)\}_{h \in H}$, $\{W(h_i)\}_i$, such that $\bigcup_i W(h_i) = H$. Now, $U'' = \bigcap_i U(h_i)$ is an open neighbourhood of x_0 in M, and defining U as the image of $H \times U''$ under the group action of H on M gives a H-invariant neighbourhood of x_0 in M and $U \subset U'$.

Let $F : \tilde{U} \to T_{x_0}M$ be a smooth map from some open neighbourhood $\tilde{U}$ of x_0 in M, whose differential at x_0 is the identity map. This can be done using the exponential map from some Riemannian metric on M, assuming that M is Hausdorff and paracompact, allowing the existence of such a Riemannian metric (for details, see for example [22]). Clearly, using the first part of this proof, we can restrict this F to an H-invariant open neighbourhood contained in $\tilde{U}$. This is needed, because we want an H-equivariant map.

Now the action of H on $T_{x_0}M$ is the isotropy action as above, because at x_0 we have that if $L_h : M \to M$ by $x \mapsto h \cdot x$, then the differential of this map maps from $T_{x_0}M$ to $T_{x_0}M$.

Then, this F can be made H-equivariant by averaging over the group using the Haar integral. Let $\tilde{F}$ be this new H-equivariant function. This $\tilde{F}$ is smooth, H-equivariant and $d\tilde{F}_m = \mathrm{id}_{T_{x_0}\tilde{U}}$. (For these details we refer to sections 2.2 and 4.2 in [8].)

By the implicit function theorem we can invert $\tilde{F}$ on a neighbourhood of x_0 to obtain the desired H-equivariant diffeomorphism. ☐

The Riemannian metric used in the proof of Theorem 2.2.3, can be adjusted, using, again, averaging over the group via the Haar integral, to make it into an H-invariant Riemannian metric on the tangent space $T_{x_0}M$ at x_0. As a consequence, the group $\{d(L_h)_{x_0} : h \in H\}$ of the isotropy action is a compact, thus closed Lie subgroup of the orthogonal group of $T_{x_0}M$ (see the comment after Theorem 2.2.1 in [8]).

Next, we show the existence of a slice at x_0 for the G-action.

Theorem 2.2.4 (Slice Theorem; existence of a slice)**.** Let M be a manifold and G a Lie group acting smoothly and properly on M. Then, for any point x_0 in M, there exists a slice at x_0 for the G-action.

Proof. Since the action is proper, we see that the isotropy group G_{x_0} is compact, because it is the image of the projection onto G of the preimage $\varphi^{-1}((x_0, x_0))$, where $\varphi : G \times M \to M \times M$ is the map $(g, m) \mapsto (g \cdot m, m)$. Therefore, we can apply Theorem 2.2.3 and the comments after the proof to get a diffeomorphism $F : U \to T_{x_0}M$, where U is some G_{x_0}-invariant neighbourhood of x_0 in M and the image $F(U)$ is an open ball B_ε of radius $\varepsilon > 0$ with center $0 \in T_{x_0}M$. Moreover, F is G_{x_0}-equivariant for the G_{x_0}-action on U and the isotropy action of G_{x_0} on $T_{x_0}M$. The map $\theta^{(x_0)} : G \to M$ by $g \mapsto g \cdot x_0$ induces a map $\alpha_{x_0} : \mathfrak{g} \to T_{x_0}M$, where $\mathfrak{g}$ is the Lie algebra of G, T_eG, by the differential $d(\theta^{(x_0)})_e$ of $\theta^{(x_0)}$ at the identity element e of G. Because of this map, which maps onto the orbit tangent space $T_{x_0}(G\cdot x_0)$, we get that $T_{x_0}(G\cdot x_0)$ is invariant under the isotropy action of G_{x_0}, i.e. $G_{x_0} \cdot T_{x_0}(G \cdot x_0) \subseteq T_{x_0}(G \cdot x_0)$. We refer to the proof of Theorem 2.3.3 in [8] for details on this point.

The orthogonal complement with respect to the G_{x_0}-invariant metric on $T_{x_0}M$, as discussed in the comment after the proof of Theorem 2.2.3, of this space, $(T_{x_0}(G\cdot x_0))^\perp$, is also G_{x_0}-invariant. Therefore, the submanifold $S_\varepsilon = F^{-1}((T_{x_0}(G \cdot x_0))^\perp \cap B_\varepsilon)$ is also a G_{x_0}-invariant submanifold of M, because of the G_{x_0}-equivariance of F. This proves that S_ε satisfies condition 3 for a slice. For condition 1 for a slice, we note that $T_{x_0}S_\varepsilon = (T_{x_0}(G \cdot x_0))^\perp$, and thus $T_{x_0}M = T_{x_0}(G \cdot x_0) \oplus T_{x_0}S_\varepsilon$.

For conditions 2 and 4 for a slice, we refer to the proof of Theorem 2.3.3 in [8]. □

Next, we show that for each point x_0 in M there is an open neighbourhood of the orbit of x_0 which is diffeomorphic to an orbit space $G \times_{G_{x_0}} V$ for some manifold V, as in Definition 2.2.2. This is the so-called local model for each orbit of each point x_0 in M.

Theorem 2.2.5. Let G be a Lie group acting on a manifold M, such that the action is smooth and proper. Then, there is a vector space V on which G_{x_0} acts, such that V can be identified with the isotropy action of G_{x_0} on $(T_{x_0}(G \cdot x_0))^\perp$ and such that there is a G-equivariant smooth embedding

$$\Phi : G \times_{G_{x_0}} V \to M$$

onto an open G-invariant neighbourhood of x_0.

Proof. We can take $V = (T_{x_0}(G\cdot x_0))^\perp$ with the G_{x_0}-action isotropy action and construct the orbit space $G \times_{G_{x_0}} V$ with the G-action $G \times (G \times_{G_{x_0}} V) \to G \times_{G_{x_0}} V$ by $(g, [g', v]) \mapsto [gg', v]$. Using the map $F : U \to T_{x_0}M$ of the proof of Theorem 2.2.4, we extend the image of U, an open ball B_ε, to the entirety of $T_{x_0}M$ by the smooth map $B_\varepsilon \to T_{x_0}M$ with $v \mapsto \tan(\frac{\pi}{2\varepsilon^2}\|v\|^2)v$, where $\| - \|$ is the norm associated to the G_{x_0}-invariant Riemannian metric. Then, F is still a G_{x_0}-equivariant diffeomorphism, because $\| - \|$ is G_{x_0}-invariant. Hence, we can define $\Phi : G \times_{x_0} V \to M$ by $[g, v] \mapsto gF^{-1}(v)$. This map is well-defined because if $(gh^{-1}, hv) = (g, v)$ for some $h \in G_{x_0}$, using the G_{x_0}-equivariance of F, we get $gh^{-1}F^{-1}(hv) = gF^{-1}(v)$. This map is also smooth because of Theorem 2.1.7 and the fact that the quotient map $G \times V \to G \times_{G_{x_0}} V$ is a smooth submersion. Moreover, G-equivariance is also immediate by construction. Finally, we claim that Φ is also a diffeomorphism for appropriate $\varepsilon > 0$. For details on this last point, we refer to the proof of Theorem 2.4.1 in [8]. □

2.3 Strata

As already mentioned, the Slice Theorem provides us with the tools that the partitioning of M into its strata is a partitioning into separate embedded submanifolds. The next theorem gives that each (H)-stratum indeed is an embedded submanifold of M. This allows us to restrict smooth functions to each (H)-stratum $M_{(H)}$, if the original domain is M.

For this proof, we will make use of the following lemma.

Lemma 2.3.1. Suppose X_1 can be topologically embedded into X_2 by $\iota : X_1 \to X_2$, i.e. ι is a homeomorphism onto the image equipped with the subspace topology inherited from X_2, and let $\pi_1 : X_1 \to Y_1$ and $\pi_2 : X_2 \to Y_2$ be two surjective continuous open maps. Then, let $f : Y_1 \to Y_2$ be a continuous injection onto $\pi_2(\iota(X_1))$, such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \overset{\iota}{\hookrightarrow} & X_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ Y_1 & \underset{f}{\longrightarrow} & Y_2. \end{array}$$

Then, f is a topological embedding as well.

Proof. For the well-defined inverse $f^{-1} : \pi_2(\iota(X_1)) \to Y_1$, where $\pi_2(\iota(X_1))$ is equipped with the subspace topology inherited from Y_2, we have the commutative diagram:

$$\begin{array}{ccc} \iota(X_1) & \overset{\iota^{-1}}{\longrightarrow} & X_1 \\ {\scriptstyle\pi_2|_{\iota(X_1)}}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ \pi_2(\iota(X_1)) & \underset{f^{-1}}{\longrightarrow} & Y_1. \end{array}$$

Here, we write $\pi_2|_{\iota(X_1)}$ for π_2 restricted to the subspace $\iota(X_1)$. As we have $f^{-1} \circ \pi_2|_{\iota(X_1)} = \pi_1 \circ \iota^{-1}$, and since $\pi_2|_{\iota(X_1)} : \iota(X_1) \to \pi_2(\iota(X_1))$ is a quotient map, f^{-1} is continuous. For this, we use the universal property of the quotient topology for $\pi_2|_{\iota(X_1)}$ (for a proof, see for example Theorem 9.4 in [36]). Clearly, $f \circ f^{-1} = \mathrm{id}_{\pi_2(\iota(X_1))}$ and $f^{-1} \circ f = \mathrm{id}_{X_1}$. Therefore, f is indeed a topological embedding. □

Theorem 2.3.2. Let G be a Lie group that acts smoothly and properly on a manifold M. Then, the (H)-stratum $M_{(H)}$ is an embedded submanifold of M.

Proof. Any subspace $A \subseteq M$ is an embedded submanifold of M if and only if each point $m \in A$ has an open neighbourhood U such that $A \cap U$ is an embedded submanifold of U. Fix a Lie subgroup H of G. By Theorem 2.2.5, we can take an open neighbourhood lying in M of a point m that lies in $M_{(H)}$ to be diffeomorphic to $U = G \times_H V$, via a G-equivariant diffeomorphism, for an appropriate V. Here, we may assume that $H = G_m$. We calculate the isotropy group of a point $[g, v]$ in $G \times_H V$. We have $g'[g, v] = [g, v] \iff (g'g, v) = (gh^{-1}, hv)$ for some $h \in H$. Clearly, this h must lie in the isotropy group of v under the isotropy group action of H on V: $h \in H_v$. Therefore, g' being an element of the isotropy group of $[g, v]$ under the G-action is equivalent to $g' \in gH_vg^{-1}$. Hence, if this isotropy group is to be conjugate with H (we are looking at $U_{(H)}$), we see that H_v has to be conjugate with H, but this implies $H = H_v$ and thus v is an element of the fixed point set of V under H: $v \in V^H$. The latter is a consequence of the following. We know that for the isotropy group of $[e, v]$ under the G-action we have $G_{[e,v]} = H_v$, as $[e, v] = [h^{-1}, hv]$ for any $h \in H$, and since Φ from Theorem 2.2.5 maps

$[e, v]$ G-equivariantly to m and $G_m = H$, we must have $H \subseteq H_v$. Of course, $H_v \subseteq H$ is always true. Hence, we may conclude that $U_{(H)}$ is equal to $G \times_H V^H$. Now, V^H is a linear subspace of V and hence a submanifold of V. The inclusion $G \times_H V^H \hookrightarrow G \times_H V$ is a smooth embedding. This is because, first, by Theorem 2.1.7, the inclusion is smooth and it can be seen to be injective and of constant rank. So indeed $G \times_H V^H \to G \times_H V$ is a smooth immersion. Secondly, this inclusion is also a topological embedding which can be seen by applying Lemma 2.3.1 to the following diagram.

$$\begin{array}{ccc} G \times V^H & \hookrightarrow & G \times V \\ \downarrow & & \downarrow \\ G \times_H V^H & \hookrightarrow & G \times_H V. \end{array}$$

Taking this all into account, we have proven that $U_{(H)}$ is a submanifold of U, and therefore, $M_{(H)}$ is a submanifold of M. □

See also Lemma 2.6.4 (ii) in [8].

We illustrate the partitioning of a manifold into its strata with an example.

Example 2.3.3. Consider the action of $G = U(1)$ on $M = \mathbb{R}^3$ by rotation in the xy-plane. See also the Figure 2.1 below. Then, the points on the z-axis have the entire group as their isotropy group and the points in M with the z-axis excluded are only fixed by the identity element of G. Therefore, we have two strata, $M_{(G)}$ and $M_{\{e\}}$. The orbit space of this action is a manifold with boundary diffeomorphic to $\mathbb{R} \times [0, \infty)$.

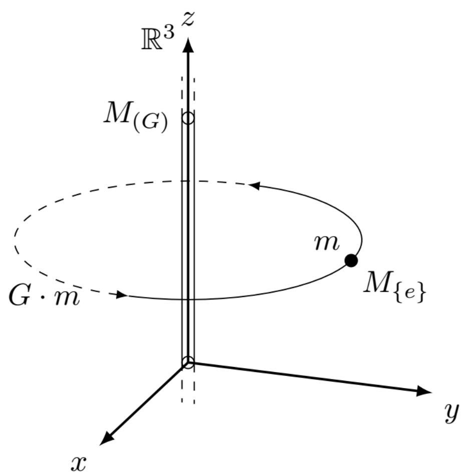


Figure 2.1: The strata for the action of $U(1)$ on the xy-plane of $\mathbb{R}^3$.

We are interested in the orbit spaces where the action is quotiented out, and while it happens so that the orbit space M/G is not always a smooth manifold on its own, we are still able to have a smooth structure on the orbit spaces of individual strata, $M_{(H)}/G$. This works, loosely speaking, because the action of G on such a (H)-stratum is 'the same' for every point in the stratum, whereas in the entirety of M we can have points that result in pathologies that impede smoothness (in every direction of the tangent space). For example, we can have so-called corners, such as, in the case, if we replaced $\mathbb{R}^3$ by $[0, \infty)$ for the z-axis in Example 2.3.3, such that the orbit space would be homeomorphic to $[0, \infty) \times [0, \infty)$. (For the definition of a manifold with corners, see for example Chapter 16 in [21].)

Theorem 2.3.4. Let G be a Lie group that acts smoothly and properly on a manifold M. For every Lie subgroup H of G we have that the orbit space $M_{(H)}/G$ can be endowed with a unique smooth structure such that $M_{(H)}/G$ is a manifold and $M_{(H)} \to M_{(H)}/G$ is a smooth submersion.

Proof. Inspired by the proof of Theorem 2.3.2, we show that the submanifold $G \times_H V^H$ of $G \times_H V$ is diffeomorphic to the manifold $(G/H) \times V^H$. Define the map $\phi : G \times_H V^H \to (G/H) \times V^H$ by $[g, v] \mapsto (gH, hv) = (gH, v)$, where the latter equality is because $v \in V^H$. Smoothness of ϕ and its inverse can be shown by the use of the following diagram:

$$\begin{array}{ccc} G \times V^H & & \\ \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ G \times_H V^H & \xrightarrow{\phi} & G/H \times V^H, \end{array}$$

where π_1 and π_2 are quotient maps. Indeed, by Theorem 2.1.7, ϕ is smooth. Similarly, ϕ^{-1} that sends (gH, v) to $[g, v]$ is also smooth. The map ϕ is also a topological homeomorphism by application of Lemma 2.3.1. Furthermore, by construction, the G-action acts on $G \times_H V^H$ only on the left entry. Now, the orbit space $(G \times_H V^H)/G$ can be mapped bijectively onto $((G/H) \times V^H)/G$, and this space is diffeomorphic to V^H, where we use the fact that $(G/H)/G = \{e\}$. Again, smoothness of the bijection of $(G \times_H V^H)/G$ onto V^H and its inverse can be checked by similar arguments as for the smoothness of ϕ. Hence, we have that every neighbourhood of $M_{(H)}/G$ is diffeomorphic to some neighbourhood in V^H, and so $M_{(H)}/G$ has a smooth structure and it is clear that $M_{(H)} \to M_{(H)}/G$ is a smooth submersion. This smooth structure is also unique because if $(M_{(H)}/G)'$ would be the same orbit space but with a different smooth structure, by Theorem 2.1.7, the identity map between $M_{(H)}/G$ and $(M_{(H)}/G)'$ would be a diffeomorphism. □

Proposition 2.3.5. Let G be a Lie group that acts smoothly and properly on a manifold M. Then, the orbit space $M_{(H)}/G$ of each stratum is topologically embedded into the orbit space M/G.

Proof. We apply Lemma 2.3.1 to the diagram:

$$\begin{array}{ccc} M_{(H)} & \xhookrightarrow{\iota} & M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_{(H)}/G & \xrightarrow{\kappa} & M/G, \end{array}$$

where π_1 and π_2 are quotient maps, ι the inclusion of $M_{(H)}$ into M and κ is such that this diagram commutes. As a consequence, κ is a topological embedding. □

For the next section, it is worthwhile to investigate the tangent space of the local model $G \times_H V$. We have a decomposition into the tangent space with respect to the stratum by

$$T_{x_0}M \cong T_{\tilde{x}_0}(G/H \times V^H) \oplus K \cong T_{x_0}(G \cdot x_0) \oplus V^H \oplus K,$$

where the signs '$\cong$' denote linear isomorphisms, $\tilde{x}_0$ denotes the part of x_0 in $(G/H) \times V^H$ and K is the orthogonal complement with respect to the G_{x_0}-invariant Riemannian metric on $T_{x_0}M$ to the stratum tangent space $T_{\tilde{x}_0}(G/H \times V^H)$.

2.4 Morse functions, critical points and critical orbits

This section will look into the critical points of smooth functions $f : M \to \mathbb{R}$, where M is a manifold. Specifically, we will look at functions that are also invariant under a smooth G-action of a Lie group G on the domain M, as mentioned in the introduction. As we know from the previous sections, the orbit space M/G does not necessarily enjoy a natural smooth structure. (See, for example, Example 2.1.19.) However, each (H)-stratum is an embedded submanifold of M and their orbit spaces $M_{(H)}/G$ are natural manifolds. The map $f : M \to \mathbb{R}$ induces a continuous map $\tilde{f} : M/G \to \mathbb{R}$ by passing through the orbit space of M under the G-action, which is, at least, a natural topological space. Note that $\tilde{f}$ is well-defined, because f is assumed to be G-invariant. By Proposition 2.3.5, we can restrict the induced function $\tilde{f}$ to each $M_{(H)}/G$. We will prove in this section that these restricted functions are smooth. Lastly, we will show how the critical points of $f : M \to \mathbb{R}$ can be characterised in terms of the critical points of each $\tilde{f}_{(H)} : M_{(H)}/G \to \mathbb{R}$ for each Lie subgroup H of G.

First, we recall the definition of critical points and of non-degenerate critical points for smooth maps of the type $f : M \to \mathbb{R}$.

Definition 2.4.1. Suppose $F : M \to N$ is a smooth map between manifolds, then we say that a point p in M is a *critical point* of F if dF_p is not surjective. For a smooth function of the type $f : M \to \mathbb{R}$ this means that some coordinate representation $\hat{f}$ of f has gradient equal to zero in the point which is the coordinate representation of p. We write $\mathrm{Cr}(f)$ for the set of critical points of f.

We can define the Hessian of a smooth function $f : M \to \mathbb{R}$ by looking at the Lie derivatives with respect to vector fields that run over the tangent space in a point p in M.

Definition 2.4.2. Suppose that M is a manifold and $f : M \to \mathbb{R}$ is a smooth function. Then, we define the *Hessian* of f at a point $p \in M$ to be the map $H_{f,p} : T_pM \times T_pM \to \mathbb{R}$ by $(X_p, Y_p) \mapsto (XYf)(p)$, where $X, Y \in \mathfrak{X}(M)$ are smooth vector fields over M, such that $X_p, Y_p \in T_pM$.

A critical point $x \in \mathrm{Cr}(f)$ is said to be *non-degenerate* if $\forall Y_p \in T_pM : H_{f,p}(X_p, Y_p) = 0 \iff X_p = 0$.

The definition of the Hessian is independent of the choice of vector fields. (For a proof, see for example Lemma 1.6. in [27].) Moreover, the Hessian can be represented by the bilinear form induced by the Hessian matrix of a coordinate representation $\hat{f}$ of f.

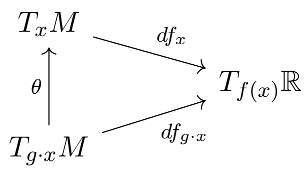
Definition 2.4.3. Let G be a Lie group that acts smoothly on a manifold M. A point $p \in M$ is *G-non-degenerate* if $\forall Y_p \in T_pM : H_{f,p}(X_p, Y_p) = 0 \iff X_p \in T_p(G \cdot p)$.

This definition is related to the similar definition for so-called Morse-Bott functions. We refer to section 2.6 in [27] for more details on Morse-Bott functions.

The following proposition says that each point on the orbit of a critical point is critical. This proposition can also be read as 'critical points of a G-invariant smooth function come in orbits'.

Proposition 2.4.4. Let M be a manifold, G a Lie group that acts smoothly on M and $f : M \to \mathbb{R}$ a smooth map, that is G-invariant. Then, $\mathrm{Cr}(f)$ is G-invariant, i.e. $G \cdot \mathrm{Cr}(f) \subseteq \mathrm{Cr}(f)$.

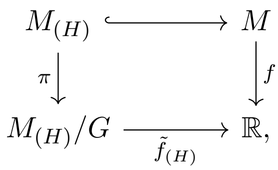
Proof. Suppose x is a critical point in $\mathrm{Cr}(f)$. Then, df_x is not surjective, and $df_{g\cdot x} : T_{g\cdot x}M \to (T_{f(g\cdot x)}\mathbb{R} = T_{f(x)}\mathbb{R})$. We consider the following diagram. The existence of a map θ such that this diagram commutes would show that $\mathrm{Cr}(f)$ is G-invariant.



We construct θ as follows. An element $v \in T_{g\cdot x}M$ is a derivation $v : C^\infty(M) \to \mathbb{R}$ at the point $g\cdot x$ in M. For such a v, if $h, k \in C^\infty(M)$, we have $v(hk) = h(g\cdot x)v(k)+k(g\cdot x)v(h)$, because it is a derivation at $g \cdot x$. To turn this into a derivation at x, i.e. mapping v to some element in T_xM, we can define $\theta : T_{g\cdot x}M \to T_xM$ with $\theta(v)$ such that $\theta(v)(h) = v(h \circ L_{g^{-1}})$, with $L_{g^{-1}} : M \to M$ such that $x \mapsto g^{-1} \cdot x$. Clearly, $h \circ L_{g^{-1}}$ lies in $C^\infty(M)$, since the G-action is smooth. Also, it is clear that $\theta(v)$ is a derivation at the point x. Therefore, for any $v \in T_{g\cdot x}M$ and for any $w \in C^\infty(\mathbb{R})$ we get $df_x(\theta(v))(w) = \theta(v)(w \circ f) = v(w \circ f \circ L_g) = v(w \circ f) = df_{g\cdot x}(v)(w)$. This shows that $df_{g\cdot x} = df_x \circ \theta$, so the diagram commutes and $\mathrm{Cr}(f)$ is G-invariant. □

Proposition 2.4.5. Let G be a Lie group that acts smoothly and properly on a manifold M. If $f : M \to \mathbb{R}$ is a G-invariant smooth function, then, the induced maps $\tilde{f}_{(H)} : M_{(H)}/G \to \mathbb{R}$ for each Lie subgroup H that send the equivalence class of x in $M_{(H)}$ in $M_{(H)}/G$ (its orbit) to $f(x)$, i.e. $[x] \mapsto f(x)$, are smooth.

Proof. Consider the following diagram,



where π is the quotient map. Note that $\tilde{f}_{(H)}$ is well-defined because of G-invariance. Applying Theorem 2.1.7 shows that $\tilde{f}_{(H)}$ is smooth. □

Now, it could be that the critical orbits are dense in the domain M, such that $\mathrm{Cr}(f)$ will not necessarily be a disjoint union of isolated orbits, in which case the topologies of the orbits do not interfere. In this case, the set of critical points is not necessarily an embedded submanifold. For example, this is the case for the smooth function $f(x, y) = \exp(-r^{-2})\sin^2(r^{-2})$ if $r > 0$ and $f(0, 0) = 0$, where $r = \sqrt{x^2 + y^2}$ from $\mathbb{R}^2$ to $\mathbb{R}$. See also Figure 2.2. To circumvent these type of situations, we consider some further requirements on the critical points. For this, we need some elementary results about non-degenerate critical points from Morse theory.

Definition 2.4.6. A smooth function $f : M \to \mathbb{R}$ from a manifold M to $\mathbb{R}$ is called a *Morse* function if all the critical points of f are non-degenerate. If, in adddition, G is a Lie group that acts smoothly on M, it is called a *G-Morse-Bott* function if all critical points are G-non-degenerate.

In the theory of Morse theory, a fundamental result is the Morse Lemma.

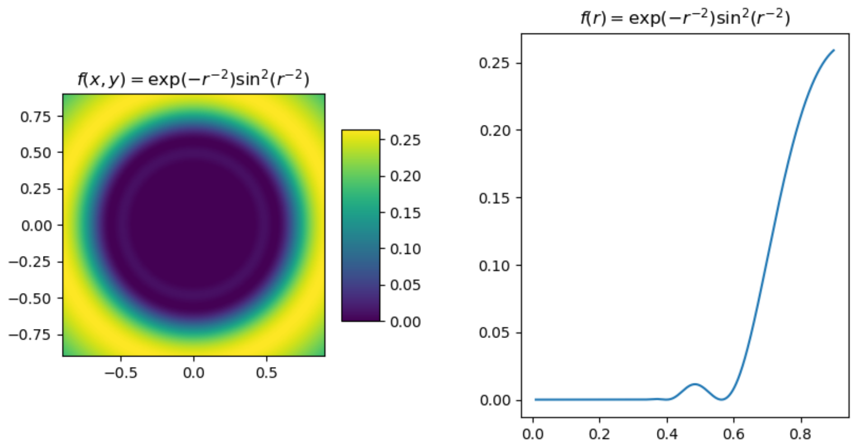


Figure 2.2: Illustrations for the graph of $f(x, y) = \exp(-r^{-2}) \sin^2(r^{-2})$.

Theorem 2.4.7 (Morse Lemma)**.** Let M be a manifold. If a point p in M is a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$, then there exists a chart (U, x) around p, with U an open neighbourhood of x in M and $x : U \to \mathbb{R}^{\dim(M)}$ is a diffeomorphism, such that the individual coordinates satisfy $x^i(p) = 0$ for $1 \leq i \leq \dim(M)$ and such that for the coordinate representation of f, say $\hat{f} = f \circ x^{-1}$, we have:

$$\hat{f}(x^i) = \hat{f}(x(p)) + \sum_i \pm(x^i)^2.$$

For a proof, see for example Corollary 1.1.17 in [27].

The non-degenerate critical points happen to be isolated and are therefore useful in excluding situations such as in the case of the example of Figure 2.2. In that example, the critical point at $r = 0$, which is also an accumulation point of the other critical orbits that lie around the origin, is not non-degenerate.

Proposition 2.4.8. Let M be a manifold. If $f : M \to \mathbb{R}$ is a smooth function, then non-degenerate critical points are isolated.

Proof. This is a direct consequence of the Morse Lemma. □

Proposition 2.4.9. Let M be a manifold. If f $f : M \to \mathbb{R}$ is a smooth Morse function and M is compact, then f has finitely many critical points.

Proof. Suppose not, then there is sequence $(a_n)_{n=1}^{\infty} \subseteq \mathrm{Cr}(f)$, which has a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$, say, converging to $a \in M$. Then a is also a non-degenerate critical point. Indeed, it is critical because the first derivative is a continuous function and it is non-degenerate because we assume that f is Morse. However, a is not an isolated non-degenerate critical point in this case, so we have a contradiction. □

Notice, that as every smooth manifold is locally compact, the critical points of a smooth Morse function f are locally finite.

With regard to Morse functions, we would like to know how $f : M \to \mathbb{R}$ being G-Morse-Bott relates to $\tilde{f}_{(H)} : M_{(H)}/G \to \mathbb{R}$ being Morse.

Proposition 2.4.10. Let G be a Lie group that acts smoothly and properly on a manifold M. If $f : M \to \mathbb{R}$ is smooth and G-invariant and each critical point of f is G-non-degenerate, i.e. f is G-Morse-Bott, then each induced map $\tilde{f}_{(H)}$ is Morse for each Lie subgroup H.

Proof. Let $[p]$ be a critical point of $\tilde{f}_{(H)}$, then $H_{\tilde{f}_{(H)},[p]}(X_{[p]}, Y_{[p]}) = H_{f,p}(X_p, Y_p)$ if $d\pi_p(X_p) = X_{[p]}$ and analogously for Y_p, where $\pi : M_{(H)} \to M_{(H)}/G$ is the quotient map. Therefore, if f is G-non-degenerate, then $\tilde{f}_{(H)}$ is Morse. □

We note that the converse is not necessarily true. Still, $\tilde{f}_{(H)}$ being Morse implies $f_{(H)}$ being G-Morse-Bott, if f is G-invariant, but this implying that f on all of M being G-Morse-Bott is not true because the tangent space of $M_{(H)}$ can have a smaller dimension than M.

We note that if $f : M \to \mathbb{R}$ is a G-invariant smooth function, then, $H_{f,p}(X_p, Y_p) = H_{f,g\cdot p}(X_{g\cdot p}, Y_{g\cdot p})$, where $X_{g\cdot p} = d(L_g)_p(X_p)$ and $Y_{g\cdot p}$ similarly. Therefore, one critical point p being G-non-degenerate implies that each point along p's orbit is critical and G-non-degenerate.

For G-Morse-Bott smooth functions $f : M \to \mathbb{R}$, we have the following analogy for the Morse Lemma.

Lemma 2.4.11 (G-Morse-Bott Lemma)**.** Let G be a Lie group that acts smoothly and properly on a manifold M. If a point p in M is a G-non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$ that is G-invariant and k is the dimension of p's orbit, then there exists a chart (U, x) around p, with U an open neighbourhood of x in M and $x : U \to \mathbb{R}^{\dim(M)}$ a diffeomorphism, such that the individual coordinates satisfy $x^i(p) = 0$ for $1 \le i \le \dim(M)$ and such that for the coordinate representation of f, say $\hat{f} = f \circ x^{-1}$, we have:

$$\hat{f}(x^i) = \hat{f}(x(p)) + \sum_{1\le i\le m-k} \pm(x^i)^2.$$

Proof. The critical orbit of p is an embedded submanifold of M by Proposition 2.1.13. The so-called Morse-Bott Lemma gives such a coordinate representation for general submanifolds consisting of critical points that satisfy a similar G-non-degeneracy condition, i.e. $G \cdot p$ interchanged with a submanifold C in Definition 2.4.3. For a proof of the Morse-Bott Lemma see, for example, [1]. □

It follows, in analogy to non-degenerate critical points being isolated, that critical orbits for which the points are G-non-degenerate are isolated. That is, if f is a G-Morse-Bott function, then, for each critical orbit there is a G-invariant open neighbourhood containing only one critical orbit.

We also have the subsequent theorem, which reduces analysis of the differential of a smooth G-invariant real-valued function on M, $f : M \to \mathbb{R}$, to the differential of f restricted to $M_{(H)}$. It is sometimes called Michel's theorem, originating in L. Michel's 1971 article [24].

First we notice, if $p \in M$, that the differential at p, $df_p : T_pM \to \mathbb{R}$, is invariant under the G_p-action. This is by, if $h \in G_p$, $df_p(h\cdot v) = df_p(d(L_h)_p(v)) = d(f \circ L_h)_p(v) = df_p(v)$. Moreover, the dual with respect to the inner product on T_pM by the G_p-invariant Riemannian metric as used in Theorem 2.2.5 of the differential df_p is an element of T_pM, because $df_p : T_pM \to \mathbb{R}$ is continuous. This is called the gradient of f at p. That is, df_p is an element of the continuous linear dual of T_pM, write $(T_pM)^*$, and so the dual of df_p is an element of T_pM itself.

If we use the following decomposition of the tangent space

$$T_pM = \underbrace{T_p(G\cdot p) \oplus F}_{T_pM_{(H)}} \oplus K,$$

where K is the orthogonal complement with respect to the stratum tangent space and for an appropriate vector space F (see also the comment after Proposition 2.3.5), we have the following proposition.

Theorem 2.4.12 (Michel's Theorem)**.** Let G be a Lie group that acts smoothly and properly on a manifold M and let $f : M \to \mathbb{R}$ be a smooth G-invariant function. The dual of the differential df_p, with respect to the G_p-invariant Riemannian metric, as used in Theorem 2.2.5, the gradient at p, lies in the vector subspace F.

Proof. We have that $F \cong V^H$ by a linear isomorphism as in Theorem 2.3.2 and the comment after Proposition 2.3.5. Here, the group H is the isotropy group of p, G_p. Since the vector space dual of $(T_pM)^*$ is G_p-equivariantly isomorphic to T_pM and as df_p is a fixed vector of G_p in $(T_pM)^*$, the dual is a fixed vector of T_pM and because the differential already vanishes along the orbit tangent space, indeed, the dual must lie in F. □

Using Michel's theorem and G-Morse-Bott functions, we can derive the main result of this chapter. For notation, let $\mathrm{Cr}(f)/G$ be the set of 'critical' orbits, not necessarily equipped with any topology.

Theorem 2.4.13. Let G be a Lie group that acts smoothly and properly on a manifold M. Let $f : M \to \mathbb{R}$ be a smooth G-invariant function. If f is G-Morse-Bott, then

$$\mathrm{Cr}(f) = \bigcup_{x\in \mathrm{Cr}(f)/G} G\cdot x \cong \bigcup_{x\in \mathrm{Cr}(f)/G} G/G_x = \bigcup_{(H)\leq(G)} \bigcup_{x\in \mathrm{Cr}(\tilde{f}_{(H)})} G/H,$$

where each critical orbit $G \cdot x$ is an embedded submanifold of M, $\mathrm{Cr}(f)$ is an embedded submanifold of M as a disjoint union of isolated critical orbits, $\cong$ denotes a diffeomorphism and $(H) \leq (G)$ denotes the ranging over all conjugate equivalence classes of subgroups of G.

Proof. The first equality is a direct consequence of Proposition 2.4.4. Each critical orbit is an embedded submanifold of M, by Proposition 2.1.13. If the critical orbits are isolated, the disjoint union of the critical orbits is still an embedded submanifold of M. Since we assume f is G-Morse-Bott, the critical orbits are isolated in M. Lastly, for the diffeomorphism, we use Corollary 2.1.11. □

2.5 Applications

In this section we discuss the potential of section 1.2.3 and a similar potential but on the configuration space S^3 with a different group action to illustrate Theorem 2.4.13.

Example 2.5.1. Let $M = \mathbb{C}^2$ and $G = U(1) \times SU(2)$. Then, as G is a product of two compact Lie groups G is a compact Lie group. For this G, we have that every element g in G is of the form:

$$g = \left(e^{ix}, \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right), \; x \in \mathbb{R}, \; \alpha, \beta \in \mathbb{C} : |\alpha|^2 + |\beta|^2 = 1.$$

We define the action $G \times M \to M$ by $(g, m) \mapsto gm$, by matrix-vector multiplication:

$$gm = e^{ix} \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \; z_1, z_2 \in \mathbb{C}.$$

We note that this is a smooth action and, as G is compact, it is also a proper action by Proposition 2.1.15. We define the norm on $\mathbb{C}^2$ by $\|z\| = \sqrt{|z_1|^2 + |z_2|^2}$, if $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then, $\|gz\| = \|z\|$ for any $g \in U(1) \times SU(2)$, so the action is norm-preserving.

Inspecting the orbits of any $(z_1, z_2) \in \mathbb{C}^2$ excluding the origin, we find that they come from some $(r, 0)$ with $r \in \mathbb{R}$:

$$\begin{aligned}
&\text{if } e^{ix}(\alpha r) = z_1 \text{ and} \\
&e^{ix}(\beta r) = z_2, \text{ then} \\
&\alpha = \sqrt{1 - \frac{|z_2|^2}{r^2}},\ e^{ix} = \frac{z_1}{\alpha r},\ \beta = \frac{z_2 \alpha}{z_1}, \\
&|\alpha|^2 + |\beta|^2 = 1 - \frac{|z_2|^2}{r^2} + \frac{|z_2|^2}{|z_1|^2}\left(1 - \frac{|z_2|^2}{r^2}\right) = 1.
\end{aligned}$$

Therefore, the orbit of any $z \in \mathbb{C}^2$ which is not the origin is the set $\{w \in \mathbb{C}^2 : \|w\| = r\}$ for some $r > 0$. We see that this subspace is diffeomorphic to the 3-sphere S^3.

We compute the isotropy group for $(r, 0)$ with $r \in \mathbb{R}_{>0}$:

$$\begin{aligned}
e^{ix}\alpha r = r,\ e^{ix}\beta r = 0 &\iff \\
&\beta = 0,\ \alpha = e^{-ix}.
\end{aligned}$$

So the isotropy group of $(r, 0)$ is the set of matrices for $x \in \mathbb{R}$:

$$\left(e^{ix}, \begin{pmatrix} e^{-ix} & 0 \\ 0 & e^{ix} \end{pmatrix}\right) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{ix} \end{pmatrix} \cong U(1),$$

where $\cong$ denotes a Lie group isomorphism.

Note that for $(0, 0)$ the isotropy group is the entire group. Furthermore, for the partitioning of M into its strata we get $\{\{0\}, \mathbb{C}^2 \setminus \{0\}\}$. We see that $(\mathbb{C}^2 \setminus \{0\})/G \cong \mathbb{R}_{>0}$ as a diffeomorphism.

If we define a potential on $\mathbb{C}^2$ as $z = (z_1, z_2) \mapsto -\|z\|^2 + \|z\|^4$, then this potential has critical points in $z = 0$ and all z with $\|z\| = \frac{1}{2}\sqrt{2}$, which is a submanifold of M diffeomorphic to S^3.

We can consider $\tilde{f}_{(H)} : \mathbb{R}_{>0} \to \mathbb{R}$ by $r \mapsto -r^2 + r^4$, for $H = U(1)$, and this is a Morse function with one critical point at $r = \frac{1}{2}\sqrt{2}$.

We show that $G/G_x \cong S^3$ as a diffeomorphism. We do this by identifying the equivalence classes (cosets) $[(\lambda, u)]$ in G/G_x for $\lambda \in U(1)$ and $u \in SU(2)$ with the matrix $u\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right)$, which is an element of $SU(2)$. This identification is well-defined and smooth. Smoothness can be shown by applying Theorem 2.1.7 to the diagram:

$$\begin{array}{ccc}
U(1) \times SU(2) & & \\
\downarrow \pi & \searrow^{m} & \\
G/G_x & \xrightarrow{\kappa} & SU(2),
\end{array}$$

where π is the quotient map, m sends (λ, u) to $u\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix}\right)$ and κ is such that $\kappa \circ \pi = m$. Lastly, using $SU(2) \cong S^3$ as a diffeomorphism, we get $G/G_x \cong S^3$ as a diffeomorphism.

Example 2.5.2. Consider the 3-sphere $M = S^3$ parametrized by $(z_1, z_2) \in \mathbb{C}^2$ such that $|z_1|^2 + |z_2|^2 = 1$. Define a $G = U(1) \times U(1)$-action on S^3 by

$$((e^{i\phi}, e^{i\theta}), (z_1, z_2)) \mapsto (e^{i\phi} z_1, e^{i\theta} z_2).$$

If we define the potential $f : S^3 \to \mathbb{R}$ by $(z_1, z_2) \mapsto |z_1|^2$ we get the following potential in Figure 2.3 for $\mathrm{Im}(z_1) = 0$ with the stereographic projection. (The mexican hat on S^3!)

More specifically, if we define the inverse charts

$$\pi_\pm : \mathbb{R}^3 \to S^3 \setminus \{(\pm 1, 0)\}, \ \pi_\pm(x, y, z) = \Bigg(\overbrace{\pm\frac{r^2-1}{r^2+1}, \frac{2x}{r^2+1}}^{z_1}, \underbrace{\frac{2y}{r^2+1}, \frac{2z}{r^2+1}}_{z_2} \Bigg),$$

where $r = \sqrt{x^2 + y^2 + z^2}$, then we have the coordinate representation of f:

$$\hat{f}(x, y, z) = f \circ \pi_\pm(x, y, z) = \frac{(r^2-1)^2 + 4x^2}{(r^2+1)^2}.$$

Now, we can compute the x, y and z partial derivatives of $\hat{f}$ and find the critical points of f, by solving $\nabla \hat{f} = 0$. We see

$$\frac{\partial \hat{f}}{\partial x} = \frac{(r^2+1)^2(2(r^2-1)2x + 8x) - ((r^2-1)^2 + 4x^2)(2(r^2+1)2x)}{(r^2+1)^4}$$

and

$$\frac{\partial \hat{f}}{\partial y} = \frac{(r^2+1)^2(2(r^2-1)2y) - ((r^2-1)^2 + 4x^2)(2(r^2+1)2y)}{(r^2+1)^4},$$

where the z partial derivative is analogous to the y partial derivative. Setting $x = 0$, or $\mathrm{Im}(z_1) = 0$, and $r^2 = y^2 + z^2 = 1$, such that $z_1 = 0$, we get $\nabla \hat{f} = 0$. Also, we have $y = z = 0, x = \pm 1$ as critical points, which by G-invariance imply that $(e^{i\phi}, 0)$ for each $\phi \in \mathbb{R}$ is a critical point. This verifies the critical points $(0, e^{i\theta}) \in S^3$ and $(e^{i\phi}, 0) \in S^3$ as the critical points of f. Now, we see that $\mathrm{Cr}(f) \cong S^1 \times \{0, 1\}$, where $\{0, 1\}$ has the discrete topology.

We can calculate the isotropy groups of this action in $z = (0, e^{i\theta})$, indeed $G_z = U(1) \times \{e\}$ and so $G/G_z = U(1) \times U(1)/(U(1) \times \{e\}) \cong U(1) \cong S^1$ as diffeomorphisms.

Also, we note that the only possible isotropy groups are of type $U(1)$ or $\{e\}$ and $S^3_{\{e\}} = S^3 \setminus \{(0, e^{i\theta}), (e^{i\phi}, 0)\}_{\theta, \phi \in [0, 2\pi)} \cong S^1 \times S^1 \times (0, 1)$, via

$$(z_1, z_2) \mapsto (z_1/|z_1|, z_2/|z_2|, |z_1|), \ (e^{i\phi}, e^{i\theta}, |z_1|) \mapsto (|z_1| e^{i\phi}, (\sqrt{1 - |z_1|^2}) e^{i\theta}).$$

This is well-defined because $|z_1| = 0, |z_2| = 0$ is excluded. Then $S^3_{\{e\}}/(U(1) \times U(1)) \cong S^1 \times S^1 \times (0, 1)/(U(1) \times U(1)) \cong (0, 1)$. The induced function becomes $\tilde{f}_{\{e\}} : (0, 1) \to \mathbb{R}$ where $t \mapsto t^2$, which indeed has no critical points on its domain.

Alternatively, if we write for $\psi, \varphi \in [0, 2\pi]$ and $\theta \in [0, \pi/2]$ (Hopf coordinates)

$$\begin{aligned} x_0 &= \cos\psi \sin\theta \\ x_1 &= \sin\psi \sin\theta \\ x_2 &= \cos\varphi \cos\theta \\ x_3 &= \sin\varphi \cos\theta, \end{aligned}$$

such that $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$, we have another means of calculating the orbit space. The isolated (critical) orbits correspond to $\theta \in \{0, \pi/2\}$. Then $M_{(H)}/G = (0, \pi/2)$ by $(\varphi, \psi, \theta) \to \theta$ such that we get $\widetilde{f} : (0, \pi/2) \to \mathbb{R}$ by $\theta \mapsto \sin^2 \theta$.

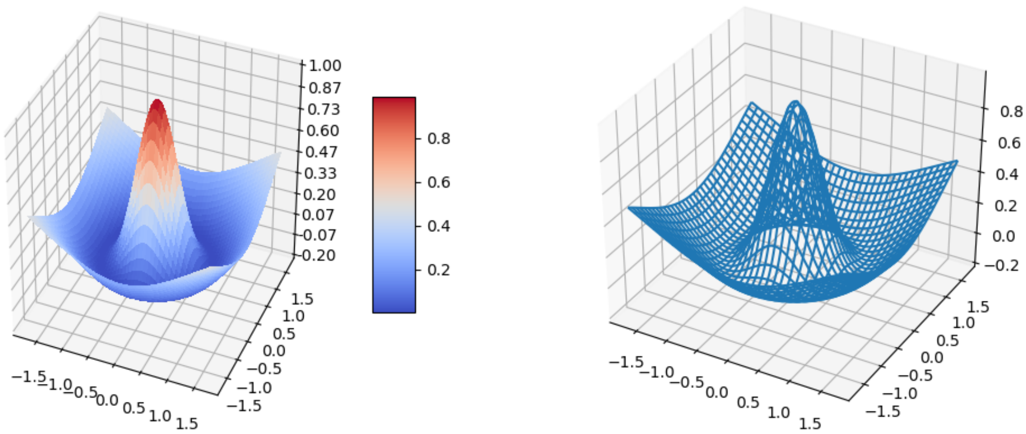


Figure 2.3: The potential $(z_1, z_2) \to\mapsto |z_1|^2$ at $\mathrm{Im}(z_1) = 0$ with stereographic projection.

Chapter 3

Fréchet spaces, Fréchet manifolds and critical fields

Introduction

As explained in section 1.3, this chapter will study two maps: the composition map and integration map. The aim is to study the critical fields for these maps. For this, we need a smooth structure on the set of smooth functions $C^\infty(M, N)$ between two manifolds M and N. This function space will be locally modelled on a topological vector space with some smooth structure. These type of spaces will be Fréchet spaces. Specifically, for each function f in $C^\infty(M, N)$ we will construct an open neighbourhood which will be bijectively mapped onto an open subset of the space of smooth sections of the so-called pullback bundle of f. This space will be denoted by $\Gamma^\infty(M, f^*TN)$, where f^*TN is the pullback bundle, as will be explained in section 3.2.2.

The first section of this chapter, section 3.1, will expound the theory of Fréchet spaces, with a focus on the topology and the calculus of Fréchet spaces. The second section, section 3.2, discusses the definition of a Fréchet manifold and shows that the set of smooth functions $C^\infty(M, N)$ for M compact is a Fréchet manifold. Lastly, section 3.3 shows how the critical fields, if appropriately defined and for M orientable, of the composition map and integration map are the same and gives a consideration on how this applies to the situation where the configuration manifold N is equipped with a smooth and proper G-action.

Fréchet spaces are but one of many types of infinite-dimensional spaces with some smooth structure. Many first constructions could be taken as the starting point of infinite-dimensional calculus. Notably, we have the Euler-Lagrange equation from the 18th century and the Dirichlet principle for PDEs in the 19th century. At the beginning of the 20th century, Fréchet himself constructed some of the first examples of what we now call Fréchet spaces. At the beginning of the second half of the 20th century, Banach manifolds (spaces locally modelled on Banach spaces) were defined. Later, due to some 'limitations' of Banach manifolds (see for example [26] for historical context), interest grew for other categories of infinite-dimensional manifolds. Notably, Hamilton's formulation of the Nash-Moser theorem from 1982 (an inverse function theorem beyond Banach spaces important for existence theorems in the theory of PDEs) uses tame Fréchet spaces [16]. Furthermore, in the 1980s, Kriegl and Michor began studying the so-called convenient setting for infinite-dimensional manifolds. In addition, Schwartz spaces and Sobolev spaces are important examples of infinite-dimensional spaces with a certain type of differentiability and are commonly used in mathematical physics.

3.1 Fréchet spaces

3.1.1 The topology and structure of a Fréchet space

There are multiple equivalent definitions of a Fréchet space. For this section, we mainly follow the parts in [16, 17, 26, 3] that are concerned with Fréchet spaces. We will discuss each part of the definition in detail and will gather all of them in Definition 3.1.7.

As a start, Fréchet spaces are vector spaces and it will turn out that Fréchet spaces are more general than Banach spaces, in the sense that each Banach space is naturally a Fréchet space. A good model to have in mind for defining what a Fréchet space is, is the set $C^\infty([0,1],\mathbb{R})$. Of course, sequences of smooth functions in this set that do not converge pointwise to an element in this set are well known. Therefore, pointwise convergences is not a guarantee for the conservation of smoothness of the limit function.

Example 3.1.1. Consider the set of functions $C^\infty([0,1],\mathbb{R}) = \{f : [0,1] \to \mathbb{R} |$ f is smooth on $(0,1)$ and all k-th left- and right-derivatives at 0 and 1 exist and are finite$\}$, where $[0,1]$ and $\mathbb{R}$ have the usual Euclidean smooth structure and k ranges over $1,2,3,\ldots$. Let $(f_n)_{n=1}^\infty$ be the sequence of smooth functions in this set such that $f_n(x) = x^n$. Then, if $x \in [0,1)$ we have $\lim_{n\to\infty} f_n(x) = 0$ and this limit is 1 for $x = 1$. Clearly, the resulting pointwise limit function is not smooth.

However, if $(f_n)_{n=1}^\infty$ is a different sequence in this set such that we can guarantee that the sequence of each k-th derivative converges uniformly on $[0,1]$ to some continuous function, we can show that the pointwise limit of $(f_n)_{n=1}^\infty$ will be smooth. Specifically, this is done using a family of seminorms, such that for each k-th derivative we have the seminorm $p_k(f) = \sup_{x\in[0,1]} |f^{(k)}(x)|$, where $f^{(k)}$ denotes the k-th derivative.

Let us recall the definition of a seminorm.

Definition 3.1.2. A *seminorm* $\|\cdot\| : X \to \mathbb{R}$ on a real vector space X satisfies the following conditions. (1) For all $v \in X: \ \|v\| \geq 0$; (2) for all $u, v \in X: \ \|v + u\| \leq \|v\| + \|u\|$ and (3) for all $c \in \mathbb{R}: \ \|cv\| = |c|\|v\|$.

The idea of using a *countable* family of seminorms serves as an inspiration for one of the equivalent definitions of a Fréchet space.

Moving on, we note that Fréchet spaces are topological spaces. Therefore, we have to define their topology. More so, Fréchet spaces are *topological vector* spaces, which means that addition and scalar multiplication are continuous as well. That is, the maps

$$+ : X \times X \to X, \ (x,y) \mapsto x + y, \ \cdot : \mathbb{R} \times X \to X, \ (c,x) \mapsto cx$$

are continuous, with respect to the respective product topologies and the Euclidean topology on $\mathbb{R}$. Notably, if a topology $\mathcal{T}$ makes X into a topological vector space, then, the topology $\mathcal{T}$ must be *translation-invariant*. That is, all maps $x \mapsto x + x_0$ for each fixed $x_0 \in X$ are homeomorphisms. As a check, we show that normed vector spaces are naturally topological vector spaces.

First, we recall the definition of topological bases and subbases.

Definition 3.1.3. A *subbase* for a topology on X is a collection of subsets $B_0 = (U_i)_{i\in I}$ that cover X. This topology generated by B_0 is the unique coarsest topology containing B_0, that is guaranteed to exist.

A *base* for a topology on X is a collection of subsets $B = (U_\alpha)_{\alpha \in A}$ of X satisfying: (1) B is a subbase on X and (2) if $U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, then there exists a $U_3 \in B$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Then, the topology that this base generates is the unique coarsest topology containing B, that is guaranteed to exist.

(For more details, see for example section 13 in [25].)

Using these (sub)bases, we can check continuity. That is, if B_Y generates the topology of Y as a (sub)base, a map $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open for each U in B_Y.

Proposition 3.1.4. Let X be a normed vector space with norm $\| - \|$. Then the balls $B(x, \epsilon) := \{y \in X : \|x - y\| < \epsilon\}$ form a base for a topology turning X into a topological vector space.

Proof. Let X be equipped with the topology generated by this base. Then, for any $z \in X$ and $\epsilon > 0$ we have that the inverse image $+^{-1}(B(z, \epsilon))$ is open. To see this, if (x, y) is in this set, setting $\delta = \epsilon - \|x + y - z\| > 0$, we have $(x, y) \in B(x, \delta/2) \times B(y, \delta/2) \subset +^{-1}(B(z, \epsilon))$, so (x, y) lies in an open in this inverse image.

The argument for scalar multiplication is a little bit more convoluted: suppose $(r, x) \in \cdot^{-1}(B(z, \epsilon))$. Let $\delta = \epsilon - \|rx - z\| > 0$. Then, let $\delta' = \frac{\delta}{2r}$ and $\delta'' = \frac{\delta}{2(\delta' + \|x\|)}$. Consequently, $(r, x) \in (r - \delta'', r + \delta'') \times B(x, \delta') \subset \cdot^{-1}(B(z, \epsilon))$, because for $(s, y) \in (r - \delta'', r + \delta'') \times B(x, \delta')$ we have $\|sy - z\| = \|sy - ry + ry - rx + rx - z\| \leq |s - r|\|y\| + |r|\|y - x\| + \|rx - z\|$ and noting that $\|y\| \leq \delta' + \|x\|$ we see that $\|sy - z\|$ is smaller than ϵ. □

Now, the topology of a Fréchet space may be defined by defining the open balls around any center point. That is, if X a vector space and $\{p_k : k \in \mathbb{N}\}$ a countable family of seminorms on X, the sets U, with center $x \in X$, that satisfy *finitely* many inequalities, in the sense that

$$y \in U \iff p_k(x - y) < \epsilon_k, \ \epsilon_k > 0$$

(so for at most finitely many different $k \in \mathbb{N}$), form a base for a topology on X. It is clear that the sets satisfying only one inequality form a subbase for the same topology. Let us call this the *Fréchet topology* of $\{p_k\}_{k \in \mathbb{N}}$, denote it by $\mathcal{T}_P$ and call the base elements P-sets.

Now we want to show that this topology gives a topological vector space. Therefore, we want to check that addition and multiplication are continuous. In fact, this topology is equivalent to the topology generated by the initial topology when one considers the family of maps $(\alpha_k)_{k \in \mathbb{N}}$ such that $\alpha_k : X \to X_k$, where X_k is the quotient space $X/p_k^{-1}(0)$, and $\alpha_k(x) \mapsto [x]$, where $[x]$ denotes the equivalence class $x + p_k^{-1}(0)$, such that each α_k corresponds to the quotient map onto X_k. Note that X_k comes with a norm $\|x + p_k^{-1}(0)\| = p_k(x)$, equipping X_k with a natural vector space topology. First we prove the equivalence between this initial topology and the $\mathcal{T}_P$ topology and then use the so-called universal property of the initial topology to show continuity of addition and scalar multiplication.

Let us recall the definition of an initial topology.

Definition 3.1.5. Let X be a set and $(X_\alpha)_{\alpha \in A}$ a family of topological spaces and let $f_\alpha : X \to X_\alpha$ be functions. Then, there is a unique coarsest topology on X which makes each f_α continuous for every $\alpha \in A$. We call this topology the *initial topology*.

(This topology is also called the weak topology for a set of maps $\{f_\alpha\}_{\alpha\in A}$, see for example section 8 in [36].)

Let $\mathcal{T}_{\text{init}}$ denote the initial topology for the maps $(\alpha_k)_{k\in\mathbb{N}}$ on the space X with its countable family of seminorms. Indeed, we see that the opens of X for $\mathcal{T}_{\text{init}}$ contain the sets $\{x \in X : p_k(x) < \epsilon\}$ for each $\epsilon > 0$ and each $k \in \mathbb{N}$, which are in fact open neighbourhoods of $0 \in X$. Taking finite intersections indeed gives that $\mathcal{T}_P \subseteq \mathcal{T}_{\text{init}}$. Conversely, we see that $\mathcal{T}_P$ makes each α_k continuous, so we also have $\mathcal{T}_{\text{init}} \subseteq \mathcal{T}_P$.

Now we can show that $+ : X \times X \to X$ is continuous by considering $\alpha_k \circ +$ for each $k \in \mathbb{N}$, where $X \times X$ is equipped with the product topology. The universal property of the initial topology says that $+$ is continuous if and only if each $\alpha_k \circ +$ is continuous. (For a proof, see for example Theorem 8.10 in [36].) Therefore, as $(\alpha_k \circ +) = (+_k \circ (\alpha_k \times \alpha_k))$, where $+_k$ is addition in X_k and $\alpha_k \times \alpha_k : X \times X \to X_k \times X_k$ is a combined quotient map, and as $(+_k \circ (\alpha_k \times \alpha_k))$ is continuous, we see that $\alpha_k \circ +$ is continuous for each k. The continuity of scalar multiplication goes similarly.

Moving on, we establish further conditions for the definition of a Fréchet space. Notably, one requires that a Fréchet space X is *Hausdorff*. This can be formulated in terms of the countable family of seminorms. Namely, if a family of seminorms $\{p_k\}_{k\in\mathbb{N}}$ is *separating*, then, the Fréchet topology defined by this family of seminorms is Hausdorff.

A family of seminorms is called *separating* if it satifies $p_k(x) = 0$ for all $k \in \mathbb{N}$ if and only if $x = 0$. Indeed, in this case the Fréchet topology is Hausdorff, because if $x, y \in X$ and $x \neq y$, there is a seminorm, p_k, such that $p_k(x - y) = \epsilon > 0$, such that the sets containg those points z in X that satisfy $p_k(x - z) < \epsilon/2$ and $p_k(y - z) < \epsilon/2$, respectively, are disjoint P-sets. Moreover, it is clear that if the Fréchet topology of a family of seminorms is Hausdorff, then, the family must be separating.

Furthermore, we require a Fréchet space to be *complete with respect to a compatible metric*. That is, a metric is compatible if its metric topology coincides with the Fréchet topology.

Using the countable family of seminorms, we may consider the metric

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x - y)}{1 + p_k(x - y)}.$$

Let us call this the *Fréchet metric* associated to the family of seminorms $\{p_k\}_{k\in\mathbb{N}}$. Note that d is a *translation-invariant* metric, such that $d(x + z, y + z) = d(x, y)$.

We will show that the metric topology generated by this metric is the same as the Fréchet topology of the family of seminorms. Let us recall what a metric and its metric topology are.

Definition 3.1.6. A *metric* is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ (1) $d(x, y) = 0 \iff x = y$, (2) $d(x, y) = d(y, x)$ and (3) $d(x, z) \leq d(x, y) + d(y, z)$. Then, the *metric topology* defined by d is the topology generated by the base of balls $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

So, we want to compare the Fréchet topology $\mathcal{T}_P$ and the metric topology, which we will denote by $\mathcal{T}_d$. First we check whether d, as defined above, is a metric. Indeed, the limit of the sum of d always exists because $\frac{x}{1+x}$ is bounded by 1 and the infinite sum $\sum_{k=1}^{\infty} 2^{-k}$ is finite. For condition (1), if $d(x, y) = 0$, then $p_k(x - y) = 0$ for all $k \in \mathbb{N}$ and so $x = y$, assuming the family of seminorms is separating. Condition (2) is obvious. We note that

$$\frac{p_k(x - z)}{1 + p_k(x - z)} \leq \frac{p_k(x - y) + p_k(y - z)}{1 + p_k(x - y) + p_k(y - z)} \leq \frac{p_k(x - y)}{1 + p_k(x - y)} + \frac{p_k(y - z)}{1 + p_k(y - z)},$$

because $\frac{x}{1+x}$ is an increasing function in x. Therefore, d also satisfies condition (3) for the definition of a metric.

Now, let $P(k,x,\epsilon)$ be one of the subbase elements of the Fréchet topology with center $x \in X$, such that $P(k,x,\epsilon) = \{y \in X : p_k(x-y) < \epsilon\}$. We prove that each point in $P(k,x,\epsilon)$ is contained in a metric ball that is fully contained in $P(k,x,\epsilon)$ itself. Let $y \in P(k,x,\epsilon)$ be arbitrary and define $\epsilon' = \epsilon - p_k(x-y)$. Then, if we set $\epsilon'' = \frac{1}{2^k}\frac{\epsilon'}{1+\epsilon'}$, we get that $B(y,\epsilon'') \subseteq P(k,x,\epsilon)$, because if $y' \in B(y,\epsilon'')$, then $\frac{1}{2^k}\frac{p_k(y-y')}{1+p_k(y-y')} < \epsilon'' \Rightarrow p_k(y-y') < \epsilon' \Rightarrow p_k(x-y') < \epsilon$. Here, we note that, using the Fréchet metric as above, $B(y,\varepsilon'')$ is the set $\{y' \in X : d(y,y') < \epsilon''\}$.

Conversely, let $B(x,\epsilon)$ be a ball in the metric topology and let $y \in B(x,\epsilon)$ be arbitrary. Let n be such that $\sum_{k=n+1}^{\infty} 2^{-k} < \epsilon/2$. Then, if δ is smaller than the minimum of $\epsilon/2 - \sum_{k=1}^{n} \frac{1}{2^k}\frac{p_k(x-y)}{1+p_k(x-y)}$ and $\sum_{k=1}^{n} 2^{-k}$ and if δ' is such that $\sum_{k=1}^{n} \frac{1}{2^k}\frac{\delta'}{1+\delta'} = \delta$. Note that here we need $\delta < \sum_{k=1}^{n} 2^{-k}$. In this case, the intersection $\cap_{k=1}^{n} P(k,y,\delta')$ is an open P-set contained in $B(x,\epsilon)$. So, we may conclude $\mathcal{T}_P = \mathcal{T}_d$.

Finally, we require a Fréchet space to be *complete* with respect to its compatible Fréchet metric. That is, if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to this metric, then there is an element x in X to which $(x_n)_{n=1}^{\infty}$ converges, again with respect to this metric.

Now we have discussed all the conditions for a Fréchet space. We can summarize the definition of a Fréchet space as follows.

Definition 3.1.7. Let X be a vector space equipped with a *countable family of seminorms* $\{p_k\}_{k\in\mathbb{N}}$. Let this family of seminorms be *separating* and let X be equipped with the *Fréchet topology* generated by this family of seminorms. Let

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{p_k(x-y)}{1+p_k(x-y)}$$

be the associated *Fréchet metric* on X. Then X is a *Fréchet space* if X is a *topological vector space* that is *Hausdorff* with respect to this Fréchet topology and the topology is *complete* with respect to the Fréchet metric d.

We note that X is a topological vector space and Hausdorff by construction of the Fréchet topology and the assumption that the seminorms are separating. Therefore, the fine point is whether X is complete with respect to its Fréchet metric.

Next, we show some useful notions that hold for Fréchet spaces, concerning convergence and continuity and conclude with a characterization of the Fréchet topology.

Proposition 3.1.8. Let X be a Fréchet space. Then, a sequence $(x_n)_{n=1}^{\infty}$ converges to a point x in X with respect to the Fréchet topology on X if and only if the sequence converges to x with respect to each seminorm. That is, if $\{p_k\}_{k\in\mathbb{N}}$ is the family of seminorms, that $\lim_{n\to\infty} x_n = x$ with respect to the Fréchet topology if and only if $\lim_{n\to\infty} p_k(x-x_n) = 0$ for each $k \in \mathbb{N}$.

Proof. Suppose first that $(x_n)_{n=1}^{\infty}$ converges to x in the Fréchet topology. Then, for each k and each $\epsilon > 0$ there must exist an $N \in \mathbb{N}$ such that $x_n \in P(k,x,\epsilon)$ for all $n \geq N$. Therefore, $p_k(x-x_n) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$.

Conversely, suppose that $p_k(x-x_n) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$. Then, for each k and $\epsilon > 0$, letting U be a P-set as a finite intersection of sets $P(k_i,x,\epsilon_i)$, we can choose an $N \in \mathbb{N}$ such that $x_n \in U$ for each $n \geq N$. So $x_n \to x$ as $n \to \infty$ with respect to the Fréchet topology. □

Furthermore, as each Fréchet space is first countable, we have the following lemma. First, we recall the definition of a first countable topological space.

Definition 3.1.9. Let X be a topological space. Then X is *first countable* if for each $x \in X$ there exists a countable family of open neighbourhoods of x, $B_x = \{V_n : n \in \mathbb{N}\}$, such that if U is an open neighbourhood of x, there is an $n \in \mathbb{N}$, such that $x \in V_n \subseteq U$.

In the case of Fréchet spaces, we can take the finite intersections of the P-sets $P(k, x, \varepsilon)$ for $\varepsilon \in \mathbb{Q}$ and $\varepsilon > 0$ for such a family. Using this fact, continuity of a function between two Fréchet spaces can be proven using converging sequences.

Lemma 3.1.10. Let $f : X \to Y$ be a continuous map between two Fréchet spaces. If for each converging sequence $(x_n)_{n\in\mathbb{N}}$ with limit $x \in X$ we have that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x) \in Y$, then f is continuous at x.

This is a general characteristic for first countable spaces. For a proof, see for example Theorem 30.1 in [25].

We conclude with a characterization of Fréchet topologies. We note that each separating family of seminorms, independent of its cardinality, defines an initial topology on X, but things like first countability could be lost for cardinalities greater than that of $\mathbb{N}$.

Proposition 3.1.11. Suppose E is a topological vector space defined by a family of seminorms $\{p_i\}_{i\in I}$ with I not necessarily countable. Then the following are equivalent:

- The topology of E can be defined by a countable family of seminorms.
- The topology of E can be defined by a translation-invariant metric.

For a proof, see for example Proposition 25.1 in [23]. The spaces for which the family of seminorms is not necessarily countable are called locally convex spaces. (For more details on locally convex spaces, see for example section 18 in [17].)

3.1.2 Examples of Fréchet spaces

This section is devoted to showing examples of Fréchet spaces. Almost all of our examples are function spaces. Moreover, we are interested in Fréchet spaces that are constructed with the use of manifolds. For this, the following lemma concerning the second countability of manifolds is useful. We note that we assumed manifolds to mean second countable, finite-dimensional smooth manifolds in section 1.3, so this is already included in our use of the word manifold.

Lemma 3.1.12. Every manifold M can be covered by countably many charts (U, φ) such that U is precompact and φ can be smoothly extended on an open V containing U such that (V, φ) is again a chart.

Proof. If $x \in M$ and (U, φ) is a chart around x such that $x \in U$, then there are $\delta' > \delta > 0$ such that $B(\varphi(x), \delta) \subset B(\varphi(x), \delta') \subset \varphi(U)$, where $B(x, \delta) = \{y \in \mathbb{R}^m : \|x - y\| < \delta\}$ are the usual Euclidean balls. Then, redefining U as $\varphi^{-1}(B(\varphi(x), \delta))$, φ can be extended to $\varphi^{-1}(B(\varphi(x), \delta'))$ and this contains the compact closure $\varphi^{-1}(\overline{B(\varphi(x), \delta)})$ of $\varphi^{-1}(B(\varphi(x), \delta))$. Furthermore, assuming that M is second countable, we may assume that there exists a smooth atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that I is countable and each U_i is precompact and φ_i may be smoothly extended to an open set containing the compact closure $\overline{U_i}$ of U_i. ☐

Now we can begin with a list of examples:

(a) As was noted in the beginning, each Banach space is a Fréchet space. This can be seen by taking as the family of seminorms only the norm of the Banach space. Continuity of the vector space operations, separability of the seminorms and completeness are immediate.

(b) Similar to the product of two Banach spaces, the product of two Fréchet spaces is again a Fréchet space. This can be seen as follows. Suppose X_1 and X_2 are two Fréchet spaces defined by the two countable sets of separating seminorms $\{p_k^1\}_{k\in\mathbb{N}}$ and $\{p_k^2\}_{k\in\mathbb{N}}$ respectively. Addition and scalar multiplication are componentwise on the product $X_1 \times X_2$, such that the product is a vector space: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $c \cdot (x_1, x_2) = (c \cdot x_1, c \cdot x_2)$, for $x_1, y_1 \in X_1, x_2, y_2 \in X_2$ and $c \in \mathbb{R}$. Let $(x_1, x_2) \in X_1 \times X_2$ and consider the countable set of seminorms defined as $\hat{p}_k^i(x_1, x_2) = p_k^i(x_i)$ for $i = 1, 2$ and $k \in \mathbb{N}$. We claim that the countable family of seminorms $\{\hat{p}_k^i\}_{i=1,2,k\in\mathbb{N}}$ defines a Fréchet topology that turns $X_1 \times X_2$ into a topological vector space and that the induced Fréchet metric is complete.

(c) The set $C^\infty([0, 1], \mathbb{R})$ as defined in Example 3.1.1 can be made into a Fréchet space by using the family of seminorms defined as $p_k(f) = \sup_{x\in[0,1]} |f^{(k)}(x)|$, for $k \in \{0, 1, 2, \dots\} = \mathbb{N}_0$, where the derivative at 0 and 1 is defined as the left- and right-derivative respectively and we write $f^{(k)}(x) = \frac{d^k f}{dx^k}(x)$ for the k-th derivative. It is clear that the family of seminorms is separating. In Proposition 3.1.13 below we will show that $C^\infty([0, 1], \mathbb{R})$ is complete with respect to its Fréchet metric.

(d) Let $C(\mathbb{R}, \mathbb{R})$ be the set of continuous real-valued functions on the real line. Then, the family of seminorms

$$p_n(f) = \sup_{x\in[-n,n]} |f(x)|,$$

turns $C(\mathbb{R}, \mathbb{R})$ into a Fréchet space.

(e) Let $U \subseteq \mathbb{R}^n$ be an open subset. Then, $C^\infty(U, \mathbb{R})$ is a Fréchet space with a family of seminorms defined as follows. Let $K_n = \{x \in U : \|x\| \le n, \operatorname{dist}(x, U^c) \le \frac{1}{n}\}$, such that $U = \cup_{n=1}^\infty K_n$ (this is an *exhaustion* of U by compact sets). Then, if $f \in C^\infty(U, \mathbb{R})$ and if $k = (k_1, k_2, \dots, k_n)$ with $k_i \in \mathbb{N}_0$ for each $1 \le i \le n$ and $|k| = \sum_{i=1}^n k_i$, we have the seminorms defined as

$$p_{n,k}(f) = \sup_{x\in K_n} |D^k f(x)|, \text{ where } D^k f(x) = \frac{\partial^{|k|} f}{\partial^{k_1}\partial^{k_2}\dots\partial^{k_n}}(x).$$

It is clear that this family of seminorms is separating. The completeness of the induced Fréchet metric is proven with similar arguments as in Proposition 3.1.13 below, as explained in the comments after this proposition. Furthermore, we also claim that this would work for a compact set $K \subseteq \mathbb{R}^n$, where $C^\infty(K, \mathbb{R})$ would be defined as consisting of those functions f such that for each point $x \in K$ there is an open subset of $\mathbb{R}^n$ containing x on which f can be smoothly extended.

(f) The idea of example (e) can be extended to the set $C^\infty(U, \mathbb{R}^m)$, where $U \subseteq \mathbb{R}^n$ again open, by letting

$$p_{n,k}(f) = \sup_{1\le i\le m,\ x\in K_n} |D^k f_i(x)|,$$

where $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$. This countable family of seminorms turns $C^\infty(U, \mathbb{R}^m)$ into a Fréchet space. We note that it is homeomorphic to the product of Fréchet spaces $\prod_{i=1}^m C^\infty(U, \mathbb{R})$.

(g) The Schwartz space $\mathcal{S}(\mathbb{R})$ that one encounters in Fourier theory is a Fréchet space. (For more on Schwartz spaces and Fourier theory, see for example [32].) Specifically, $\mathcal{S}(\mathbb{R})$ is a subset of $C^\infty(\mathbb{R}, \mathbb{C})$ (functions for which the imaginary and real parts are smooth) such that for each element f of $\mathcal{S}(\mathbb{R})$ the supremum

$$\sup_{x\in\mathbb{R}} |x|^n |f^{(k)}(x)|, \ n, k \in \mathbb{N}_0$$

is finite for each $n, k \in \mathbb{N}_0$. Accordinly, the seminorms on $\mathcal{S}(\mathbb{R})$ are indexed by the pairs $(n, k) \in \mathbb{N}_0^2$, such that

$$p_{n,k}(f) = \sup_{x\in\mathbb{R}} |x|^n |f^{(k)}(x)|$$

are the seminorms. The Fréchet topology generated by these seminorms makes $\mathcal{S}(\mathbb{R})$ into a topological vector space and the induced Fréchet metric is complete. For a proof of completeness, see for example Theorem V.9 in [28]. Similarly, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space for any integer d greater than 0.

(h) Let M be a compact manifold with a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. By compactness, we have a finite number of charts $\{(U_i, \varphi_i)\}_{i=1}^N$ that cover M such that $\cup_{i=1}^N U_i = M$. Then, let $\{K_n^i\}_{n\in\mathbb{N}}$ be an exhaustion of $\varphi(U_i)$ as in example (e) for each $1 \leq i \leq N$. Then, the family of seminorms defined by

$$p_{i,n,k}(f) = \sup_{x\in K_n^i} |D^k(f \circ \varphi_i^{-1})(x)|,$$

where $1 \leq i \leq N, n \in \mathbb{N}, k \in \mathbb{N}_0^m$ and $m = \dim(M)$, turns $C^\infty(M, \mathbb{R})$ into a Fréchet space. In the proof of Proposition 3.1.14 we show how the Fréchet topology is independent of the choice of charts and that the Fréchet metric is complete.

(i) The idea of example (h) can also be generalized to general manifolds by the use of their second countability, by using a countable open cover and a countable exhaustion by compact sets for each covering open set. On the other hand, we can invoke Lemma 3.1.12 as an alternative to using the exhaustions of possible unbounded chart codomains. Indeed, let M be a manifold and $\{(U_i, \varphi_i)\}_{i\in I}$ a countable smooth atlas of M, such that, for each $i \in I$, U_i is precompact and φ_i can be extended on an open set containing $\overline{U_i}$. Then, we may note that for any $i \in I$ and any $k \in \mathbb{N}_0^m$ we have

$$\sup_{x\in\varphi_i(U_i)} |D^k(f \circ \varphi_i^{-1})(x)| \leq \sup_{x\in\varphi_i(\overline{U_i})} |D^k(f \circ \varphi_i^{-1})(x)|,$$

and the latter is seen to be finite. Using this, we can define a countable family of seminorms by

$$p_{i,k}(f) = \sup_{x\in\varphi(U_i)} |D^k(f \circ \varphi_i^{-1})(x)|,$$

for $i \in I, k \in \mathbb{N}_0^m$ and $m = \dim(M)$. This family of seminorms turns $C^\infty(M, \mathbb{R})$ into a Fréchet space.

(j) Suppose M is a compact manifold and E is a smooth vector bundle over M, then, the space of smooth sections, denoted by $\Gamma^\infty(M, E)$, can be made into a Fréchet space as follows. Let M be of dimension m, such that there is a smooth vector bundle of rank n, $p : E \to M$. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M such that there are smooth local trivializations, Φ_α, of the vector bundle p,

$$\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n.$$

Using smooth charts of M, written as $(U_\alpha, \varphi_\alpha)$, we can also cover M by an open cover $\{U_\alpha\}_{\alpha\in A}$ such that each U_α is contained in an open subset of M associated to a local trivialization Φ_α. Similar to Lemma 3.1.12, we may assume each U_α to be precompact, such that Φ_α can be extended on an open set containing the closure of $p^{-1}(U_\alpha)$. As M is compact we can cover M by finitely many such precompact open sets, say $\{U_i\}_{i=1}^N$. Using this and writing ψ_i^j for the projection on the j-th coordinate of Φ_i in $\mathbb{R}^n$ we have the seminorms on the set of smooth sections of p defined as

$$p_{i,k}(f) = \sup_{1\le j\le n,\ x\in\varphi_i(U_i)} |D^k(\psi_i^j \circ f \circ \varphi_i^{-1})(x)|,$$

where $1 \le i \le N$ and $k = (k_1, k_2, \ldots, k_m) \in \mathbb{N}_0^m$. This family of seminorms turns the set of sections $\Gamma^\infty(M, E)$ into a Fréchet space. (Furthermore, this construction can be generalized to general manifolds M by the use of second countability, by covering M with countable such open sets as in example (i).)

The following are some elaborating remarks, propositions and lemmas.

As noted in example (c), we show that the Fréchet space in that example is complete.

Proposition 3.1.13. $C^\infty([0,1], \mathbb{R})$ is complete with respect to the metric induced by the family of seminorms defined as $p_k(f) = \sup_{x\in[0,1]} |f^{(k)}(x)|$ for each $k \in \mathbb{N}_0$.

Proof. Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the Fréchet metric. By the general principle of uniform convergence for continuous bounded functions (for a proof, see for example Corollary 14.1.5 in [11]), for each $k \in \mathbb{N}_0$, we have a continuous function g_k on $[0,1]$ such that $f_n^{(k)} \to g_k$ in the supremum norm over $[0,1]$. Out of all the g_k's we want to construct one f such that $p_k(f_n - f) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}_0$. This would mean that $f_n \to f$ as $n \to \infty$ with respect to the Fréchet metric as well (see Proposition 3.1.8).

We first show, for each $k \in \mathbb{N}$ and $x \in [0,1]$, that

$$\int_0^x g_k(t)dt + g_{k-1}(0) = g_{k-1}(x). \tag{3.1}$$

For $k = 1$ we get $f_n \to g_0$ and $f_n^{(1)} \to g_1$, with respect to the seminorms p_0 and p_1 respectively, such that for each $x \in [0,1]$ and each $\varepsilon > 0$

$$\left|\int_0^x g_1 dt + g_0(0) - g(x)\right| \le \left|\int_0^x g_1 dt + g_0(0) - \int_0^x f_n^{(1)} dt - f_n(0)\right| + \\ \underbrace{\left|\int_0^x f_n^{(1)} dt + f_n(0) - f_n(x)\right|}_{=0} + |f_n(x) - g_0(x)| \le \varepsilon,$$

for approriate $n \in \mathbb{N}$. So indeed we have equality in (3.1) and the same reasoning can be applied to $k > 1$. By the validity of (3.1), we also have $g_k(x) = g'_{k-1}(x)$ for each $x \in [0,1]$ and each $k \in \mathbb{N}$. Therefore, by induction, for any $k \in \mathbb{N}_0$, we have $g_k(x) = g_{k-m}^{(m)}(x)$, for $m \le k$. So, letting f be g_0 we indeed have $f_n \to f$ as $n \to \infty$ with respect to the Fréchet topology. ☐

We note that this proof is similar to the proof of Theorem V.9 in [28].

For the function space with an open subset $U \subseteq \mathbb{R}^n$ as its domain as in example (e), one can use the gradient theorem and its converse for each $k \in \mathbb{N}_0^n$:

$$\int_\gamma \nabla D^k f_n(\mathbf{r}) \cdot d\mathbf{r} + D^k f_n(\mathbf{q}) = D^k f_n(\mathbf{p}), \tag{3.2}$$

for a smooth path γ contained in a compact K_n of the exhaustion of U going from $\mathbf{p}$ to $\mathbf{q}$. The converse of the gradient theorem allows us to find that $\nabla g_k = (g_{k+(1,0,\dots,0)}, \dots, g_{k+(0,\dots,0,1)})$, where g_k is the continuous limit of $D^k f_n$ on K_n and $k+(1,0,\dots,0) = (k_1+1, k_2, \dots, k_n)$. That is, by equality in equation (3.2) with $D^k f_n(\mathbf{q})$ and $D^k f_n(\mathbf{p})$ interchanged with $g_k(\mathbf{q})$ and $g_k(\mathbf{p})$, respectively, and $\nabla D^k f_n$ in the integral along γ with $(g_{k+(1,0,\dots,0)}, \dots, g_{k+(0,\dots,0,1)})$. For a of the proof of the converse of the gradient theorem, see for example Theorem 2.1 in [37].

Proposition 3.1.14. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^\infty(M,\mathbb{R})$ equipped with the topology as in example (h). Then, there is a unique $f \in C^\infty(M,\mathbb{R})$, such that $f_n \to f$ as $n \to \infty$ with respect to each $p_{i,k}$ for $1 \leq i \leq N$ and $k \in \mathbb{N}_0^m$. Moreover, the topology on $C^\infty(M,\mathbb{R})$ is independent of the choice of coordinate charts.

Proof. Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the Fréchet metric of example (h). By using similar arguments as in Proposition 3.1.13 and the comments after it for the multi-dimensional case, we get, for each chart index i, a unique g_i such that $f_n \to g_i$ as $n \to \infty$ with respect to $p_{i,k}$ for each $k \in \mathbb{N}_0^m$. Suppose there are i, j such that $U_i \cap U_j \neq \emptyset$, then we want to show that $g_i(x) = g_j(x)$ for each $x \in U_i \cap U_j$. We have that for each $k \in \mathbb{N}_0^m$ and for each $\varepsilon > 0$

$$p_{i,k}(g_i - g_j) \leq p_{i,k}(g_i - f_n) + p_{i,k}(f_n - g_j) \leq \varepsilon,$$

for some appropriate $n \in \mathbb{N}$. Therefore, g_i and g_j coincide on the intersection of their domains. Hence, the Fréchet topology on $C^\infty(M,\mathbb{R})$ is independent of the choice of coordinate charts. □

The following is a lemma, which we will state now for later reference, and which we claim generalizes to $\Gamma^\infty(M,E)$ of example (j) for M compact.

Lemma 3.1.15. Let $K \subseteq \mathbb{R}^n$ be compact and let $C^\infty(K,\mathbb{R}^m)$ be equipped with the Fréchet topology as defined in (e). Then, for each open subset $V \subseteq \mathbb{R}^m$, the set $C^\infty(K,V) = \{g \in C^\infty(K,\mathbb{R}^m) : g(K) \subseteq V)\}$ is open in the Fréchet topology of $C^\infty(K,\mathbb{R}^m)$.

Proof. We show that each function in $C^\infty(K,V)$ is contained in an open subset (with respect to the Fréchet topology on $C^\infty(K,\mathbb{R}^m)$), and that this open subset is wholly contained in $C^\infty(K,V)$. Let $f \in C^\infty(K,V)$. Then, the set $\{g \in C^\infty(K,\mathbb{R}^m) :$ for all $x \in K \ \|f(x) - g(x)\| < \varepsilon\}$ is an open subset (with respect to the Fréchet topology) contained in the set $C^\infty(K,V)$ for an appropriate $\varepsilon > 0$ if it exists. This ε exists, because we can cover $f(K)$ with finitely many open balls $B(x_i, r_i)$ (with respect to the Euclidean metric on $\mathbb{R}^m$) with $x_i \in f(K)$ and $r_i > 0$, such that $f(K) \subseteq \cup_i B(x_i, r_i) \subseteq V$. Then, setting $\varepsilon = \min_i r_i$ gives the desired ε. □

The following is a remark in response to the previous lemma when the source space is not compact.

Remark 3.1.16 (The compact-open topology)**.** Concerning example (e), for $U = \mathbb{R}$, we have that $C^\infty(\mathbb{R},(-1,1))$ is not an open subset of $C^\infty(\mathbb{R},\mathbb{R})$. For if it was, using that the sequence $(x \mapsto \frac{1}{n}x^2)$ (with index $n \in \mathbb{N}$) converges to 0, the sequence elements should be contained in $C^\infty(\mathbb{R},(-1,1))$ for all n greater than some $N \in \mathbb{N}$. But this is clearly not the case.

This is related to the so-called compact-open topology for a space of functions between two topological spaces. In general, if $C^\infty(M,\mathbb{R})$ is the Fréchet space from example (h), we could ask if $C^\infty(M,U)$ is open in the Fréchet topology if U is open. In general, if the source space is compact, the Fréchet topology will coincide with the so-called compact-open topology (with respect to each k-th derivative), such that $C^\infty(M,U)$ is an open subset of $C^\infty(M,\mathbb{R})$. For more details on this matter and the compact-open topology, see for example section 2.3.2 in [35] and section 41 in [18].

3.1.3 Calculus on Fréchet spaces

Having defined the topology of a Fréchet space, we are in a position to study the smooth aspects of Fréchet spaces, such that we can do calculus on Fréchet spaces.

Suppose X and Y are Fréchet spaces, $U \subseteq X$ is an open subset and $F : U \to Y$ is a map. Let $f \in U$ and $h \in X$. Then, the derivative of F *at* f *in the direction of* h is defined by

$$DF(f,h) = \lim_{t\to 0} \frac{F(f+th) - F(f)}{t},$$

if the limit exists. If this limit exists for all $h \in X$, then, we say that F is *differentiable* at f. We say that F is *continuously differentiable* on U if F is differentiable at each $f \in U$ and if $DF : U \times X \to Y$ is continuous for the product topology on $U \times X$. We say that F is of type C^1 if F is also continuous.

The definition of the second derivative naturally follows. If $F : U \to Y$ is of type C^1, then

$$D^2F(f,h,k) = \lim_{t\to 0} \frac{DF(f+tk,h) - DF(f,h)}{t}$$

is the second derivative of F if the limit exists. Again, if $D^2F : U \times X \times X \to Y$ is continuous and F is already of type C^1, then F is of type C^2. This is equivalent to DF being of type C^1. Therefore, we can generalize and define that F is of type C^n if DF is of type C^{n-1}. Accordingly, F is of type C^∞ or (Fréchet) *smooth* if F is of type C^n for each $n \in \mathbb{N}$. Moreover, a function between two Fréchet spaces $F : X \to Y$ is called a (Fréchet) *diffeomorphism* if F is bijective and both F and its inverse F^{-1} are of type C^∞.

With regard to continuous differentiability, if F is continuously differentiable, F itself is also continuous.

Proposition 3.1.17. If F is continuously differentiable on U, then F is continuous on U.

For a proof, see for example Proposition I.2.3 (iii) in [26].

We also have the following propositions, similar to finite-dimensional calculus.

Proposition 3.1.18. $DF : U \times X \to Y$ is $\mathbb{R}$-linear in X. That is, $DF(f) : X \to Y$ by $h \mapsto DF(f,h)$ is $\mathbb{R}$-linear.

For a proof, see for example Proposition I.2.3 (i) in [26].

Proposition 3.1.19. Let X, Y and Z be Fréchet spaces and let $U \subseteq X$ and $V \subseteq Y$ be open. Let $F_1 : U \to V$ and $F_2 : V \to Z$ be continuously differentiable. Then, $F_2 \circ F_1 : U \to Z$ is continuously differentiable and $D(F_2 \circ F_1)(f,h) = DF_2(F_1(f), DF_1(f,h))$.

For a proof, see for example Proposition I.2.3 (vi) in [26], which also generalizes to higher orders of differentiability.

For the Fréchet space of example (c), we also have the following lemma.

Lemma 3.1.20. Let $X = C^\infty([0,1],\mathbb{R})$ be the Fréchet space of example (c) and let $V : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function. Then, the maps

- Composition with a fixed map: $V_* : X \to X$ by $f \mapsto V \circ f$;
- Pointwise multiplication: $X \times X \to X$ by $(f, g) \mapsto f \cdot g$,

are continuous. Furthermore, the composition map is smooth.

Proof. We may use Lemma 3.1.10. For composition, suppose that $(f_n)_{n\in\mathbb{N}}$ is a converging sequence with respect to the Fréchet topology of X, with limit f in X. Then, $p_k(V \circ f_n - V \circ f) \to 0$ if $n \to \infty$ for each $k \in \mathbb{N}_0$. One can use here that for each k the pointwise limit of $(\frac{d^k}{dx^k}(V \circ f_n))_{n\in\mathbb{N}}$ is known (using the chain rule for differentiation), and this pointwise limit is also continuous.

For multiplication, we have that if $(f_n, h_n) \to (f, h)$ as $n \to \infty$ for a sequence $((f_n, h_n))_{n\in\mathbb{N}}$ in X, then, the pointwise limit for each k can be calculated (using the product rule) and is continuous. Therefore, we also have $p_k(f_n \cdot h_n - f \cdot h) \to 0$ if $n \to \infty$.

We only show that the composition map is continuously differentiable (of type C^1), and claim that the argument for higher orders of differentiability is similar. The pointwise limit of the difference quotient $\frac{1}{t}(V_*(f + th) - V_*(f))$ as $t \to 0$ gives $DV_*(f, h)(x) = V'(f(x))h(x)$. This function is also smooth and therefore it is the actual limit. It can be seen that the map $(f, h) \mapsto DV_*(f, h)$ is a chain of maps such that $(f, h) \mapsto (V' \circ f, h) \mapsto (V' \circ f) \cdot h$. As these intermediate maps are either a composition with a fixed map or pointwise multiplication, the map DV_* is continuous. □

We claim that a generalization to the space $\Gamma^\infty(M, V)$ from example (j) of this lemma concerning the composition map is also true. For a proof, see for example 3.1.7, 3.3.3 and 3.6.6 in [16].

Lastly, the following are two examples of how the differential of a map between two Fréchet spaces can give rise to a (partial) differential equation. First we recall (a version of) the fundamental lemma of the calculus of variations.

Lemma 3.1.21 (Fundamental lemma of the calculus of variations)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}$ a continuous function. If for each compactly supported smooth function $h : U \to \mathbb{R}$ we have $\int_U fh = 0$, then, $f(x) = 0$ for each $x \in U$.

For a proof for $n = 2$, to which the proof for general n is similar, see for example, the Lemma on page 22 in section 1.5 of [12].

Example 3.1.22. Consider the map $A : C^\infty([-1,1],\mathbb{R}) \to \mathbb{R}$ by $f \mapsto \int_{-1}^{1} 2\pi f(x)\sqrt{1 + f'(x)^2}dx$. We look for those f such that for each h with $h(\pm 1) = 0$ we have $DA(f, h) = 0$. This reflects finding the minimal (critical) surface area of a bubble of soap between two parallel circular rings. Calculating the differential using differentiation under the integral sign, we get

$$DA(f,h) = \int_{-1}^{1} 2\pi \left[h(x)\sqrt{1 + f'(x)^2} + \frac{f(x)f'(x)h'(x)}{\sqrt{1 + f'(x)^2}} \right] dx.$$

Using $h(\pm 1) = 0$ and integration by parts we get

$$\int_{-1}^{1} 2\pi h(x) \left[\frac{1 + f'(x)^2 - f(x)f''(x)}{(1 + f'(x)^2)^{3/2}} \right] dx.$$

Therefore, by the fundamental lemma of the calculus of variations, on $(-1,1)$, we get the differential equation $ff'' = 1 + f'^2$. (This is 3.1.9 in [16].)

Example 3.1.23 (Plateau's problem)**.** Consider the Fréchet space $X = C^\infty(D, \mathbb{R})$ where $D = \overline{B_1(0)} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ (where $\| - \|$ is the Euclidean norm on $\mathbb{R}^2$) and the map $A : X \to \mathbb{R}$ by $f \mapsto \int_D \sqrt{1 + \nabla f \cdot \nabla f}$. We want to find a PDE for f such that for those h which vanish on the boundary of D, we have $DA(f, h) = 0$. This reflects the idea of finding the minimal surface area for some continuous function $g : \partial D \to \mathbb{R}$, which serves as the boundary value.

We may differentiate after the integral sign to get $DA(f, h) = \int_D \nabla f \cdot \nabla h(1 + \nabla f \cdot \nabla f)^{-1/2}$. Using integration by parts and $h(x) = 0$ for x on the boundary of D, we get $DA(f, h) = -\int_D \nabla \cdot (\nabla f(1+\nabla f \cdot \nabla f)^{-1/2})h = 0$. Therefore, the PDE for f that satisfies $DA(f, h) = 0$ with h as before is $\nabla \cdot (\nabla f(1+\nabla f \cdot \nabla f)^{-1/2}) = 0$ on the interior of D. We note that going from the integral to the PDE follows from the the fundamental lemma of the calculus of variations.

3.2 Fréchet manifolds

3.2.1 Manifold structure, tangent spaces and submanifolds

The notion of a *Fréchet manifold* can be seen as a generalization of the theory of finite-dimensional manifolds to infinite-dimensional manifolds. In other words, a Fréchet manifold is a topological space equipped with a smooth structure modelled on the smooth structure of a Fréchet space. This will allows us to speak of smooth maps between two Fréchet manifolds. Similarly, we also have Banach manifolds modelled on Banach spaces and Hilbert spaces modelled on Hilbert spaces, using the notion of taking derivatives in Banach spaces and Hilbert spaces, respectively. For more details on Banach and Hilbert manifolds, we refer to [20] and Chapter VII A in [3]. For our exposition of the theory of Fréchet manifolds we largely draw from [26], Chapter 3 in [35], section 4 in [16] and Chapter 1 in [30] and for the comparisons with the finite-dimensional case to [21].

Let us recall that coordinate charts are homeomorphisms between open sets, that map into some model space. In the case of finite-dimensional manifolds, this model space is $\mathbb{R}^n$ for some some positive integer n.

Definition 3.2.1. A *Fréchet manifold* $\mathcal{M}$ is a topological space, such that this space is Hausdorff and there exists a set of charts $\varphi_\alpha : U_\alpha \to X_\alpha$, such that each U_α is an open subset of $\mathcal{M}$, the family of all sets U_α cover $\mathcal{M}$, each X_α is a Fréchet space, each φ_α is a homeomorphism and each composition on a non-empty intersection of the domains $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a Fréchet diffeomorphism.

Similar to finite-dimensional manifolds, a set of charts turning a topological space into a Fréchet manifold is called an *atlas* and if each transition function $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth for any two charts φ_α and φ_β it is called a *smooth atlas*. Furthermore, for each smooth atlas there exists a unique *maximal smooth atlas*. (For a proof, see for example Proposition 1.17 in [21] to which the proof is similar.)

Moreover, it is clear that, if $\mathcal{M}$ and $\mathcal{N}$ are two Fréchet manifolds, the product $\mathcal{M} \times \mathcal{N}$ can be equipped with a natural Fréchet manifold structure as follows. If $\{(\varphi_\alpha, U_\alpha)\}_{\alpha \in A}$ is the smooth atlas of $\mathcal{M}$ and $\{(\psi_\beta, V_\beta)\}_{\beta \in B}$ of $\mathcal{N}$, then $\{(\varphi_\alpha \times \psi_\beta, U_\alpha \times V_\beta)\}_{(\alpha,\beta) \in A \times B}$ turns $\mathcal{M} \times \mathcal{N}$ into a Fréchet manifold.

As announced in the introduction of this chapter, the Fréchet manifold with which we are mainly concerned is the set of smooth functions between two (finite-dimensional) manifolds M and N, with the additional assumption that M is compact.

Now, we will discuss the tangent space structure of a Fréchet manifold, similar to the tangent spaces of finite-dimensional manifolds. For this, we will need the notion of a smooth function between two Fréchet manifolds, which will be a carbon copy of smoothness on finite-dimensional manifolds.

Definition 3.2.2. A function $F : \mathcal{M} \to \mathcal{N}$ between two Fréchet manifolds is (Fréchet) *smooth*, if for each $m \in \mathcal{M}$ there are coordinate charts φ on $\mathcal{M}$ and ψ on $\mathcal{N}$ with m and $F(m)$ in their domain, respectively, such that $\psi \circ F \circ \varphi^{-1}$ is Fréchet smooth as a map between two Fréchet spaces.

Moreover, it is not difficult to reconcile this definition for maps between a Fréchet space and a finite-dimensional manifold. For example, it is clear what the smoothness of a path $\gamma : \mathbb{R} \to \mathcal{M}$ means. Using these smooth paths, we can construct the tangent space structure on a Fréchet manifold, similar to one of the equivalent tangent space structures on finite-dimensional manifolds using smooth curves with 'the same velocity' at a point.

Definition 3.2.3. Let $\mathcal{M}$ be a Fréchet manifold and let m be a point in $\mathcal{M}$. Let $S_m\mathcal{M}$ be the set of smooth curves $c : \mathbb{R} \to \mathcal{M}$ such that $c(0) = m$. We define the equivalence relation $c_1 \sim c_2$ on $S_m\mathcal{M}$ if for any chart φ on $\mathcal{M}$ with m in the domain, we have that the coordinate derivative at zero is equal: $\varphi \circ c_1'(0) = \varphi \circ c_2'(0)$. Let $T_m\mathcal{M}$ be the set of equivalence classes under this equivalence relation. Then, $T_m\mathcal{M}$ is the (Fréchet) *tangent space* at m and $T_m\mathcal{M}$ is a Fréchet space.

To show that $T_m\mathcal{M}$ is a Fréchet space, we see that if X_α is the Fréchet space on which $\mathcal{M}$ is modelled at m, that is, $m \in U_\alpha$ and there is a chart $\varphi_\alpha : U_\alpha \to X_\alpha$, we have a bijective map between $T_m\mathcal{M}$ and X_α by a chart the chart φ_α, such that $[c] \mapsto (\varphi_\alpha \circ c)'(0)$, where $[c]$ is an equivalence class in $T_m\mathcal{M}$. The inverse will be $v \mapsto [\varphi_\alpha^{-1}(tv + \varphi_\alpha(m))]$ for $v \in X_\alpha$. We note that even though $tv + \varphi_\alpha(m)$ could lie outside the domain of φ_α^{-1} for some $t \in \mathbb{R}$ this does not make a difference to $(\varphi_\alpha \circ c)'(0)$. Moreover, this bijection is also a linear map. Using this linearity, $T_m\mathcal{M}$ can also be equipped with the topology it inherits from this map onto X_α.

Furthermore, we may define the *tangent bundle* of a Fréchet manifold as the disjoint union of tangent spaces:

$$T\mathcal{M} = \coprod_{m \in \mathcal{M}} T_m\mathcal{M}.$$

This tangent bundle can be made into a Fréchet manifold with charts mapping into the topological product of the model Fréchet space with itself, similar to finite-dimensional tangent bundles (see Chapter 3 in [21]).

Next, using the tangent spaces we may define the differential of a smooth map between two Fréchet manifolds. Again, this will follow the same lines when defining the differential of a smooth map between two finite-dimensional manifolds.

Definition 3.2.4. Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between two Fréchet manifolds and m a point in $\mathcal{M}$. Then, we define the *differential at a point* m as $dF_m : T_m\mathcal{M} \to T_{F(m)}\mathcal{N}$ by $[c] \mapsto [F \circ c]$. The *total differential* will be $dF : T\mathcal{M} \to T\mathcal{N}$, where dF restricted to $T_m\mathcal{M}$ will be dF_m.

We note that if $(\varphi_\alpha, U_\alpha)$ is chart around m and (ψ_β, V_β) a chart around $F(m)$, then the differential of the local representation $\hat{F} = \psi_\beta \circ F \circ \varphi_\alpha^{-1}$ at $\varphi_\alpha(m)$ can be identified with the differential of F at m. To make this precise, we have a commutative diagram:

$$\begin{array}{ccc} T_m\mathcal{M} & \xrightarrow{dF_m} & T_{F(m)}\mathcal{N} \\ {\scriptstyle d(\varphi_\alpha)_m}\updownarrow & & \updownarrow{\scriptstyle d(\psi_\beta)_{F(m)}} \\ X_\alpha & \xrightarrow[d\hat{F}_{\varphi_\alpha(m)}]{} & Y_\beta, \end{array}$$

where we use that $d(\varphi_\alpha)_m$ and $d(\psi_\beta)_{F(m)}$ are Fréchet diffeomorphisms between two Fréchet spaces. Note that $d\hat{F}_{\varphi_\alpha(m)} = D\hat{F}(\varphi_\alpha(m), -)$ as a Fréchet derivative and that if F is smooth the total differential dF is also smooth.

Furthermore, we can also define submanifolds for the Fréchet manifold structure. Although this definition is similar to the definition in the finite-dimensional setting, the equivalence with the existence of a smooth embedding (a topological embedding and smooth immersion) might not necessarily be true in the infinite-dimensional setting.

Definition 3.2.5. Suppose $\mathcal{N}$ is a subset of $\mathcal{M}$. If $\mathcal{N}$ can be covered by a set of chart domains $(U_\alpha, \varphi_\alpha)$ (from the maximal smooth atlas on $\mathcal{M}$) such that $\varphi_\alpha : U_\alpha \to X_\alpha \times Y_\alpha$, and $\varphi_\alpha(\mathcal{N} \cap U_\alpha) = \varphi_\alpha \cap (X_\alpha \times \{0\})$, then $\mathcal{N}$ is a (Fréchet) *submanifold* of $\mathcal{M}$.

It is clear that $\mathcal{M} \times \{y\}$ and $\{x\} \times \mathcal{N}$ are submanifolds of $\mathcal{M} \times \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ are Fréchet manifolds, $\mathcal{M} \times \mathcal{N}$ inherits its natural structure from $\mathcal{M}$ and $\mathcal{N}$ and if $x \in \mathcal{M}, y \in \mathcal{N}$.

3.2.2 The Fréchet manifold $C^\infty(M, N)$

In this section, we show how to construct a smooth structure on the set of smooth functions between two finite-dimensional manifolds. To start, let M and N be two finite-dimensional manifolds, such that M is compact. The model Fréchet spaces at any $f \in C^\infty(M, N)$ will be the space of smooth sections $\Gamma^\infty(M, f^*TN)$, where f^*TN is the pullback bundle of f, which will be defined below. The following construction is a generalization of the construction in [31] for smooth loop spaces and was further inspired by Chapter 1 in [30], section 4 in [16] and section 42 in [18]. For an alternative approach, using a more specific category of so-called tame manifolds, see for example Corollary 3.2.6 and the results leading to it in [35].

The pullback bundle and local additions

First, we will show that the pullback bundle f^*TN is a smooth vector bundle over M. Let $\pi_N : TN \to N$ be the projection of the tangent bundle of N onto N and $\dim(N) = n$. The set f^*TN is defined as

$$f^*TN = \{(m, v) \in M \times TN : f(m) = \pi_N(v)\}.$$

Let $p^* : f^*TN \to M$ be the map $(m, v) \mapsto m$. This will be the vector bundle map for f^*TN over M. Because π_N is a smooth vector bundle over N, there are local trivializations of π_N such that for each $p \in N$ there is an open neighbourhood V of p, such that there is a diffeomorphism

$$\psi : \pi_N^{-1}(V) \to V \times \mathbb{R}^n$$

(remember $\dim(N) = n$), and $\pi_V \circ \psi = \pi_N$, where $\pi_V : V \times \mathbb{R}^n \to V$ is the projection onto V and ψ restricted to T_qN for $q \in V$ is a linear isomorphism onto $\{q\} \times \mathbb{R}^n$. Suppose

$m \in M$ such that $m \in f^{-1}(V)$. We note that $f^{-1}(V)$ is open, because f is continuous. Then, the map $\psi^* : (p^*)^{-1}(f^{-1}(V)) \to f^{-1}(V) \times \mathbb{R}^n$ by $(m, v) \mapsto (m, \pi_{\mathbb{R}^n}(\psi(f(m), v)))$, where $\pi_{\mathbb{R}^n} : f^{-1}(V) \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto $\mathbb{R}^n$, is a local trivialization of f^*TN if it is equipped with the smooth structure that these local trivializations induce. It can be shown that f^*TN is also an embedded submanifold of $M \times TN$. The proof for this uses the map $F : M \times TN \to N \times N$ by $(m, v) \mapsto (f(m), \pi(v))$. This map is *transversal* to the diagonal of $N \times N$, which is an embedded submanifold of $N \times N$. By Theorem 6.30 in [21] it follows that the inverse image of the diagonal in $N \times N$ under F is an embedded submanifold of $M \times TN$. (For more details on transversality see the section "Transversality" in Chapter 6 of [21].)

Note that we have the following commutative diagram:

$$\begin{array}{ccc} f^*TN & \longrightarrow & TN \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_N} \\ M & \underset{f}{\longrightarrow} & N, \end{array}$$

where the above arrow is the projection onto TN. Finally, using example (j) from section 3.1.2, we conclude that $\Gamma^\infty(M, f^*TN)$ is also a Fréchet space.

Next, we will turn our attention to a so-called *local addition* for N. A local addition for a manifold N is a map $\eta : U \subseteq TN \to N$ on an open neighbourhood U of the zero section of TN, such that $\eta(0_p) = p$ for any $p \in N$ and such that $\pi_N \times \eta : U \to N \times N$, where $v \mapsto (\pi_N(v), \eta(v))$ if $v \in U$, is a diffeomorphism onto an open neighbourhood of the diagonal in $N \times N$. We will show that there is a local addition for N with TN as its domain.

Let g be a Riemannian metric on N, such that we have the exponential map $\exp : U \subseteq TN \to N$, where U is an open neighbourhood of the zero section such that exp restricted to $U \cap T_pN$ maps geodesics to geodesics for each $p \in N$. If we inspect the map $\pi_N \times \exp : U \to N \times N$, it turns out that $\pi_N \times \exp$ is a local diffeomorphism for each element 0_p of the zero section. After redefining U as a union of open neighbourhoods of each 0_p on which the restriction of $\pi_N \times \exp$ is a diffeomorphism, $\pi_N \times \exp$ on U is also bijective and therefore a (global) diffeomorphism. For more details on the exponential map of a Riemannian metric, see [22].

Now, we want a diffeomorphism between U and TN. Suppose first that $U \neq TN$. Then, the wanted diffeomorphism can be constructed using a smooth map $\epsilon : N \to (0, \infty)$, such that $U' = \{v \in TN : \sqrt{g(v, v)} < \epsilon(\pi_N(v))\}$ is an open subset of U (see for example 42.4 in [18]). In this case, there is a diffeomorphism $h : U' \to TN$, such that $(\pi_N \times \exp) \circ h^{-1} : TN \to N \times N$ is a diffeomorphism onto an open neighbourhood of the diagonal in $N \times N$. For later reference, let $\eta = \exp \circ h^{-1}$ be the local addition for which this is achieved and let $e : TN \to V$ be the diffeomorphism onto V, the image of $\pi_N \times \eta$.

Charts and transition functions

At this place, we are ready to start defining the charts on $C^\infty(M, N)$. For this, we construct charts for each $f \in C^\infty(M, N)$ mapping into $\Gamma^\infty(M, f^*TN)$. Now, let $U_f = \{g \in C^\infty(M, N) : \text{for all } m \in M \ (f(m), g(m)) \in V\}$, where V is the image of the map e above, which will be the domain of the chart at f. In this case, we have a diagram:

$$\begin{array}{ccc} C^\infty(M,TN) & \xrightarrow{e_*} & C^\infty(M,N\times N) \\ \uparrow{\scriptstyle (\pi_{TN})_*} & & \uparrow{\scriptstyle f\times} \\ \Gamma^\infty(M,f^*TN) & & U_f, \end{array}$$

where $e_* : C^\infty(M,TN) \to C^\infty(M,N\times N)$ by $\beta \mapsto e\circ\beta$. The $(\pi_{TN})_*$ map is the map $\gamma \mapsto \pi_{TN}\circ\gamma$ and the map $f\times$ is the map $g\mapsto f\times g$, where $f\times g : M \to N\times N$ is by $m\mapsto (f(m), g(m))$.

It is clear that e_* is a bijection, because e is. Moreover, the map $(\pi_{TN})_*$ gives a bijection of $\Gamma^\infty(M, f^*TN)$ onto the set $\{\beta \in C^\infty(M,TN) : p^*\circ\beta = f\}$ and the map $f\times$ is clearly injective. Further inspection gives that the image of $\Gamma^\infty(M,f^*TN)$ under $e_*\circ(\pi_{TN})_*$ is mapped onto the set of functions of the form $f\times g$ in $C^\infty(M,N\times N)$, such that $(f\times g)(M)\subseteq V$. This gives a bijection with the image of U_f under the right-up map. Therefore, there is a bijection $\Psi_f : U_f \to \Gamma^\infty(M,f^*TN)$. It can be seen that this is by $g\mapsto (\mathrm{id}_M \times (e^{-1}\circ(f\times g)))$ and the inverse is by $\gamma\mapsto \eta\circ\widetilde{\gamma}$ if $\gamma(m) = (m,\widetilde{\gamma}(m))$ for each $m\in M$, such that $\widetilde{\gamma}\in C^\infty(M,TN)$. Note that $\widetilde{\gamma} = \pi_{TN}\circ\gamma$. In conclusion, we have

$$\Psi_f(g) = (\mathrm{id}_M \times (e^{-1}\circ(f\times g))) \text{ and } \Psi_f^{-1}(\gamma) = \eta\circ\widetilde{\gamma}.$$

Next, we show that the transition functions are smooth. Let $f_1, f_2 \in C^\infty(M,N)$, and write $U_1 = U_{f_1}$ and $\Psi_1 = \Psi_{f_1}$ and for f_2 similarly. Denote U_{12} for the intersection $U_1\cap U_2$. We will make use of the set

$$W_{12} = \{(m,v)\in f_1^*TN : (f_2(m), \eta(v))\in V\},$$

which is open because it is the inverse image of V under $f_2\times\eta : f_1^*TN \to N\times N$ by $(m,v)\mapsto (f_2(m),\eta(v))$. In this case, we will show that $\Psi_1^{-1}(\Gamma^\infty(M,W_{12})) = U_{12}$.

Specifically, we show that $\gamma\in\Gamma^\infty(M,W_{12})$ if and only if $\Psi_1^{-1}(\gamma)\in U_2$. Note that we already have $\Psi_1^{-1}(\gamma)\in U_1$. As before, let $\gamma(m) = (m,\widetilde{\gamma}(m))$, such that the image of γ under $(\pi_{TN})_*$ as in the diagram above is $\widetilde{\gamma}$. For $\gamma\in\Gamma^\infty(M,W_{12})$ we see that for each $m\in M$ that $(f_2(m),\eta(\widetilde{\gamma}(m)))\in V$. Therefore, $f_2\times(\eta\circ\widetilde{\gamma})\in C^\infty(M,V)$. But because of the bijections in the diagram above, we have that $\eta\circ\widetilde{\gamma}\in U_2$. Also, we see that $\eta\circ\widetilde{\gamma} = \Psi_1^{-1}(\gamma)$, which shows that $\Psi_1^{-1}(\gamma)\in U_2$. Similarly, we have that $\Psi_2^{-1}(\Gamma^\infty(M,W_{21})) = U_{12}$, where W_{21} is the set analogous to W_{12} but with the 1 and 2 interchanged.

Moving on, we show that $\Psi_2\circ\Psi_1^{-1} : \Gamma^\infty(M,W_{12})\to\Gamma^\infty(M,W_{21})$ is smooth. To do this, we will employ the map $\theta_1 : W_{12}\to TN$ by $(m,v)\mapsto e^{-1}(f_2(m),\eta(v))$. Note that for $(m,v)\in W_{12}$ we may use that $(f_2(m),\eta(v))\in V$. This map is clearly smooth, and we define θ_2 similarly, such that $\theta_2(m,v) = e^{-1}(f_1(m),\eta(v))$. For later use, we note that $\eta\circ e^{-1}$ is the projection of the open neighbourhood of the diagonal $V\subseteq N\times N$ onto the second N in the product.

Now, we can define $\phi_{12} : W_{12}\to W_{21}$ by $(m,v)\mapsto (m,\theta_1(m,v))$. This is well-defined because $(f_1(m),\eta(\theta_1(m,v))) = (f_1(m),\eta(v)) = e(v)$ lies in V (remember $\pi_N\times\eta = e$), so for each $m\in M$ we have that $(m,\theta_1(m,v))\in W_{21}$. It is also clear that ϕ_{12} is smooth. Similarly, we can also define $\phi_{21} : W_{21}\to W_{12}$, which is also well-defined and smooth.

Consider $(\phi_{21} \circ \phi_{12})(m, v)$ for any $(m, v) \in \Gamma^\infty(M, W_{12})$. We see that:

$$\begin{aligned} \phi_{21}(\phi_{12}(m, v)) &= \phi_{21}(m, \theta_1(m, v)) \\ &= (m, \theta_2(m, \theta_1(m, v))) \\ &= (m, e^{-1}(f_1(m), \eta(\theta_1(m, v)))) \\ &= (m, e^{-1}(f_1(m), \eta(v))) \\ &= (m, v), \end{aligned}$$

where we use in the step to the last line that $(m, v) \in f_1^* TN$. Therefore, we can conclude that $\phi_{21} \circ \phi_{12} = \mathrm{id}_{W_{12}}$. Similarly, we also have $\phi_{12} \circ \phi_{21} = \mathrm{id}_{W_{21}}$. Accordingly, ϕ_{12} and ϕ_{21} are diffeomorphisms between W_{12} and W_{21} and each others inverse.

Furthermore, we want to show that $\phi_{12*} : \Gamma^\infty(M, W_{12}) \to \Gamma^\infty(M, W_{21})$ by $\gamma \mapsto \phi_{12} \circ \gamma$ is smooth, as a map between two open subsets of two Fréchet spaces. Generalizing Lemma 3.1.15, we indeed have that $\Gamma^\infty(M, W_{12})$ is an open subset of $\Gamma^\infty(M, f_1^* TN)$. Note that we use the compactness of M here. Following the comments after Lemma 3.1.20 and referring to 3.6.6. in [16], we claim that ϕ_{12*} is indeed smooth. In addition, it is clear that ϕ_{12*} is a bijection.

Now, let $\gamma \in \Gamma^\infty(M, W_{12})$ be fixed and let $h \in \Gamma^\infty(M, f_1^* TN)$, then for $t > 0$ large enough we inspect

$$\frac{1}{t}(\phi_{12*}(\gamma + th) - \phi_{12*}(\gamma)).$$

Evaluating in an arbitrary $m \in M$, we get

$$\frac{1}{t}(\phi_{12}(\gamma(m) + th(m)) - \phi_{12}(\gamma(m))),$$

which is seen to be the directional derivative of ϕ_{12} at $\gamma(m)$ in the direction of $h(m)$, if we take the limit $t \to 0$. Therefore, using $d\phi_{12} : TW_{12} \to TW_{21}$ as the total derivative of ϕ_{12}, we get that the Fréchet derivative of ϕ_{12*} at γ is $D\phi_{12*}(\gamma, h)(m) = d\phi_{12}(\gamma(m), h(m))$.

Lastly, we show that $\phi_{12*} = \Psi_2 \circ \Psi_1^{-1}$. Indeed, we have

$$\begin{aligned} \phi_{12*}(\gamma)(m) &= (m, e^{-1}(f_2(m), \eta(\widetilde{\gamma}(m))), \\ \Psi_2 \circ \Psi_1^{-1}(\gamma)(m) &= \Psi_2(\eta(\widetilde{\gamma}))(m) \\ &= (m, e^{-1}(f_2(m), \eta(\widetilde{\gamma}(m)))). \end{aligned}$$

Taking all these steps into account, we have found a smooth atlas on $C^\infty(M, N)$, such that it can be turned into a Fréchet manifold.

A type of submanifold for $C^\infty(M, N)$

We end with the following proposition. If S is an embedded submanifold of N, we have that $C^\infty(M, S) \subseteq C^\infty(M, N)$ is a Fréchet submanifold.

Proposition 3.2.6. Let S be an embedded submanifold of N. Then, the subset $C^\infty(M, S)$ of $C^\infty(M, N)$ is a Fréchet submanifold.

For a proof, see for example 4.2.2 and 4.2.3 in [16] and, for a proof with more details, see Corollary 3.2.12 and the results leading to it, in [35].

3.3 Critical fields

3.3.1 The composition map

Suppose M and N are manifolds, and M is compact. Let $V : N \to \mathbb{R}$ be a smooth map. In this case, we can define a map from a Fréchet manifold into a Fréchet space by

$$F : C^\infty(M, N) \to C^\infty(M, \mathbb{R}) \text{ by } f \mapsto V \circ f.$$

The aim of this section is to show that the differential of F at f, $dF_f(h)$, for $f \in C^\infty(M, N)$ and $h \in \Gamma^\infty(M, f^*TN)$ is the zero function for each h if and only if $f \in C^\infty(M, \mathrm{Cr}(V))$. We identify the tangent space at f with the model space at f, $\Gamma^\infty(M, f^*TN)$.

Indeed, if $\hat{F} = F \circ \Psi_f^{-1}$, where Ψ_1 is a chart as in section 3.2.2, we can identify the differential dF_f as $D\hat{F}(\Psi_f(f), -)$. Notice that F is indeed a smooth map between two Fréchet manifolds, by the same arguments as for the smoothness of ϕ_{12*}. The map $\hat{F}$ goes from $\Gamma^\infty(M, f^*TN)$ to $C^\infty(M, \mathbb{R})$. Specifically, if $\gamma \in \Gamma^\infty(M, f^*TN)$, then if $x \in M$ we have

$$(F \circ \Psi_f^{-1}(\gamma))(x) = (F(\eta \circ \widetilde{\gamma}))(x) = (V \circ \eta \circ \widetilde{\gamma})(x),$$

where $\widetilde{\gamma}$ is such that $\gamma = \mathrm{id}_N \times \widetilde{\gamma}$ and $\eta : TN \to N$ the local addition of N, as in section 3.2.2. Therefore, similar to the Fréchet derivative of ϕ_{12*}, we get

$$D\hat{F}(\gamma, h)(x) = d(V \circ \eta)(\widetilde{\gamma}(x), \widetilde{h}(x)).$$

As η is a smooth submersion, for $D\hat{F}(\gamma, h)$ to be the zero function on M for each h, we must have that $\eta \circ \widetilde{\gamma}$ maps only into $\mathrm{Cr}(V)$. Interchanging $\eta \circ \widetilde{\gamma}$ with f, we get the following result.

Theorem 3.3.1. Let M and N be two manifolds, such that M is compact. In addition, let $V : N \to \mathbb{R}$ be a smooth map such that the set of critical points $\mathrm{Cr}(V)$ is an embedded submanifold of M. Moreover, let

$$F : C^\infty(M, N) \to C^\infty(M, \mathbb{R}) \text{ by } f \mapsto V \circ f,$$

as a smooth map between a Fréchet manifold and a Fréchet space. Then, the functions f in $C^\infty(M, N)$ for which the image of the Fréchet differential dF_f is the set only containing the zero function on M (the critical functions) are those contained in the Fréchet submanifold $C^\infty(M, \mathrm{Cr}(V))$ of $C^\infty(M, N)$.

We note that $C^\infty(M, \mathrm{Cr}(V))$ is a Fréchet submanifold of $C^\infty(M, N)$ because of Proposition 3.2.6.

As an example of some conditions for which $\mathrm{Cr}(V)$ is an embedded submanifold of N, using the main result from the previous chapter, Theorem 2.4.13, we can take V to be a smooth, G-invariant and G-Morse-Bott function for a smooth and proper G-action, for a Lie group G. In this case, the critical functions have the critical orbits as their image. In section 3.3.3, we will get back to this example.

3.3.2 The integration map

With regard to the theory of integration for finite-dimensional orientable manifolds as in Chapter 16 in [21], we could also consider the integration map

$$S : C^\infty(M, N) \to \mathbb{R} \text{ by } f \mapsto \int_M V \circ f \, d\mu,$$

where $d\mu$ is the measure generated by the combination of a partition of the unity subordinated to a set of domains of charts and the pullback measure of the Lebesgue measure on the image of the charts that lies in $\mathbb{R}^{\dim(M)}$. We note that a map $V \circ f$ on M indeed produces a $\dim(M)$-form, such that we can integrate it over M. So, let M be an orientable compact manifold. Again, using Ψ_f^{-1} we get a map $\hat{S} = S \circ \Psi_f^{-1}$. Unravelling this further, we get that

$$\hat{S}(\gamma + th) = \sum_i \int_M \psi_i \cdot V \circ \eta \circ (\widetilde{\gamma} + t\widetilde{h}) d\mu,$$

where $\{\psi_i\}$ is a finite partition of unity subordinate to the domains of a collection of charts (φ_i, U_i) such that $\mathrm{supp}(\psi_i) \subseteq U_i$. Taking the limit $t \to 0$, we use that for a compact integration domain we may differentiate under the integral sign, such that

$$\lim_{t \to 0} \frac{1}{t} \left(\hat{S}(\gamma + th) - \hat{S}(\gamma) \right) = \sum_i \int_M \frac{d}{dt} \psi_i \cdot V \circ \eta \circ (\widetilde{\gamma} + t\widetilde{h})|_{t=0} d\mu.$$

Similarly as above, we have that

$$\frac{d}{dt} V \circ \eta \circ (\widetilde{\gamma} + t\widetilde{h})|_{t=0}(x) = d(V \circ \eta)(\widetilde{\gamma}(x), \widetilde{h}(x)).$$

In the coordinate representation of $V \circ \eta$ we therefore can apply the fundamental lemma of the calculus of variations on each domain $\varphi_i(U_i)$ to get that $dV_{f(x)}$ must be the zero function for each $x \in M$. Therefore, we must have $f \in C^\infty(M, \mathrm{Cr}(V))$. Consequently, this gives the same result as in Theorem 3.3.1. Note that we can differentiate $\hat{S}$ to each order and using the Lebesgue dominated convergence theorem for continuity, we find that $\hat{S}$ and therefore S are Fréchet smooth maps. (For the Lebesgue dominated convergence theorem, we refer to Theorem 2.4.5 in [4].)

Theorem 3.3.2. Let M and N be two manifolds, where M is compact and orientable. In addition, let $V : N \to \mathbb{R}$ be a smooth map such that the critical points $\mathrm{Cr}(V)$ is an embedded submanifold of N. Then, let

$$S : C^\infty(M, N) \to \mathbb{R} \text{ by } f \mapsto \int_M V \circ f d\mu,$$

be the smooth integration map with respect to the measure as above. Then, the differential at $f \in C^\infty(M, N)$, dS_f, maps to zero if and only if $f \in C^\infty(M, \mathrm{Cr}(V))$ and this set is a Fréchet submanifold of $C^\infty(M, N)$.

3.3.3 Representatives for the critical fields

We consider the situation where we have a smooth proper group action of a Lie group G on manifold N, such that the results of the chapter 2 are applicable. With the idea of gauge independent variables in mind as discussed in section 1.2, we might be tempted to ask if it is possible to mod out the local gauge transformations, for the critical fields (functions) of the composition and integration map. We know that for each connected component of M, the image of a critical field is in some critical orbit of V, so we could ask if it is possible to bring back a critical field to a constant field on each connected component of M. More specifically, if $f \in C^\infty(M, \mathrm{Cr}(V))$ is a critical field, we want to know if there is a local gauge transformation, i.e. a function $g \in C^\infty(M, G)$, such that $g \cdot f$ (with pointwise group action) is constant on each connected component of M.

Let us first check that the function $g \cdot f$ is a smooth function or, in other words, an element of $C^\infty(M, N)$. Indeed, this map can be seen as a chain of compositions. That is, if $m : G \times N \to N$ is the group action map and $g \times f : M \to N \times N$ is the smooth concatenation of g and f, then $g \cdot f$ can be written as $m \circ (g \times f)$. Moreover, if M is a compact manifold, $C^\infty(M, G)$ carries a Fréchet manifold structure by applying section 3.2.2 and this structure also makes it that the group multiplication map $C^\infty(M, G) \times C^\infty(M, G) \to C^\infty(M, G)$ by $(g, h) \mapsto g \cdot h$ (pointwise multiplication) and pointwise inversion map are Fréchet smooth. This way, $C^\infty(M, G)$ is also a *Fréchet-Lie group* and the action of this group on $C^\infty(M, N)$ by $(g, f) \mapsto g \cdot f$ if $(g, f) \in C^\infty(M, G) \times C^\infty(M, N)$ (pointwise multiplication) is also smooth. Therefore, this group action is also a Fréchet smooth group action. For more on Fréchet-Lie groups, see for example [26].

To address the question raised above, we turn to the theory of lifts and covering spaces. For simplicity, suppose M is connected, such that for a critical field f we have a map $f : M \to G \cdot x$ into a connected component of an orbit, by restricting the codomain, and where x is some critical point of the potential V. A possible approach could be to use the identification $G \cdot x \cong G/G_x$ and the quotient map $\pi : G \to G/G_x$. This way, if there is a lift of $\phi \circ f : M \to G/G_x$ to $\widetilde{f} : M \to G$, where $\phi : G \cdot x \to G/G_x$ is a diffeomorphism, such that $\pi \circ \widetilde{f} = \phi \circ f$, we could invert $\widetilde{f}$ to get a function $g : M \to G$, such that the pointwise application of g to f, $g \cdot f$, is constant, assuming the value x, on all of M.

For this, the following theorem about lifts and covering spaces, adapted from Theorem 3.5.2 in [6], does the job. First, we recall the definition of a *covering space*.

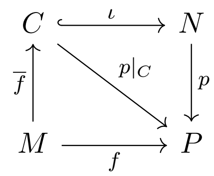
Definition 3.3.3. Let X be a topological space. A pair of a topological space Y and a map $p : Y \to X$ is called a *covering space* of X if $p : Y \to X$ is a continuous surjection, such that the space X can be covered by open sets U, such that $p^{-1}(U)$ is a disjoint union of open sets that are all homeomorphic to U via p. In this case p is called the *covering map*.

Naturally, if X and Y are manifolds and p is also a smooth surjection, the pair of Y and p is a *covering manifold* if X can be covered by open sets U, such that $p^{-1}(U)$ is a disjoint union of open sets that are all diffeomorphic to U via p.

Theorem 3.3.4. Let M, N and P be manifolds, and let $f : M \to P$ be a smooth map. Suppose that $p : N \to P$ is a smooth surjection, such that N has a submanifold, C, for which the restriction $p|_C$ is a smooth covering map of P. Let m and c be points in M and C, respectively, such that $f(m) = p(c)$. Then, if M is path connected, locally path connected and simply connected, there is a smooth lift $\widetilde{f} : M \to N$ of f, i.e. $p \circ \widetilde{f} = f$ and $\widetilde{f}(m) = c$.

Proof. Fix $m \in M$ and $c \in C$ such that $f(m) = p(c)$. As the fundamental group of M is trivial, Theorem 3.5.2 in [6] gives us a continuous lift $\overline{f} : M \to C$, such that $p|_C \circ \overline{f} = f$ and $\overline{f}(m) = c$. We claim that this lift is also smooth. Indeed, if m is any point in M and W a path connected open neighbourhood of m, the lift property $p|_C \circ \overline{f} = f$ gives that the restriction of $\overline{f}$ to W is equal to $w \mapsto (f(w), j)$ where j is a fixed index such that an open neighbourhood of $f(w)$, say V, is covered by a submanifold of C diffeomorphic to $V \times J$ where J is a discrete index space, such that j is an element of J. Therefore, $\overline{f}$ is indeed smooth.

Lastly, as the inclusion of C into N, say ι, is smooth, we get a lift $\widetilde{f} : M \to N$ by defining $\widetilde{f} = \iota \circ \overline{f}$.



□

How this relates to lifting $f : M \to G \cdot x$ is as follows. Let $\phi \circ f : M \to G/G_x$ be as above and let $p : G \to G/G_x$ be the quotient map. To apply the previous theorem, we see that we need a few assumptions. That is, we assume that M is path connected, locally connected and simply connected and that G has a submanifold, C, such that $p|_C$ is a covering map of G/G_x. Below, we will discuss an example where these assumptions hold.

Theorem 3.3.5. Let M be a compact, path connected, locally path connected and simply connected manifold and let N be a manifold. Let G be a Lie group that acts smoothly and properly on N and $V : N \to \mathbb{R}$ a smooth, G-invariant and G-Morse-Bott map. Let f be a critical field as in Theorem 3.3.1 or 3.3.2, such that x is a point in the image of f. Suppose that G has a submanifold C, such that the quotient map $p : G \to G/G_x$, restricted to C, $p|_C$, is a smooth covering map. Then, there exists a function g in $C^\infty(M, G)$ such that $g \cdot f$ is constant and equal to x on M.

Proof. Since M is connected, the image of f lies in the connected component of a critical orbit of the potential V, such that f can be written as a map $f : M \to G \cdot x$. Theorem 3.3.4 gives a lift of $\phi \circ f$, $\widetilde{f} : M \to G$, such that $p \circ \widetilde{f} = f$. Inverting $\widetilde{f}$ pointwise, which is a smooth operation, gives us a map $g : M \to G$, such that $g \cdot (\phi \circ f) = G_x = [e]$ as a left coset in G/G_x, which is the image of x under ϕ, i.e. $\phi(x) = [e]$. Therefore, $(g \cdot f)(m) = x$ for all m in M. □

Example 3.3.6. Let M be a compact, path connected and simply connected submanifold of $\mathbb{R}^4$ (spacetime), and let N be the manifold $\mathbb{C}^2$. Let G be the Lie group $U(1) \times SU(2)$ that acts on N by matrix multiplication and let $V : N \to \mathbb{R}$ be the Mexican hat potential, $z \mapsto -|z|^2 + |z|^4$. Consider critical fields f that map into the critical orbit which is not the origin of $\mathbb{C}^2$. To apply Theorem 3.3.5, we must show that $U(1) \times SU(2)$ has a submanifold C, such that the quotient map $p : U(1) \times SU(2) \to U(1) \times SU(2)/G_x$ restricted to this submanifold is a covering space. For the critical point x in N, we can take $(\frac{1}{2}\sqrt{2}, 0)$, for which we found that the isotropy group, G_x, is the set of pairs $(e^{i\theta}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix})$ for $\theta \in (-\pi, \pi)$. We saw that $U(1) \times SU(2)/G_x$ is diffeomorphic to $SU(2)$ in Example 2.5.1. In this case, p can be identified by the projection map $U(1) \times SU(2) \to SU(2)$. Now, by restricting p to the submanifold $C = \{1\} \times SU(2)$, we get a covering map of $SU(2)$. Therefore, there is a map $g : M \to G$ such that $g \cdot f$ is constant on M.

More explicitly, f can be identified as a function $(a(m), b(m))$ that maps from M to $\mathbb{C}^2$, where a and b are smooth functions on M that map into $\mathbb{C}$, such that $|a|^2 + |b|^2 = \frac{1}{2}$ for each m. Using these functions, we have that

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix},$$

for each $m \in M$, and $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix}$ is clearly a smooth function from M to $SU(2)$.

Chapter 4

Conclusion and outlook

4.1 Conclusion

As a conclusion, we recap the main theorems of this thesis.

In chapter 2 the main theorem is Theorem 2.4.13. This theorem gives conditions for how the critical points of a smooth map $f : M \to \mathbb{R}$, where M is a manifold, can be an embedded submanifold of M, which can be described in terms of the embedded orbits of a smooth group action by a Lie group G. For this, we used proper group actions and the Slice Theorem that holds for such proper group actions.

In chapter 3 we have two main results. That is, Theorems 3.3.2 and 3.3.1 show that the critical fields of the composition map and the integral map coincide, for an appropriate definition of critical fields. For this, we showed how to construct a smooth structure on the set of smooth functions $C^\infty(M, N)$, for M a compact manifold and N a manifold, using Fréchet spaces. Secondly, Theorem 3.3.5 uses the description of the critical points in Theorem 2.4.13 and gives conditions such that for each critical field there is a local gauge transformation that transforms the critical field into a constant field.

4.2 Outlook

The established framework is still an interesting source of possible questions, such as:

- As the spacetime manifold M is usually non-compact in theoretical physics, it could be an interesting question whether the manifold structure of $C^\infty(M, N)$ can be generalized for M non-compact. For example, one could try the convenient setting of Kriegl and Michor [18].
- It could be worth investigating how the quotient space $C^\infty(M, N)/C^\infty(M, G)$ relates to the stratum orbit spaces $N_{(H)}/G$.
- It could be interesting to generalize Michel's theorem to the infinte-dimensional setting, for an infinite-dimensional manifold and an infinite-dimensional group. For example, Gaeta and Morando [10] establish Michel's theorem for a specific type of Hilbert manifolds. Furthermore, Diez and Rudolph [7] give conditions for the validity of the Slice Theorem in the Fréchet setting. Specifically, consider a Fréchet manifold $\mathcal{M}$ and a Fréchet-Lie group $\mathcal{G}$ that acts smoothly on $\mathcal{M}$. Let $F : \mathcal{M} \to \mathbb{R}$ be a smooth $\mathcal{G}$-invariant function. Consider the differential of F at m, dF_m. In this case, we have a $\mathcal{G}_m$-action on $T_m\mathcal{M}$ by the differential of the group action map, just as in the finite-dimensional case. Then, we can ask

if dF_m always vanishes on the part of the tangent space that lies outside of the stratum. (The analogue of the vector space K above Theorem 2.4.12.) However, it is not immediately clear how to define this part of the tangent space in the Fréchet setting and if it is possible.

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