Quantisation versus lattice gauge theory

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Preface

Before you lies the report of my findings that I accumulated as part of my PhD project over a period of four and a half years. Although it certainly has some weight to it in a strictly physical sense, the matter of whether this is true in the figurative sense is, of course, up to the scientific community to decide. Irrespective of its verdict, however, since I myself have invested a significant portion of my time in this work, I find it only appropriate to acknowledge the contributions of the people who did likewise.

First off, I would like to thank my advisor and promotor Walter van Suijlekom for providing me with the opportunity to work on this topic and to learn more about mathematics in general, and for always being available to answer any questions that I had.

I would also like to express my gratitude towards Klaas Landsman, who was not merely the chair of the manuscript committee that approved this thesis, but who introduced me to the topic of quantisation in the first place during my master's. Having been a teaching assistant to his courses on mathematical physics greatly contributed to my understanding of the subject, which was especially useful during my work on the second part of this thesis.

Next, I would like to thank my (remaining) coauthors, starting with Francesca Arici, who, aside from her work on the paper on which chapter 4 is based, was instrumental in communicating our work to the mathematical community and returning with invaluable feedback from prominent researchers. Furthermore, I want to thank Teun van Nuland, with whom I collaborated on the work on which chapters 5 and 7 are based, and who provided some key ideas regarding the resolvent algebra without which the second part of this thesis would not have been possible at all.

Having thanked all of the people whose scientific contribution to this thesis is directly observable, I would like to mention those whose input is not immediately measurable but nonzero nonetheless. At the beginning of my PhD, Jord Boeijink helped me on my way by sharing with me some of his own findings on the topic of "quantisation commutes with reduction". Although this line of research does not feature prominently in this thesis, it was certainly useful to be acquainted with it, specifically in regard to chapters 3 and 8. The next person that I want to mention here is Alexander Stottmeister, who pointed out an important, very concrete

discrepancy between the results of chapter 4 and the physics literature, and whose remark became a guiding principle in part II of this thesis. Moreover, I wish to thank Abel Stern and Chris Ripken for enlightening discussions on the topics of regularisation and renormalisation.

Moving on to more senior researchers in this category, I want to thank prof. dr. Gerd Rudolph and prof. dr. Rainer Verch for their hospitality and for the discussions that I had with them during my stay at the *Institut für Theoretische Physik* at the *Universität Leipzig*. I would also very much like to express my appreciation for the time and effort that the members of the manuscript committee invested in reading and correcting this thesis.

A pleasant work environment is paramount to being able to carry out one's duties and I am happy to report that overall, the department of mathematics of the Radboud University does an excellent job at creating one of these. This also includes providing welcome distractions every once in a while, be it work-related in the form of marking sessions, or otherwise.

There are a few colleagues specifically whom I would still like to mention by name because they had a very positive impact on my life as a PhD student. One of them is Frank Roumen, who frequently hosted board game days during which he supplied many games from his own collection, and with whom I have had numerous interesting discussions. Board game days became a tradition that continues to this day, and many of them have also been hosted or attended by Julius Witte and Milan Lopuhaä. On the subject of interesting discussions, the name of our former local philosophical anarchist Henrique Tavares cannot go unmentioned.

With mathematicians spending much of their time inside their own heads or in the ivory tower of the university, it is sometimes easy to forget that there exist people outside of the realm of academia as well. My parents are two examples of such people. This does not diminish the value that their continuing support holds to me in the slightest; it is safe to say that it is at the very least on par with that of any of the individuals mentioned above.

Eindhoven, October 2019 Ruben Stienstra

Contents

Chapter 1

General introduction

This thesis is concerned with the interplay between quantisation and lattice gauge theory. As we will see, there is considerable tension between these two subjects, at least in so far as their current formulations in the literature are concerned, which motivates our choice of the word 'versus' in the title. In this thesis, we will bring to light the point of contention, and indicate how it might be resolved. To put our work into perspective, we first discuss the two subjects separately, giving a brief historical overview of the relevant topics. In particular, in the case of quantisation, we will present the considerations that led to the groupoid approach, which will be recalled as well. Afterwards, we will motivate the main problems encountered in this thesis, and present its outline.

1.1 Quantisation

Much of the information in this section regarding the early history is found in greater detail in [68], and references therein. *Quantisation* is essentially the translation of the formalism of classical mechanics into the formalism of quantum mechanics. Classical mechanics is the physical theory that describes the motions of objects that are macroscopic and whose velocity is small relative to the speed of light, and the way in which these motions are affected by forces acting on these object. Its original mathematical formulation is due to Newton, and alternative formulations of the theory were obtained by Lagrange and Hamilton.

Quantum mechanics is the physical theory that describes the motion of microscopically small objects such as electrons, atoms and molecules. The word 'quantum' refers to the discrete rather than continuous nature of the spectrum, or set of energy levels of a physical system, and that to pass from a given energy level to a higher or lower one, energy packets, or *quanta*, are absorbed or emitted by the system, respectively. This idea was used in 1900 by Planck to improve on a model for black body radiation developed by Wien, by Einstein in 1905 to explain the photoelectric effect, and by Bohr in 1913 to improve on Rutherford's model of the atom by accounting for the observed lines in the spectrum of light emitted by the hydrogen atom. The theory of nonrelativistic quantum mechanics was subsequently developed by physicists including but not limited to Schrödinger, Heisenberg, Dirac and Born, and mathematicians such as Hilbert, von Neumann and Weyl, mostly in the 1920's.

Central to the formulation of quantum mechanics is the canonical commutation relation (CCR)

$$[\hat{p}, \hat{x}] = -i\hbar,$$

which is attributed to Born, making its first appearance in Born's paper with Jordan [21, equation (38)]. Here, $\hbar$ denotes the reduced Planck constant, also known as Dirac's constant, and we have used physicists' notation for operators on $L^2(\mathbb{R})$. Furthermore, the operators $\hat{p}$ and $\hat{x}$ in the above equation are both unbounded, and the equation should be interpreted accordingly. Any quantum theory should include such a relation in one way or another. A bounded version known as the *Weyl form of the CCR* reads

$$e^{is\hat{p}}e^{it\hat{x}} = e^{ist\hbar}e^{it\hat{x}}e^{is\hat{p}}, \quad s, t \in \mathbb{R}.$$

The families of operators $(e^{is\hat{p}})_{s\in\mathbb{R}}$ and $(e^{it\hat{x}})_{t\in\mathbb{R}}$ are strongly continuous one-parameter groups of unitary operators, and can be constructed from (self-adjoint extensions of) the operators $\hat{p}$ and $\hat{x}$ using spectral theory. Alternatively, they may be defined directly as groups of translation and multiplication operators, respectively. Stone [108] stated that up to unitary equivalence, there is a unique irreducible representation on a Hilbert space of a pair of strongly continuous one-parameter groups satisfying the Weyl form of the CCR. Von Neumann [114] carried out the proof of this theorem and sharpened the result, thereby demonstrating that any two formulations of quantum mechanics in terms of such an irreducible representation on a Hilbert space, for instance Schrödinger's wave mechanics and Heisenberg's matrix mechanics, are in fact equivalent. Moreover, the Stone–von Neumann theorem shows that irreducibility can serve as a natural requirement of quantum mechanical formulations of physical systems.

On a historical note, the formulations of quantum mechanics by Schrödinger and Heisenberg can be shown to be equivalent without appealing to the Stone–von Neumann theorem, and such attempts were made by various people, including Schrödinger himself in 1926 [104], who showed that operators in his theory could be mapped to matrices in Heisenberg's theory using an injective algebra homomorphism. However, Schrödinger was unable to establish surjectivity, due to the absence of a proper functional-analytic framework, which was established by von Neumann in 1932 [115]. For a more extensive discussion of the history of the equivalence of the two formulations, we refer to Muller [85, 86].

Thus group theory and representation theory play an important role in the formulation of quantum mechanics. The next significant advance on this front was made by Mackey, who generalised the Stone–von Neumann theorem in the form of his imprimitivity theorem [79]. This formalism was later cast into the language of C*-dynamical systems by various authors; for an overview, we refer to the notes at the end of [90, section 7.6]. One of the assumptions on these systems is that they satisfy a certain covariance condition that can be regarded as the abstract version of the Weyl form of the CCR, and this requirement is used to define the multiplication on a C*-algebra called the *crossed product algebra* associated to the dynamical system. In that way, the image of this algebra under any *-representation on a Hilbert space will have a built-in version of the canonical commutation relation(s).

Another major aspect to quantisation is that of *deformations* of the classical theory. This is motivated by two observations. The first one stems from the *correspondence principle*, originally due to Bohr, who observed that the difference between the frequencies associated to the excited states of an electron orbiting a nucleus are multiples of some fundamental frequency for large energies compared to the energy of the ground state. The fundamental frequency is the inverse of the period of the periodic motion that the electron allegedly carries out, and is the lowest frequency in a Fourier expansion that corresponds to the classical description of the motion. Nowadays, the term ‘correspondence principle’ entails the more general idea that physical systems of which associated quantities such as the total number of constituents or the total energy approximate macroscopic sizes, will behave like classical systems. This can often be simulated by considering very small effective values of $\hbar$, i.e., $\hbar \approx 0$, or even taking the limit $\hbar \to 0$, which is known as the *classical limit*.

The second observation is due to Dirac, who realised that the canonical commutation relation has a counterpart in classical mechanics, specifically in its Hamiltonian formulation, namely $\{p, x\} = 1$, where $\{\cdot, \cdot\}$ denotes the usual Poisson bracket on $\mathbb{R}^2$. Further evidence of a connection between the Poisson bracket and the commutator bracket is provided by the equations that govern the time evolution of systems in both formulations (where on the quantum mechanical side, one should consider the reformulation of the Schrödinger equation in the Heisenberg picture).

Groenewold [45] and Moyal [83] independently investigated the possibility of formulating quantum mechanics directly in terms of the classical phase space, which is now referred to as *phase space quantisation*. Their work paved the way for the field known as *deformation quantisation*, of which the objective is to deform the commutative pointwise product of smooth functions on phase space (which are the classical observables) into a noncommutative product, in such a way that the Poisson brackets of the functions correspond to $-i\hbar$ times the commutator of the functions with respect to the deformed product. It was shown by Groenewold [45] and Van Hove [111] that, in addition to a number of other algebraic requirements, one cannot simultaneously have irreducibility of a quantum system, as well as an exact correspondence between the Poisson bracket and the commutator. For this reason, the latter assumption is dropped, and one instead requires the correspondence between the Poisson bracket and the commutator to become exact in the classical limit only.

There are two styles of deformation quantisation. The first one is *formal deformation quantisation*, which was pioneered by Berezin [16, 17] and by Flato, Lichnerowicz and Sternheimer [39], and revolves around the construction of deformations of the usual product on the algebra $C^\infty(M)[[\hbar]]$ of formal power series in $\hbar$ that take coefficients in the ring $C^\infty(M)$ of smooth functions on the phase space M.

The second approach, known as *strict deformation quantisation*, was defined by Rieffel [97], who substituted the ring of formal power series by a family of C*-algebras $(A_\hbar)_{\hbar\in I}$ called a *continuous field of C*-algebras*, where $I \subseteq [0,\infty)$ is a set that contains 0 as an accumulation point, A_0 is a commutative algebra, and $A_\hbar$ is its (noncommutative) *quantisation* for $\hbar > 0$. In the examples of interest, all $A_\hbar$ are isomorphic to each other for $\hbar \in I\backslash\{0\}$. Berezin proposed a similar definition in the papers cited above, discussing families of *-algebras, but he did not (explicitly) endow the elements of his families with norms. By contrast, the norms on the algebras appearing in Rieffel's work are an essential ingredient of his notion of a strict deformation quantisation, as they facilitate a precise definition of a classical limit. Strict deformation quantisation is much closer to the usual formulation of quantum mechanics in terms of Hilbert spaces and operator algebras, and as a result it is able to address questions of convergence more readily than formal deformation quantisation. However, it demands from its practitioners a substantially greater effort with regard to the analysis involved.

The representation-theoretic and deformational aspects of quantisation are brought together in the groupoid formulation by Landsman, which is expounded in his monograph [65], which also contains many references to the literature. This formulation is the point of departure of this thesis, and forms the basis of part I, in particular chapter 4. We will elaborate on the groupoid approach in that chapter, so we will only give a brief account here. First of all, to any groupoid endowed with a Haar system, a C*-algebra can be associated, which was done by Renault [93]. With regard to the representation theory, in many cases of interest, the crossed product algebra associated to a C*-dynamical system is canonically isomorphic to the C*-algebra of some groupoid.

Furthermore, it is not uncommon for the sets of objects and morphisms that comprise a groupoid to be endowed with smooth structures with respect to which all of the groupoid operations are smooth, and the source and target maps are submersions; such groupoids are called *Lie groupoids*. As their name already suggests, they generalise Lie groups, and similar to how every Lie group has an associated Lie algebra, every Lie groupoid has an associated *Lie algebroid*.

The idea behind the groupoid formulation of deformation quantisation is that the family of algebras $(A_\hbar)_{\hbar\in I}$ arises as a family of C*-algebras associated to the fibres of a bundle of geometric objects with base space I. Specifically, the fibre over $\hbar \in I\backslash\{0\}$ of this bundle is a given Lie groupoid, which is the same for all $\hbar > 0$, while the fibre over 0 is the Lie algebroid associated to the Lie groupoid. The bundle is endowed with a smooth structure in such a way that the Lie algebroid is smoothly deformed into the given Lie groupoid. This fibre bundle is called the *normal groupoid*, and was introduced by Hilsum and Skandalis in [54] in the context of KK-theory. Hilsum and Skandalis generalised a construction known as the *tangent groupoid*, which had been constructed earlier by Connes. (In their paper, Hilsum and Skandalis refer to a preprint of [30], but the tangent groupoid seems to not have made it into the published version, and can instead be found in Connes' monograph [31, section 2.5], where it is used to prove the Atiyah–Singer index theorem.)

For the purpose of this thesis, it suffices to briefly discuss the tangent groupoid. Given a smooth Riemannian manifold Q, which we view as the configuration space of a classical system, its tangent bundle TQ is smoothly deformed into the pair groupoid $Q \times Q$; the manifold TQ, endowed with some additional structure, is the Lie algebroid associated to the Lie groupoid $Q\times Q$. Functions on the former space correspond to fibrewise Fourier transforms of classical observables, i.e., functions on the classical phase space T^*Q, while functions on the latter occur as integral kernels of integral operators on $L^2(Q)$, thereby yielding quantum observables. The associated C*-algebras are $A_0 = C_0(T^*Q)$ and $A_\hbar = B_0(L^2(Q))$, where the latter algebra denotes the space of compact operators on $L^2(Q)$. Elements of A_0 can be explicitly deformed into elements of $A_\hbar$ using a generalisation of Weyl quantisation, see [66]. Thus the tangent groupoid provides a very appealing geometric picture of deformation quantisation, in that it makes precise the idea of deforming a Lie algebroid of a Lie groupoid into that Lie groupoid.

Note that up to this point we have only discussed aspects of quantisation that concern the observables, i.e., the relationship between classical observables, which are functions on phase space, and quantum mechanical observables, which are operators on the Hilbert space. This is the main subject of study in this thesis; we ignore questions regarding the origin of the Hilbert space on which the operators are defined, some of which are addressed by geometric quantisation.

When discussing quantisation, we will always have in mind the definition of Landsman of a *strict quantisation* [65, Definition II.1.1.1], who takes inspiration from Rieffel's definition of a strict deformation quantisation. This will be discussed in greater detail in section 7.3, but let us already mention that like Rieffel, Landsman includes a family $(A_\hbar)_{\hbar\in I}$ of C*-algebras into his definition, where $A_\hbar$ is commutative for $\hbar = 0$ and noncommutative for $\hbar > 0$. In addition to the above, a strict quantisation consists of a family of maps

$$\mathcal{Q}_\hbar \colon \mathcal{A}_0 \to A_\hbar, \quad \hbar \in I,$$

called *quantisation maps*, that satisfy certain conditions. Here, $I \subseteq [0,\infty)$ has the properties mentioned earlier, and $\mathcal{A}_0$ denotes a dense *-subalgebra of A_0 that at the same time is a Poisson subalgebra of $C^\infty(M)$, where M denotes the phase space of the classical system. Moreover, in the groupoid examples we may assume that the algebra $A_\hbar$ does not depend on the particular value of $\hbar > 0$. For $\hbar = 0$, the map $\mathcal{Q}_\hbar$ is simply the inclusion map.

The most notable requirement on these maps is Rieffel's axiom

$$\lim_{\hbar\to 0} \left\| [\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] - (-i\hbar)^{-1}\mathcal{Q}_\hbar(\{f,g\}) \right\| = 0,$$

for each $f, g \in \mathcal{A}_0$, which makes precise Dirac's observation, and shows that $\mathcal{Q}_\hbar$ should be thought of as a right-inverse to the operation of taking the classical limit. Interestingly, Landsman uses the term *strict deformation quantisation* for a strict quantisation that satisfies some additional properties [65, Definition 1.1.2], thus deviating from the established notion of deformation quantisation that emphasises the deformation of the product, and underlining the role of the quantisation maps instead. A version of this definition is already mentioned by Rieffel in his review of the subject [99, section 4], where he explicitly refrains from calling such maps deformation quantisations for the reason just mentioned. Nevertheless, the formulation in terms of quantisation maps is closer to the everyday practice of physicists, and probably also to the ideas of the founding fathers of quantum mechanics, especially Heisenberg. We will return to the topic of quantisation after the introduction of the other main topic of this thesis.

1.2 Gauge theory and regularisation

The second principal topic of this thesis is *lattice gauge theory*. However, we will not discuss it in much detail in this introduction, leaving it for chapter 2 instead. Rather, we will discuss the main ideas behind and reasons for its development, which come from quantum field theory (QFT).

We will start by discussing gauge theory. As with lattice gauge theory, we focus on the motivation for its study in physics. Gauge fields made their first appearance in Maxwell's theory of electromagnetism. Their importance was not recognised at that time, though, since they only appear in Maxwell's equations through the electric and magnetic fields, which make up the electromagnetic field tensor, and are therefore strictly speaking not necessary to formulate the theory. That being said, they can be (and were) used to simplify computations. On the other hand, the situation in quantum mechanics is quite different; for example, one cannot understand the Aharonov–Bohm effect without some notion of gauge fields.

Weyl is credited with the discovery of the principle of gauge invariance, which he first used in an attempt to unify electromagnetism with general relativity [117]. Although his attempt failed, he later used a similar idea to perform what is nowadays known as *minimal substitution* or *minimal coupling*, by replacing the operation of differentiation in the Schrödinger equation with its gauge covariant derivative [119]. His work, among other discoveries such as the Dirac equation by the eponymous physicist, led to the development of *quantum electrodynamics* (QED), which was the first example of a quantum field theory.

The underlying structure group $\mathrm{U}(1)$ (or $\mathbb{T}$, as we will write in the second part of this thesis,) is an abelian Lie group. Yang and Mills [123] famously realised that gauge theories with other gauge groups, in particular nonabelian ones, could be formulated as well, and tried to use such a theory with structure group $\mathrm{SU}(2)$ to explain the *strong interaction* or *strong force*, which is the force that binds nuclei of atoms together. While it was later realised that this interaction is more accurately described by $\mathrm{SU}(3)$, their work eventually made possible the standard model of particle physics, whose structure group is given by $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$. The first and third factor correspond to the forces mentioned above, while the factor $\mathrm{SU}(2)$ corresponds to the *weak interaction*, which is the force responsible for nuclear fission and radioactive decay of atoms.

Quantum field theory is the most notorious - in both the positive and the negative sense of the word - theory of physics to date. On the one hand, it is extremely succesful from an experimental point of view. QED has made theoretical predictions for a multitude of phenomena that have been measured experimentally and whose outcomes have been compared to each other through the (effective) values of the inverse α^{-1} of the fine-structure constant α that the theoretical predictions require to agree with the measurements in the corresponding experiment (cf. [91, pp. 197–198]). In the cases where higher order terms do not introduce any significant corrections in the theoretical models used to describe the phenomena, these values agree with each other very well, up to the seventh significant digit in the most accurate experiments.

On the other hand, from its conception, physicists have struggled with its mathematical formulation. The earliest calculations of physical quantities yielded infinite values; these were made finite through a process dubbed *renormalisation* devised by Feynman, Schwinger and Tomonaga, for which they received the Nobel prize in physics in 1965, and subsequently streamlined by Dyson. Their approach is now called *perturbative renormalisation*.

Renormalisation is needed due to the fact that field theories have an infinite number of degrees of freedom: they have finitely many for each point in spacetime. Another consequence of this abundance of freedom is that Feynman's path integral, which is essentially a quantisation procedure based on the Lagrangian framework for classical mechanics, is an integral over an infinite dimensional space, and these are generally very hard if not impossible to define rigorously. A notable exception is the Feynman–Kac formula, which is based on stochastic calculus, and holds in Euclidean time.

In gauge theory, this problem is exacerbated by the physically redundant degrees of freedom introduced by gauge symmetry [91, section 9.4]. Faddeev and Popov introduced a procedure to remove these degrees of freedom from the path integral by adding an additional term to the Lagrangian [37] that fixes a gauge, introducing new fields called *ghost fields* in the nonabelian case. The procedure eventually led to the Batalin–Vilkovisky (BV) formalism, via the BRST formalism.

Despite significant efforts to put them on firm mathematical grounds

- such as the constructive quantum field theory program, see e.g. [13, 14] and references therein for the part of that program focussed on gauge theory, including lattice gauge theory - quantum field theories based on Yang–Mills theory have eluded rigour thus far. Defining a rigorous framework for these theories is one of the six unsolved Clay Millennium problems.

One of the ingredients of any quantum field theory, be it in its formulation or its application to concrete problems, is a form of *regularisation*. Regularisation is a rather loose term by mathematical standards, but it typically refers to the first step in a three-step procedure:

1. A mathematical entity that is for some reason intractible due to its infinite or otherwise unbounded nature is converted into a net of finite or bounded entities through some process of truncation. This process is referred to as the *introduction of a regulator* or *cutoff* into the problem.

The terms *regulator* and *cutoff* can refer to the upward directed set $(J, \leq)$ that parametrises the net, or to a net of other mathematical entities that are used to obtain the truncations. In the former case, the term cutoff is sometimes also used to refer to an element of the directed set; another common term for such an element is a *scale*. In the physics literature, the most common examples of directed sets mentioned in this context are $((0, \infty), \leq)$ (for momentum and energy scales) and $((0, \infty), \geq)$ (for length scales). Other directed sets are possible, however; see e.g. section 5.1 for the directed set used in loop quantum gravity, which is much closer to what we have in mind when considering regulators in this thesis.

The result of the previous step is a net of *effective field theories* parametrised by the directed set, or a net of objects that can be thought of as being associated to a collection of effective field theories. The word 'effective' signifies that the field theories are approximations to some 'true' or 'fundamental' field theory.

Let us assume that the net of objects is given by $(X_i)_{i \in J}$. Given two elements $i, j \in J$ such that $i < j$, the object X_j should provide more information about the 'true' theory than the object X_i. Within the context of this thesis, this is understood to mean one of the following two things: Depending on the category in which the objects reside, there is either an embedding

$$\iota_{i,j} \colon X_i \hookrightarrow X_j,$$

or a surjective morphism

$$\pi_{i,j} \colon X_j \twoheadrightarrow X_i.$$

Examples of the first type include nets of observable algebras (in some category of C*-algebras) and nets of Hilbert spaces, while examples of the second type include nets of configuration spaces (in a category of topological spaces carrying regular probability measures) and pair groupoids (in the category of groupoids). The occurrence of the second type of morphism between configuration spaces is an indication that one is dealing with a form of *coarse graining.*

In all of our examples, the family of maps $(\iota_{i,j})_{i,j\in J,\, i\leq j}$ corresponding to the first scenario satisfies two conditions:

(1) For each $i \in J$, we have $\iota_{i,i} = \mathrm{Id}_{X_i}$;

(2) For each $i, j, k \in J$ such that $i \leq j \leq k$, we have $\iota_{i,k} = \iota_{j,k} \circ \iota_{i,j}$.

The pair of families $((X_i)_{i\in J}, (\iota_{i,j})_{i,j\in J,\, i\leq j})$ is called a *direct* or *injective system* in the pertinent category. The dual notion of a family $((X_i)_{i\in J}, (\pi_{i,j})_{i,j\in J,\, i\leq j})$ corresponding to the second scenario is called an *inverse* or *projective system.*

The next step in the procedure is renormalisation; we will sketch what it means in the language introduced above.

2. Let $((A_i)_{i\in J}, (\iota_{i,j})_{i,j\in J,\, i\leq j})$ be a direct family of C*-algebras, where for each $i \in J$, the algebra A_i is the observable algebra corresponding to the scale i. Furthermore, for each $i \in J$, let $\omega_i \in A_i^*$ be a state on A_i that represents the state of the system under consideration at the scale i. Finally, fix $i_0 \in J$, and suppose that i_0 is large enough to ensure that at present, no measurement apparatus has a resolution that allows it to measure a discrepancy between the two values of the physical quantity predicted by the theories associated to i_0 and any $i > i_0$.

Then we may as well assume that the expectation value of any observable corresponding to the theory associated to i_0 is equal to the expectation value of its counterpart for i, i.e.,

$$\omega_i \circ \iota_{i_0,i} = \omega_{i_0},$$

for each $i \geq i_0$, thereby rendering the theories mutually consistent. This is the basic idea behind renormalisation; we use consistency to select a net of states $(\omega_i)_{i \in J}$, where ω_i denotes a state on A_i for each $i \in J$.

In practice, consistency is obtained only for a finite set of functions of expectation values of a subset of the observable algebra: one selects functions $f_1, \ldots, f_m \colon \mathbb{C}^n \to \mathbb{C}$, and $a_1, \ldots, a_n \in A_{i_0}$, and demands that

$$f_l(\omega_i \circ \iota_{i_0,i}(a_1), \ldots, \omega_i \circ \iota_{i_0,i}(a_n)) = f_l(\omega_{i_0}(a_1), \ldots, \omega_{i_0}(a_n)),$$

for $l = 1, 2, \ldots, m$ and $i \geq i_0$. Moreover, one does not consider the full state space associated to an algebra of observables for some $i \geq i_0$, but merely a finite-dimensional submanifold. This submanifold is parametrised by a finite set of so-called *coupling constants*, which are functions $J \to \mathbb{R}$, and are therefore not really constant. By requiring that the observables in the finite set do not depend on $i \in J$, and expressing their expectation values in terms of the coupling constants (a procedure that usually involves perturbation theory and other types of approximations), one obtains a system of equations called *renormalisation group equations*. One subsequently solves these equations for the coupling constants to obtain the *renormalisation group flow*.

A consequence of the restrictions and approximations made in these calculations is that the passage from elements of state spaces to coupling constants is not functorial. Renormalisation nonetheless remains a very powerful technique, as evidenced by the succesful applications of its perturbative variant to quantum field theory (see above), and its nonperturbative variant (which was developed by Wilson, who was awarded the Nobel prize in 1982 for his work) in statistical mechanics to account for power laws that occur near second-order phase transitions (cf. e.g. [44, section 1.2]).

3. The final step in the procedure consists of taking the appropriate limit of the net of truncated entities, which is referred to as *removing the regulator* (or *cutoff*). This limit is then taken to correspond to the entity that one started out with, provided that it is independent of the particular form of regularisation.

In the context of this thesis, this means that for direct systems, we consider their *direct* (or *injective*) *limit*

$$\left(\varinjlim_{i\in J} X_i, (\iota_{i,\infty})_{i\in J}\right),$$

which (if it exists) is uniquely determined up to unique isomorphism by a universal property. The *inverse* (or *projective*) *limit*

$$\left(\varprojlim_{i\in J} X_i, (\pi_{i,\infty})_{i\in J}\right),$$

is the dual notion for inverse systems.

With regard to the former type of limit, we only consider limits of direct systems of various types of Banach spaces with contractions, such as systems of Hilbert spaces in which the morphisms are given by partial isometries, and systems of C*-algebras in which the morphisms are *-homomorphisms. To see how the direct limit is constructed in the latter case with $(J, \leq) = (\mathbb{N}, \leq)$, we refer the reader to [87, section 1.1], which is readily generalised to other categories of interest and upward directed sets. As for the latter type of limit, we merely deal with limits of inverse systems of compact Hausdorff spaces. The reader can consult [95, section 1.1] for the construction of these limits, as well as their main properties.

Algebraic quantum field theory (AQFT) provides an excellent illustration of the mathematical concepts mentioned thus far (though it is up for debate to what extent it provides an example of regularisation), from which a lot of the mathematical literature on lattice gauge theory and loop quantum gravity (including this thesis) draws inspiration. In AQFT, the net J consists of open subsets of Minkowski space with compact closure, and the relation is given by inclusion of sets. Viewing $(J, \subseteq)$ as a category, one postulates the existence of a covariant functor from $(J, \subseteq)$

to the category of unital C*-algebras that assigns to each open subset in Minkowski space an algebra that should be regarded as the local observable algebra associated to that open subset, thereby obtaining a direct system of C*-algebras. The image of a morphism (U, V) under this functor is the embedding of the observable algebra associated to the open set U into the algebras associated to the open set $V \supseteq U$. One obtains maps between the corresponding state spaces by taking the transposes of the *-homomorphisms, thus defining a contravariant functor from $(J, \subseteq)$ to the category of compact Hausdorff spaces, and obtaining an inverse system in the latter category. The direct and inverse limits of the aforementioned systems are the 'full' observable algebra and state space, respectively. (The former is called the *algebra of quasilocal observables.*) Standard references on the subject of AQFT are Araki [6] and Haag [49].

It is worth noting that AQFT assumes the existence of quantum field theories that satisfy certain assumptions (the Wightman axioms) and establishes properties of such QFTs rather than constructing them in the first place. Nevertheless, it has inspired an approach known as *perturbative algebraic quantum field theory* or *pAQFT* for short, which combines ideas from both AQFT and 'ordinary' QFT as it is mostly practiced by physicists, and manages to produce quantum field theories in a mathematically rigourous way. pAQFT is forced to abandon the C*-algebraic framework, however, relying on formal deformation quantisation to quantise classical field theories instead. For introductions to the subject of pAQFT, we refer to the books by Dütsch [36] (for physicists) and Rejzner [92] (for mathematicians), both of which contain many references to the literature. We refer to [55] for the construction of quantum gauge theories on curved spacetimes using pAQFT.

The fact that many physical systems have an infinite number of degrees of freedom is due to one or both of the following features of the system:

- The system occupies an infinite volume in space or spacetime;
- The system has at least one degree of freedom associated to each point in a continuum.

Regularisation as described above then consists of reformulating the system as a net of effective theories, such that each of these theories has a finite number of degrees of freedom. This is accomplished as follows:

- If one is dealing with an infinite volume type problem, then the truncation typically consists of restriction to a bounded subset, which consequently has finite volume/measure. Removing the regulator is referred to as *taking the thermodynamic* or *infinite volume limit*. This type of situation is common in condensed matter physics in problems in which one considers infinite lattices of particles.

- If the continuum is a source of an infinite number of degrees of freedom, as it is in field theories, then one usually divides up the continuum into an at most countable set of subsets such that the intersection of any two distinct subsets has measure zero. Let us assume that the continuum itself has finite volume (or measure), so that it can be divided into a finite number of such regions; otherwise, one first restricts to a bounded set as mentioned in the previous point. One then assigns to each region a value that the original field can take at a point. The idea is that the resulting map on the set of these bounded subsets is an integrated version of the original field, where the value assigned to a particular subset corresponds to the average of the values that the field takes at the points in that subset. In this situation, removing the regulator is referred to as *taking the continuum* or *ultraviolet (UV) limit*.

As will be discussed in chapter 2, lattice gauge theory uses both of these ideas to reduce the infinite dimensional spaces such as the space of connections and the gauge group to finite-dimensional manifolds. This makes rigorous definitions of the path integral possible, and for small lattices even computable using numerical simulations. Furthermore, there is also a Hamiltonian version of the theory, which opens up the way to the application of techniques such as the ones mentioned in the previous section to quantise the system.

1.3 This thesis

1.3.1 The main problem

The principal question that we try to answer in this thesis, is the following:

How does one quantise a net of classical lattice gauge theories?

This question is addressed in chapters 4 and 8, which for this reason form the heart of this thesis.

In chapter 4, we attempt to answer this problem from a groupoid perspective. Although we are able to construct a direct system of quantum observable algebras that is natural from this perspective, we find that in the case relevant to the thermodynamic limit, the *-homomorphisms between the observable algebras are not the ones that are used by physicists in this context. On the other hand, if we try to replace these maps by the maps that physics dictates, then the observable algebras are not mapped into each other; in some sense, they are 'too small'. We also consider the direct limit of the observable algebras, which does not allow for interesting dynamics, thereby providing further evidence that the groupoid formalism is not compatible with lattice regularisation.

This leads us to part II of the thesis. Here, inspired by the work by Buchholz and Grundling on the resolvent algebra for $\mathbb{R}^{2n}$ equipped with the standard symplectic form, we propose a definition of the resolvent algebra of $T^*\mathbb{T}^n$ that extends the observable algebras in Part I (for $G = \mathbb{T}^n$) in a way such that the maps that are correct from the point of view of physics, are well-defined. This part of the thesis has its own introduction, which can be found in section 5.1. We just mention here that the basic idea is to first define classical versions of the field and observable algebras, and subsequently quantise them.

With the aid of our new algebra, in chapter 8, we return to the problem concerning the *-homomorphisms that embed the various observable algebras into each other, which was encountered at the end of chapter 4. We devise an alternative procedure that yields different embedding maps. We accomplish this by first constructing various functors from a modified version of the category of graphs introduced in chapter 4 to a number of categories associated to classical objects, one of which is a category containing pairs of classical observable algebras with their corresponding dense Poisson subalgebra as objects. We subsequently look for a functor with the same source that maps to a category containing the quantum observable algebras. The guiding principle here is that the family of quantisation maps $(\mathcal{Q}_\hbar)_{\hbar\in I}$ should form some kind of natural transformation between the two functors. We show that this idea motivates the aforementioned map between algebras dictated by physics from the point of view of quantisation, which is the only type of map that occurs in direct systems of algebras relevant to the thermodynamic limit. We note that while the naive version of this construction works in this case, in order to tackle the case relevant to the continuum limit, reduction by the gauge group is necessary. We finish with a discussion in which we indicate how the formalism might be modified so that both cases may be treated on equal footing.

It is worth noting that in this thesis we are mainly concerned with kinematics; questions regarding dynamics, in particular renormalisation, are mostly ignored. See however sections 5.3 and 7.4.

1.3.2 Outline

Let us give a brief overview of the chapters in this thesis. We have included chapters 4 and 8 in this outline for the sake of completeness.

Chapter 2 provides preliminaries regarding lattice gauge theory, focussing on its Hamiltonian incarnation.

In chapter 3, we examine how the quantum mechanical version of the Gauss law in lattice gauge theory, which is formulated in terms of unbounded operators, can be recast into a formulation in terms of bounded operators using representation theory. Imposing the Gauss law is part of the reduction of the gauge theory by the gauge group, both for the classical and for the quantum mechanical formulation of lattice gauge theory. While the main result of this chapter is unrelated to the principal question of this thesis, reduction by the gauge group comes up in subsequent chapters, in particular chapters 4 and 8.

As already mentioned in the previous subsection, in chapter 4 we study the interplay between quantisation from the perspective of groupoids, and lattice gauge theory, and point out several problems.

Chapter 5 provides an introduction to part II of this thesis, and gives a motivation and definition for our version of the classical resolvent algebra of the space $T^*\mathbb{T}^n$. Furthermore, we give a more elementary characterisation of the algebra, and study some of its properties, including closure under time evolution for a large class of Hamiltonians for $n = 1$. We finish by commenting on the general case of arbitrary $n \in \mathbb{N}$.

Chapter 6 is an intermezzo to the main problem of this thesis. Here, we study the Gelfand spectrum of an algebra that is closely related to the classical resolvent algebra of $T^*\mathbb{T}^n$, which is effectively equivalent to the problem of determining the spectrum of the resolvent algebra itself. This is arguably the most technical chapter of this thesis, but it stands alone and may be skipped by the reader who is primarily interested in the principal question of this thesis.

In chapter 7, we quantise the classical resolvent algebra using Weyl quantisation, realising it as a set of operators on $L^2(\mathbb{T}^n)$. Except for continuity at values $\hbar > 0$, we show that the quantisation is strict in the sense of Rieffel. Similar to chapter 5, we show that the quantised algebra is closed under time evolution for $n = 1$ for a large class of Hamiltonians, and comment on the higher dimensional case.

Finally, in chapter 8, we modify the category of graphs defined in chapter 4, and define a functor from this category to various categories associated to classical objects, in particular a category containing the classical observable algebras. We subsequently use the idea that quantisation should play the role of a natural transformation to define a functor from the category of graphs to a category containing the quantum observable algebras, and finish by suggesting directions for future research.

1.3.3 Prerequisites

On the mathematical side, we expect the reader to be familiar with differential geometry, including the basics of symplectic geometry, Riemannian geometry and the mathematical formulation of gauge theory, as well as the theory of Lie groups, and functional analysis, specifically the theory of operators on Hilbert spaces. Needless to say, the reader is expected to know any subject that is required to have a workable understanding of the aforementioned fields. We assume that the reader is comfortable with the notions of a category, a functor, and a natural transformation, but we do not assume any in-depth knowledge of category theory in general.

On the side of physics, we assume familiarity with classical mechanics, quantum mechanics, and the application of gauge theory to concrete physical theories such as the covariant formulation of Maxwell's theory of electromagnetism. Quantum field theory is not required, though knowledge of this topic helps to appreciate the material presented here.

1.4 Some remarks on notation and conventions

• $\mathbb{N}$ denotes the set of natural numbers including 0.

• Given a Lie group G and an element $g \in G$, the maps $G \to G$ corresponding to left and right multiplication by g are denoted by L_g and R_g, respectively.

• The tangent map $TM \to TN$ of a map $f\colon M \to N$ between smooth manifolds are denoted by Tf, the tangent map at a point $x \in M$ by $T_x f$. However, N will occasionally be a vector space, which means that for each point $y \in N$, there is a canonical identification of the tangent space $T_y N$ of N at y with N, so that images of Tf correspond to elements of N. The map $TM \to N$ thus obtained is denoted by df, and its restriction to the fibre of TM over a point $x \in M$ by df_x. More generally, if $f \in \Omega^\bullet(M, N) \cong \Omega^\bullet(M) \otimes N$, i.e., f is an N-valued differential form, then df denotes its exterior derivative.

• The structure group G (with Lie algebra $\mathfrak{g}$) of a principal fibre bundle (P, M, G, π) will act on the total space P from the left. We thereby deviate from the convention in the literature, in which G acts from the right. The definition of a connection 1-form is changed accordingly, i.e., such forms are assumed to be invariant under the canonical action of G on $\Omega^1(P, \mathfrak{g})$ induced by the action of G on P and the adjoint representation of G on its Lie algebra. The reason for this choice can be found in section 2.3.

• In the setting of the previous point, given a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ and a smooth local section $\sigma\colon M \supseteq U \to P$ of π, then the corresponding gauge field $\sigma^*(\omega)$ is denoted by A, and the corresponding field tensor $\sigma^*(D\omega)$ is denoted by F, regardless of the gauge group.

• Given a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, we follow the convention from physics with regard to the inner product, assuming it to be linear in its second argument and conjugate linear in its first argument.

Part I

Lattice gauge theory and groupoid C*-algebras

Chapter 2

Classical lattice gauge theory

The purpose of this chapter is to bring the reader up to speed with the basic formulation of lattice gauge theory, in particular its Hamiltonian incarnation. After a brief review of the motivation for its introduction, we indicate how one arrives at the Hamiltonian formulation of Yang–Mills theory. We then explain how one passes from the continuum formulation to the discretised one. We will give the Hamiltonian of the discretised system, and indicate how it corresponds to the Hamiltonian of the continuous system.

A few remarks on the choices made in this chapter concerning the material and its presentation are in order:

- We only consider 'pure' gauge theory, i.e., our formulation only includes gauge fields and no matter fields;

- In our presentation of the justification of the Hamiltonian of the discrete system in the final two sections, we have made an effort to use differential-geometric arguments to relate the magnetic and electric fields to their discrete counterparts wherever possible, rather than relying on arguments that involve Taylor expansions such as those found in the original papers [120, 62];

- With regard to the magnetic field (section 2.4): although we disregard its associated term in the Hamiltonian in the rest of this thesis, the author is unaware of an exposition of a derivation of this term that meets the standard set in the previous point, which is why we have included it here. Furthermore, to our knowledge, Proposition 2.3 does not appear elsewhere as such, but is otherwise easy to derive from known results.

- With regard to the electric field (section 2.5): unlike the magnetic field, it will play a role in chapters 4 and 8. In the former chapter, it will only enter the discussion through its corresponding term in the Hamiltonian. However, in the latter chapter, more specifically in Example 8.14, the results in this chapter will provide the physical justification for the definition of the map between phase spaces.

2.1 Introduction

Lattice gauge theory was introduced by the physicist K.G. Wilson [120] (who was already mentioned in the introduction to this thesis because of his work on the renormalisation group) in an attempt to explain the phenomenon known as *confinement of quarks*. Quarks are the subatomic constituents of protons and neutrons, which in turn are the building blocks of nuclei of atoms. The term *confinement* refers to the observation that quarks do not occur in free states, i.e., as single particles, but only in bound states together with other quarks.

Wilson's original model was a discretisation of the Yang–Mills action, and was therefore based on a Lagrangian theory. As mentioned in section 1.2, Lagrangian field theories are quantised by means of a path integral, but from a mathematical perspective, these are not well defined, and on top of this, gauge theories come with the additional challenge of controlling the gauge freedom. On the other hand, the lattice fields form a finite-dimensional space, which makes it easier to define a notion of integration on this space, and thereby a path integral. Although even in this setting, path integrals may still be difficult or even impossible to calculate explicitly, it is possible to at least approximate such integrals by means of numerical simulations, and this has indeed been done for small lattices. For an overview of the history of QCD that includes a discussion on numerical simulations, we refer to [41].

Another upside of working with lattices is the fact that they form a natural ultraviolet (UV) cutoff. Thirdly, if one assumes the lattice to be contained within a compact region of spacetime, which for the moment we will assume to be $\mathbb{R}^4$ with the standard Lorentzian metric, then they simultaneously serve as an infrared (IR) cutoff. This means that computation of correlation functions yields finite quantities, although such quantities should still be subjected to the process of renormalisation when taking the appropriate limits. This is typically done by working with a cubic lattice with a certain lattice spacing, ℓ say, writing down all of the quantities in terms of ℓ, and finally take the limit $\ell \to 0$ to remove the UV cutoff, i.e., take the continuum limit. Similarly, by assuming that the cubic lattice itself forms a large cube of which each side consists of N edges, one may remove the IR cutoff, i.e., take the thermodynamic or infinite volume limit, by taking the limit $N \to \infty$. One may take the thermodynamic limit without taking the continuum limit by keeping ℓ fixed as in Figure 2.1, or one can take both limit simultaneously by taking the limit $\ell \to 0$ and $N\ell \to \infty$, as depicted in Figure 2.2:

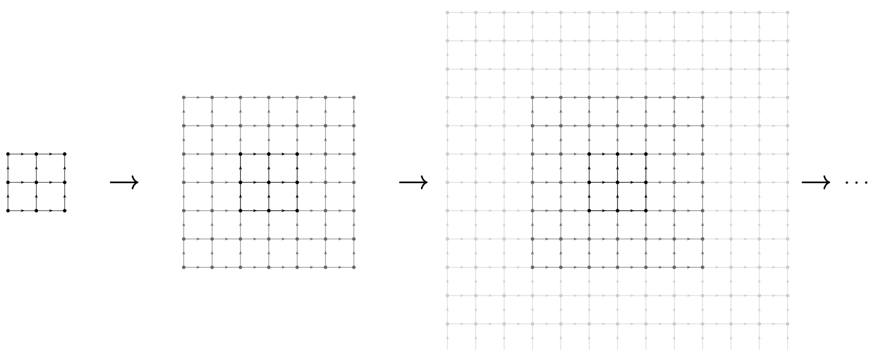


Figure 2.1: Taking the thermodynamic without taking the continuum limit.

Rather than using path integrals to quantise the field theory, one can also first consider the (canonical) Hamiltonian version of the Yang–Mills theory, which is a gauge theory of which the base manifold of the underlying principal bundle is a time slice rather than a spacetime. One can perform

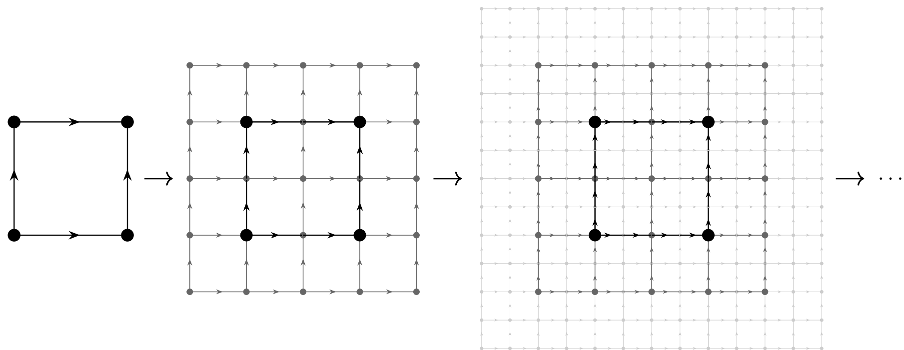


Figure 2.2: Simultaneously taking the thermodynamic and continuum limits.

the discretisation on the time slice to arrive at a description that resembles the Hamiltonian description of systems encountered in classical mechanics, and attempt to quantise this version of the system. This approach was pioneered by Kogut and Susskind [62]. From a mathematical viewpoint, the relationship between the Hamiltonian formulation of a classical mechanical system and its quantum mechanical counterpart is much better understood, and the formulation of Kogut and Susskind allows us to take advantage of this fact.

2.2 Gauge theory on the continuum

We will now discuss the Hamiltonian formulation of Yang–Mills theory without matter, starting from the Lagrangian version. Let (P, M, G, π) be a principal fibre bundle. Let us assume for simplicity that the four-dimensional base manifold M endowed with a Lorentzian metric α representing spacetime is contractible. The standard example of such a manifold is of course Minkowski space $(\mathbb{R}^4, \eta)$ We will adopt the particle physicists' convention, assuming that α has signature $- + ++$. Since M is contractible, the principal fibre bundle P is trivialisable, so we may assume without loss of generality that $P = M \times G$. The space of connections may be identified with $\Omega^1(M, \mathfrak{g})$. The Lagrangian density is then given by

$$\begin{aligned} \mathcal{L}\colon J^1(T^*M \otimes (M \times \mathfrak{g})) &\to \left|\bigwedge^4\right|(T^*M), \\ j^1_m A &\mapsto -\frac{1}{2g^2}\langle F, F\rangle_{\bigwedge,\mathfrak{g},m} \cdot \sqrt{|\det \alpha_m|}. \end{aligned}$$

Here,

- $J^1(T^*M \otimes (M \times \mathfrak{g}))$ denotes the first jet prolongation of the tensor product of vector bundles over M of the cotangent bundle T^*M with the trivial bundle $M \times \mathfrak{g}$;
- $\left|\bigwedge^4\right|(T^*M)$ denotes the bundle of densities on M;
- $j^1_m A$ denotes an element in the fibre of $J^1(T^*M \otimes (M \times \mathfrak{g}))$ over $m \in M$;
- g denotes a coupling constant;
- As mentioned in section 1.4, F is the field strength tensor associated to the gauge field A. Note that the value of F at m depends on the value of A at m, as well as the values of its first order derivatives at m, all of which are encoded by $j^1_m A$;
- $\langle\cdot,\cdot\rangle_{\bigwedge,\mathfrak{g}}$ denotes the nondegenerate symmetric bilinear form on the exterior algebra of the bundle $T^*M \otimes (M \times \mathfrak{g})$ induced by a nondegenerate symmetric bilinear form on $T^*M \otimes (M \times \mathfrak{g})$. The latter bilinear form is in turn canonically induced by two other nondegenerate symmetric bilinear forms, the first one being the Lorentzian metric h, which has an associated bilinear form on T^*M, and the second one being an Ad-invariant inner product on $\mathfrak{g}$. In the examples of interest, the Lie group G is defined as a subgroup of a group of unitaries on an inner product space V, so we have a Lie group representation $\rho\colon G \to U(V) \subset \mathrm{End}(V)$, which has an associated Lie algebra representation $d\rho_{\mathbf{1}_G}$, and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ is defined by

  $$\langle X, Y\rangle_{\mathfrak{g}} := \mathrm{Tr}(d\rho_{\mathbf{1}_G}(X)^* \cdot d\rho_{\mathbf{1}_G}(Y)).$$

  Here, * denotes the adjoint with respect to the inner product on V;

- $\sqrt{|\det k_m|}$ denotes the density on T_mM corresponding to α_m.

Given the behaviour of F under gauge transformations, it follows from Ad-invariance of the inner product on $\mathfrak{g}$ that the Lagrangian density is invariant under gauge transformations, and therefore the same is true for the corresponding action. It follows that the general form of the Euler–Lagrange equations is independent of the chosen gauge.

In order to pass to the canonical Hamiltonian formulation, one assumes that M can be written as $\mathbb{R} \times M'$, where $\mathbb{R}$ represents time, and is endowed with minus the standard Riemannian metric, while M' is (isomorphic to) a Cauchy surface in M representing space, and is endowed with a Riemannian metric β. To simplify the exposition, we will assume that $(M, \alpha) = (\mathbb{R}^4, \eta)$, i.e., (M, α) is standard Minkowski space, and $M' = \mathbb{R}^3$ endowed with the standard Riemannian metric.

We must now extract the part of the Lagrangian density that contains time derivatives. As is customary in the physics literature, we use (x^0, x^1, x^2, x^3) to denote the canonical chart on $\mathbb{R}^4$. Furthermore, we use the notation

$$A_\mu := A\left(\frac{\partial}{\partial x^\mu}\right), \quad F_{\mu\nu} := F\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right),$$

where $\mu, \nu \in \{0, 1, 2, 3\}$. We then have

$$\langle F, F\rangle_{\wedge,\mathfrak{g}} = -\sum_{\nu=1}^{3} \langle F_{0\nu}, F_{0\nu}\rangle_{\mathfrak{g}} + \sum_{1\le\mu<\nu\le 3} \langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}.$$

Here, the first sum contains terms with time derivatives, since

$$F_{0\nu} = \frac{\partial A_\nu}{\partial x^0} - \frac{\partial A_0}{\partial x^\nu} + [A_0, A_\nu].$$

In the canonical Hamiltonian formulation, field theories, in this case a gauge theory, are viewed in a way analogous to systems in classical mechanics; First, a configuration space Q is identified, in this case a subspace of the space of (smooth) sections of the bundle $T^*M' \otimes (M' \otimes \mathfrak{g})$ endowed with a topology that turns it into a Fréchet manifold. A suitable subspace is the space $\Omega^1_c(M, \mathfrak{g})$ of compactly supported smooth sections of the bundle. We refer to [3, section 5.5] for a more general introduction to infinite dimensional systems.

A Hamiltonian is then defined on the tangent bundle of this space. In the case at hand, this means that we have to write down a Hamiltonian in terms of the spatial components of the gauge field and their time derivatives; in particular, we must get rid of the second and third term in the above formula for $F_{0\nu}$. This is accomplished by imposing the *temporal gauge*, which is the condition

$$A_0 = 0.$$

It is always possible to impose this condition:

2.1 Proposition. *Let $A \in \Omega^1(M, \mathfrak{g})$, assume that $M = \mathbb{R} \times M'$, and let $\frac{d}{dt} \in \Gamma^\infty(TM)$ be the vector field on M given by*

$$\left.\frac{d}{dt}\right|_{(s,x)} = \frac{d}{dr}(s+r, x)|_{r=0} \in T_{(s,x)}M.$$

Then there exists a $g \in C^\infty(M, G)$ such that

$$\big(\mathrm{Ad}(g(m)) \circ A_m + (T_{\mathbf{1}_G} R_{g(m)})^{-1} \circ T_m g\big)\left(\left.\frac{d}{dt}\right|_m\right) = 0,$$

for each $m \in M$. Moreover, $g_1 \in C^\infty(M, G)$ is a map that satisfies the above differential equation if and only if there exists a unique element $h \in C^\infty(M', G)$ such that for each $(s, x) \in \mathbb{R} \times M'$, we have

$$g_1(s, x) = h(x) \cdot g(s, x).$$

Proof. The differential equation can be rewritten as

$$T_m g\left(\left.\frac{d}{dt}\right|_m\right) = -T_{\mathbf{1}_G} L_{g(m)} \circ A_m\left(\left.\frac{d}{dt}\right|_m\right).$$

By working in local coordinates on G and invoking the Picard–Lindelöf theorem, it can be shown that for each $x \in M'$, there exists a smooth map

$g_x\colon \mathbb{R} \to G$ such that

$$\begin{cases} T_s g_x \left(\left.\dfrac{d}{dt}\right|_s \right) = -T_{\mathbf{1}_G} L_{g_x(s)} \circ A_{(s,x)} \left(\left.\dfrac{d}{dt}\right|_{(s,x)} \right), & s \in \mathbb{R} \\ \qquad g_x(0) = \mathbf{1}_G & \end{cases}.$$

By writing down the corresponding differential equation for the curve $s \mapsto (g_x(s), x) \in G \times M'$ and using the smoothness of A, we can argue from the smooth dependence of the solution of an ODE on the initial condition that the map

$$g\colon \mathbb{R} \times M' \to G, \quad (s,x) \mapsto g_x(s),$$

is smooth, and it is readily seen that g solves the original differential equation.

With regard to the final assertion, let $h \in C^\infty(M', G)$, and define $g_1 \in C^\infty(M, G)$ as in the statement of the proposition. Since

$$\begin{aligned} T_{(s,x)} g_1 \left(\left.\frac{d}{dt}\right|_{(s,x)} \right) &= T_{g(s,x)} L_{h(x)} \circ T_s g_x \left(\left.\frac{d}{dt}\right|_{(s,x)} \right) \\ &= -T_{g(s,x)} L_{h(x)} \circ T_{\mathbf{1}_G} L_{g_x(s)} \circ A_m \left(\left.\frac{d}{dt}\right|_m \right) \\ &= -T_{\mathbf{1}_G} L_{g_1(s,x)} \circ A_m \left(\left.\frac{d}{dt}\right|_m \right), \end{aligned}$$

the map g_1 is a solution too.

Conversely, suppose g_1 is a solution, and define the map

$$h\colon M' \to G, \quad x \mapsto g_1(0,x) \cdot g(0,x)^{-1}.$$

Then h is smooth, because g and g_1 are smooth and G is a Lie group. Furthermore, we just argued that the map

$$g_2\colon M \to G, \quad (s,x) \mapsto h(x) \cdot g(s,x),$$

is a solution to the differential equation. Now define $g_{1,x}$ and $g_{2,x}$ in terms of g_1 and g_2, respectively, in the same way in which g_x depends on g for each $x \in M'$, and observe that $g_{1,x}$ and $g_{2,x}$ are both solutions to the ODE that we used to define g_x, and that they satisfy the same initial condition. Uniqueness of the solution implies $g_{1,x} = g_{2,x}$ for each $x \in M'$, hence $g_1 = g_2$, which concludes the proof of the final assertion. ∎

Since the temporal gauge does not uniquely determine a section σ, but only a family of sections that are equal up to a time independent gauge transformation, it is referred to as a *partial gauge*.

By imposing the temporal gauge, the factor in front of the density $\sqrt{|\det \alpha|}$ in the expression for the Lagrangian density becomes

$$-\frac{1}{2g^2}\langle F, F\rangle_{\bigwedge,\mathfrak{g}} = \frac{1}{2g^2}\left(\sum_{\nu=1}^{3}\left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0}\right\rangle_{\mathfrak{g}} - \sum_{1\leq\mu<\nu\leq 3}\langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\right).$$

Pulling back the right-hand side along an inclusion of M' in M, we obtain an expression of the form

$$K\left(\frac{\partial A}{\partial x^0}\right) - V(A),$$

where $K\colon TQ \to C_c^\infty(M')$ and $V\colon Q \to C_c^\infty(M')$ denote the first and second term within parentheses multiplied by the factor $1/(2g^2)$, respectively. We now define a *Hamiltonian density*

$$\begin{aligned}
\mathscr{H}\colon TQ &\to \Gamma^\infty\left(\left|\textstyle\bigwedge^3\right|(T^*M')\right),\\
\left(A, \frac{\partial A}{\partial x^0}\right) &\mapsto \left(K\left(\frac{\partial A}{\partial x^0}\right) + V(A)\right)\cdot\sqrt{|\det\beta|}\\
&= \frac{1}{2g^2}\left(\sum_{\nu=1}^{3}\left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0}\right\rangle_{\mathfrak{g}} + \sum_{1\leq\mu<\nu\leq 3}\langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\right) dx,
\end{aligned}$$

where $\sqrt{|\det\beta|}$ denotes the density corresponding to β, and dx denotes the canonical density on $\mathbb{R}^3$. The Hamiltonian H is obtained by integrating the Hamiltonian density over M', i.e.,

$$H\colon TQ \to \mathbb{R}, \quad H = \int_{\mathbb{R}^3}\mathscr{H}.$$

We leave it to the reader to check that this Hamiltonian is invariant under time independent gauge transformations.

2.3 Discretisation

We now show how to pass from the continuous version to the lattice version of Hamiltonian gauge theory. Let M and M' be as in the previous section, and let $P' := M' \times G$ with the obvious projection map π and group action be the trivial principal fibre bundle over M' with structure group G. Let ω be a connection on P, and consider a smooth curve $c\colon [0,1] \to M'$. Let $S_c\colon \pi^{-1}(\{c(0)\}) \to \pi^{-1}(\{c(1)\})$ be the corresponding parallel transport map, and let $a \in G$ be the unique element such that $P_c(c(0), \mathbf{1}_G) = (c(1), a)$. By G-equivariance of the parallel transport map, we have

$$S_c(c(0), g) = g \cdot S_c(c(0), \mathbf{1}_G) = g \cdot (c(1), a) = (c(1), g \cdot a),$$

so with respect to the canonical trivialisation, the parallel transport map S_c can be thought of as right multiplication with a.

Now suppose that we have two smooth curves $c_1, c_2\colon [0,1] \to M'$, and suppose that $c_1(1) = c_2(0)$, i.e., the end point of the first curve is the starting point of the second. Let a_1 and a_2 be their corresponding group elements, and let $c_2 \circ c_1$ be a concatenation of these two curves. (A reparametrisation of a curve does not result in a change of its parallel transport map.) Then

$$S_{c_2 \circ c_1}(c(0), \mathbf{1}_G) = S_{c_2} \circ S_{c_1}(c(0), \mathbf{1}_G) = S_{c_2}(c(0), a_1) = (c(0), a_1 \cdot a_2),$$

so $a_1 \cdot a_2$ is the group element corresponding to $c_2 \circ c_1$, i.e., concatenation of curves corresponds to multiplication of the corresponding group elements. Note that this is a consequence of the assumption that G acts from the left on the principal bundle; if G acts from the right, which is most often the case in the literature, then parallel transport corresponds to left multiplication with a group element, and concatenation of curves corresponds to multiplication in the opposite group. Since we prefer to work with ordinary multiplication, this motivates our deviation from the convention. An argument similar to the one above shows that if $c_2 = c_1^{-1}$, then $a_2 = a_1^{-1}$.

Next, we discuss gauge transformations. Before, we worked in the canonical trivialisation, which corresponds to the section $x \mapsto (x, \mathbf{1}_G)$. Now let $c\colon [0,1] \to M'$ again be a curve in M', let $g \in C^\infty(M', G)$, and consider the trivialisation Φ associated to the section $x \mapsto (x, g(x))$. Let a and a' be the group elements associated to the parallel transport map with respect to the first and second trivialisation, respectively. Then

$$\begin{aligned}(c(1), a') &= \Phi \circ S_c \circ \Phi^{-1}(c(0), \mathbf{1}_G) = \Phi \circ S_c(c(0), g \circ c(0)) \\ &= (g \circ c(0)) \cdot \Phi \circ S_c(c(0), \mathbf{1}_G) = (g \circ c(0)) \cdot \Phi(c(1), a) \\ &= (g \circ c(0)) \cdot a \cdot \Phi(c(1), \mathbf{1}_G) = (g \circ c(0)) \cdot a \cdot (c(1), (g \circ c(1))^{-1}),\end{aligned}$$

hence

$$a' = (g \circ c(0)) \cdot a \cdot (g \circ c(1))^{-1}.$$

We are now ready to introduce a lattice. Fix a finite set of points $\Lambda^0 \subset M'$, and let Λ^1 be a finite set of piecewise smooth paths between elements of Λ^0. The pair $\Lambda := (\Lambda^0, \Lambda^1)$ is then a finite, oriented graph. The starting and end points of an edge $e \in \Lambda^1$ will be denoted by $s(e)$ and $t(e)$, respectively. The idea behind lattice gauge theory is that the set of maps

$$G^{\Lambda^1} := \{f \mid f \colon \Lambda^1 \to G\},$$

can serve as an approximation to the space of connections on the principal fibre bundle P' over M'. The gauge group is given by G^{Λ^0}, and gauge transformations are implemented by means of the group action

$$G^{\Lambda^0} \times G^{\Lambda^1} \to G^{\Lambda^1}, \quad ((g_x)_{x \in \Lambda^0}, (a_e)_{e \in \Lambda^1}) \mapsto (g_{s(e)} a_e g^{-1}_{(t(e))})_{e \in \Lambda^1},$$

which is motivated by the calculation in the previous paragraph.

Thus far, the formalism in the Lagrangian case is the same as it is in the Hamiltonian case, the only difference being that in the Lagrangian case, Λ^1 corresponds to edges in M as opposed to M'. In the Hamiltonian case, we must define a notion of phase space, which is typically done by taking the cotangent bundle of the configuration space. The configuration space is given by G^{Λ^0}, hence phase space is given by $T^*(G^{\Lambda^0})$ endowed with its canonical symplectic form. We can identify $T^*(G^{\Lambda^0})$ with $(T^*G)^{\Lambda^0}$ using the corresponding isomorphism between tangent spaces. The action of the gauge group on the configuration space induces an action on phase space, which is given by

$$\begin{gathered}G^{\Lambda^0} \times (T^*G)^{\Lambda^1} \to (T^*G)^{\Lambda^1}, \\ ((g_x)_{x \in \Lambda^0}, (a_e, \xi_e)_{e \in \Lambda^1}) \mapsto \left(g_{s(e)} a_e g^{-1}_{t(e)}, \xi_e \circ (T_{a_e}(L_{g_{s(e)}} \circ R_{g^{-1}_{t(e)}}))^{-1} \right)_{e \in \Lambda^1}.\end{gathered}$$

2.2 Remark. The above action preserves the canonical symplectic form and there is a canonical momentum map for this phase space. However, since the action of the gauge group on the configuration and/or phase space is not free, the associated Marsden–Weinstein quotient is not a manifold. The analysis of the reduced phase space in a simple example of a lattice consisting of one plaquette can be found in [38, 57, 56]. The analysis of the reduced phase space for the general case can be done along the same lines using spanning trees in the graph Λ, at least to the extent that one is able to describe the Marsden–Weinstein quotient as a topological space; describing the various strata is a more difficult problem. This is beyond the scope of this thesis, however.

In order to write down a Hamiltonian, we impose the additional condition on Λ that it is a cubic lattice in M' with lattice spacing ℓ, and that each edge is parallel to one of the coordinate axes of $M' = \mathbb{R}^3$. The Hamiltonian of the corresponding lattice gauge theory is given by

$$
\begin{aligned}
&H \colon (T^*G)^{\Lambda^1} \to \mathbb{R}, \\
&(a_e, \xi_e)_{e\in\Lambda^1} \mapsto \frac{\ell^3}{2g^2} \sum_{e\in\Lambda^1} \beta^*_{G,a_e}(\xi_e, \xi_e) + \frac{1}{g^2\ell} \sum_{p\in\Lambda^2} \mathrm{Re}\left(\mathrm{Tr}(1-\rho(a_p))\right).
\end{aligned}
$$

Here,

- β_G denotes the bi-invariant Riemannian metric
$$\beta_{G,a}(v,w) := \mathrm{Tr}(d\rho_a(v)^* \cdot d\rho_a(w)),$$
on G, and β^*_G is its pushforward under the musical isomorphism $TG \to T^*G$;

- Λ^2 denotes the set of *plaquettes* of Λ, which are the loops in Λ that are the concatenation of four distinct edges. The group element associated to such a loop p is labelled a_p. Although a_p depends on the orientation, and, in the nonabelian case, on the base point as well, the expression $\mathrm{Re}(\mathrm{Tr}(\rho(a_p)))$ does not. Indeed, it is independent of the orientation of the loop, because a reversal of the orientation changes a_p into a_p^{-1}, and since ρ is a unitary representation, we find that
$$\mathrm{Tr}(\rho(a_p^{-1})) = \mathrm{Tr}(\rho(a_p)^*) = \overline{\mathrm{Tr}(\rho(a_p))},$$

hence the real parts of $\mathrm{Tr}(\rho(a_p^{-1}))$ and $\mathrm{Tr}(\rho(a_p))$ are equal. It is independent of the base point of p, since a different choice of base point results in conjugation of a_p with an element $b \in G$, and we have

$$\mathrm{Tr}(\rho(ba_pb^{-1})) = \mathrm{Tr}(\rho(b)\rho(a_p)\rho(b)^{-1}) = \mathrm{Tr}(\rho(a_p)).$$

This argument can also be used to show that $\mathrm{Tr}(\rho(a_p))$ is gauge invariant.

The first instance of a Hamiltonian for lattice gauge theory can be found in the original paper by Kogut and Susskind in [62]. The Hamiltonian above however resembles more closely that of Rudolph and Schmidt in [103, section 10.7], who give the Hamiltonian for the case in which $G = \mathrm{SU}(3)$ and ρ is the defining representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$.

The first and second sum in the formula for the Hamiltonian are called the *electric* and *magnetic term*, respectively. We have just argued that the magnetic term is gauge invariant. The electric term is gauge invariant, since β_G is a bi-invariant Riemannian metric, thus the Hamiltonian is gauge invariant. Both terms and their names will be motivated in the following two sections. Let us already mention that in this expression for H, in the case of electromagnetism ($G = U(1)$), ξ_e should be thought of as the average of the electric field on the path e.

2.4 The magnetic term

We start by motivating the magnetic term in the Hamiltonian. Its continuum counterpart is the term

$$\frac{1}{2g^2}\int_{\mathbb{R}^3}\sum_{1\leq\mu<\nu\leq 3}\langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\, dx.$$

First of all, let us mention that in the case of electromagnetism, $F_{\mu\nu} = \varepsilon^{\mu\nu j}B_j$, where $\mu < \nu$, the number j is such that $\{\mu,\nu,j\} = \{1,2,3\}$, and $\varepsilon^{\mu\nu j}$ is the Levi-Civita tensor, i.e., the sign of the permutation that maps 1, 2 and 3 to μ, ν and j, respectively. Here, we assume that the magnetic field B_j in the j-th direction takes values in $\mathfrak{g} = i\mathbb{R}$; to obtain the corresponding real number, which is the quantity that physicists work with, one should multiply by $-ig^{-1}$ (where g denotes the coupling constant that appears in the Lagrangian density mentioned at the beginning of section 2.2).

We return to the general setting, dropping the assumption that $G = U(1)$. To approximate the integral, we fix a compact subset $C \subseteq \mathbb{R}^3$, and a cubic lattice Λ that is embedded in C in such a way that the small closed cubes of which each edge is (the image of) an element of Λ^1 are contained in C, and the sum of their volumes is approximately equal to the volume of C. We assume in addition that the field F is approximately constant over a distance of the order of the lattice spacing ℓ, and that it is negligible outside of C, so that we have

$$\begin{aligned}\int_{\mathbb{R}^3} \sum_{1\leq\mu<\nu\leq 3} \langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\, dx &\approx \int_C \sum_{1\leq\mu<\nu\leq 3} \langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\, dx \\ &\approx \sum_{x\in X} \ell^3 \sum_{1\leq\mu<\nu\leq 3} \langle F_{\mu\nu}(x), F_{\mu\nu}(x)\rangle_{\mathfrak{g}},\end{aligned}$$

where

$$X := \{x \in \Lambda^0 \colon x + \ell e_\mu, x + \ell e_\nu, x + \ell(e_\mu + e_\nu) \in \Lambda^0 \text{ for } 1 \leq \mu < \nu \leq 3\},$$

where $e_\mu \in \mathbb{R}^3$ denotes the μ-th standard basis vector.

Before we move on to the next step, we prove the following result:

2.3 Proposition. *Let (P, M, G, π) be a principal fibre bundle, let $\omega \in \Omega^1(P, \mathfrak{g}^*)$ be a connection 1-form on this bundle, and let $D\omega \in \Omega^2(P, \mathfrak{g}^*)$ be its covariant derivative, i.e., the curvature 2-form associated to ω. Let $p \in P$, let $v, w \in T_{\pi(p)}M$, let (U, φ) be a chart on M such that $\pi(p) \in U$ and $\varphi \circ \pi(p) = 0$, and let $\theta := d\varphi_{\pi(p)} \colon T_{\pi(p)}M \to \mathbb{R}^n$. Furthermore, fix $\delta > 0$ such that the convex hull of the points 0, $\delta\theta(v)$, $\delta\theta(w)$ and $\delta\theta(v+w)$ is a subset of $\varphi(U)$. For each $t \in (0, \delta]$, let $c_t \colon [0, 4t] \to \varphi(U)$ be the piecewise smooth loop based at $\varphi \circ \pi(p)$ given by*

$$c_t(s) := \begin{cases} s\theta(v) & s \in [0, t] \\ t\theta(v) + (s - t)\theta(w) & s \in (t, 2t] \\ (3t - s)\theta(v) + t\theta(w) & s \in (2t, 3t] \\ (4t - s)\theta(w) & s \in (3t, 4t] \end{cases}.$$

Finally, for any curve $c \colon [a, b] \to M$, let $S_c \colon \pi^{-1}(\{c(a)\}) \to \pi^{-1}(\{c(b)\})$ be the associated parallel transport map (for the connection ω). Then for *each smooth local section σ of π defined in a neighbourhood of $\pi(p)$, we have*

$$D\omega_p(T_{\pi(p)}\sigma(v), T_{\pi(p)}\sigma(w)) = -\omega_p\left(\frac{d}{dt}\, S_{\varphi^{-1}\circ c_{\sqrt{t}}}(p)\Big|_{t=0}\right).$$

Proof. First, we note that the vectors $v, w \in T_{\pi(p)}M$ can be extended to vector fields V_0 and W_0 on M such that they are constant with respect to the chart (U, φ) on the convex hull of the images of the points 0, $\delta\theta(v)$, $\delta\theta(w)$ and $\delta\theta(v+w)$ under φ^{-1}. Now let V and W be the unique horizontal lifts to P of V_0 and W_0, respectively, i.e., V is the unique vector field on P such that

$$V(p) \in \ker\omega_p \cap T_p\pi^{-1}(\{V_0 \circ \pi(p)\}),$$

for each $p \in P$, and W is defined similarly. Let Φ_V and Φ_W be their flows on P. Then for any local section σ of π defined in a neighbourhood of $\pi(p)$, the vectors $T_{\pi(p)}\sigma(v) - V(p)$ and $T_{\pi(p)}\sigma(w) - W(p)$ are vertical, hence we obtain

$$\begin{aligned} D\omega_p(T_{\pi(p)}\sigma(v), T_{\pi(p)}\sigma(w)) &= D\omega_p(V(p), W(p)) = d\omega_p(V(p), W(p)) \\ &= -\omega([V, W])(p), \end{aligned}$$

where in the final step, we used the invariant formula

$$d\omega(V, W) = V(\omega(W)) - W(\omega(V)) - \omega([V, W]).$$

Furthermore, we have

$$S_{\varphi^{-1}\circ c_t}(p) = \Phi_{W,-t} \circ \Phi_{V,-t} \circ \Phi_{W,t} \circ \Phi_{V,t}(p),$$

for each $t \in (0, \delta]$, which yields

$$\begin{aligned} \frac{d}{dt}\, S_{\varphi^{-1}\circ c_{\sqrt{t}}}(p)\Big|_{t=0} &= \frac{d}{dt}\, \Phi_{W,-\sqrt{t}} \circ \Phi_{V,-\sqrt{t}} \circ \Phi_{W,\sqrt{t}} \circ \Phi_{V,\sqrt{t}}(p)\Big|_{t=0} \\ &= [V, W](p); \end{aligned}$$

the second step is a consequence of the discussion in [106, pp. 159–163]. Applying the map ω_p to both sides of this equation and comparing it to the expression for $D\omega_p(T_{\pi(p)}\sigma(v), T_{\pi(p)}\sigma(w))$, we obtain the desired result. ■

We now resume the derivation of the magnetic term in the Hamiltonian. Fix $x \in X$ and $\mu, \nu \in \{1, 2, 3\}$, with $\mu < \nu$. There is an associated plaquette p of which the four corners are given by x, $x + \ell e_\mu$, $x + \ell(e_\mu + e_\nu)$, and $x + \ell e_\nu$. We claim that

$$\langle F_{\mu\nu}(x), F_{\mu\nu}(x)\rangle_{\mathfrak{g}} \approx \frac{2}{\ell^4}\mathrm{Re}\left(\mathrm{Tr}(1 - \rho(a_p))\right). \tag{2.1}$$

Let σ be the canonical section of P', let ω be the connection on P' such that $F = \sigma^*(D\omega)$. Applying Proposition 2.3, we find that

$$F_{\mu\nu}(x) = -\omega_{\sigma(x)}\left(\frac{d}{dt}\, P_{\phi^{-1}\circ c_{\sqrt{t}}} \circ \sigma(x)\Big|_{t=0}\right),$$

where the vectors v and w that are used to define the curve $c_{\sqrt{t}}$ in the proposition are $\frac{\partial}{\partial x^\mu}\big|_x$ and $\frac{\partial}{\partial x^\nu}\big|_x$, respectively. For each $t > 0$, the parallel transport map $P_{\phi^{-1}\circ c_{\sqrt{t}}}$ can be viewed as right multiplication with a group element; for each $t > 0$, let $b(t)$ be this group element, and let $a(t) := b(t^2)$. Note that $a_p = a(\ell)$. Furthermore, we have

$$P_{\phi^{-1}\circ c_{\sqrt{t}}} \circ \sigma(x) = (x, b(t))$$

The right-hand side is a curve in the image of the fibre of P' over x, with tangent vector

$$\frac{d}{dt}(x, b(t))|_{t=0} = \frac{d}{dt}\,\left(x, \exp(t \cdot b'(0)) \cdot \mathbf{1}_G\right) \cdot\Big|_{t=0}$$

at $t = 0$, hence

$$F_{\mu\nu}(x) = -b'(0),$$

and therefore

$$d\rho_{\mathbf{1}_G}(F_{\mu\nu}(x)) = -d\rho_{\mathbf{1}_G}(b'(0)).$$

Next, let us approximate $b'(0)$ in terms of a_p. Applying the chain rule to $\rho(a(t))$, we obtain

$$\frac{d}{dt}\rho(a(t))|_{t=0} = 0, \quad \frac{d^2}{dt^2}\rho(a(t))|_{t=0} = 2d\rho_{\mathbf{1}_G}(b'(0)),$$

hence, by L'Hôpital's rule,

$$d\rho_{\mathbf{1}_G}(b'(0)) = \lim_{t\to 0}\frac{\rho(a(t)) - \rho(a(0))}{t^2}.$$

Since $a(0) = \mathbf{1}_G$, we thus find

$$d\rho_{\mathbf{1}_G}(b'(0)) \approx \frac{\rho(a(t)) - 1}{t^2},$$

for $t \approx 0$. In particular, taking $t = \ell$, we obtain

$$\begin{aligned}
\langle F_{\mu\nu}(x), F_{\mu\nu}(x)\rangle_{\mathfrak{g}} &= \mathrm{Tr}(d\rho_{\mathbf{1}_G}(b'(0))^* \cdot d\rho_{\mathbf{1}_G}(b'(0))) \\
&\approx \mathrm{Tr}\left(\frac{\rho(a_p)^* - 1}{\ell^2} \cdot \frac{\rho(a_p) - 1}{\ell^2}\right) \\
&= \frac{2}{\ell^4}\mathrm{Re}\left(\mathrm{Tr}(1 - \rho(a_p))\right),
\end{aligned}$$

which is equation (2.1).

To arrive at the expression for the magnetic term in the Hamiltonian, we must make the additional assumption that the number of small cubes with volume ℓ^3 enclosed by the edges of Λ is large; if Λ encloses a large cube, as sketched in section 2.1, then this translates to the assumption that N is large. This is necessary to show that

$$\begin{aligned}
\sum_{x\in X} \ell^3 \sum_{1\leq\mu<\nu\leq 3} \langle F_{\mu\nu}(x), F_{\mu\nu}(x)\rangle_{\mathfrak{g}} &\approx \sum_{p\in\Lambda^2} \ell^3 \cdot \frac{2}{\ell^4}\mathrm{Re}\left(\mathrm{Tr}(1 - \rho(a_p))\right) \\
&= \frac{2}{\ell} \sum_{p\in\Lambda^2} \mathrm{Re}\left(\mathrm{Tr}(1 - \rho(a_p))\right),
\end{aligned}$$

since in the present case, X is a proper subset of Λ^0, and not every plaquette formed by Λ occurs as a plaquette p in the previous paragraph. The assumption guarantees that the error in the magnetic term due to these facts remains small compared to the magnetic term itself. Putting all of the approximations together, we obtain

$$\frac{1}{2g^2} \int_{\mathbb{R}^3} \sum_{1\leq\mu<\nu\leq 3} \langle F_{\mu\nu}, F_{\mu\nu}\rangle_{\mathfrak{g}}\, dx \approx \frac{1}{g^2\ell} \sum_{p\in\Lambda^2} \mathrm{Re}\left(\mathrm{Tr}(1 - \rho(a_p))\right),$$

as desired.

2.5 The electric term

The continuum counterpart of the electric term is

$$\frac{1}{2g^2}\int_{\mathbb{R}^3}\sum_{\nu=1}^{3}\left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0}\right\rangle_{\mathfrak{g}} dx.$$

Again, in the case of electromagnetism, under the additional assumption that the temporal gauge has been imposed, we have

$$\frac{\partial A_\nu}{\partial x^0} = -\frac{E_\nu}{c},$$

where c denotes the speed of light, and E_ν the electric field in the ν-th direction, which explains the use of the term ‘electric’, and our fields take values in $\mathfrak{g}$; one has to multiply by $-ig^{-1}$ to obtain the quantity physicists work with.

The first steps to arrive at the lattice approximation of this expression, namely choosing a compact subset $C \subseteq \mathbb{R}^3$ and a lattice Λ, are the same as those in the magnetic case. The approximation obtained from these steps is

$$\int_{\mathbb{R}^3}\sum_{\nu=1}^{3}\left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0}\right\rangle_{\mathfrak{g}} dx \approx \sum_{x\in X}\ell^3\sum_{\nu=1}^{3}\left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0}\right\rangle_{\mathfrak{g}}.$$

Our next objective is to show how a change in the group element associated to an edge in Λ^1 corresponding to a curve that goes into the positive ν-direction is related to $\frac{\partial A_\nu}{\partial x^0}$. To this end, it is useful to first prove a more general result.

2.4 Lemma. *Let (P, M, G, π) be a principal fibre bundle, let ω be a connection on P and let $c\colon [0,1] \to M$ be a smooth curve. Furthermore, let σ be a local section of P whose domain contains the image of c, let $A := \sigma^*(\omega)$, and let $a\colon [0,1] \to G$ be the curve on G determined by the requirement that for each $s \in [0,1]$, the element $a(s)$ is the group element corresponding to the parallel transport map of ω along the curve $c|_{[0,s]}$ in the gauge σ; in particular, $a(0) = \mathbf{1}_G$ for each $t \in \mathbb{R}$. Then for each $s_0 \in [0,1]$, we have*

$$\frac{d}{ds}(a(s)\cdot a(s_0)^{-1})|_{s=s_0} = -\mathrm{Ad}(a(s_0)) \circ A_{c(s_0)}\left(c'(s_0)\right).$$

or equivalently,

$$\frac{d}{ds}(a(s_0)^{-1} \cdot a(s))|_{s=s_0} = -A_{c(s_0)}\left(c'(s_0)\right).$$

Proof. let Φ be the local trivialisation corresponding to σ, and let $\gamma\colon [0,1] \to P$ be the unique lift of c to P determined by

$$\begin{cases} \omega_{\gamma(s)}(\gamma'(s)) = 0, & s \in [0,1] \\ \gamma(0) = \sigma \circ c(0) \end{cases}.$$

By definition of a, we have

$$\Phi \circ \gamma(s) = (c(s), a(s)),$$

hence, by equivariance of Φ, we have

$$\gamma(s) = a(s) \cdot \Phi^{-1}(c(s), \mathbf{1}_G) = a(s) \cdot \sigma \circ c(s).$$

It follows that for each $s_0 \in [0,1]$, we have

$$\begin{aligned} 0 &= \omega_{\gamma(s_0)}(\gamma'(s_0)) \\ &= \omega_{\gamma(s_0)}\left(\frac{d}{ds}(a(s) \cdot a(s_0)^{-1}) \cdot \gamma(s_0)|_{s=s_0}\right) \\ &\quad + \omega_{a(s_0)\cdot\sigma\circ c(s_0)}\left(T_{\sigma\circ c(s_0)}L_{a(s_0)}\frac{d}{ds}\sigma \circ c(s)|_{s=s_0}\right) \\ &= \frac{d}{ds}(a(s) \cdot a(s_0)^{-1})|_{s=s_0} + \mathrm{Ad}(a(s_0)) \circ A_{c(s_0)}\left(c'(s_0)\right), \end{aligned}$$

from which the assertion readily follows. ∎

2.5 Proposition. *Let (P, M, G, π) be a principal fibre bundle over M with structure group G, let ω be a connection on P. Let*

$$c\colon (-\varepsilon, \varepsilon) \times [0,1] \to M,$$

be a smooth map, which we view as a family of smooth curves $[0,1] \to M$. Furthermore, let σ be a local section of P whose domain contains the image of c, and let

$$a\colon (-\varepsilon, \varepsilon) \times [0,1] \to G,$$

be the associated smooth family of smooth curves $[0,1] \to G$ determined by the requirement that for each $(t,s) \in (-\varepsilon,\varepsilon) \times [0,1]$, the element $a(t,s)$ is the group element corresponding to the parallel transport map of ω along the curve $c|_{\{t\}\times[0,s]}$ with respect to the chosen gauge σ; in particular, $a(t,0) = \mathbf{1}_G$ for each $t \in (-\varepsilon,\varepsilon)$. Then, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\, a(t,1)\cdot a(0,1)^{-1}\big|_{t=0} \\
&\qquad = -\int_0^1 \operatorname{Ad}(a(0,s)) \left(\frac{\partial}{\partial t}\, A_{c(t,s)} \left(\frac{\partial}{\partial s'} c(s',t)|_{s'=s} \right)\bigg|_{t=0} \right) ds.
\end{aligned}$$

Proof. Let $s_0 \in [0,1]$, and let (U,φ) be a chart on G such that $\mathbf{1}_G \in U$. Then

$$\begin{aligned}
&\frac{\partial}{\partial s}\frac{\partial}{\partial t}\, a(t,s)\cdot a(0,s)^{-1}\big|_{\substack{s=s_0\\t=0}} \\
&\quad = \frac{\partial}{\partial s}\, d\varphi^{-1}_{\mathbf{1}_G} \circ d\varphi_{\mathbf{1}_G} \left(\frac{\partial}{\partial t}\, (a(t,s)\cdot a(0,s)^{-1})\big|_{t=0} \right)\bigg|_{s=s_0} \\
&\quad = d\varphi^{-1}_{\mathbf{1}_G} \left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\, \varphi(a(t,s)\cdot a(0,s)^{-1})\big|_{\substack{s=s_0\\t=0}} \right) \\
&\quad = d\varphi^{-1}_{\mathbf{1}_G} \left(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\, \varphi(a(t,s)\cdot a(0,s)^{-1})\big|_{\substack{s=s_0\\t=0}} \right) \\
&\quad = d\varphi^{-1}_{\mathbf{1}_G} \left(\frac{\partial}{\partial t}\, d\varphi \left(T_{\mathbf{1}_G} R_{a(t,s_0)\cdot a(0,s_0)^{-1}} \left(\frac{\partial}{\partial s}\, a(t,s)\cdot a(t,s_0)^{-1}\big|_{s=s_0} \right) \right.\right. \\
&\qquad\quad \left.\left. + T_{\mathbf{1}_G} L_{a(t,s_0)\cdot a(0,s_0)^{-1}} \left(\frac{\partial}{\partial s}\, \left(a(0,s)\cdot a(0,s_0)^{-1}\right)^{-1}\Big|_{s=s_0} \right) \right)\bigg|_{t=0} \right).
\end{aligned}$$

We now apply Lemma 2.4 twice; first, note that

$$\begin{aligned}
&T_{\mathbf{1}_G} R_{a(t,s_0)\cdot a(0,s_0)^{-1}} \left(\frac{\partial}{\partial s}\, a(t,s)\cdot a(t,s_0)^{-1}\big|_{s=s_0} \right) \\
&\quad = -T_{\mathbf{1}_G} R_{a(t,s_0)\cdot a(0,s_0)^{-1}} \circ \operatorname{Ad}(a(t,s_0)) \circ A_{c(t,s_0)} \left(\frac{\partial}{\partial s}\, c(t,s)|_{s=s_0} \right) \\
&\quad = -T_{\mathbf{1}_G} \left(L_{a(t,s_0)} \circ R_{a(0,s_0)^{-1}} \right) \circ A_{c(t,s_0)} \left(\frac{\partial}{\partial s}\, c(t,s)|_{s=s_0} \right),
\end{aligned}$$

and similarly, one finds that

$$
\begin{aligned}
&T_{\mathbf{1}_G} L_{a(t,s_0)\cdot a(0,s_0)^{-1}} \left(\frac{\partial}{\partial s} \left(a(0,s)\cdot a(0,s_0)^{-1}\right)^{-1}\Big|_{s=s_0} \right) \\
&\quad = T_{\mathbf{1}_G} L_{a(t,s_0)\cdot a(0,s_0)^{-1}} \circ \mathrm{Ad}(a(0,s_0)) \circ A_{c(0,s_0)} \left(\frac{\partial}{\partial s}\, c(0,s)|_{s=s_0} \right) \\
&\quad = T_{\mathbf{1}_G} \left(L_{a(t,s_0)} \circ R_{a(0,s_0)^{-1}} \right) \circ A_{c(0,s_0)} \left(\frac{\partial}{\partial s}\, c(0,s)|_{s=s_0} \right),
\end{aligned}
$$

hence

$$
\begin{aligned}
&\frac{\partial}{\partial s}\frac{\partial}{\partial t}\, a(t,s)\cdot a(0,s)^{-1}\big|_{\substack{s=s_0\\ t=0}} \\
&\quad = -d\varphi_{\mathbf{1}_G}^{-1} \Bigg(\frac{\partial}{\partial t}\, d\varphi \circ T_{\mathbf{1}_G} \left(L_{a(t,s_0)} \circ R_{a(0,s_0)^{-1}} \right) \Bigg(\\
&\qquad\qquad A_{c(t,s_0)} \left(\frac{\partial}{\partial s}\, c(t,s)|_{s=s_0} \right) - A_{c(0,s_0)} \left(\frac{\partial}{\partial s}\, c(0,s)|_{s=s_0} \right) \Bigg)\Bigg|_{t=0} \Bigg) \\
&\quad = -d\varphi_{\mathbf{1}_G}^{-1} \left(\frac{\partial}{\partial t}\, d\varphi \circ T_{\mathbf{1}_G} \left(L_{a(t,s_0)} \circ R_{a(0,s_0)^{-1}} \right) (0)\big|_{t=0} \right) \\
&\qquad - \mathrm{Ad}(a(0,s_0)) \left(\frac{\partial}{\partial t} A_{c(t,s_0)} \left(\frac{\partial}{\partial s}\, c(t,s)|_{s=s_0} \right) \right) \\
&\quad = -\mathrm{Ad}(a(0,s_0)) \left(\frac{\partial}{\partial t}\, A_{c(t,s_0)} \left(\frac{\partial}{\partial s} c(s,t)|_{s=s_0} \right)\Bigg|_{t=0} \right).
\end{aligned}
$$

Here, in the second step, we used the chain rule. Applying the fundamental theorem of calculus and using the fact that $a(t,0) = \mathbf{1}_G$ for each $t \in (-\varepsilon,\varepsilon)$, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\, a(t,1)\cdot a(0,1)^{-1}\big|_{t=0} \\
&\quad = \frac{\partial}{\partial t}\, a(t,0)\cdot a(0,0)^{-1}\big|_{t=0} + \int_0^1 \frac{\partial}{\partial s}\frac{\partial}{\partial t}\, a(t,s)\cdot a(0,s)^{-1}\big|_{t=0}\, ds \\
&\quad = -\int_0^1 \mathrm{Ad}(a(0,s)) \left(\frac{\partial}{\partial t}\, A_{c(t,s)} \left(\frac{\partial}{\partial s'} c(s',t)|_{s'=s} \right)\Bigg|_{t=0} \right) ds,
\end{aligned}
$$

as desired. ■

We now apply this proposition as follows. Let $M := \mathbb{R} \times M'$, let $P := M \times G$, let ω be a connection on P, let $\sigma \in \Gamma^\infty(P)$ be a section such that $A := \sigma^*(D\omega)$ is in the temporal gauge, and let $x \in M'$. Consider the map

$$c\colon \mathbb{R} \times [0,1] \to M, \quad (t,s) \mapsto (t,x) + s\ell \cdot e_\nu,$$

that represents a smooth family of smooth curves $[0,1] \to M$. Let

$$a\colon \mathbb{R} \times [0,1] \to G,$$

be the map that has the property that for each $(t,s) \in \mathbb{R} \times [0,1]$, $a(t,s)$ is the group element corresponding to the parallel transport map of ω along the curve $c|_{\{t\}\times[0,s]}$ with respect to the gauge σ. Proposition 2.5 then yields

$$\frac{\partial}{\partial t}\, a(t,1) \cdot a(t_0,1)^{-1}\big|_{t=t_0} = -\ell \int_0^1 \mathrm{Ad}(a(t_0,s)) \circ \frac{\partial A_\nu}{\partial x^0} \circ c(t_0,s)\, ds,$$

for each $t_0 \in \mathbb{R}$. This formula can be simplified somewhat by imposing a constraint in addition to the temporal gauge. By using an argument similar to the one in Proposition 2.1, we may find a gauge σ' that is obtained from σ by applying a gauge transformation that does not vary in the time direction in such a way that, with respect to this gauge, we have

$$A_{(t_0,x)}\left(\frac{\partial}{\partial x^\nu}\bigg|_{(t_0,x)}\right) = 0,$$

for each $x \in M'$. Lemma 2.4 now implies that $a(t_0,s) = \mathbf{1}_G$ for each $s \in [0,1]$, therefore

$$\frac{\partial}{\partial t}\, a(t,1)|_{t=t_0} = -\ell \int_0^1 \frac{\partial A_\nu}{\partial x^0} \circ c(t_0,s)\, ds, \tag{2.2}$$

and hence

$$\frac{\partial A_\nu}{\partial x^0} \approx -\ell^{-1} \cdot \frac{\partial}{\partial t}\, a(t,1)|_{t=t_0}, \tag{2.3}$$

with respect to σ' in some neighbourhood of the image of the curve $c|_{\{t_0\}\times[0,1]}$, provided that ℓ is small and one has some control over the variation of $\dfrac{\partial A_\nu}{\partial x^0}$. Note that the norm of $\dfrac{\partial A_\nu}{\partial x^0}$ does not change when passing from σ to σ'.

Thus, in the lattice approximation, if the state of the system is described by the element $(a_e, v_e)_{e\in\Lambda^1} \in (TG)^{\Lambda^1} \cong T(G^{\Lambda^1})$, then we have

$$\sum_{x\in X} \ell^3 \sum_{\nu=1}^{3} \left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0} \right\rangle_{\mathfrak{g}} \approx \sum_{e\in\Lambda^1} \ell^3 \cdot \beta_{G,a_e}(\ell^{-1}v_e, \ell^{-1}v_e),$$

where we make the same assumption on the lattice Λ as at the end of the last section to motivate this approximation. For reasons that will be discussed in chapter 8, and in particular in Example 8.14, a natural Riemannian metric on G^{Λ^1} is

$$\begin{aligned} T_aG^{\Lambda^1} \times T_aG^{\Lambda^1} &\to \mathbb{R}, \\ ((v_e)_{e\in\Lambda^1}, (w_e)_{e\in\Lambda^1}) &\mapsto \sum_{e\in\Lambda^1} \ell_e^{-1} \beta_{G,a_e}(v_e, w_e). \end{aligned}$$

Here, $a = (a_e)_{e\in\Lambda^1} \in G^{\Lambda^1}$, and ℓ_e denotes the length of an edge $e \in \Lambda^1$; in the case of a cubic lattice with lattice spacing ℓ, all lengths are of course equal to ℓ. We use the musical isomorphism corresponding to this inner product to establish an isomorphism $TG^{\Lambda^1} \to T^*G^{\Lambda^1}$. If $(a_e, \xi_e)_{e\in\Lambda^1} \in (T^*G)^{\Lambda^1} \cong T^*G^{\Lambda^1}$ denotes the image of $(a_e, v_e)_{e\in\Lambda^1}$ under this isomorphism, then

$$\sum_{e\in\Lambda^1} \ell^3 \cdot \beta_{G,a_e}(\ell^{-1}v_e, \ell^{-1}v_e) \approx \sum_{e\in\Lambda^1} \ell^3 \cdot \beta^*_{G,a_e}(\xi_e, \xi_e),$$

and in the case of electromagnetism, ξ_e represents the average electric field on e. Thus we obtain

$$\frac{1}{2g^2} \int_{\mathbb{R}^3} \sum_{\nu=1}^{3} \left\langle \frac{\partial A_\nu}{\partial x^0}, \frac{\partial A_\nu}{\partial x^0} \right\rangle_{\mathfrak{g}} dx \approx \frac{\ell^3}{2g^2} \sum_{e\in\Lambda^1} \beta^*_{G,a_e}(\xi_e, \xi_e),$$

which justifies the electric term in the Hamiltonian for the lattice gauge theory.

Chapter 3

Gauss's law and reduction

In this chapter, we give an operator-algebraic interpretation of the notion of an ideal generated by the unbounded operators associated to the elements of the Lie algebra of a Lie group that implements the symmetries of a quantum system. We use this interpretation to establish a link between Rieffel induction and the implementation of a local Gauss law in lattice gauge theories similar to the method discussed by Kijowski and Rudolph in [60, 61]. This chapter is based on [107].

3.1 Introduction

There are well-developed theories of reduction of both classical and quantum mechanical systems that possess symmetries. The study of reduction of classical systems was initiated by Dirac in [32] with his theory of first and second order constraints, and later put into the language of symplectic manifolds by Arnold and Smale. The reduction of a symplectic manifold with respect to an equivariant moment map was described by Marsden and Weinstein in their paper [81]. For a more detailed account of the history of symplectic reduction, we refer to [82] and references therein. A procedure known as Rieffel induction, developed in [96], appears to be a good candidate for a quantum version of Marsden–Weinstein reduction [64] (cf. [65, IV.2]).

The primary aim of this chapter is to compare two different ways to reduce the quantum mechanical observable algebra. The first one is the method of Rieffel induction mentioned above. The second one was outlined by Kijowski and Rudolph in [60, 61] in the context of a quantum lattice gauge theory, in which they explicitly implement a constraint, the local Gauss law, by ensuring that the operators associated to the generators of the gauge group vanish in the observable algebra of the reduced system. The corresponding operators on the unreduced Hilbert space are unbounded, however, which requires them to appeal to the theory of C*-algebras generated by unbounded operators as developed by Woronowicz in [121], something that is not necessary for Rieffel induction. Nevertheless, both procedures yield the same reduced observable algebra. In this chapter, we modify the latter method so that it is formulated entirely in terms of bounded operators, and show that it agrees with the final step in the process of Rieffel induction.

This chapter is organised as follows. In section 2, we briefly recall the process of Rieffel induction. In section 3, we formulate and prove the main theorem that establishes the link. In section 4, we discuss some examples, including the lattice gauge theory mentioned above.

3.2 Reduction of quantum systems using Rieffel induction

The kinematical data of a quantum system consists of a Hilbert space $\mathcal{H}$ and a faithful representation π of a C*-algebra A on $\mathcal{H}$. A continuous symmetry of a quantum system typically (but, in accordance with Wigner's celebrated theorem, not exclusively,) corresponds to a continuous unitary representation $\rho\colon K \to U(\mathcal{H})$ of some Lie group K on $\mathcal{H}$. We are interested in studying the reduction of the kinematical data with respect to such symmetries in the case in which K is both compact and connected. A systematic way to obtain this reduction, known as Rieffel induction, was proposed by Landsman in [64] using an induction procedure for representations of C*-algebras developed by Rieffel in [96].

Let us first briefly recall the process of Rieffel induction; we refer the reader who wants more detail to [65, Part IV]. The exposition provided by Wren [122] is a nice introduction to the subject. Starting from the above representation of the group K, one endows $\mathcal{H}$ with the structure of a right Hilbert $C^*(K)$-module, where $C^*(K)$ denotes the group C*-algebra of K.

One subsequently takes the quotient of $\mathcal{H}$ with respect to the null space of a bilinear form on $\mathcal{H}$, which yields a space naturally isomorphic to $\mathcal{H}^K$, the subspace of $\mathcal{H}$ of K-invariant elements. Thus we obtain the Hilbert space of the reduced system.

At the level of the observable algebra, one first considers the algebra A^K of elements of A that are invariant with respect to the given unitary representation. The space $\mathcal{H}^K$ is invariant under these observables, yielding a representation π^K of the C*-algebra A^K on $\mathcal{H}^K$. The image of this representation is isomorphic to $A^K / \ker(\pi^K)$, and hence one obtains a faithful representation of $A^K / \ker(\pi^K)$ on $\mathcal{H}^K$, which forms the remaining part of the kinematical data of the reduced system.

Motivated by the theory of strict quantization of observable algebras as described extensively in [65, Part II], we are interested in the case where $A = B_0(\mathcal{H})$, the space of compact operators, and its representation on $\mathcal{H}$ is the obvious one. It can then be shown that $A^K / \ker(\pi^K)$ is isomorphic to $B_0(\mathcal{H}^K)$, and that the representation of this algebra on the reduced Hilbert space $\mathcal{H}^K$ is again the obvious one.

3.3 Associating algebras to infinitesimal generators

The main purpose of this section is to discuss a possible interpretation of an observation made by Kijowski and Rudolph in [61, section 3] in the case of a quantum lattice gauge theory, namely that the kernel of the representation $\pi^K : A^K \to B(\mathcal{H}^K)$, where as before $A = B_0(\mathcal{H})$, is in some sense generated by the elements of the Lie algebra $\mathfrak{k}$ of the symmetry group K. Using Stone's theorem on strongly continuous one-parameter groups of unitary operators, the representation of the group K on $\mathcal{H}$ can be used to associate anti-self adjoint operators on $\mathcal{H}$ to the elements of $\mathfrak{k}$. In the examples that we are interested in, these anti-self adjoint operators are differential operators and the Hilbert space on which they are defined is infinite-dimensional, which results in the fact that the operators are unbounded. If instead the representation space is finite-dimensional then the representation of the Lie algebra $\mathfrak{k}$ is bounded. Using this fact and other standard results from the representation theory of Lie groups, we will show how the operators associated to the elements of the Lie algebra generate $\ker(\pi^K)$. In addition, we need the following preparatory lemma, which can be found in [87, Exercise 4.2(c)]:

3.1 Lemma. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space, let a be a compact operator on $\mathcal{H}$, and suppose that $(b_j)_{j\in J}$ is a bounded net of bounded operators that converges strongly to $b \in B(\mathcal{H})$. Then the net $(b_j a)_{j\in J}$ converges in norm to ba. If in addition the operator b_j is hermitian for each $j \in J$, then the net $(ab_j)_{j\in J}$ converges in norm to ab.*

The following result shows how $\ker(\pi^K)$ can be generated by unbounded operators. Throughout the rest of this chapter, given a continuous representation σ of a Lie group with unit element $\mathbf{1}$, the associated Lie algebra representation $d\sigma_{\mathbf{1}}$ will be denoted by σ as well.

3.2 Theorem. *Let K, π, A, $\mathcal{H}$ and π^K be as above. Let S be a collection of finite-dimensional subrepresentations of a continuous representation $\rho\colon K \to U(\mathcal{H})$, and for each $\sigma \in S$, let $\mathcal{H}_\sigma \subseteq \mathcal{H}$ be the subspace on which σ is represented. Suppose that these representation spaces form an orthogonal decomposition of $\mathcal{H}$, i.e.,*

$$\mathcal{H} = \overline{\bigoplus_{\sigma\in S} \mathcal{H}_\sigma}.$$

Then $\ker(\pi^K)$ is the closed, two-sided ideal generated by the set

$$\left\{ \int_K \rho(k)\sigma(X)^n\rho(k)^{-1}\,\mathrm{d}k \colon \sigma \in S,\ X \in \mathfrak{k},\ n \geq 1 \right\}. \tag{3.1}$$

3.3 Remark. In the set of generators above, $\sigma(X)$ is regarded as the compression of $\rho(X)$ to $\mathcal{H}_\sigma$. Moreover, we note that the integrals of vector-valued functions can be defined using Bochner integration.

Proof of Theorem 3.2. Let I be the ideal in A^K generated by the set in equation (3.1). We first show that $I \subseteq \ker(\pi^K)$. Indeed, $\sigma(X)^n$ maps $\mathcal{H}$ into $\mathcal{H}_\sigma$ for each $\sigma \in S$, each $X \in \mathfrak{k}$ and each $n \geq 1$, hence so does $\int_K \rho(k)\sigma(X)^n\rho(k)^{-1}\,\mathrm{d}k$, which implies that it is a finite rank operator. In particular, it is compact. Moreover, it follows from left invariance of the Haar measure that $\int_K \rho(k)\sigma(X)^n\rho(k)^{-1}\mathrm{d}k$ is equivariant with respect to ρ, so it is an element of A^K. Finally, to show that it is an element of $\ker(\pi^K)$, let $p_\sigma : \mathcal{H} \to \mathcal{H}_\sigma$ be the orthogonal projection onto the representation space of σ. For each $v \in \mathcal{H}^K$ we have $p_\sigma v \in \mathcal{H}^K$ and $\sigma(X)v = 0$. Hence

$$\int_K \rho(k)\sigma(X)^n \rho(k)^{-1}(v)\,\mathrm{d}k = \int_K \rho(k)\sigma(X)^n(v)\,\mathrm{d}k = 0,$$

and therefore $\int_K \rho(k)\sigma(X)^n\rho(k)^{-1}\,\mathrm{d}k \in \ker(\pi^K)$. Thus the generators of I are contained in $\ker(\pi^K)$. Since $\ker(\pi^K)$ is a closed, two sided ideal, it follows that $I \subseteq \ker(\pi^K)$.

We turn to the proof of the reverse inclusion. Let $b \in \ker(\pi^K)$, let $p_{\mathcal{H}^K}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^K$. It is easy to see that

$$p_{\mathcal{H}^K} = \int_K \rho(k)\,\mathrm{d}k.$$

Since $b \in \ker(\pi^K)$, it follows that

$$b = b(\mathrm{Id}_{\mathcal{H}} - p_{\mathcal{H}^K}) = b\int_K (\mathrm{Id}_{\mathcal{H}} - \rho(k))\,\mathrm{d}k = \sum_{\sigma\in S} b\int_K (p_\sigma - \sigma(k))\,\mathrm{d}k.$$

By the preceding lemma, the series on the right-hand side is norm-convergent, hence to show that $b \in I$, it suffices to show that

$$b\int_K (p_\sigma - \sigma(k))\,\mathrm{d}k \in I,$$

for each $\sigma \in S$. Since I is closed under multiplication with elements of A^K, we are done if we can show that

$$\int_K (p_\sigma - \sigma(k))\,\mathrm{d}k \in I.$$

From bi-invariance of the Haar measure and Fubini's theorem, we infer that

$$\int_K (p_\sigma - \sigma(k))\,\mathrm{d}g = \int_K\int_K \rho(h)(p_\sigma - \sigma(k))\rho(h)^{-1}\,\mathrm{d}h\,\mathrm{d}k.$$

The norm topology and the strong topology coincide on the finite-dimensional algebra $B(\mathcal{H}_\sigma)$, so the first integral on the right-hand side is a norm limit of Riemann sums, i.e. for each $\varepsilon > 0$, there exist $k_j \in K$ and $c_j \geq 0$ for $j = 1, \dots, n$, such that

$$\Bigg\| \int_K \int_K \rho(h)(p_\sigma - \sigma(k))\rho(h)^{-1}\,\mathrm{d}h\,\mathrm{d}k - \sum_{j=1}^{n} c_j \int_K \rho(h)(p_\sigma - \sigma(k_j))\rho(h)^{-1}\,\mathrm{d}h \Bigg\| < \varepsilon.$$

Since I is closed by definition, it suffices to show that

$$\sum_{j=1}^{n} c_j \int_K \rho(h)(p_\sigma - \sigma(k_j))\rho(h)^{-1}\,dh \in I.$$

We prove this by showing that

$$(*) \qquad \int_K \rho(h)(p_\sigma - \sigma(k))\rho(h)^{-1}\,\mathrm{d}h \in I,$$

for each $k \in K$. Now fix such a k. Because K is both compact and connected, the exponential map $\exp\colon \mathfrak{k} \to K$ is surjective, so there exists an $X \in \mathfrak{k}$ such that $k = \exp(X)$. But σ is a homomorphism of Lie groups, so

$$\sigma(k) = \sigma \circ \exp(X) = \exp \circ \sigma(X) = p_\sigma \sum_{j=0}^{\infty} \frac{\sigma(X)^j}{j!}.$$

Thus

$$p_\sigma - \sigma(k) = -\sum_{j=1}^{\infty} \frac{\sigma(X)^j}{j!}.$$

The map

$$B(\mathcal{H}_\sigma) \to B(\mathcal{H}_\sigma), \quad a \mapsto \int_K \rho(h) a \rho(h)^{-1}\,\mathrm{d}h,$$

is a linear operator on the finite-dimensional algebra $B(\mathcal{H}_\sigma)$, hence it is norm-continuous, so

$$\int_K \rho(h)(p_\sigma - \sigma(k))\rho(h)^{-1}\,\mathrm{d}h = -\sum_{j=1}^{\infty} \frac{1}{j!} \int_K \rho(h)\sigma(X)^j \rho(h)^{-1}\,\mathrm{d}h,$$

and the series on the right-hand side converges with respect to the norm on $B(\mathcal{H})$. Each of the partial sums is an element of I, which implies that $(*)$ holds, as desired. ■

In general, the set S in the above theorem will not be unique. Suppose that we are in the situation of the theorem, and that we are given a set S satisfying the assumption. If the Hilbert space $\mathcal{H}$ is infinite-dimensional, there are infinitely many different sets like S that satisfy the assumption. Indeed, S is an infinite set because $\mathcal{H}$ is infinite-dimensional, so we can take any finite subset $F \subseteq S$ containing at least two representations, define the subrepresentation $\sigma_F := \bigoplus_{\sigma\in F}\sigma$, and the set $S' = (S\backslash F) \cup \{\sigma_F\}$. Then $S' \neq S$, and it satisfies the assumption of the theorem.

The last argument can be formulated slightly more generally as follows: Suppose that S_1 and S_2 are sets of orthogonal finite-dimensional subrepresentations, and that S_1 satisfies the assumption of the theorem. If each element of S_1 is a subrepresentation of S_2, then S_2 also satisfies the assumption. If $\mathcal{H}$ is infinite-dimensional, then from any set S_1 one can always construct a different set S_2 with these properties. Thus one can always make the set S 'arbitrarily coarse', which is another reason why we view Theorem 3.2 as a possible way to make the idea of 'the ideal generated by unbounded operators' rigorous.

The fact that a set S like the one in Theorem 3.2 always exists, is a consequence of the following result. Recall that for any representation ρ of a group K on a space V, a vector v is called *K-finite* if and only if the smallest subspace containing v that is invariant under ρ, i.e., the span of $\{\rho(k)v\colon k \in K\}$, is finite-dimensional. We let V^{fin} denote the subspace of K-finite vectors of V.

3.4 Proposition. *Let ρ be a continuous representation of a compact Lie group K in a complete locally convex topological vector space V. Then V^{fin} is dense in V.*

This result can be found in [35] as part of Corollary 4.6.3. Using this result and Zorn's lemma, one can now readily show that there exists a set S that satisfies the assumption of our theorem. Needless to say, explicitly exhibiting such a set might be impossible. However, as we shall see in the next section, there are situations in which there is a canonical choice for S.

To prepare for the examples in the final section, we briefly recall some other notions from representation theory. Let $\widehat{K}$ be the set of equivalence classes of irreducible representations of K, and let $[\delta] \in \widehat{K}$. The *isotypical component of type* $[\delta]$ is the set $V[\delta]$ of elements $v \in V^{\mathrm{fin}}$ such that the subrepresentation generated by v is equivalent to the representation $\delta \oplus \cdots \oplus \delta$ (n copies) for some $n \in \mathbb{N}$.

3.4 Examples

3.5 Example. *(Local Gauss law in quantum lattice gauge theories)*
We start with the motivating example for this chapter, namely the local Gauss law discussed by Kijowski and Rudolph in [61] in the context of quantum lattice gauge theories. Suppose that we have a finite, oriented, connected graph Λ embedded in the base manifold of a principal fibre bundle with compact structure group G, and consider the corresponding lattice gauge theory, as in section 2.3. Let $\mathcal{G} := G^{\Lambda^0}$, and let $\mathcal{A} := G^{\Lambda^1}$, and consider the action of $\mathcal{G}$ on $\mathcal{A}$. This action induces a continuous unitary representation ρ of $\mathcal{G}$ on $\mathcal{H} := L^2(\mathcal{A})$ by $(\rho(g)(\psi))(a) := \psi(g^{-1} \cdot a)$, where $g \in \mathcal{G}$, $\psi \in \mathcal{H}$ and $a \in G^{\Lambda^1}$. The Hilbert space $\mathcal{H}$ is considered to be the quantisation of the classical phase space $T^*\mathcal{A}$, and the algebra of compact operators on $\mathcal{H}$ is the associated field algebra; this will be elaborated on in the next chapter.

Baez already noted in [12] that the action of $\mathcal{G}$ restricts to the isotypical components of $\mathcal{H}$ with respect to the left regular representation of G^{Λ^1} on this space —in fact, the $\mathcal{G}$-invariant subspaces of the isotypical components form the basis for spin networks as introduced in [101]. Indeed, the representation ρ can be regarded as the composition of two group homomorphisms; first, we have a homomorphism

$$\iota\colon \mathcal{G} \to G^{\Lambda^1} \times G^{\Lambda^1} \simeq (G \times G)^{\Lambda^1}, \quad (g_x)_{x\in\Lambda^0} \mapsto (g_{s(e)}, g_{t(e)})_{e\in\Lambda^1},$$

which by connectedness of the graph is an injection if and only if Λ has more than one vertex. The second homomorphism is the product representation $L \times R\colon G^{\Lambda^1} \times G^{\Lambda^1} \to U(\mathcal{H})$ of the left and right regular representations L and R, respectively. It follows that each subspace of $\mathcal{H}$ that is invariant under the representation $L \times R$, is also invariant under ρ. The

Peter–Weyl theorem asserts that

$$\mathcal{H}^{\mathrm{fin}} = \bigoplus_{[\delta]\in\widehat{G^{\Lambda^1}}} \mathcal{H}[\delta],$$

and that the isotypical components $\mathcal{H}[\delta]$ are irreducible subrepresentations of the representation $L \times R$ of dimension $\dim(\delta)^2$. Here, $\mathcal{H}^{\mathrm{fin}}$ denotes the set of G^{Λ^1}-finite vectors with respect to the left regular representation of G^{Λ^1} on $L^2(\mathcal{A})$. Thus we may take the set S in Theorem 3.2 to be the collection of subrepresentations obtained by restricting ρ to $\mathcal{H}[\delta]$ for each $\delta \in \widehat{G^{\Lambda^1}}$.

Since the elements of the Lie algebra of the gauge group $\mathcal{G}$ generate the gauge group, Theorem 3.2 provides a link between two different methods of reduction of the quantum observable algebra, the first being Rieffel induction, and the second being the implementation of a local Gauss law by taking the quotient with respect to an ideal generated by unbounded operators associated to Lie algebra elements, as mentioned by Kijowski and Rudolph in [61].

3.6 Example. *(Hamiltonian symmetries)*
The second example that we discuss is really a class of examples, namely that of quantum systems with a given Hamiltonian that possesses a certain symmetry. Let $\mathcal{H}$ be a Hilbert space, let H be a (possibly unbounded) self-adjoint operator on $\mathcal{H}$, and suppose $\rho\colon K \to U(\mathcal{H})$ is a continuous unitary representation of a compact connected Lie group K on $\mathcal{H}$ with the property that $\rho(k)$ preserves $\mathrm{Dom}(H)$ and $[\rho(k), H] = 0$ for each $k \in K$. Moreover, let $\sigma_p(H)$ be the point spectrum of H, and for each $\lambda \in \sigma_p(H)$, let $\mathcal{H}_\lambda$ be the eigenspace corresponding to λ. Suppose that $\mathcal{H}_\lambda$ is finite dimensional for each $\lambda \in \sigma_p(\lambda)$, and that $\mathcal{H} = \overline{\bigoplus_{\lambda\in\sigma_p(H)} \mathcal{H}_\lambda}$. Then ρ restricts to a representation ρ_λ on $\mathcal{H}_\lambda$ for each $\lambda \in \sigma_p(H)$, and we may set $S := \{\rho_\lambda \colon \lambda \in \sigma_p(H)\}$.

A notable subclass of examples satisfying the above conditions is the class of quantum systems in which $\mathcal{H} = L^2(Q)$, where Q is a compact smooth Riemannian manifold that admits a Lie group of isometries, and $H = \Delta$ is the Laplacian on Q. In particular, the lattice gauge theories in Example 3.5 can be studied in this way if one endows G^{Λ^1} with a bi-invariant Riemannian metric. It is a result from representation theory (cf.

[110, Theorem 3.3.5]) that $\mathcal{H}[\delta]$ is a subspace of an eigenspace of Δ for each $\delta \in \widehat{G^{\Lambda^1}}$, so the decomposition obtained in the previous example is finer than the decomposition into eigenspaces of Δ.

Chapter 4

Groupoids and refinements

In this chapter, which is based on [8], we present an operator-algebraic approach to the quantisation and reduction of lattice field theories. Our approach uses groupoid C*-algebras to describe the observables. We introduce direct systems of Hilbert spaces and direct systems of (observable) C*-algebras, and, dually, corresponding inverse systems of configuration spaces and (pair) groupoids. We then take their corresponding limits. Since all constructions are equivariant with respect to the gauge group, the reduction procedure as described in the previous chapter applies in the limit as well.

We have already briefly sketched in section 1.1 how groupoids can be used to quantise a classical system. For this reason, when investigating the interplay between quantisation and regularisation as we do in this chapter in the context of lattice gauge theory, it is natural to ask whether (and if so, how) they are compatible with regularisation. In this chapter, we find maps between the relevant (pair) groupoids that at first sight appear to be good candidates, i.e., they can be used to define a direct system in a mathematically natural way. However, upon examination of the *-homomorphisms that these maps induce between the corresponding groupoid C*-algebras, and the limits of the corresponding direct systems, we find that there are several problems:

- Their transposes do not induce maps between the corresponding state spaces;

- The map relevant to the thermodynamic limit is not the one that is used in the physics literature;
- The C*-algebra that we find for the continuum limit does not admit the sort of dynamics that one would expect to be able to define on such an algebra.

These problems are discussed in the final section of this chapter and the first section of the next one, and show that the groupoid approach to quantisation (at least in its current formulation) is incompatible with the type of regularisation discussed in this thesis. This is the point of contention mentioned at the very beginning of chapter 1. The second part of this thesis, specifically chapter 8, attempts to solve the first two problems by using different maps in the direct system of observable algebras, which in turn requires that we work with larger observable algebras than the ones used in this chapter, which is where the classical and quantum resolvent algebra on $T^*\mathbb{T}^n$, defined in chapters 5 and 7, respectively, come in.

4.1 Introduction

For reasons already mentioned in section 2.1, from a mathematical perspective, the Hamiltonian approach to lattice gauge theory is a good starting point for the quantisation of gauge theories. In this chapter, we give a novel operator algebraic approach to the quantization of Hamiltonian lattice gauge theories using groupoid C*-algebras. We discuss how gauge theories corresponding to 'finer' lattices, or more generally, to graphs, are related to coarser ones. At the Hilbert space level, this was described mathematically by Baez in [12], whose results we extend to the field algebras, obervable algebras and Hamiltonians. Baez's paper is part of a research program initiated by Ashtekar now referred to as loop quantum gravity (LQG), and his construction of the limit Hilbert space is based primarily on work by Ashtekar et al. [10]. It is worth mentioning that analogous constructions of the observable algebra are still an active area of research within the LQG community; see for instance [2] and [1] for an approach using noncommutative geometry.

Aside from constructing the field and observable algebras, we also study their relation to certain groupoids, and show that this relation is preserved in the relevant limits, thus providing a geometric picture of the kinematical framework for constructing the infinite volume and continuum limits of such theories. The first, also called the thermodynamic limit, has recently been studied along similar lines on a lattice in [47, 48], but without the groupoid description. It should be noted that we restrict ourselves to 'pure gauge theories', i.e., we do not consider the interaction of gauge fields with matter fields, and that we only consider the electric part of such fields in our Hamiltonians. The electric term in the Hamiltonian is essentially the 'free' problem, while the inclusion of the magnetic term would introduce an interaction. The study of the system with interactions is much more involved since it requires some form of renormalisation, and will therefore be left as the subject of future research.

This chapter is organized as follows. In section 4.2 we review the classical Hamiltonian lattice gauge theory, and its quantum mechanical counterpart. In sections 4.3, 4.4 and 4.5, we recall some old results and develop some new methods to relate lattices with different lattice spacings, as well as the corresponding classical and quantum systems. This is necessary for constructing the thermodynamic and continuum limits, which may be treated on equal footing in this formalism. We also describe the behaviour of groupoid C*-algebras associated to refinements of graphs. In section 4.6 we describe the behaviour of the system with respect to the limit and also identify the groupoid that describes this limit. We finish the chapter by pointing out a number of problems with the limit observable algebra.

4.2 The quantum system

Since we have already discussed the classical formulation of lattice gauge theory in chapter 2, we will focus on the mathematical setup for the corresponding quantum system, along the lines of strict deformation quantization (cf. [100] and [65]). We also describe the field and observable algebras as groupoid C*-algebras and discuss reduction of the quantum system.

We use the same notation as in chapter 2. The structure group of the (lattice) gauge theory will be denoted by G, which we assume to be a compact Lie group. Furthermore, given a finite oriented graph $\Lambda =$

(Λ^0, Λ^1), let $\mathcal{G} := G^{\Lambda^0}$ with the obvious group operation be the group of gauge transformations of the lattice gauge theory, and let $\mathcal{K} := G^{\Lambda^1}$ be its configuration space.

To obtain a quantisation of the canonical system $T^*(\mathcal{K}) = T^*(G^{\Lambda^1})$, we adopt the C*-algebraic approach to quantisation of the cotangent bundle as described in [65, section II.3]. In line with Weyl quantisation of $T^*\mathbb{R}^n$, the quantisation of T^*Q for any compact Riemannian manifold Q is given there by the observable algebra $B_0(L^2(Q))$, the space of compact operators on $L^2(Q)$. Since the compact Lie group $\mathcal{K}$ is naturally a compact Riemannian manifold, we find that the quantised observable algebra of $T^*(\mathcal{K})$ is given by $A := B_0(L^2(\mathcal{K}))$ and the Hilbert space is $\mathcal{H} = L^2(\mathcal{K})$. Note that this is in line with the finite-lattice approximation of Hamiltonian QCD in [60, 61].

Geometrically, we can also realize this C*-algebra as a groupoid C*-algebra. The construction is based on the pair groupoid $\mathsf{G} = \mathcal{K} \times \mathcal{K}$ so we first recall its general definition (cf. [27, section 3]).

4.1 Definition. Let X be a set. The *pair groupoid* associated to X has object space X and space of morphism $X \times X$, with source and target maps given by the projections onto the first and the second factor respectively. Composition of morphism is given by concatenation and the inverse by $(x, y)^{-1} = (y, x)$. Note that all free and transitive groupoids are necessarily pair groupoids.

Now suppose that X is a locally compact Hausdorff space endowed with a Radon measure μ of full support X. Recall that this is a measure on the Borel σ-algebra of X that is locally finite and inner regular. The *-algebra $C_c(X \times X)$, with (kernel) product

$$(\phi_1 * \phi_2)(x_1, x_2) := \int_X \phi_1(y, x_2)\phi_2(x_1, y)\, \mathrm{d}\mu(y),$$

and involution $\phi^*(x_1, x_2) := \overline{\phi(x_2, x_1)}$, is then represented by compact operators on $L^2(X, \mu)$. Indeed, given $h \in C_c(X \times X)$, the associated integral operator T_h on $L^2(X)$ is given by

$$T_h\psi(x) := \int_X h(y, x)\psi(y)\, \mathrm{d}\mu(y). \tag{4.1}$$

By definition the *reduced groupoid C^*-algebra* $C^*_{\mathrm{r}}(X \times X)$ is the closure in $B(L^2(X))$ of the image of the above representation. This is actually isomorphic to the *full* groupoid C*-algebra and one has

$$C^*_{\mathrm{r}}(X \times X) \simeq C^*(X \times X) \simeq B_0(L^2(X)). \tag{4.2}$$

We refer to [93] for full details on the construction of groupoid C*-algebras, see also [65, III.3.4 and III.3.6]. The relation of this construction to strict quantization can be found in [65, III.3.12].

If we specialise to our case for which $\mathcal{K} = G^{\Lambda^1}$ is our configuration space, this leads us to consider the pair groupoid $\mathsf{G} := \mathcal{K} \times \mathcal{K}$, whose space of morphism is $\mathsf{G}^{(1)} = \mathcal{K} \times \mathcal{K}$ and whose space of objects is $\mathsf{G}^{(0)} = \mathcal{K}$. Thus the observable algebra A is isomorphic to $C^*(\mathsf{G})$.

It is possible to include matter fields in the formalism. In [61] (cf. [48]), Kijowski and Rudolph extend the algebra A by considering its algebraic tensor product with the CAR-algebra associated to the classical space of the matter fields, and complete it with respect to its unique C*-norm. As mentioned in the introduction to this chapter, we will restrict our attention to the gauge fields, ignoring the matter fields and all objects associated to it.

4.2.1 Gauge symmetries and reduction of the quantized system

Let us recall from the previous chapter that the action of the gauge group $\mathcal{G}$ on the configuration space $\mathcal{K}$ induces a unitary representation U of the gauge group $\mathcal{G} = G^{\Lambda^0}$ on $\mathcal{H} = L^2(\mathcal{K})$:

$$U((g_x)_{x\in\Lambda^0})\psi((a_e)_{e\in\Lambda^1}) = \psi\left((g_{s(e)}a_e g_{t(e)}^{-1})_{e\in\Lambda^1}\right), \tag{4.3}$$

for all $\psi \in \mathcal{H}$. The reduced Hilbert space is the subspace $\mathcal{H}^{\mathcal{G}}$ of $\mathcal{G}$-invariant vectors in $\mathcal{H}$, and the observable algebra of the reduced system is the space of compact operators $B_0(\mathcal{H}^{\mathcal{G}})$ on the reduced Hilbert space. In view of equation (4.2), it is natural to associate the pair groupoid $(\mathcal{G}\backslash\mathcal{K}) \times (\mathcal{G}\backslash\mathcal{K})$ to the reduced system.

4.2 Remark. In the literature, the algebra A of the system without reduction of gauge symmetries is called the *field algebra*, whereas the algebra corresponding to the reduced system is typically referred to as the observable algebra. We shall adopt this terminology throughout the rest of the chapter.

4.2.2 The quantum Hamiltonian

Recall from sections 2.3 and 2.5 that the electric part H_e of the classical Hamiltonian for a cubic lattice with lattice spacing ℓ in three spatial dimensions, evaluated at $(a_e, \xi_e)_{e\in\Lambda^1} \in (T^*G)^{\Lambda^1} \cong T^*\mathcal{K}$ is given by

$$H_e = \frac{\ell^3}{2g^2} \sum_{e\in\Lambda^1} \beta^*_{G,a_e}(\xi_e, \xi_e).$$

We now write this expression as

$$H_e = c(\ell) \cdot \frac{1}{2} \sum_{e\in\Lambda^1} I_e \beta^*_{G,a_e}(\xi_e, \xi_e),$$

where $c(\ell) := \ell^2/g(\ell)^2$ should be thought of as a coupling constant depending on ℓ, and for each $e \in \Lambda^1$, we have $I_e := \ell$. If Λ is a more general lattice, then I_e denotes the length of the edge ℓ_e. The notation I_e originates from [62], in which Kogut and Susskind draw an analogy with a classical system that consists of a lattice that on each link e has a rigid rotor (or rotator) whose moment of inertia is given by I_e.

We now forget about the coupling constant c, and consider the quantisation of the expression that remains, which is the following differential operator on $C^\infty(G^{\Lambda^1})$:

$$H_0 = \sum_{e\in\Lambda^1} -\frac{1}{2} I_e \Delta_e \tag{4.4}$$

where Δ_e is the Laplacian on G with respect to the bi-invariant Riemannian metric β_G, or, which is the same (up to a positive scalar multiple), the quadratic Casimir element of G. The operator H_0 is essentially self-adjoint on $C^\infty(G^{\Lambda^1}) \subset L^2(G^{\Lambda^1})$; we let H_0 denote its closure with domain $\mathrm{Dom}(H_0) \subset L^2(G^{\Lambda^1})$. Since H_0 is the differential operator associated to the quadratic Casimir element of G, it is well-behaved with respect to the action of the gauge group:

4.3 Proposition. *Let $\mathcal{H} := L^2(\mathcal{K})$. The operator H_0 is equivariant with respect to the action of the gauge group defined in equation* (4.3)*. Its restriction $H_{0,\mathrm{red}}$ to* $\mathrm{Dom}(H_0) \cap \mathcal{H}^{\mathcal{G}}$ *is a self-adjoint operator on $\mathcal{H}^{\mathcal{G}}$, and the following diagram*

$$\begin{array}{ccc} \mathrm{Dom}(H_0) & \xrightarrow{\quad H_0 \quad} & \mathcal{H} \\ {\scriptstyle p_{\mathcal{H}^{\mathcal{G}}}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\mathcal{H}^{\mathcal{G}}}} \\ \mathrm{Dom}(H_0) \cap \mathcal{H}^{\mathcal{G}} & \xrightarrow{\quad H_{0,\mathrm{red}} \quad} & \mathcal{H}^{\mathcal{G}} \end{array}$$

is commutative.

Proof. Note that $H_0|_{C^\infty(\mathsf{G}^{(0)})}$ is equivariant with respect to the left-regular representation since it is a left-invariant differential operator. It is also equivariant with respect to the right regular representation, since the quadratic Casimir element Ω_e lies in the center of the universal enveloping algebra $\mathfrak{A}(\mathfrak{g}^{\Lambda^1})$ for each $e \in \Lambda^1$. Thus $H_0|_{C^\infty(\mathsf{G}^{(0)})}$ is equivariant with respect to the action of the product of the two aforementioned representations, so in particular, it is equivariant with respect to the action of $\mathcal{G}$.

An immediate consequence of this equivariance is that $H_0|_{C^\infty(\mathsf{G}^{(0)})}$ leaves $\mathcal{H}^{\mathcal{G}}$ invariant. Because H_0 is by definition the closure of $H_0|_{C^\infty(\mathsf{G}^{(0)})}$, the space $\mathcal{H}^{\mathcal{G}}$ is also an invariant subspace for H_0. In addition, since the orthogonal projection $p_{\mathcal{H}^{\mathcal{G}}}$ onto $\mathcal{H}^{\mathcal{G}}$ is a strong limit of linear combinations of unitary operators associated to elements of H, we have $p_{\mathcal{H}^{\mathcal{G}}}(\mathrm{Dom}(H_0)) \subseteq \mathrm{Dom}(H_0) \cap \mathcal{H}^{\mathcal{G}}$, and the above diagram is indeed commutative.

Finally, we prove that $H_{0,\mathrm{red}}$ is self-adjoint. Let $J\colon \mathcal{H}^2 \to \mathcal{H}^2$ be the operator given by $(x, y) \mapsto (-y, x)$. Then we have $\mathcal{H}^2 = \mathrm{Graph}(H_0) \oplus J(\mathrm{Graph}(H_0))$ by self-adjointness of H_0, cf. [102, Theorem 13.10]. From the fact that $\mathrm{Graph}(H_{0,\mathrm{red}}) \subseteq \mathrm{Graph}(H_0)$, we infer that $\mathrm{Graph}(H_{0,\mathrm{red}}) \perp J(\mathrm{Graph}(H_{0,\mathrm{red}}))$. On the other hand, it follows from our discussion in the previous paragraph that

$$\mathrm{Graph}(H_{0,\mathrm{red}}) = \{(p_{\mathcal{H}^{\mathcal{G}}}(x), p_{\mathcal{H}^{\mathcal{G}}}(y))\colon (x, y) \in \mathrm{Graph}(H_0)\},$$

so $\mathrm{Graph}(H_{0,\mathrm{red}}) + J(\mathrm{Graph}(H_{0,\mathrm{red}})) = (\mathcal{H}^{\mathcal{G}})^2$, hence

$$\mathrm{Graph}(H_{0,\mathrm{red}}) \oplus J(\mathrm{Graph}(H_{0,\mathrm{red}})) = (\mathcal{H}^{\mathcal{G}})^2,$$

which shows that $\mathrm{Graph}(H_{0,\mathrm{red}})$ is indeed self-adjoint. ■

4.3 Refinements of the quantum system

Our approach towards formulating a continuum limit from a gauge theory on a graph is based on a suitable notion of embeddings of graphs, referred to as 'refinements'.

We follow Baez [12] in his description of an inverse system of configuration spaces and a direct system of Hilbert spaces, both indexed over the set of graphs with partial order given by refinement. After reviewing this construction, we will extend this description to the level of the pair groupoids, the corresponding field and observable C*-algebras and the (free) Hamiltonians.

4.3.1 Refinements of graphs

We start by recalling the following notion (cf. [78, Theorem II.7.1]):

4.4 Definition. Let $\Lambda = (\Lambda^0, \Lambda^1)$ be an oriented graph. The *free* or *path category generated by* Λ, denoted by C_Λ, is defined as follows:

- Its set of objects is Λ^0;
- Let $x, y \in \Lambda^0$. The set of morphisms from x to y is given by the collection of orientation respecting paths in Λ with starting point x and end point y;
- Composition of morphisms is given by concatenation of paths;
- The identity element of each object $x \in \Lambda^0$ is the path of length 0 starting and ending at x.

This predicates the following formulation of embedding a graph into another one:

4.5 Definition. Let Λ_i and Λ_j be two oriented graphs with corresponding free categories C_{Λ_i} and C_{Λ_j}. Suppose in addition that there exists a functor $\iota_{i,j} \colon \mathsf{C}_{\Lambda_i} \to \mathsf{C}_{\Lambda_j}$ such that:

(1) The map $\iota_{i,j}^{(0)} \colon \Lambda_i^0 \to \Lambda_j^0$ between the sets of objects is an injection;

(2) The map $\iota_{i,j}^{(1)}$ between the sets of morphisms maps elements of Λ_i^1 (identified with their corresponding paths) to paths in Λ_j such that

- Each edge $e \in \Lambda_i^1$ is mapped to a nontrivial path under the map $\iota_{i,j}^{(1)}$;
- For each $e \in \Lambda_i^1$, the path $\iota_{i,j}^{(1)}(e)$ does not intersect itself;
- If e and e' are distinct elements of Λ_i^1, then the paths $\iota_{i,j}^{(1)}(e)$ and $\iota_{i,j}^{(1)}(e')$ have no common vertices except perhaps for their starting points or end points.

We call the triple $(\Lambda_i, \Lambda_j, \iota_{i,j})$ a *refinement* of the graph Λ_i. Given such a refinement, we say that Λ_i is *coarser* than Λ_j, and that Λ_j is *finer* than Λ_i.

When no confusion arises, we will omit the subscript $_{i,j}$ from ι.

4.6 Remark. Given three graphs Λ_i, Λ_j and Λ_k, and refinements $(\Lambda_i, \Lambda_j, \iota_{i,j})$ and $(\Lambda_j, \Lambda_k, \iota_{j,k})$, then there exists a canonical refinement $(\Lambda_i, \Lambda_k, \iota_{i,k})$, where we have $\iota_{i,k} = \iota_{j,k} \circ \iota_{i,j}$.

This allows us to define another category:

4.7 Definition. We let Refine denote the category with the following properties:

- Its objects are oriented graphs;
- Given two oriented graphs Λ_i and Λ_j, then the set of morphisms from Λ_i to Λ_j is given by the set of refinements $(\Lambda_i, \Lambda_j, \iota)$.
- Composition is given by composition of refinement functors.
- For each oriented graph Λ, there is a canonical refinement $(\Lambda, \Lambda, \mathrm{Id})$, where $\mathrm{Id}^{(0)}$ and $\mathrm{Id}^{(1)}$ are the identity maps on the spaces of objects and morphisms in C_Λ.

Given a refinement $\iota : \Lambda_i \to \Lambda_j$ of two graphs Λ_i, Λ_j, we introduce a map $R_{i,j} \colon \mathcal{K}_j \to \mathcal{K}_i$ between the corresponding configuration spaces as follows. Given an edge $e \in \Lambda_i$, we let

$$R_{i,j}(a)_e = a_{e_1} \cdots a_{e_n}, \tag{4.5}$$

where $\iota^{(1)}_{i,j}(e) = (e_1, \dots, e_n)$. The compatibility of these maps under composition is readily checked and we arrive at the following result.

4.8 Proposition. *There exists a canonical contravariant functor from* Refine *to the category of compact Hausdorff spaces that sends a graph* Λ_i *to the space* $\mathcal{K}_i$*, and a refinement* $(\Lambda_i, \Lambda_j, \iota_{i,j})$ *to the map* $R_{i,j}$*.*

4.9 Remark. (1) A particular consequence of the above proposition is that a direct system $((\Lambda_i)_{i\in I}, (\iota_{i,j})_{i,j\in I,\, i\leq j})$ in Refine induces an inverse system
$((\mathcal{K}_i)_{i\in I}, (R_{i,j})_{i,j\in I,\, i\leq j})$ in the category of compact Hausdorff spaces. In what follows, we will construct various other co- and contravariant functors from Refine to certain categories, which induce direct and inverse systems in these categories, respectively. For the sake of brevity, we will write the above direct system as $(\Lambda_i, \iota_{i,j})$, and do the same with other direct and inverse systems.

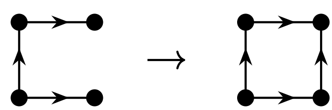
(2) Actually, more information can be encoded into the above functor as follows. Consider the category of which the objects are compact Hausdorff spaces endowed with continuous group actions. Let us write (X, H) for such an object if the space and group are given by X and H, respectively; additional structures such as topologies and the group action will be implicit. A natural notion of a morphism between two objects (X, H) and (Y, K) in this category is a pair (f, ϕ) consisting of a continuous map $f \colon X \to Y$ and a (continuous) group homomorphism $\phi \colon H \to K$ such that f is equivariant with respect to the action of H on both X and Y, i.e., $f(h \cdot x) = \phi(h) \cdot f(x)$ for each $x \in X$ and each $h \in H$.

In addition to the space of parallel transporters $\mathcal{K}$, the image of a graph Λ under the modified functor would encode the gauge group $\mathcal{G} = G^{\Lambda^0}$ of Λ, along with the action of the group on $\mathcal{K}$. Furthermore, the image of a refinement $(\Lambda_i, \Lambda_j, \iota)$ would contain the restriction map $G^{\Lambda_i^0} \to G^{\Lambda_j^0}$ that arises as the pullback of $\iota^{(0)}$; see subsection 4.3.3. As with the previous point in this remark, a similar statement holds for the other functors in this chapter that map Λ to some object that has not yet been reduced with respect to the action of the gauge group.

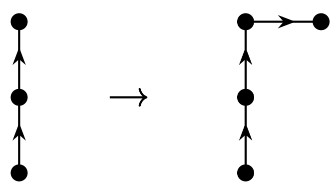
4.3.2 Elementary refinements

In what follows we need to carry out a number of computations, some of which are rather tedious to write out for arbitrary refinements. We simplify our computations by making use of the fact that any refinement can be decomposed into the composition of *elementary refinements*. This is in line with [12, Lemma 4], although we do not admit the reversal of the orientation of an edge. More precisely, given an arbitrary refinement $(\Lambda_i, \Lambda_j, \iota)$, there exists a sequence $(\Lambda_k, \Lambda_{k+1}, \iota_{k,k+1})_{k=0}^{n-1}$ of refinements such that $\Lambda_0 = \Lambda_i$, $\Lambda_n = \Lambda_j$, $\iota = \iota_{n-1,n} \circ \cdots \circ \iota_{0,1}$, and for each $i \in \{0, \ldots, n-1\}$, the refinement $(\Lambda_k, \Lambda_{k+1}, \iota_{k,k+1})$ falls into one of the following two classes of examples:

- The graph Λ_{k+1} is obtained from Λ_k by adding an extra edge



  or by adding an extra vertex and an extra edge:

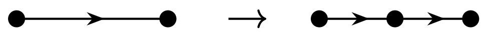


  At the level of configuration spaces, both of these embeddings induce the map

$$R_{k,k+1}\colon \mathcal{K}_{k+1} \to \mathcal{K}_k, \quad ((a_e)_{e\in\Lambda_k^1}, a_{e_0}) \mapsto (a_e)_{e\in\Lambda_k^1}, \tag{4.6}$$

  where $e_0 \in \Lambda_{k+1}^{(1)}$ denotes the 'added' edge.

- The graph Λ_{k+1} is obtained from Λ_k by subdividing an edge into two edges:



This type of embedding induces the following map between configuration spaces:
(4.7)
$$R_{k,k+1}\colon \mathcal{K}_{k+1} \to \mathcal{K}_k, \quad ((a_e)_{e\in\Lambda_k^1-\{e_0\}}, a_{e_1}, a_{e_2}) \mapsto ((a_e)_{e\in\Lambda_k^1-\{e_0\}}, a_{e_1}a_{e_2}),$$

where $e_0 \in \Lambda_k^1$ denotes the edge that is 'subdivided' into e_1 and e_2.

It follows from Proposition 4.10 that

$$R_{i,j} = R_{0,1} \circ \cdots \circ R_{n-1,n},$$

and hence that the composition on the right-hand side is independent of the choice of the sequence $(\Lambda_k, \Lambda_{k+1}, \iota_{k,k+1})_{k=0}^{n-1}$.

Throughout the rest of the text, whenever we discuss a refinement $(\Lambda_i, \Lambda_j, \iota)$ of graphs, we shall only discuss the cases in which the refinements are elementary refinements, and use the above observation to extend statements to the general case.

4.3.3 The action of the gauge group

Let us fix two graphs Λ_i and Λ_j together with a refinement $(\Lambda_i, \Lambda_j, \iota)$.

The map $\iota^{(0)}$ induces a surjective group homomorphism between the gauge groups given by pull-back:

$$(\iota^{(0)})^*\colon \mathcal{G}_j \to \mathcal{G}_i, \quad g = (g_x)_{x\in\Lambda_j^0} \mapsto (g_{\iota^{(0)}(x)})_{x\in\Lambda_i^0}, \tag{4.8}$$

Clearly, this map can be directly factorized into products of maps corresponding to elementary refinements. Moreover, one readily verifies that $R_{i,j}\colon \mathcal{K}_j \to \mathcal{K}_i$ satisfies the equivariance condition

$$(\iota^{(0)})^*(g) \cdot R_{i,j}(a) = R_{i,j}(g\cdot a),$$

for all $g \in \mathcal{G}_j$ and $a \in \mathcal{K}_j$, hence it descends to a map $R_{i,j}^{\mathrm{red}}\colon \mathcal{G}_j\backslash\mathcal{K}_j \to \mathcal{G}_i\backslash\mathcal{K}_i$.

If we let $\pi_i \colon \mathcal{K}_i \to \mathcal{G}_i \backslash \mathcal{K}_i$ denote the canonical projection, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{K}_j \arrow[r, "R_{i,j}"] \arrow[d, "\pi_j"'] & \mathcal{K}_i \arrow[d, "\pi_i"] \\
\mathcal{G}_j \backslash \mathcal{K}_j \arrow[r, "R^{\mathrm{red}}_{i,j}"] & \mathcal{G}_i \backslash \mathcal{K}_i
\end{tikzcd}$$

Figure 4.1

This fact and Proposition 4.8 yield the following result.

4.10 Proposition. *There exists a canonical contravariant functor from* Refine *to the category of compact Hausdorff spaces that sends a graph Λ_i to the space $\mathcal{G}_i \backslash \mathcal{K}_i$, and a refinement $(\Lambda_i, \Lambda_j, \iota_{i,j})$ to the map $R^{\mathrm{red}}_{i,j}$.*

4.4 Hilbert spaces

Next, we construct the Hilbert spaces of square integrable functions with respect to the normalised Haar measure, and define the corresponding maps between them. We start by recalling some results for the (Haar) measures on the configuration spaces, originally derived in [10, 11, 76] (see also [12, 50]).

4.11 Lemma. *On the inverse system of Hausdorff spaces $(\mathcal{K}_i, R_{i,j})$ we have an* exact *inverse system of measures for $(\mathcal{K}_i, R_{i,j})$, i.e., a collection of Radon measures μ_i on $\mathcal{K}_i$ such that for $i \leq j$ one has $(R_{i,j})_*(\mu_j) = \mu_i$. In particular, the image of the Haar measure on $G^{\Lambda^1_j}$ under the map induced by the map $R_{i,j}$ is the Haar measure on $G^{\Lambda^1_i}$.*

Proof. The first part of the theorem follows from the Riesz–Markov representation theorem. We will check the second part of the statement for the elementary refinements of subsection 4.3.2. Let ϕ be a continuous function on $G^{\Lambda^1_i}$ and let μ_{i+1} be the Haar measure on $G^{\Lambda^1_{i+1}}$. By definition

$$\int_{G^{\Lambda^1_i}} \phi \, d(R_{i,i+1})_*(\mu_{i+1}) := \int_{G^{\Lambda^1_{i+1}}} (R_{i,i+1})^* (\phi) \, d\mu_{i+1}.$$

We will show that $(R_{i,i+1})_*(\mu_{i+1})$ is left-invariant, i.e. that

$$\int_{G^{\Lambda_i^1}} L_h\phi \, d(R_{i,i+1})_*(\mu_{i+1}) = \int_{G^{\Lambda_i^1}} \phi \, d(R_{i,i+1})_*(\mu_{i+1}),$$

for any $h \in G^{\Lambda_i^1}$. Since the Haar measure on $G^{\Lambda_i^1}$ is the product of $|\Lambda_i^1|$ Haar measures on G it follows that

$$\int_{G^{\Lambda_i^1}} L_h\phi \, d(R_{i,i+1})_*(\mu_{i+1}) = \int_{G^{\Lambda_{i+1}^1}} \phi(h^{-1} \cdot R_{i,i+1}(a)) \, d\mu_{i+1}(a).$$

An elementary refinement consisting of the addition of an edge amounts to forgetting an integration variable so there is nothing to prove. For the subdivision of an edge e_0 into (e_1, e_2) we have

$$\begin{aligned}
&\int_{G^{\Lambda_{i+1}^1}} \phi(h^{-1} \cdot R_{i,i+1}(a)) \, d\mu_{i+1}(a) \\
&= \int_{G^{\Lambda_{i+1}^1}} \phi\left(h^{-1} \cdot \left((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1}a_{e_2}\right)\right) \, d\mu_{i+1}(a) \\
&= \int_G \int_{G^{\Lambda_i^1}} \phi \circ R_{i,i+1}((h^{-1} \cdot ((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1})), a_{e_2}) \, d\mu_i((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1}) \\
&\qquad\qquad d\mu(a_{e_2}) \\
&= \int_G \int_{G^{\Lambda_i^1}} \phi \circ R_{i,i+1}\left(((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1}), a_{e_2}\right) \, d\mu_i((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1}) \\
&\qquad\qquad d\mu(a_{e_2}) \\
&= \int_{G^{\Lambda_{i+1}^1}} \phi(R_{i,i+1}(a)) \, d\mu_{i+1}(a),
\end{aligned}$$

where in the second last equality we have used left-invariance of the Haar measure $\mathrm{d}\mu_i$. ■

4.12 Proposition. *On the inverse system of Hausdorff spaces* $(\mathcal{G}_i\backslash\mathcal{K}_i, R_{i,j}^{\mathrm{red}})$ *we have an* exact *inverse system of measures.*

Proof. By the Riesz–Markov representation theorem, the projection $\pi_i : \mathcal{K}_i \to \mathcal{G}_i\backslash\mathcal{K}_i$ induces a map from the space of Radon measures on $\mathcal{K}_i$ to the space of Radon measures on $\mathcal{G}_i\backslash\mathcal{K}_i$. Figure 4.1 then implies the existence of a commutative diagram between the corresponding spaces of Radon measures. ■

We now dualise this construction on the measure spaces $\mathcal{K}_i$ and construct a direct system of Hilbert spaces $L^2(\mathcal{K}_i)$. We write $R : \mathcal{K}_j \to \mathcal{K}_i$ for a map between configuration spaces induced by an arbitrary refinement $\iota : \Gamma_i \to \Gamma_j$. We then set

$$u := (R)^* \colon L^2(\mathcal{K}_i) \to L^2(\mathcal{K}_j), \quad \psi \mapsto \psi \circ R. \tag{4.9}$$

Moreover, we define

$$u^{\mathrm{red}} := (R^{\mathrm{red}})^* \colon L^2(\mathcal{G}_i \backslash \mathcal{K}_i) \to L^2(\mathcal{G}_j \backslash \mathcal{K}_j), \quad \psi \mapsto \psi \circ R^{\mathrm{red}}.$$

4.13 Proposition. *If p_i is the map given by*

$$p_i \colon L^2(\mathcal{K}_i) \mapsto L^2(\mathcal{G}_i \backslash \mathcal{K}_i), \quad \psi \mapsto \left(\mathcal{G}_i a \mapsto \int_{\mathcal{G}_i} \psi(g \cdot a)\, d\mu_{\mathcal{G}_i}(g) \right),$$

for all i, where $\mu_{\mathcal{G}_i}$ denotes the Haar measure on $\mathcal{G}_i$, then:

(1) The pullback $\pi_i^* : L^2(\mathcal{G}_i \backslash \mathcal{K}_i) \to L^2(\mathcal{K}_i)$ of π_i is the adjoint of p_i (for all i).

(2) The following squares

$$\begin{tikzcd} L^2(\mathcal{K}_i) \arrow[r, "u"] & L^2(\mathcal{K}_j) \\ L^2(\mathcal{G}_i \backslash \mathcal{K}_i) \arrow[u, "\pi_i^*"] \arrow[r, "u^{\mathrm{red}}"] & L^2(\mathcal{G}_j \backslash \mathcal{K}_j) \arrow[u, "\pi_j^*"'] \end{tikzcd}$$

and

$$\begin{tikzcd} L^2(\mathcal{K}_i) \arrow[r, "u"] \arrow[d, "p_i"'] & L^2(\mathcal{K}_j) \arrow[d, "p_j"] \\ L^2(\mathcal{G}_i \backslash \mathcal{K}_i) \arrow[r, "u^{\mathrm{red}}"] & L^2(\mathcal{G}_j \backslash \mathcal{K}_j) \end{tikzcd}$$

commute.

(3) The maps u and u^{red} are isometries.

Proof. (1) For $\psi \in L^2(\mathcal{K}_i), \varphi \in L^2(\mathcal{G}_i \backslash \mathcal{K}_i)$ we have that

$$\begin{aligned}
\langle \varphi, p_i \psi \rangle_{L^2(\mathcal{G}_i \backslash \mathcal{K}_i)} &= \int_{\mathcal{G}_i \backslash \mathcal{K}_i} \overline{\varphi(\mathcal{G}_i a)} \int_{\mathcal{G}_i} \psi(g \cdot a)\, d\mu_{\mathcal{G}_i}(g)\, d\mu_{\mathcal{G}_i \backslash \mathcal{K}_i}(a) \\
&= \int_{\mathcal{K}_i} \overline{\varphi \circ \pi_i(a)} \int_{\mathcal{G}_i} \psi(g \cdot a)\, d\mu_{\mathcal{G}_i}(g)\, d\mu_i(a) \\
&= \int_{\mathcal{K}_i} \int_{\mathcal{G}_i} \overline{\varphi \circ \pi_i(g^{-1} \cdot a)} \psi(a)\, d\mu_{\mathcal{G}_i}(g)\, d\mu_i(a) \\
&= \int_{\mathcal{K}_i} \overline{\varphi \circ \pi_i(a)} \psi(a)\, d\mu_i(a) = \langle \pi_i^*(\varphi), \psi \rangle_{L^2(\mathcal{K}_i)},
\end{aligned}$$

where we have used bi-invariance of the Haar measure on $\mathcal{K}_i$ in the fourth step.

Commutativity of the first square in (2) follows directly from the fact that $\pi_i \circ R = R^{\text{red}} \circ \pi_j$, which holds by definition of R^{red}. For the second square to commute, we let $a \in \mathcal{K}_j, \psi \in L^2(\mathcal{K}_i)$ and compute that indeed

$$\begin{aligned}
(p_j \circ u(\psi))(\mathcal{G}_j a) &= \int_{\mathcal{G}_j} (u(\psi))(g \cdot a)\, d\mu_{\mathcal{G}_j}(g) = \int_{\mathcal{G}_j} \psi \circ R(g \cdot a)\, d\mu_{\mathcal{G}_j}(g) \\
&= \int_{\mathcal{G}_j} \psi((\iota^0)^*(g) \cdot R(a))\, d\mu_{\mathcal{G}_j}(g) \\
&= \int_{G^{\Lambda_j^0 - \iota^0(\Lambda_i^0)}} \int_{\mathcal{G}_i} \psi(g \cdot R(a))\, d\mu_{\mathcal{G}_i}(g)\, d\nu(g') \\
&= \int_{\mathcal{G}_i} \psi(g \cdot R^{\text{red}}(a))\, d\mu_{\mathcal{G}_i}(g) = p_i(\psi)(\mathcal{G}_i R(a)) \\
&= (u^{\text{red}} \circ p_i(\psi))(\mathcal{G}_j a),
\end{aligned}$$

where ν denotes the Haar measure on $G^{\Lambda_j^0 - \iota^0(\Lambda_i^0)}$.

For (3) we use that by definition of the measures on the spaces $\mathcal{G}_i \backslash \mathcal{K}_i$ and $\mathcal{G}_j \backslash \mathcal{K}_j$, the maps π_i^* and π_j^* are isometries. Thus, by commutativity of the first square in (2), it suffices to show that u is an isometry. We will prove the statement for the elementary refinements discussed in subsection 4.3.2. Let $\psi \in L^2(\mathcal{K}_i)$.

- If Λ_j is obtained from Λ_i by adding an edge $e' \in \Lambda_j^1$, then

$$\begin{aligned}
\|u(\psi)\|^2_{L^2(\mathcal{K}_j)} &= \int_{G^{\Lambda_j^1}} |u(\psi)((a_e)_{e\in\Lambda_j^1})|^2 \, d\mu_j((a_e)_{e\in\Lambda_j^1}) \\
&= \int_G \int_{G^{\Lambda_i^1}} |\psi((a_e)_{e\in\Lambda_i^1})|^2 \, d\mu_i((a_e)_{e\in\Lambda_j^1}) \, d\mu(a_{e'}) \\
&= \int_{G^{\Lambda_i^1}} |\psi((a_e)_{e\in\Lambda_i^1})|^2 \, d\mu_i((a_e)_{e\in\Lambda_i^1}) \\
&= \|\psi\|^2_{L^2(\mathcal{K}_i)},
\end{aligned}$$

- If Λ_j is obtained from Λ_i by subdividing an edge $e_0 \in \Lambda_i^1$ into two edges $e_1, e_2 \in \Lambda_j^1$, then

$$\begin{aligned}
&\|u(\psi)\|^2_{L^2(\mathcal{K}_j)} = \int_{G^{\Lambda_j^1}} |u(\psi)((a_e)_{e\in\Lambda_j^1})|^2 \, d\mu_j((a_e)_{e\in\Lambda_j^1}) \\
&= \int_G \int_G \int_{G^{\Lambda_i^1\setminus\{e_0\}}} |\psi((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_1}a_{e_2})|^2 \, d\nu((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\mu(a_{e_1}) d\mu(a_{e_2}) \\
&= \int_G \int_{G^{\Lambda_i^1\setminus\{e_0\}}} |\psi((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}, a_{e_2})|^2 \, d\nu((a_e)_{e\in\Lambda_i^1\setminus\{e_0\}}) \, d\mu(a_{e_2}) \\
&= \int_{G^{\Lambda_i^1}} |\psi((a_e)_{e\in\Lambda_i^1})|^2 \, d\mu_i((a_e)_{e\in\Lambda_i^1}) = \|\psi\|^2_{L^2(\mathcal{K}_i)},
\end{aligned}$$

  since the Haar measure is left-invariant and normalized. Here, μ and ν denote the Haar measures on G and $G^{\Lambda_i^1\setminus\{e_0\}}$, respectively. ∎

4.14 Proposition. *There exist two canonical covariant functors from* Refine *to the category of Hilbert spaces that send a graph Λ_i to the spaces $L^2(\mathcal{K}_i)$ and $L^2(\mathcal{G}_i\backslash\mathcal{K}_i)$, and a refinement $(\Lambda_i, \Lambda_j, \iota_{i,j})$ to the linear isometries $u_{i,j}$ and $u^{\mathrm{red}}_{i,j}$, respectively.*

Proof. Let Λ_i, Λ_j and Λ_k be three graphs, with corresponding spaces of connections $\mathcal{K}_i$, $\mathcal{K}_j$ and $\mathcal{K}_k$ and gauge groups $\mathcal{G}_i$, $\mathcal{G}_j$ and $\mathcal{G}_k$. Suppose in addition that we are given refinements $(\Lambda_i, \Lambda_j, \iota_{i,j})$ and $(\Lambda_j, \Lambda_k, \iota_{j,k})$.

We need to prove that the corresponding maps between Hilbert spaces satisfy

$$u_{i,k} = u_{j,k} \circ u_{i,j};$$

$$u_{i,k}^{\mathrm{red}} = u_{j,k}^{\mathrm{red}} \circ u_{i,j}^{\mathrm{red}};$$

The fact that $u_{i,k} = u_{j,k} \circ u_{i,j}$ follows from Remark 4.6 and the definition of the map R. To prove $u_{i,k}^{\mathrm{red}} = u_{j,k}^{\mathrm{red}} \circ u_{i,j}^{\mathrm{red}}$, note that for $\Lambda_i \leq \Lambda_j$, the maps p_i^* and p_j^* are isometries by definition of the measure on $\mathcal{G}_i \backslash \mathcal{K}_i$ and $\mathcal{G}_j \backslash \mathcal{K}_j$. Thus $p_i p_i^* = \mathrm{Id}_{L^2(\mathcal{G}_i \backslash \mathcal{K}_i)}$ and $p_j p_j^* = \mathrm{Id}_{L^2(\mathcal{G}_j \backslash \mathcal{K}_j)}$.

Commutativity of the first square in Proposition 4.13 and the fact that p_i^* and p_j^* are sections of p_i and p_j, respectively, imply that $p_i \circ u \circ p_i^* = u^{\mathrm{red}}$, and that u maps $\mathcal{G}_i$-invariant functions to $\mathcal{G}_j$-invariant functions. Observing that $p_i^* p_i$ and $p_j^* p_j$ are the orthogonal projections onto the spaces of $\mathcal{G}_i$- and $\mathcal{G}_j$-invariant functions, respectively, we infer that

$$\begin{aligned} u_{i,k}^{\mathrm{red}} &= p_k \circ u_{i,k} \circ p_i^* = p_k \circ u_{j,k} \circ u_{i,j} \circ p_i^* = p_k \circ u_{j,k} \circ p_j^* p_j \circ u_{i,j} \circ p_i^* \\ &= u_{j,k}^{\mathrm{red}} \circ u_{i,j}^{\mathrm{red}}, \end{aligned}$$

which proves the claim. ■

4.5 Field algebras and observable algebras

The isometries between the Hilbert spaces constructed in the previous subsection naturally induce maps between the field algebras and between the observable algebras. In fact, we have the following:

4.15 Proposition. *The maps*

$$\begin{aligned} &v\colon B_0(L^2(\mathcal{K}_i)) \to B_0(L^2(\mathcal{K}_j)), && b \mapsto ubu^*; \\ &v^{\mathrm{red}}\colon B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i)) \to B_0(L^2(\mathcal{G}_j \backslash \mathcal{K}_j)), && b \mapsto u^{\mathrm{red}} b (u^{\mathrm{red}})^*, \end{aligned}$$

are injective **-homomorphisms.*

Proof. It is clear that v and v^{red} respect its linear structures as well as the involutions. Since u and u^{red} are isometries, we have

$$u^* u = \mathrm{Id}_{L^2(\mathcal{K}_i)}, \quad \text{and} \quad (u^{\mathrm{red}})^* u^{\mathrm{red}} = \mathrm{Id}_{L^2(\mathcal{G}_i \backslash \mathcal{K}_i)},$$

from which it readily follows that the maps v and v^{red} are injective and respect the algebra structures. ■

Thus the maps u and v are embeddings of the 'coarse' Hilbert space and field algebra into the corresponding 'finer' structures, respectively, and the maps u^{red} and v^{red} are their 'reduced' counterparts. We can now formulate the analogue of Proposition 4.13 for the field algebras and the observable algebras:

4.16 Proposition. *Define the maps* P_i, P_j, Π_i *and* Π_j *by*

$$\begin{aligned}
P_i &\colon B_0(L^2(\mathcal{K}_i)) \to B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i)), && b \mapsto p_i b p_i^*;\\
P_j &\colon B_0(L^2(\mathcal{K}_j)) \to B_0(L^2(\mathcal{G}_j \backslash \mathcal{K}_j)), && b \mapsto p_j b p_j^*;\\
\Pi_i &\colon B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i)) \to B_0(L^2(\mathcal{K}_i)), && b \mapsto p_i^* b p_i;\\
\Pi_j &\colon B_0(L^2(\mathcal{G}_j \backslash \mathcal{K}_j)) \to B_0(L^2(\mathcal{K}_j)), && b \mapsto p_j^* b p_j.
\end{aligned}$$

Then the following squares

$$\begin{tikzcd}
B_0(L^2(\mathcal{K}_i)) \arrow[r, "v"] & B_0(L^2(\mathcal{K}_j)) \\
B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i)) \arrow[u, "\Pi_i^*"] \arrow[r, "v^{\mathrm{red}}"] & B_0(L^2(\mathcal{G}_j \backslash \mathcal{K}_j)) \arrow[u, "\Pi_j^*"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
B_0(L^2(\mathcal{K}_i)) \arrow[r, "v"] \arrow[d, "P_i"'] & B_0(L^2(\mathcal{K}_j)) \arrow[d, "P_j"] \\
B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i)) \arrow[r, "v^{\mathrm{red}}"] & B_0(L^2(\mathcal{G}_j \backslash \mathcal{K}_j))
\end{tikzcd}$$

commute.

Proof. We shall only present a proof of commutativity of the first square; commutativity of the second square can be proved in a similar fashion. Let $b \in B_0(L^2(\mathcal{G}_i \backslash \mathcal{K}_i))$. Then using the commutativity of the first square in Proposition 4.13, we obtain

$$\begin{aligned}
v \circ \Pi_i(b) &= u p_i^* b p_i u^* = (u p_i^*) b (u p_i^*)^* \\
&= (p_j^* u^{\mathrm{red}}) b (p_j^* u^{\mathrm{red}})^* = p_j^* u^{\mathrm{red}} b (u^{\mathrm{red}})^* p_j = \Pi_j \circ v^{\mathrm{red}}(b),
\end{aligned}$$

as desired. ∎

4.17 Proposition. *There exist two canonical covariant functors from* Refine *to the category of* C^**-algebras that send a graph* Λ_i *to the spaces* $B_0(L^2(\mathcal{K}_i))$ *and* $B_0(L^2(\mathcal{G}_i\backslash\mathcal{K}_i))$*, and a refinement* $(\Lambda_i, \Lambda_j, \iota_{i,j})$ *to the injective* **-homomorphisms* $v_{i,j}$ *and* $v^{\mathrm{red}}_{i,j}$*, respectively. The collections* $(B_0(L^2(\mathcal{K}_i)), v_{i,j})$ *and* $(B_0(L^2(\mathcal{G}_i\backslash\mathcal{K}_i)), v^{\mathrm{red}}_{i,j})$ *form direct systems of* C^**-algebras.*

Proof. This follows from the fact that

$$v_{i,k} = v_{j,k} \circ v_{i,j}; \quad v^{\mathrm{red}}_{i,k} = v^{\mathrm{red}}_{j,k} \circ v^{\mathrm{red}}_{i,j},$$

which is a direct consequence of Proposition 4.14. ∎

We are also interested in describing the refinements of the field algebras and the observable algebras in purely geometric terms, that is to say, in terms of the pair groupoids $\mathsf{G}_i = \mathcal{K}_i \times \mathcal{K}_i$ that we associated to a graph Λ_i. A map $R_{i,j} \colon \mathcal{K}_j \to \mathcal{K}_i$ canonically gives rise to a groupoid morphism $\mathsf{R}_{i,j} = \left(\mathsf{R}^{(0)}_{i,j}, \mathsf{R}^{(1)}_{i,j}\right) : \mathsf{G}_j \to \mathsf{G}_i$, where $\mathsf{R}^{(0)}_{i,j} = R_{i,j}$ and $\mathsf{R}^{(1)}_{i,j} = R_{i,j} \times R_{i,j}$. Similarly, we obtain a groupoid morphism $\mathsf{R}^{\mathrm{red}}_{i,j} \colon \mathsf{G}^{\mathrm{red}}_j \to \mathsf{G}^{\mathrm{red}}_i$ between the pair groupoids associated to the reduced configuration spaces. The following proposition is then an immediate consequence of Proposition 4.8:

4.18 Proposition. *There exist contravariant functors from* Refine *to the category of groupoids that send a graph* Λ_i *to the groupoids* G_i *and* $\mathsf{G}^{\mathrm{red}}_i$*, and a refinement* $(\Lambda_i, \Lambda_j, \iota_{i,j})$ *to the groupoid morphisms* $\mathsf{R}_{i,j}$ *and* $\mathsf{R}^{\mathrm{red}}_{i,j}$*.*

More interestingly, the maps $\mathsf{R}_{i,j}$ induce a map $\mathsf{R}^*_{i,j}$ between the groupoid C*-algebras $C^*(\mathsf{G}_i)$ and $C^*(\mathsf{G}_j)$, given simply by pullback. We will show that it coincides with $v_{i,j} = v$ from Proposition 4.15, after identifying $C^*(\mathsf{G}_i) \simeq B_0(L^2(\mathcal{K}_i))$, using the isomorphism induced by the map defined in Equation (4.1).

4.19 Proposition. *The following diagram*

$$\begin{tikzcd} C^*(\mathsf{G}_i) \arrow[r, "\mathsf{R}^*_{i,j}"] \arrow[d, "\cong"'] & C^*(\mathsf{G}_j) \arrow[d, "\cong"] \\ B_0(L^2(\mathcal{K}_i)) \arrow[r, "v_{i,j}"] & B_0(L^2(\mathcal{K}_j)) \end{tikzcd}$$

commutes.

Proof. With $u_{i,j} : L^2(\mathcal{K}_i) \to L^2(\mathcal{K}_j)$ as defined in Equation (4.9), we have to establish that

$$u_{i,j}(T_h)u_{i,j}^* = T_{R_{i,j}^*(h)}.$$

By the disintegration theorem, there exists a family of measures $(\nu_b)_b$ on $\mathcal{K}_j$ for almost every $b \in \mathcal{K}_i$ such that ν_b is supported in $R_{i,j}^{-1}(\{a\})$, and satisfies

$$\int_{\mathcal{K}_j} f(a)\, d\mu_j(a) = \int_{\mathcal{K}_i} \int_{R_{i,j}^{-1}(\{b\})} f(a)\, d\nu_b(a)\, d\mu_j(b).$$

It follows that for each $\psi \in L^2(\mathcal{K}_j)$ and each $\varphi \in L^2(\mathcal{K}_i)$, we have

$$\begin{aligned}
\langle \varphi, u_{i,j}^*\psi\rangle_{L^2(\mathcal{K}_i)} = \langle u_{i,j}\varphi, \psi\rangle_{L^2(\mathcal{K}_j)} &= \int_{\mathcal{K}_j} \overline{\varphi \circ R_{i,j}(a)}\psi(a)\, d\mu_j(a) \\
&= \int_{\mathcal{K}_i} \int_{R_{i,j}^{-1}(\{b\})} \overline{\varphi \circ R_{i,j}(a)}\psi(a)\, d\nu_b(a)\, d\mu_i(b) \\
&= \int_{\mathcal{K}_i} \overline{\varphi(b)} \int_{R_{i,j}^{-1}(\{b\})} \psi(a)\, d\nu_b(a)\, d\mu_i(b),
\end{aligned}$$

so

$$(u^*\psi)(b) = \int_{R_{i,j}^{-1}(\{b\})} \psi(a)\, \mathrm{d}\nu_b(a),$$

for almost every $b \in \mathcal{K}_i$. Next, let $h \in C(\mathsf{G}_i^{(1)})$. Then for each $b \in \mathcal{K}_i$, we have

$$(T_h u^*\psi)(b) = \int_{\mathcal{K}_i} h(b', b) \int_{R_{i,j}^{-1}(\{b'\})} \psi(a)\, d\nu_{b'}(a)\, d\mu_i(b'),$$

hence, for each $a \in \mathcal{K}_j$, we have

$$\begin{aligned}
(uT_h u^*\psi)(a) &= \int_{\mathcal{K}_i} h(b', R_{i,j}(a)) \int_{R_{i,j}^{-1}(\{b'\})} \psi(a')\, d\nu_{b'}(a')\, d\mu_i(b') \\
&= \int_{\mathcal{K}_j} h(R_{i,j}(a'), R_{i,j}(a))\psi(a')\, d\mu_j(a') = (T_{R_{i,j}^*(h)}\psi)(a).
\end{aligned}$$

Since a and ψ were arbitrary, this completes the proof of the proposition. ∎

The statement can readily be modified for the groupoid C*-algebras of the reduced groupoids. In fact, it is true for any two compact spaces carrying Radon probability measures compatible with the map $R_{i,j}$, and their corresponding pair groupoids. We summarise the results obtained in this subsection in the following:

4.20 Theorem. *The collections* $\left(C^*(\mathsf{G}_i), \mathsf{R}^*_{i,j}\right)$ *and* $\left(C^*(\mathcal{G}_i \backslash \mathsf{G}_i), \mathsf{R}^{\mathrm{red}*}_{i,j}\right)$ *with connecting maps induced by the maps* $R_{i,j}$ *and* $R^{\mathrm{red}}_{i,j}$*, respectively, form direct systems of C*-algebras. Moreover, these direct systems of C*-algebras are isomorphic to the direct systems described in Proposition 4.17.*

4.6 Limits

We will now consider both the thermodynamic and the continuum limit of our theory by considering the limit objects of the inverse and direct systems constructed in the previous section. This includes inverse limits of measure spaces and groupoids, and the direct limits of Hilbert spaces and (groupoid) C*-algebras. In particular, we will identify a limit pair groupoid G_∞ for which the groupoid C*-algebra $C^*(\mathsf{G}_\infty)$ is isomorphic to the limit of the field algebras $\varinjlim_{i\in I} A_i$.

First of all, fix a direct system $(\Lambda_i, \iota_{i,j})$ such as the one depicted in Figure 2.2 in section 2.1. Applying the contravariant functor mentioned in Proposition 4.10, we obtain an inverse system $(\mathcal{K}_i, R_{i,j})$ of compact Hausdorff spaces. This inverse system has a limit in the category of topological spaces, which is unique up to unique isomorphism, and which can be realised as follows:

$$\mathcal{K}_\infty = \varprojlim_{i\in I} \mathcal{K}_i := \left\{ a = (a_i)_{i\in I} \in \prod_{i\in I} \mathcal{K}_i \mid a_i = R_{i,j}(a_j) \text{ for all } i \le j \right\}$$

together with maps

$$R_{i,\infty} : \mathcal{K}_\infty \to \mathcal{K}_i,$$

which are given by the projection. Note that since the maps $R_{i,j}$ are not group homomorphism, the limit space $\mathcal{K}_\infty$ does not automatically possess a group structure.

By [95, Lemma 1.1.10], since $\mathcal{K}_\infty$ is an inverse limit of compact Hausdorff spaces, the maps $R_{i,\infty}$ are surjective for all $i \in I$. Moreover, since the spaces involved are compact, the maps $R_{i,j}$ are automatically proper and so are the structure maps $R_{i,\infty}$. In addition, by Lemma 4.11, $(\mathcal{K}_i, R_{i,j})$ is an inverse system of probability spaces. The existence of a probability measure on the limit space is then a consequence of Prokhorov's theorem ([105, Theorem 21]).

4.21 Proposition. *Let $\mathcal{K}_\infty$ denote the limit of the inverse system of measurable topological spaces $((\mathcal{K}_i, \mu_i), R_{i,j})$. Then there exists a Radon measure μ_∞ on $\mathcal{K}_\infty$ such that $R_{i,\infty}(\mu_\infty) = \mu_i$.*

4.22 Remark. The measure on the inverse limit constructed in this fashion is referred to in the LQG literature as the *Ashtekar–Lewandowski measure*, as its construction was described in [9] (cf. [11]). Now suppose the direct system of graphs has been embedded into some smooth manifold. (In LQG, these manifolds are typically assumed to be analytic, see section 5.1.) The space of smooth connections of the trivial principal G-bundle of this manifold has a natural embedding into $\mathcal{K}_\infty$, provided that the graphs that constitute the directed system are in some sense dense in the manifold; we refer to [2, Definition 2.1.7 and Proposition 2.2.4], and note that in point (2) of this definition, the condition of linear independence should be replaced by that of completeness. Moreover, Rendall [94] has shown that the smooth connections are dense in the direct limit. Marolf and Mourão, however, have shown in [80] that the space of smooth connections has Ashtekar–Lewandowski measure 0.

Thus, although $\mathcal{K}_\infty$ provides a good approximation to the space of smooth connections, it is impossible to restrict the Ashtekar–Lewandowski measure on the former space to a nontrivial measure on the latter. This may be an artifact of the graph, which disregards the topology of the space in which it is embedded, in this case the smooth manifold. It is at present unknown how to take into account such topological data in the above construction, although it is interesting to mention that Lüscher [77] appears to use some version of nonabelian Čech cohomology to implement such data. A proposal to patch together classical field configurations of abelian gauge theories and associated observables - both defined on contractible parts of a possibly noncontractible base manifold - to their corresponding global versions using homotopy theory is given in [15], which is similarly motivated by the possible nontriviality of the principal fibre bundle, as well as the treatment of such systems in the presence of topological charges.

By Proposition 4.14 we have a direct system of Hilbert spaces $(\mathcal{H}_i, u_{i,j})$, where $\mathcal{H}_i := L^2(\mathcal{K}_i, \mu_i)$. Its direct limit is nothing but the space of L^2 functions on the inverse limit of the spaces of connections with respect to the inverse limit measure:

$$\mathcal{H}_\infty := \varinjlim_{i\in I} \mathcal{H}_i \cong L^2(\mathcal{K}_\infty, \mu_\infty).$$

This was originally proved by Baez in [12]. In fact, the result is not merely true for L^2-spaces, but for L^p-spaces for any $p \in [1,\infty)$ as well, . However, the result is generally false for L^∞-spaces. The following proposition relates the inverse limit of Hilbert spaces with the direct limit of their algebras of observables.

4.23 Proposition. *Let $((\mathcal{H}_i, \langle\cdot,\cdot\rangle_i), u_{i,j})$ be a direct system of Hilbert spaces such that each map $u_{i,j}$ is an isometry. Let $(\mathcal{H}_\infty, \langle\cdot,\cdot\rangle, (u_{i,\infty})_{i\in I})$ be its direct limit. For each $i, j \in I$ with $i \le j$, define the map $v_{i,j}$ by*

$$v_{i,j}\colon B_0(\mathcal{H}_i) \to B_0(\mathcal{H}_j), \quad a \mapsto u_{i,j} a u_{i,j}^*.$$

Then $v_{i,j}$ is an injective *-homomorphism, hence it is an isometry. Furthermore, $(B_0(\mathcal{H}_i), v_{i,j})$ is a direct system of C-algebras, and we have*

$$\varinjlim_{i\in I} B_0(\mathcal{H}_i) \simeq B_0(\mathcal{H}_\infty).$$

Proof. Let $(v_{i,\infty})_{i\in I}$ be the collection of *-homomorphisms associated to $\varinjlim_{i\in I} B_0(\mathcal{H}_i)$. For each $i \in I \cup \{\infty\}$, let $F(\mathcal{H}_i)$ be the space of finite rank operators on $\mathcal{H}_i$. We first show that the set

$$A := \{v_{i,\infty}(a_i)\colon i \in I,\ a_i \in F(\mathcal{H}_i)\},$$

is a subset of $F(\mathcal{H}_\infty)$ that is dense in $B_0(\mathcal{H}_\infty)$.

Let $i \in I$, and let $a_i \in F(\mathcal{H}_i)$. Since $u_{i,\infty}$ is an isometry, the operator is bounded and $v_{i,\infty}(a_i)$ has the same rank as a_i, so in particular, it is a finite rank operator, hence $A \subseteq F(\mathcal{H})$. To show that A is dense in $B_0(\mathcal{H}_\infty)$, it suffices to show that for each rank 1 operator $a \in B_0(\mathcal{H}_\infty)$ and each $\varepsilon > 0$, there exists an $i \in I$ and an $a_i \in A$ such that $\|a - v_i(a_i)\| < \varepsilon$.

Fix such a and ε. Without loss of generality, we may assume that $\|a\| \leq 1$. Because a is a rank 1 operator, in physicists' bra-ket notation it is of the form $a = |\psi\rangle\langle\phi|$ for some nonzero $\psi, \phi \in \mathcal{H}_\infty$, and we may assume that $\|\psi\|, \|\phi\| \leq 1$. It follows that there exist $j, k \in I$, $\psi_j \in \mathcal{H}_j$ and $\phi_k \in \mathcal{H}_k$ such that

$$\|\psi - u_{j,\infty}(\psi_j)\|, \|\phi - u_{k,\infty}(\phi_k)\| \leq \min\left(1, \frac{\varepsilon}{3}\right).$$

Let $i \in I$ be an element such that $i \geq j, k$, let $\psi_i := u_{j,i}(\psi_j)$ and $\phi_i := u_{k,i}(\psi_k)$, and let $a_i := |\psi_i\rangle\langle\phi_i|$. Then

$$\begin{aligned}
\|a - v_{i,\infty}(a_i)\| &\leq \||\psi\rangle\langle\phi - u_{i,\infty}(\phi_i)|\| + \||\psi - u_{i,\infty}(\psi_i)\rangle\langle u_{i,\infty}(\phi_i)|\| \\
&\leq \|\psi\| \cdot \|\phi - u_{i,\infty}(\phi_i)\| \\
&\quad + \|\psi - u_{i,\infty}(\psi_i)\| \cdot (\|\phi - u_{i,\infty}(\phi_i)\| + \|\phi\|) \\
&< 3 \cdot \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which shows that A is dense in $B_0(\mathcal{H}_\infty)$.

For each $i \in I$, $v_{i,\infty}$ is an isometry since $v_{i,j}$ is an isometry for each $j \geq i$. It follows from the universal property of the direct limit that there exists an isometric *-homomorphism

$$\varinjlim_{i \in I} B_0(\mathcal{H}_i) \to B_0(\mathcal{H}_\infty),$$

and this *-homomorphism has dense range by the discussion above, hence it is surjective. We conclude that it is a *-isomorphism, finishing the proof. ■

In our case of interest, we have:

4.24 Corollary. *The direct limit of the field algebras is given by*

$$\varinjlim_{i \in I} A_i \cong B_0(L^2(\mathcal{K}_\infty)).$$

Next, we determine the inverse limit of the groupoids G_i and show that the direct limit C*-algebra $A_\infty = B_0(L^2(\mathcal{K}_\infty, \mu_\infty))$ agrees with the C*-algebra $C^*(\mathsf{G}_\infty)$ of the inverse limit groupoid G_∞.

Given the simple structure of the groupoid morphisms $\mathsf{R}_{i,j} : \mathsf{G}_j \to \mathsf{G}_i$ one easily checks that the limit groupoid G_∞ is also a pair groupoid and is given by

$$\mathsf{G}_\infty = \mathcal{K}_\infty \times \mathcal{K}_\infty.$$

It is by definition a free and transitive groupoid.

4.25 Remark. More generally, the limit of an inverse family of compact *transitive* groupoids such that all groupoid homomorphisms are surjective is also transitive. Moreover, for inverse families of compact *free* groupoids, the limit is also a free groupoid. The proofs rely on the fact that the source and target maps on the limit groupoid are defined componentwise.

On the groupoid $\mathsf{G}_\infty = \mathcal{K}_\infty \times \mathcal{K}_\infty$ we have a natural Haar system given by

$$\{\mu \times \delta_x \colon x \in \mathcal{K}_\infty\},$$

where δ_x denotes the Dirac measure at x and μ is a positive Radon measure on $\mathcal{K}_\infty$ of full support.

4.26 Theorem. *The groupoid C^*-algebra $C^*(\mathcal{K}_\infty \times \mathcal{K}_\infty)$ is isomorphic to the limit field algebra A_∞, which in turn is isomorphic to $B_0(L^2(\mathcal{K}_\infty, \mu))$, where μ is the injective limit of the measures on $\mathcal{K}_i$.*

Proof. Since the limit measure μ_∞ on the space $\mathcal{K}_\infty$ is a positive Radon measure of full support, the result follows from the second isomorphism in Equation (4.2) and Corollary 4.24 above. ∎

4.27 Remark. The question whether the C*-algebras associated with two Haar system on a given groupoid are isomorphic was answered positively by Muhly, Renault and Williams for the case of transitive groupoids (cf. [84, Theorem 3.1]), of which pair groupoids are a special case. Hence the choice of Haar system does not affect, in our setting, the structure of the groupoid C*-algebra. For a more in-depth discussion on the dependence of the groupoid C*-algebra on the choice of Haar system we refer the reader to [27, section 5] and [89, section 3.1].

Thus we have

$$C^*(\mathsf{G}_\infty) = C^*(\varprojlim_{i\in I} \mathsf{G}_i) \cong B_0(L^2(\mathcal{K}_\infty, \mu)) \cong \varinjlim_{i\in I} B_0(L^2(\mathcal{K}_i, \mu_i)) \cong \varinjlim_{i\in I} C^*(\mathsf{G}_i),$$

justifying the idea that the quantised field algebra on the inverse limit $\mathcal{K}_\infty$ is the direct limit of the quantised field algebras on the spaces $(\mathcal{K}_i)_{i\in I}$.

4.6.1 Quantum gauge symmetries and the limit

We finish this section by discussing the reduction of the quantum system in the limit.

By equivariance of the maps involved in the refinement procedure as described in Figure (4.1), the results of Proposition 4.21 holds *mutatis mutandis* for the inverse family of quotient measure spaces with respect to the action of the gauge group. There exists a Radon measure $\mu_\infty^{\mathrm{red}}$ such that $R_{i,\infty}^{\mathrm{red}}(\mu_\infty^{\mathrm{red}}) = \mu_i^{\mathrm{red}}$ on $\mathcal{K}_\infty^{\mathrm{red}}$, where $\mathcal{K}_\infty^{\mathrm{red}}$ denotes the limit of the inverse system of topological measure spaces $((\mathcal{G}_i\backslash\mathcal{K}_i, \mu_i^{\mathrm{red}}), (R_{i,j}^{\mathrm{red}})_{i,j\in I})$.

Next, we can consider the space of square integrable functions on $\mathcal{K}_\infty^{\mathrm{red}}$ with respect to the Radon measure $\mu_\infty^{\mathrm{red}}$. Then the direct limit of the direct system of Hilbert spaces $(L^2(\mathcal{G}_j\backslash\mathcal{K}_j), u_{j,k}^{\mathrm{red}})$ of Proposition 4.14 is given by

$$\mathcal{H}_\infty^{\mathrm{red}} \cong L^2(\mathcal{K}_\infty^{\mathrm{red}}, \mu_\infty^{\mathrm{red}}).$$

An application of Proposition 4.23 yields

$$\varinjlim_{i\in I} B_0(\mathcal{H}_i^{\mathcal{G}_i}) \cong B_0(\mathcal{H}_\infty^{\mathrm{red}}).$$

As above we may then infer that the underlying groupoid for the observable algebra is a direct limit of pair groupoids so that

$$C^*(\mathsf{G}_\infty^{\mathrm{red}}) \cong B_0(L^2(\mathcal{K}_\infty^{\mathrm{red}}, \mu_\infty^{\mathrm{red}})).$$

In other words, we have arrived at the reduced analogue of Theorem 4.26.

4.7 The Hamiltonian

Suppose again that we have fixed two graphs Λ_i, Λ_j together with a refinement $(\Lambda_i, \Lambda_j, \iota)$. Consider the Hamiltonians

$$H_{0,i} = \sum_{e \in \Lambda_i^1} -\frac{1}{2} I_{i,e} \Delta_e, \quad \text{and} \quad H_{0,j} = \sum_{e \in \Lambda_j^1} -\frac{1}{2} I_{j,e} \Delta_e,$$

let $\mathcal{H}_i := L^2(\mathcal{K}_i)$, let $\mathcal{H}_j := L^2(\mathcal{K}_j)$ and let $u \colon \mathcal{H}_i \to \mathcal{H}_j$ and $u_{\mathrm{red}} \colon \mathcal{H}_i^{\mathcal{G}_i} \to \mathcal{H}_j^{\mathcal{G}_j}$ be the maps between the corresponding Hilbert spaces.

4.28 Proposition. *Suppose that for each $e \in \Lambda_i^1$, we have*

$$I_{i,e} = \sum_{k=1}^{n} I_{j,e_k}, \tag{4.10}$$

where $\iota^{(1)}(e) = (e_1, e_2, \ldots, e_n)$.

(1) We have $u(\mathrm{Dom}(H_{i,0})) \subseteq \mathrm{Dom}(H_{j,0})$, and the following diagram

$$\begin{array}{ccc} \mathrm{Dom}(H_{0,i}) & \xrightarrow{u} & \mathrm{Dom}(H_{0,j}) \\ \Big\downarrow{\scriptstyle H_{0,i}} & & \Big\downarrow{\scriptstyle H_{0,j}} \\ \mathcal{H}_i & \xrightarrow{u} & \mathcal{H}_j \end{array}$$

is commutative.

(2) We have $u(\mathrm{Dom}(H_{0,i}) \cap \mathcal{H}_i^{\mathcal{G}_i}) \subseteq \mathrm{Dom}(H_{0,j}) \cap \mathcal{H}_j^{\mathcal{G}_j}$, and the following diagram

$$\begin{array}{ccc} \mathrm{Dom}(H_{0,i}) \cap \mathcal{H}_i^{\mathcal{G}_i} & \xrightarrow{u^{\mathrm{red}}} & \mathrm{Dom}(H_{0,j}) \cap \mathcal{H}_j^{\mathcal{G}_j} \\ \Big\downarrow{\scriptstyle H_{0,i}} & & \Big\downarrow{\scriptstyle H_{0,j}} \\ \mathcal{H}_i^{\mathcal{G}_i} & \xrightarrow{u^{\mathrm{red}}} & \mathcal{H}_j^{\mathcal{G}_j} \end{array}$$

is commutative, where $H^{\mathrm{red}}_{0,i}$ *denotes the restriction of* $H_{0,i}$ *to* $\mathrm{Dom}(H_{0,i}) \cap \mathcal{H}_i^{\mathcal{G}_i}$, *and* $H^{\mathrm{red}}_{0,j}$ *is defined analogously.*

Proof.

(1) As before, we shall provide a proof of the proposition for the elementary refinements discussed in subsection 4.3.2, and for the sake of simplicity, we assume that Λ_i is the graph consisting of one edge e. It is clear that $u(C^\infty(\mathcal{K}_i)) \subseteq C^\infty(\mathcal{K}_j)$. Now let $\psi \in C^\infty(\mathcal{K}_i)$, let $X \in \mathfrak{g}$, and let $(a_1, a_2) \in \mathsf{G}_j^{(0)}$.

- If Λ_j is obtained from Λ_i by adding the edge $e_2 \in \Lambda_j^1$ then $I_{j,(s(e_1),t(e_1))} = I_{i,(s(e),t(e))}$ and we have trivially that

$$H_{j,0}(u(\psi))(a_1, a_2) = -\frac{1}{2} I_{j,(s(e_1),t(e_1))} \Delta_{e_1} \psi(a_1) = (u \circ H_{i,0}(\psi))\,(a_1, a_2).$$

- If Λ_j is obtained from Λ_i by subdividing the edge $e \in \Lambda_i^1$ into the two edges e_1 and $e_2 \in \Lambda_j^1$ then $I_{j,(s(e_1),t(e_1))} + I_{j,(s(e_2),t(e_2))} = I_{i,(s(e),t(e))}$ and

$$\begin{aligned} &H_{0,j}(u(\psi))(a_1, a_2)\\ &\quad= -\frac{1}{2}\left(I_{j,(s(e_1),t(e_1))}\Delta_{e_1}(u(\psi))(a_1, a_2)\right.\\ &\qquad \left.+ I_{j,(s(e_2),t(e_2))}\Delta_{e_2}(u(\psi))(a_1, a_2)\right)\\ &\quad= -\frac{1}{2}\left(I_{j,(s(e_1),t(e_1))} + I_{j,(s(e_2),t(e_2))}\right)\Delta_e(\psi)(a_1 a_2)\\ &\quad= -\frac{1}{2} I_{i,(s(e),t(e))}(u \circ \Delta_e(\psi))(a_1, a_2)\\ &\quad= (u \circ H_{0,i}(\psi))(a_1, a_2). \end{aligned}$$

  using invariance of the Laplacian on $L^2(G)$ with respect to the left and right action of G in going to the third line.

This proves commutativity of the diagram for the restrictions of the operators to the spaces of smooth functions. The assertion now follows from the fact that u is a bounded operator and the fact that $H_{0,i}$ and $H_{0,j}$ are the closures of their restrictions to $C^\infty(\mathcal{K}_i)$ and $C^\infty(\mathcal{K}_j)$, respectively.

(2) The inclusion is a consequence of the first part of this proposition, and the definition of u_{red}. Now let $p_i := p_{\mathcal{H}_i^{\mathcal{G}_i}}$, let $p_j := p_{\mathcal{H}_j^{\mathcal{G}_j}}$, and consider the following cube:

$$\begin{tikzcd}
 & \operatorname{Dom}(H_{0,j}) \arrow[rr, "H_{0,j}"] \arrow[dd, "p_j"'] & & \mathcal{H}_j \arrow[dd, "p_j"] \\
\operatorname{Dom}(H_{0,i}) \arrow[ur, "u"] \arrow[rr, "H_{0,i}"] \arrow[dd, "p_i"'] & & \mathcal{H}_i \arrow[ur, "u"] \arrow[dd, "p_i"'] & \\
 & \operatorname{Dom}(H_{0,j}) \cap \mathcal{H}_j^{\mathcal{G}_j} \arrow[rr, "H_{0,j}"'] & & \mathcal{H}_j^{\mathcal{G}_j} \\
\operatorname{Dom}(H_{0,i}) \cap \mathcal{H}_i^{\mathcal{G}_i} \arrow[ur, "u^{\text{red}}"'] \arrow[rr, "H_{0,i}"] & & \mathcal{H}_i^{\mathcal{G}_i} \arrow[ur, "u^{\text{red}}"'] &
\end{tikzcd}$$

The top face is commutative by the previous part of the proposition. The side faces of the cube are commutative by Proposition 4.13. The front and rear faces of the cube are commutative by Proposition 4.3, and by the same proposition, the map $p_i \colon \operatorname{Dom}(H_{0,i}) \to \operatorname{Dom}(H_{0,i}) \cap \mathcal{H}_i^{\mathcal{G}_i}$ is surjective. It follows that the bottom face of the cube is commutative, which is what we wanted to show. ■

If we take I_e to be the length of the edge e, as already mentioned in subsection 4.2.2, then the condition in Proposition 4.28 is satisfied, and we can define a Hamiltonian on the limits of both the unreduced and the reduced Hilbert spaces, which is what we show next.

4.7.1 The free Hamiltonian in the limit

Next, we show that if we take I_e to be the length of the edge e as already mentioned in subsection 4.2.2 then the condition in Proposition 4.28 is satisfied, and the free Hamiltonians on the Hilbert spaces that we have been investigating thus far give rise to a Hamiltonian on the direct limit. We accomplish this by studying more general systems of (possibly unbounded) self-adjoint operators on direct systems of Hilbert spaces and their spectral resolutions. The resolutions discussed in our proofs are obtained using the method descibed in [102, chapter 13], which involves taking the Cayley transform of the self-adjoint operator. In preparation for the manipulation of spectral resolutions of self-adjoint operators, we state and prove the following result from measure theory, which is essentially a version of the Borel–Cantelli lemma for integrable functions.

4.29 Lemma. *Let $(X, \mathcal{A}, \mu)$ a measure space, and let $(f_n)_{n=0}^{\infty}$ be a sequence of $\mathcal{A}$-measurable representatives of elements of $L^1(X, \mu)$ such that $\sum_{n=0}^{\infty} \|f_n\|_1 < \infty$. Then $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ for almost every $x \in X$. In particular, the sequence $(f_n)_{n=0}^{\infty}$ converges pointwise almost everywhere to the zero function.*

Proof. For each $N \in \mathbb{N}$, let $g_N := \sum_{n=0}^{N} |f_n|$. We claim that the set

$$Y := \{x \in X : \forall M > 0\, \exists N \geq 0 : g_N(x) \geq M\},$$

which is the set of all $x \in X$ on which $\sum_{n=0}^{\infty} |f_n(x)| = \infty$, is $\mathcal{A}$-measurable and has measure zero. Indeed, measurability follows from measurability of f_n for each $n \in \mathbb{N}$, and the fact that

$$Y = \bigcap_{M=1}^{\infty} \bigcup_{N \in \mathbb{N}} g_N^{-1}([M, \infty)).$$

Moreover, note that for each $M \in \mathbb{N} \backslash \{0\}$, we have

$$\begin{aligned} \mu(Y) &\leq \mu\left(\bigcup_{N \in \mathbb{N}} g_N^{-1}([M, \infty)) \right) = \sup_{N \in \mathbb{N}} \mu(g_N^{-1}([M, \infty))) \\ &\leq \sup_{N \in \mathbb{N}} \frac{1}{M} \int_X \sum_{n=0}^{N} |f_n| \, d\mu \leq \frac{1}{M} \sum_{n=0}^{\infty} \|f_n\|_1. \end{aligned}$$

Now taking $M \to \infty$, we conclude that $\mu(Y) = 0$, thereby proving the first assertion. For the second assertion, we note that the set

$$Z := \{x \in X : \exists \varepsilon > 0\, \forall N \in \mathbb{N}\, \exists n \geq N : |f_n(x)| \geq \varepsilon\},$$

of all points in X where the sequence $(f_n)_{n=0}^{\infty}$ does not converge to zero, is a subset of Y. Futhermore, Z is measurable, since

$$Z = \bigcup_{m=1}^{\infty} \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} |f_n|^{-1}([1/m, \infty)),$$

and f_n is measurable for each $n \in \mathbb{N}$, hence $\mu(Z) = 0$, which concludes our proof of the lemma. ■

4.30 Lemma. *For $j \in \{1, 2\}$, let $(\mathcal{H}_j, \langle \cdot, \cdot \rangle_j)$ be a Hilbert space, and let $T_j : \mathrm{Dom}(T_j) \to \mathcal{H}_j$ be a self-adjoint operator on $\mathcal{H}_j$. Suppose that $u : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map such that $u(\mathrm{Dom}(T_1)) \subseteq \mathrm{Dom}(T_2)$ and that the following diagram*

$$\begin{tikzcd}
\mathrm{Dom}(T_1) \arrow[r, "T_1"] \arrow[d, "u"'] & \mathcal{H}_1 \arrow[d, "u"] \\
\mathrm{Dom}(T_2) \arrow[r, "T_2"] & \mathcal{H}_2
\end{tikzcd}$$

commutes. Then for each bounded Borel-measurable function $f : \mathbb{R} \to \mathbb{C}$, the following diagram

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "f(T_1)"] \arrow[d, "u"'] & \mathcal{H}_1 \arrow[d, "u"] \\
\mathcal{H}_2 \arrow[r, "f(T_2)"] & \mathcal{H}_2
\end{tikzcd}$$

commutes.

Proof. For $j \in \{1,2\}$, let U_j be the Cayley transform of T_j. The Cayley transform of a self-adjoint operator is a unitary map by [102, Theorem 13.19(c)], hence its spectrum is a subset of the circle $S^1 \subset \mathbb{C}$. Using Gelfand duality, the C*-algebra $C(\sigma(U_j))$ is isomorphic to the C*-subalgebra $C^*(U_j)$ of $B(\mathcal{H}_j)$ generated by C_j, and this isomorphism is uniquely determined by the images of the elements $z \mapsto 1$ and $z \mapsto z$ (which are $\mathrm{Id}_{\mathcal{H}_j}$ and U_j, respectively). We claim that for each $f \in C(S^1)$, we have

$$u \circ f|_{\sigma(U_1)}(U_1) = f|_{\sigma(U_2)}(U_2) \circ u. \tag{4.11}$$

To see this, let $\psi \in \mathrm{Dom}(T_1)$. Then $u\psi \in \mathrm{Dom}(T_2)$, and

$$\begin{aligned} u \circ U_1((T_1 + i\mathrm{Id}_{\mathcal{H}_1})(\psi)) &= u((T_1 - i\mathrm{Id}_{\mathcal{H}_1})(\psi)) = (T_2 - i\mathrm{Id}_{\mathcal{H}_2}) \circ u(\psi) \\ &= U_2 \circ (T_2 + i\mathrm{Id}_{\mathcal{H}_2}) \circ u(\psi) \\ &= U_2 \circ u((T_1 + i\mathrm{Id}_{\mathcal{H}_1})(\psi)). \end{aligned}$$

Since $T_1 + i\mathrm{Id}_{\mathcal{H}_1}$ is surjective, it follows that $u \circ U_1 = U_2 \circ u$. A similar argument shows that $u \circ U_1^{-1} = U_2^{-1} \circ u$. It follows that equation (4.11) holds when f is a trigoniometric polynomial, i.e., f is of the form $\sum_{n=-N}^{N} c_n z^n$. Now let $f \in C(S^1)$ be arbitrary, and let $\varepsilon > 0$. It is a well-known fact that the trigoniometric polynomials are dense in $C(S^1)$, so we may fix such a polynomial p such that $\|f - p\|_\infty < \varepsilon/2$, which implies

$$\begin{aligned} &\|u \circ f|_{\sigma(U_1)}(U_1) - f|_{\sigma(U_2)}(U_2) \circ u\| \\ &\quad \leq |u \circ p|_{\sigma(U_1)}(U_1) - p|_{\sigma(U_2)}(U_2) \circ u\| \\ &\qquad + \|u \circ f|_{\sigma(U_1)}(U_1) - u \circ p|_{\sigma(U_2)}(U_2)\| \\ &\qquad + \|f|_{\sigma(U_2)}(U_2) \circ u - p|_{\sigma(U_2)}(U_2) \circ u\| \\ &\quad \leq \|u\|(\|f|_{\sigma(U_1)}(U_1) - p|_{\sigma(U_1)}(U_1)\| + \|f|_{\sigma(U_2)}(U_2) - p|_{\sigma(U_2)}(U_2)\|) \\ &\quad < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we infer that equation (4.11) holds for general $f \in C(S^1)$.

Next, we extend equation (4.11) to the case where f is a bounded Borel-measurable function. Let f be such a function. We are going to approximate f by a sequence of continuous functions on S^1 that is bounded with respect to the sup-norm and that converges rapidly to f in the L^1-space associated to a particular measure space, which we define first.

For $j = 1, 2$, let $\tilde{E}_j$ be the spectral resolution of the Cayley transform U_j, and let $\psi_j \in \mathcal{H}_j$. Now let $\mu_1 := \tilde{E}_{1,u^*(\psi_2),\psi_1}$ and $\mu_2 := \tilde{E}_{2,\psi_2,u(\psi_1)}$ be the complex measures on $S^1 \subset \mathbb{C}$ associated to the spectral resolutions and vectors, where

$$\mu_1(X) = \tilde{E}_{1,u^*(\psi_2),\psi_1}(X \cap \sigma(U_1)) = \langle u^*(\psi_2), \tilde{E}_1(X \cap \sigma(U_1))\psi_1\rangle,$$

for each Borel-measurable subset $X \subseteq S^1$, and μ_2 and $\tilde{E}_{2,\psi_2,u(\psi_1)}$ are defined similarly. Let $|\mu_1|$ and $|\mu_2|$ be the variations of μ_1 and μ_2, respectively. The measure space of which we want to consider its corresponding L^1-space is S^1 endowed with its Borel σ-algebra and the measure $\mu := |\mu_1| + |\mu_2|$.

Both $|\mu_1|$ and $|\mu_2|$ are finite measures. It follows that μ is finite as well, and since it is a measure on a Borel-subset of $\mathbb{C}$, it is regular (cf. [29, Proposition 1.5.6]), hence $C(S^1)$ is dense in $L^1(S^1, \mu)$. Now choose a bounded sequence $(f_n)_{n=0}^\infty$ of elements in $C(S^1)$ such that $\|f_n - f\|_1 < 2^{-n}$ for each $n \in \mathbb{N}$. By Lemma 4.29, the sequence $(f_n)_{n=0}^\infty$ converges pointwise to f μ-almost everywhere. Equivalently, it converges pointwise to f $|\mu_j|$-almost everywhere for $j = 1, 2$.

By definition of the measures μ_1 and μ_2 as well as the already established result that equation (4.11) holds for continuous functions, we find that for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_{S^1} f_n \, d\mu_1 &= \langle u^*(\psi_2), f_n|_{\sigma(U_1)}(U_1)(\psi_1)\rangle_1 = \langle \psi_2, u \circ f_n|_{\sigma(U_1)}(U_1)(\psi_1)\rangle_2 \\ &= \langle \psi_2, f_n|_{\sigma(U_2)}(U_2) \circ u(\psi_1)\rangle_2 = \int_{S^1} f_n \, d\mu_2,\end{aligned}$$

hence, noting that the sequence $(f_n)_{n=0}^\infty$ is bounded from above by a constant function, which is both integrable with respect to both μ_1 and μ_2, we may invoke the dominated convergence to obtain

$$\begin{aligned}\langle \psi_2, u \circ f|_{\sigma(U_1)}(U_1)(\psi_1)\rangle_2 &= \int_{S^1} f \, d\mu_1 = \lim_{n\to\infty} \int_{S^1} f_n \, d\mu_1 \\ &= \lim_{n\to\infty} \int_{S^1} f_n \, d\mu_2 = \int_{S^1} f \, d\mu_2 = \langle \psi_2, f|_{\sigma(U_2)}(U_2) \circ u(\psi_1)\rangle_2.\end{aligned}$$

Since $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$ were arbitrary, it follows that equation (4.11) also holds if f is a bounded Borel-measurable function.

The spectral resolutions E_j of T_j and $\tilde{E}_j$ of U_j, $j = 1, 2$, are related as follows. First, one notes that $\tilde{E}_j(\{1\}) = 0$ (cf. the proof of [102, Theorem 13.30]), so that the resolution $\tilde{E}_j$ can be restricted to a resolution on $\sigma(U_j)\backslash\{1\}$, and subsequently, one defines the resolution E_j on $\mathbb{R}$ by setting $E_j(X) := \tilde{E}_j(\varphi^{-1}(X))$, where φ is the map

$$\sigma(U_j)\backslash\{1\} \to \mathbb{R}, \quad z \mapsto i\frac{1+z}{1-z}.$$

Thus for each bounded Borel function f on $\mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} f\, dE_2 \circ u &= \left(\int_{\sigma(U_2)\backslash\{0\}} f \circ \varphi\, d\tilde{E}_2\right) \circ u = u \circ \int_{\sigma(U_1)\backslash\{0\}} f \circ \varphi_1\, d\tilde{E}_1 \\ &= u \circ \int_{\mathbb{R}} f\, dE_1,\end{aligned}$$

as desired. ∎

4.31 Proposition. *Let $((\mathcal{H}_i, \langle\cdot,\cdot\rangle_i)_{i\in I}, (u_{i,j})_{i,j\in I, i\leq j})$ be a direct system of Hilbert spaces with direct limit $((\mathcal{H}_\infty, \langle\cdot,\cdot\rangle), (u_{i,\infty})_{i\in I})$. For each $i \in I$, let $T_i\colon \mathrm{Dom}(T_i) \to \mathcal{H}_i$ be a (possibly unbounded) self-adjoint operator on $\mathcal{H}_i$. Assume that for each $i, j \in I$ with $i \leq j$, we have $u_{i,j}(\mathrm{Dom}(T_i)) \subseteq \mathrm{Dom}(T_j)$, and that the following diagram*

$$\begin{array}{ccc}\mathrm{Dom}(T_i) & \xrightarrow{T_i} & \mathcal{H}_i \\ {\scriptstyle u_{i,j}}\downarrow & & \downarrow{\scriptstyle u_{i,j}} \\ \mathrm{Dom}(T_j) & \xrightarrow{T_j} & \mathcal{H}_j\end{array}$$

commutes. Then

(1) There exists a unique self-adjoint operator $T_\infty\colon \mathrm{Dom}(T_\infty) \to \mathcal{H}$ on $\mathcal{H}$ such that for each $i \in I$, we have $u_{i,\infty}(\mathrm{Dom}(T_i)) \subseteq \mathrm{Dom}(T_\infty)$, and the following diagram

$$\begin{tikzcd} \mathrm{Dom}(T_i) \arrow[r, "T_i"] \arrow[d, "u_{i,\infty}"'] & \mathcal{H}_i \arrow[d, "u_{i,\infty}"] \\ \mathrm{Dom}(T_\infty) \arrow[r, "T_\infty"] & \mathcal{H}_\infty \end{tikzcd}$$

commutes.

(2) For each $i \in I$ and for each bounded Borel function $f\colon \mathbb{R} \to \mathbb{C}$, the following diagram

$$\begin{tikzcd} \mathcal{H}_i \arrow[r, "f(T_i)"] \arrow[d, "u_{i,\infty}"'] & \mathcal{H}_i \arrow[d, "u_{i,\infty}"] \\ \mathcal{H}_\infty \arrow[r, "f(T_\infty)"] & \mathcal{H}_\infty \end{tikzcd}$$

commutes.

Proof.

(1) Note that for each $i, j \in I$ with $i \leq j$, the solid arrows in the diagram in Figure 4.2 commute, hence there exists a unique linear map $\tilde{T}_\infty\colon \bigcup_{k\in I} u_{k,\infty}(\mathrm{Dom}(T_k)) \to \mathcal{H}$ that makes the diagram commutative. The operator $\tilde{T}$ is densely defined since $\bigcup_{k\in I} u_{k,\infty}(\mathrm{Dom}(T_k))$ is dense in $\mathcal{H}_\infty$.

Next, we claim that $\tilde{T}_\infty$ is hermitian, i.e., $\tilde{T}_\infty \subseteq \tilde{T}_\infty^*$. Note that we have direct system of Hilbert spaces $((\mathrm{Graph}(T_i))_{i\in i}, ((u_{i,j} \oplus u_{i,j})|_{\mathrm{Graph}(T_i)})_{i,j\in I, i\leq j})$, where $u_{i,j} \oplus u_{i,j}$ denotes the map

$$\mathcal{H}_i^2 \to \mathcal{H}_j^2, \quad (\psi, \phi) \mapsto (u_{i,j}(\psi), u_{i,j}(\phi)),$$

Identifying $\lim_{i\in I} \mathcal{H}_i \oplus \mathcal{H}_i$ with $\mathcal{H}_\infty^2$ in the canonical way, we can identify the canonical maps $\mathcal{H}_i^2 \to \lim_{j\in I} \mathcal{H}_j^2$ with the maps $u_i \oplus u_i$. It is now readily seen that $\mathrm{Graph}(\tilde{T}_\infty) = \bigcup_{i\in I} u_i \oplus u_i(\mathrm{Graph}(T_i))$. For each $i \in I$,

$$
\begin{tikzcd}
 & \bigcup_{k\in I} u_{k,\infty}(\mathrm{Dom}(T_k)) \arrow[dr, dashed, "\tilde{T}_\infty"] & \\
 & \mathrm{Dom}(T_j) \arrow[u, "u_{j,\infty}"] \arrow[r, "u_j \circ T_j"] & \mathcal{H}_\infty \\
\mathrm{Dom}(T_i) \arrow[uur, "u_{i,\infty}"] \arrow[ur, "u_{i,j}"'] \arrow[urr, "u_i \circ T_i"'] & &
\end{tikzcd}
$$

Figure 4.2: The essentially self-adjoint operator $\tilde{T}_\infty$ induced by the self-adjoint operators $(T_i)_{i\in I}$.

we have a map $J_i \in B(\mathcal{H}_i^2)$ given by $(a,b) \mapsto (-b,a)$, and similarly, we have a map $J_\infty \in B(\mathcal{H}_\infty^2)$ defined by the same formula, and

$$J_\infty(\mathrm{Graph}(\tilde{T}_\infty)) = \bigcup_{i\in I}(u_{i,\infty} \oplus u_{i,\infty})(J_i(\mathrm{Graph}(T_i))).$$

Now let $i,j \in I$. Then there exists $k \in I$ such that $k \geq i,j$. Since T_k is selfadjoint, we have

$$\mathcal{H}_k^2 = \mathrm{Graph}(T_k) \oplus J_k(\mathrm{Graph}(T_k)),$$

so in particular $\mathrm{Graph}(T_k) \perp J_k(\mathrm{Graph}(T_k))$. Since $(u_{l,k} \oplus u_{l,k})(\mathrm{Graph}(T_l)) \subseteq \mathrm{Graph}(T_k)$ for $l = i,j$, we have

$$(u_{i,k} \oplus u_{i,k})(\mathrm{Graph}(T_i)) \perp (u_{j,k} \oplus u_{j,k})(J_j(\mathrm{Graph}(T_j))),$$

and hence

$$(u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)) \perp (u_{j,\infty} \oplus u_{j,\infty})(J_j(\mathrm{Graph}(T_j))).$$

Since i and j were arbitrary, it follows that

$$(u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)) \perp \bigcup_{j\in I}(u_{j,\infty} \oplus u_{j,\infty})(J_j(\mathrm{Graph}(T_j))),$$

and hence

$$\bigcup_{i\in I}(u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)) \perp \bigcup_{j\in I}(u_{j,\infty} \oplus u_{j,\infty})(J_j(\mathrm{Graph}(T_j))),$$

i.e., $\mathrm{Graph}(\tilde{T}_\infty) \perp J_\infty(\mathrm{Graph}(\tilde{T}_\infty))$. This shows that $\tilde{T}_\infty$ is hermitian.

Next, we show that $\tilde{T}_\infty$ is essentially selfadjoint. Let T_∞ be the closure of $\tilde{T}_\infty$. Note that the closure exists by virtue of the fact that $\tilde{T}_\infty$ is hermitian. Now observe that

$$\begin{aligned}
\mathcal{H}^2_\infty &= \varinjlim_{i\in I} \mathcal{H}^2_i = \varinjlim_{i\in I}(\mathrm{Graph}(T_i) \oplus J_i(\mathrm{Graph}(T_i))) \\
&= \overline{\bigcup_{i\in I}(u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i) \oplus J_i(\mathrm{Graph}(T_i)))} \\
&= \overline{\bigcup_{i\in I}((u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)) \oplus J_\infty \circ (u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)))} \\
&= \overline{\bigcup_{i\in I}((u_{i,\infty} \oplus u_{i,\infty})(\mathrm{Graph}(T_i)))} \oplus J_\infty \left(\overline{\bigcup_{j\in I}(u_{j,\infty} \oplus u_{j,\infty})(\mathrm{Graph}(T_j))}\right) \\
&= \mathrm{Graph}(T_\infty) \oplus J_\infty(\mathrm{Graph}(T_\infty)).
\end{aligned}$$

Thus T_∞ is selfadjoint, and consequently, $\tilde{T}_\infty$ is essentially selfadjoint. We conclude that T_∞ is the unique operator on $\mathcal{H}_\infty$ that has all of the desired properties.

(2) This is an immediate consequence of the first part of the proposition and Lemma 4.30. ■

Combining Propositions 4.28 and 4.31, we infer that there exist canonical self-adjoint operators $H_{0,\infty}$ and $H^{\mathrm{red}}_{0,\infty}$ on the unreduced Hilbert space $L^2(\mathcal{K}_\infty)$ and the reduced Hilbert space $L^2(\mathcal{K}^{\mathrm{red}}_\infty)$, respectively, that we may regard as Hamiltonians. Moreover, the spectral decomposition of $H_{0,\infty}$ is well-behaved with respect to the spectral decompositions of each of the members of the family of Hamiltonians $(H_{0,i})_{i\in I}$, and an analogous statement holds for the reduced Hamiltonian in the limit. In particular, this implies

$$\exp\left(\frac{itH_{0,k}}{\hbar}\right) \circ u_{j,k} = u_{j,k} \circ \exp\left(\frac{itH_{0,j}}{\hbar}\right),$$

for each $t \in \mathbb{R}$, each $j \in I$, and each $k \in I \cup \{\infty\}$ with $k \geq j$, and where $i = \sqrt{-1}$. In other words, the families of isometries $(u_{j,k})_{j\leq k}$ are compatible with the free time evolutions of the quantum systems, including the one on the direct limit. In contrast with the spectral properties of $H_{0,i}$, it is less clear what the summability properties of $H_{0,\infty}$ are, as for instance infinite multiplicities will appear. We leave the analysis of this aspect of the limit Hamiltonian for future work.

4.8 Problems with the groupoid approach

For the quantization of the configuration space we have followed the approach of [64] and defined the quantized field algebra and observable algebra as groupoid C*-algebras. The merit of this approach is that it is fully compatible with the natural maps between configuration spaces induced by graph refinements. Hence it allowed us to concretely describe the field algebras and the observable algebras in both the continuum and the themodynamic limit.

However, when we want to extend the above kinematical description of the limiting quantum gauge system to incorporate the Hamiltonian dynamics for the interacting system, we run into the following problems. Namely, since our limit observable algebra is given by the space of compact operators, it does not really capture the infinite number of degrees of freedom that one would expect for an interacting quantum field theory (cf. [124] for a nice overview of this point), or in the description of the statistical physics of an infinite system at finite temperature [7]. As such, our limit observable algebra only admits KMS-states that are associated to inner automorphisms of the algebra, which prompts the question whether it is the right algebra for the description of a nontrivial quantum field theory.

The reason for this lack of interesting states might be that even though our choice of maps between configuration spaces is natural from a *classical* point of view, the induced maps $v_{i,j}$ between the different observable algebras defined in Proposition 4.16 do not induce maps between the state spaces of the algebras.

It is in this context interesting to mention that there are other approaches to the construction of the limit observable algebra, one of which was developed by Kijowski in [58], and later by Okołów in [88] (cf. [59]), and recently explored in depth by Lanéry and Thiemann in a series of papers [72, 73, 74, 75], see [71] for a summary of these papers. The main point where their approach differs from ours is that they assume the existence of a canonical unitary map between Hilbert spaces, which they use to define injective *-homomorphism between the corresponding algebras of bounded operators, and which ensures that the transpose of this homomorphism maps states to states, i.e. preserves the normalisation of positive functionals. However, in their approach, the maps at the level of bounded operators do not reduce to maps between the algebras of compact operators, a problem that was already observed by Stottmeister and Thiemann [109]. This suggests that the framework C*-algebraic quantisation of groupoids described in e.g. [65] should at least be modified (if not abandoned) to ensure that there is an induced map between state spaces.

In [48], Grundling and Rudolph include matter fields, and go beyond the kinematic picture by proving that the dynamics on certain algebras associated to finite lattices converge to a group of automorphisms on a corresponding limit algebra in the thermodynamic limit; this is the first known rigorous result on global dynamics for lattice gauge theory. They subsequently identify a subalgebra of this limit algebra that is closed under the global time evolution as the field algebra of the system. Interestingly, they note that in their earlier paper [47], the algebra that they constructed there, which is different from the one in [48], does not admit interesting dynamics. Our limit algebra may suffer from the same problem, indicating that already in the thermodynamic limit, a different algebra may be required, such as the one in [48]. We address some of these problems in the second part of this thesis.

Part II

A resolvent algebra for the cylinder

Chapter 5

The classical resolvent algebra

The purpose of this chapter is to introduce a notion of a classical/commutative resolvent algebra for the cotangent bundle $T^*\mathbb{T}^n$ of the n-dimensional torus $\mathbb{T}^n$, and serves as the foundation for the second part of this thesis. We begin by reviewing the resolvent algebra on a symplectic vector space as conceived by Buchholz and Grundling [26], in which we focus on the particular properties of this algebra that are of interest to us. We then discuss the work of van Nuland on the commutative version of this algebra [112]; his characterisation of the generators is readily generalised to $T^*\mathbb{T}^n$. We prove that the *-algebra generated by these elements is a Poisson subalgebra of $C^\infty(T^*\mathbb{T}^n)$, and show that its closure is a C*-algebra that is closed under time evolution for $n = 1$ (although it is worth noting that the result is true for arbitrary n).

The work presented in chapters 5 and 7 of this thesis was carried out in collaboration with Teun van Nuland.

5.1 Introduction

The resolvent algebra $\mathcal{R}(V, \omega)$ on a symplectic vector space (V, ω) is a C*-algebra introduced by Buchholz and Grundling in [25], and subsequently studied in greater detail by the same authors in [26, 22]. The resolvent algebra is defined as a completion of a *-algebra defined through generators and relations; the generators of the algebra are denoted by $\mathcal{R}(\lambda, v)$, where $\lambda \in \mathbb{R}\backslash\{0\}$ and $v \in V$, and should be thought of as representing resolvents of certain unbounded operators in the following way.

Suppose for simplicity that $V = \mathbb{R}^{2n}$, and that ω is the standard symplectic form on this vector space. Thus, writing $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\omega((q_1, p_1), (q_2, p_2)) = p_1 \cdot q_2 - p_2 \cdot q_1,$$

where $\cdot$ denotes the standard inner product. Now we consider the usual quantum-mechanical counterparts of elements $(a, 0)$ and $(0, b)$ for $a = (a_1, \ldots, a_n)$ and $b = (b^1, \ldots, b^n)$ in $\mathbb{R}^n$, which are unbounded operators on $L^2(\mathbb{R}^n)$. They are given by

$$a \cdot Q \colon \psi \mapsto \left(x = (x^1, \ldots, x^n) \mapsto \sum_{j=1}^{n} a_j x^j \psi(x) \right),$$

and

$$b \cdot P \colon \psi \mapsto \left(x \mapsto -i\hbar \cdot \sum_{j=1}^{n} b^j \frac{\partial \psi}{\partial x^j}(x) \right),$$

respectively, both of which (as well as their sum) can be defined on suitable dense domains, such as $C_c^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$, and subsequently extended to self-adjoint operators, which will also be denoted by $a \cdot Q$ and $b \cdot P$. One can now consider their resolvents $(i\lambda - a \cdot Q)^{-1}$ and $(i\lambda - b \cdot P)^{-1}$ (where $\lambda \in \mathbb{R}\backslash\{0\}$ as above), both of which are bounded operators on $L^2(\mathbb{R}^n)$. The idea is that the resolvent algebra should admit a faithful representation on $L^2(\mathbb{R}^n)$ that maps the generators $\mathcal{R}(\lambda, (a, 0))$ and $\mathcal{R}(\lambda, (0, b))$ to the above resolvents, respectively, hence the name 'resolvent algebra'. There is indeed such a representation; this follows from [26, Corollary 4.4] and the fact that the usual representation of the Weyl algebra corresponding to $\mathbb{R}^{2n}$ on $L^2(\mathbb{R}^n)$ is regular. For future reference, let us briefly recall the definition of the Weyl algebra:

5.1 Definition. Let $\hbar \in \mathbb{R}\backslash\{0\}$. The *Weyl algebra* $\mathcal{W}^\hbar(\mathbb{R}^n)$ *(on* $\mathbb{R}^{2n}$*(!))* is the C*-subalgebra of $B(L^2(\mathbb{R}^n))$ generated by the operators of the form

$$e^{i(a \cdot Q + b \cdot P)},$$

where $a, b \in \mathbb{R}^n$, and $a \cdot Q$ and $b \cdot P$ are as above.

The C*-algebra that is obtained by taking $\hbar = 0$ in this definition is the set of multiplication operators associated to the algebra of *almost periodic functions on* $\mathbb{R}^n$, which is the C*-subalgebra of $C_b(\mathbb{R}^n)$ generated by the functions

$$\mathbb{R}^n \to \mathbb{C}, \quad x \mapsto e^{ia \cdot x}, \quad a \in \mathbb{R}^n.$$

This algebra of functions will accordingly be denoted by $\mathcal{W}^0(\mathbb{R}^n)$.

5.2 Remark. The term 'Weyl algebra' is sometimes also used for the infinitesimal counterpart of the above algebra, i.e., the complex algebra generated by the operators $a \cdot Q$ and $b \cdot P$ defined earlier, viewed as linear operators on $C^\infty(\mathbb{R}^n)$.

As mentioned above, a certain set of relations (cf. [26, Definition 3.1]) is imposed on the generators $\mathcal{R}(\lambda, v)$ of the resolvent algebra. These relations together encode two properties of the generators. First of all, the relations ensure that each $\mathcal{R}(\lambda, v)$ is a resolvent of some unbounded operator $\phi(v)$ corresponding to v, where ϕ is some map that assigns to each element of V an unbounded operator and is linear in the sense that

$$\phi(\mu v + w) = \mu\phi(v) + \phi(w),$$

for each $\mu \in \mathbb{C}$ and $v, w \in V$, provided that both sides of the above equation are restricted to the intersection of the domains of the three unbounded operators in this formula. Secondly, the relations encode the canonical commutation relation $[\phi(v), \phi(w)] = -i\hbar\omega(v, w)$; here, $\hbar$ can be removed from the equation by redefining ω.

The relations imply that the image of a generator $\mathcal{R}(\lambda, v)$ under a *-representation of the *-algebra generated by these elements has the property that its norm is bounded by $|\lambda|^{-1}$, in addition to the well-definedness of the GNS representations of this *-algebra and the fact that its image consists of bounded operators [26, Proposition 3.3]. These facts are used to define a C*-seminorm on the *-algebra, which, by taking the quotient with respect to its null space and subsequently taking the completion of the quotient space, yields the resolvent algebra $\mathcal{R}(V, \omega)$.

As the motivation for their study of the resolvent algebra, Buchholz and Grundling mention some of the problems that the Weyl algebra has, one of which is the lack of interesting dynamics that it admits. The resolvent algebra is better behaved in this respect, at least in the case where the vector space V is two-dimensional; cf. [26, Proposition 6.1]. Buchholz has shown that the resolvent algebra is also stable under large classes of dynamics in the context of oscillating lattice systems [23] and nonrelativistic Bose fields [24]. We want our analogue of the resolvent algebra for $T^*\mathbb{T}^n$ to have a similar property.

Stability under time evolution is very much desirable, although it is not the primary motivation behind our interest in the resolvent algebra, which we will discuss next. Suppose $U \subseteq V$ is a symplectic subspace of V, i.e., the restriction of ω to U is nondegenerate, and let

$$U^\omega := \{v \in V \mid \forall u \in U : \omega(u, v) = 0\},$$

be the associated complementary subspace. (Buchholz and Grundling call this space $U^\perp$.) By abuse of notation, we write ω for the restrictions of the symplectic form ω on V to U and U^ω. Then there is a canonical embedding map $\mathcal{R}(U, \omega) \hookrightarrow \mathcal{R}(V, \omega)$ that maps every generator $\mathcal{R}(\lambda, w)$ of the domain to the generator of the codomain denoted by the same expression. Since symplectic bases of U and U^ω can always be found in principle, we may assume without loss of generality that (V, ω) is $\mathbb{R}^{2n}$ endowed with the standard symplectic form, that

$$U = (\mathbb{R}^{n_1} \times \{0\})^2 \subseteq \mathbb{R}^{2n},$$

and that

$$U^\omega = (\{0\} \times \mathbb{R}^{n_2})^2 \subseteq \mathbb{R}^{2n},$$

where $n_1 + n_2 = n$. In terms of the canonical representations of $\mathcal{R}(U, \omega)$, $\mathcal{R}(U^\omega, \omega)$ and $\mathcal{R}(V, \omega)$ on $L^2(\mathbb{R}^{n_1})$, $L^2(\mathbb{R}^{n_2})$ and $L^2(\mathbb{R}^n)$, respectively (which are faithful by Corollary 4.4 and Theorem 5.1 in [26]), this embedding map reads

$$a \mapsto a \otimes \mathbf{1}_{\mathbb{R}^{n_2}} \in B(L^2(\mathbb{R}^{n_1}) \mathbin{\hat{\otimes}} L^2(\mathbb{R}^{n_2})) \cong B(L^2(\mathbb{R}^n)),$$

where $\mathbf{1}_{\mathbb{R}^{n_2}} \in B(L^2(\mathbb{R}^{n_2}))$ denotes the unit element, the symbol $\hat{\otimes}$ denotes the completion of algebraic tensor product with respect to (the norm corresponding to) the canonical inner product, and the isomorphism between the spaces of bounded operators is the one induced by the canonical isomorphism between the corresponding Hilbert spaces. Buchholz and Grundling themselves already mention this in [26, Remark 5.5(d)] without referring to the aforementioned representation of these algebras.

Embeddings of this type appear in the work Stottmeister and Thiemann in [109], and the transposes of such maps form the basis of the work of Lanéry and Thiemann mentioned at the end of the previous chapter. It is standard doctrine that, given two quantum systems with Hilbert spaces $\mathcal{H}_j$ and observable algebras $A_j \subseteq B(\mathcal{H}_j)$ for $j = 1, 2$, the composite system is given by their Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, with observable algebra $A_{1,2} \subseteq B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. (To simplify the discussion, we will not bother with the distinction between field and observable algebra in this section). Furthermore, the embedding of the observable algebra of the first quantum system into the composite system is given by

$$A_1 \hookrightarrow A_{1,2}, \quad a \mapsto a \otimes \mathbf{1}_{\mathcal{H}_2},$$

and the embedding of the observables of the second system is defined similarly. (The author thanks Alexander Stottmeister for pointing this out to him.) This will be motivated more thoroughly in chapter 8. Note that if we take the observable algebra of each of the above three systems to be space of compact operators on the corresponding Hilbert space, and if $\mathcal{H}_j$ has infinite dimension for $j = 1$ or $j = 2$, then the corresponding embedding of A_j into $A_{1,2}$ is ill-defined, since each nonzero element of A_j will be mapped to an operator that is not compact. By contrast, families of resolvent algebras corresponding to the family of nondegenerate subspaces of a symplectic vector space are closed under the above map.

This type of embedding appears in situations where one wishes to take the thermodynamic limit. For example, in the setting of chapter 4, this is the map between the observable algebras of the corresponding graphs that we expect based on the physics literature when adding an edge (with Hilbert space $\mathcal{H}_2 = L^2(G)$) to a graph (with Hilbert space $\mathcal{H}_1 = L^2(G^{\Lambda^1})$). Thus, if we are able to find an analogue of the resolvent algebra for T^*G that has the analogue of the property of the algebra of Buchholz and Grundling mentioned above, then we will have found a better candidate for the observable algebra of the system than the space of compact operators. This the main reason for the present investigation.

Since a definition of such an algebra for compact connected Lie groups G, let alone general Lie groups, appears to be far away, we have restricted ourselves to the case $G = \mathbb{T}^n$, in which we can draw analogies with the established case $G = \mathbb{R}^n$ to motivate our definition of the resolvent algebra.

While embedding maps for observable algebras are necessary to form a direct system of C*-algebras and take its corresponding limit, there is another way to obtain an algebra that *a priori* could be considered an observable algebra of the system in the limit. Indeed, one could start with a direct system of Hilbert spaces with isometries $((\mathcal{H}_i)_{i\in I}, (u_{i,j})_{i,j\in I, i\leq j})$, along with a family of Hilbert spaces $(\mathcal{H}_{i,j})_{i,j\in I,i\leq j}$ and a family of isomorphisms $\mathcal{H}_j \cong \mathcal{H}_i \hat{\otimes} \mathcal{H}_{i,j}$ that satisfy certain conditions, as described in [73, Definition 2.1]. The example in the previous paragraph provides an illustration of this: take $\mathcal{H}_i = L^2(G^{\Lambda^1})$, take $\mathcal{H}_{i,j} = L^2(G)$, and take $\mathcal{H}_j = L^2(G^{\Lambda^1\cup\{e\}})$, which are the Hilbert spaces associated of the smaller graph, the edge that is being added to the smaller graph, and the resulting larger graph, respectively.

Returning to the construction, let $(A_i)_{i\in I}$ be a family of C*-algebras parametrised by the same set I, and let $(\pi_i)_{i\in I}$ be a family of *-representations, where π_i is a representation of A_i on $\mathcal{H}_i$ for each $i \in I$. One then forms the direct limit $\mathcal{H}_\infty := \varinjlim_{i\in I} \mathcal{H}_i$. Now for each $i \in I$, there exists a Hilbert space $\mathcal{H}_i'$ and an isomorphism $\mathcal{H}_\infty \cong \mathcal{H}_i \hat{\otimes} \mathcal{H}_i'$, and $\pi_i(A_i)$ can be embedded into $B(\mathcal{H}_\infty)$ by taking the tensor product with $\mathbf{1}_{\mathcal{H}_i'}$; to avoid cumbersome notation in what follows, let us identify A_i with the image of its embedding in $B(\mathcal{H}_\infty)$. The C*-algebra generated by the union of the family of algebras $(A_i)_{i\in I}$ is taken to be the limit algebra $A \subseteq B(\mathcal{H}_\infty)$. This is roughly the approach taken by Grundling and Rudolph in [48], who, after constructing A as described above, close it with respect to the time evolution. More precisely, they first construct a strongly continuous one-parameter group of unitary operators on $\mathcal{H}_\infty$ as a limit of a family of one-parameter groups of unitaries, each of which is generated by the Hamiltonian on $\mathcal{H}_i$, thereby obtaining dynamics on the limit. They subsequently define their limit field algebra as the C*-algebra generated by the orbits of the elements of $\bigcup_{i\in I} A_i \subseteq B(\mathcal{H}_\infty)$ under the adjoint action of the one-parameter group.

This construction has the following drawback. Given $i, j \in I$ such that

$i \leq j$, in general, it cannot be expected that A_i is a subset of A_j, and this is indeed not the case in [48], as the authors themselves note. Thus, if $I' \subset I$ is a proper subset that is cofinal in I, then we have the following inclusion

$$C^* \left(\bigcup_{i \in I'} A_i \right) \subseteq C^* \left(\bigcup_{i \in I} A_i \right),$$

of C*-algebras generated by the unions of the corresponding subalgebras of $B(\mathcal{H}_\infty)$, but the inclusion will in general not be an equality. This means that the limit algebra depends on the index set, which goes against the philosophy that the limit should be independent of the particular choice of regulator.

Of course, in certain systems, one could exclude other index sets than a given set I on physical grounds. For example, in the context of condensed matter physics, if one is interested in modeling a material of which the atoms form a lattice, then it makes sense to take I to be as large as possible, i.e., by taking I to be the set of all finite subgraphs of this lattice. On the other hand, in the context of gauge theory and particle physics, one is interested in taking the continuum limit, and a choice such as the one just mentioned seems arbitrary. It is however worth noting that in loop quantum gravity, I is taken to be the set of all finite graphs, of which the edges correspond to piecewise analytic (compact) paths in spacetime. The implicit assumption here is that spacetime is described by an analytic manifold. The condition of analyticity is imposed to ensure that the paths only intersect each other at finitely many points at most, which in turn is necessary to make sure that I is upward directed, but it is unclear what it means from a physical point of view.

For us, the great thing about the resolvent algebra is that it already appears to contain all of the resolvent algebras associated to subsystems or coarser descriptions of the system of which we are considering the resolvent algebra, hence one has more freedom in choosing the index set. In this context, it is also worth mentioning the work of van Nuland [112], who considered the classical/commutative counterpart $C_{\mathcal{R}}(\mathbb{R}^{2n})$ of the resolvent algebra - which is also defined for odd-dimensional vector spaces - and showed that $\mathcal{R}(\mathbb{R}^{2n}, \omega)$ is a quantisation of this commutative C*-algebra. The algebra $C_{\mathcal{R}}(\mathbb{R}^{2n})$ is in some sense the *smallest* commutative C*-algebra containing $C_0(\mathbb{R}^{2n})$, as well as the embeddings of each algebra

$C_{\mathcal{R}}(U)$ that corresponds to a subspace U of $\mathbb{R}^{2n}$, in particular those subspaces U on which ω is nondegenerate.

To put it more formally (and more generally), let $\mathcal{P}_0$ be the set of linear subspaces of $\mathbb{R}^n$, and regard the poset $(\mathcal{P}_0, \subseteq)$ as a category in the usual way. We can now consider the covariant functor C_b from this category to the category of commutative C*-algebras that is defined as follows:

- An object U in $(\mathcal{P}_0, \subseteq)$, i.e., a subspace U of $\mathbb{R}^n$, is mapped to its space of continuous bounded functions $C_b(U)$;

- A morphisms (U, V), i.e., a pair of subspaces of $\mathbb{R}^n$ such that $U \subseteq V$, is mapped to the pullback of the orthogonal projection of V onto U to the corresponding function spaces.

We can now consider the smallest subfunctor F of C_b with the property that $C_0(U) \subseteq F(U)$ for each $U \in \mathcal{P}_0$. We then have $C_{\mathcal{R}}(\mathbb{R}^n) = F(\mathbb{R}^n)$. Note that $C_{\mathcal{R}}(\mathbb{R}^n)$ is unital since it contains the embedding of $C_0(\{0\}) \cong \mathbb{C}$ into $C_b(\mathbb{R}^n)$.

This suggests that the quantum counterpart $\mathcal{R}(\mathbb{R}^{2n}, \omega)$ of $C_{\mathcal{R}}(\mathbb{R}^{2n})$, specifically its image under the canonical representation on $L^2(\mathbb{R}^n)$, is the smallest quantised algebra that allows embeddings of resolvent algebras associated to subsystems that in addition contains the quantisation $B_0(L^2(\mathbb{R}^n))$ of $C_0(\mathbb{R}^{2n})$. This is a feature that we want our resolvent algebra for the cylinder $T^*\mathbb{T}^n$ to exhibit as well, in addition to being invariant under the time evolutions associated to a large class of Hamiltonians.

We will return to the matter of embedding algebras of subsystems in chapter 8, both in the classical and in the quantum sense, where we will see that in the quantum case, this is a rather delicate matter. Before that, in this chapter, we will define our classical resolvent algebra for the cylinder, of which we will determine the Gelfand spectrum in chapter 6, and which we will quantise in chapter 7 using Weyl quantisation.

5.2 Definition and motivation

To motivate the definition of the classical resolvent algebra for $T^*\mathbb{T}^n$, it is useful to first discuss the motivation for van Nuland's definition [112, Definition 2.1] of $C_{\mathcal{R}}(\mathbb{R}^n)$. Given $q \in \mathbb{R}^n$, the image of generator

$\mathcal{R}(\lambda,(q,0))$ of the resolvent algebra corresponding to $(q,0)\in\mathbb{R}^{2n}$ and $\lambda\in\mathbb{R}\backslash\{0\}$ under the canonical regular representation on $L^2(\mathbb{R}^n)$ is the operator

$$\psi\mapsto\left(x\mapsto\left(i\lambda-\sum_{j=1}^{n}x^jq^j\right)^{-1}\psi(x)\right),$$

which is the multiplication operator corresponding to the function

$$\mathbb{R}^n\to\mathbb{C},\quad x\mapsto(i\lambda-x\cdot q)^{-1}. \tag{5.1}$$

For this reason, van Nuland defines $C_{\mathcal{R}}(\mathbb{R}^n)$ as the C*-subalgebra of $C_b(\mathbb{R}^n)$ generated by functions of this form. Furthermore, van Nuland shows that the space of functions

$$\mathcal{S}_{\mathcal{R}}(\mathbb{R}^n):=\operatorname{span}_{\mathbb{C}}\{g\circ r_U\colon g\in\mathcal{S}(U),\ U\text{ is a subspace of }\mathbb{R}^n\}.$$

is a dense *-subalgebra of $C_{\mathcal{R}}(\mathbb{R}^n)$, where r_U denotes the orthogonal projection of $\mathbb{R}^n$ onto U, and $\mathcal{S}(U)$ denotes the space of Schwartz functions on U. This algebra is also a Poisson algebra, i.e., in addition to being an algebra, it is closed under the Poisson bracket. It is a natural choice for the domain of a quantisation map for $C_{\mathcal{R}}(\mathbb{R}^n)$ in view of the fact that the domain of the Weyl quantisation map in [65, II.2.6] is $\mathcal{S}(\mathbb{R}^n)$, which is the dense subalgebra of the quantised $C_0(\mathbb{R}^n)$.

In order to define a resolvent algebra for the cylinder $T^*\mathbb{T}^n$, we will start by identifying generators similar to those in equation (5.1). Note that since $\mathbb{T}^n$ is a Lie group and $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, we have canonical identifications

$$T^*\mathbb{T}^n\cong\mathbb{T}^n\times\mathbb{R}^n\cong\mathbb{R}^{2n}/\mathbb{Z}^n,$$

where the action of $\mathbb{Z}^n$ on $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ is given by translation in the first n components. Inspired by Rieffel's approach to deformation quantisation in [98], we consider the action of $\mathbb{R}^n$ on itself by translation, and note that the generator of $C_{\mathcal{R}}(\mathbb{R}^n)$ in equation (5.1) is continuous and bounded, is invariant under the restriction of this action to the subgroup $\{q\}^\perp$ of $\mathbb{R}^n$ (where $\perp$ denotes the orthogonal complement with respect to the standard inner product on $\mathbb{R}^n$), and has the property that the induced map

$$\mathbb{R}\cdot q\cong\mathbb{R}^n/\{q\}^\perp\to\mathbb{C},$$

vanishes at infinity. Since the action of $\mathbb{R}^{2n}$ on itself by translation induces an action of this group on $\mathbb{R}^{2n}/\mathbb{Z}^n$, we can similarly look for functions on $\mathbb{R}^{2n}/\mathbb{Z}^n$ that have these properties.

5.3 Definition. Let $n \in \mathbb{N}$. The *classical resolvent algebra of the cylinder* $T^*\mathbb{T}^n$ is the C*-subalgebra of $C_b(T^*\mathbb{T}^n)$ generated by functions f with the property that there exists a $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that f is constant on the orbits of the restriction to $\{(x, y)\}^\perp$ (where $\perp$ denotes the orthogonal complement with respect to the standard inner product on $\mathbb{R}^{2n}$) of the action of $\mathbb{R}^{2n}$ on $T^*\mathbb{T}^n$, and such that the induced map

$$(\mathbb{T}^n \times \mathbb{R}^n)/\{(x, y)\}^\perp \to \mathbb{C},$$

vanishes at infinity. Here, the domain of the induced map carries the quotient topology. The classical resolvent algebra will be denoted by $C_\mathcal{R}(T^*\mathbb{T}^n)$.

The definition of the classical resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$ is clearly motivated, but very unwieldy. Our first task is therefore to find an alternative, more elementary characterisation of $C_\mathcal{R}(T^*\mathbb{T}^n)$.

5.4 Lemma. *Let $n \geq 1$, let $x \in \mathbb{R}^n \backslash \{0\}$, and let*

$$H := \{y + \mathbb{Z}^n : y \in \mathbb{R}^n,\ x \cdot y = 0\} \subseteq \mathbb{T}^n,$$

be the image of $\{x\}^\perp$ under the canonical projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. Then either H is dense in $\mathbb{T}^n$, or H is a Lie subgroup of $\mathbb{T}^n$ of codimension 1. In the latter case, the set

$$\{t \in (0, \infty) : tx + \mathbb{Z}^n \in H\},$$

has a minimum T, the map

$$\varphi : \mathbb{T}^n/H \to \mathbb{T}, \quad tx + \mathbb{Z}^n + H \mapsto \frac{t}{T} + \mathbb{Z},$$

is a well-defined Lie group isomorphism, and $x \in T\|x\|^2 \cdot \mathbb{Z}^n$.

5.5 Remark. Note that for $n = 2$, the statement in the proposition essentially boils down to the well-known result that a line with irrational slope on the torus $\mathbb{T}^2$ is dense in the torus.

Proof. Throughout this proof, the canonical projection $\mathbb{R}^n \to \mathbb{T}^n$ will be denoted by π. Note that π is a Lie group homomorphism that is a local diffeomorphism, and that H is the image of a subgroup of $\mathbb{R}^n$ of codimension 1, hence it is a subgroup of $\mathbb{T}^n$, but not necessarily a Lie subgroup; the point is that H is an immersed but not necessarily embedded submanifold of $\mathbb{T}^n$ of codimension 1. The closure $\overline{H}$ of H however is a closed subgroup of $\mathbb{T}^n$, hence it is a Lie subgroup of $\mathbb{T}^n$ (cf. [35, Corollary 1.10.7]) of codimension ≤ 1.

Suppose $\overline{H}$ is a codimension 0 submanifold of $\mathbb{T}^n$: in this case, it follows that $\overline{H}$ is open in $\mathbb{T}^n$. Thus $\overline{H}$ is clopen, and it is evident that it is nonempty, so $\overline{H} = \mathbb{T}^n$ by connectedness of the latter space, i.e., H is dense in $\mathbb{T}^n$.

Now suppose $\overline{H}$ is a codimension 1 submanifold of $\mathbb{T}^n$. Using the inverse function theorem and the fact that H and $\overline{H}$ have the same codimension, one can show that the inclusion map of H into $\overline{H}$ is open. Since $\overline{H}$ is an embedded submanifold of $\mathbb{T}^n$, the manifold H is an embedded submanifold of $\mathbb{T}^n$ as well, which, together with the fact that it is also a subgroup of $\mathbb{T}^n$, implies that H is a Lie subgroup; in particular, H is closed. It follows that $\mathbb{T}^n/H$ has a natural Lie group structure of which the topology coincides with the quotient topology of $\mathbb{T}^n/H$. This Lie group is compact, connected, and one-dimensional, hence it is isomorphic to $\mathbb{T}$.

We claim that φ is an isomorphism that demonstrates this explicitly. We first show that the number T with the desired property exists, and that the set

$$X := \{t \in \mathbb{R} \colon tx + \mathbb{Z}^n \in H\},$$

is equal to $T\mathbb{Z}$. Recall that H is the image of $\{x\}^\perp$ under π. Since x and $\{x\}^\perp$ together span $\mathbb{R}^n$, for each $k \in \mathbb{Z}^n$, there is a $t \in \mathbb{R}$ and an $x' \in \{x\}^\perp$ such that $k = tx + x'$. Moreover, since $\mathbb{Z}^n$ contains a basis of $\mathbb{R}^n$ and $\{x\}^\perp$ is a proper subspace, k may be chosen so that $t \neq 0$; by replacing k by $-k$ if necessary, we may assume without loss of generality that $t > 0$. This shows that $X \cap (0, \infty)$ is nonempty.

Now note that π is a local diffeomorphism, and that for each $x' \in \{x\}^\perp$, we have $T_{x'}\pi(x) \notin T_{\pi(x')}H$, where x is viewed as an element of $T_{x'}\mathbb{R}^n$. This, together with the fact that H is an embedded submanifold of $\mathbb{T}^n$, implies that X is a discrete subset of $\mathbb{R}$. Moreover, it is easy to see that X is a subgroup of $\mathbb{R}$. Thus X is a nontrivial discrete subgroup. It follows that

$X \cap (0, \infty)$ has a minimum T, and that $X = T\mathbb{Z}$.

Next, consider the smooth map

$$\mathbb{R} \to \mathbb{T}^n/H, \quad s \mapsto sTx + \mathbb{Z}^n + H.$$

This is an immersive Lie group homomorphism, and since $X = T\mathbb{Z}$, this map factors through $\mathbb{T}$ to a smooth injection $\psi\colon \mathbb{T} \to \mathbb{T}^n/H$. Since both the domain and codomain of the displayed map are 1-dimensional, we may invoke the inverse function theorem to conclude that both the displayed map and the map ψ are local diffeomorphisms. In particular, their image is a clopen subgroup of $\mathbb{T}^n/H$. Since $\mathbb{T}^n/H$ is connected, it follows that ψ is surjective, hence ψ is a Lie group isomorphism. Its inverse is the Lie group isomorphism φ.

It remains to prove the final assertion. Note that

$$\varphi(q + \mathbb{Z}^n + H) = \frac{q \cdot x}{T\|x\|^2} + \mathbb{Z},$$

for each $q \in \mathbb{R}^n$. Taking q to be a standard basis vector e_j of $\mathbb{R}^n$, we obtain

$$\frac{e_j \cdot x}{T\|x\|^2} \in \mathbb{Z},$$

hence $x \in T\|x\|^2 \cdot \mathbb{Z}^n$, as desired. ■

5.6 Definition. Let $n \in \mathbb{N}$. We define the algebra $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as the C*-subalgebra of $C_b(\mathbb{R}^n)$ generated by the classical resolvent algebra $C_{\mathcal{R}}(\mathbb{R}^n)$ and the algebra of almost periodic functions $\mathcal{W}^0(\mathbb{R}^n)$ on $\mathbb{R}^n$.

Next up is the main result of this chapter. Recall that commutative C*-algebras are nuclear (cf. [87, Theorem 6.4.15]).

5.7 Theorem. *For each $n \in \mathbb{N}$, we have*

$$C_{\mathcal{R}}(T^*\mathbb{T}^n) = C(\mathbb{T}^n) \mathbin{\hat{\otimes}} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n).$$

Here, $\hat{\otimes}$ indicates that the right-hand side is the completion of the algebraic tensor product of the two factors with respect to the unique C-norm on that *-algebra. Furthermore, we identify both the left and the right-hand side of the above equation with their respective canonical embeddings in $C_b(\mathbb{T}^n \times \mathbb{R}^n)$.*

Proof. The statement is trivial for $n = 0$, so suppose $n \geq 1$. We first prove the inclusion $C_{\mathcal{R}}(T^*\mathbb{T}^n) \subseteq C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ by showing that the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ are contained in the right-hand side. Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and let f be one of the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that is constant on each of the orbits of the restriction to $\{(x, y)\}^\perp$ of the action of $\mathbb{R}^{2n}$ on $\mathbb{T}^n \times \mathbb{R}^n$. We define H to be the image of $\{x\}^\perp$ under the canonical projection map $\mathbb{R}^n \to \mathbb{T}^n$, and we define H' to be the image of $\{(x, y)\}^\perp$ under the canonical projection $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$. Note that both of these sets are subgroups of the respective groups in which they are contained. By Lemma 5.4, exactly one of the following three statements holds true:

(i) $x = 0$: in this case, we have $H' = \mathbb{T}^n \times \{y\}^\perp$; in particular, it is a Lie subgroup of $\mathbb{T}^n \times \mathbb{R}^n$, and the map

$$(\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{R} \cdot y, \quad (q + \mathbb{Z}^n, p) + H' \mapsto (p \cdot y)y,$$

is an isomorphism of Lie groups. It follows that f is the pullback of a function in $C_0(\mathbb{R} \cdot y)$ along the above map, from which it is readily seen that

$$f \in \mathbb{C} \cdot \mathrm{Id}_{\mathbb{T}^n} \hat{\otimes} C_{\mathcal{R}}(\mathbb{R}^n) \subseteq C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n);$$

In particular, note that f is constant iff $y = 0$.

To handle the remaining two cases in which $x \neq 0$, we first note that the map

$$\begin{aligned} \theta_0 \colon \mathbb{T}^n \times \mathbb{R}^n &\to \mathbb{T}^n/H, \\ (q + \mathbb{Z}^n, p) &\mapsto \frac{q \cdot x + p \cdot y}{\|x\|^2} x + \mathbb{Z}^n + H = q + \frac{p \cdot y}{\|x\|^2} x + \mathbb{Z}^n + H, \end{aligned}$$

is a well-defined continuous group homomorphism whose kernel contains the subgroup H'. Thus the above map factors through the quotient of the domain by H', yielding a continuous group homomorphism

$$\theta \colon (\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{T}^n/H.$$

In fact, θ is a group isomorphism and a homeomorphism, since the map

$$\mathbb{T}^n/H \to (\mathbb{T}^n \times \mathbb{R}^n)/H', \quad q + \mathbb{Z}^n + H \mapsto (q + \mathbb{Z}^n, 0) + H',$$

is a well-defined countinuous group homomorphism that can be checked to be the inverse of θ. As we will see below, θ need not be an isomorphism of topological groups if we require such groups to be T_1-spaces. We proceed with the remaining two cases:

(ii) $x \neq 0$ and H is dense in $\mathbb{T}^n$: in this case, the quotient topology on $\mathbb{T}^n/H$ is the indiscrete topology, hence $(\mathbb{T}^n \times \mathbb{R}^n)/H'$ is also indiscrete by our discussion above. It follows that the function f is constant, so $f \in C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

(iii) $x \neq 0$ and H is a Lie subgroup of $\mathbb{T}^n$ of codimension 1: then the map θ_0 defined above is a surjective Lie group homomorphism of which the codomain is one-dimensional, hence its kernel H' is a Lie subgroup of $\mathbb{T}^n \times \mathbb{R}^n$ of codimension 1, and the map θ is a Lie group isomorphism. Composing θ with the map φ from Lemma 5.4, we obtain the Lie group isomorphism

$$\varphi \circ \theta \colon (\mathbb{T}^n \times \mathbb{R}^n)/H' \to \mathbb{T}, \quad (q + \mathbb{Z}^n, p) + H' \mapsto \frac{q \cdot x + p \cdot y}{T\|x\|^2} + \mathbb{Z},$$

where T is defined as in the previous lemma. Now let

$$\pi' \colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n/H',$$

be the quotient map. Then $f = g \circ \varphi \circ \theta \circ \pi'$ for some $g \in C(\mathbb{T})$; let us first assume that g is of the form

$$e_k \colon x + \mathbb{Z} \mapsto \exp(2\pi i k x),$$

for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} f(q + \mathbb{Z}^n, p) &= \exp\left(2\pi i k \frac{q \cdot x + p \cdot y}{T\|x\|^2}\right) \\ &= \exp\left(2\pi i k \frac{q \cdot x}{T\|x\|^2}\right) \cdot \exp\left(2\pi i k \frac{p \cdot y}{T\|x\|^2}\right), \end{aligned}$$

which shows that $f \in C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Since the family of exponential functions $(e_k)_{k \in \mathbb{T}}$ generate $C(\mathbb{T})$, and since pullback along the map

$$\varphi \circ \theta \circ \pi' \colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T},$$

is a homomorphism of C*-algebras, it follows that

$$f = g \circ \varphi \circ \theta \circ \pi' \in C(\mathbb{T}^n) \mathbin{\hat{\otimes}} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n),$$

for arbitrary $g \in C(\mathbb{T})$.

This establishes the inclusion $C_{\mathcal{R}}(T^*\mathbb{T}^n) \subseteq C(\mathbb{T}^n) \mathbin{\hat{\otimes}} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. The reverse inclusion is a consequence of the following three observations:

- From case (i) in the previous part of this proof, we readily obtain $\mathbb{C} \cdot \mathbf{1}_{\mathbb{T}^n} \mathbin{\hat{\otimes}} C_{\mathcal{R}}(\mathbb{R}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$;
- From case (iii), setting $y = 0$ and taking x to be a standard basis vector of $\mathbb{R}^n$, we obtain $C(\mathbb{T}^n) \mathbin{\hat{\otimes}} \mathbb{C} \cdot \mathbf{1}_{\mathbb{R}^n} \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$.
- Finally, by considering case (iii) again, but now with x the first standard basis vector and $y \in \mathbb{R}^n$ arbitrary, we see that $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ contains functions of the form

  $$(q + \mathbb{Z}^n, p) \mapsto \exp(2\pi i k q^1) \exp(i\xi \cdot p),$$

  where $k \in \mathbb{Z}\backslash\{0\}$, and $\xi \in \mathbb{R}^n$ is arbitrary. The previous point now implies that functions of the form

  $$(q + \mathbb{Z}^n, p) \mapsto \exp(i\xi \cdot p),$$

  are elements of the resolvent algebra, so $\mathbb{C} \cdot \mathbf{1}_{\mathbb{T}^n} \mathbin{\hat{\otimes}} \mathcal{W}^0(\mathbb{R}^n) \subseteq C_{\mathcal{R}}(T^*\mathbb{T}^n)$. ■

We finish this section by defining the analogue of the space of Schwartz functions of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. This allows us to introduce the notation $h_{U,\xi,g}$ for the generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, which will be used extensively in the next chapter.

5.8 Definition. For each $k \in \mathbb{Z}^n$, let

$$e_k \colon \mathbb{T}^n \to \mathbb{C}, \quad q + \mathbb{Z}^n \mapsto e^{2\pi i k \cdot q}.$$

For each subspace $U \subseteq \mathbb{R}^n$, for each $\xi \in U^\perp$, and for each Schwartz function $g \in \mathcal{S}(U)$, let

$$h_{U,\xi,g} \colon \mathbb{R}^n \to \mathbb{C}, \quad p \mapsto e^{i\xi \cdot p} g \circ r_U(p),$$

where $r_U\colon \mathbb{R}^n \to U$ denotes the orthogonal projection onto U. We define the set $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ as the linear subspace of $C_b(T^*\mathbb{T}^n)$ spanned by the elements

$$e_k \otimes h_{U,\xi,g}\colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}, \quad (q+\mathbb{Z}^n, p) \mapsto e_k(q+\mathbb{Z}^n)h_{U,\xi,g}(p), \tag{5.2}$$

viewed as functions on $T^*\mathbb{T}^n$ using the canonical identification $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$.

5.9 Proposition.

(1) Let $\mathcal{B}$ be the linear span of the functions of the form $h_{U,\xi,g} \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Then $\mathcal{B}$ is closed under multiplication and partial differentiation. Moreover, $\mathcal{B}$ is a norm-dense *-subalgebra of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

(2) The vector space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a subspace of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that is closed under multiplication and partial differentiation, and is consequently a Poisson subalgebra of $C^\infty(T^*\mathbb{T}^n)$. Moreover, $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a norm-dense *-subalgebra of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$.

Proof.

(1) For any $h_{U,\xi,g}$ as in Definition 5.8,

$$h^*_{U,\xi,g} = \overline{h_{U,\xi,g}} = h_{U,-\xi,\overline{g}} \in \mathcal{B},$$

hence $\mathcal{B}$ is closed under the *-operation.

Assume for the moment that $\mathcal{B}$ is closed under multiplication. To see that $\mathcal{B}$ is invariant under partial differentiation, it suffices to show that partial derivatives of functions of the form $h_{U,\xi,g}$ are elements of $\mathcal{B}$. Any partial derivative can be written as a sum of two directional derivatives; one in a direction lying in U, and one in a direction lying in $U^\perp$. It is readily seen that both of these directional derivatives are elements of $\mathcal{B}$, hence so is their sum.

To show that $\mathcal{B}$ is closed under multiplication, it suffices to show that the product of two functions h_{U_1,ξ_1,g_1} and h_{U_2,ξ_2,g_2} as in Definition 5.8, is an element of $\mathcal{B}$. Let

$$U := U_1 + U_2,$$

$$\xi := \xi_1 + \xi_2 - r_U(\xi_1 + \xi_2) \in U^\perp$$
$$\tilde{g} := (g_1 \circ r_{U_1})(g_2 \circ r_{U_2}).$$

Note that the restrictions of $\tilde{g}$ to U and $U^\perp$ are Schwartz and constant, respectively. Setting

$$g\colon U \to \mathbb{C}, \quad p \mapsto e^{ir_U(\xi_1+\xi_2)\cdot p}\tilde{g}|_U \circ r_U(p) = e^{i(\xi_1+\xi_2)\cdot p}\tilde{g}|_U \circ r_U(p),$$

we see that $h_{U_1,\xi_1,g_1} \cdot h_{U_2,\xi_2,g_2} = h_{U,\xi,g}$, which establishes that $\mathcal{B}$ is closed under multiplication.

Thus $\mathcal{B}$ is a *-subalgebra of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. In addition to this fact, the elements of the form $h_{\{0\},\xi,1}$ generate $\mathcal{W}^0(\mathbb{R}^n)$, while the elements of the form $h_{U,0,g}$ generate $C_{\mathcal{R}}(\mathbb{R}^n)$, hence $\mathcal{B}$ generates $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as a C*-algebra. We infer that $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ is the closure of $\mathcal{B}$.

(2) For each $k \in \mathbb{Z}^n$, define e_k as in Definition 5.8. It is a trivial matter to check that the linear span of $\{e_k\colon k \in \mathbb{Z}^n\}$ is a *-subalgebra of $C(\mathbb{T}^n)$ that is closed with respect to partial differentiation, and it is a result from Fourier analysis that this linear subspace is dense in $C(\mathbb{T}^n)$. Using these facts in conjunction with part (1) of this proposition and Theorem 5.7, it is readily seen that all of the assertions are true. ■

5.3 Invariance under time evolution

We finish this chapter by showing that for $n = 1$, for a reasonably large class of Hamiltonians, the algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is closed under the corresponding classical time evolution. The result actually holds for any $n \in \mathbb{N}$ for a large class of potentials; the proof of this fact will appear in a forthcoming paper of van Nuland and the author [113]. The case $n = 1$ can be proved using a simpler and consequently shorter argument, and understanding why it does not extend to arbitrary n allows us to better appreciate the proof of the general case.

To simplify the exposition, we will set all physical constants such as the mass of the object moving on the cylinder equal to 1. The proof consists of two steps: first we discuss the free case $H_0 := \frac{1}{2}p^2$ for arbitrary n, then we extend the result to the interacting case $H = H_0 + V$ for $n = 1$, where $V\colon \mathbb{T} \to \mathbb{R}$, and we comment on the proof for $n > 1$. As before, we identify $T^*\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$.

5.10 Lemma. *The algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is closed under the classical time evolution corresponding to the free Hamiltonian $H_0(q+\mathbb{Z}^n,p)=\frac{1}{2}p^2$.*

Proof. The classical time evolution of the free Hamiltonian is given by the one-parameter group of automorphisms

$$\begin{aligned}\tau_t\colon C_b(\mathbb{T}^n\times\mathbb{R}^n)&\to C_b(\mathbb{T}^n\times\mathbb{R}^n),\\ f&\mapsto \left((q+\mathbb{Z}^n,p)\mapsto f\left(q+tp+\mathbb{Z}^n,p\right)\right),\quad t\in\mathbb{R}.\end{aligned}$$

In the notation of Definition 5.8, we have

$$\tau_t(e_k\otimes h_{U,\xi,g})(q+\mathbb{Z}^n,p)=e^{2\pi ik\cdot(q+tp)}e^{i\xi\cdot p}g\circ r_U(p)=e_k\otimes h_{U,\xi_1,g_1}(q+\mathbb{Z}^n,p),$$

where

$$\xi_1:=\xi+2\pi t(\mathrm{Id}_{\mathbb{R}^n}-r_U)(k),\quad g_1(p):=e^{2\pi itr_U(k)}g(p).$$

Thus τ_t maps generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ to scalar multiples of generators, hence $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is an invariant subspace. ∎

5.11 Proposition. *Let $V\in C^1(\mathbb{T})$, and suppose that*

$$V'\colon\mathbb{T}\to\mathbb{R},\quad q+\mathbb{Z}\mapsto\frac{d}{dt}V(t+q+\mathbb{Z})|_{t=0},$$

is Lipschitz continuous. Then the algebra $C_{\mathcal{R}}(T^*\mathbb{T})$ is closed under the classical time evolution corresponding to the interacting Hamiltonian $H(q+\mathbb{Z},p)=\frac{1}{2}p^2+V(q+\mathbb{Z})$.

5.12 Remark. Note that the condition of Lipschitz continuity requires a notion of a metric in the analytical sense, i.e., a function $d\colon\mathbb{T}\times\mathbb{T}\to[0,\infty)$ that satisfies various conditions including the triangle inequality. We will refer to such a function as a distance function to avoid confusion with the notion of a Riemannian metric. Every smooth Riemannian manifold has an associated distance function, and thereby a notion of Lipschitz continuity. If the manifold is compact, which is the case for $\mathbb{T}$, then this notion is independent of the specific Riemannian metric. However, in what follows, as the Riemannian metric on $\mathbb{T}$, we will take the one such that the canonical projection $\mathbb{R}\to\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is a local isometry, where the Riemannian metric on $\mathbb{R}$ is the standard one.

Given any smooth Riemannian manifold Q, there is a canonical Riemannian metric on T^*Q. Indeed, the Riemannian metric induces a smooth pointwise inner product on each of the fibres of T^*Q, and since each fibre is a vector space, there is a canonical pointwise inner product on the distribution of vertical subspaces of T^*Q. Moreover, the Levi-Civita connection on TQ has a dual connection on T^*Q, of which the corresponding Ehresmann connection is a distribution of horizontal subspaces, which inherits a pointwise inner product from the Riemannian metric Q. The Riemannian metric now arises as the pointwise inner product associated to the direct sum of the horizontal and vertical distributions with their respective pointwise inner products.

In the case $Q = \mathbb{T}$, the associated metric on $T^*\mathbb{T}$ corresponds to the metric on $\mathbb{T}\times\mathbb{R}$ inherited from the canonical projection $\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}$. The notion of Lipschitz continuity on $T^*\mathbb{T}$ is taken to be the one associated to this metric.

Proof. First, note that the Hamiltonian vector field

$$X_H\colon \mathbb{T}\times\mathbb{R} \mapsto T(\mathbb{T}\times\mathbb{R}), \quad (q+\mathbb{Z}, p) \mapsto p\, \frac{\partial}{\partial q}\bigg|_{(q+\mathbb{Z},p)} - V'(q+\mathbb{Z})\, \frac{\partial}{\partial p}\bigg|_{(q+\mathbb{Z},p)},$$

of H is Lipschitz continuous, so we may use the Picard–Lindelöf theorem to establish the existence of the flow $\Phi\colon \mathcal{D} \to \mathbb{T}\times\mathbb{R}$ of X_H, where $\mathcal{D} \subseteq \mathbb{R}\times(\mathbb{T}\times\mathbb{R})$ is an open neighbourhood of $\{0\}\times(\mathbb{T}\times\mathbb{R})$.

Now let $(q_0+\mathbb{Z}, p_0) \in \mathbb{T}\times\mathbb{R}$, and let

$$(q(t)+\mathbb{Z}, p(t)) := \Phi_t(q_0+\mathbb{Z}, p_0),$$

for each $t \in S$, where $S := \{t \in \mathbb{R}\colon (t, q_0+\mathbb{Z}, p_0) \in \mathcal{D}\}$; we will see that $S = \mathbb{R}$ shortly. We regard q as a function $S \to \mathbb{R}$ in what follows, thereby implicitly fixing a representative q_0. Because $\{H, H\} = 0$, the composition of H with the map $t \mapsto (q(t)+\mathbb{Z}, p(t))$ is a constant map, hence

$$\begin{aligned} p_0^2 + 2V(q_0+\mathbb{Z}) &= H(q_0+\mathbb{Z}, p_0) = H(q(t)+\mathbb{Z}, p(t)) \\ &= p(t)^2 + 2V(q(t)+\mathbb{Z}). \end{aligned}$$

In particular, we see that

$$H(q_0+\mathbb{Z}, p_0) - 2V_{\max} \le p(t)^2 \le H(q_0+\mathbb{Z}, p_0) - 2V_{\min}\,,$$

where

$$V_{\max} := \max_{q_1+\mathbb{Z}\in\mathbb{T}} V(q_1+\mathbb{Z}), \quad V_{\min} := \min_{q_1+\mathbb{Z}\in\mathbb{T}} V(q_1+\mathbb{Z}).$$

Using the upper bound for $p(t)^2$ and a standard compactness argument, one can now show that $\mathcal{D} = \mathbb{R}\times(\mathbb{T}\times\mathbb{R})$ (and consequently $S = \mathbb{R}$). Thus we have a well-defined one-parameter group τ_t of automorphisms of $C_b(\mathbb{T}\times\mathbb{R})$, given by $\tau_t(f) = f\circ\Phi(t,\cdot)$.

To show that τ_t preserves $C_{\mathcal{R}}(T^*\mathbb{T})$, we first note that $\Phi(t,\cdot)$ is a homeomorphism for each $t\in\mathbb{R}$ with inverse $\Phi(-t,\cdot)$; in particular, $\Phi(t,\cdot)$ is continuous and proper. It follows that τ_t preserves $C_0(\mathbb{T}\times\mathbb{R})$.

To prepare for the next part of the proof, we need some estimates. Using the lower bound for $p(t)^2$, we see that if

$$p_0^2 > 2(V_{\max} - V(q_0+\mathbb{Z})),$$

then $p(t) > 0$ for each $t\in\mathbb{R}$ or $p(t) < 0$ for each $t\in\mathbb{R}$. Suppose now that the above inequality for p_0^2 holds. Then, we have

$$\begin{aligned}|p(t)+p_0|\cdot|p(t)-p_0| = |p(t)^2-p_0^2| &= 2|V(q_0+\mathbb{Z}) - V(q(t)+\mathbb{Z})|\\ &\leq 2(V_{\max}-V_{\min}),\end{aligned}$$

which yields

$$|p(t)-p_0| \leq \frac{2(V_{\max}-V_{\min})}{|p(t)+p_0|} < \frac{2(V_{\max}-V_{\min})}{|p_0|},$$

for each $t\in\mathbb{R}$, hence

$$\begin{aligned}|q(t)-(q_0+p_0t)| = \left|\int_0^t p(s)-p_0\,ds\right| &\leq \int_0^t |p(s)-p_0|\,ds\\ &\leq \frac{2(V_{\max}-V_{\min})t}{|p_0|},\end{aligned}$$

for each $t\in[0,\infty)$. This argument can easily be modified to show that in fact

$$|q(t)-(q_0+tp_0)| \leq \frac{2(V_{\max}-V_{\min})|t|}{|p_0|},$$

for each $t\in\mathbb{R}$.

It is readily seen from the generators of $C_{\mathcal{R}}(T^*\mathbb{T})$ that it is generated by $C_0(\mathbb{T}\times\mathbb{R})$ and $C(\mathbb{T})\,\hat{\otimes}\,\mathcal{W}^0(\mathbb{R})$ as a C*-algebra (even as a vector space). It is also easy to check that the family of functions of the form

$$\mathbb{T}\times\mathbb{R}\to\mathbb{C},\quad (q_1+\mathbb{Z},p_1)\mapsto g(q_1+cp_1+\mathbb{Z}),$$

where $g\in C(\mathbb{T})$ and $c\in\mathbb{R}$, is dense in $C(\mathbb{T})\,\hat{\otimes}\,\mathcal{W}^0(\mathbb{R}^n)$. Thus it remains to show that $\tau_t(g)\in C_{\mathcal{R}}(T^*\mathbb{T})$ for each $t\in\mathbb{R}$.

Now fix $g\in C(\mathbb{T})$, fix $c\in\mathbb{R}$, let $f\in C_b(\mathbb{T}\times\mathbb{R})$ be the function given by $(q+\mathbb{Z},p)\mapsto g(q+cp+\mathbb{Z})$, and let $(\tau_t^0)_{t\in\mathbb{R}}$ be the one-parameter group corresponding to the free Hamiltonian. In order to show that $\tau_t(f)\in C_{\mathcal{R}}(T^*\mathbb{T})$, we show that $\tau_t(f)-\tau_t^0(f)\in C_0(\mathbb{T}\times\mathbb{R})\subset C_{\mathcal{R}}(T^*\mathbb{T})$. Since it was already shown in the previous lemma that $\tau_t^0(f)\in C_{\mathcal{R}}(T^*\mathbb{T})$, this will imply that $\tau_t(f)\in C_{\mathcal{R}}(T^*\mathbb{T})$, which is the desired result.

Note that f can be written as a composition of two maps: the map

$$\mathbb{T}\times\mathbb{R}\to\mathbb{T},\quad (q_1+\mathbb{Z},p_1)\mapsto q_1+cp_1+\mathbb{Z},$$

and the map g. Let d be the distance function associated to the Riemannian metric on $T^*\mathbb{T}$. The first map is Lipschitz continuous with Lipschitz constant $1+|c|$, which implies that it is uniformly continuous. The map g is by assumption a continuous map on a compact metric space, so it is uniformly continuous as well. It follows that g is uniformly continuous.

Now fix $t\in\mathbb{R}$, and fix $\varepsilon>0$. By uniform continuity of f, there exists a $\delta>0$ such that for each $(q_1+\mathbb{Z},p_1)$ and $(q_2+\mathbb{Z},p_2)\in\mathbb{T}\times\mathbb{R}$ with the property that

$$d((q_1+\mathbb{Z},p_1),(q_2+\mathbb{Z},p_2))<\delta,$$

we have

$$|f(q_1+\mathbb{Z},p_1)-f(q_2+\mathbb{Z},p_2)|<\varepsilon.$$

Let

$$M:=\max\left(\sqrt{2(V_{\max}-V_{\min})},\frac{4(|t|+1)(V_{\max}-V_{\min})}{\delta}\right).$$

Then for each $q_0+\mathbb{Z}\in\mathbb{T}$ and each $p_0\in\mathbb{R}$ such that $|p_0|>M$, we have $p(s)>0$ for each $s\in\mathbb{R}$ or $p(s)<0$ for each $s\in\mathbb{R}$, and

$$d\left((q(t)+\mathbb{Z},p(t)),(q_0+tp_0+\mathbb{Z},p_0)\right)\leq|q(t)-(q_0+tp_0)|+|p(t)-p_0|<\delta,$$

so that

$$\begin{aligned}
&|\tau_t(f)(q_0 + \mathbb{Z}, p_0) - \tau_t^0(f)(q_0 + \mathbb{Z}, p_0)| \\
&\quad = |f(q(t) + \mathbb{Z}, p(t)) - f\,(q_0 + tp_0 + \mathbb{Z}, p_0)| < \varepsilon,
\end{aligned}$$

which proves that $\tau_t(f) - \tau_t^0(f) \in C_0(\mathbb{T} \times \mathbb{R})$, as desired. ∎

5.13 Remark.

(1) The proof of the proposition is not readily generalisable to cotangent bundles of tori of dimension greater than 1. There are two reasons for this:

- The estimate for $|p(t) - p_0|$ can be generalised to higher dimensions to an estimate of $|\|p(t)\| - \|p_0\||$, where $p(t), p_0 \in \mathbb{R}^n$, but in order for the proof to work, one requires an estimate of the form $\|p(t) - p_0\| = \mathcal{O}(\|p_0\|^{-1})$. Such an estimate does not exist, however. The physical intuition behind this is the following: if a moving object with nonzero mass accelerates in its direction of motion, then, given a fixed amount of added kinetic energy, the resulting difference in momentum is much smaller when the object already has a relatively high momentum, than if the object has a relatively low momentum. This is a consequence of the fact that the kinetic energy of the object is quadratic in its momentum.

  If, on the other hand, the moving object accelerates in a direction perpendicular to its direction of motion, then the difference in momentum is independent of the momentum of the object prior to acceleration (or, more accurately, it only depends on it through the composition of the potential with the particular path in configuration space that the object travels along by virtue of its initial momentum).
- Let D_n be the closed n-dimensional unit ball. The above proof uses the fact that for $n = 1$, the set $\mathbb{R}^n \backslash \lambda D_n$, with $\lambda = \sqrt{2(V_{\max} - V(q_0 + \mathbb{Z}))}$, is not path-connected. This is clearly no longer true when $n > 1$.

As already mentioned at the beginning of this section, the general case requires a different approach, which will appear in a forthcoming paper. The proof of the general case bears an interesting similarity to the proof of invariance of the quantisation of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ under the quantum time evolution, which is the content of Proposition 7.13 (for $n = 1$). Namely, to prove the general case, one first proves invariance of Hamiltonians $H = H_0 + V$ for V a finite linear combination of sines and cosines, and subsequently extends this result by means of a density argument.

(2) Suppose A is a C*-subalgebra of $C_b(\mathbb{T}^n \times \mathbb{R}^n)$ that contains $C(\mathbb{T}^n) \hat{\otimes} \mathbb{C} \cdot \mathbf{1}_{\mathbb{R}^n}$. Then closure of A under the time evolution generated by the free Hamiltonian H_0 already yields $C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0(\mathbb{R}^n) \subseteq A$. This shows that $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is the smallest C*-subalgebra A of $C_b(\mathbb{T}^n \times \mathbb{R}^n)$ that contains both $C(\mathbb{T}^n) \hat{\otimes} \mathbb{C} \cdot \mathbf{1}_{\mathbb{R}^n}$ and $\mathbb{C} \cdot \mathbf{1}_{\mathbb{T}^n} \hat{\otimes} C_{\mathcal{R}}(\mathbb{R}^n)$, and is closed under the time evolution generated by H_0.

Chapter 6

The Gelfand spectrum of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$

6.1 Introduction

The classical resolvent algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a commutative, unital C*-algebra, so it is natural to ask what its Gelfand spectrum $\Omega(C_{\mathcal{R}}(T^*\mathbb{T}^n))$ is. Also, an alternative description of the Gelfand spectrum might yield a way to generalise the concept of the (classical) resolvent algebra, namely by first generalising this alternative description to obtain a class of compact Hausdorff spaces with some additional structure, of which the space of continuous functions corresponds to the algebra of observables of some classical system.

According to Theorem 5.7, we have $C_{\mathcal{R}}(T^*\mathbb{T}^n) = C(\mathbb{T}^n) \hat{\otimes} \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ when both algebras are identified with their embedding in $C_b(\mathbb{T}^n \times \mathbb{R}^n)$. Gelfand duality now implies

$$\Omega(C_{\mathcal{R}}(T^*\mathbb{T}^n)) \cong \mathbb{T}^n \times \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)),$$

where $\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ denotes the Gelfand spectrum of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and the product on the right-hand side is endowed with the product topology. Thus the problem of determining the spectrum of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ reduces to the problem of determining that of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, which is the subject of this chapter.

As we will see, this is a rather nontrivial matter. However, some results in this direction already exist; van Nuland [112, Theorem 5.6] has shown that, as a set, the Gelfand spectrum of $C_{\mathcal{R}}(\mathbb{R}^n)$ is equal to

$$\{w + V : V \text{ is a linear subspace of } \mathbb{R}^n,\ w \in V^\perp\},$$

and carries a certain topology. Fleischhack [40] has already described the Gelfand spectrum of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ for $n = 1$. In this chapter, we extend Fleischhack's results to arbitrary $n \in \mathbb{N}$ in such a way that each of the arguments can also be used to determine the Gelfand spectrum of $C_{\mathcal{R}}(\mathbb{R}^n)$, in so far as it is relevant to that problem. Our treatment differs notably from van Nuland's approach, allowing us to describe the Gelfand spectrum as a set before studying its topology, which is treated in significantly more detail. This however comes at the expense of brevity.

For the reader's convenience, we list the three main results of this chapter, and use them to outline its structure:

(1) The first result is Theorem 6.20, which states (among other things) that

$$\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) = \mathcal{W}^0(\mathbb{R}^n) \oplus I,$$

where $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and $\mathcal{W}^0(\mathbb{R}^n)$ were already defined in the previous chapter, and I is the closed *-ideal of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ generated by all of the generators $h_{U,\xi,g}$ from Definition 5.8 of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ with $U \neq \{0\}$, i.e., the ones that are not elements of $\mathcal{W}^0(\mathbb{R}^n)$.

In order to obtain this decomposition, we study the Bohr topology on $\mathbb{R}^n$ in 6.2, establishing a useful property of nonempty Bohr open subsets in part (2) of Lemma 6.10. We subsequently recall the relation between the Bohr topology on $\mathbb{R}^n$ and almost periodic functions on this space, i.e., the elements of $\mathcal{W}^0(\mathbb{R}^n)$, in section 6.3. We combine the results from these two sections in section 6.4, where the aforementioned theorem can be found. As an immediate corollary, we obtain a canonical surjective *-homomorphism $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0(\mathbb{R}^n)$.

(2) The second result is Theorem 6.31, which requires Definition 6.24 to be understood, and shows that there is a canonical bijection from the set

$$\Omega^0_{\mathcal{R},n} := \{(V, w, \zeta) : V \subseteq \mathcal{P}_0,\ w \in V^\perp,\ \zeta \in bV\},$$

to the Gelfand spectrum of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, where $\mathcal{P}_0$ denotes the set of linear subspaces of $\mathbb{R}^n$, and bV denotes the Bohr compactification of V. An element (V, w, ζ) can be used to define a character on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ by composing the four maps below:

$$\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(w + V) \to \mathcal{W}^0(w + V) \to \mathcal{W}^0(V) \to \mathbb{C},$$

where $\mathcal{W}^0_{\mathcal{R}}(w + V)$ and $\mathcal{W}^0(w + V)$ are the analogues of the algebras $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and $\mathcal{W}^0(\mathbb{R}^n)$ for functions on $w + V$, and the maps from left to right are: restriction of functions on $\mathbb{R}^n$ to $w+V$, the analogue of the map mentioned above, the pullback of translation by w to function spaces, and the element ζ viewed as a character on $\mathcal{W}^0(V)$ (which will be dicussed in section 6.3).

Apart from defining the characters, we show in section 6.5 that the first of the four maps is well-defined. In section 6.6, we introduce the notion of the support of a character, which is similar to the support of a distribution, and prove its most important properties, which will allow us to subsequently prove Theorem 6.31 with ease.

(3) In the final section of this chapter, we endow the set $\Omega^0_{\mathcal{R},n}$ with a topology and prove Theorem 6.41, which states that with respect to this topology (and the weak*-topology on the Gelfand spectrum), the bijection mentioned in the previous point is a homeomorphism. This is the third and final main result of this chapter.

To define the topology on $\Omega^0_{\mathcal{R},n}$, we note that there is a natural partial order $\leq$ on $\Omega^0_{\mathcal{R},n}$, given by

$$\begin{aligned}&(V_1, w_1, \zeta_1) \leq (V_2, w_2, \zeta_2) \Leftrightarrow \\ &V_1 \subseteq V_2 \text{ and } (w_2, \zeta_2) = (r_{V_2^\perp}(w_1), \iota_{V_1,V_2}(\zeta_1) + \iota_{V_2,bV_2} \circ r_{V_2}(w_2)),\end{aligned}$$

where for any subspace $V \subseteq \mathbb{R}^n$, the map r_V denotes the orthogonal projection onto V, the map $\iota_{V,bV}$ denotes the map from V into its Bohr compactification, and for any subspace V' such that $V \subseteq V' \subseteq \mathbb{R}^n$, the map $\iota_{V,V'}$ denotes the map $bV \hookrightarrow bV'$ induced by the inclusion of V into V'. A base of the topology on $\Omega^0_{\mathcal{R},n}$ is then given by sets of the form

$$\downarrow(\{V\} \times U) \backslash \bigcup_{V' \in F} \downarrow(\{V'\} \times K_{V'}),$$

where $V \in \mathcal{P}_0$ is as above, F is a finite subset of $\mathcal{P}_0$ that contains proper subspaces of V, the set $U \subseteq V^\perp \times bV$ is open, for each $V' \in F$, the set $K_{V'} \subseteq (V')^\perp \times b(V')$ is compact, and $\downarrow X$ denotes the lower set generated by a subset $X \subseteq \Omega^0_{\mathcal{R},n}$ (cf. part (7) of Proposition 6.40). We view this as a generalisation of the Lawson topology from order theory (after reversing the partial order).

6.2 The Bohr compactification of $\mathbb{R}^n$

We recall that a topological group is a group carrying a topology with respect to which the inversion is continuous, and the multiplication is jointly continuous. We assume our topological groups to be Hausdorff.

6.1 Definition. Let G be a topological group. The *Bohr compactification* of G is a pair (bG, ι) consisting of a compact topological group bG and a continuous group homomorphism $\iota \colon G \to bG$ satisfying the following universal property:
For each compact topological group H and each continuous group homomorphism $\phi \colon G \to H$, there exists a unique continuous group homomorphism $\psi \colon bG \to H$ such that the following diagram

$$\begin{tikzcd} bG \arrow[dr, dashed, "\psi"] & \\ G \arrow[u, "\iota"] \arrow[r, "\phi"'] & H \end{tikzcd}$$

is commutative.

6.2 Remark. The Bohr compactification, as well as the Bohr topology which will be defined below, is named after the Danish mathematician Harald Bohr (1887–1951), not to be confused with his brother Niels Bohr, who we already mentioned in chapter 1. H. Bohr defined and studied the class of almost periodic functions $\mathcal{W}^0(\mathbb{R})$ on $\mathbb{R}$ in [20], but did not define the notions named after him. The characterisation of the Bohr compactification in terms of its defining universal property appears in the work of Weil [116, chapter VII].

The following result can be found in [4].

6.3 Theorem. *The Bohr compactification of a topological group exists and is unique up to unique isomorphism. Moreover, $\iota(G)$ is dense in bG.*

6.4 Definition. Given a topological group G with Bohr compactification (bG, ι), let τ_b be the initial topology on G with respect to the map $\iota\colon G \to bG$. Then τ_b is called the *Bohr topology on* G. Accordingly, an element of τ_b will be referred to as a Bohr open subset of G.

6.5 Proposition. *Let G be a topological group.*

(1) Each continuous group endomorphism Φ of G is continuous with respect to τ_b;

(2) Translations by fixed elements of G are continuous with respect to τ_b.

Proof.

(1) Since $\iota \circ \Phi\colon G \to bG$ is a continuous group homomorphism from G to a compact topological group, by the universal property of the Bohr compactification there exists a unique continuous group endomorphism $\tilde{\Phi}$ of bG such that $\iota \circ \Phi = \tilde{\Phi} \circ \iota$. From this identity, it is readily seen that $\tilde{\Phi}$ restricts to a continuous endomorphism of $\iota(G)$ endowed with its subspace topology, and consequently, that Φ is continuous with respect to the Bohr topology on G.

(2) The proof of this statement is similar to the proof in part (1) with the difference that, rather than invoking the universal property, one uses the fact that translations by a group element in bG are continuous since bG is a topological group. ∎

6.6 Proposition. *Let G be a topological group with topology τ. Suppose G satisfies one of the following conditions:*

- *G is locally compact, but not compact;*
- *G contains a complete, noncompact subgroup;*

Then $\tau_b \subset \tau$ (the symbol $\subset$ denotes proper inclusion).

Proof. Since ι is continuous by definition of the Bohr compactification, we have $\tau_b \subseteq \tau$. Suppose for the sake of contradiction that equality holds. Then $\tau = \{\iota^{-1}(U) \colon U \subseteq bG \text{ is open}\}$, and since (G, τ) is Hausdorff, we infer that ι is injective, hence it must be a homeomorphism onto its image relative to the subspace topology on $\iota(G)$.

- Suppose G is locally compact, but not compact. Since ι is a homeomorphism from a locally compact Hausdorff space onto a dense subspace of a Hausdorff space, its image $\iota(G)$ is open in bG. Since $\iota(G)$ is a subgroup of the topological group bG, it follows that $\iota(G)$ is also closed in bG. Since $\iota(G)$ is dense in bG, we obtain $\iota(G) = bG$, hence G must be compact, which contradicts our original assumption.

- Suppose G contains a complete, noncompact subgroup H. Then, like ι, the restriction $\iota|_H$ must be a homeomorphism onto its image. Since ι is also a group homomorphism, we infer that the map $\iota|_H$ is an isomorphism of topological groups. The uniform structure of a topological group is completely determined by its group structure and topology, so $\iota|_H$ also induces an isomorphism of uniform structures. Thus $\iota(H)$ is both complete and noncompact, since H is. However, completeness implies that $\iota(H)$ is closed in bG, which in turn implies that $\iota(H)$ is compact, yielding a contradiction.

■

6.7 Corollary. *Let $n \in \mathbb{N}\backslash\{0\}$. The Bohr topologies on $\mathbb{R}^n$, $\mathbb{Z}^n$ and $\mathbb{Q}^n$ are strictly coarser than their usual metric topologies.*

Proof. The assertions for $\mathbb{R}^n$ and $\mathbb{Z}^n$ can be proved using either of the two criteria given in the previous proposition. The second criterion can be used to prove the assertion for $\mathbb{Q}^n$ by noting that $\mathbb{Z}^n \subset \mathbb{Q}^n$. ■

6.8 Remark.

(1) If G is compact, then G is (isomorphic to) its own Bohr compactification.

(2) It follows from the previous proposition that, even when the map ι is injective - which it need not be - it is not necessarily a homeomorphism onto its image, thus showing that the Bohr compactification is not a compactification in the purely topological sense.

Recall that a character on a group G is a continuous group homomorphism $G \to \mathbb{T}$. We shall refer to such characters as *group characters* to distinguish them from characters on C*-algebras. If G is locally compact abelian, then the space of group characters endowed with the compact-open topology, and pointwise multiplication, is again a locally compact abelian group, known as the *Pontryagin dual of* G, and is denoted by $\hat{G}$.

The following result is due to Anzai and Kakutani [5, Theorem 4]:

6.9 Theorem. *Let G be a locally compact abelian group. Then its Bohr compactification bG is isomorphic to the Pontryagin dual of the topological group that as a group is the Pontryagin dual $\hat{G}$, but whose topology is the discrete topology. The map $\iota\colon G \to bG$ associated to the Bohr compactification is injective (hence a continuous group isomorphism onto its image), and is given by $g \mapsto (\phi \mapsto \phi(g))$.*

We finish this section by applying Corollary 6.7 to prove a technical lemma that will be instrumental in unraveling some of the structure of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. For each $r > 0$ and each $x \in \mathbb{R}^n$, let $B_r(x)$ be the open ball with radius r and center x. The metric topology on $\mathbb{R}^n$ will be denoted by τ_d.

6.10 Lemma. *Let $U \in \tau_b$ be nonempty.*

(1) For each $r > 0$, the set $V_{U,r} := \{x \in S^{n-1} \mid \exists s > r \colon sx \in U\}$ is open and dense in S^{n-1};

(2) The set $V_U := \{x \in S^{n-1} \mid \forall r > 0\, \exists s > r \colon sx \in U\}$ is dense in S^{n-1}.

Proof.

(1) We show that the assertion is true in three steps, proving consecutively stronger statements:

(a) If U is a Bohr open neighbourhood of 0, then it is an unbounded subset of $\mathbb{R}^n$.

Indeed, suppose this is not the case, i.e., there exists a Bohr open neighbourhood U of 0 and an $R > 0$ such that $U \subseteq B_R(0)$. Now fix $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Applying part (1) of Proposition 6.5 to the automorphism $y \mapsto \varepsilon^{-1} R y$ on $\mathbb{R}^n$, we see that it is continuous with respect to the Bohr topology on (both the domain and the codomain) $\mathbb{R}^n$, from which it follows that $R^{-1}\varepsilon U$ is a Bohr open neighbourhood of 0 that is a subset of

$B_\varepsilon(0)$. Next, applying part (2) of the same proposition to the translation $y \mapsto y - x$, we see that it is continuous with respect to the Bohr topology on $\mathbb{R}^n$, which implies that $x + R^{-1}\varepsilon U$ is a Bohr open neighbourhood of x that is a subset of $B_\varepsilon(x)$. Thus τ_b contains a base of τ_d on $\mathbb{R}^n$, implying that $\tau_d \subseteq \tau_b$. This contradicts Corollary 6.7, hence each Bohr open neighbourhood U of 0 is unbounded.

(b) If U is a Bohr open neighbourhood of 0, then for each $r > 0$, the set $V_{U,r}$ is dense in S^{n-1}.

Fix such a U and r, and suppose for the sake of contradiction that there exists a nonempty open subset $W \subseteq S^{n-1}$ disjoint from $V_{U,r}$. Since $\mathrm{O}(n)$ acts transitively on S^{n-1}, the family of sets $(R(W))_{R\in\mathrm{O}(n)}$ is an open cover of S^{n-1}, and since S^{n-1} is compact, it has a finite subcover indexed by elements $R_1, \ldots, R_m \in \mathrm{O}(n)$. It follows from this and part (1) of Proposition 6.5 that the set $\bigcap_{j=1}^m R_j(U)$ is a Bohr open neighbourhood of 0 that is a subset of $\overline{B_r(0)}$ (and hence of $B_r(0)$). This contradicts the claim proved in (a), hence $V_{U,r}$ is dense in S^{n-1}.

(c) (The original assertion.)

First, we observe that the map

$$f\colon \mathbb{R}^n \backslash \overline{B_r(0)} \to S^{n-1}, \quad x \mapsto \|x\|^{-1}x,$$

can be written as a composition of two open (and continuous) maps, namely the homeomorphism

$$\mathbb{R}^n \backslash \overline{B_r(0)} \to S^{n-1} \times (r, \infty), \quad x \mapsto (\|x\|^{-1}x, \|x\|),$$

and the canonical projection $S^{n-1} \times (r,\infty) \to S^{n-1}$, so f is open. Since $U \in \tau_b \subset \tau_d$, we have $V_{U,r} = f(U)$, hence $V_{U,r}$ is open.

It remains to be shown that $V_{U,r}$ is dense in S^{n-1}. Fix $x_0 \in U\backslash\{0\}$ (note that U is an infinite set, so this is possible), let $\varepsilon > 0$, and let $x_1 \in S^{n-1}$. We claim that $B_\varepsilon(x_1) \cap V_{U,r}$ is nonempty. To see this, we define

$$R := \max\left(r, \frac{3}{\varepsilon}\|x_0\|\right) + \|x_0\|,$$

and note that $-x_0 + U$ is a Bohr open neighbourhood of 0 (by part (2) of Proposition 6.5), so that by the claim proved in (b), the set $B_{\varepsilon/3}(x_1) \cap$

$V_{-x_0+U,R}$ is nonempty. This means that there exists an $x \in \mathbb{R}^n$ such that $x + x_0 \in U$, $\|x\| > R$, and $\|\|x\|^{-1}x - x_1\| < \varepsilon/3$. It follows that

$$\|x + x_0\| \geq \|x\| - \|x_0\| > R - \|x_0\| \geq \max\left(r, \frac{3}{\varepsilon}\|x_0\|\right),$$

so $\|x + x_0\| > r$, and

$$(\|x + x_0\|)^{-1}\|x_0\| < \frac{\varepsilon}{3}.$$

The latter inequality can be used to show that

$$\begin{aligned} &|\|x + x_0\|^{-1} - \|x\|^{-1}| \\ &\quad \leq \|x + x_0\|^{-1}\|x\|^{-1}|\|x + x_0\| - \|x\|| \\ &\quad \leq \|x + x_0\|^{-1}\|x\|^{-1}\|x_0\| < \frac{\varepsilon}{3}\|x\|^{-1}, \end{aligned}$$

so that

$$\begin{aligned} &\|\|x + x_0\|^{-1}(x + x_0) - \|x\|^{-1}x\| \\ &\quad = \|(\|x + x_0\|^{-1} - \|x\|^{-1})x + \|x + x_0\|^{-1}x_0\| \\ &\quad \leq |\|x + x_0\|^{-1} - \|x\|^{-1}| \cdot \|x\| + \|x + x_0\|^{-1}\|x_0\| < \frac{2\varepsilon}{3}, \end{aligned}$$

which yields

$$\begin{aligned} &\|\|x + x_0\|^{-1}(x + x_0) - x_1\| \\ &\quad \leq \|\|x\|^{-1}x - x_1\| + \|\|x + x_0\|^{-1}(x + x_0) - \|x\|^{-1}x\| \\ &\quad < \varepsilon. \end{aligned}$$

We conclude that $\|x + x_0\|^{-1}(x + x_0) \in B_\varepsilon(x_1) \cap V_{U,r}$, proving the claim.

(2) Note that $V_U = \bigcap_{m=1}^{\infty} V_{U,m}$, and apply part (1) of this lemma and the Baire category theorem. ■

6.3 Almost periodic functions

We begin by recalling the notion of an almost periodic function on $\mathbb{R}^n$, which was already mentioned in section 5.1.

6.11 Definition. An *almost periodic function on* $\mathbb{R}^n$ is an element in the closed linear span of the set of functions

$$e_p \colon \mathbb{R}^n \to \mathbb{C}, \quad x \mapsto e^{ip\cdot x}, \quad p \in \mathbb{R}^n.$$

The closure is taken with respect to the sup-norm.
The space of almost periodic functions is a C*-algebra, and will be denoted by $\mathcal{W}^0(\mathbb{R}^n)$.

It is worth noting that this is different from Bohr's original definition in [20] (for $n = 1$), although Bohr shows that they are in fact equivalent.

The following result can be found in [53, Theorem 1.3]:

6.12 Theorem. *For each $p \in \mathbb{R}^n$, let e_p be the function defined above, and let*

$$\tilde{e}_p \colon b\mathbb{R}^n \to \mathbb{C}, \quad \phi \mapsto \phi(e_p),$$

where we regard $b\mathbb{R}^n$ as a Pontryagin dual of a certain topological group as in Theorem 6.9, and note that e_p is an element of that topological group. The map defined on generators of $\mathcal{W}^0(\mathbb{R}^n)$ by $e_p \mapsto \tilde{e}_p$ extends in a unique way to an isomorphism $\mathcal{W}^0(\mathbb{R}^n) \to C(b\mathbb{R}^n)$ of C-algebras.*

6.13 Corollary. *For each $x \in \mathbb{R}^n$, let $\delta_x \colon \mathcal{W}^0(\mathbb{R}^n) \to \mathbb{C}$ be the character $f \mapsto f(x)$, let $\Delta := \{\delta_x \colon x \in \mathbb{R}^n\}$, and endow Δ with the weak*-topology τ_w induced by $\mathcal{W}^0(\mathbb{R}^n)$. Then the map*

$$\Phi \colon (\mathbb{R}^n, \tau_b) \to (\Delta, \tau_w), \quad x \mapsto \delta_x,$$

is a homeomorphism.

Proof. Let $F \colon \mathcal{W}^0(\mathbb{R}^n) \to C(b\mathbb{R}^n)$ be the isomorphism of the previous theorem, and let $\Omega(\mathcal{W}^0(\mathbb{R}^n))$ and $\Omega(C(b\mathbb{R}^n))$ be the Gelfand spectra of the corresponding spaces. Then the map

$$F^* \colon \Omega(C(b\mathbb{R}^n)) \to \Omega(\mathcal{W}^0(\mathbb{R}^n)), \quad \omega \mapsto \omega \circ F,$$

and the natural inclusion map

$$\Theta \colon b\mathbb{R}^n \to \Omega(C(b\mathbb{R}^n)), \quad x \mapsto \tilde{\delta}_x,$$

are both homeomorphisms. Moreover, by Theorem 6.9, the map $\iota\colon \mathbb{R}^n \to b\mathbb{R}^n$ is a homeomorphism onto its image with respect to the Bohr topology on $\mathbb{R}^n$.

We claim that

$$\Phi(x) = F^* \circ \Theta \circ \iota(x)$$

for each $x \in \mathbb{R}^n$. Indeed, let $x, p \in \mathbb{R}^n$, and let e_p and $\tilde{e}_p$ be as in the above definition and theorem. Then

$$\begin{aligned}(F^* \circ \Theta \circ \iota(x))(e_p) &= \tilde{\delta}_{\iota(x)} \circ F(e_p) = F(e_p)(\iota(x)) = \tilde{e}_p(\iota(x)) \\ &= \iota(x)(e_p) = e_p(x) = \delta_x(e_p).\end{aligned}$$

Since the functions e_p generate $\mathcal{W}^0(\mathbb{R}^n)$ as a C*-algebra, and both $F^* \circ \Theta \circ \iota(x)$ and δ_x are characters, we obtain $F^* \circ \Theta \circ \iota(x) = \delta_x = \Phi(x)$, which proves the claim.

Since F^* and Θ are homeomorphisms, the map $F^* \circ \Theta|_{\iota(\mathbb{R}^n)}$ is a homeomorphism onto its image Δ, and since ι is a homeomorphism onto its image, we conclude that Φ is a homeomorphism. ■

6.14 Definition. Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be a function, let τ_b be the Bohr topology on $\mathbb{R}^n$, and let τ_d be the usual metric topology on $\mathbb{C}$. We say that f is *Bohr continuous* if the map $f\colon (\mathbb{R}^n, \tau_b) \to (\mathbb{C}, \tau_d)$ is continuous.

6.15 Corollary. *Let $f \in \mathcal{W}^0(\mathbb{R}^n)$. Then f is Bohr continuous.*

Proof. Let $U \subseteq \mathbb{C}$ be open, and let $\hat{f}$ be the Gelfand transform of f. Since

$$\hat{f}^{-1}(U) \cap \Delta = \{\delta_x \colon x \in f^{-1}(U)\} = \Phi(f^{-1}(U)),$$

it follows from the above corollary that the set $f^{-1}(U)$ is Bohr open if (and only if) $\hat{f}^{-1}(U) \cap \Delta$ is (weak*-)open in Δ, and this is the case since $\hat{f} \in C(\Omega(\mathcal{W}^0(\mathbb{R}^n)))$. ■

6.16 Remark. Some authors (e.g. Kunen and Rudin [63]) take the opposite approach to the Bohr topology on abelian groups, defining it as the weakest topology on the group with respect to which all of the characters on the original topological group are continuous. In the case of $\mathbb{R}^n$, continuity of elements of $\mathcal{W}^0(\mathbb{R}^n)$ is then an elementary consequence of the definition.

6.4 Extracting $\mathcal{W}^0(\mathbb{R}^n)$ from $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$

As already noted in the proof of part (1) of Proposition 5.9, the generators of the C*-algebra $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ of the form $h_{\{0\},\xi,1}$ generate $\mathcal{W}^0(\mathbb{R}^n)$, so $\mathcal{W}^0(\mathbb{R}^n)$ is a closed *-subalgebra of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Moreover, it is readily seen that

$$I_0 = \operatorname{span}\{h_{U,\xi,g} \colon U \text{ is a subspace of } \mathbb{R}^n,\ U \neq \{0\},\ \xi \in \mathbb{R}^n,\ g \in \mathcal{S}(U)\}$$

is a *-ideal of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and that its closure I is a closed *-ideal. From these considerations, it seems reasonable to conjecture that $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) = \mathcal{W}^0(\mathbb{R}^n) \oplus I$, and we shall see that this is indeed the case.

To accomplish this, we shall define a function that is capable of sifting elements of $\mathcal{W}^0(\mathbb{R}^n)$ from elements in I. Here, we adopt a similar strategy to the one used in [112, Lemma 2.5], notably the idea that for any generator f of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, one can examine for each $x \in S^{n-1} \subset \mathbb{R}^n$ the behaviour of $f(sx)$ as $s \to \infty$. The main difference is that, since we have more knowledge of the topological properties of functions in $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ than their measure-theoretic properties, it makes more sense to consider open and dense subsets of S^{n-1} than subsets of full measure.

Indeed, if $U = \{0\}$ so that $f = h_{U,\xi,g}$ is almost periodic and hence Bohr continuous, there will be a dense subset of S^{n-1} for which each element x has the property that $|f(sx)|$ will come arbitrarily close to $\|f\|_\infty$ as $s \to \infty$. However, if $U \neq \{0\}$, then there will be an open and dense subset of S^{n-1} of vectors x such that $f(sx) \to 0$ as $s \to \infty$. This motivates the following definition, which should be thought of as a topological analogue of an essential supremum norm.

6.17 Definition. Let $\mathscr{D}$ be the set of open and dense subsets of S^{n-1}. We define the function $\|\cdot\|_0$ by

$$\|\cdot\|_0 \colon \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to [0,\infty), \quad f \mapsto \|f\|_0 := \inf_{V\in\mathscr{D}} \sup_{x\in V} \inf_{r>0} \sup_{s>r} |f(sx)|.$$

6.18 Remark. Since the intersection of two open and dense subsets of any topological space is again open and dense, the set $\mathscr{D}$ is naturally an upward directed set with respect to the partial order $\supseteq$. Furthermore, for any bounded function $g \colon S^{n-1} \to [0,\infty)$, the function

$$\mathscr{D} \to [0,\infty), \quad V \mapsto \sup_{x\in V} g(x),$$

is a monotone decreasing function with respect to this partial order, hence by a standard result from analysis, we have

$$\|f\|_0 = \lim_{V\in\mathscr{D}} \sup_{x\in V} \inf_{r>0} \sup_{s>r} |f(sx)|$$

for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

6.19 Lemma. *The function $\|\cdot\|_0$ is a seminorm on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Moreover, for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, we have $\|f\|_0 \le \|f\|_\infty$, and equality holds if $f \in \mathcal{W}^0(\mathbb{R}^n)$. Finally, for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and each $g \in I$, we have $\|f + g\|_0 = \|f\|_0$.*

Proof. From the definition of $\|\cdot\|_0$, it is readily seen that it is well-defined, and that is satisfies

(a) For each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, we have $\|f\|_0 \le \|f\|_\infty$.

Moreover, it is elementary to show that

(b) For each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and each $\lambda \in \mathbb{C}$, we have $|\lambda|\|f\|_0 = \|\lambda f\|_0$;
(c) For each $f, g \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, we have $\|f + g\|_0 \le \|f\|_0 + \|g\|_\infty$.

Indeed, the latter statement is a simple extension of (a) that can be obtained by noting that $|f(x) + g(x)| \le |f(x)| + \|g\|_\infty$ for each $x \in \mathbb{R}^n$, and one by one applying suprema and infinima, similar to what we do in part (e) below. From (c), we get

(d) The function $\|\cdot\|_0$ is Lipschitz continuous with respect to $\|\cdot\|_\infty$, with Lipschitz constant 1.

Indeed, let $f, g \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Then substituting f and g in the above inequality for $f+g$ and $-g$ respectively, we obtain $\|f\|_0 \le \|f+g\|_0+\|g\|_\infty$. This inequality and the one in (c) together are equivalent to

$$|\|f + g\|_0 - \|f\|_0| \le \|g\|_\infty,$$

which proves (d).

(e) For each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and each $g \in I$, we have $\|f + g\|_0 = \|f\|_0$.

We first prove the statement for $g \in I_0$. Each element of I_0 can be written as a finite sum of generators of I_0, so we may assume without loss of generality that $g = h_{U,\xi,k}$ for some nonzero subspace $U \subseteq \mathbb{R}^n$, some $\xi \in U^\perp$

and $k \in \mathcal{S}(U)$. Now let $V_0 := S^{n-1} \backslash U^\perp$. The subspace $U^\perp$ is a proper subspace of $\mathbb{R}^n$ since $U \neq \{0\}$, so V_0 is an open and dense subset of S^{n-1}.

Next, fix $x \in V_0$, and fix $\varepsilon > 0$. Since $x \notin U^\perp$, we have $r_U(x) \neq 0$. Since $k \in \mathcal{S}(U)$, it vanishes at infinity, so there exists an $r > 0$ such that for each $s > r$, we have $|g(sx)| = |k \circ r_U(sx)| < \varepsilon$, so

$$|f(sx) + g(sx)| - \varepsilon < |f(sx)| < |f(sx) + g(sx)| + \varepsilon.$$

Thus

$$\sup_{s>r} |f(sx) + g(sx)| - \varepsilon \leq \sup_{s>r} |f(sx)| \leq \sup_{s>r} |f(sx) + g(sx)| + \varepsilon,$$

for each $r > 0$, which implies

$$\begin{aligned} \inf_{r>0} \sup_{s>r} |f(sx) + g(sx)| - \varepsilon &\leq \inf_{r>0} \sup_{s>r} |f(sx)| \\ &\leq \inf_{r>0} \sup_{s>r} |f(sx) + g(sx)| + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\inf_{r>0} \sup_{s>r} |f(sx)| = \inf_{r>0} \sup_{s>r} |f(sx) + g(sx)|$$

for each $x \in V_0$, so for each $V \in \mathscr{D}$ satisfying $V \subseteq V_0$, we have

$$\sup_{x \in V} \inf_{r>0} \sup_{s>r} |f(sx)| = \sup_{x \in V} \inf_{r>0} \sup_{s>r} |f(sx) + g(sx)|.$$

Taking the limit over $V \in \mathscr{D}$ then yields

$$\begin{aligned} \|f\|_0 &= \lim_{\substack{V \in \mathscr{D} \\ V_0 \supseteq V}} \sup_{x \in V} \inf_{r>0} \sup_{s>r} |f(sx)| = \lim_{\substack{V \in \mathscr{D} \\ V_0 \supseteq V}} \sup_{x \in V} \inf_{r>0} \sup_{s>r} |f(sx) + g(sx)| \\ &= \|f + g\|_0, \end{aligned}$$

which proves the statement for $g \in I_0$.

To extend the statement to $g \in I$, we note that by (d), the map

$$F \colon \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathbb{R}, \quad g \mapsto \|f + g\|_0 - \|f\|_0.$$

is continuous, so $F^{-1}(\{0\})$ is closed, and we have just shown that $I_0 \subseteq F^{-1}(\{0\})$, so $I = \overline{I_0} \subseteq F^{-1}(\{0\})$.

(f) For each $f \in \mathcal{W}^0(\mathbb{R}^n)$, we have $\|f\|_0 = \|f\|_\infty$.

Let $f \in \mathcal{W}^0(\mathbb{R}^n)$ and fix $\varepsilon > 0$. Then there exists $x_0 \in \mathbb{R}^n$ such that $\|f\|_\infty < |f(x_0)| + \varepsilon/2$. Now let $W := \{z \in \mathbb{C}\colon |f(x_0) - z| < \varepsilon/2\}$. Then W is open, and since $f \in \mathcal{W}^0(\mathbb{R}^n)$, the function f is Bohr continuous by Corollary 6.15, so that $U := f^{-1}(W)$ is a Bohr open neighbourhood of x_0. By part (2) of Lemma 6.10, the set V_U defined there is dense in S^{n-1}. Thus for each $V \in \mathscr{D}$, the set $V \cap V_U$ is nonempty, which means that there exists $x \in V$ such that for each $r > 0$, there exists an $s > r$ such that $sx \in U$, i.e., $f(sx) \in W$, so

$$|f(sx)| \geq |f(x_0)| - |f(x_0) - f(sx)| > \|f\|_\infty - \varepsilon.$$

Hence $\sup_{s>r} |f(x)| > \|f\|_\infty - \varepsilon$, and it follows that $\inf_{r>0} \sup_{s>r} |f(x)| \geq \|f\|_\infty - \varepsilon$, which in turn implies

$$\sup_{x\in V} \inf_{r>0} \sup_{s>r} |f(sx)| \geq \|f\|_\infty - \varepsilon.$$

Since $V \in \mathscr{D}$ was arbitrary, we obtain $\|f\|_0 \geq \|f\|_\infty - \varepsilon$, and since ε was arbitrary, we get $\|f\|_0 \geq \|f\|_\infty$. Combining this with (a) yields the desired result.

(g) For each $f, g \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, we have $\|f + g\|_0 \leq \|f\|_0 + \|g\|_0$.

We first prove the statement for $g = g_1 + g_2$, with $g_1 \in \mathcal{W}^0(\mathbb{R}^n)$ and $g_2 \in I$. Applying (e), (c), (f) and (e) again, we obtain

$$\|f + g\|_0 = \|f + g_1\|_0 \leq \|f\|_0 + \|g_1\|_\infty = \|f\|_0 + \|g_1\|_0 = \|f\|_0 + \|g\|_0.$$

We now prove this inequality for general $g \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Since the union of the sets of generators of $\mathcal{W}^0(\mathbb{R}^n)$ and I is the set of generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, the subspace $\mathcal{W}^0(\mathbb{R}^n) + I$ is dense in $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. Now fix $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and consider the function

$$F\colon \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathbb{R}, \quad g \mapsto \|f\|_0 + \|g\|_0 - \|f + g\|_0.$$

It follows from (d) that F is continuous, so $F^{-1}([0,\infty))$ is closed, and by the inequality we just proved, we have $\mathcal{W}^0(\mathbb{R}^n) + I \subseteq F^{-1}([0,\infty))$, so $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) = \overline{\mathcal{W}^0(\mathbb{R}^n) + I} \subseteq F^{-1}([0,\infty))$, which proves (g).

Note that (b) and (g) together imply that $\|\cdot\|_0$ is a seminorm on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and that the remaining assertions in the lemma were proved under (a), (e) and (f). ∎

We now arrive at the main result of this section:

6.20 Theorem. *We have* $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) = \mathcal{W}^0(\mathbb{R}^n) \oplus I$. *Moreover, the seminorm* $\|\cdot\|_0$ *is the composition of the quotient norm on* $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)/I$ *with the canonical projection map* $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)/I$.

Proof. It is easy to see that $\mathcal{W}^0(\mathbb{R}^n) \cap I = \{0\}$. Indeed, if f is an element of the intersection, then using the previous lemma, we see that $\|f\|_\infty = \|f\|_0 = 0$, so $f = 0$.

Next, we claim that $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is closed. We will give an elementary proof of this fact using the norm $\|\cdot\|_0$ that we defined in this section. One can also invoke a result from the general theory of C*-algebras to obtain a proof of this claim; see the remark below.

Let $h \in \overline{\mathcal{W}^0(\mathbb{R}^n) \oplus I}$. Then there exists a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{W}^0(\mathbb{R}^n)$ and a sequence of functions $(g_n)_{n\in\mathbb{N}}$ in I such that $f_n + g_n \to h$ uniformly as $n \to \infty$. Now fix $\varepsilon > 0$. Then there exists an $N_0 \in \mathbb{N}$ such that for each $n \geq N_0$, we have $\|f_n + g_n - h\|_\infty < \varepsilon/2$. For each $m, n \geq N_0$, we obtain

$$\begin{aligned}
\|f_n - f_m\|_\infty = \|f_n - f_m\|_0 &= \|(f_n + g_n - h) - (f_m + g_m - h)\|_0 \\
&\leq \|(f_n + g_n - h) - (f_m + g_m - h)\|_\infty \\
&\leq \|(f_n + g_n - h)\|_\infty + \|(f_m + g_m - h)\|_\infty < \varepsilon,
\end{aligned}$$

where the first three steps follow from the preceding lemma, so $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}^0(\mathbb{R}^n)$. The space of almost periodic functions $\mathcal{W}^0(\mathbb{R}^n)$ is closed with respect to the sup-norm, so it is complete, hence the Cauchy sequence has a limit $f \in \mathcal{W}^0(\mathbb{R}^n)$. We may therefore fix an $N \geq N_0$ such that for each $n \geq N$, we have $\|f_n - f\|_\infty < \varepsilon/2$, which implies

$$\|g_n - (h - f)\|_\infty \leq \|f_n + g_n - h\|_\infty + \|f_n - f\|_\infty < \varepsilon,$$

for each $n \geq N$, hence the sequence $(g_n)_{n\in\mathbb{N}}$ in I converges to $h - f$. Because I is by definition closed, we get $h - f \in I$. Thus $h = f + (h - f) \in \mathcal{W}^0(\mathbb{R}^n) \oplus I$, which proves that $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is closed. We have already noted in the proof of the previous lemma that $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is dense in $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, hence the first assertion is true.

It follows that each element $f + I \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)/I$ has a unique representative in $\mathcal{W}^0(\mathbb{R}^n)$. Now fix such an element $f + I$, and suppose that $f \in \mathcal{W}^0(\mathbb{R}^n)$. Then for each $g \in I$, we have

$$\|f\|_\infty = \|f\|_0 = \|f + g\|_0 \leq \|f + g\|_\infty,$$

so

$$\|f + I\| = \inf_{g \in I} \|f + g\|_\infty = \|f\|_0,$$

and both the left and the right-hand side of this equation do not depend on the chosen representative f. This proves the second assertion. ■

6.21 Remark.

(1) There exists a shorter proof of the fact that $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is closed in $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. One can note that $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is the preimage of the forward image of $\mathcal{W}^0(\mathbb{R}^n)$ under the quotient map $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)/I$. From the theory of C*-algebras, it is known that the image of a C*-algebra under a *-homomorphism between C*-algebras is closed (cf. [87, Theorem 3.1.6]), and it is trivial that the preimage of a C*-algebra under such a morphism is a C*-subalgebra of the domain, therefore $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ is closed.

(2) Note that in the proof of Theorem 6.20, we never used the fact that $\|\cdot\|_0$ satisfies the triangle inequality, and that this inequality could also be proved using the final assertion of the theorem, and the fact that the quotient norm on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)/I$ satisfies the triangle inequality.

6.22 Corollary. *The decomposition $\mathcal{W}^0(\mathbb{R}^n) \oplus I$ has an associated projection map*

$$\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) = \mathcal{W}^0(\mathbb{R}^n) \oplus I \to \mathcal{W}^0(\mathbb{R}^n), \quad f = g + h \mapsto g,$$

*and this map is a surjective *-homomorphism.*

6.5 Constructing the set of characters

We almost have everything in place to identify the characters of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. We require one other minor result, for which we note that it is possible to define algebras $\mathcal{W}^0_{\mathcal{R}}(w + V)$ and $\mathcal{W}^0(w + V)$ for any subspace V of $\mathbb{R}^n$

and any element $w \in V^\perp$. Indeed, since V is a vector space that inherits an inner product from $\mathbb{R}^n$, we let $\mathcal{W}^0_{\mathcal{R}}(w+V)$ be the C*-subalgebra of $C_b(w+V)$ generated by functions of the form

$$h'_{U,\xi,g} \colon w+V \to \mathbb{C}, \quad p \mapsto e^{i\xi\cdot p} \cdot g \circ r_U(p),$$

where

- U is a subspace of V;
- $\xi \in U^\perp \cap V$;
- $g \in \mathcal{S}(U)$.

The algebra $\mathcal{W}^0(w+V)$ is the C*-subalgebra generated by the subset of generators of $\mathcal{W}^0_{\mathcal{R}}(w+V)$ with $U = \{0\}$. It is easy to see that we have a decomposition

$$\mathcal{W}^0_{\mathcal{R}}(w+V) = \mathcal{W}^0(w+V) \oplus I_{w+V},$$

similar to the one from Theorem 6.20.

6.23 Proposition. *Let $V \subseteq \mathbb{R}^n$ be a subspace, and let $w \in V^\perp$. The inclusion map*

$$\mathcal{W}^0_{\mathcal{R}}(w+V) \to \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n), \quad f \mapsto (p \mapsto f(r_V(p)+w)),$$

*is a well-defined isometric *-homomorphism. This map is a section for the restriction map*

$$\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(w+V), \quad f \mapsto f|_{w+V},$$

*which is a well-defined, continuous *-homomorphism, and hence the restriction map is surjective.*

Proof. The inclusion and the restriction maps are the pullbacks ρ^* and σ^* to the corresponding algebras $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and $\mathcal{W}^0_{\mathcal{R}}(w+V)$ of the projection

$$\rho \colon \mathbb{R}^n \to w+V, \quad p \mapsto r_V(p)+w,$$

and the inclusion map $\sigma \colon w+V \hookrightarrow \mathbb{R}^n$, respectively. We first note that ρ and σ can also be pulled back to the spaces $C_b(\mathbb{R}^n)$ and $C_b(w+V)$, where they induce norm-decreasing *-homomorphisms, which will also be denoted by ρ^* and σ^*. Moreover, the pullback of ρ is an isometry since ρ is surjective. Furthermore, we have

$$\rho^* \circ \sigma^* = (\sigma \circ \rho)^* = \mathrm{Id}^*_{w+V},$$

which shows that, if the pullbacks define maps between $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and $\mathcal{W}^0_{\mathcal{R}}(w+V)$, then ρ^* is a section for σ^*.

It remains to be shown that $\rho^*(\mathcal{W}^0_{\mathcal{R}}(w+V)) \subseteq \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ and $\sigma^*(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)) \subseteq \mathcal{W}^0_{\mathcal{R}}(w+V)$. We show that ρ^* maps the set of functions generating $\mathcal{W}^0_{\mathcal{R}}(w+V)$ into the set generating $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and that σ^* does the same thing with the two algebras reversed; this will imply the desired statement, since ρ^* and σ^* are norm-decreasing $*$-homomorphisms.

We start with ρ^*. Note that for each subspace $U \subseteq V$, each $\xi \in U^\perp \cap V$ and each $g \in \mathcal{S}(U)$, we have

$$\begin{aligned}\rho^*(h'_{U,\xi,g})(p) = h'_{U,\xi,g}(r_V(p)+w) &= e^{i\xi\cdot r_V(p)} g \circ r_U \circ r_V(p) = e^{i\xi\cdot p} g \circ r_U(p)\\ &= h_{U,\xi,g}(p),\end{aligned}$$

hence $\rho^*(h'_{U,\xi,g}) = h_{U,\xi,g}$, so ρ^* indeed maps generators of $\mathcal{W}^0_{\mathcal{R}}(w+V)$ to generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

We turn to σ^*. Consider a generator $h_{U,\xi,g}$ of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as in Definition 5.8. Now let

$$U' := (\ker(r_U|_V))^\perp \cap V = (U^\perp \cap V)^\perp \cap V = (U + V^\perp) \cap V.$$

Then by the first isomorphism theorem from linear algebra, the map $r_U|_{U'}$ is an isomorphism onto its image. Moreover, let

$$\xi' := r_V(\xi - r_{U'}(\xi)) = r_V(\xi) - r_{U'}(\xi) \in (U')^\perp \cap V,$$

and define the function g' as follows:

$$\begin{aligned} g' \colon U' \to \mathbb{C}, \quad p \mapsto\ & e^{i(\xi - r_V(\xi))\cdot w} e^{i\xi\cdot p} g \circ r_U(p+w)\\ &= e^{i(\xi - r_V(\xi))\cdot w} e^{i r_{U'}(\xi)\cdot p} g \circ r_U(p+w).\end{aligned}$$

Then g' is a product of the smooth function $p \mapsto e^{i\xi\cdot p}$ whose partial derivatives of all orders are bounded, and the function $e^{i(\xi - r_V(\xi))\cdot w} g \circ r_U|_{U'}$, which is a composition of an isomorphism onto its image which is a subspace of U, and a Schwartz function on U, so it is Schwartz. It follows that $g' \in \mathcal{S}(U')$.

We claim that the generator $h'_{U',\xi',g'}$ of $\mathcal{W}^0_{\mathcal{R}}(w+V)$ corresponding to these U', ξ' and g' is the restriction to $w+V$ of the generator $h_{U,\xi,g}$ of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ that we started out with. First note that

$$V = U' \oplus ((U')^\perp \cap V).$$

Now fix $u \in U'$ and $v \in (U')^\perp \cap V$. Then

$$\begin{aligned} g' \circ r_{U'}(u+v) &= g'(u) = e^{i(\xi - r_V(\xi))\cdot w} e^{i r_{U'}(\xi)\cdot u} g \circ r_U(u+w) \\ &= e^{i(\xi - r_V(\xi))\cdot (u+v+w)} e^{i r_{U'}(\xi)\cdot (u+v+w)} g \circ r_U(u+v+w), \end{aligned}$$

where the last step follows from the fact that $v \in \ker(r_U|_V)$. Furthermore, since $\xi' \in (U')^\perp \cap V$, we have

$$\begin{aligned} \xi' \cdot (u+v) &= (r_V - r_{U'})(\xi) \cdot (u+v) = \xi \cdot (r_V - r_{U'})(u+v) = \xi \cdot v \\ &= \xi \cdot (u+v), \end{aligned}$$

and $r_{U'}(w) = 0$. Thus for each $p \in w + V$, we have

$$\begin{aligned} &h'_{U',\xi',g'}(p) \\ &= e^{i\xi'\cdot p} g' \circ r_{U'}(p) = e^{i(r_V - r_{U'})(\xi)\cdot p} g' \circ r_{U'}(p-w) \\ &= e^{i(r_V - r_{U'})(\xi)\cdot (p-w+w)} e^{i(\xi - r_V(\xi))\cdot (p-w+w)} e^{i r_{U'}(\xi)\cdot (p-w+w)} g \circ r_U(p-w+w) \\ &= e^{i\xi\cdot p} g \circ r_U(p) = h_{U,\xi,g}(p). \end{aligned}$$

This proves our claim, and thereby our statement that σ^* maps the generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ to generators of $\mathcal{W}^0_{\mathcal{R}}(w+V)$. ■

6.24 Definition. Let

$$\Omega^0_{\mathcal{R},n} := \{(V, w, \zeta) \colon V \text{ is a subspace of } \mathbb{R}^n,\ w \in V^\perp,\ \zeta \in bV\}.$$

For each $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, define the map $\chi_{V,w,\zeta} \colon \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathbb{C}$ as the composition

$$\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(w+V) \to \mathcal{W}^0(w+V) \to \mathcal{W}^0(V) \to \mathbb{C},$$

where

- The first map is the restriction map σ^* from Proposition 6.23;
- The second map is analogous to the one from Corollary 6.22;
- The third map is the pullback of translation by w;

- The fourth map is the element ζ viewed as a character on $\mathcal{W}^0(V)$ obtained by using Gelfand duality in conjunction with the isomorphism in Theorem 6.12.

6.25 Proposition. *For each $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, the map $\chi_{V,w,\zeta}$ is a character on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.*

Proof. The map $\chi_{V,w,\zeta}$ defined above is a composition of four maps, each of which is a surjective $*$-homomorphism. Thus $\chi_{V,w,\zeta}$ is a surjective *-homomorphism as well, so it is a character on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. ∎

6.6 The support of a character

Having constructed the character corresponding to an element $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, we will show that all such characters are distinct, and that all characters on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ are of this form. In other words, we mean to show that the map

$$\Omega^0_{\mathcal{R},n} \to \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)), \quad (V, w, \zeta) \mapsto \chi_{V,w,\zeta},$$

is a bijection. We will accomplish this by first showing that there is a well-defined notion of the support of a character $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, which roughly corresponds to the smallest subset of $\mathbb{R}^n$ such that the value of $\chi(f)$, where $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, depends only on the restriction of f to that subset. After proving the existence of the support and some properties related to it, it then becomes straightforward to prove the main result of this section, which is Theorem 6.31.

6.26 Definition. Let

$$\mathcal{P} := \{w + V : V \subseteq \mathbb{R}^n \text{ is subspace, } w \in V^{\perp}\},$$

and regard it as a poset with its natural partial ordering $\subseteq$. For each $w + V \in P$, let

$$\rho_{w+V} : \mathcal{W}^0_{\mathcal{R}}(w + V) \to \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n), \quad \sigma_{w+V} : \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \to \mathcal{W}^0_{\mathcal{R}}(w + V),$$

be the inclusion and restriction map from Proposition 6.23, respectively. (We have omitted the *-symbol denoting pullback, and will continue to do so in what follows.) For each $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, let

$$\mathcal{P}_\chi := \{w + V \in \mathcal{P} \colon \chi = \chi \circ \rho_{w+V} \circ \sigma_{w+V}\}.$$

If $\mathcal{P}_\chi$ has a minimum, then we call this minimum the *the support of* χ; in this case, the support will be denoted by $\operatorname{supp}(\chi)$.

6.27 Lemma. *Let $U, V \subseteq \mathbb{R}^n$ be two subspaces, and suppose that $U \neq \{0\} \neq V$, but $U \cap V = \{0\}$. For each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $x \in \mathbb{R}^n \backslash B_\varepsilon(0)$, the element x has distance greater than or equal to δ to at least one of the subspaces U and V.*

Proof.

Let $d \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, $(x, y) \mapsto \|x - y\|$ be the Euclidean distance function, and recall that for any $x \in \mathbb{R}^n$ and any nonempty subset $X \subseteq \mathbb{R}^n$, we have $d(x, X) := \inf_{y \in X} d(x, y)$, and that the function $\mathbb{R}^n \to [0, \infty)$, $x \mapsto d(x, X)$ is continuous. Now consider the map

$$f \colon S^{n-1} \to [0, \infty), \quad x \mapsto \max(d(x, U), d(x, V)).$$

Then f is a continuous, nonnegative function on a compact set, so it must attain a minimum at some point $x_0 \in S^{n-1}$. If $f(x_0) = 0$, then $d(x_0, U) = 0 = d(x_0, V)$, hence $x_0 \in U$ and $x_0 \in V$, since U and V are closed. Since $U \cap V = \{0\}$, this implies $0 = x_0 \in S^{n-1}$, a contradiction. It follows that $f(x_0) > 0$.

Now fix $\varepsilon > 0$, and let $\delta := \varepsilon f(x_0)$. We claim that δ has the desired property. Indeed, let $x \in \mathbb{R}^n \backslash B_\varepsilon(0)$. Then $\|x\| \geq \varepsilon$, and $\|x\|^{-1} x \in S^{n-1}$, so

$$\begin{aligned} \max(d(x, U), d(x, V)) &= \|x\| \cdot \max(d(\|x\|^{-1} x, U), d(\|x\|^{-1} x, V)) \\ &\geq \|x\| f(x_0) \geq \varepsilon f(x_0) = \delta, \end{aligned}$$

so x has distance greater than or equal to δ to at least one of the subspaces U or V. ■

6.28 Lemma. *Let $V \subseteq \mathbb{R}^n$ be a subspace, and let*

$$X_V := \{h_{U,\xi,g} \colon U \subseteq V,\ \xi \in U^\perp \cap V,\ g \in \mathcal{S}(U)\},$$

$$Y_V := \{h_{U,0,g} \colon U \nsubseteq V,\ g \in \mathcal{S}(U)\},$$
$$Z_V := \{h_{\{0\},\xi,1} \colon \xi \in V^{\perp}\},$$

where the notation of the generators $h_{U,\xi,g}$ of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ is the same as before. Then $X_V \cup Y_V \cup Z_V$ generates $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ as a C-algebra.*

Proof. Let A be the C*-algebra generated by $X_V \cup Y_V \cup Z_V$. Clearly, $X_V \cup Y_V \cup Z_V$ is a subset of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, so $A \subseteq \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

We now prove the reverse inclusion. If $V = \mathbb{R}^n$, then X_V is precisely the set of generators that we used to define $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and the statement is trivial. Suppose that $V \neq \mathbb{R}^n$, let $U \subseteq \mathbb{R}^n$ be a subspace, let $\xi \in U^{\perp}$, and let $g \in \mathcal{S}(U)$. There are two cases:

- $U \subseteq V$; then we have

$$h_{U,\xi,g} = h_{U,r_{U^{\perp}\cap V}(\xi),g} \cdot h_{\{0\},r_{U^{\perp}\cap V^{\perp}}(\xi),1}.$$

  The first and second factor on the right-hand side are contained in X_V and Z_V, respectively, hence $h_{U,\xi,g} \in A$.

- $U \nsubseteq V$; then we have

$$h_{U,\xi,g} = h_{\{0\},r_V(\xi),1} \cdot h_{U,0,g} \cdot h_{\{0\},r_{V^{\perp}}(\xi),1}.$$

  The first, second and third factor on the right-hand side are contained in X_V, Y_V and Z_V, respectively, hence $h_{U,\xi,g} \in A$.

It follows that all generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ are contained in A, hence $A = \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, which is what we wanted to show. ∎

6.29 Proposition. *Let $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$.*

(1) The set $\mathcal{P}_{\chi}$ is a filter and has a minimum. Thus the support of χ is well-defined;

(2) If $\mathcal{W}^0_{\mathcal{R}}(\operatorname{supp}(\chi)) = \mathcal{W}^0(\operatorname{supp}(\chi)) \oplus I_{\operatorname{supp}(\chi)}$ denotes the decomposition obtained in Theorem 6.20, then $\chi \circ \rho_{\operatorname{supp}(\chi)}$ vanishes on $I_{\operatorname{supp}(\chi)}$;

(3) If $\chi = \chi_{V,w,\zeta}$ for some $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, then $\operatorname{supp}(\chi) = w + V$.

6.30 Remark. Before giving the proof, we introduce some notation that will streamline the process. Suppose we have $w+V, w'+V' \in \mathcal{P}$ such that $w+V \subseteq w'+V'$. We then note that Proposition 6.23 is still true when we replace $\mathbb{R}^n$ with $w'+V'$, so that we obtain two maps

$$\rho^{w'+V'}_{w+V} : \mathcal{W}^0_{\mathcal{R}}(w+V) \to \mathcal{W}^0_{\mathcal{R}}(w'+V'),$$
$$\sigma^{w'+V'}_{w+V} : \mathcal{W}^0_{\mathcal{R}}(w'+V') \to \mathcal{W}^0_{\mathcal{R}}(w+V).$$

Moreover, introducing a third element $w''+V'' \in \mathcal{P}$ such that $w'+V' \subseteq w''+V''$, it is readily seen that

$$\rho^{w''+V''}_{w+V} = \rho^{w''+V''}_{w'+V'} \circ \rho^{w'+V'}_{w+V}, \quad \sigma^{w''+V''}_{w+V} = \sigma^{w'+V'}_{w+V} \circ \sigma^{w''+V''}_{w'+V'},$$

by noting that the maps in the above formulas are pullbacks of functions between the above three elements of $\mathcal{P}$, and that it is easy to verify the corresponding identities for these functions.

On a similar note, Lemma 6.28 can be generalised to show that certain sets $X^{w+V}_{w+V'}$, $Y^{w+V}_{w+V'}$ and $Z^{w+V}_{w+V'}$ defined in a way similar to X_V, Y_V and Z_V in the lemma, generate $\mathcal{W}^0_{\mathcal{R}}(w+V)$ for some subspaces $V, V' \subseteq \mathbb{R}^n$ such $V' \subseteq V$, and $w \in V^\perp$.

Proof.

(1) We first show that $\mathcal{P}_\chi$ is an upper set (see the next section for the definition of this term), i.e.:

(a) Let $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, let $w+V, w'+V' \in \mathcal{P}$, and suppose that $w+V \in \mathcal{P}_\chi$ and $w+V \subseteq w'+V'$. Then $w'+V' \in \mathcal{P}_\chi$.

Indeed, we have

$$\begin{aligned}
\chi \circ \rho_{w'+V'} \circ \sigma_{w'+V'} &= \chi \circ \rho_{w+V} \circ \sigma_{w+V} \circ \rho_{w'+V'} \circ \sigma_{w'+V'} \\
&= \chi \circ \rho_{w+V} \circ \sigma^{w'+V'}_{w+V} \circ \sigma_{w'+V'} \circ \rho_{w'+V'} \circ \sigma_{w'+V'} \\
&= \chi \circ \rho_{w+V} \circ \sigma^{w'+V'}_{w+V} \circ \sigma_{w'+V'} = \chi \circ \rho_{w+V} \circ \sigma_{w+V} \\
&= \chi,
\end{aligned}$$

which proves (a).

(b) For each $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ and each $w+V, w'+V' \in \mathcal{P}_\chi$, the set $(w+V) \cap (w'+V')$ is nonempty.

Suppose for the sake of contradiction that this is not the case. Then $w + (V + V')$ and $w' + (V + V')$ are disjoint. In particular, this means that $V + V' \neq \mathbb{R}^n$, so $U := (V + V')^{\perp} \neq \{0\}$, and $u := r_U(w' - w) \neq 0$. Now let $f_0 \colon \mathbb{R} \to [0, 1]$ be a smooth, compactly supported function such that $f_0(0) = 0$ and $f_0(1) = 1$, and consider the function

$$f \colon \mathbb{R}^n \to \mathbb{C}, \quad p \mapsto f_0\left(\frac{u \cdot (p - w)}{\|u\|^2}\right).$$

Then $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and we have $\sigma_{w+(V+V')}(f) \equiv 0$ and $\sigma_{w'+(V+V')}(f) \equiv 1$. From this, it is readily seen that $\rho_{w+V} \circ \sigma_{w+V}(f) \equiv 0$, and $\rho_{w'+V'} \circ \sigma_{w'+V'}(f) \equiv 1$, so

$$\chi(f) = \chi \circ \rho_{w+V} \circ \sigma_{w+V}(f) = 0 \neq 1 = \chi \circ \rho_{w'+V'} \circ \sigma_{w'+V'}(f) = \chi(f),$$

which is the desired contradiction.

(c) For each $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, and each $w + V, w' + V' \in \mathcal{P}_\chi$, we have $(w + V) \cap (w' + V') \in \mathcal{P}_\chi$.

From (b), we know that $(w + V) \cap (w' + V')$ contains some element w''. Then $w'' + V = w + V$ and $w'' + V' = w' + V'$, so

$$(w + V) \cap (w' + V') = w'' + (V \cap V') \in \mathcal{P}.$$

Moreover, we may now assume without loss of generality that $w = w' = w''$.

We show that $w + (V \cap V') \in \mathcal{P}_\chi$. Note that this statement is trivial if either of the subspaces V or V' is a subspace of the other, so we may assume that this is not the case. To simplify the notation somewhat, we define

$$\begin{aligned} \tau &:= \rho_{w+V} \circ \sigma_{w+V}, \\ \tau' &:= \rho_{w+V'} \circ \sigma_{w+V'}, \\ \tau'' &:= \rho_{w+(V \cap V')} \circ \sigma_{w+(V \cap V')}. \end{aligned}$$

Now let $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and fix $\varepsilon > 0$. All elements of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ are uniformly continuous, as is easily seen by looking at the generators, hence there exists a $\delta > 0$ such that for each $x, y \in \mathbb{R}^n$, if $\|x - y\| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Since neither V nor V' is a subspace of the other, we have $V \neq V \cap V' \neq V'$, so both spaces

$$U := V \cap (V \cap V')^\perp, \quad U' := V' \cap (V \cap V')^\perp,$$

are nonzero. In addition, since their intersection is trivial, we may invoke Lemma 6.27 to obtain a $\gamma > 0$ such that for each $x \in \mathbb{R}^n \backslash B_\delta(0)$, x has distance greater than or equal to γ to at least one of the subspaces U and U'.

Next, fix a smooth function $g_0 \colon w + V^\perp \to [0,1]$ such that $g_0(w) = 1$, and $g_0(w + x) = 0$ for each $x \in V^\perp$ with $\|x\| \geq \gamma$, and similarly, fix a smooth function g_0' satisfying the same conditions with $V^\perp$ replaced by $(V')^\perp$. Observe that $g_0 \in \mathcal{W}^0_{\mathcal{R}}(w + V^\perp)$ and $g_0' \in \mathcal{W}^0_{\mathcal{R}}(w + (V')^\perp)$. We now define

$$g := \rho_{w+V^\perp}(g_0), \quad g' := \rho_{w+(V')^\perp}(g_0').$$

Then g and g' are constant and equal to 1 on $w+V$ and $w+V'$, respectively, so $\tau(g) \equiv 1 \equiv \tau'(g')$.

Moreover, let $x \in w + (V \cap V')$. Then the product gg' of g and g' satisfies $gg'(x) = 1$. Now let $y \in (V \cap V')^\perp$ be an element such that $\|y\| \geq \delta$. Then y has distance greater than or equal to γ to U or U'. Furthermore, for each $u \in U \cup U'$, we have

$$\|(x + y) - (w + u)\|^2 = \|(x - w) + (y - u)\|^2 = \|x - w\|^2 + \|y - u\|^2,$$

since $x - w \in V \cap V'$ and $y - u \in (V \cap V')^\perp$, so $\|(x+y) - (w+u)\| \geq \|y - u\|$. It follows that $x + y$ has distance greater than or equal to γ to $w + U$ or $w + U'$. Since $x \in w + (V \cap V')$ was arbitrary, and $V = U \oplus (V \cap V')$ and $V' = U' \oplus (V \cap V')$, we infer that $x + y$ has distance greater than or equal to γ to $w + V$ or $w + V'$. In the first case, we have $g(x + y) = 0$, and in the second case, we have $g'(x + y) = 0$. Thus in both cases, we have $(gg')(x + y) = 0$. Summarising, we have $gg' \equiv 1$ on $w + (V \cap V')$ and gg' vanishes on $\mathbb{R}^n \backslash (w + (V \cap V') + B_\delta(0))$.

We now obtain the following facts:

- It is readily seen from the definition of τ'' that $(\tau''(f) - f)(x) = 0$ for each $x \in w + (V \cap V')$, hence $(gg'(\tau''(f) - f))(x) = 0$ for each $x \in w + (V \cap V')$;

- For each $x \in w + (V \cap V') + B_\delta(0)$, we have $\|w + r_{V\cap V'}(x) - x\| < \delta$, hence

$$|(gg'(\tau''(f) - f))(x)| \leq |(\tau''(f) - f)(x)| = |f(w + r_{V\cap V'}(x)) - f(x)| < \varepsilon;$$

- For each $x \in \mathbb{R}^n \backslash (w + (V \cap V') + B_\delta(0))$, we have $gg'(x) = 0$, hence $(gg'(\tau''(f) - f))(x) = 0$.

It follows that $\|gg'(\tau''(f) - f)\|_\infty \leq \varepsilon$. Furthermore, since $\tau(g) \equiv 1 \equiv \tau'(g')$, and since $w + V, w + V' \in \mathcal{P}_\chi$, we have

$$\begin{aligned}\chi(gg'(\tau''(f) - f)) &= \chi(g)\chi(g')\chi(\tau''(f) - f) \\ &= \chi \circ \tau(g) \cdot \chi \circ \tau'(g') \cdot \chi(\tau''(f) - f) \\ &= \chi(\tau''(f) - f),\end{aligned}$$

hence

$$\begin{aligned}|\chi \circ \tau''(f) - \chi(f)| = |\chi(\tau''(f) - f)| &= |\chi(gg'(\tau''(f) - f))| \\ &\leq \|gg'(\tau''(f) - f)\|_\infty \leq \varepsilon,\end{aligned}$$

and since $\varepsilon > 0$ and $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ were arbitrary, we obtain $\chi = \chi \circ \tau''$. It now follows from the definition of τ'' that $w + (V \cap V') \in \mathcal{P}_\chi$, as desired.

This concludes our proof of (c), and together with (a), this implies that $\mathcal{P}_\chi$ is a filter. In fact, under the assumption that (a) holds, (c) is equivalent to the statement that $\mathcal{P}_\chi$ is a filter.

(d) For each $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, the filter $\mathcal{P}_\chi$ has a minimum.

Fix χ, and let $w + V \in \mathcal{P}_\chi$ be an element such that $\dim V$ is minimal. Note that this is possible since $\mathbb{R}^n \in \mathcal{P}_\chi$, so $\mathcal{P}_\chi$ is nonempty. Now for any $w' + V' \in \mathcal{P}_\chi$, we have $(w + V) \cap (w' + V') \in \mathcal{P}_\chi$. As in (c), we may assume without loss of generality that $w = w'$, so that $(w + V) \cap (w' + V') = w + (V \cap V')$. Clearly, $\dim(V \cap V') \leq \dim V$, and since the dimension of V was assumed to be minimal, it follows that $\dim(V \cap V') = \dim V$. Since $V \cap V' \subseteq V$ and V is finite dimensional, we obtain $V \cap V' = V$. This is of course equivalent to $V \subseteq V'$, which implies that $w + V \subseteq w + V' = w' + V'$, so $w + V$ is the minimum of $\mathcal{P}_\chi$.

(2) Let $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$, let $w + V := \operatorname{supp}(\chi)$ (we assume that $w \in V^\perp$), and let $\mathcal{W}^0_{\mathcal{R}}(w + V) = \mathcal{W}^0(w + V) \oplus I_{w+V}$ be the decomposition obtained in Theorem 6.20. Suppose for the sake of contradiction that χ does not vanish on I_{w+V}. Then $I_{w+V} \neq \{0\}$, so $V \neq \{0\}$, and there exists a generator of I_{w+V} that is not mapped to 0 by $\chi \circ \rho_{w+V}$. Since the generators of I_{w+V} are of the form

$$h = h'_{U,\xi,g} \colon w + V \to \mathbb{C}, \quad p \mapsto e^{i\xi \cdot p} g \circ r_U(p),$$

for some nonzero subspace $U \subseteq V$, some $\xi \in U^\perp \cap V$ and some $g \in \mathcal{S}(U)$, we may fix such a generator h along with its corresponding U, ξ and g. Without loss of generality, we may assume that the dimension of U is maximal with respect to the above property, and that $\xi = 0$.

Consider the C*-algebra A generated by the functions $f \in \mathcal{W}^0_{\mathcal{R}}(w+V)$ with the property that the map

$$V \to \mathbb{C}, \quad p \mapsto f(w + p),$$

factors through $V/(U^\perp \cap V)$, and that the resulting map $\tilde{f} \colon V/(U^\perp \cap V) \to \mathbb{C}$ is an element of $C_0(V/(U^\perp \cap V))$. Then A is isomorphic to $C_0(U)$; the canonical isomorphism is given by

$$A \to C_0(U), \quad f \mapsto (p \mapsto \tilde{f}(p + (U^\perp \cap V))),$$

and A contains the generator h. Thus the character $\chi \circ \rho_{w+V}$ on $\mathcal{W}^0_{\mathcal{R}}(w + V)$ restricts to a character on A, and we can push it forward along the above isomorphism to obtain a character $\tilde{\chi}$ on $C_0(U)$. Since U is a locally compact Hausdorff space, it follows from the Gelfand–Naimark theorem that $\tilde{\chi} = \delta_u$ for some $u \in U$, where δ_u denotes the evaluation of a function at u.

Let $w' := w+u$, and define sets $X^{w+V}_{w'+U}$, $Y^{w+V}_{w'+U}$ and $Z^{w+V}_{w'+U}$ of generators of $\mathcal{W}^0_{\mathcal{R}}(w + V)$ as in Lemma 6.28 and Remark 6.30. We claim that

$$\chi \circ \rho_{w+V}(h') = \chi \circ \rho_{w'+(U^\perp \cap V)} \circ \sigma^{w+V}_{w'+(U^\perp \cap V)}(h'), \tag{6.1}$$

for each $h' \in X^{w+V}_{w'+U} \cup Y^{w+V}_{w'+U} \cup Z^{w+V}_{w'+U}$. We prove this statement in three steps:

(a) Equation (6.1) *holds for each* $h' \in X^{w+V}_{w'+U}$.

Indeed, let $h' \in X^{w+V}_{w'+U}$. Since $h \in A$, we have

$$h(w') = \chi \circ \rho_{w+V}(h) \neq 0,$$

and since $hh' \in A$, we see that

$$\begin{aligned} h(w')h'(w') = (hh')(w') = \chi \circ \rho_{w+V}(hh') &= \chi \circ \rho_{w+V}(h) \cdot \chi \circ \rho_{w+V}(h') \\ = h(w')\chi \circ \rho_{w+V}(h'), \end{aligned}$$

for each generator $h' \in X^{w+V}_{w'+U}$. Since $h(w') \neq 0$, it follows that

$$\chi \circ \rho_{w+V}(h') = h'(w'),$$

and from the fact that h' is constant on $w' + (U^\perp \cap V)$, we conclude that equation (6.1) holds for each $h' \in X^{w+V}_{w'+U}$.

(b) Equation (6.1) *holds for each* $h' \in Y^{w+V}_{w'+U}$.

First, let $h' = h'_{U',\xi',g'}$ be any generator of $\mathcal{W}^0_{\mathcal{R}}(w+V)$, and suppose that $\chi \circ \rho_{w+V}(h') \neq 0$. Then $\chi \circ \rho_{w+V}(hh') \neq 0$. Clearly, $\dim(U) \leq \dim(U+U')$, and since hh' is a generator that is Schwartz on $U + U'$, maximality of the dimension of U implies $\dim(U) = \dim(U + U')$, which in turn implies $U = U + U'$, hence $U' \subseteq U$.

Now let $h' = h'_{U',\xi',g'} \in Y^{w+V}_{w'+U}$. Then by contraposition we obtain $\chi \circ \rho_{w+V}(h') = 0$. Moreover, we have

$$\begin{aligned} \rho_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)}(h')(p) &= h'_{U',0,g'}(r_{U^\perp\cap V}(p) + w') \\ &= g' \circ r_{U'}(r_{U^\perp\cap V}(p) + w'), \end{aligned}$$

for each $p \in \mathbb{R}^n$. Now let

$$U'' := \ker(r_{U'} \circ r_{U^\perp\cap V})^\perp \cap V,$$

and let

$$g'' \colon U'' \to \mathbb{C}, \quad p \mapsto g' \circ r_{U'}(r_{U^\perp\cap V}(p) + w').$$

Then $g'' \in \mathcal{S}(U'')$. Suppose for the sake of contradiction that $U'' \subseteq U$. Then

$$U^\perp \subseteq \ker(r_{U'} \circ r_{U^\perp\cap V}) + V^\perp = \ker(r_{U'} \circ r_{U^\perp\cap V}),$$

and the right-hand side also contains U and is a linear subspace of $\mathbb{R}^n$, hence it is equal to $\mathbb{R}^n$. This implies

$$U' = U' \cap V \subseteq (U^\perp \cap V)^\perp \cap V = (U + V^\perp) \cap V = U,$$

which contradicts $U' \nsubseteq U$ (which holds true since $h' \in Y^{w+V}_{w'+U}$). Thus $U'' \nsubseteq U$, and this fact together with our computations above implies

$$\rho^{w+V}_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)}(h') = h_{U'',0,g''} \in Y^{w+V}_{w'+U},$$

so that, by the same argument as for h', we have

$$\begin{aligned}
&\chi \circ \rho_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)}(h') \\
&\quad = \chi \circ \rho_{w+V} \circ \rho^{w+V}_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)}(h') = 0.
\end{aligned}$$

Thus equation (6.1) also holds for each $h' \in Y^{w+V}_{w'+U}$.

(c) Equation (6.1) *holds for each* $h' \in Z^{w+V}_{w'+U}$.

This statement follows from the fact that each $h' \in Z^{w+V}_{w'+U}$ is constant on $w'' + U$ for each $w'' \in w' + (U^\perp \cap V)$.

Since equation (6.1) holds for each $h' \in X^{w+V}_{w'+U} \cup Y^{w+V}_{w'+U} \cup Z^{w+V}_{w'+U}$, since this set generates $\mathcal{W}^0_{\mathcal{R}}(w+V)$ by Lemma 6.28, and since both sides are *-homomorphisms, we obtain

$$\chi \circ \rho_{w+V} = \chi \circ \rho_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)}.$$

It follows that

$$\begin{aligned}
\chi \circ \rho_{w'+(U^\perp\cap V)} \circ \sigma_{w'+(U^\perp\cap V)} &= \chi \circ \rho_{w'+(U^\perp\cap V)} \circ \sigma^{w+V}_{w'+(U^\perp\cap V)} \circ \sigma_{w+V} \\
&= \chi \circ \rho_{w+V} \circ \sigma_{w+V} = \chi,
\end{aligned}$$

hence $w' + (U^\perp \cap V) \in \mathcal{P}_\chi$. But $U \neq \{0\}$ by assumption, so $U^\perp \cap V$ is a proper subspace of V, hence $w' + (U^\perp \cap V) \subset w + V$, which contradicts the assumption that $w + V$ is the minimum of $\mathcal{P}_\chi$. Thus χ vanishes on I_{w+V}.

(3) By construction of $\chi := \chi_{V,w,\zeta}$, we have $\chi = \chi_{w+V} \circ \sigma_{w+V}$ for some character $\chi_{w+V} \in \mathcal{W}^0_{\mathcal{R}}(w+V)$. It follows that

$$\chi \circ \rho_{w+V} \circ \sigma_{w+V} = \chi_{w+V} \circ \sigma_{w+V} \circ \rho_{w+V} \circ \sigma_{w+V} = \chi_{w+V} \circ \sigma_{w+V} = \chi,$$

hence $w + V \in \mathcal{P}_\chi$.

It remains to be shown that $w+V$ is the minimum of $\mathcal{P}_\chi$. Suppose for the sake of contradiction that it is not, i.e., there exists $w' + V' \in \mathcal{P}_\chi$ such that $w' + V' \subset w + V$. Then $\chi = \chi \circ \rho_{w'+V'} \circ \sigma_{w'+V'}$, from which it follows that $\chi \circ \rho_{w'+V'}$ is nonzero, so it is a character on $\mathcal{W}^0_{\mathcal{R}}(w'+V')$. This implies $\chi \circ \rho_{w'+V'}(1) = 1$, where 1 on the left-hand side of this equation denotes the constant function on $w' + V'$ that is equal to 1 everywhere.

Now let $U := V \cap (V')^\perp$. Then $U \neq \{0\}$ since $V' \subset V$. Fix a function $g_0 \in \mathcal{S}(U) \subset \mathcal{W}^0_{\mathcal{R}}(U)$ such that $g_0(0) = 1$. Pulling it back along the translation by $-w'$, we obtain a function $g_1 \in \mathcal{W}^0_{\mathcal{R}}(w' + U)$ such that $g_1(w') = 1$. Next, we set $g := \rho_{w'+U}(g_1)$. Then $g \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, and $g \equiv 1$ on $w' + V'$, so on the one hand, we have

$$\chi(g) = \chi \circ \rho_{w'+V'} \circ \sigma_{w'+V'}(g) = 1. \tag{6.2}$$

On the other hand, note that

$$g = \rho_{w'+U}(g_1) = \rho_{w+V} \circ \rho^{w+V}_{w'+U}(g_1),$$

Now recall Theorem 6.20, which says that we have a decomposition $\mathcal{W}^0_{\mathcal{R}}(w+V) = \mathcal{W}^0(w+V) \oplus I_{w+V}$, and its corollary, which says that the decomposition induces a canonical map $P\colon \mathcal{W}^0_{\mathcal{R}}(w+V) \to \mathcal{W}^0(w+V)$ with kernel I. By our construction of g_1, we have $\rho^{w+V}_{w'+U}(g_1) \in I$. Moreover, the character $\chi_{w+V}\colon \mathcal{W}^0_{\mathcal{R}}(w+V) \to \mathbb{C}$ is by definition of the form $\chi^0_{w+V} \circ P$, where χ^0_{w+V} is a character on $\mathcal{W}^0(w+V)$. It follows that

$$\chi(g) = \chi_{w+V} \circ \sigma_{w+V} \circ \rho_{w+V} \circ \rho^{w+V}_{w'+U}(g_1) = \chi^0_{w+V} \circ P \circ \rho^{w+V}_{w'+U}(g_1) = 0,$$

which contradicts equation (6.2). We conclude that such a $w' + V'$ does not exist, so $w + V$ is the minimum of $\mathcal{P}_\chi$. ■

It is now straightforward to prove the following theorem, which is the main result of this section.

6.31 Theorem. *The map*

$$\Omega^0_{\mathcal{R},n} \to \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)), \quad (V, w, \zeta) \mapsto \chi_{V,w,\zeta},$$

is a bijection.

Proof. Let us call the map in the statement of the theorem F. We first prove that F is injective. Let $(V, w, \zeta), (V', w', \zeta') \in \Omega^0_{\mathcal{R},n}$, let $\chi := \chi_{V,w,\zeta}$, let $\chi' := \chi_{V',w',\zeta'}$, and suppose that $\chi = \chi'$. By part (3) of Proposition 6.29, we have

$$w + V = \operatorname{supp}(\chi) = \operatorname{supp}(\chi') = w' + V',$$

which is equivalent to $V = V'$ and $w = w'$. Now let $P\colon \mathcal{W}^0_{\mathcal{R}}(w + V) \to \mathcal{W}^0(w + V)$ be the map from Corollary 6.22, and let $\tau_w\colon \mathcal{W}^0(w + V) \to \mathcal{W}^0(V)$ be the pullback of translation by w. Then

$$\zeta \circ \tau_w \circ P \circ \sigma_{w+V} = \chi = \chi' = \zeta' \circ \tau_w \circ P \circ \sigma_{w+V},$$

and since each of the maps τ_w, P and σ_{w+V} is surjective, we obtain $\zeta = \zeta'$, which proves that F is injective.

Next, we prove that F is surjective. Let $\chi \in \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ be arbitrary. Let $w + V := \operatorname{supp}(\chi)$. We may assume without loss of generality that w is the unique element in this equivalence class such that $w \in V^\perp$. Moreover, let I, P and τ_w be as in the previous paragraph. By part (2) of Proposition 6.29, the character $\chi \circ \rho_{w+V}$ vanishes on I, so the map $\chi^0 := \chi \circ \rho_{w+V}|_{\mathcal{W}^0(w+V)}$ is a character on $\mathcal{W}^0(w + V)$ that satisfies $\chi^0 \circ P = \chi \circ \rho_{w+V}$. It follows that $\zeta := \chi^0 \circ \tau_w^{-1}$ is a character on $\mathcal{W}^0(V)$, which we may regard as an element of bV. We now have $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, and

$$\chi_{V,w,\zeta} = \zeta \circ \tau_w \circ P \circ \sigma_{w+V} = \chi^0 \circ P \circ \sigma_{w+V} = \chi \circ \rho_{w+V} \circ \sigma_{w+V} = \chi,$$

so F is indeed surjective. ∎

6.32 Remark. A curious consequence of Theorem 6.31 is that for each $x \in \mathbb{R}^n$, there exists a different extension of the character

$$\delta_x\colon \mathcal{W}^0(\mathbb{R}^n) \to \mathbb{C}, \quad f \mapsto f(x),$$

to $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ for each subspace $V \subseteq \mathbb{R}^n$, namely the character $\chi_{V,w,\zeta}$ with

$$(V, w, \zeta) = (V, r_{V^\perp}(x), \iota_V \circ r_V(x))$$

where ι_V denotes the canonical map from V to its Bohr compactification bV.

6.7 The topology on the Gelfand spectrum

Now that we know what the characters on $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ are, we wish to give an alternative characterisation of the natural topology on $\Omega^0_{\mathcal{R},n}$, i.e., the initial topology with respect to the weak*-topology on $\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ and the bijection in Theorem 6.31. We will first construct a topology on $\Omega^0_{\mathcal{R},n}$, and show how it is related to the Lawson topology from order theory. After proving some of the properties of $\Omega^0_{\mathcal{R},n}$ endowed with this topology, we prove that it coincides with the aforementioned initial topology.

We proceed with the construction. First, let $\mathcal{P}_0$ be the set of linear subspaces of $\mathbb{R}^n$. Inclusion of sets defines a partial order on $\mathcal{P}_0$, and thus turns $\mathcal{P}_0$ into a small category whose objects are the elements of $\mathcal{P}_0$, and whose morphisms are pairs $(V, V') \in \mathcal{P}_0 \times \mathcal{P}_0$ with the property that $V \subseteq V'$.

Next, we construct a functor $\mathcal{F}$ from $\mathcal{P}_0$ to the category of locally compact abelian groups as follows:

- For each object $V \in \mathcal{P}_0$, let $\mathcal{F}(V) := V^\perp \times bV$;
- For each morphism (V, V'), let

  $$\begin{aligned}\mathcal{F}(V, V') \colon V^\perp \times bV &\to (V')^\perp \times bV',\\ (w, \zeta) &\mapsto (r_{(V')^\perp}(w), \iota_{V,V'}(\zeta) + \iota_{V',bV'} \circ r_{V'}(w)),\end{aligned}$$

  where maps of the form r_W with $W \in \mathcal{P}_0$ denote both the orthogonal projection of $\mathbb{R}^n$ onto W, as well as its canonically induced map $b\mathbb{R}^n \to bW$ between Bohr compactifications. Moreover, $\iota_{V',bV'}$ denotes the canonical map $V' \to bV'$, and $\iota_{V,V'}$ denotes the map $bV \hookrightarrow bV'$ that is canonically induced by the inclusion map $V \hookrightarrow V'$. Regarding the spaces $\mathcal{F}(V)$ and $\mathcal{F}(V')$ as subsets of $b\mathbb{R}^n$, we see that $\mathcal{F}(V, V')$ is the natural inclusion of the former subset into the latter.

6.33 Remark. Note that the group multiplication in bV' has been denoted by $+$; we are justified in doing so by virtue of the fact that bV' is abelian, which follows from the fact that it contains a dense abelian subgroup. It is worth noting however that when realised as a Pontryagin dual, the group multiplication is given by pointwise multiplication of group characters; we will use this fact in the proof of Theorem 6.41 below.

It is readily checked that $\mathcal{F}$ is indeed a functor. As a set, we have

$$\Omega^0_{\mathcal{R},n} = \bigsqcup_{V \in \mathcal{P}_0} \mathcal{F}(V),$$

and we use this to identify the images of the elements of $\mathcal{P}_0$ under $\mathcal{F}$ as subsets of $\Omega^0_{\mathcal{R},n}$. Moreover, the functor $\mathcal{F}$ induces a partial order $\leq$ on $\Omega^0_{\mathcal{R},n}$ as follows:

$$(V, w, \zeta) \leq (V', w', \zeta') \Leftrightarrow V \subseteq V' \text{ and } \mathcal{F}(V, V')(w, \zeta) = (w', \zeta').$$

It is now convenient to recall some basic notions from order theory:

6.34 Definition. Let $(X, \leq)$ be a partially ordered set (poset). Let $Y \subseteq X$.

- The set Y is said to be a *lower set in X* iff for each $x \in X$, if there exists a $y \in Y$ such that $x \leq y$, then $x \in Y$;
- The set
$$\downarrow\! Y := \{x \in X \mid \exists\, y \in Y : x \leq y\},$$
is the lower set generated by Y. In particular, if $Y = \{y\}$ for some $y \in X$, then one writes $\downarrow\! y$ instead of $\downarrow\!\{y\}$. Lower sets generated by singletons are called *principal ideals*;
- We define
$$\Downarrow\! Y := \{x \in \downarrow\! Y \mid \forall y \in Y : (x \leq y \vee y \leq x) \Rightarrow x < y\}.$$
Thus $\Downarrow\! Y$ consists of the elements of X that are strictly below Y.

The dual notions of *upper set* and *principal filter* and the notation associated with it can be obtained by replacing the symbols $\leq$, $<$, $\downarrow$ and $\Downarrow$ by $\geq$, $>$, $\uparrow$ and $\Uparrow$, respectively.

We are now ready to define a topology on $\Omega^0_{\mathcal{R},n}$. For each $V \in \mathcal{P}_0$, endow $\{V\} \times \mathcal{F}(V)$ with the topology of $\mathcal{F}(V)$ using the obvious identification, and let $\mathcal{K}_V$ be the set of compact subsets of $\{V\} \times \mathcal{F}(V)$. Moreover, for each $x \in \{V\} \times \mathcal{F}(V) \subseteq \Omega^0_{\mathcal{R},n}$, let $\mathcal{N}_x$ be the set of open neighbourhoods in $\{V\} \times \mathcal{F}(V)$.

The main idea behind the definition of our topology is that for each $x = (V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, the sets of the form

$$\downarrow U \setminus \bigcup_{V' \in F} \downarrow K_{V'},$$

where

- $U \in \mathcal{N}_x$;
- $F \subseteq \Downarrow V$ is finite;
- For each $V' \in F$, we have $K_{V'} \in \mathcal{K}_{V'}$;

form a neighbourhood base of x.

Now let $\mathscr{C}$ be the set of subsets $C \subseteq \Omega^0_{\mathcal{R},n}$ with the following property:

$$\begin{aligned} &\forall V \in \mathcal{P}_0 \quad \forall x \in \{V\} \times \mathcal{F}(V) \colon \\ &\Bigg(\forall U \in \mathcal{N}_x \quad \forall \text{ finite } F \subseteq \Downarrow V \quad \forall (K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'} \colon \\ &\quad C \cap \downarrow U \not\subseteq \bigcup_{V' \in F} \downarrow K_{V'} \Bigg) \Rightarrow x \in C, \end{aligned}$$

i.e., if each element of the neighbourhood base of x has nonempty intersection with C, then $x \in C$. Note that if $F = \emptyset$, then following the convention in set theory, we define $\prod_{V' \in F} \mathcal{K}_{V'}$ to be the set containing the single map $\emptyset \to \emptyset$.

6.35 Proposition. *The set $\mathscr{C}$ is the set of closed subsets relative to some topology on X, i.e.,*

(1) $\emptyset, X \in \mathscr{C}$;

(2) $\mathscr{C}$ is closed under arbitrary intersections;

(3) $\mathscr{C}$ is closed under finite unions.

Proof.

(1) For each $V \in \mathcal{P}_0$ and each $x \in \{V\} \times \mathcal{F}(V)$, the statement

$$\forall U \in \mathcal{N}_x \quad \forall \text{finite } F \subseteq \Downarrow V \quad \forall (K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}:$$
$$\emptyset \cap \downarrow U \nsubseteq \bigcup_{V' \in F} \downarrow K_{V'},$$

is false (take $F = \emptyset$), so $\emptyset \in \mathscr{C}$, whereas the statement $x \in X$ is true, so $X \in \mathscr{C}$.

(2) Let $(C_i)_{i \in I}$ be a family of elements in $\mathscr{C}$, and let $C := \bigcap_{i \in I} C_i$. Let $V \in \mathcal{P}_0$, let $x \in \{V\} \times \mathcal{F}(V)$, and suppose that for each $U \in \mathcal{N}_x$, each finite $F \subseteq \Downarrow V$ and each $(K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}$, we have $C \cap \downarrow U \nsubseteq \bigcup_{V' \in F} \downarrow K_{V'}$. Now let $i \in I$, and fix such U, F, and $(K_{V'})_{V' \in F}$. Since $C \subseteq C_i$, it follows that $C_i \cap \downarrow U \nsubseteq \bigcup_{V' \in F} \downarrow K_{V'}$. Since U, F and $(K_{V'})_{V' \in F}$ were arbitrary, we have $x \in C_i$, and since i was arbitrary, this implies $x \in C$, so $C \in \mathscr{C}$.

(3) Let $(C_i)_{i=1}^m$ be a family of elements in $\mathscr{C}$, and let $C := \bigcup_{i=1}^m C_i$. Let $V \in \mathcal{P}_0$, let $x \in \{V\} \times \mathcal{F}(V)$, and suppose that $x \notin C$. Then for $i = 1, \ldots, m$, we have $x \notin C_i$, and since $C_i \in \mathscr{C}$, we infer that for $i = 1, \ldots, m$, there exist an open neighbourhood $U_i \in \mathcal{N}_x$, a finite set $F_i \subseteq \Downarrow V$ and a family of compact sets $(K_{V',i})_{V' \in F_i} \in \prod_{V' \in F_i} \mathcal{K}_{V'}$ such that $C_i \cap \downarrow U_i \subseteq \bigcup_{V' \in F_i} \downarrow K_{V',i}$.

Now let $U := \bigcap_{i=1}^m U_i$, let $F := \bigcup_{i=1}^m F_i$ and for each $V' \in F$, let

$$K_{V'} := \bigcup_{i=1}^m K_{V',i},$$

where it should be understood that $K_{V',i} = \emptyset$ if $V' \notin F_i$. Then $U \in \mathcal{N}_x$, and the set F is finite and satisfies $F \subseteq \Downarrow V$. In addition, each $K_{V'}$ is a compact subset of $\{V'\} \times \mathcal{F}(V')$, and we have

$$C \cap \downarrow U = \left(\bigcup_{i=1}^m C_i \right) \cap \left(\bigcap_{i=1}^m \downarrow U_i \right) \subseteq \bigcup_{i=1}^m C_i \cap \downarrow U_i \subseteq \bigcup_{i=1}^m \bigcup_{V' \in F_i} \downarrow K_{V',i}$$
$$= \bigcup_{V' \in F} \downarrow K_{V'}.$$

We conclude that $C \in \mathscr{C}$. ■

6.36 Definition. We call the topology $\{\Omega^0_{\mathcal{R},n} \backslash C \colon C \in \mathscr{C}\}$ on $\Omega^0_{\mathcal{R},n}$ the *generalised inverted Lawson topology*, or *GIL topology* on $\Omega^0_{\mathcal{R},n}$. The GIL topology will be denoted by τ_{GIL}.

6.37 Remark.

(1) Identifying the set $\mathcal{P}$ from the previous subsection with the set

$$\{(V, w) \colon V \text{ is a subspace of } \mathbb{R}^n,\ w \in V^\perp\}, \tag{6.3}$$

we can define a functor, a partial order, and consequently, the GIL topology on $\mathcal{P}$ by removing all components in the constructions above related to the Bohr compactification. In this case, $\mathcal{F}(V) = V^\perp$, and the map $\mathcal{F}(V, V')$ is simply the orthogonal projection $r_{(V')^\perp} \colon V^\perp \to (V')^\perp$.

In the remaining part of this subsection, we will prove statements that are true for both $\Omega^0_{\mathcal{R},n}$ and $\mathcal{P}$, endowed with their respective GIL topologies, but we shall only prove them for $\Omega^0_{\mathcal{R},n}$; the proofs for $\mathcal{P}$ can be readily obtained from these.

(2) To motivate the name of our topology, we first recall the notion of the *Scott topology* on a poset $(X, \leq)$; this is the topology whose closed sets are lower sets $C \subseteq X$ with the property that if $Y \subseteq C$ is an upwards directed subset in C that has a (unique) supremum in X, then $\sup Y \in C$. The *Lawson topology* on $(X, \leq)$ is the smallest topology with respect to which the Scott closed subsets of $(X, \leq)$ and the principal filters in X are closed. For more information on these topologies, we refer to [43].

If we reverse the order in the above definitions, then we see that the upper sets from part (2) of Proposition 6.40 below are Scott closed subsets. Indeed, since $(\mathcal{P}_0, \subseteq)$ is an Artinian poset, sets in $(\Omega^0_{\mathcal{R},n}, \leq)$ are closed under infima of downwards directed subsets. This holds in particular for the upper sets in part (2) of Proposition 6.40. Furthermore, the sets of the form $\downarrow Y$ from part (4) are reminiscent of the principal filters that one has to include to obtain the Lawson topology. The sets mentioned in part (7) of Proposition 6.40 can be obtained as finite intersections of complements of these sets. Since the sets from part (7) form a base of τ_{GIL}, the complements of the sets from parts (2) and (4) together constitute a subbase of the topology.

6.38 Definition. Let $(X, \leq)$ be a poset, and let

$$D := \{(x, y) \in X \times X : \downarrow x \cap \downarrow y \neq \emptyset\}.$$

We say that $(X, \leq)$ *admits the partial meet operation* iff for each $(x, y) \in D$, the set $\downarrow x \cap \downarrow y$ has a maximum. In that case, we define the partial meet operation $\wedge$ to be the map that assings to a pair $(x, y) \in D$ the maximum $x \wedge y$ of $\downarrow x \cap \downarrow y$.

6.39 Lemma. *Let $(X, \leq)$ be either of the posets $(\Omega^0_{\mathcal{R},n}, \leq)$ or $(\mathcal{P}, \leq)$, and let $\mathcal{F}$ be its associated functor. Then $(X, \leq)$ admits the partial meet operation. Furthermore, for each $V_1, V_2 \in \mathcal{P}_0$, the map*

$$\begin{aligned} \mathcal{F}(V_1 \cap V_2) &\to \mathcal{F}(V_1) \times \mathcal{F}(V_2), \\ x &\mapsto (\mathcal{F}(V_1 \cap V_2, V_1)(x), \mathcal{F}(V_1 \cap V_2, V_2)(x)), \end{aligned}$$

is continuous and proper.

Proof. We only prove the statement for $(X, \leq) = (\Omega^0_{\mathcal{R},n}, \leq)$; the proof for $(\mathcal{P}, \leq)$ can be obtained by disregarding the third component of the triples in $\Omega^0_{\mathcal{R},n}$.

Let $x_1 = (V_1, w_1, \zeta_1), x_2 = (V_2, w_2, \zeta_2) \in X$, and suppose that $\downarrow x_1 \cap \downarrow x_2$ contains an element $y = (V, w, \zeta)$. We claim that

$$x := (V_1 \cap V_2, r_{(V_1 \cap V_2)^\perp}(w), \iota_{V, V_1 \cap V_2}(\zeta) + \iota_{V_1 \cap V_2, b(V_1 \cap V_2)} \circ r_{V_1 \cap V_2}(w)),$$

is the maximum of $\downarrow x_1 \cap \downarrow x_2$. It is easy to see that $x \in \downarrow x_1 \cap \downarrow x_2$. Now let $y' = (V', w', \zeta') \in \downarrow x_1 \cap \downarrow x_2$.

- Since $V' \subseteq V_1, V_2$, we have $V' \subseteq V_1 \cap V_2$;
- To see that

  $$r_{(V_1 \cap V_2)^\perp}(w) = r_{(V_1 \cap V_2)^\perp}(w'), \tag{6.4}$$

  first note that

  $$\begin{aligned} (w_1, w_2) &= (r_{V_1^\perp}(w), r_{V_2^\perp}(w)) \\ &= (r_{V_1^\perp} \circ r_{(V_1 \cap V_2)^\perp}(w), r_{V_2^\perp} \circ r_{(V_1 \cap V_2)^\perp}(w)), \end{aligned}$$

and the same identity holds if we replace y by y' and w by w'. Moreover, observe that the linear map

$$f_1\colon (V_1 \cap V_2)^\perp \to V_1^\perp \times V_2^\perp, \qquad w'' \mapsto (r_{V_1^\perp}(w''), r_{V_2^\perp}(w'')), \tag{6.5}$$

is injective; indeed, if w'' is an element of the kernel, then

$$w'' \in (V_1^\perp)^\perp \cap (V_2^\perp)^\perp = V_1 \cap V_2,$$

and since $w'' \in (V_1 \cap V_2)^\perp$, it follows that $w'' = 0$. Injectivity of the above map now implies that equation (6.4) holds.

- Note that

$$\begin{aligned}\zeta_1 &= \iota_{V,V_1}(\zeta) + \iota_{V_1,bV_1} \circ r_{V_1}(w)\\ &= \iota_{V_1\cap V_2,V_1} \circ \iota_{V,V_1\cap V_2}(\zeta) + \iota_{V_1\cap V_2,V_1} \circ \iota_{V_1\cap V_2,b(V_1\cap V_2)} \circ r_{V_1\cap V_2}(w)\\ &\quad + \iota_{V_1,bV_1} \circ r_{V_1} \circ r_{(V_1\cap V_2)^\perp}(w),\end{aligned}$$

and that the same identity holds if we replace the components of y by those of y'. Applying the map $r_{V_1\cap V_2}\colon bV_1 \to b(V_1 \cap V_2)$ to these equations, we obtain

$$\begin{aligned}&\iota_{V,V_1\cap V_2}(\zeta) + \iota_{V_1\cap V_2,b(V_1\cap V_2)} \circ r_{V_1\cap V_2}(w)\\ &\quad = r_{V_1\cap V_2}(\zeta_1) = \iota_{V',V_1\cap V_2}(\zeta') + \iota_{V_1\cap V_2,b(V_1\cap V_2)} \circ r_{V_1\cap V_2}(w').\end{aligned}$$

Thus we have $y' \leq x$, so x is indeed the maximum of $\downarrow x_1 \cap \downarrow x_2$, which proves the first assertion.

Let f be the map $\mathcal{F}(V_1 \cap V_2) \to \mathcal{F}(V_1) \times \mathcal{F}(V_2)$ as defined above. It is clear that f is continuous. To show that f is proper, it suffices to show that f has a continuous left inverse, g, say.

We construct g. We have already shown that the map f_1 in equation (6.5) is linear and injective. We can define a linear map $g_1\colon V_1^\perp \times V_2^\perp \to (V_1 \cap V_2)^\perp$ by requiring that it is the inverse of f_1 on the image of f_1, and that it vanishes on the orthogonal complement of the image of f_1. Since g_1 is a linear map between finite dimensional vector spaces, it is continuous. Now define g by

$$g\colon \mathcal{F}(V_1) \times \mathcal{F}(V_2) \to \mathcal{F}(V_1 \cap V_2)$$

$$((w_1, \zeta_1), (w_2, \zeta_2)) \mapsto (g_1(w_1, w_2), r_{V_1 \cap V_2}(\zeta_1)).$$

Then g is a continuous left inverse of f, as desired. ■

6.40 Proposition. *Let $(X, \leq)$ be either of the posets $(\Omega^0_{\mathcal{R},n}, \leq)$ or $(\mathcal{P}, \leq)$, let $\mathcal{F}$ be its associated functor, and endow X with the GIL topology. For each $V \in \mathcal{P}_0$, let $T_V \colon \mathcal{F}(V) \hookrightarrow X$ be the canonical inclusion map.*

(1) For each $V \in \mathcal{P}_0$, the map T_V is continuous. (Equivalently, the τ_{GIL} on $(X, \leq)$ is equal to or coarser than the disjoint union topology.)

Let $Y \subseteq X$.

(2) If Y is an upper set, and Y has the property that for each $V \in \mathcal{P}_0$, the set $T_V^{-1}(Y)$ is closed, then Y is closed.

Now suppose that Y is a set with the property that there exists a $V \in \mathcal{P}_0$ and a set $Y_0 \subseteq \mathcal{F}(V)$ such that $Y = T_V(Y_0)$.

(3) The set Y_0 is open if and only if $\downarrow Y$ is open;

(4) The following are equivalent:

(i) Y_0 is compact;

(ii) Y is closed in X;

(iii) $\downarrow Y$ is closed in X;

(5) X is Hausdorff;

(6) X is compact;

(7) The family of sets of the form

$$\downarrow U \setminus \bigcup_{V' \in F} \downarrow K_{V'},$$

form a base of τ_{GIL}, where $V \in \mathcal{P}_0$, the set $U \subseteq X$ is of the form $T_V(U_0)$ for some open subset $U_0 \subseteq \mathcal{F}(V)$, the set $F \subseteq \Downarrow V$ is finite, and $(K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}$ is a finite family of compact sets.

(8) *Let $V \in \mathcal{P}_0$, let $y \in \mathcal{F}(V)$, and let $x := T_V(y)$. Similarly, let $(V_i)_{i\in I}$ be a net in $\mathcal{P}_0$, and for each $i \in I$, let $y_i \in \mathcal{F}(V_i)$, and let $x_i := T_{V_i}(y_i)$. Then the following are equivalent:*

(i) *The net $(x_i)_{i\in I}$ converges to x with respect to τ_{GIL};*

(ii) *The nets $(V_i)_{i\in I}$ and $(y_i)_{i\in I}$ satisfy the following conditions:*

- *There exists an $i_0 \in I$ such that for each $i \geq i_0$, we have $V_i \subseteq V$;*
- *The net $(\mathcal{F}(V_i, V)(y_i))_{i\geq i_0}$ converges to y;*

Moreover, if x is replaced by an element $x' = T_{V'}(y') < x$, and the net $(x_i)_{i\in I}$ by a subnet $(x_j)_{j\in J}$ (and ditto for its corresponding nets $(V_j)_{j\in J}$ and $(y_j)_{j\in J}$), then the two above conditions are not both satisfied.

Proof.

(1) Let $Y \subseteq X$ be a closed subset, and let $(w,\zeta) \in \overline{T_V^{-1}(V)}$. Then each open neighbourhood of (w,ζ) has nonempty intersection with $T_V^{-1}(Y)$. Now let $x := (V,w,\zeta) = T_V(w,\zeta)$. Moreover, let $U \in \mathcal{N}_x$, let $F \subseteq \Downarrow V$ be finite, and let $(K_{V'})_{V'\in F} \in \prod_{V'\in F} \mathcal{K}_{V'}$ be any family (note that the product is always nonempty). Then we have

$$T_V^{-1}(Y \cap {\downarrow}U) = T_V^{-1}(Y) \cap T_V^{-1}(U) \neq \emptyset,$$

since $T_V^{-1}(U)$ is an open neighbourhood of (w,ζ), but

$$T_V^{-1}\left(\bigcup_{V'\in F} {\downarrow}K_{V'}\right) = \emptyset.$$

Thus

$$Y \cap {\downarrow}U \nsubseteq \bigcup_{V'\in F} {\downarrow}K_{V'}.$$

and from the closedness of Y, it follows that $x \in Y$, so $(w,\zeta) \in T_V^{-1}(Y)$. Hence we have shown that $T_V^{-1}(Y)$ is closed, and we conclude that T_V is continuous.

(2) Let Y be a set as in the statement of part (2) of the proposition, let $V \in \mathcal{P}_0$, and let $x = (V, w, \zeta) \in \{V\} \times \mathcal{F}(V)$ be an element such that for each $U \in \mathcal{N}_x$, for each finite subset $F \subseteq \Downarrow V$, and each family $(K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}$, we have

$$Y \cap \downarrow U \nsubseteq \bigcup_{V' \in F} \downarrow K_{V'}.$$

Taking $F = \emptyset$, we see that the left-hand side is nonempty, so we may fix an element $y_0 \in Y \cap \downarrow U$, and there exists a unique $V_0 \in \mathcal{P}_0$ such that $y_0 = (V_0, w_0, \zeta_0)$, where $(w_0, \zeta_0) \in \mathcal{F}(V_0)$. Since $y_0 \in \downarrow U$, there exists an element $y_1 \in U$ such that $y_0 \leq y_1$, namely $y_1 = (V, \mathcal{F}(V_0, V)(w_0, z_0))$, and this element is unique by definition of the partial order. Because Y was assumed to be an upper set, we also have $y_1 \in Y$, so $y_1 \in Y \cap U$. Thus each open neighbourhood $U_0 \subseteq \mathcal{F}(V)$ of (w, ζ) has nonempty intersection with $T_V^{-1}(Y)$. Because $T_V^{-1}(Y)$ was assumed to be closed, it follows that $(w, \zeta) \in T_V^{-1}(Y)$, so $x = T_V(w, \zeta) \in Y$. We conclude that Y is closed.

(3) The "if" part of the statement is a trivial consequence of part (1) of this proposition

We turn to the "only if" part. Let V and Y_0 be as in the statement of part (3), and suppose that Y_0 is open in $\mathcal{F}(V)$. By part (1) of the proposition and the fact that the map $\mathcal{F}(V', V)$ is continuous, we see that this implies that for each $V' \in \mathcal{P}_0$, the set

$$T_{V'}^{-1}(\downarrow Y) = \begin{cases} T_{V'}^{-1}(\mathcal{F}(V', V)^{-1}(\downarrow Y)) = (\mathcal{F}(V', V))^{-1}(Y_0) & \text{if } V' \subseteq V \\ \emptyset & \text{if } V' \nsubseteq V \end{cases},$$

is open in $\mathcal{F}(V')$.

Since $T_{V'}^{-1}(X \backslash \downarrow Y) = V' \backslash T_{V'}^{-1}(\downarrow Y)$, and since $X \backslash \downarrow Y$ is an upper set in X, we can apply part (2) of the proposition to see that $X \backslash \downarrow Y$ is closed, so $\downarrow Y$ is open.

(4) **(i) $\Rightarrow$ (iii):** Suppose Y_0 is compact. The same argument that showed that $T_{V'}^{-1}(\downarrow Y)$ is open in part (3) can be used in the present setting to show that $T_{V'}^{-1}(\downarrow Y)$ is closed for each $V' \in \mathcal{P}_0$.

Let $V_x \in \mathcal{P}_0$, let $x = (V_x, w_x, \zeta_x) \in X$, and suppose that for each $U \in \mathcal{N}_x$, each finite subset $F \subseteq \Downarrow V_x$, and each family $(K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}$, we have

$$\downarrow Y \cap \downarrow U \nsubseteq \bigcup_{V' \in F} \downarrow K_{V'}. \tag{6.6}$$

Fix such a U. Since $\mathcal{F}(V_x)$ is a locally compact Hausdorff space, we may assume without loss of generality that $U_0 := T_{V_x}^{-1}(U)$ has compact closure.

We claim that $T_{V \cap V_x}^{-1}(\downarrow Y \cap \downarrow U)$ has compact closure. To see this, we note that

$$\begin{aligned}
&T_{V \cap V_x}^{-1}(\downarrow Y \cap \downarrow U) \\
&\quad = \mathcal{F}(V \cap V_x, V)^{-1}(T_V^{-1}(\downarrow Y)) \cap \mathcal{F}(V \cap V_x, V_x)^{-1}(T_{V_x}^{-1}(\downarrow U)) \\
&\quad = \mathcal{F}(V \cap V_x, V)^{-1}(Y_0) \cap \mathcal{F}(V \cap V_x, V_x)^{-1}(U_0) \\
&\quad = f^{-1}(Y_0 \times U_0),
\end{aligned}$$

where

$$\begin{aligned}
f \colon \mathcal{F}(V \cap V_x) &\to \mathcal{F}(V) \times \mathcal{F}(V_x), \\
(w', \zeta') &\mapsto (\mathcal{F}(V \cap V_x, V)(w', \zeta'), \mathcal{F}(V \cap V_x, V_x)(w', \zeta')).
\end{aligned}$$

Since f is proper by Lemma 6.39, and $Y_0 \times \overline{U_0}$ is compact, the set $f^{-1}(Y_0 \times \overline{U_0})$ is a compact set containing $f^{-1}(Y_0 \times U_0)$, so $f^{-1}(Y_0 \times U_0) = T_{V \cap V_x}^{-1}(\downarrow Y \cap \downarrow U)$ has compact closure, K, say.

Returning to equation (6.6), we see that if $V \cap V_x \subset V_x$, then we can take $F = \{V \cap V_x\}$ and $K_{V \cap V_x} = T_{V \cap V_x}(K)$ to obtain a contradiction, so $V \cap V_x = V_x$, i.e., $V_x \subseteq V$.

Similarly, taking $F = \emptyset$, we see that there exist $y \in Y$ and $z \in U$ such that $\downarrow y \cap \downarrow z \neq \emptyset$. By Lemma 6.39, $y \wedge z = (V', w', \zeta')$ exists and is an element of $\downarrow Y \cap \downarrow U$. This implies $V' \subseteq V_x$, and $z = (V_x, \mathcal{F}(V', V_x)(w', \zeta'))$, and

$$y = (V, \mathcal{F}(V', V_x)(w', \zeta')) = (V, \mathcal{F}(V_x, V) \circ \mathcal{F}(V', V_x)(w', \zeta')),$$

from which we obtain $z \leq y$. This yields $y \wedge z = z$, hence $\downarrow Y \cap U \neq \emptyset$, and we infer that each open neighbourhood U_0 of (w_x, ζ_x) has nonempty intersection with $T_{V_x}^{-1}(\downarrow Y)$. Furthermore, $T_{V_x}^{-1}(\downarrow Y) = \mathcal{F}(V_x, V)^{-1}(Y_0)$ is closed, so $(w_x, \zeta_x) \in \mathcal{F}(V_x, V)^{-1}(Y_0)$, hence $x \in \downarrow Y$. We conclude that

$\downarrow Y$ is closed, as desired.

(iii) $\Rightarrow$ (ii): Fix Y, Y_0 and V. By part (2) of this proposition, the set $\bigcup_{V' \in \uparrow V} \{V'\} \times \mathcal{F}(V')$ is closed in X. It follows that the set

$$Y = \downarrow Y \cap \bigcup_{V' \in \uparrow V} \{V'\} \times \mathcal{F}(V'),$$

is closed in X.

(ii) $\Rightarrow$ (i): We show by induction on m that for $m = 0, 1, \ldots, n - \dim V$, there exist finite subsets $F_m \subseteq \{V' \in \uparrow V \colon \dim V' = n - m\}$ and compact sets $K'_V \in \mathcal{K}_{V'}$ for each $V' \in F_m$ such that

$$Y \subseteq \bigcup_{V' \in F_m} \downarrow K_{V'}.$$

For $m = 0$, we take $F_0 = \{\mathbb{R}^n\}$ and $K_{\mathbb{R}^n} = \{\mathbb{R}^n\} \times \{0\} \times b\mathbb{R}^n$.

Now suppose we have found F_m and $(K_{V'})_{V' \in F_m}$ for some $m \in \mathbb{N}$, $0 \leq m < n - \dim V$. Then for each $V' \in F_m$, we have $\dim V' > \dim V$, so the sets $K_{V'}$ have empty intersection with Y. Since Y is closed, for each $V' \in F_m$, we can use the axiom of choice to find open neighbourhoods $U_x \in \mathcal{N}_x$, finite sets $F_x \subseteq \Downarrow V' \cap \uparrow V$ satisfying $\dim V'' = n - (m+1)$ for each $V'' \in F_x$, and families of compact sets $(K_{x,V''})_{V'' \in F_x} \in \prod_{V'' \in F_x} \mathcal{K}_x$ such that

$$Y \cap \downarrow U_x \subseteq \bigcup_{V'' \in F_x} \downarrow K_{x,V''},$$

for each $x \in K_{V'}$. By compactness of $K_{V'}$, the open cover $(U_x)_{x \in K_{V'}}$ of $K_{V'}$ has a finite subcover $(U_{x_j})_{j=1}^k$. Now for each $V' \in F_m$, let

$$F_{m+1,V'} := \bigcup_{j=1}^k F_{x_j},$$

(note: the F_{x_j} depend on V' through $K_{V'}$) and for each $V'' \in F_{m+1,V'}$, let

$$F_{m+1,V',V''} := \bigcup_{j=1}^k K_{x_j,V''}.$$

As in the proof of part (3) of Proposition 6.35, if $V'' \notin F_{x_j}$, then it should be understood that $K_{x_j,V''} = \emptyset$. Finally, let

$$F_{m+1} := \bigcup_{V' \in F_m} F_{m+1,V'},$$

and for each $V'' \in F_{m+1}$, let

$$K_{V''} := \bigcup_{V' \in F_m} \bigcup_{K' \in F_{m+1,V',V''}} K'.$$

Then the set F_{m+1} is finite, and for each $V'' \in F_{m+1}$, the set $K_{V''}$ is compact. Furthermore, using the induction hypothesis, one can check that

$$Y \subseteq \bigcup_{V'' \in F_{m+1}} {\downarrow} K_{V''}.$$

This completes the induction step, and thereby our proof by induction.

Since by construction, each element $V' \in F_{\dim V}$ satisfies $V \subseteq V'$, and $\dim V = \dim V'$, we have in fact $V = V'$, so $F_{\dim V} \subseteq \{V\}$. Moreover, we have

$$Y_0 \subseteq \begin{cases} T_V^{-1}(K') & \text{if } V \in F_{\dim V} \\ \emptyset & \text{if } V \notin F_{\dim V} \end{cases},$$

where K' denotes the unique element in $F_{\dim V,V}$, so Y_0 is contained in a compact set. It follows from part (1) that $Y_0 = T_V^{-1}(Y)$ is closed in $\mathcal{F}(V)$, so Y_0 is compact, as desired.

(5) Let $x, y \in X$, and suppose that $x \neq y$. Then $x \nleq y$ or $y \nleq x$. Assume $x \nleq y$; the other case is similar. We know that $x \in \{V_x\} \times \mathcal{F}(V_x)$, and that $y \in \{V_y\} \times \mathcal{F}(V_y)$ for some $V_x, V_y \in \mathcal{P}_0$. Since $x \nleq y$, the set ${\uparrow}x$ does not contain y. Moreover, ${\uparrow}x$ intersects the sets of the form $\{V\} \times \mathcal{F}(V)$ only if $V_x \subseteq V \in \mathcal{P}_0$, and the intersections contain a single element. Using the fact that $\mathcal{F}(V)$ is Hausdorff for each $V \in \mathcal{P}_0$, we can now fix an open set $U \in \mathcal{F}(V_y)$ such that $T_{V_y}(U) \cap {\uparrow}x = \emptyset$, and since $\mathcal{F}(V)$ is also locally compact, we may assume that $\overline{U}$ is compact, and that $T_{V_y}(\overline{U}) \cap {\uparrow}x = \emptyset$. Then by part (3), the set ${\downarrow}U$ is an open neighbourhood of y in X, and by part (4), the set $X \backslash {\downarrow}T_{V_y}(\overline{U})$ is an open neighbourhood of x in X. It is clear from the definition that these two neighbourhoods are disjoint, so we have shown that X is Hausdorff.

(6) Let Y_0, Y and V be as in part (4) of the proposition. We prove by induction on $m = \dim V$ that $\downarrow Y$ is compact in X. Suppose that we have proved this statement. Then we can take $Y_0 = \mathcal{F}(\mathbb{R}^n) = \{0\} \times b\mathbb{R}^n$, which is compact, and we have $\downarrow Y = X$, so this implies that X is compact.

It remains to carry out the induction. If $m = 0$, then we have $\downarrow Y = Y = T_{\{0\}}(Y_0)$, and part (1) implies that $T_{\{0\}}(Y_0)$ is compact.

Next, suppose we know the statement holds for some $m \in \{0, 1, \ldots, n-1\}$, and that $\dim V = m + 1$. Let $(C_i)_{i \in I}$ be a family of closed subsets of $\downarrow Y$ (with respect to the subspace topology) such that $\bigcap_{i \in I} C_i = \emptyset$. To prove compactness, we must show that there exists a finite subset $J \subseteq I$ such that $\bigcap_{i \in J} C_i = \emptyset$.

First, by part (4), the set $\downarrow Y$ is closed in X, so the sets $(C_i)_{i \in I}$ are also closed with respect to the topology on X. By part (1), the set $T_V^{-1}(C_i) \subseteq Y_0$ is closed in $\mathcal{F}(V)$ for each $i \in I$. Since $\bigcap_{i \in I} T_V^{-1}(C_i) = \emptyset$, and since Y_0 is compact, there exists a finite set $J_0 \subseteq I$ such that $\bigcap_{i \in J_0} T_V^{-1}(C_i) = \emptyset$.

Since $\bigcap_{i \in J_0} C_i$ is closed and has empty intersection with Y, we can invoke the axiom of choice to find open neighbourhoods $U_x \in \mathcal{N}_x$, finite sets $F_x \subseteq \Downarrow V$, and families of compact sets $(K_{x,V'})_{V' \in F_x} \in \prod_{V' \in F_x} \mathcal{K}_x$ such that

$$\bigcap_{i \in J_0} C_i \cap \downarrow U_x \subseteq \bigcup_{V' \in F_x} \downarrow K_{x,V'},$$

for each $x \in Y$, and we may assume that $\dim V' = m$ for each $V' \in F_x$. By compactness of Y, the open cover $(U_x)_{x \in Y}$ of Y has a finite subcover $(U_{x_j})_{j=1}^k$. Let $F := \bigcup_{j=1}^k F_{x_j}$, and for each $V' \in F$, let $K_{V'} := \bigcup_{j=1}^k K_{x_j,V'}$. (A remark on the definition of $K_{x_j,V'}$ similar to the ones in the proofs of part (3) of Proposition 6.35 and of part (4) of the current proposition applies here as well.) Then F is finite, and $K_{V'}$ is compact (or equivalently, $T_{V'}^{-1}(K_{V'})$ is compact) for each $V' \in F$. By the induction hypothesis, the set $\downarrow K_{V'}$ is compact for each $V' \in F$, so there exists a family of finite subsets $(J_{V'})_{V' \in F}$ of I such that $\bigcap_{i \in J_{V'}} C_i \cap \downarrow K_{V'} = \emptyset$. Now define

$$J := J_0 \cup \bigcup_{V' \in F} J_{V'}.$$

Then J is a finite subset of I, and it can be checked that $\bigcap_{i \in J} C_i = \emptyset$, as desired. This completes the induction step.

(7) The sets of the form

$$\downarrow U \backslash \bigcup_{V' \in F} \downarrow K_{V'},$$

are open in X since they are the difference of the set $\downarrow U$ which is open in X by part (3), and the set $\bigcup_{V' \in F} \downarrow K_{V'}$, which is closed in X by part (4). Since sets of this form were used to define neighbourhood bases of points in X, it is immediate that they constitute a base of τ_{GIL}.

(8) (i) ⇒ (ii): Suppose $(x_i)_{i \in I}$ converges to x. Let $U \in \mathcal{N}_x$. Then the set $\downarrow U$ is an open neighbourhood of x by part (3) of this proposition, so there exists an $i_U \in I$ such that $x_i \in \downarrow U$ for each $i \geq i_U$. In particular, taking $U = \{V\} \times \mathcal{F}(V)$, we see that with $i_0 := i_{\{V\} \times \mathcal{F}(V)}$, we have $V_i \subseteq V$ for each $i \geq i_0$. Furthermore, for any arbitrary $U \in \mathcal{N}_x$, we can now fix $i_U \in I$ so that $i_U \geq i_0$, which implies that $\mathcal{F}(V_i, V)(y_i)$ is well defined, and an element of $T_V^{-1}(U)$ for each $i \geq i_U$, hence $(\mathcal{F}(V_i, V)(y_i))_{i \geq i_0}$ converges to y. Thus the two conditions mentioned under (ii) are satisfied.

Now fix $x' \in \Downarrow x$, and let $V' \in \mathcal{P}_0$ and $y' \in \mathcal{F}(V')$ be the unique elements such that $x' = T_{V'}(y')$. In addition, find a $U \in \mathcal{N}_{x'}$ with compact closure with respect to the topology on $\mathcal{F}(V')$. Then by part (4) of the proposition, the set $X \backslash \downarrow \overline{U}$ is an open neighbourhood of x in X, so there exists an $i' \in I$ such that for each $i \geq i'$, we have $x_i \notin \downarrow \overline{U} \supseteq \downarrow U$. Thus at least one of the two conditions mentioned under (ii) is not satisfied if we replace x by x' and the net $(x_i)_{i \in I}$ by a subnet.

(ii) ⇒ (i): Suppose $(x_i)_{i \in I}$ satisfies both conditions with respect to some element $x \in X$, but no subnet of $(x_i)_{i \in I}$ satisfies this condition with respect to any element of $\Downarrow x$. It suffices to show that for each element Y of some neighbourhood base of x with respect to τ_{GIL}, there exists $i_Y \in I$ such that for each $i \geq i_Y$, we have $x_i \in Y$. By part (7) of this proposition, a neighbourhood base of x of open neighbourhoods is given by sets of the form

$$\downarrow U \backslash \bigcup_{V' \in F} \downarrow K_{V'},$$

where $U \in \mathcal{N}_x$, the set $F \subseteq \Downarrow V$ is finite, and $(K_{V'})_{V' \in F} \in \prod_{V' \in F} \mathcal{K}_{V'}$ is a finite family of compact sets.

Now fix such U, F, and $(K_{V'})_{V' \in F}$, let Y be the corresponding neighbourhood of x in X, and fix $i_0 \in I$ such that $V_i \subseteq V$ for each $i \geq i_0$. Then there exists an $i_U \geq i_0$ such that for each $i \geq i_U$, we have $\mathcal{F}(V_i, V)(x_i) \in U$, so $x_i \in {\downarrow}U$.

Suppose now for the sake of contradiction that there exists a $V' \in F$ such that the set $I_{V'} := \{i \in I \colon x_i \in {\downarrow}K_{V'}\}$ is cofinal in I. Then $(x_i)_{i \in I_{V'}}$ is a subnet of $(x_i)_{i \in I}$ contained in ${\downarrow}K_{V'}$. By part (4) of the proposition, the set ${\downarrow}K_{V'}$ is closed in X, and part (6) now implies that ${\downarrow}K_{V'}$ is compact, so the subnet $(x_i)_{i \in I_{V'}}$ has a subnet $(x_j)_{j \in J}$ that converges to some $x' \in K_{V'}$. We distinguish between the following two cases:

- $x' < x$: then by the implication (i) $\Rightarrow$ (ii), the net $(x_j)_{j \in J}$ satisfies both conditions in (ii), which contradicts our original assumption on $(x_i)_{i \in I}$;
- $x' \not< x$: then $\mathcal{F}(V', V)(x') \neq x$, and since $\mathcal{F}(V)$ is Hausdorff, we can find open neighbourhoods $W \in \mathcal{N}_x$ and $W' \in \mathcal{N}_{T_V \circ \mathcal{F}(V', V)(x')}$ such that $W \cap W' = \emptyset$, and it follows that ${\downarrow}W \cap {\downarrow}W' = \emptyset$. Just like we did for ${\downarrow}U$, we can argue that there exists an $i_W \in I$ such that for each $i \geq i_W$, we have $x_i \in {\downarrow}W$. The subset $Y' := {\downarrow}T_{V'}((T_V \circ \mathcal{F}(V', V))^{-1}(W')) \subseteq {\downarrow}W'$ is open in X by part (3) of this proposition, and from our discussion, we infer that it has the property that $x_i \notin Y'$ for each $i \geq i_W$. But this contradicts the statement that $(x_j)_{j \in J}$ converges to x'.

It follows that for each $V' \in F$, $I_{V'}$ is not cofinal in I, i.e., there exists an $i_{V'} \in I$ such that for each $i \geq i_{V'}$, we have $x_i \notin {\downarrow}K_{V'}$. Since I is an upwards directed set and F is finite, we can find an element $i_Y \in I$ such that $i_Y \geq i_U$ and $i_Y \geq i_{V'}$ for each $V' \in F$. Then for each $i \geq i_Y$, we have $x_i \in Y$, so $(x_i)_{i \in I}$ does indeed converge to x. ■

Note that the final part of the above proposition establishes that the topology on $\mathcal{P}$ as we have defined it in this subsection coincides with the topology in [112, Definition 5.2]. All that remains is to prove the main result, which is the analogue of [112, Theorem 5.6]. The next theorem can also be formulated and proved in the same way for $\mathcal{P}$, thus providing an alternative proof of the cited theorem.

6.41 Theorem. *Let τ_w be the weak topology on the Gelfand spectrum*

$\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. The map

$$(\Omega^0_{\mathcal{R},n}, \tau_{\mathrm{GIL}}) \to (\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)), \tau_w), \quad (V, w, \zeta) \mapsto \chi_{V,w,\zeta},$$

from Theorem 6.31 is a homeomorphism.

Proof. As in the proof of Theorem 6.31, the map will be denoted by F. The domain of F is compact by part (6) of Proposition 6.40, and the codomain of F is Hausdorff by definition of the weak*-topology. The map F is a bijection by Theorem 6.31, so to prove that F is a homeomorphism, it suffices to show that F is continuous. Since τ_w is by definition the coarsest topology on $\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ such that the map

$$\hat{f} \colon \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)) \to \mathbb{C}, \quad \chi \mapsto \chi(f),$$

is continuous for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, it suffices to show that the map $\hat{f} \circ F$ is continuous for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. In fact, since $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n) \subseteq C_b(\mathbb{R}^n)$, it suffices to show that $\hat{f} \circ F$ is continuous for each f in some set of generators of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

As the set of generators, we take the functions of the form $h_{U,\xi,g}$, where $h_{U,\xi,g}$ is defined as in Lemma 6.28. Let $(V, w, \zeta) \in \Omega^0_{\mathcal{R},n}$, and let $\mathcal{W}^0_{\mathcal{R}}(w + V) = \mathcal{W}^0(w + V) \oplus I_{w+V}$ be the decomposition obtained in Theorem 6.20. Then the following are equivalent:

- $h_{U,\xi,g}|_{w+V} \in \mathcal{W}^0(w + V)$;
- $V \subseteq \ker r_U$;
- $V \subseteq U^\perp$;

Similarly, the following are equivalent as well:

- $h_{U,\xi,g}|_{w+V} \in I_{w+V}$;
- $V \nsubseteq U^\perp$.

Thus for $V \subseteq U^\perp$, we have

$$\begin{aligned}
\zeta \circ \tau_w \circ P(h_{U,\xi,g}|_{w+V}) &= \zeta \circ \tau_w(h_{U,\xi,g}|_{w+V}) \\
&= \zeta(p \mapsto e^{i\xi\cdot(p+w)} g \circ r_U(p + w)) \\
&= g \circ r_U(w) \cdot \zeta(e_\xi|_V) \cdot e^{i\xi\cdot w} \\
&= g \circ r_U(w) \cdot \iota_{V,\mathbb{R}^n}(\zeta)(e_\xi) \cdot \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w)(e_\xi) \\
&= g \circ r_U(w) \cdot (\iota_{V,\mathbb{R}^n}(\zeta) + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w))(e_\xi) \\
&= g \circ r_U(w) \cdot \tilde{e}_\xi(\iota_{V,\mathbb{R}^n}(\zeta) + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w)),
\end{aligned}$$

where τ_w and P are as in the proof of Theorem 6.31, e_ξ denotes the group character $\mathbb{R}^n \to \mathbb{T}$, $p \mapsto e^{i\xi\cdot p}$, and $\tilde{e}_\xi$ is the associated map $b\mathbb{R}^n \to \mathbb{T} \subset \mathbb{C}$. Furthermore, in the last three lines, we regard elements in $b\mathbb{R}^n$ as group characters on the Pontryagin dual of $\mathbb{R}^n$, and we note that pointwise multiplication of such characters is the group operation in $b\mathbb{R}^n$, see Remark 6.33. It follows that for arbitrary $V \in \mathcal{P}_0$, we have

$$\begin{aligned} &\hat{h}_{U,\xi,g} \circ F(V, w, \zeta) \\ &\quad = \chi_{V,w,\zeta}(h_{U,\xi,g}) = \zeta \circ \tau_w \circ P(h_{U,\xi,g}|_{w+V}) \\ &\quad = \begin{cases} g \circ r_U(w) \cdot \tilde{e}_\xi(\iota_{V,\mathbb{R}^n}(\zeta) + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w)) & \text{if } V \subseteq U^\perp \\ 0 & \text{if } V \not\subseteq U^\perp \end{cases}. \end{aligned} \tag{6.7}$$

In particular, we can now read off that for each $V \in \mathcal{P}_0$, the function $\hat{h}_{U,\xi,g} \circ F \circ T_V$ is continuous.

Next, we show that for each subset $Z \subseteq \mathbb{C}\backslash\{0\}$, we have

$$(\hat{h}_{U,\xi,g} \circ F)^{-1}(Z) = \downarrow\left(\{U^\perp\} \times T^{-1}_{U^\perp}((\hat{h}_{U,\xi,g} \circ F)^{-1}(Z))\right). \tag{6.8}$$

Note that from equation (6.7), we immediately get $(\hat{h}_{U,0,g} \circ F)^{-1}(Z) \subseteq \downarrow(\{U^\perp\} \times \mathcal{F}(U^\perp))$.

To prove the inclusion $\supseteq$, we show that the set $(\hat{h}_{U,0,g} \circ F)^{-1}(Z)$ is a lower set. Suppose $(V, w, \zeta), (V', w', \zeta') \in X$ are two elements such that $(V, w, \zeta) \le (V', w', \zeta')$ and suppose that $\hat{h}_{U,0,g} \circ F(V', w', \zeta') \in Z$. Then $V \subseteq V' \subseteq U^\perp$, and this implies $r_U(w) = r_U \circ r_{(V')^\perp}(w) = r_U(w')$, so $g \circ r_U(w) = g \circ r_U(w')$. Moreover, we have $\zeta' = \iota_{V,V'}(\zeta) + \iota_{V',bV'} \circ r_{V'}(w)$, hence

$$\begin{aligned} &(\iota_{V',\mathbb{R}^n}(\zeta') + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w'))(e_\xi) \\ &\quad = \Big(\iota_{V',\mathbb{R}^n} \circ \iota_{V,V'}(\zeta) + \iota_{V',\mathbb{R}^n} \circ \iota_{V',bV'} \circ r_{V'}(w) \\ &\qquad + \iota_{\mathbb{R}^n,b\mathbb{R}^n} \circ r_{(V')^\perp}(w)\Big)(e_\xi) \\ &\quad = (\iota_{V,\mathbb{R}^n}(\zeta) + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(r_{V'}(w) + r_{(V')^\perp}(w)))(e_\xi) \\ &\quad = (\iota_{V,\mathbb{R}^n}(\zeta) + \iota_{\mathbb{R}^n,b\mathbb{R}^n}(w))(e_\xi). \end{aligned}$$

This yields

$$\hat{h}_{U,0,g} \circ F(V, w, \zeta) = \hat{h}_{U,0,g} \circ F(V', w', \zeta') \in (\hat{h}_{U,0,g} \circ F)^{-1}(Z),$$

therefore $(\hat{h}_{U,\xi,g} \circ F)^{-1}(Z)$ is indeed a lower set.

To prove the inclusion $\subseteq$, let $(V, w, \zeta) \in (\hat{h}_{U,\xi,g} \circ F)^{-1}(Z)$. We have already noted that this implies $V \subseteq U^{\perp}$. Furthermore, a computation similar to the one above will show that

$$\hat{h}_{U,\xi,g} \circ F(U^{\perp}, \mathcal{F}(V, U^{\perp})(w, \zeta)) = \hat{h}_{U,0,g} \circ F(V, w, \zeta) \in Z,$$

from which the inclusion readily follows, and thereby equation 6.8.

Now fix a closed subset $C \subseteq \mathbb{C}$, and let $C_0 := (\hat{h}_{U,\xi,g} \circ F)^{-1}(C)$. We distinguish between two cases:

- $0 \in C$: in this case, the set $\mathbb{C} \backslash C$ is an open subset of $\mathbb{C} \backslash \{0\}$, so taking $Z = \mathbb{C} \backslash C$, we can apply equation (6.8) and part (3) of Proposition 6.40 to see that $\Omega^0_{\mathcal{R},n} \backslash C_0$ is open, hence C_0 is closed in $\Omega^0_{\mathcal{R},n}$.

- $0 \notin C$: in this case, we first show that $T^{-1}_{U^{\perp}}(C_0)$ is compact. This set is closed since $\hat{h}_{U,\xi,g} \circ F \circ T_V$ is continuous and C is closed, so we only need to show that $T^{-1}_{U^{\perp}}(C_0)$ is a subset of a compact set.

  Since C is closed and does not contain 0, there exists an $r > 0$ such that for each $z \in C$, we have $|z| \geq r$. Using this fact and equation (6.7), we see that for each $(w, \zeta) \in T^{-1}_{U^{\perp}}(C_0)$, we have

  $$|g(w)| = |\hat{h}_{U,\xi,g} \circ F(U^{\perp}, w, \zeta)| \geq r.$$

  Furthermore, since $g \in \mathcal{S}(U)$, the function g must vanish at infinity, so there exists a compact set $K \subseteq U$ such that for each $w \in U \backslash K$, we have $|g(w)| < r$. It follows that

  $$T^{-1}_{U^{\perp}}(C_0) \subseteq K \times b(U^{\perp}),$$

  and the right-hand side is a product of two compact sets, hence it is compact. We now apply equation (6.8) with $Z = C$ and part (4) of Proposition 6.40 to conclude that C_0 is closed in $\Omega^0_{\mathcal{R},n}$.

Thus in both cases, C_0 is closed, and we conclude that $\hat{h}_{U,\xi,g} \circ F$ is continuous. ■

6.42 Corollary. *The topological space* $(\Omega^0_{\mathcal{R},n}, \tau_{\mathrm{GIL}})$*, together with the map*

$$\alpha\colon \mathbb{R}^n \mapsto \Omega^0_{\mathcal{R},n}, \quad x \mapsto (\{0\}, x, 0),$$

is a compactification of $\mathbb{R}^n$ *endowed with the usual metric topology.*

Proof. It is readily seen from part (3) of Proposition 6.40 that α is a homeomorphism onto its image. To see that its image is dense in $\Omega^0_{\mathcal{R},n}$, note that the composition of α with the map $F\colon \Omega^0_{\mathcal{R},n} \to \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$ from the previous theorem is the map

$$\mathbb{R}^n \to \Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)), \quad x \mapsto \delta_x,$$

where $\delta_x(f) = f(x)$ for each $f \in \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$. The image of $F \circ \alpha$ separates the elements of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, so using Urysohn's lemma and the Gelfand–Naimark theorem, it can be shown to be dense in $\Omega(\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n))$. Since F is a homeomorphism, it follows that the image of α is dense in $\Omega^0_{\mathcal{R},n}$. We conclude that $(\Omega^0_{\mathcal{R},n}, \tau_{\mathrm{GIL}})$ together with α is a compactification of $\mathbb{R}^n$. ■

Chapter 7

Quantisation of the resolvent algebra

7.1 Introduction

Having determined the Gelfand spectrum of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ in the previous chapter, and thereby the spectrum of our classical resolvent algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$, we now return to the matter of chapter 5, and ask whether there exists a quantum version of this algebra. A complicating factor in this context is that, contrary to the resolvent algebra $\mathcal{R}(\mathbb{R}^{2n}, \omega)$ of Buchholz and Grundling, our resolvent algebra is not defined in terms of generators and relations implementing canonical commutation relations, and it is also not clear how to do this, or whether it is even possible in the first place. Thus we must take a different approach.

Van Nuland [112] shows that $\mathcal{R}(\mathbb{R}^{2n}, \omega)$ arises as the quantisation of $C_{\mathcal{R}}(\mathbb{R}^{2n})$ using results by Rieffel [98]. Rieffel uses a version of Weyl quantisation to deform the algebra $C_u(\mathbb{R}^{2n})$ of uniformly continuous, bounded functions on $\mathbb{R}^{2n}$ into a family of noncommutative algebras parametrised by $\hbar > 0$, and shows that this quantisation is in fact a strict deformation quantisation. Rieffel even discusses how to pass to the cylinder in chapter 2 and Examples 10.5 and 10.6 in [98].

We will take a similar approach in this chapter, with one notable difference: we will define our quantisation of the algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as an algebra represented on $L^2(\mathbb{T}^n)$, similar to what we did in chapter 4, and to the definition of Landsman [65, section II.3.4] for general Riemannian manifolds. By contrast, Rieffel's algebra in [98, Example 10.6], apart from being a quantisation of $C_0(T^*\mathbb{T})$ which is too small for our purposes, is in some sense a universal object from which a quantisation of a physical system is obtained as the image of this object under one of its irreducible representations on a Hilbert space. The corresponding algebra has many inequivalent irreducible representations due to the fact that $\mathbb{T}$ is not simply connected; see the discussion in [70, section 7.7]. By no means do we intend to discount such universal objects; on the contrary, we will see in chapter 8 that such objects are likely better suited to fill the role of objects in the quantum category in the framework presented there. The main advantage of quantising $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ as an algebra of operators on $L^2(\mathbb{T}^n)$ lies in the explicit formula for the quantisations of the generators of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ that we are able to derive, which will be of use in this chapter and the next one.

This chapter is structured as follows. In section 7.2, we define the Weyl quantisation map and prove the aforementioned explicit formula. In section 7.3, we show that, except for continuity of the map $\hbar \mapsto \|\mathcal{Q}_\hbar^W(f)\|$ at $\hbar > 0$ for fixed $f \in C_{\mathcal{R}}(T^*\mathbb{T}^n)$, the quantisation is a strict quantisation. The chapter ends with section 7.4 with a proof that in the case $n = 1$, our algebra is stable under the quantum time evolution for a large class of Hamiltonians, and we comment on the higher dimensional case.

7.2 Definition of the quantisation map

Let us first recall the basics of Weyl quantisation in $\mathbb{R}^{2n}$, the quantisation procedure in [118] conceived by H. Weyl. Given say, a Schwartz function $f \in \mathcal{S}(\mathbb{R}^{2n})$, one associates an operator $\mathcal{Q}_\hbar^W(f) \in B(L^2(\mathbb{R}^n))$ to it as follows. First, one views f as a linear combination of functions of the form

$$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}, \quad (q,p) \mapsto e^{i(a\cdot q + b\cdot p)},$$

where $a, b \in \mathbb{R}^n$, by considering the Fourier transform of f. One subsequently substitutes these exponential functions with the operators

$$e^{i(a\cdot Q + b\cdot P)},$$

from section 5.1, thereby obtaining

$$
\begin{aligned}
&(2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(q,p) e^{ia\cdot(Q-q)+ib\cdot(P-p)}\, dq\, dp\, da\, db \\
&\quad = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(q,p) e^{i\hbar\frac{a\cdot b}{2}} e^{ia\cdot(Q-q)} e^{ib\cdot(P-p)}\, dq\, dp\, da\, db,
\end{aligned}
$$

where we take $\hbar > 0$. To define the above integrals rigorously, we can insert a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ on the right-hand side of the integrand, and check that the resulting expression is well-defined and that it defines a bounded operator on $\mathcal{S}(\mathbb{R}^n)$ viewed as a subspace of $L^2(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the operator has a unique bounded extension to $L^2(\mathbb{R}^n)$, which we define to be $\mathcal{Q}_\hbar^W(f)$. Using standard identities for Fourier transforms of functions and performing a number of substitutions, it can be shown that

$$
(\mathcal{Q}_\hbar^W(f)\psi)(x) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(x + \frac{y}{2}, p\right) e^{-i\frac{y\cdot p}{\hbar}} \psi(x+y)\, dp\, dy,
$$

for each $\psi \in \mathcal{S}(\mathbb{R}^n)$ and each $x \in \mathbb{R}^n$. Weyl quantisation of functions on $\mathbb{R}^{2n}$ is noted for its excellent symmetry properties compared to other quantisation schemes [68, section 4.3], being equivariant with respect to the actions of the symplectic group on $\mathbb{R}^{2n}$ on the classical side, and the metaplectic group that arises as a central extension of the symplectic group on the quantum side [33, section 8.7.2].

We now adapt the Weyl quantisation formula to $T^*\mathbb{T}^n$ in such a way that we can quantise elements of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. We already identified an analogue of the space of Schwartz functions in section 5.2, namely the space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ of finite linear combinations of functions of the form $e_k \otimes h_{U,\xi,g}$; see Proposition 5.9. These are the functions that we will quantise. Not all of these functions vanish at infinity, and are therefore not Schwartz in the conventional sense. To handle such functions, we take inspiration from Rieffel's work [98], regarding the integrals in the above formula as oscillatory integrals, and regularising the expression by inserting a factor in the integrand in the form of a member of a net of functions that converges pointwise to the constant function on $\mathbb{R}^n$ that is equal to 1 everywhere, as in part (1) of the next proposition. Part (2) of this proposition is the analogue of [98, Proposition 1.11].

7.1 Proposition.

(1) *Let* $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$*, let* $\hbar > 0$*, and let* $\psi \in C(\mathbb{T}^n)$*. Then for each* $x + \mathbb{Z}^n \in \mathbb{T}^n$*, the limit*

$$\lim_{\delta\to 0}(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(x + \frac{y}{2} + \mathbb{Z}^n, p\right) e^{-\frac{\delta}{2}p^2} e^{-i\frac{y\cdot p}{\hbar}} \cdot \psi(x + y + \mathbb{Z}^n)\, dp\, dy, \tag{7.1}$$

exists.

(2) *Now assume* $f = e_k \otimes h_{U,\xi,g}$ *is a function as described in equation* (5.2) *in Proposition 5.9. Then the expression in equation* (7.1) *is equal to*

$$(2\pi\hbar)^{-\dim(U)} e^{\pi i k\cdot\hbar\xi} e^{2\pi i k\cdot x} \cdot \int_U\int_U g\left(p + \pi\hbar r_U(k)\right) e^{-i\frac{y\cdot p}{\hbar}} \psi(x + y + \hbar\xi + \mathbb{Z}^n)\, dp\, dy.$$

For each $l \in \mathbb{Z}^n$*, let* ψ_l *be the function*

$$\mathbb{T}^n \to \mathbb{C}, \quad x + \mathbb{Z}^n \mapsto e^{2\pi i l\cdot x},$$

and regard it as an element of $L^2(\mathbb{T}^n)$*. Let* L *be the linear span of all such elements.*

(3) *In addition to the assumptions in the previous part of the proposition, suppose that* $\psi = \psi_l$ *for some* $l \in \mathbb{Z}^n$*. Then the expression in equation* (7.1) *is equal to*

$$h_{U,\xi,g}(\pi\hbar(k + 2l))\psi_{k+l}(x),$$

and the map defined on L *sending* ψ *to the function on* $\mathbb{T}^n$ *that assigns to a point* $x + \mathbb{Z}^n \in \mathbb{T}^n$ *the limit in* (7.1) *extends in a unique way to a bounded linear operator on* $L^2(\mathbb{T}^n)$ *with norm* $\leq \|g\|_\infty$*.*

7.2 Remark. Note that the integrand in equation 7.1 is in general not integrable as a function on $\mathbb{R}^n \times \mathbb{R}^n$, and that the integral is only well-defined as an iterated integral. Indeed, the inner integral is the Fourier transform of a Schwartz function on $\mathbb{R}^n$, and since $\mathcal{S}(\mathbb{R}^n)$ is stable under Fourier transformation, the integrand of the outer integral is again an integrable function on $\mathbb{R}^n$.

Proof. We first show that for functions f of the form $e_k \otimes h_{U,\xi,g}$, i.e. f as in part (2) of the proposition, the limit in equation (7.1) exists, and is equal to the formula in part (2) of the proposition. Since $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is by definition the linear span of such functions, part (1) will follow from this.

Thus, take such an f, and note that for any $\delta > 0$, we have

$$\begin{aligned}
&(2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(x + \frac{y}{2} + \mathbb{Z}^n, p\right) e^{-\frac{\delta}{2}p^2} e^{-i\frac{y\cdot p}{\hbar}} \psi(x+y+\mathbb{Z}^n)\, dp\, dy \\
&\quad = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\left(\xi\cdot p - \frac{y\cdot p}{\hbar}\right)} g\circ r_U(p) e^{-\frac{\delta}{2}p^2}\, dp \\
&\qquad\qquad\qquad \cdot e^{2\pi i k\cdot\left(x+\frac{y}{2}\right)} \psi(x+y+\mathbb{Z}^n)\, dy \\
&\quad = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\frac{y\cdot p}{\hbar}} g\circ r_U(p) e^{-\frac{\delta}{2}p^2}\, dp \\
&\qquad\qquad\qquad \cdot e^{2\pi i k\cdot\left(x+\frac{y+\hbar\xi}{2}\right)} \psi(x+y+\hbar\xi+\mathbb{Z}^n)\, dy.
\end{aligned}$$

The inner integral over p can be written as a product of two integrals; one over U and one over $U^\perp$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-i\frac{y\cdot p}{\hbar}} g\circ r_U(p) e^{-\frac{\delta}{2}p^2}\, dp \\
&\quad = \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{r_U(y)\cdot p_1}{\hbar}}\, dp_1 \cdot \int_{U^\perp} e^{-\frac{\delta}{2}p_2^2} e^{-ip_2\cdot\frac{y-r_U(y)}{\hbar}}\, dp_2 \\
&\quad = \int_U g(p_1) e^{-\frac{\delta}{2}p_1^2} e^{-i\frac{r_U(y)\cdot p_1}{\hbar}}\, dp_1 \cdot (2\pi\delta^{-1})^{\frac{\dim(U^\perp)}{2}} e^{-\frac{1}{2\delta\hbar^2}(y-r_U(y))^2}.
\end{aligned}$$

Inserting this back into the previous displayed formula, and splitting the outer integral in that formula into an integral over U and an integral over $U^\perp$, we obtain

$$\begin{aligned}
&(2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(x + \frac{y}{2} + \mathbb{Z}^n, p\right) e^{-\frac{\delta}{2}p^2} e^{-i\frac{y\cdot p}{\hbar}} \psi(x+y+\mathbb{Z}^n)\, dp\, dy \\
&\quad = (2\pi\hbar)^{-\dim(U)} \int_U h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1, y_2)\, dy_2\, dy_1,
\end{aligned}$$

where

$$h_{1,\delta} \colon U \to \mathbb{C},$$

$$y_1 \mapsto e^{2\pi i k\cdot\left(x+\frac{y_1+\hbar\xi}{2}\right)} \int_U g(p_1)e^{-\frac{\delta}{2}p_1^2}e^{-i\frac{y_1\cdot p_1}{\hbar}}\,dp_1,$$

and

$$h_{2,\delta}\colon U\times U^\perp \to \mathbb{C},$$
$$(y_1,y_2)\mapsto (2\pi\delta\hbar^2)^{\frac{-\dim(U^\perp)}{2}}e^{-\frac{1}{2\delta\hbar^2}y_2^2}\cdot\psi(x+y_1+y_2+\hbar\xi+\mathbb{Z}^n)e^{\pi i k\cdot y_2}.$$

Now note that the family of functions

$$U^\perp\to\mathbb{R},\quad y_2\mapsto(2\pi\delta\hbar^2)^{\frac{-\dim(U^\perp)}{2}}e^{-\frac{1}{2\delta\hbar^2}y_2^2},$$

indexed by $\delta>0$ is an approximation to the identity for functions on $U^\perp$. By continuity of ψ, it follows that the functions

$$h_{3,\delta}\colon U\to\mathbb{C},\quad y_1\mapsto\int_{U^\perp}h_{2,\delta}(y_1,y_2)\,dy_2$$

converge pointwise to the function

$$U\to\mathbb{C},\quad y_1\mapsto\psi(x+y_1+\hbar\xi+\mathbb{Z}^n),$$

as $\delta\to 0$. Moreover, they are bounded, with $\|h_{3,\delta}\|_\infty\le\|\psi\|_\infty$ for each $\delta>0$. In addition, by the dominated convergence theorem, the functions $h_{1,\delta}$ converge pointwise to the function

$$U\to\mathbb{C},\quad y_1\mapsto e^{2\pi i k\cdot\left(x+\frac{y_1+\hbar\xi}{2}\right)}\int_U g(p_1)e^{-i\frac{y_1\cdot p_1}{\hbar}}\,dp_1,$$

as $\delta\to 0$. Indeed, the integrands defining these functions are all dominated by the integrable function $|g|$. Furthermore, note that

$$\begin{aligned}&\int_U g(p_1)e^{-\frac{\delta}{2}p_1^2}e^{-i\frac{y_1\cdot p_1}{\hbar}}\,dp_1\\ &=\frac{(1+\|y_1\|^2)^{\dim(U)}}{(1+\|y_1\|^2)^{\dim(U)}}\int_U g(p_1)e^{-\frac{\delta}{2}p_1^2}e^{-i\frac{y_1\cdot p_1}{\hbar}}\,dp_1\\ &=\frac{1}{(1+\|y_1\|^2)^{\dim(U)}}\int_U(1-\hbar^2\Delta_U)^{\dim(U)}(g(p')e^{-\frac{\delta}{2}(p')^2})|_{p'=p_1}e^{-i\frac{y_1\cdot p_1}{\hbar}}\,dp_1,\end{aligned}$$

where Δ_U denotes the standard Laplacian on U, and that for the family of the functions

$$U \to \mathbb{C}, \quad p_1 \mapsto (1 - \hbar^2 \Delta_U)^{\dim(U)} (g(p') e^{-\frac{\delta}{2}(p')^2})|_{p'=p_1},$$

indexed by $\delta \in (0, C]$, where C is an arbitrary positive real number, there exists a positive function $H_C \in L^1(U)$ dominating the entire family. It follows that for each $\delta \in (0, C]$ and each $y_1 \in U$, we have

$$|h_{1,\delta}(y_1)| \leq \frac{\|H_C\|_1}{(1 + \|y_1\|^2)^{\dim(U)}}.$$

The (absolute values of the) functions

$$U \to \mathbb{C}, \quad y_1 \mapsto h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1, y_2)\, dy_2,$$

are therefore dominated by the integrable function

$$y_1 \mapsto \frac{\|H_C\|_1 \|\psi\|_\infty}{(1 + \|y_1\|^2)^{\dim(U)}},$$

so we may again invoke the dominated convergence theorem to find that

$$\begin{aligned}
&\lim_{\delta \to 0} (2\pi\hbar)^{-\dim(U)} \int_U h_{1,\delta}(y_1) \int_{U^\perp} h_{2,\delta}(y_1, y_2)\, dy_2\, dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} \int_U \left(\lim_{\delta \to 0} h_{1,\delta}(y_1) \right) \left(\lim_{\delta \to 0} \int_{U^\perp} h_{2,\delta}(y_1, y_2)\, dy_2 \right) dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} \\
&\qquad \cdot \int_U \int_U g(p_1) e^{-i\frac{y_1 \cdot p_1}{\hbar}}\, dp_1\, e^{2\pi i k \cdot \left(x + \frac{y_1 + \hbar\xi}{2}\right)} \psi(x + y_1 + \hbar\xi + \mathbb{Z}^n)\, dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} e^{\pi i k \cdot \hbar\xi} e^{2\pi i k \cdot x} \\
&\qquad \cdot \int_U \int_U g(p_1) e^{-i y_1 \cdot \left(\frac{p_1}{\hbar} - \pi k\right)}\, dp_1\, \psi(x + y_1 + \hbar\xi + \mathbb{Z}^n)\, dy_1 \\
&\quad = (2\pi\hbar)^{-\dim(U)} e^{\pi i k \cdot \hbar\xi} e^{2\pi i k \cdot x} \\
&\qquad \cdot \int_U \int_U g(p_1 + \pi\hbar r_U(k)) e^{-i\frac{y_1 \cdot p_1}{\hbar}}\, dp_1\, \psi(x + y_1 + \hbar\xi + \mathbb{Z}^n)\, dy_1,
\end{aligned}$$

which completes our proof of part (2).

For part (3), we simply take $\psi = \psi_l \in C(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$, with $l \in \mathbb{Z}^n$, and apply the formula we just found:

$$\begin{aligned}
&(2\pi\hbar)^{-\dim(U)} e^{\pi i k\cdot\hbar\xi} e^{2\pi i k\cdot x} \\
&\quad \cdot \int_U \int_U g(p_1 + \pi\hbar r_U(k)) e^{-i\frac{y_1\cdot p_1}{\hbar}}\, dp_1\, e^{2\pi i l\cdot(x+y_1+\hbar\xi)}\, dy_1 \\
&= (2\pi\hbar)^{-\dim(U)} e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} \\
&\quad \cdot \int_U \int_U g(p_1 + \pi\hbar r_U(k)) e^{-iy_1\cdot\left(\frac{p_1}{\hbar}-2\pi l\right)}\, dp_1\, dy_1 \\
&= (2\pi)^{-\dim(U)} e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} \\
&\quad \cdot \int_U \int_U g(p_1 + \pi\hbar r_U(k+2l)) e^{-iy_1\cdot p_1}\, dp_1\, dy_1 \\
&= e^{\pi i(k+2l)\cdot\hbar\xi} e^{2\pi i(k+l)\cdot x} g \circ r_U(\pi\hbar(k+2l)) \\
&= h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l}(x),
\end{aligned}$$

which proves the formula in part (3).

We thus see that the linear map on L uniquely determined by

$$\psi_l \mapsto h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l},$$

maps an orthonormal basis to an orthogonal system of vectors in $L^2(\mathbb{T}^n)$, and the norm of the image of such a vector ψ_l is less than or equal to $\|g\|_\infty = \|h_{U,\xi,g}\|_\infty = \|f\|_\infty$. (Note that the suprema defining these sup-norms are taken over U, $\mathbb{R}^n$ and $\mathbb{T}^n \times \mathbb{R}^n$, respectively.) Because of this and the fact that L is dense in $L^2(\mathbb{T}^n)$, the map extends in a unique way to a bounded operator on $L^2(\mathbb{T}^n)$ with norm $\leq \|g\|_\infty$, which proves the final assertion. ■

The proposition justifies the following definitions:

7.3 Definition. For each $\hbar > 0$ and each $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, we define the *Weyl quantisation $\mathcal{Q}_\hbar^W(f)$ of f* to be the unique bounded linear extension of the operator on $L \subset L^2(\mathbb{T}^n)$ defined by the formula

$$(\mathcal{Q}_\hbar^W(f)\psi)(x) := \lim_{\delta\to 0}(2\pi\hbar)^{-n} \cdot \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(x + \frac{y}{2} + \mathbb{Z}^n, p\right)$$

$$\cdot\, e^{-\frac{\delta}{2}p^2} e^{-i\frac{y\cdot p}{\hbar}} \psi(x+y+\mathbb{Z}^n)\, dp\, dy.$$

We thus obtain a map, the *Weyl quantisation map*

$$\mathcal{Q}_\hbar^W : \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \to B(L^2(\mathbb{T}^n)),$$

for each $\hbar > 0$. We define the *quantum resolvent algebra $A_\hbar$ on $\mathbb{T}^n \times \mathbb{R}^n$* to be the C*-subalgebra of $B(L^2(\mathbb{T}^n))$ generated by the image of $\mathcal{Q}_\hbar^W$.

7.4 Proposition. *Let $\hbar > 0$.*

(1) The Weyl quantisation map is linear and respects the involutions on both spaces;

(2) For each $\hbar' > 0$, we have $A_\hbar = A_{\hbar'}$;

(3) The image of

$$\mathrm{span}_\mathbb{C}\{e_k \otimes g \colon k \in \mathbb{Z}^n,\ g \in \mathcal{S}(\mathbb{R}^n)\} \subset \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n) \cap C_0(T^*\mathbb{T}^n)$$

under $\mathcal{Q}_\hbar^W$ is a dense subspace of $B_0(L^2(\mathbb{T}^n))$;

(4) Consider the group representation ρ_0 of $\mathbb{T}^n$ on $C_\mathcal{R}(T^*\mathbb{T}^n)$ given by

$$\rho_0(x+\mathbb{Z}^n)f := ((q+\mathbb{Z}^n, p) \mapsto f(-x+q+\mathbb{Z}^n, p)).$$

The space $S_\mathcal{R}(T^*\mathbb{T}^n)$ is an invariant subspace of this representation. Moreover, consider the group representation $\rho_\hbar$ of $\mathbb{T}^n$ on $B(L^2(\mathbb{T}^n))$ given by

$$\begin{aligned}\rho_\hbar(x+\mathbb{Z}^n)a &:= L^*(x+\mathbb{Z}^n)a(L^*(x+\mathbb{Z}^n))^* \\ &= L^*(x+\mathbb{Z}^n)aL^*(-x+\mathbb{Z}^n),\end{aligned}$$

where $L^* \colon \mathbb{T}^n \to U(L^2(\mathbb{T}^n))$ denotes the left regular representation of $\mathbb{T}^n$. The Weyl quantisation map is equivariant with respect to these representations.

7.5 Remark. Because of part (2) of this proposition, we will write $A_\hbar$ for the C*-algebra generated by $\mathcal{Q}_{\hbar'}^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n))$ for any value of $\hbar' > 0$ without specifying $\hbar$. Part (3) is the analogue of the first part of [65, Corollary II.2.5.4] in the present setting, while part (4) is the analogue of [65, Theorem II.2.5.1].

Proof.

(1) Linearity of $\mathcal{Q}_\hbar^W$ is obvious from the definition. Now let $e_k \otimes h_{U,\xi,g}$ be a generator of $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, and let

$$\mathcal{F}\colon L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n), \quad \psi' \mapsto (a \mapsto \langle \psi_a, \psi' \rangle),$$

be the Fourier transform. It follows from part (3) of Proposition 7.1 that

$$\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g}) = \mathcal{F}^{-1} S^k M_{h_1} \mathcal{F},$$

where S^k denotes the shift operator

$$\ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n), \quad \psi \mapsto (l \mapsto \psi(-k+l)),$$

and M_{h_1} denotes the multiplication operator on $\ell^2(\mathbb{Z}^n)$ associated to the function

$$h_1\colon \mathbb{Z}^n \to \mathbb{C}, \quad l \mapsto h_{U,\xi,g}(\pi\hbar(k+2l)).$$

Next, for each $l \in \mathbb{Z}$, we have

$$\begin{aligned}(S^k M_{h_1})^* \psi_l = M_{\overline{h_1}} S^{-k} \psi_l &= \overline{h_{U,\xi,g}}(\pi\hbar(k+2(l-k)))\psi_{l-k} \\ &= \overline{h_{U,\xi,g}}(\pi\hbar(-k+2l))\psi_{l-k} = S^{-k} M_{h_2} \psi_l,\end{aligned}$$

where

$$h_2\colon \mathbb{Z}^n \to \mathbb{C}, \quad l \mapsto \overline{h_{U,\xi,g}}(\pi\hbar(-k+2l)).$$

Now note that

$$\mathcal{Q}_\hbar^W(\overline{e_k \otimes h_{U,\xi,g}}) = \mathcal{Q}_\hbar^W(e_{-k} \otimes \overline{h_{U,\xi,g}}) = \mathcal{F}^{-1} S^{-k} M_{h_2} \mathcal{F},$$

so by unitarity of the Fourier transform, we have

$$\mathcal{Q}_\hbar^W(\overline{e_k \otimes h_{U,\xi,g}}) = \mathcal{F}^{-1}(S^k M_{h_1})^* \mathcal{F} = (\mathcal{F}^{-1} S^k M_{h_1} \mathcal{F})^* = \mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g})^*,$$

hence $\mathcal{Q}_\hbar^W$ is indeed compatible with the involutions.

(2) For each $\hbar > 0$, each $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ and each $\psi \in L$, we have

$$(\mathcal{Q}_\hbar^W(f)\psi)(x) = \lim_{\delta \to 0} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(x + \frac{y}{2} + \mathbb{Z}^n, \hbar p'\right)$$

$$\cdot\, e^{-\frac{\delta}{2}(p')^2} e^{-iy\cdot p'} \psi(x+y+\mathbb{Z}^n)\, dp'\, dy,$$

where we have made the substitution $p = \hbar p'$ in the formula defining $\mathcal{Q}^W_\hbar(f)\psi$, and absorbed a factor $\hbar^2$ in δ. Next, we observe that $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ is closed under the map

$$f \mapsto ((q+\mathbb{Z}^n, p) \mapsto f(q+\mathbb{Z}^n, Cp))$$

for each $C \in \mathbb{R}$, in particular for $C = \hbar'/\hbar$ for any $\hbar, \hbar' > 0$. It follows that $\mathcal{Q}^W_\hbar(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)) = \mathcal{Q}^W_{\hbar'}(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n))$, hence $A_\hbar = A_{\hbar'}$, as desired.

(3) Let B be the left-hand side of the displayed formula in the statement. Now let $k \in \mathbb{Z}^n$, and let $g \in \mathcal{S}(\mathbb{R}^n)$. Using notation from the proof of part (1) of this proposition, we have

$$\mathcal{Q}^W_\hbar(e_k \otimes g) = \mathcal{F}^{-1} S^k M_{g_k} \mathcal{F},$$

where g_k denotes the function

$$\mathbb{Z}^n \to \mathbb{C}, \quad l \mapsto g(\pi\hbar(k+2l)).$$

This function vanishes at infinity, so its corresponding multiplication operator M_{g_k} is compact. All of the other operators that we compose to obtain $\mathcal{Q}^W_\hbar(e_k \otimes g)$ are bounded, hence $\mathcal{Q}^W_\hbar(e_k \otimes g)$ is compact. Since $\mathcal{Q}^W_\hbar$ is a linear map and $B_0(L^2(\mathbb{T}^n))$ is a linear subspace of $B(L^2(\mathbb{T}^n))$, it follows that $\mathcal{Q}^W_\hbar(B) \subseteq B_0(L^2(\mathbb{T}^n))$.

To prove the assertion that $\mathcal{Q}^W_\hbar(B)$ is in fact a dense subspace of $B_0(L^2(\mathbb{T}^n))$, we note that, given a and $b \in \mathbb{Z}^n$, we can fix a $g \in \mathcal{S}(\mathbb{R}^n)$ such that

$$g(\pi\hbar(a-b+2l)) = \delta_{l,b},$$

for each $l \in \mathbb{Z}^n$. It follows that, in bra-ket notation,

$$\mathcal{Q}^W_\hbar(e_{a-b} \otimes g) = |\psi_a\rangle\langle\psi_b|,$$

and from the fact that a and $b \in \mathbb{Z}^n$ were arbitrary and that the family of vectors $(\psi_l)_{l\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$, we infer that $\mathcal{Q}^W_\hbar(B)$ is dense in $B_0(L^2(\mathbb{T}^n))$.

(4) Suppose f is of the form $e_k \otimes h_{U,\xi,g}$. Then it is readily seen that

$$\rho_0(x+\mathbb{Z}^n)(e_k \otimes h_{U,\xi,g}) = e^{-2\pi i k\cdot x} e_k \otimes h_{U,\xi,g} \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n),$$

for each $x+\mathbb{Z}^n \in \mathbb{T}^n$, from which it follows that $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is an invariant subspace of the representation ρ_0. Furthermore, for each $l \in \mathbb{Z}^n$, we have

$$\begin{aligned}
&(\rho_\hbar(x+\mathbb{Z}^n)(\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g})))\psi_l \\
&\quad = L^*(x+\mathbb{Z}^n)\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g})L^*(-x+\mathbb{Z}^n)\psi_l \\
&\quad = e^{2\pi i l\cdot x} L^*(x+\mathbb{Z}^n)\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g})\psi_l \\
&\quad = e^{2\pi i l\cdot x} h_{U,\xi,g}(\pi\hbar(k+2l))L^*(x+\mathbb{Z}^n)\psi_{k+l} \\
&\quad = e^{2\pi i l\cdot x} e^{-2\pi i (k+l)\cdot x} h_{U,\xi,g}(\pi\hbar(k+2l))\psi_{k+l} \\
&\quad = \mathcal{Q}_\hbar^W(e^{-2\pi i k\cdot x} e_k \otimes h_{U,\xi,g})\psi_l,
\end{aligned}$$

from which we conclude that

$$\rho_\hbar(x+\mathbb{Z}^n)(\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g})) = \mathcal{Q}_\hbar^W(\rho_0(x+\mathbb{Z}^n)(e_k \otimes h_{U,\xi,g})),$$

for each $x+\mathbb{Z}^n$ and each generator $e_k \otimes h_{U,\xi,g}$ of $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$. Since these generators span $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, and the quantisation map and the maps $\rho_0(x+\mathbb{Z}^n)$ and $\rho_\hbar(x+\mathbb{Z}^n)$ are linear, we may substitute for $e_k \otimes h_{U,\xi,g}$ any element of $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ in the above equation. ■

7.3 Proof of strict quantisation

We now show that Weyl quantisation as defined in the previous section yields a strict quantisation of the dense Poisson subalgebra $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ of the classical resolvent algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ on $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$, see [65, section II.1.1.1] or Theorem 7.8 below. Of these properties, the most difficult one to prove is Rieffel's condition, i.e., convergence of the operator norms of $\mathcal{Q}_\hbar^W(f)$ to the sup-norm of $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, which we discuss separately before showing that the other conditions hold. To prepare for the proof, we first make the following observation:

7.6 Lemma. *Let $K_1, \ldots, K_n \in \mathbb{N}\backslash\{0\}$. Then let $K := (K_1, \ldots, K_n) \in \mathbb{N}^n$, let $K\mathbb{Z}^n := K_1\mathbb{Z} \times \cdots \times K_n\mathbb{Z}$, and let $\mathbb{Z}^n_K := \mathbb{Z}^n/K\mathbb{Z}^n$. Finally, for each $k \in \mathbb{Z}^n_K$, let $S^k_{\mathrm{per}} \colon \ell^2(\mathbb{Z}^n_K) \to \ell^2(\mathbb{Z}^n_K)$ be the operator given by*

$$\phi \mapsto (l \mapsto \phi(-k+l)).$$

Then for any $f \in \ell^\infty(\mathbb{Z}_K^n)$, we have

$$\left\| \sum_{k \in \mathbb{Z}_K^n} f(k) S_{\mathrm{per}}^k \right\| = \max_{l \in \mathbb{Z}_K^n} \left| \sum_{k \in \mathbb{Z}_K^n} f(k) e^{2\pi i \sum_{j=1}^n \frac{k_j l_j}{K_j}} \right|.$$

Proof. This is readily seen by conjugating the operator $\sum_{k \in \mathbb{Z}_K^n} f(k) S_{\mathrm{per}}^k$ with the discrete Fourier transform

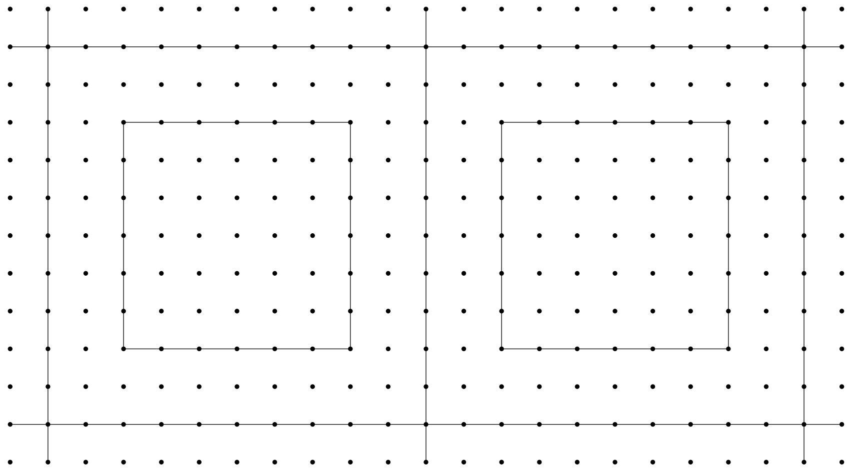
$$\begin{aligned} &\ell^2(\mathbb{Z}_K^n) \to \ell^2(\mathbb{Z}_K^n), \\ &\phi \mapsto \left(l \mapsto \left(\prod_{j=1}^n K_j \right)^{-\frac{1}{2}} \cdot \sum_{m \in \mathbb{Z}_K^n} \phi(m) e^{-2\pi i \sum_{j=1}^n \frac{l_j m_j}{K_j}} \right), \end{aligned}$$

yielding the multiplication operator of which the corresponding function is the one within absolute value strokes. ■

7.7 Proposition. (Rieffel's condition) *For each $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, we have*

$$\lim_{\hbar \to 0} \| \mathcal{Q}_\hbar^W(f) \| = \|f\|_\infty.$$

Before we give a precise proof of this proposition, it is instructive to first give a sketch of the underlying idea. To relate the norm of $\mathcal{Q}_\hbar^W(f)$ to that of f, we conjugate the quantised function with the Fourier transform to obtain an operator on $\ell^2(\mathbb{Z}^n)$. We visualise $\mathbb{Z}^n$ as a lattice of points in $\mathbb{R}^n$, and divide it into identical boxes. In each of these boxes, we identify a slightly smaller box such that all of the smaller boxes are translates of each other in the same way that the larger boxes that contain them are translates of each other. Part of the lattice with two such boxes that are adjacent, and each of which contains a smaller box, have been depicted below (for $n = 2$).



The difference between the sizes of the small boxes and the sizes of the larger boxes is determined by the values of the various k_j that appear in the function

$$f = \sum_{j=1}^{m} e_{k_j} \otimes h_{U_j,\xi_j,g_j},$$

of which we consider the quantisation; specifically, the shift $S_{k_j} \in B(\ell^2(\mathbb{Z}^n))$ should always map elements on $\ell^2(\mathbb{Z}^n)$ supported on points inside the smaller box to functions supported on points inside the larger box containing the small one. The size of the larger box is determined by a chosen value of $\varepsilon > 0$ and a crude estimate of $\|\mathcal{Q}_\hbar^W(f)\|$.

Given a function $\phi \in \ell^2(\mathbb{Z}^n)$, we can now estimate the norm of its image under the conjugated quantised function as follows. First, we consider the projection of ϕ onto the subspace of $\ell^2(\mathbb{Z}^n)$ of elements supported on the set of points inside one of the smaller boxes, and use the fact that its image under the operator consists of elements supported on the set of points inside the larger box. We can then consider a periodic version of the operator, and use the preceding lemma to get an estimate on its norm and relate it to the norm of f. Finally, we sum the contributions of all projections of ϕ onto the subspaces corresponding to the smaller boxes to obtain an estimate on the difference of the norm of f and that of the conjugated version of its quantisation. To control the difference between ϕ and its projection onto the space corresponding to the union of all of the smaller boxes, we note that the partition into boxes can always be offset by some element of $\mathbb{Z}^n$ in such a way that the part of ϕ supported on the complement of this union is small.

Proof. Fix $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ and $\varepsilon > 0$. We first prove the following statement:

(a) *There exists an $\hbar_1 \in (0, \infty)$ such that for each $\hbar \in (0, \hbar_1]$, we have*

$$\|\mathcal{Q}_\hbar^W(f)\| < \|f\|_\infty + \varepsilon.$$

Since $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, there exist functions $f_1, \dots, f_m$, where

$$f_j = e_{k_j} \otimes h_{U_j,\xi_j,g_j},$$

is a generator of $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ for $j = 1, \dots, m$, and there exist $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, such that $f = \sum_{j=1}^m \lambda_j f_j$. By absorbing each of the constants λ_j into the function g_j corresponding to f_j, we may assume without loss of generality that $\lambda_j = 1$ for $j = 1, \dots, m$, so that $f = \sum_{j=1}^m f_j$. For $j = 1, \dots, m$, fix the corresponding $k_j \in \mathbb{Z}^n$, $U_j \subseteq \mathbb{R}^n$, $\xi_j \in U_j^\perp$ and $g_j \in \mathcal{S}(U_j)$, and let

$$h_j := h_{U_j,\xi_j,g_j}.$$

Note that by part (3) of Proposition 7.1, we have a uniform bound on the norms of the operators $(\mathcal{Q}_\hbar^W(f))_{\hbar>0}$, namely

$$\|\mathcal{Q}_\hbar^W(f)\| \leq \sum_{j=1}^m \|h_j\|_\infty = \sum_{j=1}^m \|g_j\|_\infty =: C.$$

Since the case $C = 0$ is trivial, we assume that $C > 0$ (which also implies that $m > 0$). Now for $l = 1, \dots, n$, let $k^{(l)} := \max_{1\leq j\leq m} |k_j^{(l)}|$, and fix $K_1, \dots, K_n \in \mathbb{N}\backslash\{0\}$ such that $K_l \geq 2k^{(l)}$, and

$$\prod_{l=1}^n \left(1 - \frac{2k^{(l)}}{K_l}\right) > 1 - \left(\frac{\varepsilon}{4C}\right)^2 \tag{7.2}$$

Moreover, for $j = 1, \dots, m$, the function h_j is differentiable and its first order Fréchet derivative is bounded, which implies that it is Lipschitz continuous, hence there exists $\hbar_1 \in (0,\infty)$ such that for each $\hbar \in (0,\hbar_1]$, each $a \in \mathbb{Z}^n$ and each $b \in \mathbb{Z}^n$ with $|b_l| < K_l$ for $l = 1,\dots,n$, we have

$$|h_j(2\pi\hbar a) - h_j(\pi\hbar(k_j + 2(a+b)))| < \frac{\varepsilon}{4m}. \tag{7.3}$$

Now fix $\hbar \in (0,\hbar_1]$, fix $\psi \in L^2(\mathbb{T}^n)$ with $\|\psi\| = 1$, and let ϕ be the image of ψ under the Fourier transform $\mathcal{F}\colon L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$, which we already defined in part (1) of the proof of Proposition 7.4. Furthermore, we define the set

$$X := \{a \in \mathbb{Z}^n \colon k^{(l)} \leq a_l < K_l - k^{(l)} \text{ for } l = 1,\dots,n\},$$

and we define $K\mathbb{Z}^n$ and $\mathbb{Z}^n_K$ as in the previous lemma. Then, we have

$$\begin{aligned}
&\sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a \in X + K\mathbb{Z}^n} |\phi(a+b)|^2 \\
&\quad= \sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a \in X} \sum_{a' \in K\mathbb{Z}^n} |\phi(a+a'+b)|^2 \\
&\quad= \sum_{a \in X} \sum_{b+K\mathbb{Z}^n \in \mathbb{Z}^n_K} \sum_{a' \in K\mathbb{Z}^n} |\phi(a+a'+b)|^2 \\
&\quad= \sum_{a \in X} \sum_{b \in \mathbb{Z}^n} |\phi(a+b)|^2 = |X| \cdot \sum_{b \in \mathbb{Z}^n} |\phi(b)|^2 = |X|,
\end{aligned}$$

where

$$|X| = \prod_{l=1}^n (K_l - 2k^{(l)}),$$

is the cardinality of the set X. It follows that there exists a $b \in \mathbb{Z}^n$ with $0 \leq b_l < K_l$ for $l = 1,\dots,n$ such that

$$\begin{aligned}
\sum_{a \in X + K\mathbb{Z}^n} |\phi(a+b)|^2 &\geq |\mathbb{Z}^n_K|^{-1} \prod_{l=1}^n (K_l - 2k^{(l)}) = \prod_{l=1}^n \left(1 - \frac{2k^{(l)}}{K_l}\right) \\
&> 1 - \left(\frac{\varepsilon}{4C}\right)^2.
\end{aligned}$$

Let $P_{X,b}$ be the orthogonal projection of $\ell^2(\mathbb{Z}^n)$ onto the subspace

$$\{\phi' \in \ell^2(\mathbb{Z}^n) \colon \operatorname{supp}(\phi') \subseteq b + X + K\mathbb{Z}^n\},$$

so that by the above inequality, we have

$$\begin{aligned}\|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}(1-P_{X,b})\mathcal{F}\psi\| &\leq \|\mathcal{Q}_\hbar^W(f)\|\|\mathcal{F}^{-1}(1-P_{X,b})\phi\| \\ &\leq C\|(1-P_{X,b})\phi\| \\ &= C\left(1-\|P_{X,b}\phi\|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{4}.\end{aligned} \tag{7.4}$$

For each $a \in K\mathbb{Z}^n$, let

$$P_{a,b}\colon \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n_K), \quad \phi' \mapsto (a' + K\mathbb{Z}^n \mapsto \phi'(a + a' + b)),$$

where the representative $a' \in \mathbb{Z}^n$ has been chosen so that $0 \leq a'_l < K_l$ for $l = 1, \ldots, n$. Furthermore, for each $a \in \mathbb{Z}^n$, we have a corresponding shift operator

$$S^a\colon \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n), \quad \phi' \mapsto (a' \mapsto \phi'(-a + a')),$$

and for each $a + K\mathbb{Z}^n \in \mathbb{Z}^n_K$, we define the shift operator $S_{\mathrm{per}}^{a+K\mathbb{Z}^n}$ as in the previous lemma. Finally, for each $a \in K\mathbb{Z}^n$, we define

$$A_{a,b} := \sum_{j=1}^{k} h_j(2\pi\hbar(a+b))S_{\mathrm{per}}^{k_j+K\mathbb{Z}^n}.$$

Using Lemma 7.6, we obtain

$$\begin{aligned}\|A_{a,b}\| &= \max_{a'+K\mathbb{Z}^n\in\mathbb{Z}^n_K}\left|\sum_{j=1}^{m} e^{2\pi i\sum_{l=1}^{n}\frac{k_j^{(l)}a'}{K_j}} h_j(2\pi\hbar(a+b))\right| \\ &\leq \sup_{q+\mathbb{Z}^n\in\mathbb{T}^n}\left|\sum_{j=1}^{m} e^{2\pi i k_j\cdot q}h_j(2\pi\hbar(a+b))\right| \\ &\leq \sup_{(q+\mathbb{Z}^n,p)\in\mathbb{T}^n\times\mathbb{R}^n}\left|\sum_{j=1}^{m} e^{2\pi i k_j\cdot q}h_j(p)\right| \\ &= \|f\|_\infty.\end{aligned} \tag{7.5}$$

Moreover, using part (3) of Proposition 7.1, we find that

$$P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi$$

$$
\begin{aligned}
&= P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\sum_{a'\in\mathbb{Z}^n}\phi(a')\delta_{a'}\\
&= P_{a,b}\sum_{a'\in b+X+K\mathbb{Z}^n}\sum_{j=1}^m h_j(\pi\hbar(k_j+2a'))\phi(a')\delta_{a'+k_j}\\
&= \sum_{a'\in X}\sum_{j=1}^m h_j(\pi\hbar(k_j+2(a+b+a')))\phi(a+b+a')\delta_{a'+k_j+K\mathbb{Z}^n}\\
&= \sum_{j=1}^m S_{\mathrm{per}}^{k_j+K\mathbb{Z}^n}\sum_{a'\in X} h_j(\pi\hbar(k_j+2(a+b+a')))\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n},
\end{aligned}
$$

where in the second step, we have used the fact that $0 \leq a'_l + k_j^{(l)} < K_l$ for each $a' \in X$, for $j = 1, \ldots, m$ and $l = 1, \ldots, n$. On the other hand, we have

$$
\begin{aligned}
A_{a,b}P_{a,b}P_{X,b}\phi &= A_{a,b}P_{a,b}P_{X,b}\sum_{a'\in\mathbb{Z}^n}\phi(a')\delta_{a'}\\
&= A_{a,b}\sum_{a'\in X}\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}\\
&= \sum_{j=1}^m S_{\mathrm{per}}^{k_j+K\mathbb{Z}^n}\sum_{a'\in X} h_j(2\pi\hbar(a+b))\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}.
\end{aligned}
$$

Writing

$$\mu_{a',j} := h_j(2\pi\hbar(a+b)) - h_j(\pi\hbar(k_j+2(a+b+a'))),$$

for $j = 1, \ldots, m$ and $a' \in X$, we obtain

$$
\begin{aligned}
&\|(A_{a,b}P_{a,b}P_{X,b} - P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b})\phi\|\\
&\quad= \left\|\sum_{j=1}^m S_{\mathrm{per}}^{k_j+K\mathbb{Z}^n}\sum_{a'\in X}\mu_{a',j}\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}\right\|\\
&\quad\leq \sum_{j=1}^m\left\|\sum_{a'\in X}\mu_{a',j}\phi(a+b+a')\delta_{a'+K\mathbb{Z}^n}\right\|\\
&\quad= \sum_{j=1}^m\left(\sum_{a'\in X}|\mu_{a',j}|^2|\phi(a+b+a')|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq m \cdot \max_{a'' \in X} |\mu_{a'',j}| \left(\sum_{a' \in X} |\phi(a+b+a')|^2 \right)^{\frac{1}{2}}.$$

Using the inequality from equation (7.3) to estimate the right-hand side, we obtain

$$\|(A_{a,b}P_{a,b}P_{X,b} - P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b})\phi\| < \frac{\varepsilon}{4}\|P_{a,b}P_{X,b}\phi\|. \tag{7.6}$$

Equations (7.5) and (7.6) together yield

$$\begin{aligned}
&\|P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\| \\
&\quad \leq \|A_{a,b}P_{a,b}P_{X,b}\phi\| + \|(A_{a,b}P_{a,b}P_{X,b} - P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b})\phi\| \\
&\quad < \left(\|f\|_\infty + \frac{\varepsilon}{4}\right)\|P_{a,b}P_{X,b}\phi\|,
\end{aligned}$$

for each $a \in K\mathbb{Z}^n$. It is straightforward to see that for each $\phi' \in \ell^2(\mathbb{Z}^n)$, we have

$$\sum_{a \in K\mathbb{Z}^n} \|P_{a,b}\phi'\|^2 = \|\phi'\|^2,$$

so

$$\begin{aligned}
\|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 &= \|\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 \\
&= \sum_{a \in K\mathbb{Z}^n} \|P_{a,b}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\phi\|^2 \\
&< \sum_{a \in K\mathbb{Z}^n} \left(\|f\|_\infty + \frac{\varepsilon}{4}\right)^2 \|P_{a,b}P_{X,b}\phi\|^2 \\
&= \left(\|f\|_\infty + \frac{\varepsilon}{4}\right)^2 \|P_{X,b}\phi\|^2 \leq \left(\|f\|_\infty + \frac{\varepsilon}{4}\right)^2,
\end{aligned}$$

which together with equation (7.4) implies

$$\begin{aligned}
\|\mathcal{Q}_\hbar^W(f)\psi\| &\leq \|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,b}\mathcal{F}\psi\| + \|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}(1-P_{X,b})\mathcal{F}\psi\| \\
&< \|f\|_\infty + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \|f\|_\infty + \frac{\varepsilon}{2},
\end{aligned}$$

and since $\psi \in L^2(\mathbb{T}^n)$ was an arbitrary vector with norm 1, we obtain

$$\|\mathcal{Q}_\hbar^W(f)\| \leq \|f\|_\infty + \frac{\varepsilon}{2} < \|f\|_\infty + \varepsilon,$$

for each $\hbar \in (0, \hbar_1]$ which proves (a).

We now turn to the reverse inequality:

(b) There exists an $\hbar_2 \in (0, \infty)$ such that for each $\hbar \in (0, \hbar_2]$, we have

$$\|f\|_\infty < \|\mathcal{Q}_\hbar^W(f)\| + \varepsilon.$$

Let $(q_0, p_0) \in [0,1)^n \times \mathbb{R}^n$ be a point such that

$$\|f\|_\infty < |f(q_0 + \mathbb{Z}^n, p_0)| + \frac{\varepsilon}{8}.$$

By Lipschitz continuity of f, there exists a $\delta > 0$ such that for each $(q, p) \in (-1, 1)^n \times \mathbb{R}^n$ with $\sum_{l=1}^n |q_l - q_{0,l}| + |p_l - p_{0,l}| < \delta$, we have

$$|f(q_0 + \mathbb{Z}^n, p_0) - f(q + \mathbb{Z}^n, p)| < \frac{\varepsilon}{8}.$$

Now fix $k \in \mathbb{N}^n$ as in the proof of part (a), and fix $K \in \mathbb{N}^n$ in such a way that equation (7.2) holds, and that we have

$$K_l > \max\left(2k^{(l)}, \frac{2n}{\delta}\right), \tag{7.7}$$

for $l = 1, \ldots, n$. Furthermore, fix $\hbar_2 > 0$ such that equation (7.3) holds for each $\hbar \in (0, \hbar_2]$, and that we have

$$2\pi\hbar_2 K_l < \frac{\delta}{2n}. \tag{7.8}$$

Now fix such an $\hbar \in (0, \hbar_2]$. Next, we note that by equation (7.8) there exists an $a \in K\mathbb{Z}^n$ such that

$$p_{0,l} - \frac{\delta}{2n} < 2\pi\hbar a_l \leq p_{0,l},$$

and that by equation (7.7), there exists a $b \in \mathbb{Z}^n$ such that $0 \leq b_l < K_l$, and

$$\left| q_{0,l} - \frac{b_l}{K_l} \right| < \frac{\delta}{2n},$$

for $l = 1, \ldots, n$. Fix such a and b. It follows that

$$\sum_{l=1}^n \left| \frac{b_l}{K_l} - q_{0,l} \right| + |2\pi\hbar a_l - p_{0,l}| < \delta,$$

so that
$$\left|\sum_{j=1}^{m} e^{2\pi i \sum_{l=1}^{n} \frac{k_j^{(l)} b_l}{K_l}} h_j(2\pi\hbar a) - f(q_0+\mathbb{Z}^n, p_0)\right| < \frac{\varepsilon}{8},$$
and therefore, by the triangle inequality and our choice of $(q_0+\mathbb{Z}^n, p_0)$,
$$\left|\left|\sum_{j=1}^{m} e^{2\pi i \sum_{l=1}^{n} \frac{k_j^{(l)} b_l}{K_l}} h_j(2\pi\hbar a)\right| - \|f\|_\infty\right| < \frac{\varepsilon}{4}.$$
Now define $\phi \in \ell^2(\mathbb{Z}^n)$ by
$$\phi(a') := \begin{cases} \prod_{l=1}^{n} K_l^{-\frac{1}{2}} e^{-2\pi i \frac{a_l' b_l}{K_l}} & \text{if } 0 \le a_l' - a_l < K_l \text{ for } l = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases},$$
and let $\psi := \mathcal{F}^{-1}\phi \in L^2(\mathbb{T}^n)$. Then $\|\psi\| = \|\phi\| = 1$, and
$$A_{a,0}P_{a,0}\phi = \sum_{j=1}^{m} e^{2\pi i \sum_{l=1}^{n} \frac{k_j^{(l)} b_l}{K_l}} h_j(2\pi\hbar a) P_{a,0}\phi,$$
where $A_{a,0}$ and $P_{a,0}$ are defined in the same way as $A_{a,b}$ and $P_{a,b}$ were defined in part (a), respectively, so
$$\|A_{a,0}P_{a,0}\phi\| = \left|\sum_{j=1}^{m} e^{2\pi i \sum_{l=1}^{n} \frac{k_j^{(l)} b_l}{K_l}} h_j(2\pi\hbar a)\right| > \|f\|_\infty - \frac{\varepsilon}{4}.$$
Defining X in the same way as we did in the proof part (a), it follows that
$$\begin{aligned} \|A_{a,0}P_{a,0}P_{X,0}\phi\| &\ge \|A_{a,0}P_{a,0}\phi\| - \|A_{a,0}\|\|(1-P_{X,0})\phi\| \\ &> \|f\|_\infty - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \|f\|_\infty - \frac{\varepsilon}{2}. \end{aligned}$$
Next, we note that the vector $\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,0}\phi$, viewed as a function on $\mathbb{Z}^n$, is supported in the set of $a' \in \mathbb{Z}^n$ satisfying $a_l \le a_l' < a_l + K_l$ for $l = 1, \ldots, n$. Combining this observation with the estimate just obtained and equation (7.6) yields
$$\|\mathcal{F}\mathcal{Q}_\hbar^W \mathcal{F}^{-1} P_{X,0}\phi\| = \|P_{a,0}\mathcal{F}\mathcal{Q}_\hbar^W \mathcal{F}^{-1} P_{X,0}\phi\|$$

$$\begin{aligned}
&\geq \|A_{a,0}P_{a,0}P_{X,0}\phi\| \\
&\quad - \|(A_{a,0}P_{a,0}P_{X,0} - P_{a,0}\mathcal{F}\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}P_{X,0})\phi\| \\
&> \|f\|_\infty - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \|f\|_\infty - \frac{3\varepsilon}{4}.
\end{aligned}$$

We use this together with equation (7.4) to obtain

$$\begin{aligned}
\|\mathcal{Q}_\hbar^W(f)\psi\| &= \|\mathcal{F}\mathcal{Q}_\hbar^W(f)\psi\| \\
&\geq \|\mathcal{F}\mathcal{Q}_\hbar^W\mathcal{F}^{-1}P_{X,0}\phi\| - \|\mathcal{Q}_\hbar^W(f)\mathcal{F}^{-1}(1-P_{X,0})\mathcal{F}\psi\| \\
&> \|f\|_\infty - \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \|f\|_\infty - \varepsilon.
\end{aligned}$$

Since $\|\psi\| = 1$, this establishes (b).

Finishing up the proof, taking $\hbar_0 := \min(\hbar_1, \hbar_2)$, we infer that for each $\hbar \in (0, \hbar_0]$, we have $|\|\mathcal{Q}_\hbar^W(f)\| - \|f\|_\infty| < \varepsilon$, hence $\lim_{\hbar\to 0}\|\mathcal{Q}_\hbar^W(f)\| = \|f\|_\infty$, as desired. ■

We are now ready to prove the main result of this section. Let $\mathcal{Q}_0^W := \mathrm{Id}_{\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)}$, let A_0 be the C*-algebra $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. In the following theorem, it should be understood that $\|\mathcal{Q}_\hbar^W(f)\| := \|f\|_\infty$ for $\hbar = 0$.

7.8 Theorem. *Let $I \subset [0, \infty)$ be a subset containing 0 as an accumulation point. Then, except for continuity at $\hbar > 0$, the triple*

$$(I, (A_\hbar)_{\hbar\in I}, (\mathcal{Q}_\hbar^W : \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n) \to A_\hbar)_{\hbar\in I})$$

is a strict quantisation of the Poisson algebra $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, i.e., it satisfies

(1) Rieffel's condition at $\hbar = 0$: for each $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, the function $\hbar \mapsto \|\mathcal{Q}_\hbar^W(f)\|$ is continuous at 0.

(2) Von Neumann's condition: for each $f, g \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, we have

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|\mathcal{Q}_\hbar^W(f)\mathcal{Q}_\hbar^W(g) - \mathcal{Q}_\hbar^W(fg)\| = 0.$$

(3) Dirac's condition: for each $f, g \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, we have

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|(-i\hbar)^{-1}[\mathcal{Q}_\hbar^W(f), \mathcal{Q}_\hbar^W(g)] - \mathcal{Q}_\hbar^W(\{f, g\})\| = 0.$$

(4) *Completeness: for each* $\hbar \in I$*, the set* $\mathcal{Q}_\hbar^W(\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n))$ *is dense in* $A_\hbar$*.*

Proof.

(1) This was shown in Proposition 7.7.

(2) First suppose that f_j is a generator $e_{k_j} \otimes h_{U_j,\xi_j,g_j}$ of $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ for $j = 1, 2$. As in Proposition 7.7, we will write h_j instead of h_{U_j,ξ_j,g_j}. Let $k := k_1 + k_2$. Then

$$f_1 \cdot f_2 = e_{k_1} \otimes h_1 \cdot e_{k_2} \otimes h_2 = e_k \otimes (h_1 \cdot h_2)$$

Applying part (3) of Proposition 7.1 yields

$$\mathcal{Q}_\hbar^W(f_1 f_2)\psi_a = (h_1 \cdot h_2)(\pi\hbar(k + 2a))\psi_{k+a}.$$

for each $a \in \mathbb{Z}^n$. On the other hand, we have

$$\begin{aligned}\mathcal{Q}_\hbar^W(f_1)\mathcal{Q}_\hbar^W(f_2)\psi_a &= h_2(\pi\hbar(k_2 + 2a))\mathcal{Q}_\hbar^W(f_1)\psi_{k_2+a} \\ &= h_1(\pi\hbar(k_1 + 2(k_2 + a)))h_2(\pi\hbar(k_2 + 2a))\psi_{k+a} \\ &= h_1(\pi\hbar(k + k_2 + 2a)) \cdot h_2(\pi\hbar(k - k_1 + 2a)) \cdot \psi_{k+a},\end{aligned} \tag{7.9}$$

so

$$\begin{aligned}&(\mathcal{Q}_\hbar^W(f_1)\mathcal{Q}_\hbar^W(f_2) - \mathcal{Q}_\hbar^W(f_1 f_2))\psi_a \\ &\quad = (h_1(\pi\hbar(k + k_2 + 2a)) \cdot h_2(\pi\hbar(k - k_1 + 2a))) \\ &\qquad -(h_1 \cdot h_2)(\pi\hbar(k + 2a)))\, \psi_{k+a},\end{aligned}$$

for each $a \in \mathbb{Z}^n$. Now let $c_{a,\hbar}^{(1)}$ be the scalar in front of ψ_{k+a} on the right-hand side of the last equation. It is not hard to see from this equation that

$$\|\mathcal{Q}_\hbar^W(f_1)\mathcal{Q}_\hbar^W(f_2) - \mathcal{Q}_\hbar^W(f_1 f_2)\| \leq \sup_{a\in\mathbb{Z}^n} |c_{a,\hbar}^{(1)}|,$$

for each $\hbar > 0$. Now note for $j = 1, 2$, the Fréchet derivative Dh_j of h_j is bounded, i.e.,

$$\|Dh_j\|_\infty := \sup_{p\in\mathbb{R}^n} \|Dh_j(p)\| < \infty,$$

so that

$$|h_j(p) - h_j(p')| \leq \|Dh_j\|_\infty \cdot \|p - p'\|,$$

for each $p, p' \in \mathbb{R}^n$. In particular, h_j is Lipschitz continuous, a fact that we already used in the proof of Proposition 7.7. (We have made the Lipschitz constant explicit, since we will be using similar notation in a slightly different setting in the proof of part (3) of this theorem.) Using the triangle inequality, we obtain

$$\begin{aligned}|c^{(1)}_{a,\hbar}| &\leq |h_1(\pi\hbar(k+2a))(h_2(\pi\hbar(k+2a)) - h_2(\pi\hbar(k-k_1+2a)))| \\ &\quad + |(h_1(\pi\hbar(k+k_2+2a)) - h_1(\pi\hbar(k+2a))) \cdot h_2(\pi\hbar(k-k_1+2a))| \\ &\leq \pi\hbar\left(\|h_1\|_\infty \cdot \|Dh_2\|_\infty \cdot \|k_1\| + \|Dh_1\|_\infty \cdot \|h_2\|_\infty \cdot \|k_2\|\right),\end{aligned}$$

for each $\hbar > 0$ and each $a \in \mathbb{Z}^n$. The right-hand side of this inequality is independent of a, and converges to 0 as $\hbar \to 0$, hence

$$\lim_{\substack{\hbar\to 0\\ \hbar\in I}} \|\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2) - \mathcal{Q}^W_\hbar(f_1 f_2)\| = 0.$$

By bilinearity, this result extends to arbitrary $f_1, f_2 \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$.

(3) As in the previous part of the proof, we prove the statement for $f_j = e_{k_j} \otimes h_j$, from which the general case readily follows. We have

$$\begin{aligned}&\{f_1, f_2\} \\ &\quad = \sum_{l=1}^n \left(\frac{\partial f_1}{\partial p_l}\frac{\partial f_2}{\partial q^l} - \frac{\partial f_1}{\partial q^l}\frac{\partial f_2}{\partial p_l}\right) \\ &\quad = \sum_{l=1}^n \left(e_{k_1} \otimes \frac{\partial h_1}{\partial p_l}\right) \cdot \left(\frac{\partial e_{k_2}}{\partial q^l} \otimes h_2\right) - \left(\frac{\partial e_{k_1}}{\partial q^l} \otimes h_1\right) \cdot \left(e_{k_2} \otimes \frac{\partial h_2}{\partial p_l}\right) \\ &\quad = 2\pi i e_k \otimes \left(Dh_1(\cdot)(k_2) \cdot h_2 - h_1 \cdot Dh_2(\cdot)(k_1)\right),\end{aligned}$$

where $k = k_1 + k_2$, as in part (2) of this theorem, and $Dh_j(\cdot)(v)$ denotes the map

$$\mathbb{R}^n \to \mathbb{C}, \quad p \mapsto Dh_j(p)(v),$$

for each $v \in \mathbb{R}^n$ and $j = 1, 2$. Applying part (3) of Proposition (7.1) yields

$$\mathcal{Q}^W_\hbar(\{f_1, f_2\})\psi_a$$

$$= 2\pi i\,(Dh_1(\cdot)(k_2)\cdot h_2 - h_1\cdot Dh_2(\cdot)(k_1))\,(\pi\hbar(k+2a))\psi_{k+a},$$

while equation (7.9) yields

$$\begin{aligned}
&[\mathcal{Q}_\hbar^W(f_1), \mathcal{Q}_\hbar^W(f_2)]\psi_a \\
&= (h_1(\pi\hbar(k+k_2+2a))\cdot h_2(\pi\hbar(k-k_1+2a)) \\
&\quad - h_1(\pi\hbar(k-k_2+2a))\cdot h_2(\pi\hbar(k+k_1+2a)))\,\psi_{k+a}.
\end{aligned}$$

It follows that

$$\left((-i\hbar)^{-1}[\mathcal{Q}_\hbar^W(f_1), \mathcal{Q}_\hbar^W(f_2)] - \mathcal{Q}_\hbar^W(\{f_1,f_2\})\right)\psi_a = c^{(2)}_{a,\hbar}\psi_{k+a},$$

where for each $a\in\mathbb{Z}^n$ and each $\hbar>0$, we define

$$\begin{aligned}
c^{(2)}_{a,\hbar} := (-i\hbar)^{-1}\,&(h_1(\pi\hbar(k+k_2+2a))\cdot h_2(\pi\hbar(k-k_1+2a)) \\
&\quad - h_1(\pi\hbar(k-k_2+2a))\cdot h_2(\pi\hbar(k+k_1+2a))) \\
&- 2\pi i\,(Dh_1(\cdot)(k_2)\cdot h_2 - h_1\cdot Dh_2(\cdot)(k_1))\,(\pi\hbar(k+2a)).
\end{aligned}$$

It is readily seen that

$$\left\|(-i\hbar)^{-1}[\mathcal{Q}_\hbar^W(f_1), \mathcal{Q}_\hbar^W(f_2)] - \mathcal{Q}_\hbar^W(\{f_1,f_2\})\right\| \leq \sup_{a\in\mathbb{Z}^n}|c^{(2)}_{a,\hbar}|.$$

We claim that the right-hand side of this inequality converges to 0 as $\hbar\in I$ goes to 0; evidently, this will show that Dirac's condition holds.

By Taylor's theorem, we have

$$\begin{aligned}
&|h_j(\pi\hbar(k+v+2a)) - (h_j(\pi\hbar(k+2a)) + \pi\hbar Dh_j(\pi\hbar(k+2a))(v))| \\
&\quad \leq (\pi\hbar)^2\|v\|^2\|D^2h_j\|_\infty,
\end{aligned} \tag{7.10}$$

for each $v\in\mathbb{R}^n$ and $j=1,2$. Here, $D^2h_j\in\mathbb{R}^n\to S^2(\mathbb{R}^n)^*$ denotes the second order Fréchet derivative of h_j, which assigns to each element $x\in\mathbb{R}^n$ the linear map

$$D^2h_j(x)\colon (\mathbb{R}^n)^{\otimes 2}\supseteq S^2\mathbb{R}^n\to\mathbb{C},\quad v\otimes w\mapsto \frac{\partial^2}{\partial s\partial t}h_j(x+sv+tw)|_{s,t=0},$$

and the norm of this map is the operator norm with respect to the unique cross norm on $(\mathbb{R}^n)^{\otimes 2}$ corresponding to the euclidean norm on $\mathbb{R}^n$. It can be checked that the map $x \mapsto D^2 h_j(x)$ is uniformly bounded with respect to the operator norm, and we define

$$\|D^2 h_j\|_\infty := \sup_{x\in\mathbb{R}^n} \|D^2 h_j(x)\|.$$

Returning to equation (7.10), dividing the expression on the left-hand side within absolute value strokes by $-i\hbar$ and modifying the right-hand side accordingly yields

$$\begin{aligned}
&|(-i\hbar)^{-1} h_j(\pi\hbar(k+v+2a)) \\
&\quad - ((-i\hbar)^{-1} h_j(\pi\hbar(k+2a)) + \pi i D h_j(\pi\hbar(k+2a))(v))| \\
&\quad \le \pi^2 \hbar \|v\|^2 \|D^2 h_j\|_\infty,
\end{aligned}$$

and taking the limit $\hbar \to 0$, we see that the left-hand side converges to 0. This can be used to show that $|c^{(2)}_{a,\hbar}| \to 0$ as $\hbar \to 0$ uniformly in $a \in \mathbb{Z}^n$, which proves the claim.

(4) According to part (2) of Proposition 5.9, the space $\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$ is a *-subalgebra of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. According to part (1) of Proposition 7.4 the Weyl quantisation map is linear and compatible with the involutions on the algebras involved. Moreover, it is readily seen from our computation of $\mathcal{Q}^W_\hbar(f_1)\mathcal{Q}^W_\hbar(f_2)$ in the proof of part (2) of this theorem that $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$ is closed under multiplication. Thus $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$ is a *-algebra. It follows that $A_\hbar$, which is by definition the smallest C*-algebra generated by $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$, is the closure of $\mathcal{Q}^W_\hbar(\mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n))$. ∎

7.9 Remark. The statement that for arbitrary $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, the map

$$[0,\infty) \to [0,\infty), \quad \hbar \mapsto \|\mathcal{Q}^W_\hbar(f)\|,$$

is continuous at points other than $\hbar = 0$ is false. As a counterexample, let $\hbar_0 > 0$ be arbitrary, and consider the function $f = e_0 \otimes h$, where the function h is defined as follows:

$$h\colon \mathbb{R}^n \to \mathbb{R}, \quad p = (p_1, p_2, \ldots, p_n) \mapsto \sin\left(\frac{p_1}{\hbar_0}\right).$$

Note that h can be written as the sum of two generators of $\mathcal{W}^0(\mathbb{R}^n) \subseteq \mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$, so $f \in \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$. Futhermore, h vanishes at each point in

$2\pi\hbar_0 \cdot \mathbb{Z}^n$, hence $\mathcal{Q}^W_{\hbar_0}(f) = 0$ by part (3) of Proposition 7.1, or equivalently, $\|\mathcal{Q}^W_{\hbar_0}(f)\| = 0$. On the other hand, for each $N \in \mathbb{N}\backslash\{0\}$, let

$$\hbar_N := \hbar_0 \left(1 + \frac{1}{4N}\right).$$

Then $\|\mathcal{Q}^W_{\hbar_N}(f)\| = 1$; indeed, we have $\|\mathcal{Q}^W_{\hbar_N}(f)\| \leq \|h\|_\infty = 1$, and equality holds since

$$\mathcal{Q}^W_{\hbar_0}(f)\psi_{(N,0,0,\ldots,0)} = \psi_{(N,0,0,\ldots,0)}.$$

Thus we have $\lim_{N\to\infty} \hbar_N = \hbar_0$, while

$$\lim_{N\to\infty} \|\mathcal{Q}^W_{\hbar_N}(f)\| = 1 \neq 0 = \|\mathcal{Q}^W_{\hbar_0}(f)\|,$$

so the function $\hbar \to \|\mathcal{Q}^W_\hbar(f)\|$ fails to be continuous at $\hbar_0$.

The issue of continuity of the norm of the quantisation of a given function at points $\hbar \neq 0$ is often sidestepped in the literature for reasons related to geometric quantisation, which imposes the condition that $\hbar$ be of the form $\hbar_0/m$, $m \in \mathbb{N}\backslash\{0\}$ for some fixed $\hbar_0 > 0$ (cf. [52] for a discussion of this point, and also a nice overview of the various notions of quantisation throughout the literature). It follows that in such cases the set $I\backslash\{0\}$ in the above theorem is a discrete subset of $(0,\infty)$, so the restriction of $\hbar \to \|\mathcal{Q}^W_\hbar(f)\|$ to I is trivially continuous at all points outside of 0, and the family of quantisation maps constitutes an actual strict quantisation.

Despite the fact that the norm of the quantisation of a function is not continuous for $\hbar > 0$, we still have continuity of quantisation in another way:

7.10 Proposition. *Let $f \in \mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$. Then the map*

$$(0,\infty) \to A_\hbar \subseteq B(L^2(\mathbb{T}^n)), \quad \hbar \mapsto \mathcal{Q}^W_\hbar(f),$$

is continuous with respect to the strong operator topology on the codomain.

Proof. By linearity of the quantisation map and the fact that $\mathcal{S}_\mathcal{R}(T^*\mathbb{T}^n)$ is the linear span of generators of $\mathcal{C}_\mathcal{R}(T^*\mathbb{T}^n)$, we may assume without loss of generality that there exists a $k \in \mathbb{Z}^n$ and a generator h of $\mathcal{W}^0_\mathcal{R}(\mathbb{R}^n)$

such that $f = e_k \otimes h$. Furthermore, it is readily seen from part (3) of Proposition 7.1 that there exists a generator $\tilde{h}$ of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$ such that

$$\mathcal{Q}^W_\hbar(f) = M_{e_k}\mathcal{Q}^W_\hbar(e_0 \otimes \tilde{h}),$$

where $M_{e_k} \in B(L^2(\mathbb{T}^n))$ denotes the multiplication operator associated to the function e_k, and does not depend on $\hbar$. Regarding the quantised functions on both sides of this equations as maps in $\hbar$, continuity (with respect to the strong operator topology) of the map on the right implies continuity of the map on the left. Thus we may assume without loss of generality that $k = 0$, i.e., $f = e_0 \otimes h$ for some generator h of $\mathcal{W}^0_{\mathcal{R}}(\mathbb{R}^n)$.

Now fix $\hbar_0 > 0$, let $\psi \in L^2(\mathbb{T}^n)$ be such that $\|\psi\| \leq 1$, and let $\varepsilon > 0$. Then there exists a subset $X \subset \mathbb{Z}^n$ that is bounded with respect to any norm $\|\cdot\|$ on $\mathbb{R}^n$ such that

$$\|\psi - P_X\psi\| < \frac{\varepsilon}{4(\|h\|_\infty + 1)},$$

where $P_X \in B(L^2(\mathbb{T}^n))$ denotes the orthogonal projection

$$\psi \mapsto \sum_{a\in X}\langle\psi_a, \psi\rangle\psi_a,$$

and the assumption that X is bounded means that there exists $R > 0$ such that $\|a\| \leq R$ for each $a \in X$. The function h is Lipschitz continuous with Lipschitz constant M, say, so if we set

$$\delta := \min\left(\hbar_0, \frac{\varepsilon}{4\pi R(M+1)}\right),$$

then for each $\hbar \in (\hbar_0 - \delta, \hbar_0 + \delta)$ and each $a \in X$, we have

$$\|2\pi\hbar a - 2\pi\hbar_0 a\| < 2\pi\delta\|a\| \leq \frac{\varepsilon}{2(M+1)},$$

hence

$$|h(2\pi\hbar a) - h(2\pi\hbar_0 a)| < \frac{\varepsilon}{2},$$

and consequently

$$\left\|\left(\mathcal{Q}^W_\hbar(f) - \mathcal{Q}^W_{\hbar_0}(f)\right)\psi\right\|$$

$$
\begin{aligned}
&\leq \left\| \left(\mathcal{Q}_\hbar^W(f) - \mathcal{Q}_{\hbar_0}^W(f) \right) P_X \psi \right\| + \left(\left\| \mathcal{Q}_\hbar^W(f) \right\| + \left\| \mathcal{Q}_{\hbar_0}^W(f) \right\| \right) \|\psi - P_X \psi\| \\
&\leq \left\| \sum_{a \in X} \langle \psi_a, \psi \rangle (h(2\pi\hbar a) - h(2\pi\hbar_0 a)) \psi_a \right\| + 2\|h\|_\infty \cdot \|\psi - P_X \psi\| \\
&< \frac{\varepsilon}{2} \|\psi\| + 2\|h\|_\infty \cdot \frac{\varepsilon}{4(\|h\|_\infty + 1)} < \varepsilon,
\end{aligned}
$$

which shows that the map $\hbar \mapsto \mathcal{Q}_\hbar^W(f)$ is strongly continuous at $\hbar_0$. ∎

7.4 Invariance under time evolution

We now show that the quantisation $A_\hbar$ of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ is invariant under the quantum time evolution for a large class of Hamiltonians, in a way analogous to the discussion for the classical case in section 5.3. First we discuss the free case for general n, then the interacting one for $n = 1$, and we end the section with a discussion on the higher dimensional case. Our proof strategy is basically the method of Buchholz and Grundling [26, Proposition 6.1 and pp. 40–41] adapted to the cylinder. As in section 5.3, a proof of the general case of $n \in \mathbb{N}$ will appear in a forthcoming paper of van Nuland and the author [113]. In our exposition, we set all physical constants in the Hamiltonian equal to 1, except for $\hbar$.

7.11 Lemma. *Let $\hbar > 0$. The algebra $A_\hbar$ is closed under the quantum time evolution corresponding to the free Hamiltonian H_0 that is the unique self-adjoint extension of the essentially self-adjoint operator $-\frac{\hbar^2}{2}\Delta$ with domain $C^\infty(\mathbb{T}^n)$.*

7.12 Remark. The fact that for any compact Riemannian manifold M, the Laplace–Beltrami operator on $C^\infty(M)$ has a unique self-adjoint extension, is due to Gaffney [42].

Proof. We show that the quantum time evolution corresponding to H_0 maps the set of quantisations of the generators $e_k \otimes h_{U,\xi,g}$ of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$ into itself; since the time evolution consists of a family of automorphisms of C*-algebras, the lemma will follow from this.

Let $f = e_k \otimes h_{U,\xi,g}$ be such a generator. Note that for each $a \in \mathbb{Z}^n$, we have

$$
e^{-\frac{itH_0}{\hbar}} \psi_a = e^{-2\pi^2 it\hbar\|a\|^2} \psi_a.
$$

Using part (3) of Proposition 7.1, we obtain

$$\begin{aligned}
&e^{\frac{itH_0}{\hbar}}\mathcal{Q}_\hbar^W(f)e^{-\frac{itH_0}{\hbar}}\psi_a \\
&\quad= e^{2\pi^2 it\hbar(\|a+k\|^2-\|a\|^2)}e^{\pi\hbar i(k+2a)\cdot\xi}g\circ r_U(\pi\hbar(k+2a))\psi_{k+a} \\
&\quad= e^{\pi i\hbar(k+2a)\cdot(\xi+2\pi tk)}g\circ r_U(\pi\hbar(k+2a))\psi_{k+a} \\
&\quad= \mathcal{Q}_\hbar^W(e_k\otimes h_{U,\tilde{\xi},\tilde{g}})\psi_a,
\end{aligned}$$

for each $a\in\mathbb{Z}^n$, where

$$\tilde{\xi} := \xi + 2\pi t(I-r_U)(k)\in U^\perp,$$

and

$$\tilde{g}\colon U\to\mathbb{C},\quad p\mapsto e^{2\pi itr_U(k)\cdot p}g(p),$$

is again a Schwartz function on U, so $\tilde{f} := e_k\otimes h_{U,\tilde{\xi},\tilde{g}}$ is a generator of $C_{\mathcal{R}}(T^*\mathbb{T}^n)$. Since both $e^{\frac{itH_0}{\hbar}}\mathcal{Q}_\hbar^W(f)e^{-\frac{itH_0}{\hbar}}$ and $\mathcal{Q}_\hbar^W(\tilde{f})$ are bounded by definition of $\mathcal{Q}_\hbar^W$, it follows from the above computation that they must be equal, so the set of images of generators of the resolvent algebra under $\mathcal{Q}_\hbar^W$ is indeed invariant under the quantum time evolution. ∎

7.13 Proposition. *Let $\hbar>0$, let $V\in C(\mathbb{T})$, let $M(V)$ be its corresponding multiplication operator on $L^2(\mathbb{T})$, and let $H=-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+M(V)$ be the operator with domain $\mathcal{D}(H_0)$. The algebra $A_\hbar$ is closed under the quantum time evolution corresponding to the Hamiltonian H.*

7.14 Remark. The self-adjointness of H is a consequence of the Kato–Rellich theorem.

Proof. We claim that for each $t\in\mathbb{R}$, we have

$$e^{\frac{itH_0}{\hbar}}e^{\frac{-itH}{\hbar}}\in A_\hbar.$$

Suppose for the moment that this claim holds true. Let τ^0 and $\tau\colon\mathbb{R}\to\operatorname{Aut}(B(L^2(\mathbb{T})))$ be the quantum time evolutions (in the Heisenberg picture) corresponding to the Hamiltonians H_0 and H, respectively; for each $t\in\mathbb{R}$, we write τ_t^0 for $\tau^0(t)$, and τ_t for $\tau(t)$. Then for each $a\in A_\hbar$ and each $t\in\mathbb{R}$, we have

$$\tau_t(a)=e^{\frac{itH}{\hbar}}ae^{\frac{-itH}{\hbar}}=\left(e^{\frac{itH}{\hbar}}e^{\frac{-itH_0}{\hbar}}\right)^*(\tau_t(a))\left(e^{\frac{itH_0}{\hbar}}e^{\frac{-itH}{\hbar}}\right).$$

By assumption, the first and the third factors within parentheses are elements of $A_\hbar$, and the second factor is an element of $A_\hbar$ by the preceding lemma. It then follows that $\tau_t(a) \in A_\hbar$.

Thus it remains to prove the claim. First note that without loss of generality, we may assume that $\int_{\mathbb{T}} V(x+\mathbb{Z})\,dx = 0$. As in [26], we use the fact that the product of two of the elements of the different one parameter groups can be written as a norm-convergent Dyson series, i.e.,

$$\begin{aligned}
&e^{\frac{itH_0}{\hbar}} e^{\frac{-itH}{\hbar}} \\
&= \sum_{k=0}^{\infty} (i\hbar)^{-k} \\
&\qquad \cdot \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} (\tau^0_{t_1} \circ M(V)) \circ \cdots \circ (\tau^0_{t_k} \circ M(V))\, dt_k \, \ldots \, dt_2\, dt_1.
\end{aligned}$$

The integrals in the above expression can be defined in the following way. First, observe that the function

$$\mathbb{R} \to B(L^2(\mathbb{T})), \quad t \mapsto \tau^0_t \circ M(V),$$

is bounded and strongly continuous. It follows that the function

$$\mathbb{R}^k \to B(L^2(\mathbb{T})), \quad (t_1, \ldots, t_k) \mapsto (\tau^0_{t_1} \circ M(V)) \circ \cdots \circ (\tau^0_{t_k} \circ M(V)),$$

is bounded and strongly continuous. For each $\psi \in L^2(\mathbb{T})$, one can therefore define the integral

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} (\tau^0_{t_1} \circ M(V)) \circ \cdots \circ (\tau^0_{t_k} \circ M(V))(\psi)\, dt_k \, \ldots \, dt_2\, dt_1,$$

using Bochner integration, and it is easy to check that the norm of the corresponding operator is less than or equal to $(k!)^{-1}\|V\|^k_\infty$, so that the Dyson series is indeed norm-convergent. In addition, as in [26], it suffices to prove the claim for potentials $V \in S$, where S is any dense subset of the space

$$\left\{ g \in C(\mathbb{T})\colon \int_{\mathbb{T}} g(x+\mathbb{Z})\,dx = 0 \right\}.$$

We prove the claim for $S = \operatorname{span}_{\mathbb{C}}\{e_k \colon k \in \mathbb{Z}\backslash\{0\}\}$.

The identity operator, which is the first term in the Dyson series, is an element of $A_\hbar$. We claim that the remaining terms are all Hilbert–Schmidt operators; in particular, they are compact, which implies that their sum is compact as well, hence by part (3) of Proposition 7.4, they are elements of $A_\hbar$. First, we show that for each $t \in \mathbb{R}$ and each $k \in \mathbb{Z}$, we have

$$\int_0^t \tau_s^0 \circ M(e_k)\, ds \in L^2(L^2(\mathbb{T})). \tag{7.11}$$

From the calculation of $\tau_t^0(\mathcal{Q}_\hbar^W(e_k \otimes h_{U,\xi,g}))$ in the previous lemma, taking $U = \{0\}$, $\xi = 0$, and $g = 1$, we get

$$\tau_t^0 \circ M(e_k)\psi_a = e^{2\pi^2 it\hbar((a+k)^2-a^2)}\psi_{k+a}$$

for each $a \in \mathbb{Z}$ and each $t \geq 0$. It follows that

$$\begin{aligned}\int_0^t \tau_s^0 \circ M(e_k)\psi_a\, ds &= \frac{e^{2\pi^2 it\hbar((a+k)^2-a^2)} - 1}{2\pi^2 i\hbar((a+k)^2 - a^2)}\psi_{k+a} \\ &= a^{-1}\frac{e^{2\pi^2 it\hbar((a+k)^2-a^2)} - 1}{2\pi^2 i\hbar(2k + a^{-1}k^2)}\psi_{k+a},\end{aligned}$$

hence

$$M(e_{-k})\int_0^t \tau_s^0 \circ M(e_k)\psi_a\, ds = a^{-1}\frac{e^{2\pi^2 it\hbar((a+k)^2-a^2)} - 1}{2\pi^2 i\hbar(2k + a^{-1}k^2)}\psi_a,$$

for each $a \in \mathbb{Z}\backslash\{0\}$ and each $t \geq 0$. Now, as $a \to \pm\infty$, the denominator of the fraction on the right-hand side of this equation converges to $4\pi^2 i\hbar k$, while the absolute value of the numerator is bounded by 2, so the fraction, viewed as a function of a, is bounded. This, combined with the fact that $\sum_{a\in\mathbb{Z}\backslash\{0\}} a^{-2} < \infty$, implies that $M(e_{-k})\int_0^t \tau_s^0 \circ M(e_k)\, ds$ is a Hilbert–Schmidt operator, hence equation (7.11) holds for $t \geq 0$. A similar argument shows that the operator $\int_0^t \tau_s^0 \circ M(e_k)\, ds$ is Hilbert–Schmidt if $t < 0$. Thus $\int_0^t \tau_s^0 \circ M(V)\, ds$ is Hilbert–Schmidt for each $V \in S$.

Next, note that the map

$$\mathbb{R} \to B(L^2(\mathbb{T})), \quad t \mapsto \int_0^t \tau_s^0 \circ M(V)\, ds,$$

is norm-continuous. Using this together with the fact that the map $t \mapsto \tau_t^0 \circ M(V)$ is strongly continuous, we infer that the pointwise product of these maps is also strongly continuous, and its image lies in $L^2(L^2(\mathbb{T}))$. Moreover, from computations above, it is clear that the L^2-norms of the images of the function in the displayed formula above are uniformly bounded. Since a strong limit of Hilbert–Schmidt operators whose Hilbert–Schmidt norms are uniformly bounded, is again Hilbert–Schmidt, we have

$$\begin{aligned} &\int_0^t \int_0^{t_1} (\tau_{t_1}^0 \circ M(V)) \circ (\tau_{t_2}^0 \circ M(V))\, dt_2\, dt_1 \\ &\quad = \int_0^t (\tau_{t_1}^0 \circ M(V)) \circ \int_0^{t_1} \tau_{t_2}^0 \circ M(V)\, dt_2\, dt_1 \in L^2(L^2(\mathbb{T})). \end{aligned}$$

Using induction on the number of integrals, where the induction step is essentially the same as the one just given, it can be shown that each term in the Dyson series except for the first one lies in $L^2(L^2(\mathbb{T}))$, which is what we wanted to show. ■

7.15 Remark. As mentioned at the beginning of this section, a proof of the statement of Proposition 7.13 for arbitrary n will appear in a forthcoming paper; here, we only briefly sketch the idea behind this proof. As in the one-dimensional case, one notes that it suffices to show that the terms

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} (\tau_{t_1}^0 \circ M(V)) \circ \cdots \circ (\tau_{t_l}^0 \circ M(V))\, dt_l\, \ldots\, dt_2\, dt_1,$$

in the Dyson series are elements of $A_\hbar$. One restricts to the case that

$$V \in \operatorname{span}_{\mathbb{C}}\{e_k \colon k \in \mathbb{Z}^n \setminus \{0\}\},$$

and notes that the terms in the Dyson series are linear combinations of operators of the form

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} (\tau_{t_1}^0 \circ M(e_{k_1})) \circ \cdots \circ (\tau_{t_l}^0 \circ M(e_{k_l}))\, dt_l\, \ldots\, dt_2\, dt_1.$$

One then writes these operators as tensor products of integral operators on $L^2(K)$, where K denotes some Lie subgroup of $\mathbb{T}^n$, and the identity operator on the Hilbert space associated to the quotient $\mathbb{T}^n/K$. Such tensor products can be shown to be elements of $A_\hbar$.

Chapter 8

The embedding maps revisited

8.1 Introduction

In the final chapter of this thesis, we return to the problems discussed at the end of chapter 4 and the beginning of chapter 5, specifically the problem of the embedding maps of the observable algebras associated to refinements of graphs. We will construct different maps that are not motivated by the groupoid picture, but rather by the idea that families of quantisation maps, in our case the family of Weyl quantisation maps $(\mathcal{Q}_\hbar^W)_{\hbar\in(0,\infty)}$ from chapter 7, should constitute a weak version of a natural transformation, which we call an *approximate* natural transformation, between two functors having the same category $\mathsf{wtRefine}$ as their domain. Here, $\mathsf{wtRefine}$ is a modified version of Refine of which the objects carry weights in addition to having an orientation, and of which the morphisms respect these weights. The first functor will then point to a category $\mathsf{Classical}$ of algebras associated to classical systems, while the second functor points to a category $\mathsf{Quantum}$ of algebras associated to quantum systems.

Although to the knowledge of the author, regarding quantisation as a (type of) natural transformation is a novel idea, attempts to cast strict quantisation into a categorical framework have been made before by others, notably Landsman [67, 69], and arguably earlier by Rieffel as well

[98], albeit with a more modest scope. It is therefore only appropriate to give a brief exposition of both of these, and compare them to the point of view presented in this chapter.

We start by discussing the considerations that lead up to Landsman's proposed functor. In [67], Landsman notes that the most naive proposal, in which the classical category consists of Poisson manifolds with (the suitable notion of) isomorphisms between them, and the quantum category consists of C*-algebras with *-isomorphisms, functoriality entails equivariance of the quantisation procedure with respect to actions of some group on both sides of the functor, and that this group generally depends on the quantisation map and the space that is being quantised, with Weyl quantisation, out of all quantisations of $\mathbb{R}^{2n}$, admitting the largest group. It is therefore necessary to work with categories in which the notion of isomorphism is weaker. According to Landsman, a good choice for the classical category is the category $\mathsf{Poisson}$ that has Poisson manifolds as objects, and isomorphism classes of Weinstein dual pairs as morphisms (with an appropriate notion of composition). Landsman then discusses the possibility of taking the category in which C*-algebras are the objects and Hilbert bi-modules are the morphisms as the quantum category, and proves that there is a quantisation functor to this category whose domain is a subcategory of $\mathsf{Poisson}$, namely the one that consists of Poisson manifolds associated to Lie groupoids, with a more restrictive notion of morphism. This functor maps an object in the subcategory to the C*-algebra of its corresponding Lie groupoid.

To allow for a notion of functoriality that extends to the larger category of Poisson manifolds, one should weaken the notion of morphism from Hilbert bi-modules to KK-classes (for fixed $\hbar$). To incorporate deformation quantisation into his framework, so that we may consider quantisations of classical objects for varying $\hbar$, Landsman introduces the category RKK, which is the analogue of KK for fields of C*-algebras, and he conjectures that strict quantisation yields a functor $\mathsf{Poisson} \to \mathsf{RKK}$.

In [69], it is argued that the map between the classes of arrows defined by this functor corresponds to geometric quantisation, and that its functoriality, i.e., its compatibility with respect to composition of arrows in both classes, yields a reformulation of the Guillemin–Sternberg conjecture when applied to specific arrows in Poisson. For the reader's convenience, we briefly recall the statement of the conjecture here.

Suppose that (M, ω) is a pre-quantisable symplectic manifold that carries a smooth action of a Lie group G that acts on M by symplectomorphisms. There are now two possible ways to obtain the reduced version of the Hilbert space associated to the quantum system corresponding to (M, ω):

(1) We can first apply geometric quantisation to (M, ω) to obtain the unreduced Hilbert space, after which we consider the elements of the resulting Hilbert space that are invariant with respect to the action of G on the full Hilbert space;

(2) We can first take the Marsden–Weinstein quotient of (M, ω) with respect to the action of G to obtain the reduced classical phase space, which we subsequently quantise (again using geometric quantisation) to obtain the reduced Hilbert space.

According to the Guillemin–Sternberg conjecture, the resulting reduced Hilbert spaces should not depend on the chosen option, i.e., they should be isomorphic in a canonical way. Thus, it should not matter whether one first quantises and then reduces the classical phase space, or the other way around; for this reason, the conjecture is sometimes stated as "quantisation commutes with reduction."

Returning to the discussion of functoriality of quantisation, the conjectured functoriality of the extension of Landsman's quantisation functor provides a way to generalise the Guillemin–Sternberg conjecture. While this is very impressive in its own right, in this chapter, we are interested in the naive setting in which the quantum category consists of C*-algebras with *-homomorphisms between them, since we wish to find the *-homomorphism between our quantum resolvent algebras associated to a morphism between classical objects, which we also construct in this chapter. We already listed some problems in section 4.8 that call into question the assumption that the quantisation of the systems under consideration is given by the groupoid C*-algebra of the pair groupoid associated to each of their configuration spaces, or at least the usefulness of this assumption in the context of this thesis; this assumption is part of Landsman's functor, however. Worse still, there is the issue that the functor Poisson $\to$ RKK only records the *outcome* of quantisation in the guise of fields of C*-algebras, and not the *process* of quantisation that produces these fields, i.e., families $(\mathcal{Q}_\hbar)_{\hbar\in I}$ of quantisation maps, rendering it far too coarse for our purposes.

Since we wish to return to a slightly more general version of the naive setting mentioned earlier, we should address how we avoid the issue that forces Landsman to work with different categories. Since we regard our family of quantisation maps as a natural transformation rather than a functor, when it comes to equivariance of the quantisation maps with respect to group actions, we only need to worry about images of isomorphisms in $\mathsf{wtRefine}$ under the functor to $\mathsf{Classical}$. This already tremendously reduces the family of isomorphisms for which we have to check equivariance. Furthermore, our functor will be a composition of three functors, the first of which is a contravariant functor from $\mathsf{wtRefine}$ to a category of which the class of objects contains the configuration spaces of interest. This means that there will be no isomorphisms that 'mix' position and momentum degrees of freedom, making the requirement of equivariance with respect to isomorphisms practically trivial. In fact, to incorporate gauge transformations into our formulation, we note that the various categories other than $\mathsf{wtRefine}$ can be modified as in part (2) of Remark 4.9, thereby effectively imposing equivariance conditions ourselves.

Before discussing how we generalise the naive setting, we first recall Rieffel's functor, which can be found in in [98] and is much closer to what we are after. Let V be a finite-dimensional real vector space V and let A be a Fréchet algebra that carries an isometric action of V such that the subalgebra A^∞ of elements that are smooth with respect to this action (which are called *smooth vectors*) is dense in A. Rieffel shows how the (associative) product on A^∞ can be deformed into another associative product in a functorial way using a linear operator J on V (see section 2 and in particular proposition 2.10 in [98]). The resulting algebra is called A^∞_J; If $J = 0$, then the deformed product is the original one. Furthermore, if A^∞ carries an involution, then A^∞_J can be endowed with an involution as well provided that J is skew-symmetric.

Now let $C_u(V, A)$ be the space of uniformly continuous A-valued functions on V. The actions of V on A and on itself by translation canonically induce an action of V on $C_u(V, A)$, whose subspace of smooth vectors is denoted by $\mathcal{B}^A$. Rieffel applies the construction of the deformed product to the case $A^\infty = \mathcal{B}^A$ and shows that if A is a C*-algebra and J is skew-symmetric, then its deformation $\mathcal{B}^A_J$ can be endowed with an involution and a norm such that the completion with respect to this norm can be given the structure of a C*-algebra in a canonical way. He proceeds to embed A^∞ into this new C*-algebra, and uses the C*-norm to define a norm on A^∞, which he uses to complete A^∞ to obtain a C*-algebra. Rieffel subsequently shows that the functorial correspondence between algebras A^∞ and their deformations extends to a functorial correspondence between their respective completions [98, Theorem 5.7]. The final result that we mention is [98, Theorem 9.3], which shows that certain families of deformations form a deformation quantisation of the original C*-algebra.

The case of interest to us is the one in which $V = \mathbb{R}^{2n}$, the space A is the observable algebra of a classical system, and J is the standard symplectic matrix multiplied by $\hbar$, which is essentially Weyl quantisation (cf. [98, Examples 10.1, 10.5 and 10.6]). In this setting, functoriality of the deformations entails that there is a functor between the following categories:

- The domain of the functor is the category whose objects are commutative C*-algebras endowed with an action of $\mathbb{R}^{2n}$ whose space of smooth vectors is dense in the algebra, and whose morphisms are *-homomorphisms that are equivariant with respect to the action;

- The codomain of the functor is defined similarly to the domain, but we drop the requirement that the algebras be commutative.

Given a *-homomorphism between classical observable algebras, the functor immediately provides us with a *-homomorphism between the deformations, which is what we are after in principle. However, the image of an object under the functor need not have a canonical representation on the Hilbert space corresponding to the configuration space of the system; cf. [98, Example 10.6], where we would like to represent the deformed algebra on $L^2(\mathbb{T})$. Furthermore, the objects in Rieffel's category all carry actions of $\mathbb{R}^{2n}$, and the functor is defined for a fixed skew-symmetric matrix J, although [98, Proposition 2.7] suggests that the former point can be relaxed somewhat by appropriately modifying the defintion of the category to allow objects to carry actions of subspaces of $\mathbb{R}^{2n}$, or spaces that contain an isomorphic copy of $\mathbb{R}^{2n}$. By contrast, the objects in the category $\mathsf{Classical}$ that we introduce in this chapter do not contain an action as part of their data, but merely the Poisson structure that is obtained from this action, making it more flexible in principle.

It is also worth mentioning that we introduce a notion of morphisms between manifolds endowed with a bilinear form that generalises among other things the notion of a symplectomorphism, allowing us to not just consider diffeomorphisms between symplectic manifolds that preserve the symplectic structure, but also other maps such as immersions and submersion; this is what we were referring to when we mentioned the generalisation of the naive setting in the discussion of Landsman's work above. We are specifically interested in submersions, since these are the maps that appear in the inverse systems of the configuration spaces and phase spaces that we consider. Pullbacks of the maps between phase spaces induce Poisson maps between Poisson algebras and therefore allow us to provide a more general framework than the generalisations of Rieffel's categories.

Finally, as already mentioned at the end of section 1.1, Rieffel's notion of strict deformation quantisation focusses on deformation of the product on the classical observable algebra, whereas we wish to use Landsman's notion of quantisation in which a central role is reserved for a family of quantisation maps $(\mathcal{Q}_\hbar)_{\hbar \in I}$. We require our approximate natural transformations to become natural transformations in the classical limit $\hbar \to 0$ only (hence the use of the term 'approximate') in a sense that will be made precise in section 8.3. This notion anticipates the use of families of quantisation maps that are different from Weyl quantisation, while still giving an indication how we ought to define the arrow part of our functor from $\mathsf{wtRefine}$ to $\mathsf{Quantum}$.

We give a brief outline of this chapter. In section 8.2, we first define some classes of manifolds with morphisms between them of which certain subclasses form categories. We then define the classical category $\mathsf{Classical}$ of algebras, as well as a functor from the previous subclasses to this category, and discuss how lattice gauge theory fits into this picture.

In section 8.3, we define the quantum category $\mathsf{Quantum}$, and, in the context of lattice gauge theory with $G = \mathbb{T}^n$, attempt to define a functor from a modified version of the category of refinements to the quantum category by defining the image of refinements in such a way that $\mathcal{Q}_\hbar^W$ is a natural transformation from the classical functor to the quantum functor. We will see that the most naive attempt fails, and that one is forced to either use gauge invariance, or to work with Rieffel's deformed algebras, which we already discussed above and which were mentioned earlier in section 7.1. We end this chapter with a discussion on possible generalisations and future work.

8.2 The classical functor

In this section, we will show in Theorem 8.9 that for any two Riemannian manifolds (Q_1, β_1) and (Q_2, β_2), and a map $f\colon Q_1 \to Q_2$ between them satisfying certain conditions, the map f induces a map $F\colon T^*Q_1 \to T^*Q_2$ that is in some sense compatible with the canonical symplectic forms on both spaces. We discuss a version of this theorem for vector spaces before formulating and proving the result for general manifolds, since some of the theory in the first subsection will be used to formulate and prove the main result in the second subsection, and the linear version better illustrates the key idea behind the theorem.

In the final subsection, we construct the classical category, and show that the restriction of the pullback of F to the spaces of smooth functions on both cotangent bundles to a suitable Poisson subalgebra is compatible with the Poisson structures on these spaces.

8.2.1 The linear case

In order to fix notation, we recall some basic notions for bilinear forms:

8.1 Definition. Let V be a finite dimensional vector space over some field K, and suppose that $b\colon V \times V \to K$ is a bilinear form on V.

- The form b is said to be *reflexive* iff for each $v, w \in V$, we have $b(v, w) = 0 \Leftrightarrow b(w, v) = 0$.
- For each subspace $U \subseteq V$, let $U^b := \{v \in V \mid \forall\, u \in U\colon b(u, v) = 0\}$. Note that U^b is a subspace of V. If b is an inner product, then we

write $U^{\perp}$ instead of U^b;

- The form b is said to be *nondegenerate on U* iff $U \cap U^b = \{0\}$. The form b is said to be *nondegenerate* iff it is nondegenerate on V;
- If b is nondegenerate, then the canonical linear map $\flat\colon V \to V^*$, $v \mapsto b(\cdot, v)$ is an isomorphism. This map, as well as its inverse $\sharp$, are called *musical isomorphisms.* Since we consider musical isomorphisms associated to bilinear forms on different spaces, we will often label them with a subscript. To ensure that these subscripts remain legible, we will write $\flat(v)$ for the image of a vector $v \in V$ under the musical isomorphism $V \to V^*$, instead of $v^\flat$ which is done in the literature. A similar remark applies to $\sharp$.

Now suppose V_1 and V_2 are vector spaces over K that carry nondegenerate reflexive bilinear forms b_1 and b_2, respectively. For $j = 1, 2$, let $\flat_j$ and $\sharp_j$ be their corresponding musical isomorphisms. Moreover, suppose that $S\colon V_1 \to V_2$ is a linear map, and let $S^T\colon V_2^* \to V_1^*$, $f \mapsto f \circ S$ be its transpose.

- The map $S^b := \sharp_1 \circ S^T \circ \flat_2$ is called the *adjoint of S*; if $V_1 = V_2$ and $S = S^b$, then we say that S is *selfadjoint*; if S is selfadjoint and satisfies $S^2 = S$ (i.e., S is an idempotent), then we say that S is a *projection.*
- We say that the map S is a *partial isomorphism* iff b_1 is nondegenerate on $\ker S$, and for each $v, w \in (\ker S)^{b_1}$, we have $b_1(v, w) = b_2(S(v), S(w))$;

In addition, if b_1 and b_2 are inner products, then we write S^* for the adjoint of S, and the notion of a partial isomorphism coincides with that of a partial isometry. If b_1 and b_2 are symplectic forms, then we write S^ω for the adjoint of S.

8.2 Example. Symplectic forms and inner products form two classes of nondegenerate reflexive bilinear forms. In fact, these are the examples that we will be most concerned with.

The following lemma is meant to streamline the proof of the subsequent proposition, allowing us to simultaneously prove it for symmetric and antisymmetric bilinear forms, thereby not having to explicitly distinguish between the two classes mentioned in the previous example. For a proof, we refer to [46, Propositions 1.7.6 and 1.7.7].

8.3 Lemma. *Let V be a vector space over a field K of characteristic $\neq 2$, and let b be a nonzero reflexive bilinear form on V. Then b is either symmetric or antisymmetric.*

The following proposition contains some elementary statements from linear algebra that the reader is probably already familiar with. We have included it here mostly because of the final part, which is also given the most attention. Part (8) is somewhat delicate in that the condition of nondegeneracy that is part of the definition of a partial isomorphism requires a bit more care than the corresponding statement for the smaller class of partial isometries. Indeed, any subspace of an inner product space is nondegenerate with respect to that inner product, making partial isometries easier to handle.

8.4 Proposition. *Let V be a finite dimensional vector space over a field K of characteristic $\neq 2$, let b be a nondegenerate reflexive bilinear form on V, and let $U \subseteq V$ be a subspace. Then*

(1) $(U^b)^b = U$;

(2) The following are equivalent:

- *(i)* b *is nondegenerate on* U;
- *(ii)* b *is nondegenerate on* U^b;
- *(iii)* $V = U \oplus U^b$;

Now let V_1 and V_2 be vector spaces endowed with nondegenerate reflexive bilinear forms b_1 and b_2, respectively, and let $S\colon V_1 \to V_2$ be a linear map.

(3) Let $\varepsilon := 1$ if b_1 and b_2 are both symmetric or both antisymmetric, and let $\varepsilon := -1$ otherwise. Furthermore, let $T\colon V_2 \to V_1$ be a linear map. Then the following are equivalent:

- *(i)* $T = S^b$;
- *(ii) For each $v_1 \in V_1$ and each $v_2 \in V_2$, we have $b_1(v_1, T(v_2)) = b_2(S(v_1), v_2)$;*

(iii) *For each* $v_1 \in V_1$ *and each* $v_2 \in V_2$*, we have* $b_1(T(v_2), v_1) = \varepsilon \cdot b_2(v_2, S(v_1))$*;*

(4) $(S^b)^b = S$*;*

(5) *We have* $\ker(S) = \mathrm{Im}(S^b)^{b_1}$ *and* $\ker(S^b) = \mathrm{Im}(S)^{b_2}$*;*

(6) *If* $V_1 = V_2$ *and* S *is a projection, then* b_1 *is nondegenerate on both* $\ker S$ *and* $\mathrm{Im}(S)$*, and* $V_1 = \ker S \oplus \mathrm{Im}(S)$*;*

(7) *If* S *is a partial isomorphism, then for each* $v \in V_1$*, the following are equivalent:*

(i) $v \in (\ker S)^{b_1}$*;*

(ii) *For each* $w \in V_1$*, we have* $b_1(w, v) = b_2(S(w), S(v))$*;*

(8) *If* b_1 *and* b_2 *are both symmetric or both antisymmetric, then the following are equivalent:*

(i) S *is a partial isomorphism;*

(ii) $S^b \circ S$ *is a projection with image* $(\ker S)^{b_1}$*;*

(iii) S^b *is a partial isomorphism;*

(iv) $S \circ S^b$ *is a projection with image* $\mathrm{Im}(S)$*;*

Proof. To prove (1), we first note that $(U^b)^b$ is a subspace of V satisfying $U \subseteq (U^b)^b$ since b is reflexive, and then observe that the map

$$U \to (V/U^b)^*, \quad v \mapsto (w + U^b \mapsto b(w, v)),$$

is an isomorphism. This yields $\dim(U) = \dim((V/U^b)^*) = \dim(V/U^b) = \dim(V) - \dim(U^b)$, and a similar argument shows that $\dim(U^b) = \dim(V) - \dim((U^b)^b)$, so $\dim(U) = \dim((U^b)^b)$, and it follows that $U = (U^b)^b$.

This argument can also be used in (2) to show that (i) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is immediate from the definition of nondegeneracy of b on U. Similar arguments can be employed in conjunction with (1) to prove the equivalence of (ii) and (iii).

For (3), the implication (i) $\Rightarrow$ (ii) is simply a matter of expanding the definition of the adjoint, while the reverse implication follows from nondegeneracy of b_1. The equivalence of (ii) and (iii) follows from Lemma 8.3.

(4) and (5) are readily obtained from (3).

To prove (6), we first note that if $v \in \ker S \cap \mathrm{Im}(S)$, then $v = S(v) = 0$, hence $\ker S \cap \mathrm{Im}(S) = \{0\}$. Furthermore, (5) implies $\ker S = \mathrm{Im}(S)^{b_1}$, so b_1 is nondegenerate on both $\ker S$ and $\mathrm{Im}(S)$, and (2) now implies $V_1 = \ker S \oplus \mathrm{Im}(S)$.

(7) **(i) $\Rightarrow$ (ii):** Let $v \in (\ker S)^{b_1}$, and let $w \in V_1$. Since b_1 is nondegenerate on $(\ker S)^{b_1}$, we may apply (2) to find unique $w_1 \in \ker S$ and $w_2 \in (\ker S)^{b_1}$ such that $w = w_1 + w_2$. Then we obtain

$$b_2(S(w), S(v)) = b_2(S(w_1), S(v)) = b_1(w_1, v) = b_1(w, v),$$

which shows that (ii) holds.

(ii) $\Rightarrow$ (i): Let v and w as above. Then

$$b_1(w, v) = b_2(S(w), S(v)) = b_1(w, S^b \circ S(v)),$$

and since $w \in V_1$ was arbitrary, parts (1) and (5) of this proposition and nondegeneracy of b_1 together imply $v = S^b \circ S(v) \in \mathrm{Im}(S^b) = (\ker S)^{b_1}$, as desired.

(8) The equivalence of (iii) and (iv) can be obtained from the equivalence of (i) and (ii) by substituting S for S^b, and applying parts (1), (4) and (5) of this proposition. Similarly, to prove the equivalence of (ii) and (iv), it suffices to prove the implication (ii) $\Rightarrow$ (iv). Thus we only need to prove the following three implications:

(i) $\Rightarrow$ (ii): Suppose S is a partial isomorphism. Part (7) and nondegeneracy of b imply that $S^b \circ S(v) = v$ for each $v \in (\ker S)^{b_1}$, and it is trivial that $S^b \circ S(v) = 0$ for each $v \in \ker S$. Let $v \in V_1$, and let $w \in (\ker S)^{b_1}$. By part (2) of this proposition and nondegeneracy of $(\ker S)^{b_1}$, we have $V_1 = \ker S \oplus (\ker S)^{b_1}$. From this decomposition and our computations above, it is readily seen that $S^b \circ S$ is an idempotent with image $(\ker S)^{b_1}$.

It follows from (3) that $S^b \circ S$ is selfadjoint, hence $S^b \circ S$ is a projection.

(ii) $\Rightarrow$ (i): Suppose $S^b \circ S$ is a projection on V_1 with image $(\ker S)^{b_1}$. By part (6), b_1 is nondegenerate on $(\ker S)^{b_1}$, and by part (7), we have $b_1(v, w) = b_2(S(v), S(w))$ for each $v, w \in (\ker S)^{b_1}$, so S is a partial isomorphism.

(ii) $\Rightarrow$ (iv): Suppose $S^b \circ S$ is a projection with image $\ker(S)^{b_1}$. Again, by part (6), b_1 is nondegenerate on $(\ker S)^{b_1}$. It follows from (3) that $S \circ S^b$ is selfadjoint. Note that by (1) and (5), we have $\mathrm{Im}(S^b \circ S) = (\ker S)^{b_1} = \mathrm{Im}(S^b)$, and since $S^b \circ S$ is a projection, we have $S^b \circ S|_{\mathrm{Im}(S^b)} = \mathrm{Id}_{\mathrm{Im}(S^b)}$, hence

$$(S \circ S^b)^2 = S \circ (S^b \circ S) \circ S^b = S \circ S^b,$$

which shows that $S \circ S^b$ is also an idempotent, and therefore a projection. It is clear that $\mathrm{Im}(S \circ S^b) \subseteq \mathrm{Im}(S)$. Because b_1 is nondegenerate on $(\ker S)^{b_1}$, we have $\ker S \oplus (\ker S)^{b_1}$ by (2), hence $\mathrm{Im}(S) = \mathrm{Im}(S|_{(\ker S)^{b_1}})$. Now let $v \in (\ker S)^{b_1}$. Then

$$(S \circ S^b) \circ S(v) = S \circ (S^b \circ S)(v) = S(v),$$

which shows that $\mathrm{Im}(S|_{(\ker S)^{b_1}}) \subseteq \mathrm{Im}(S \circ S^b)$. We conclude that $\mathrm{Im}(S \circ S^b) = \mathrm{Im}(S)$. ∎

A natural question is whether the composition of two partial isomorphisms is again a partial isomorphism. The answer is, in general, no; we do however have the following sufficient conditions:

8.5 Proposition. *Let V_1, V_2, and V_3 be three vector spaces endowed with nondegenerate bilinear forms b_1, b_2, and b_3, respectively, and suppose that the three bilinear forms are either all symmetric or all antisymmetric. Furthermore, suppose $S_{12}\colon V_1 \to V_2$ and $S_{23}\colon V_2 \to V_3$ are partial isomorphisms. If S_{12} and S_{23} are both injective or both surjective, then $S_{23} \circ S_{12}$ is a partial isomorphism.*

Proof. Let $S_{13} := S_{23} \circ S_{12}$. First suppose S_{12} and S_{23} are both injective. Then S_{13} is also injective, so b_1 is trivially nondegenerate on $(\ker S_{13})^{b_1} = V_1$, and for each $v, w \in V_1$, we have

$$b_3(S_{13}(v), S_{13}(w)) = b_2(S_{12}(v), S_{12}(w)) = b_1(v, w),$$

so S_{13} is a partial isomorphism.

Now suppose S_{12} and S_{23} are both surjective. Then S_{12}^b and S_{23}^b are both injective by part (5) of Proposition 8.4, and we have $S_{13}^b = S_{12}^b \circ S_{23}^b$. Applying the first part of the current proposition, we find that S_{13}^b is a partial isomorphism, so by part (8) of Proposition 8.4, the map S_{13} is a partial isomorphism, as desired. ■

We now recall the basic construction of a symplectic vector space from a given vector space. Let V be a finite dimensional real vector space, and let V^* be its dual space. Then there is the canonical symplectic form ω on the vector space $V \times V^*$ given by

$$\omega\colon (V \times V^*) \times (V \times V^*) \to \mathbb{R}, \quad ((v, \zeta), (w, \eta)) \mapsto \zeta(w) - \eta(v),$$

and the spaces $V \times \{0\}$ and $\{0\} \times V^*$ are Lagrangian subspaces of $(V \times V^*, \omega)$.

A partial isomorphism between vector spaces canonically induces a partial isomorphism between the corresponding symplectic spaces, as is shown in the proposition below.

8.6 Proposition. *Let V_1 and V_2 be two finite dimensional real vector spaces endowed with nondegenerate reflexive bilinear forms b_1 and b_2, respectively, and suppose S_{12} is a partial isomorphism. For $j = 1, 2$, let ω_j be the canonical symplectic forms on $V_j \times V_j^*$. Then the map*

$$T_{12}\colon V_1 \times V_1^* \to V_2 \times V_2^*, \quad (v, \zeta) \mapsto (S_{12}(v), \zeta \circ S_{12}^b),$$

is a partial isomorphism between the corresponding symplectic spaces.

Moreover, this construction is functorial in the following sense: suppose V_3 is another finite dimensional real vector spaces with a nondegenerate reflexive bilinear form b_3, and suppose that $S_{23}\colon V_2 \to V_3$ and $S_{13} := S_{23} \circ S_{12}$ are partial isomorphisms. Let T_{23} and T_{13} be the respective induced maps between symplectic spaces. Then $T_{13} = T_{23} \circ T_{12}$.

Proof. Let $U := \ker S_{12}$, let $U^0 := \{\zeta \in V_1^* \mid \forall v \in U\colon \zeta(v) = 0\}$ be the annihilator of U, and let $U_{b_1}^0$ be the annihilator of U^{b_1}. Let $\flat_1$ be the musical isomorphism $V_1 \to V_1^*$ corresponding to b_1, and let $\sharp_1$ be its inverse. Note that U^0 and $U_{b_1}^0$ are the images of U^{b_1} and U under $\flat_1$, respectively. Furthermore, by parts (1) and (5) of Proposition 8.4, we have $\mathrm{Im}(S_{12}^b) = (\ker S_{12})^{b_1} = U^{b_1}$, so for each $\zeta \in V_1^*$, we have $\zeta \circ S_{12}^b = 0$ if and only if $\zeta \in U_{b_1}^0$. It follows that $\ker T_{12} = U \times U_{b_1}^0$.

We now claim that $U^{b_1} \times U^0 = (\ker T_{12})^{\omega_1}$. For each $(v, \zeta) \in V \times V^*$ and each $(w, \eta) \in U \times U_{b_1}^0$, we have

$$\omega_1((v, \zeta), (w, \eta)) = \zeta(w) - \eta(v).$$

If $(v, \zeta) \in U^{b_1} \times U^0$, then the right-hand side is equal to 0, which shows that $U^{b_1} \times U^0 \subseteq (\ker T_{12})^{\omega_1}$. On the other hand, if $(v, \zeta) \in (\ker T_{12})^{\omega_1}$, then we can take $w = 0$ and $\eta \in U^0$ arbitrary, and apply part (1) of Proposition 8.4 to see that $v \in U$, and if we take $w \in U$ arbitrary and $\eta = 0$, then we see that $\zeta \in U^0$, which establishes the reverse inclusion. This proves the claim.

Next, we show that ω_1 is nondegenerate on $(\ker T_{12})^{\omega_1}$. Let $(v, \zeta) \in U^{b_1} \times U^0$, and suppose that for each $(w, \eta) \in U^{b_1} \times U^0$, we have

$$0 = \omega_1((v, \zeta), (w, \eta)) = \zeta(w) - \eta(v).$$

In particular, taking $w = 0$, we see that $b_1(v, \sharp_1(\eta)) = 0$ for each $\eta \in U^0$, and since $\sharp_1(U^0) = U^{b_1}$ and b_1 is nondegenerate on U^{b_1}, we infer that $v = 0$. Taking $\eta = 0$ instead, we obtain $b_1(w, \sharp_1(\zeta)) = 0$ for each $w \in U^{b_1}$, so by the same argument, we have $\sharp_1(\zeta) = 0$, which is equivalent to $\zeta = 0$. Thus $(v, \zeta) = 0$, and we conclude that ω_1 is nondegenerate on $(\ker T_{12})^{\omega_1}$.

We are now ready to show that T_{12} is a partial isomorphism. Let $(v, \zeta), (w, \eta) \in U^{b_1} \times U^0$. Then

$$\begin{aligned}\omega_2(T_{12}(v, \zeta), T_{12}(w, \eta)) &= \zeta \circ S_{12}^b \circ S_{12}(w) - \eta \circ S_{12}^b \circ S_{12}(v) = \zeta(w) - \eta(v) \\ &= \omega_1((v, \zeta), (w, \eta)),\end{aligned}$$

where we used part (8) of Proposition 8.4 in the second step. Together with the nondegeneracy of ω_1 on $(\ker T_{12})^{\omega_1}$ that was established in the previous paragraph, this shows that T_{12} is a partial isomorphism.

To prove the final assertion, let $(v, \zeta) \in V \times V^*$. Then by definition, we have $S_{13}(v) = S_{23} \circ S_{12}(v)$. In addition,

$$\zeta \circ S_{13}^b = \zeta \circ (S_{23} \circ S_{12})^b = \zeta \circ S_{12}^b \circ S_{23}^b,$$

and from this, it is readily seen that $T_{13} = T_{23} \circ T_{12}$. ■

8.2.2 The general case

In this subsection, we adapt the definitions and results obtained in the previous subsection to smooth manifolds.

8.7 Definition. Let M_1 and M_2 be two smooth manifolds. For $j = 1, 2$, let $T^{0,2}M_j$ be the second tensor power of the cotangent bundle T^*M_j, and let $b_j \in \Gamma(T^{0,2}M_j)$ be a smoothly varying bilinear form on M_j that is nondegenerate and reflexive at each point of M_j. Let $F\colon M_1 \to M_2$ be a smooth map. Suppose now that

(1) F has constant rank;

(2) For each $m \in M_1$, the map T_mF is a partial isomorphism;

Then we say that F is a *bilinear morphism*. In particular,

- If b_1 and b_2 are symplectic forms, then we say that F is a *symplectic morphism*. If F is a submersion, then we say that F is a *symplectic submersion*. Similarly, if F is an immersion, then we say that F is a *symplectic immersion*.

- If b_1 and b_2 are Riemannian metrics, then we say that F is a *Riemannian morphism*. We define the notions of *Riemannian submersion* and *Riemannian immersion* analogously to their symplectic counterparts.

8.8 Remark. The notions of Riemannian immersion and submersion as defined above coincide with those found in the literature. The notion of a symplectic submersion was found by Lanéry and Thiemann: see [72, Definition 2.1], where it is simply called a “compatible map”. Our notion of a symplectic morphism thus generalises theirs. It should not be confused with the established notion of a *symplectomorphism*, which is a symplectic morphism that is also a diffeomorphism.

8.9 Theorem. *Let (Q_1, b_1) and (Q_2, b_2) be two manifolds equipped with smoothly varying nondegenerate reflexive bilinear forms, and let $f\colon Q_1 \to Q_2$ be a bilinear morphism. Define the map*

$$F\colon T^*Q_1 \to T^*Q_2, \quad (x, \xi) \mapsto (f(x), \xi \circ T_x f^b).$$

Then F is a symplectic morphism. Furthermore, F is a symplectic immersion if f is an immersion, and F is a symplectic submersion if f is a submersion.

Finally, the assignment $f \mapsto F$ is functorial in the following sense: Let $f_{12} := f$, let $F_{12} := F$, let (Q_3, b_3) be a third manifold equipped with a smoothly varying nondegenerate reflexive bilinear form, and let $f_{23}\colon Q_2 \to Q_3$ be a bilinear morphism with induced symplectic morphism F_{23}. If $f_{13} := f_{23} \circ f_{12}$ is a bilinear morphism, then the induced symplectic morphism F_{13} satisfies $F_{13} = F_{23} \circ F_{12}$.

Proof. First note that F is smooth, since it is the unique map that makes the following diagram

$$\begin{tikzcd} T^*Q_1 \arrow[r, "F"] & T^*Q_2 \\ TQ_1 \arrow[u] \arrow[r, "Tf"] & TQ_2 \arrow[u] \end{tikzcd}$$

commutative, where the vertical arrows denote the musical isomorphisms associated to b_1 and b_2.

Let $x \in Q_1$, and let $\xi \in T_x Q_1^*$. Since f is a bilinear morphism, it has constant rank, so by the constant rank theorem, we may fix coordinate neighbourhoods (U_1, φ_1) of x on Q_1 and (U_2, φ_2) of $f(x)$ on Q_2 such that $f(U_1) \subseteq U_2$, $\varphi_1(x) = 0$, and $\varphi_2 \circ f \circ \varphi_1^{-1}$ is of the form

$$\begin{aligned} \mathbb{R}^{n_1} \supseteq \varphi_1(U_1) &\to \varphi_2(U_2) \subseteq \mathbb{R}^{n_2}, \\ (x^1, \ldots, x^{n_1}) &\mapsto (x^1, \ldots, x^{\mathrm{rk}(f)}, 0, \ldots, 0), \end{aligned}$$

where $\mathrm{rk}(f)$ denotes the rank of f.

Now let $\alpha_1\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ be the linear isomorphism uniquely determined by the requirement that

$$\begin{aligned} \alpha_1|_{\{0\}\times\mathbb{R}^{n_1-\mathrm{rk}(f)}} &= \mathrm{Id}_{\{0\}\times\mathbb{R}^{n_1-\mathrm{rk}(f)}}, \\ \alpha_1|_{\mathbb{R}^{\mathrm{rk}(f)}\times\{0\}} &= d\varphi_{1,x} \circ T_x f^b \circ T_x f \circ (d\varphi_{1,x})^{-1}|_{\mathbb{R}^{\mathrm{rk}(f)}\times\{0\}}, \end{aligned}$$

and let $\alpha_2 \colon \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ be the linear isomorphism uniquely determined by the requirement that

$$\begin{aligned}
&\alpha_2|_{\mathbb{R}^{\mathrm{rk}(f)}\times\{0\}} = \mathrm{Id}_{\mathbb{R}^{\mathrm{rk}(f)}\times\{0\}}, \\
&\alpha_2|_{\{0\}\times\mathbb{R}^{n_2-\mathrm{rk}(f)}} \\
&\quad = \Big(\mathrm{Id} - d\varphi_{2,f(x)} \circ T_x f \circ T_x f^b \circ (d\varphi_{2,f(x)})^{-1}\Big)\Big|_{\{0\}\times\mathbb{R}^{n_2-\mathrm{rk}(f)}}.
\end{aligned}$$

By replacing φ_1 with $\alpha_1^{-1}\circ\varphi_1$, we can arrange that the local frame $\left(\frac{\partial}{\partial x^i}\right)_{i=1}^{n_1}$ induced by (U_1, φ_1) has the property that

$$\left.\frac{\partial}{\partial x^1}\right|_x, \dots, \left.\frac{\partial}{\partial x^{\mathrm{rk}(f)}}\right|_x \in (\ker T_x f)^{b_1},$$

and similarly, by replacing φ_2 with $\alpha_2^{-1} \circ \varphi_2$ if necessary, we can arrange that the local frame $\left(\frac{\partial}{\partial y^i}\right)_{i=1}^{n_2}$ induced by (U_2, φ_2) has the property that

$$\left.\frac{\partial}{\partial y^{\mathrm{rk}(f)+1}}\right|_{f(x)}, \dots, \left.\frac{\partial}{\partial y^{n_2}}\right|_{f(x)} \in (\mathrm{Im}\ T_x f)^{b_2}.$$

For $j = 1, 2$, the chart (U_j, φ_j) induces a chart $(\pi_j^{-1}(U_j), \Phi_j)$ of the cotangent bundle T^*Q_j of Q_j, where $\pi_j \colon T^*Q_j \to Q_j$ denotes the canonical map, and we can use these maps to obtain isomorphisms

$$T_{(x,\xi)}(T^*Q_1) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \times (\mathbb{R}^{n_1})^* \to T_x Q_1 \times T_x Q_1^*.$$

Here,

- The first isomorphism is given by $d\Phi_{1,(x,\xi)}$;
- The second isomorphism is the identity between the first factors, and the identification of $\mathbb{R}^{n_1}$ with its dual space using the standard basis in the second factors;
- The third isomorphism is given by $(d\phi_{1,x})^{-1} \times (d\phi_{1,x})^T$;

Let θ_1 be the composition of these three isomorphisms. With respect to the coordinates $(x^1, \dots, x^{n_1}, \zeta_1, \dots, \zeta_{n_1})$ induced by Φ_1, the symplectic form $\omega_{1,(x,\xi)}$ takes the usual form $\sum_{j=1}^{n_1} dx^j \wedge d\zeta_j$ and is therefore the pullback of the canonical symplectic form on $T_xQ_1 \times T_xQ_1^*$ under θ_1. Similarly, we find an isomorphism

$$\theta_2 \colon T_{F(x,\xi)}(T^*Q_2) \to T_{f(x)}Q_2 \times T_{f(x)}Q_2^*,$$

and $\omega_{2,F(x,\xi)}$ is the pullback of the canonical symplectic form on $T_{f(x)}Q_2 \times T_{f(x)}Q_2^*$ under θ_2.

We claim that the unique map T_{12} that makes the following diagram

$$\begin{tikzcd}
T_{(x,\xi)}(T^*Q_1) \arrow[r, "T_{(x,\xi)}F"] \arrow[d, "\theta_1"'] & T_{F(x,\xi)}(T^*Q_2) \arrow[d, "\theta_2"] \\
T_xQ_1 \times T_xQ_1^* \arrow[r, "T_{12}"] & T_{f(x)}Q_2 \times T_{f(x)}Q_2^*
\end{tikzcd}$$

commutative, is the one from Proposition 8.6, with $V_1 := T_xQ_1$, $V_2 := T_{f(x)}Q_2$, and $S_{12} = T_xf$.

To see this, we first note that by our assumptions on the local frames, we have

$$T_xf^b\left(\left.\frac{\partial}{\partial y^i}\right|_{f(x)}\right) = \begin{cases} \left.\dfrac{\partial}{\partial x^i}\right|_{f(x)} & \text{if } i \leq \operatorname{rk}(f) \\ 0 & \text{if } i > \operatorname{rk}(f) \end{cases},$$

for $i = 1, \ldots, n_2$, hence

$$(T_xf^b)^T(dx_x^i) = \begin{cases} dy^i_{f(x)} & \text{if } i \leq \operatorname{rk}(f) \\ 0 & \text{if } i > \operatorname{rk}(f) \end{cases},$$

for $i = 1, \ldots, n_1$. It is clear that

$$\begin{aligned}
&\Phi_2 \circ F \circ \Phi_1^{-1}((x^1, \ldots, x^{n_1}), (0, \ldots, 0)) \\
&\quad = (\phi_2 \circ f \circ \phi_1^{-1}(x^1, \ldots, x^{n_1}), (0, \ldots, 0)) \\
&\quad = ((x^1, \ldots, x^{\operatorname{rk}(f)}, 0, \ldots, 0), (0, \ldots, 0)),
\end{aligned}$$

for each $(x^1, \ldots, x^{n_1}) \in U_1$, and from our computation of $(T_xf^b)^T$, we obtain

$$\Phi_2 \circ F \circ \Phi_1^{-1}(\phi_1(x), (\xi_1, \ldots, \xi_{n_1})) = (\phi_2 \circ f(x), (\xi_1, \ldots, \xi_{\operatorname{rk}(f)}, 0, \ldots, 0)),$$

for each $(\xi_1, \dots, \xi_{n_1}) \in \mathbb{R}^{n_1}$. Thus, if

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n_1}}, \frac{\partial}{\partial \zeta_1}, \dots, \frac{\partial}{\partial \zeta_{n_1}},$$

denotes the local frame induced by $(\pi_1^{-1}(U_1), \Phi_1)$, and

$$\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{n_2}}, \frac{\partial}{\partial \eta_1}, \dots, \frac{\partial}{\partial \eta_{n_2}},$$

the local frame induced by $(\pi_2^{-1}(U_2), \Phi_2)$, then

$$T_{(x,\xi)}F\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,\xi)}\right) = \begin{cases} \left.\dfrac{\partial}{\partial y^i}\right|_{F(x,\xi)} & \text{if } i \leq \mathrm{rk}(f) \\ 0 & \text{if } i > \mathrm{rk}(f) \end{cases},$$

$$T_{(x,\xi)}F\left(\left.\frac{\partial}{\partial \zeta_i}\right|_{(x,\xi)}\right) = \begin{cases} \left.\dfrac{\partial}{\partial \eta_i}\right|_{F(x,\xi)} & \text{if } i \leq \mathrm{rk}(f) \\ 0 & \text{if } i > \mathrm{rk}(f) \end{cases}.$$

Now assume that $T_{12} = T_x f \times (T_x f^b)^T$, i.e., T_{12} is as in Proposition 8.6. We find that for $i = 1, \dots, \mathrm{rk}(f)$, we have

$$\theta_2 \circ T_{(x,\xi)}F\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,\xi)}\right) = \theta_2\left(\left.\frac{\partial}{\partial y^i}\right|_{F(x,\xi)}\right) = \left(\left.\frac{\partial}{\partial y^i}\right|_{f(x)}, 0\right),$$

On the other, hand, we have

$$T_{12} \circ \theta_1\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,\xi)}\right) = T_{12}\left(\left.\frac{\partial}{\partial x^i}\right|_{x}, 0\right) = \left(\left.\frac{\partial}{\partial y^i}\right|_{f(x)}, 0\right),$$

so $\left.\frac{\partial}{\partial x^i}\right|_{(x,\xi)}$ is mapped to the same element in both cases. If $i = \mathrm{rk}(f) + 1, \dots, n_1$, then this element is mapped to $(0,0) \in T_{f(x)}Q_2 \times T_{f(x)}Q_2^*$ in both cases.

Similarly, for $i = 1, \dots, \mathrm{rk}(f)$, we have

$$\theta_2 \circ T_{(x,\xi)}F\left(\left.\frac{\partial}{\partial \zeta_i}\right|_{(x,\xi)}\right) = \theta_2\left(\left.\frac{\partial}{\partial \eta_i}\right|_{F(x,\xi)}\right) = \left(0, dy^i_{f(x)}\right),$$

and

$$T_{12} \circ \theta_1 \left(\left.\frac{\partial}{\partial \zeta_i}\right|_{(x,\xi)} \right) = T_{12}\left(0, dx^i_x\right) = \left(0, dy^i_{f(x)}\right),$$

so $\left.\frac{\partial}{\partial \zeta_i}\right|_{(x,\xi)}$ is mapped to the same element in both cases. Again, if $i = \operatorname{rk}(f)+1, \ldots, n_1$, then this element is mapped to $(0,0) \in T_{f(x)}Q_2 \times T_{f(x)}Q_2^*$ in both cases. Since $\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,\xi)}\right)_{i=1}^{n_1}$ and $\left(\left.\frac{\partial}{\partial \zeta_i}\right|_{(x,\xi)}\right)_{i=1}^{n_1}$ together form a basis of $T_{(x,\xi)}(T^*Q_1)$ and all of the maps in the above diagram are linear, we infer that the diagram is indeed commutative. Furthermore, θ_1 and θ_2 are symplectic isomorphisms, and T_{12} is a partial isomorphism between symplectic vector spaces, therefore $T_{(x,\xi)}F$ is a partial isomorphism between symplectic vector spaces. From our computations, it is also evident that F has constant rank equal to $2 \cdot \operatorname{rk}(f)$, hence it is a symplectic morphism, and that F is an immersion or a submersion if f is an immersion or a submersion, respectively.

It remains to prove functoriality. Consider the following diagram:

$$\begin{tikzcd}
T^*Q_1 \arrow[rr, "F_{13}"] \arrow[dr, "F_{12}"'] & & T^*Q_3 \\
 & T^*Q_2 \arrow[ur, "F_{23}"'] & \\
 & TQ_2 \arrow[u] \arrow[dr, "Tf_{23}"] & \\
TQ_1 \arrow[uuu] \arrow[ur, "Tf_{12}"] \arrow[rr, "Tf_{13}"'] & & TQ_3 \arrow[uuu]
\end{tikzcd}$$

Here, the vertical arrows denote the musical isomorphisms. The bottom triangle commutes since it is obtained by applying the tangent functor to the commuting triangle

$$
\begin{tikzcd}
 & Q_2 \arrow[dr, "f_{23}"] & \\
Q_1 \arrow[ur, "f_{12}"] \arrow[rr, "f_{13}"] & & Q_3
\end{tikzcd}
$$

and the three quadrilaterals that each have two musical isomorphisms as their sides commute by definition of the maps F_{12}, F_{23} and F_{13}. It follows that the top triangle commutes, which is what we wanted to show. ■

As in the linear case, the composition of two bilinear morphisms is not necessarily a bilinear morphism, so the class of smooth manifolds endowed with smoothly varying bilinear forms and bilinear morphisms between them does not form a category. It contains subclasses that do form categories, however.

8.10 Definition.

- We define the *category* $\mathsf{RiemannIm}$ *of Riemannian immersions* as follows:
  - Its class of objects consists of all smooth Riemannian manifolds;
  - The set of morphisms from an object (Q_1, β_1) to an object (Q_2, β_2) consists of all Riemannian immersions between these manifolds.
- We define the *category* $\mathsf{RiemannSub}$ *of Riemannian submersions* to be the category with the same class of objects as $\mathsf{RiemannIm}$, but whose set of morphisms from an object (Q_1, β_1) to an object (Q_2, β_2) consists of all Riemannian submersions between these manifolds.

For symplectic manifolds, we define the *category* SympIm *of symplectic immersions* and the category of $\mathsf{SympSub}$ *of symplectic submersions* analogously.

8.11 Corollary. *The classes* $\mathsf{RiemannIm}$*,* $\mathsf{RiemannSub}$*,* SympIm*, and* $\mathsf{SympSub}$ *are categories. Furthermore, we have covariant functors*

$$\mathsf{RiemannIm} \to \mathsf{SympIm}, \quad \mathsf{RiemannSub} \to \mathsf{SympSub}.$$

that are defined as follows:

- *A Riemannian manifold (Q, β) is mapped to its cotangent bundle T^*Q, endowed with its canonical symplectic form;*
- *A Riemannian morphism $f\colon (Q_1, \beta_1) \to (Q_2, \beta_2)$ is mapped to the map F as defined in Theorem 8.9.*

Proof. This follows from Proposition 8.5 and Theorem 8.9. ∎

8.12 Example. We now discuss a couple of examples of Riemannian immersions and submersions, and compute the symplectic morphism that each of them induces.

(1) Let (Q, β_0) be a Riemannian manifold. Imagine a massive spherical object, e.g., a cannonball, of mass $m > 0$ moving on Q, and divide the ball into two smaller objects, the first being a smaller ball with the same centre as the original with mass $m_1 > 0$, and the second being the remaining spherical shell with mass $m_2 > 0$, so that $m = m_1 + m_2$. The configuration space of the entire object is Q, and we endow it with the Riemannian metric $\beta := m \cdot \beta_0$. The configuration space of the system composed of the two smaller objects is given by $Q \times Q$ that carries the Riemannian metric $\beta_{1,2}$ given by

$$\begin{aligned}\beta_{1,2,(q_1,q_2)}\colon (T_{(q_1,q_2)}(Q \times Q))^2 &\cong (T_{q_1}Q \times T_{q_2}Q)^2 \to \mathbb{R},\\ ((v_1, v_2), (w_1, w_2)) &\mapsto m_1 \cdot \beta_{0,q_1}(v_1, w_1) + m_2 \cdot \beta_{0,q_2}(v_2, w_2).\end{aligned}$$

This definition of the Riemannian metric is motivated by the kinetic energy that the objects have; it is given by

$$TQ \to \mathbb{R}, \quad (q, v) \mapsto \frac{1}{2}\beta_q(v, v),$$

in the first case, and

$$T(Q \times Q) \to \mathbb{R}, \quad ((q_1, q_2), (v_1, v_2)) \mapsto \frac{1}{2}\beta_{1,2,(q_1,q_2)}((v_1, v_2), (v_1, v_2)),$$

in the second. Since the two objects that make up the composite system both have the same centre of mass, namely the centre of the ball, and since this is also the centre of mass of the undivided object, it is natural to consider the map

$$f\colon Q \to Q \times Q, \quad q \mapsto (q, q).$$

It is easy to see that this is a Riemannian immersion; indeed, it is obvious that it is an immersion, and for each $q \in Q$ and each $v, w \in T_qQ$, we have

$$\begin{aligned}\beta_{1,2,f(q)}(T_qf(v), T_qf(w)) &= \beta_{1,2,(q,q)}((v,v),(w,w))\\ &= m_1 \cdot \beta_{0,q}(v,w) + m_2 \cdot \beta_{0,q}(v,w)\\ &= m \cdot \beta_{0,q}(v,w) = \beta_q(v,w),\end{aligned}$$

hence T_qf is a partial isometry. We compute the associated map F from Theorem 8.9. Fix $q \in Q$, let $\flat_0 \colon T_qQ \to T_qQ^*$ be the musical isomorphism associated to $\beta_{0,q}$, with inverse $\sharp_0$, let $\flat \colon T_qQ \to T_qQ^*$ be the musical isomorphism associated to β_q with inverse $\sharp$, and let $\flat_{1,2}$ and $\sharp_{1,2}$ be the musical isomorphisms associated to $\beta_{1,2,(q,q)}$. Then for each $(v,w) \in (T_qQ)^2 \cong T_{(q,q)}(Q \times Q)$, we have

$$\begin{aligned}T_qf^*(v,w) &= \sharp \circ (T_qf)^T \circ \flat_{1,2}(v,w) = \sharp \circ (T_qf)^T(m_1 \cdot \flat_0(v), m_2 \cdot \flat_0(w))\\ &= \frac{1}{m}\sharp_0(m_1 \cdot \flat_0(v) + m_2 \cdot \flat_0(w)) = \frac{m_1}{m}v + \frac{m_2}{m}w,\end{aligned}$$

which yields

$$(T_qf^*)^T \colon T_qQ^* \to T_qQ^* \times T_qQ^* \cong T_{(q,q)}(Q \times Q)^*, \quad p \mapsto \left(\frac{m_1}{m}p, \frac{m_2}{m}p\right),$$

hence

$$F(q,p) = \left((q,q), \left(\frac{m_1}{m}p, \frac{m_2}{m}p\right)\right),$$

which is consistent with the Newtonian theory of classical mechanics. Note that the map F is independent of our choice of the initial metric β_0.

(2) Suppose we are in the same setting as in the previous example, with $Q = \mathbb{R}^n$, the Riemannian metric β_0 is the standard one, but the map f is given by

$$f \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad (q_1, q_2) \mapsto \frac{m_1q_1 + m_2q_2}{m}.$$

This corresponds to the reduction of a system consisting of two objects moving on some background $\mathbb{R}^n$, one with mass m_1 and centre of mass q_1, and the other with mass m_2 and centre of mass q_2, to a system consisting of a single object with mass $m = m_1 + m_2$, and centre of mass $f(q_1, q_2)$. We claim that it is a Riemannian submersion. It is easy to see that f is a submersion. Now fix $(q_1, q_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Identifying tangent spaces of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself in the canonical way, we obtain

$$T_{(q_1,q_2)} f(v_1, v_2) = \frac{m_1}{m} v_1 + \frac{m_2}{m} v_2,$$

for each $(v_1, v_2) \in T_{(q_1,q_2)}(\mathbb{R}^n \times \mathbb{R}^n) \cong T_{q_1}\mathbb{R}^n \times T_{q_2}\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. It follows that

$$\ker T_{(q_1,q_2)} f = \{(m_2 \cdot v, -m_1 \cdot v) \in T_{(q_1,q_2)}(\mathbb{R}^n \times \mathbb{R}^n) \colon v \in T_{f(q_1,q_2)}\mathbb{R}^n\},$$

and from this, it is readily seen that

$$(\ker T_{(q_1,q_2)} f)^\perp = \{(v, v) \in T_{(q_1,q_2)}(\mathbb{R}^n \times \mathbb{R}^n) \colon v \in T_{f(q_1,q_2)}\mathbb{R}^n\},$$

where the orthogonal complement is taken with respect to $\beta_{1,2,(q_1,q_2)}$. For each $(v, v), (w, w) \in (\ker T_{(q_1,q_2)} f)^\perp$, we have

$$\begin{aligned}
&\beta_{f(q_1,q_2)}(T_{(q_1,q_2)} f(v, v), T_{(q_1,q_2)} f(w, w)) \\
&\quad = m \cdot \beta_{0,f(q_1,q_2)} \left(\frac{m_1}{m} v + \frac{m_2}{m} v, \frac{m_1}{m} w + \frac{m_2}{m} w \right) \\
&\quad = m \cdot \beta_{0,f(q_1,q_2)}(v, w) \\
&\quad = m_1 \cdot \beta_{0,q_1}(v, w) + m_2 \cdot \beta_{0,q_2}(v, w) \\
&\quad = \beta_{1,2,(q_1,q_2)}((v, v), (w, w)),
\end{aligned}$$

which shows that $T_{(q_1,q_2)} f$ is a partial isomorphism, so f is indeed a Riemannian submersion.

Furthermore, we find that

$$\begin{aligned}
T_{(q_1,q_2)} f^*(v) &= \sharp_{1,2} \circ (T_{(q_1,q_2)} f)^T \circ \flat(v) = \sharp_{1,2} \circ (T_{(q_1,q_2)} f)^T (m \cdot \flat_0(v)) \\
&= \sharp_{1,2}(m_1 \cdot \flat_0(v), m_2 \cdot \flat_0(v)) = (v, v),
\end{aligned}$$

for each $v \in T_{f(q_1,q_2)}\mathbb{R}^n$, from which it is readily seen that

$$(T_{(q_1,q_2)} f^*)^T (p_1, p_2) = p_1 + p_2,$$

for each $(p_1, p_2) \in T_{(q_1,q_2)}(\mathbb{R}^n \times \mathbb{R}^n)^* \cong (T_{q_1}\mathbb{R}^n)^* \times (T_{q_2}\mathbb{R}^n)^* \cong (\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$. Thus the induced symplectic morphism F from Theorem 8.9 is given by

$$F((q_1, q_2), (p_1, p_2)) = \left(\frac{m_1 q_1 + m_2 q_2}{m}, p_1 + p_2 \right),$$

i.e. the total momentum is the same in the original and the reduced system. Also note that the map f defined in this example is a left-inverse for the map f as it is defined in the previous example.

(3) *(Lifting and extending symmetries of the configuration space)* Let (Q, β) be an arbitrary Riemannian manifold, and suppose $f\colon Q \to Q$ is a diffeomorphism that is also an isometry. Then f is evidently a Riemannian morphism, we have $T_q f^* = (T_q f)^{-1}$ for each $q \in Q$, and the induced symplectic morphism F is given by $(q,p) \mapsto (f(q), p \circ (T_q f)^{-1})$.

It is in this context interesting to mention that the induction of a symplectic morphism by a Riemannian morphism is equivariant with respect to group actions. Indeed, suppose we are in the situation of Theorem 8.9, that for $j = 1, 2$, there is a group G_j acting by isometries on (Q_j, β_j), and we have a group homomorphism $\phi\colon G_1 \to G_2$. If f is equivariant with respect to these actions, i.e., $f(g \cdot q) = \phi(g) \cdot f(q)$ for each $q \in Q_1$ and each $g \in G_1$, then functoriality implies $F(g \cdot (q,p)) = \phi(g) \cdot F(q,p)$ for each $(q,p) \in T^*Q_1$ and each $g \in G_1$, i.e., F is equivariant with respect to the induced actions.

(4) *(Restriction to a subsystem)* Suppose (Q_1, β_1) and (Q_2, β_2) are two Riemannian manifolds, endow $Q_1 \times Q_2$ with the product metric, and consider the map

$$f\colon Q_1 \times Q_2 \to Q_1, \quad (q_1, q_2) \mapsto q_1.$$

We can view this as a system consisting of two subsystems, one with configuration space Q_1 and the other with configuration space Q_2, and we disregard the second system. It is straightforward to show that this is a Riemannian submersion, and that induced symplectic morphism is given by

$$F\colon T^*(Q_1 \times Q_2) \to T^*Q_2, \quad ((q_1, q_2), (p_1, p_2)) \mapsto (q_1, p_1),$$

where we have made the identification $T_{(q_1,q_2)}(Q_1 \times Q_2)^* \cong T_{q_1}Q_1^* \times T_{q_2}Q_2^*$.

8.13 Remark. We note that it is also possible to do the first, second and fourth examples in greater generality. For instance, in the first example, we could have divided the system into n subsystems with masses $m_1, \ldots, m_n$. In that case, the map F becomes

$$T^*Q \to T^*(Q^n), \quad (q,p) \mapsto \left((q, \ldots, q), \left(\frac{m_1}{m}p, \ldots, \frac{m_n}{m}p\right)\right).$$

This result can be obtained either by direct computation, or by repeated application of the case $n = 2$ and use of the functoriality as described in Theorem 8.9. In the latter case, one first divides the system into two subsystems, one of mass m_1, the other of mass $m - m_1$, and subsequently one divides the second subsystem into two other subsystems with mass m_2 and $m - m_1 - m_2$, and so on. One can use an analogous approach in the second and fourth examples.

In the examples that we are most interested in, namely the ones pertaining to lattice gauge theory, the approach of repeated application of the case $n = 2$ allows us to restrict to the two types of elementary refinements discussed in subsection 4.3.2. The elementary refinement of addition of a single edge is a special case of the third example above. Thus we only need to discuss the elementary refinement corresponding to subdivision of an edge into two edges.

8.14 Example. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Fix an Ad-invariant inner product on $\mathfrak{g}$, and extend it to a bi-invariant Riemannian metric β_0 on G by left translation. The map corresponding to subdivision of a directed edge into two directed edges pointing in the same direction, is

$$f\colon G \times G \to G, \quad (a_1, a_2) \mapsto a_1 a_2.$$

It is easy to see that this map is a smooth submersion; we endow both the domain and the codomain of this map with Riemannian metrics in a natural way such that this map becomes a Riemannian submersion. Suppose that the length of the undivided edge is $\ell > 0$, and that the lengths of the first and second edge into which it is divided are $\ell_1 > 0$ and $\ell_2 > 0$, respectively, so that $\ell = \ell_1 + \ell_2$. We now endow the codomain of f with the Riemannian metric $\beta := \ell^{-1} \cdot \beta_0$, and we endow the domain of f with the metric $\beta_{1,2}$ that is the product of the metric $\ell_1^{-1} \cdot \beta_0$ on the first factor, and the metric $\ell_2^{-1} \cdot \beta_0$ on the second factor. More explicitly, it is given by

$$\begin{aligned}\beta_{1,2,(a_1,a_2)}\colon T_{(a_1,a_2)}(G \times G) \times T_{(a_1,a_2)}(G \times G) &\to \mathbb{R},\\ ((v_1, v_2), (w_1, w_2)) &\mapsto \ell_1^{-1} \cdot \beta_{0,a_1}(v_1, w_1) + \ell_2^{-1} \cdot \beta_{0,a_2}(v_2, w_2),\end{aligned}$$

where we have made use of the identification $T_{(a_1,a_2)}(G \times G) \cong T_{a_1}G \times T_{a_2}G$. We show that with respect to these metrics, the map f is indeed a

Riemannian submersion. Fix $(a_1, a_2) \in G \times G$. Then the tangent map of f at (a_1, a_2) is given by

$$\begin{aligned} T_{(a_1,a_2)}f \colon T_{(a_1,a_2)}(G \times G) &\to T_{a_1a_2}G, \\ (v_1, v_2) &\mapsto T_{a_1}R_{a_2}(v_1) + T_{a_2}L_{a_1}(v_2), \end{aligned}$$

hence

$$\ker T_{(a_1,a_2)}f = \{((T_{a_1}R_{a_2})^{-1}(v), -(T_{a_2}L_{a_1})^{-1}(v)) \colon v \in T_{a_1a_2}G\},$$

and with our choice of Riemannian metric, it follows that

$$(\ker T_{(a_1,a_2)}f)^{\perp} = \{(\ell_1(T_{a_1}R_{a_2})^{-1}(v), \ell_2(T_{a_2}L_{a_1})^{-1}(v)) \colon v \in T_{a_1a_2}G\}.$$

Note that for each $v \in T_{a_1a_2}G$, we have

$$T_{(a_1,a_2)}f(\ell_1(T_{a_1}R_{a_2})^{-1}(v), \ell_2(T_{a_2}L_{a_1})^{-1}(v)) = \ell_1 v + \ell_2 v = \ell v,$$

therefore, for each $v, w \in T_{a_1a_2}G$, we have

$$\begin{aligned} &\beta_{1,2,(a_1,a_2)}\big((\ell_1(T_{a_1}R_{a_2})^{-1}(v), \ell_2(T_{a_2}L_{a_1})^{-1}(v)), \\ &\qquad\qquad (\ell_1(T_{a_1}R_{a_2})^{-1}(w), \ell_2(T_{a_2}L_{a_1})^{-1}(w))\big) \\ &= \ell_1^{-1} \cdot \ell_1^2 \cdot \beta_{0,a_1}((T_{a_1}R_{a_2})^{-1}(v), (T_{a_1}R_{a_2})^{-1}(w)) \\ &\quad + \ell_2^{-1} \cdot \ell_2^2 \cdot \beta_{0,a_2}((T_{a_2}L_{a_1})^{-1}(v), (T_{a_2}L_{a_1})^{-1}(w)) \\ &= \ell_1 \cdot \beta_{0,a_1a_2}(v, w) + \ell_2 \cdot \beta_{0,a_1a_2}(v, w) \\ &= \ell \cdot \beta_{0,a_1a_2}(v, w) \\ &= \ell^{-1} \cdot \beta_{0,a_1a_2}(\ell v, \ell w) \\ &= \beta_{a_1a_2}\big(T_{(a_1,a_2)}f(\ell_1(T_{a_1}R_{a_2})^{-1}(v), \ell_2(T_{a_2}L_{a_1})^{-1}(v)), \\ &\qquad\qquad T_{(a_1,a_2)}f(\ell_1(T_{a_1}R_{a_2})^{-1}(w), \ell_2(T_{a_2}L_{a_1})^{-1}(w))\big), \end{aligned}$$

which shows that $T_{(a_1,a_2)}f$ is a partial isometry. We conclude that f is a Riemannian submersion.

From the above computations and part (8) of Proposition 8.4, it is readily seen that

$$T_{(a_1,a_2)}f^*(v) = \left(\frac{\ell_1}{\ell}(T_{a_1}R_{a_2})^{-1}(v), \frac{\ell_2}{\ell}(T_{a_2}L_{a_1})^{-1}(v)\right),$$

hence the induced symplectic submersion $F\colon T^*(G\times G)\to T^*G$ is given by

$$((a_1,a_2),(\xi_1,\xi_2))\mapsto \left(a_1a_2,\frac{\ell_1}{\ell}\xi_1\circ(T_{a_1}R_{a_2})^{-1}+\frac{\ell_2}{\ell}\xi_2\circ(T_{a_2}L_{a_1})^{-1}\right),$$

where we have made the identification $T^*_{(a_1,a_2)}(G\times G)\cong T^*_{a_1}G\times T^*_{a_2}G$.

Note that this formula also makes sense from a physical point of view. Indeed, it was argued in section 2.5 that ξ_1 and ξ_2 can be thought of as (proportional to) the average electric field on the path corresponding to the first and second edge, respectively. Thus, in order to obtain the average electric field on the concatenation of these paths, one should take the weighted average of the averages of the electric fields on the individual paths, with the weights given by the lengths of the two paths.

Furthermore, if one takes the Laplacians corresponding to the above Riemannian metrics, then one obtains the free Hamiltonians in part (1) of Proposition 4.28.

The objects of the category Refine that we constructed earlier are oriented graphs. The above example motivates us to introduce weights on the graphs, so that there is a well-defined notion of the length of an edge, and by extension, the length of a path.

8.15 Definition. We define the category $\mathsf{wtRefine}$ as follows:

- Its objects are pairs (Λ,ℓ), where $\Lambda=(\Lambda^0,\Lambda^1)$ is an object in Refine, i.e., an oriented graph whose set of vertices is given by Λ^0, and whose set of oriented edges is given by Λ^1. Moreover, ℓ is a function $\Lambda^1\to(0,\infty)$;

- A morphism from an object (Λ_1,ℓ_1) to an object (Λ_2,ℓ_2) is a morphism $(\Lambda_1,\Lambda_2,\iota)$ in Refine, i.e., a refinement of graphs, that in addition has the property that for each $e\in\Lambda_1^1$, we have

$$\ell_1(e)=\sum_{i=1}^m \ell_2(e_i),$$

  where $\iota^{(1)}(e)=(e_1,\dots,e_m)$ is the corresponding path in Λ_2.

We can now summarise Remark 8.13 and Example 8.14 in this language:

8.16 Proposition. *Let G be a compact Lie group, and let β_0 be a bi-invariant Riemannian metric on G. Then the following assignments define a contravariant functor* $\mathsf{wtRefine} \to \mathsf{RiemannSub}$*:*

- *An object (Λ, ℓ) in* $\mathsf{wtRefine}$ *is mapped to the smooth manifold G^{Λ^1} endowed with the Riemannian metric β defined by*
$$\beta_{(a_e)_{e\in\Lambda^1}}\left((v_e)_{e\in\Lambda^1}, (w_e)_{e\in\Lambda^1}\right) := \sum_{e\in\Lambda^1} \ell(e)^{-1}\beta_{0,a_e}(v_e, w_e),$$
for each $(a_e)_{e\in\Lambda^1} \in G^{\Lambda^1}$ and each
$$(v_e)_{e\in\Lambda^1}, (w_e)_{e\in\Lambda^1} \in T_{(a_e)_{e\in\Lambda^1}} G^{\Lambda^1} \cong \prod_{e\in\Lambda^1} T_{a_e}G;$$

- *A morphism $(\Lambda_1, \Lambda_2, \iota)$ from an object (Λ_1, ℓ_1) to an object (Λ_2, ℓ_2) is mapped to the Riemannian submersion*
$$G^{\Lambda_2^1} \to G^{\Lambda_1^1}, \quad (a_{e'})_{e'\in\Lambda_2^1} \mapsto (a_{e_1}\dots a_{e_m})_{e\in\Lambda_1^1}, \quad (\iota^{(1)}(e) = (e_1, \dots, e_m)).$$

Composing this functor with the functor $\mathsf{RiemannSub} \to \mathsf{SympSub}$ *yields the following contravariant functor* $\mathsf{wtRefine} \to \mathsf{SympSub}$*:*

- *An object (Λ, ℓ) in* $\mathsf{wtRefine}$ *is mapped to the smooth manifold $T^*G^{\Lambda^1}$ endowed with the canonical symplectic form;*

- *A morphism $(\Lambda_1, \Lambda_2, \iota)$ from an object (Λ_1, ℓ_1) to an object (Λ_2, ℓ_2) is mapped to the symplectic submersion*
$$T^*G^{\Lambda_2^1} \to T^*G^{\Lambda_1^1},$$
$$\left((a_{e'})_{e'\in\Lambda_2^1}, (\xi_e)_{e'\in\Lambda_2^1}\right) \mapsto \left((a_{e_1}\dots a_{e_m})_{e\in\Lambda_1^1}, \left(\sum_{i=1}^m \frac{\ell_2(e_i)}{\ell_1(e)} \widetilde{\xi_{e_i}}\right)_{e\in\Lambda_1^1} \right),$$
where $e_1, \dots, e_m$ are related to e as above, and
$$\widetilde{\xi_{e_i}} := \xi_{e_i} \circ \left(T_{a_{e_i}}\left(L_{a_{e_1}\dots a_{e_{i-1}}} \circ R_{a_{e_{i+1}}\dots a_{e_m}}\right)\right)^{-1}.$$

This functor is independent of the particular choice of β_0.

8.17 Remark. The four categories containing manifolds mentioned in the proposition above can be modified to include group actions of the gauge groups on the manifolds as part of the data encoding an object, and morphisms can be required to be equivariant with respect to these actions, as in part (2) of Remark 4.9. It follows from our discussion in part (3) of Example 8.12 that the above proposition also has an equivariant version.

8.2.3 The classical category

Having defined suitable notions of morphisms between configuration spaces and phase spaces of systems, and having established that certain subclasses of these spaces, together with these notions of morphisms form categories, we now want to do something similar for the observable algebras associated to such systems. With the notion of a quantisation in mind, it makes sense to define a category as follows:

8.18 Definition. We define the *classical category* $\mathsf{Classical}$ as follows:

- Its objects consist of pairs $(A, \mathcal{A})$, where A is a commutative unital C*-algebra, and $\mathcal{A}$ is a dense *-subalgebra of A that is endowed with a Poisson bracket;

- The set of morphisms from an object $(A, \mathcal{A})$ to an object $(B, \mathcal{B})$ is the set of *-homomorphisms $\phi\colon A \to B$ with the property that ϕ restricts to a Poisson map $\mathcal{A} \to \mathcal{B}$.

Composition of morphisms is simply given by composition of maps, and the identity element is the identity map on the C*-algebra.

We leave it to the reader to define the equivariant version of this category.

In typical examples, the objects $(A, \mathcal{A})$ in the classical category arise as function spaces on symplectic manifolds. It should therefore not come as a surprise that a typical morphism is a pullback of a map between symplectic manifolds to function spaces. In the same way in which Definition 8.7 is a generalisation of that by Lanéry and Thiemann, the following proposition is an extension of [72, Proposition 2.2], and motivates the notions of partial isomorphism and symplectic morphism.

8.19 Proposition. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds, let $F\colon M_1 \to M_2$ be a symplectic morphism, and let*

$$\mathcal{A}_F := \{f \in C^\infty(M_2) \mid \forall\, m \in M_1\colon X_f \circ F(m) \in \operatorname{Im}(T_m F)\},$$

where $X_f = \{f, \cdot\}_{M_2}$ denotes the Hamiltonian vector field of f. Then $\mathcal{A}_F$ is a Poisson subalgebra of $C^\infty(M_2)$, and the map $F^*\colon \mathcal{A}_F \to C^\infty(M_1)$ is a Poisson map, i.e.,

$$\{F^*(f), F^*(g)\}_{M_1} = F^*(\{f, g\}_{M_2}),$$

for each $f, g \in \mathcal{A}_F$.

8.20 Remark. In the particular case of Lanery and Thiemann's version of this proposition where F is a symplectic submersion, which is of primary interest to us as well, we have $\mathcal{A}_F = C^\infty(M_2)$, which makes this the easiest case to work with.

Proof. It is clear that $\mathcal{A}_F$ is a subspace of $C^\infty(M_2)$; closure with respect to multiplication follows from the fact that the Poisson bracket is a derivation in both of its arguments. To see that $\mathcal{A}_F$ is closed under the Poisson bracket, first note that $X_{\{f,g\}_{M_2}} = [X_f, X_g]$ for each $f, g \in C^\infty(M_2)$ since the Poisson bracket is a Lie bracket. Since F has constant rank, for any $m \in M_1$, one can fix charts (U_1, ϕ_1) on M_1 and (U_2, ϕ_2) on M_2 such that $m \in U_1$, $F(U_1) \subseteq U_2$, and the map $\phi_2 \circ F \circ \phi_1^{-1}$ is of the form

$$\begin{aligned} \mathbb{R}^{n_1} \supseteq \phi_1(U_1) &\to \phi_2(U_2) \subseteq \mathbb{R}^{n_2}, \\ (x_1, \ldots, x_{n_1}) &\mapsto (x_1, \ldots, x_{\operatorname{rk}(F)}, 0, \ldots, 0) \end{aligned}$$

where n_j is the dimension of M_j for $j = 1, 2$. A computation in local coordinates will now show that $[X_f, X_g] \circ F(m) \in \operatorname{Im}(T_m F)$ for each $f, g \in \mathcal{A}_F$ and each $m \in M_1$, hence $\{f, g\}_{M_2} \in \mathcal{A}_F$. Thus $\mathcal{A}_F$ is a Poisson algebra.

It remains to show that F^* is a Poisson map. Let $m \in M_1$, let $\flat_1\colon T_m M_1 \to T_m^* M_1$ and $\flat_2\colon T_{F(m)} M_2 \to T_{F(m)}^* M_2$ be the musical isomorphisms associated to ω_1 and ω_2, with inverses $\sharp_1$ and $\sharp_2$, respectively. Since F is a morphism of symplectic manifolds, we can apply part (7) of Proposition 8.4 to find that

$$T_m F \circ \sharp_1 \circ T_m F^T \circ \flat_2|_{\operatorname{Im}(T_m F)} = \operatorname{Id}_{\operatorname{Im}(T_m F)}.$$

Moreover, for each $f \in \mathcal{A}_F$, we have $\sharp_2(df_{F(m)}) = X_f \circ F(m) \in \mathrm{Im}(T_mF)$, hence

$$\begin{aligned} T_mF(X_{F^*(f)}(m)) &= T_mF \circ \sharp_1(d(F^*(f))_m) = T_mF \circ \sharp_1(F^*(df)_m) \\ &= T_mF \circ \sharp_1 \circ T_mF^T \circ \flat_2 \circ \sharp_2(df_{F(m)}) = X_f \circ F(m). \end{aligned}$$

It follows that for each $f, g \in \mathcal{A}_F$, we have

$$\begin{aligned} &\{F^*(f), F^*(g)\}_{M_1}(m) \\ &\quad = \omega_{1,m}(X_{F^*(f)}(m), X_{F^*(g)}(m)) = d(F^*(g))_m(X_{F^*(f)}(m)) \\ &\quad = F^*(dg)_m(X_{F^*(f)}(m)) = dg_{F(m)}(T_mF(X_{F^*(f)}(m))) \\ &\quad = \omega_{F(m)}(X_f \circ F(m), X_g \circ F(m)) = \{f, g\}_{M_2} \circ F(m), \end{aligned}$$

so $\{F^*(f), F^*(g)\}_{M_1} = F^*(\{f, g\}_{M_2})$, since $m \in M_1$ was arbitrary. We conclude that F^* is a Poisson map. ■

The remaining part of this section is devoted to the construction of a functor $\mathsf{wtRefine} \to \mathsf{Classical}$ relevant to lattice gauge theory builing on the work in Example 8.14 and the subsequent text in the previous subsection. We restrict to the case in which the structure group G is the torus $\mathbb{T}^n$, and show that for both types of elementary refinements, we obtain a natural algebra morphism between the corresponding resolvent algebras.

8.21 Proposition. *Let $n \in \mathbb{N}$, let ℓ_1 and ℓ_2 be positive real numbers, let $\ell := \ell_1 + \ell_2$, and let β_0 be a bi-invariant Riemannian metric on $\mathbb{T}^n$. Define β, β_1 and $\beta_{1,2}$ as in Example 8.14. Furthermore, let $A_{0,c} := C_{\mathcal{R}}(T^*\mathbb{T}^n)$ be the resolvent algebra associated to $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ with dense Poisson subalgebra $\mathcal{A}_{0,c} := \mathcal{S}_{\mathcal{R}}(T^*\mathbb{T}^n)$, and let $A_{0,f} := C_{\mathcal{R}}(T^*(\mathbb{T}^n \times \mathbb{T}^n))$ be the resolvent algebra associated to $T^*(\mathbb{T}^n \times \mathbb{T}^n) \cong \mathbb{T}^{2n} \times \mathbb{R}^{2n}$ with dense Poisson subalgebra $\mathcal{A}_{0,f} := \mathcal{S}_{\mathcal{R}}(T^*(\mathbb{T}^n \times \mathbb{T}^n))$.*

(1) Let F be the symplectic morphism induced by the Riemannian morphism

$$f\colon (G \times G, \beta_{1,2}) \to (G, \beta_1), \quad (a_1, a_2) \mapsto a_1.$$

Then the pullback $F^*\colon C^\infty(T^*\mathbb{T}^n) \to C^\infty(T^*(\mathbb{T}^n \times \mathbb{T}^n))$ induces a Poisson map $\mathcal{A}_{0,c} \to \mathcal{A}_{0,f}$ that maps the generator $e_k \otimes h_{U,\xi,g}$ to the generator $e_{(k,0)} \otimes h_{U \times \{0\},(\xi,0),g \otimes \mathbf{1}_{\{0\}}}$ for each $k \in \mathbb{Z}^n$, each subspace $U \subseteq \mathbb{R}^n$, each $\xi \in U^\perp$ and each $g \in \mathcal{S}(U)$.

(2) Let F be the symplectic morphism induced by the Riemannian morphism

$$f\colon (G\times G,\beta_{1,2})\to (G,\beta),\quad (a_1,a_2)\mapsto a_1a_2.$$

Then the pullback $F^*\colon C^\infty(T^*\mathbb{T}^n)\to C^\infty(T^*(\mathbb{T}^n\times\mathbb{T}^n))$ induces a Poisson map $\mathcal{A}_{0,c}\to\mathcal{A}_{0,f}$ that maps the generator $e_k\otimes h_{U,\xi,g}$ to the generator $e_{(k,k)}\otimes h_{\tilde{U},\tilde{\xi},\tilde{g}}$, where

$$\begin{aligned}\tilde{U} &:= \{(\ell_1 v,\ell_2 v)\in\mathbb{R}^n\times\mathbb{R}^n\colon v\in U\},\\ \tilde{\xi} &:= \left(\frac{\ell_1}{\ell}\xi,\frac{\ell_2}{\ell}\xi\right),\\ \tilde{g}\colon \tilde{U}\to\mathbb{C},\quad (v,w) &\mapsto g\left(\frac{\ell_1^2+\ell_2^2}{\ell^2}(v+w)\right),\end{aligned}$$

for each $k\in\mathbb{Z}^n$, each subspace $U\subseteq\mathbb{R}^n$, each $\xi\in U^\perp$ and each $g\in\mathcal{S}(U)$.

In both cases, the Poisson map has a unique extension to an injective *-homomorphism $A_{0,c}\to A_{0,f}$.

Proof.

(1) The assertion that the pullbacks induce Poisson maps between the spaces of smooth functions was already proved in Proposition 8.19. Also, the assertion that the map $\mathcal{A}_{0,c}\to\mathcal{A}_{0,f}$ extends uniquely to a *-homomorphism follows from the fact that the pullback is obviously a *-homomorphism that is continuous with respect to the sup-norms, and part (2) of Proposition 5.9. Injectivity of the *-homomorphism is a consequence of surjectivity of the symplectic morphism. If we show that F^* maps any generator of $\mathcal{A}_{0,c}$ to a generator of $\mathcal{A}_{0,f}$, namely the one described in the statement of the theorem, then it follows that F^* maps $\mathcal{A}_{0,c}$ into $\mathcal{A}_{0,f}$. Thus it suffices to show that

$$F^*(e_k\otimes h_{U,\xi,g}) = e_{(k,0)}\otimes h_{U\times\{0\},(\xi,0),g\otimes\mathbf{1}_{\{0\}}};$$

note that $(\xi,0)\in U^\perp\times\{0\}\subseteq(U\times\{0\})^\perp$, and that $g\otimes\mathbf{1}_{\{0\}}\in\mathcal{S}(U\times\{0\})$, so the right-hand side is indeed a generator of $\mathcal{A}_{0,f}$.

The formula for the induced symplectic morphism can be found in part (3) of Example 8.12. Let $k \in \mathbb{Z}^n$, let $U \subseteq \mathbb{R}^n$ be a subspace, let $\xi \in U^\perp$ and let $g \in \mathcal{S}(U)$. We then find that

$$\begin{aligned} &F^*(e_k \otimes h_{U,\xi,g})((a_1, a_2), (p_1, p_2)) \\ &\quad = e_k(a_1) h_{U,\xi,g}(p_1) = e^{2\pi i k \cdot a_1} e^{i\xi \cdot p_1} g \circ r_U(p_1) \\ &\quad = e^{2\pi i (k,0)\cdot(a_1,a_2)} e^{i(\xi,0)\cdot(p_1,p_2)} (g \otimes \mathbf{1}_{\{0\}}) \circ r_{U\times\{0\}}(p_1, p_2) \\ &\quad = (e_{(k,0)} \otimes h_{U\times\{0\},(\xi,0),g\otimes\mathbf{1}_{\{0\}}})((a_1, a_2), (p_1, p_2)), \end{aligned}$$

for each $((a_1, a_2), (p_1, p_2)) \in \mathbb{T}^{2n} \times \mathbb{R}^{2n}$, which is what we wanted to show.

(2) For the same reasons as in (1), it suffices to show that

$$F^*(e_k \otimes h_{U,\xi,g}) = e_{(k,k)} \otimes h_{\tilde{U},\tilde{\xi},\tilde{g}}.$$

The formula for the induced symplectic morphism can be found in Example 8.14; with respect to the chosen trivialisations, it reads

$$((a_1, a_2), (p_1, p_2)) \mapsto \left(a_1 + a_2, \frac{\ell_1}{\ell} p_1 + \frac{\ell_2}{\ell} p_2 \right),$$

where in the first component on the right-hand side, the symbol $+$ denotes the group multiplication in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. As in part (1), let $k \in \mathbb{Z}^n$, let $U \subseteq \mathbb{R}^n$ be a subspace, let $\xi \in U^\perp$ and let $g \in \mathcal{S}(U)$. Define $\tilde{U}$ as above. It can be checked that the orthogonal projection onto $\tilde{U}$ with respect to the standard inner product is given by the map

$$\begin{aligned} r_{\tilde{U}} \colon \mathbb{R}^n \times \mathbb{R}^n &\to \tilde{U}, \\ (p_1, p_2) &\mapsto (\ell_1^2 + \ell_2^2)^{-1} \cdot (\ell_1(\ell_1 r_U(p_1) + \ell_2 r_U(p_2)), \\ &\qquad\qquad \ell_2(\ell_1 r_U(p_1) + \ell_2 r_U(p_2))), \end{aligned}$$

Define the map

$$S \colon \tilde{U} \to U, \quad (v, w) \mapsto \frac{\ell_1^2 + \ell_2^2}{\ell^2}(v + w).$$

Then a straightforward computation shows that

$$S \circ r_{\tilde{U}}(p_1, p_2) = r_U \left(\frac{\ell_1}{\ell} p_1 + \frac{\ell_2}{\ell} p_2 \right),$$

for each $p_1, p_2 \in \mathbb{R}^n$, hence

$$\tilde{g} \circ r_{\tilde{U}}(p_1, p_2) = g \circ S \circ r_{\tilde{U}}(p_1, p_2) = g \circ r_U\left(\frac{\ell_1}{\ell}p_1 + \frac{\ell_2}{\ell}p_2\right).$$

We now apply this to find that

$$\begin{aligned}
&F^*(e_k \otimes h_{U,\xi,g})((a_1, a_2), (p_1, p_2)) \\
&= e_k(a_1 + a_2) h_{U,\xi,g}\left(\frac{\ell_1}{\ell}p_1 + \frac{\ell_2}{\ell}p_2\right) \\
&= e^{2\pi i k\cdot(a_1+a_2)} e^{i\xi\cdot\left(\frac{\ell_1}{\ell}p_1 + \frac{\ell_2}{\ell}p_2\right)} g \circ r_U\left(\frac{\ell_1}{\ell}p_1 + \frac{\ell_2}{\ell}p_2\right) \\
&= e^{2\pi i (k,k)\cdot(a_1,a_2)} e^{i\left(\frac{\ell_1}{\ell}\xi, \frac{\ell_2}{\ell}\xi\right)\cdot(p_1,p_2)} \tilde{g} \circ r_{\tilde{U}}(p_1, p_2) \\
&= (e_{(k,k)} \otimes h_{\tilde{U},\tilde{\xi},\tilde{g}})((a_1, a_2), (p_1, p_2)),
\end{aligned}$$

for each $((a_1, a_2), (p_1, p_2)) \in \mathbb{T}^{2n} \times \mathbb{R}^{2n}$, as desired. Note that

$$\left(\frac{\ell_1}{\ell}\xi, \frac{\ell_2}{\ell}\xi\right) \in \tilde{U}^{\perp}, \quad \tilde{g} \in \mathcal{S}(\tilde{U}).$$

■

Similarly to how we obtained Proposition 8.16 from Example 8.14, we now get the following proposition from Proposition 8.21.

8.22 Proposition. *The following assignment defines a covariant functor* wtRefine $\to$ Classical*:*

- *An object (Λ, ℓ) is mapped to the pair*

$$(A_{0,\Lambda}, \mathcal{A}_{0,\Lambda}) = \left(C_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1}), \mathcal{S}_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})\right);$$

- *A morphism $(\Lambda_1, \Lambda_2, \iota)$ from an object (Λ_1, ℓ_1) to an object (Λ_2, ℓ_2) is mapped to the pullback of the map F to the spaces of bounded functions, restricted to A_{0,Λ_2}, where F denotes the image of $(\Lambda_1, \Lambda_2, \iota)$ under the functor* wtRefine $\to$ SympSub *from Proposition 8.16.*

8.23 Remark. As with Proposition 8.16, there is an equivariant version of this functor that maps an object in $\mathsf{wtRefine}$ to $\mathsf{Classical}$ endowed with an action of the gauge group $(\mathbb{T}^n)^{\Lambda^0}$. Indeed, it essentially follows from the first assertion in part (4) of Proposition 7.4 that gauge transformations preserve both $\mathcal{A}_{0,\Lambda}$ and $A_{0,\Lambda}$.

Now consider the subalgebras $\mathcal{A}_{0,\Lambda}^{\mathrm{red}} \subseteq \mathcal{A}_{0,\Lambda}$ and $A_{0,\Lambda}^{\mathrm{red}} \subseteq A_{0,\Lambda}$ that consist of gauge invariant elements of the field algebras. It follows from Proposition 8.16 that the image of a refinement $(\Lambda_1, \Lambda_2, \iota)$ under the functor $\mathsf{wtRefine} \to \mathsf{SympSub}$ restricts to maps

$$\mathcal{A}_{0,\Lambda_1}^{\mathrm{red}} \hookrightarrow \mathcal{A}_{0,\Lambda_2}^{\mathrm{red}}, \quad A_{0,\Lambda_1}^{\mathrm{red}} \hookrightarrow A_{0,\Lambda_2}^{\mathrm{red}},$$

between the gauge invariant subalgebras.

From the above considerations, we obtain the following result:

8.24 Proposition. *The following assignment defines a covariant functor* $\mathsf{wtRefine} \to \mathsf{Classical}$*:*

- *An object* (Λ, ℓ) *is mapped to the pair*

$$(A_{0,\Lambda}^{\mathrm{red}}, \mathcal{A}_{0,\Lambda}^{\mathrm{red}}) = \left(C_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})^{(\mathbb{T}^n)^{\Lambda^0}}, \mathcal{S}_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})^{(\mathbb{T}^n)^{\Lambda^0}} \right),$$

  where the superscript $(\mathbb{T}^n)^{\Lambda^0}$ *indicates that we consider the gauge invariant elements of the algebras.*

- *A morphism* $(\Lambda_1, \Lambda_2, \iota)$ *from an object* (Λ_1, ℓ_1) *to an object* (Λ_2, ℓ_2) *is mapped to the pullback of the map* F *to the spaces of bounded functions, restricted to* $A_{0,\Lambda_2}^{\mathrm{red}}$*, where* F *denotes the image of* $(\Lambda_1, \Lambda_2, \iota)$ *under the functor* $\mathsf{wtRefine} \to \mathsf{SympSub}$ *from Proposition 8.16.*

8.25 Definition. We call the functor $\mathsf{wtRefine} \to \mathsf{Classical}$ from Proposition 8.22 the *classical functor*, and it will be denoted by F_C. We call the functor from Proposition 8.22 between the above categories the *reduced classical functor*, and, in keeping with the notation in chapter 4, it will be denoted by $\mathsf{F}_C^{\mathrm{red}}$.

8.26 Remark. At this point, the attentive reader may object to the use of the word 'reduced' in this context, specifically to its use with regard to the algebra $A^{\mathrm{red}}_{0,\Lambda_2}$, since it does not yet correspond to the Marsden–Weinstein quotient of the phase space $T^*(\mathbb{T}^n)^{\Lambda^1}$ by the action of the gauge group G^{Λ^0}. For the moment, we will ignore this issue and continue to work with the reduced algebra as defined above in the next section, leaving it for the discussion in section 8.4 instead.

8.3 The quantum functor

Having found a functor from $\mathsf{wtRefine}$ to a suitable category of classical observables, we now investigate whether there exists a similar functor to a category of quantum mechanical observables.

8.27 Definition. We define the *quantum category* $\mathsf{Quantum}$ as follows:

- Its objects are unital C*-algebras;
- The set of morphisms from an object A_1 to an object A_2 is the set of unital *-homomorphisms $A_1 \to A_2$.

Our desired covariant functor $\mathsf{F}_Q\colon \mathsf{wtRefine} \to \mathsf{Quantum}$ should send an object (Λ, ℓ) to the quantum mechanical resolvent algebra $A_{\hbar,\Lambda}$ of $T^*(\mathbb{T}^n)^{\Lambda^1}$.

Furthermore, suppose that there exists a subset $I \subseteq \mathbb{R}\backslash\{0\}$ that has 0 as an accumulation point and such that for each object (Λ, ℓ) in $\mathsf{wtsRefine}$, we are given a family of quantisation maps $(\mathcal{Q}_{\hbar,(\Lambda,\ell)})_{\hbar\in I}$ from $\mathcal{A}_{0,\Lambda}$ into $A_{\hbar,\Lambda} = \mathsf{F}_Q(\Lambda, \ell)$, where $(A_{0,\Lambda}, \mathcal{A}_{0,\Lambda}) = \mathsf{F}_C(\Lambda, \ell)$. Then we require the family

$$((\mathcal{Q}_{\hbar,(\Lambda,\ell)})_{\hbar\in I})_{(\Lambda,\ell)\in\mathrm{Obj}(\mathsf{wtRefine})}, \tag{8.1}$$

to be an *approximate natural transformation* from the functor $\mathsf{F}_C\colon \mathsf{wtRefine} \to \mathsf{Classical}$ to the functor $\mathsf{F}_Q\colon \mathsf{wtRefine} \to \mathsf{Quantum}$, by which we mean the following:

(1) For each object (Λ, ℓ) in $\mathsf{wtRefine}$, the family $(\mathcal{Q}_{\hbar,(\Lambda,\ell)})_{\hbar\in I}$ is a strict quantisation of $\mathcal{A}_{0,\Lambda}$, except for the requirement that the map $\hbar \to \|\mathcal{Q}_{\hbar,(\Lambda,\ell)}(f)\|$ is continuous at $\hbar > 0$;

(2) For each morphism $(\Lambda_1, \Lambda_2, \ell)$ from an object (Λ_1, ℓ_1) to an object (Λ_2, ℓ_2) in $\mathsf{wtRefine}$, the following diagram

$$\begin{array}{ccc}
\mathcal{A}_{0,\Lambda_2} & \xrightarrow{\mathcal{Q}_{\hbar,(\Lambda_2,\ell_2)}} & A_{\hbar,\Lambda_2} \\
{\scriptstyle \mathsf{F}_C(\Lambda_1,\Lambda_2,\iota)}\uparrow & & \uparrow{\scriptstyle \mathsf{F}_Q(\Lambda_1,\Lambda_2,\iota)} \\
\mathcal{A}_{0,\Lambda_1} & \xrightarrow{\mathcal{Q}_{\hbar,(\Lambda_1,\ell_1)}} & A_{\hbar,\Lambda_1}
\end{array}$$

becomes commutative in the limit $\hbar \to 0$, by which we mean that for each $f \in \mathcal{A}_{0,\Lambda_1}$, we have

$$\lim_{\hbar\to 0} \left\| \mathsf{F}_Q(\Lambda_1, \Lambda_2, \iota) \circ \mathcal{Q}_{\hbar,(\Lambda_1,\ell_1)}(f) - \mathcal{Q}_{\hbar,\mathsf{F}_C(\Lambda_2,\ell_2)} \circ \mathsf{F}_C(\Lambda_1, \Lambda_2, \iota)(f) \right\| \\ = 0.$$

Of course, we will take $\mathcal{Q}_\hbar = \mathcal{Q}_\hbar^W$, the Weyl quantisation map, which has already been shown to satisfy the first requirement in Theorem 7.8. We use the second requirement to look for a suitable definition of the functor F_Q on morphisms in $\mathsf{Quantum}$. We restrict our attention to elementary refinements in the same way as in Proposition 8.21.

(1) We first consider the case of addition of an edge to a graph, i.e., part (1) of the aforementioned proposition. Using this proposition and part (3) of Proposition 7.1, we obtain

$$\begin{aligned}
&\mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)} \circ \mathsf{F}_C(\Lambda_1, \Lambda_2, \iota)(e_k \otimes h_{U,\xi,g})\psi_{(a_1,a_2)} \\
&= \mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)}(e_{(k,0)} \otimes h_{U\times\{0\},(\xi,0),g\otimes\mathbf{1}_{\{0\}}})\psi_{(a_1,a_2)} \\
&= h_{U\times\{0\},(\xi,0),g\otimes\mathbf{1}_{\{0\}}}(\pi\hbar((k,0) + 2(a_1, a_2)))\psi_{(k,0)+(a_1,a_2)} \\
&= h_{U,\xi,g}(\pi\hbar(k + 2a_1))\psi_{(k+a_1,a_2)},
\end{aligned}$$

for each $(a_1, a_2) \in \mathbb{Z}^n \times \mathbb{Z}^n$, where $e_k \otimes h_{U,\xi,g}$ is a generator of the commutative resolvent algebra. Comparing this to part (3) of Proposition 7.1, and noting that under the isomorphism $L^2(\mathbb{T}^n \times \mathbb{T}^n) \cong L^2(\mathbb{T}^n) \otimes L^2(\mathbb{T}^n)$, we have $\psi_{k+a_1,a_2} = \psi_{k+a_1} \otimes \psi_{a_2}$, we find that a natural choice for $\mathsf{F}_Q(\Lambda_1, \Lambda_2, \iota)$ is the map

$$A_{\hbar,\Lambda_1} \to A_{\hbar,\Lambda_2}, \quad a \mapsto a \otimes \mathrm{Id}_{L^2(\mathbb{T}^n)}.$$

Note that $A_{\hbar,\Lambda_2} \subseteq B(L^2(\mathbb{T}^n \times \mathbb{T}^n))$, and that $B(L^2(\mathbb{T}^n \times \mathbb{T}^n)) \cong B(L^2(\mathbb{T}^n)) \hat{\otimes} B(L^2(\mathbb{T}^n))$, where the tensor product denotes the tensor product of von Neumann algebras. The above map is a *-homomorphism, and we see that the above diagram is commutative for each $\hbar \neq 0$. More generally, suppose that (Λ_1, ℓ_1) and (Λ_2, ℓ_2) are objects in $\mathsf{wtRefine}$ for which there exists a refinement $(\Lambda_1, \Lambda_2, \iota)$ that can be written as a composition of elementary refinements, each of which corresponds to addition of an edge. Then the map

$$\mathsf{F}_Q(\Lambda_1, \Lambda_2, \iota) \colon A_{\hbar,\Lambda_1} \to A_{\hbar,\Lambda_2}, \quad a \mapsto a \otimes \mathrm{Id}_{L^2\left((\mathbb{T}^n)^{\Lambda_2^1 \setminus \iota^{(1)}(\Lambda_1^1)}\right)},$$

defines an injective *-homomorphism such that the diagram in part (2) of our definition of an approximate natural transformation is commutative. As already mentioned in section 5.1, this is consistent with the literature, which says that in the situation of part (4) of Example 8.12, the induced map at the level of observable algebras is of the form

$$a \mapsto a \otimes \mathrm{Id}_{\mathcal{H}_2},$$

where for $j = 1, 2$, $\mathcal{H}_j$ is the Hilbert space associated to the phase space T^*Q_j, and a is some operator on $\mathcal{H}_1$. We refer to [48, section 2.3] for the version of this statement in lattice gauge theory, and the first three paragraphs of [34, section 2] for its formulation in a more general setting.

(2) Next, we turn to the case of subdivision of an edge to a graph, i.e., part (2) of Proposition 8.21. In the same way as in the previous case, we find that

$$\begin{aligned}
&\mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)} \circ \mathsf{F}_C(\Lambda_1, \Lambda_2, \iota)(e_k \otimes h_{U,\xi,g})\psi_{(a_1,a_2)} \\
&= \mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)}(e_{(k,k)} \otimes h_{\tilde{U},\tilde{\xi},\tilde{g}})\psi_{(a_1,a_2)} \\
&= h_{\tilde{U},\tilde{\xi},\tilde{g}}(\pi\hbar((k,k) + 2(a_1,a_2)))\psi_{(k,k)+(a_1,a_2)} \\
&= h_{U,\xi,g}\left(\pi\hbar\left(k + 2\left(\frac{\ell_1}{\ell}a_1 + \frac{\ell_2}{\ell}a_2\right)\right)\right)\psi_{(k+a_1,k+a_2)}.
\end{aligned}$$

We now run into the following problem. In order to define a *-homomorphism $\mathsf{F}_Q(\Lambda_1, \Lambda_2, \iota) \colon A_{\hbar,\Lambda_1} \to A_{\hbar,\Lambda_2}$, we want to use the above expression to define the image of the operator $\mathcal{Q}^W_{\hbar,(\Lambda_1,\ell_1)}(e_k \otimes h_{U,\xi,g})$ under the *-homomorphism. Thus we need to somehow extract the scalar

$$h_{U,\xi,g}\left(\pi\hbar\left(k+2\left(\frac{\ell_1}{\ell}a_1+\frac{\ell_2}{\ell}a_2\right)\right)\right),$$

from this operator without explicit reference to k, U, ξ or g, since *a priori*, we can only determine the matrix elements

$$\langle\psi_{k+a}, \mathcal{Q}^W_{\hbar,(\Lambda_1,\ell_1)}(e_k \otimes h_{U,\xi,g})\psi_a\rangle = h_{U,\xi,g}(\pi\hbar(k+2a)),$$

i.e., we only have access to the values of the function $h_{U,\xi,g}$ for $p \in \pi\hbar(k+2\mathbb{Z}^n)$. However, in order to obtain the scalar, we need to know the values of this function on the set

$$\left\{\pi\hbar\left(k+2\left(\frac{\ell_1}{\ell}a_1+\frac{\ell_2}{\ell}a_2\right)\right) : a_1, a_2 \in \mathbb{Z}^n\right\},$$

of which $\pi\hbar(k+2\mathbb{Z}^n)$ is in general a proper subset.

Let us discuss some possible naive ways in which we may define linear maps

$$A_{\hbar,\Lambda_1} \to A_{\hbar,\Lambda_2},$$

in the situation of subdivision of an edge, and argue why they are unsatisfactory.

- We could drop the requirement $\ell = \ell_1 + \ell_2$ and try to do the above computations with $\ell_1 = \ell_2 = \ell = 1$ instead: in this case, we have sufficient information to define a map $A_{\hbar,\Lambda_1} \to A_{\hbar,\Lambda_2}$. However, if we modify the corresponding maps on the classical side as well, then we note that $\mathsf{F}_C(\Lambda_1, \Lambda_2, \iota)$ is no longer compatible with the Poisson structures on the classical Poisson algebras;

- We could try to use linear interpolation of known matrix elements to obtain unknown ones. Let α be the corresponding tentative map between the quantum resolvent algebras, which we define using matrix elements by requiring that

$$\langle\psi_{(k+a_1,k+a_2)}, \alpha(b)\psi_{(a_1,a_2)}\rangle = \frac{\ell_1}{\ell}\langle\psi_{k+a_1}, b\psi_{a_1}\rangle + \frac{\ell_2}{\ell}\langle\psi_{k+a_2}, b\psi_{a_2}\rangle,$$

for each $k, a_1, a_2 \in \mathbb{Z}^n$ and each $b \in A_{\hbar,\Lambda^1}$, and that all other matrix elements vanish. We now ask whether α is a *-homomorphism. The answer is no, since α is not compatible with multiplication. Indeed, take $b_1 = |\psi_0\rangle\langle\psi_0|$ and $b_2 = |\psi_k\rangle\langle\psi_0|$, where $k \in \mathbb{Z}^n\backslash\{0\}$. Then on the one hand, we have $b_1 b_2 = 0$, hence $\alpha(b_1 b_2) = 0$, while on the other hand, we have

$$\begin{aligned}
\alpha(b_1) &= \frac{\ell_1}{\ell}|\psi_0\rangle\langle\psi_0| \otimes \mathrm{Id}_{L^2(\mathbb{T}^n)} + \frac{\ell_2}{\ell}\mathrm{Id}_{L^2(\mathbb{T}^n)} \otimes |\psi_0\rangle\langle\psi_0| \\
\alpha(b_2) &= \frac{\ell_1}{\ell}|\psi_k\rangle\langle\psi_0| \otimes S^k + \frac{\ell_2}{\ell}S^k \otimes |\psi_k\rangle\langle\psi_0|,
\end{aligned}$$

where the notation is as in the proof of Proposition 7.4, and we have made the identification $L^2(\mathbb{T}^n \times \mathbb{T}^n) \cong L^2(\mathbb{T}^n) \hat{\otimes} L^2(\mathbb{T}^n)$, so that after some algebra, we find

$$\begin{aligned}
&\alpha(b_1)\alpha(b_2) \\
&\quad = \frac{\ell_1\ell_2}{\ell^2}\left(|\psi_k\rangle\langle\psi_0| \otimes |\psi_0\rangle\langle\psi_{-k}| + |\psi_0\rangle\langle\psi_{-k}| \otimes |\psi_k\rangle\langle\psi_0|\right) \neq 0.
\end{aligned}$$

These are strong indications that the problem in part (2) above is not something that can be readily solved by a good choice of parameters or maps, but is inherent to the Hilbert spaces on which we defined our quantum resolvent algebras. This may not come as a complete surprise, since in the context of QFT, physicists work with Fock spaces of one-particle spaces rather than with the one-particle Hilbert spaces themselves like we do here, for reasons that can be traced back to the nature of the solutions of the Dirac equation. It is nonetheless remarkable that an indication that those one-particle Hilbert spaces do not allow for a satisfactory formulation can already be found in the present nonrelativistic context in which there is not yet any mention of an infinite number of degrees of freedom.

It appears that the idea that $\mathcal{Q}_\hbar^W$ is an approximate natural transformation works in the case of addition of an edge, but fails in the case of subdivision of an edge, which prevents us from treating both cases on equal footing at this level. We can make the following observations, however, which show that the idea does work on a different level.

(1) If $a_1 = a_2 = a$, then $\frac{\ell_1}{\ell} a_1 + \frac{\ell_2}{\ell} a_2 = a$, hence

$$\begin{aligned}&\langle \psi_{(k+a,k+a)}, \mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)} \circ \mathsf{F}_C(\Lambda_1,\Lambda_2,\iota)(e_k \otimes h_{U,\xi,g})\psi_{(a,a)}\rangle \\ &= \langle \psi_{k+a}, \mathcal{Q}^W_{\hbar,(\Lambda_1,\ell_1)}(e_k \otimes h_{U,\xi,g})\psi_a\rangle,\end{aligned}$$

for each $a \in \mathbb{Z}^n$, which shows that the pullback of the compression of $\mathcal{Q}^W_{\hbar,(\Lambda_2,\ell_2)} \circ \mathsf{F}_C(\Lambda_1,\Lambda_2,\iota)(e_k \otimes h_{U,\xi,g})$ to the closed linear subspace generated by

$$\{\psi_{(a,a)} \in L^2(\mathbb{T}^n \times \mathbb{T}^n) \colon a \in \mathbb{Z}^n\} \tag{8.2}$$

under the isometry

$$L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n \times \mathbb{T}^n), \quad \psi_a \mapsto \psi_{(a,a)}, \tag{8.3}$$

is equal to $\mathcal{Q}^W_{\hbar,\mathsf{F}_C(\Lambda_1,\ell_1)}(e_k \otimes h_{U,\xi,g})$. This isometry between Hilbert spaces is precisely the pullback of the map $\mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T}^n$ that implements the refinement at the level of configuration spaces. Thus, informally speaking, if there had been a good map $\mathsf{F}_Q(\Lambda_1,\Lambda_2,\iota)\colon \mathcal{A}_{0,\Lambda_1} \to \mathcal{A}_{0,\Lambda_2}$, conjugation with the adjoint of the above isometry would have been a natural left inverse for it.

(2) In the setting of lattice gauge theory, the objects

$$\mathsf{F}_c(\Lambda,\ell) = (A_{0,\Lambda}, \mathcal{A}_{0,\Lambda}) = (C_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1}), \mathcal{S}_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})),$$

really correspond to the *field* algebras; we have not yet taken into account the gauge freedom. We will therefore now consider the pair

$$\mathsf{F}^{\mathrm{red}}_c(\Lambda,\ell) = \left(A^{\mathrm{red}}_{0,\Lambda}, \mathcal{A}^{\mathrm{red}}_{0,\Lambda}\right) = \left(C_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})^{\mathcal{G}}, \mathcal{S}_{\mathcal{R}}(T^*(\mathbb{T}^n)^{\Lambda^1})^{\mathcal{G}}\right),$$

where we have used the same notation for the gauge group $\mathcal{G} := (\mathbb{T}^n)^{\Lambda^0}$ as in chapter 4. In Proposition 8.24, we obtained a map $\mathsf{F}^{\mathrm{red}}_C(\Lambda_1,\Lambda_2,\iota)$ that maps the above algebras for $\Lambda = \Lambda_1$ into those for $\Lambda = \Lambda_2$. Furthermore, it is a consequence of part (4) of Proposition 7.4 that for each object (Λ,ℓ) in $\mathsf{wtRefine}$, the quantisation map $\mathcal{Q}^W_{\hbar,(\Lambda,\ell)}$ is equivariant with respect to the action of the gauge groups on the algebras that make up its domain and codomain. By density of the image of $\mathcal{Q}^W_{\hbar,(\Lambda,\ell)}$ in $A_{\hbar,\Lambda}$, if $\mathsf{F}_Q(\Lambda,\ell)$ can be defined, then it maps the image of $(\Lambda_1, \Lambda_2, \iota)$ to a map that restricts to a map

$$A^{\mathcal{G}_1}_{\hbar,\Lambda_1} \to A^{\mathcal{G}_2}_{\hbar,\Lambda_2},$$

between the gauge invariant subalgebras of the quantum resolvent algebras. (Here, $\mathcal{G}_j := (\mathbb{T}^n)^{\Lambda^0_j}$ for $j = 1, 2$.) Finally, as discussed in chapter 3, the subspaces

$$L^2((\mathbb{T}^n)^{\Lambda^1_j})^{\mathcal{G}_j} \subseteq L^2((\mathbb{T}^n)^{\Lambda^1_j}), \quad j = 1, 2,$$

of gauge invariant elements of the Hilbert spaces associated to the quantum systems, are invariant subspaces for the gauge invariant elements of the field algebras $A_{\hbar,\Lambda_j}$, $j = 1, 2$.

We can connect these two observations with the help of the following lemma.

8.28 Lemma. *Consider the unitary group representation*

$$\mathbb{T}^n \to U(L^2(\mathbb{T}^n \times \mathbb{T}^n)), \quad g \mapsto (\psi \mapsto ((a_1, a_2) \mapsto \psi(a_1 g, g^{-1} a_2))),$$

of $\mathbb{T}^n$ *on* $L^2(\mathbb{T}^n \times \mathbb{T}^n)$*. Let* $V := L^2(\mathbb{T}^n \times \mathbb{T}^n)^{\mathbb{T}^n}$ *be the subspace of invariant elements, and let* V' *be the closed linear span of the set in equation* (8.2)*. Then* $V = V'$*.*

Proof. It is readily seen that $V' \subseteq V$. For the reverse inclusion, we show that for each k_1 and $k_2 \in \mathbb{Z}^n$ such that $k_1 \neq k_2$, we have $\psi_{(k_1,k_2)} \in V^\perp$. Indeed, let $\psi \in V$, and fix k_1 and k_2 as above. Then

$$\begin{aligned}
\langle \psi_{(k_1,k_2)}, \psi \rangle &= \int_{\mathbb{T}^n \times \mathbb{T}^n} e^{-2\pi i (k_1 \cdot a_1 + k_2 \cdot a_2)} \psi(a_1, a_2)\, d(a_1, a_2) \\
&= \int_{\mathbb{T}^n \times \mathbb{T}^n} e^{-2\pi i (k_1 \cdot a_1 + k_2 \cdot (-a_1 + a_2))} \psi(a_1, -a_1 + a_2)\, d(a_1, a_2) \\
&= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} e^{-2\pi i ((k_1 - k_2) \cdot a_1 + k_2 \cdot a_2)} \psi(0 + \mathbb{Z}^n, a_2)\, da_1\, da_2 \\
&= \int_{\mathbb{T}^n} 0\, da_2 = 0,
\end{aligned}$$

as desired. ∎

Now suppose that $(\Lambda_1, \Lambda_2, \iota)$ is an elementary refinement that subdivides a single edge $e \in \Lambda_1^1$ into two edges e_1 and e_2 in Λ_2^1, i.e., $\iota^{(1)}(e) = (e_1, e_2)$. Furthermore, let

$$\mathcal{H}_j := L^2\left((\mathbb{T}^n)^{\Lambda_j^1}\right).$$

Then we have canonical isomorphisms

$$\mathcal{H}_1 \cong L^2(\mathbb{T}^n) \mathbin{\hat{\otimes}} \hat{\bigotimes_{e' \in \Lambda_1^1 \setminus \{e\}}} L^2(\mathbb{T}^n),$$

and

$$\mathcal{H}_2 \cong L^2(\mathbb{T}^n \times \mathbb{T}^n) \mathbin{\hat{\otimes}} \hat{\bigotimes_{e' \in \iota^{(1)}(\Lambda_1^1 \setminus \{e\})}} L^2(\mathbb{T}^n).$$

In these two expressions, the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{T}^n \times \mathbb{T}^n)$ are thought of as the factors of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ that are associated to e and (e_1, e_2), respectively, with the action of the gauge groups $\mathcal{G}_j$ defined accordingly. In both cases, we use these isomorphisms to transfer the actions of the gauge group $\mathcal{G}_j$ on $\mathcal{H}_j$ to an action of the same group on the right-hand side. Now let $y_0 \in \Lambda_2^0$ be the vertex connecting e_1 and e_2. Then the map

$$\begin{aligned}
&\mathcal{G}_1 \times \mathbb{T}^n \to \mathcal{G}_2,\\
&((g_x)_{x \in \Lambda_1^0}, g') \mapsto\\
&\left(y \mapsto \begin{cases} g' & \text{if } y = y_0 \\ g_x & \text{if there exists } x \in \Lambda_1^0 \text{ such that } y = \iota^{(0)}(x) \end{cases}\right),
\end{aligned}$$

is a group isomorphism, and the inclusion of $\mathcal{G}_1 \cong \mathcal{G}_1 \times \{\mathbf{1}\} \hookrightarrow \mathcal{G}_2$ is a group homomorphism with respect to which the inclusion $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ is equivariant in the sense of part (2) of Remark 4.9. Using these isomorphisms of Hilbert spaces, as well as the lemma above and the equivariance of the group action of $\mathcal{G}_1$, we find that the following spaces are all canonically isomorphic:

$$\mathcal{H}_1^{\mathcal{G}_1} \cong \left(L^2(\mathbb{T}^n) \mathbin{\hat{\otimes}} \hat{\bigotimes_{e' \in \Lambda_1^1 \setminus \{e\}}} L^2(\mathbb{T}^n)\right)^{\mathcal{G}_1}$$

$$
\begin{aligned}
&\cong \left(V' \mathbin{\hat{\otimes}} \hat{\bigotimes_{e' \in \iota^{(1)}(\Lambda_1^1 \setminus \{e\})}} L^2(\mathbb{T}^n) \right)^{\mathcal{G}_1} \\
&= \left(V \mathbin{\hat{\otimes}} \hat{\bigotimes_{e' \in \iota^{(1)}(\Lambda_1^1 \setminus \{e\})}} L^2(\mathbb{T}^n) \right)^{\mathcal{G}_1} \\
&\cong \mathcal{H}_2^{\mathcal{G}_2}.
\end{aligned}
$$

Here, V and V' are the spaces from Lemma 8.28, and the isomorphism between $L^2(\mathbb{T})$ and V is the one found in equation (8.3).

For $j = 1, 2$, let $p_j \colon \mathcal{H}_j \to \mathcal{H}_j^{\mathcal{G}_j}$ be the orthogonal projection onto $\mathcal{H}_j^{\mathcal{G}_j}$. Moreover, let

$$
A_{\hbar,\Lambda_j}^{\mathrm{red}} := \left\{ p_j a p_j^* \colon a \in A_{\hbar,\Lambda_j}^{\mathcal{G}_j} \subseteq B(\mathcal{H}_j) \right\},
$$

be the subalgebra of the compressions of gauge invariant elements of $A_{\hbar,\Lambda_j}$ to the gauge invariant subspace of $\mathcal{H}_j$; this is motivated by the second observation. Then according to the first observation, we have a well-defined embedding map

$$
A_{\hbar,\Lambda_1}^{\mathrm{red}} \hookrightarrow A_{\hbar,\Lambda_2}^{\mathrm{red}};
$$

it is even a *-isomorphism.

8.29 Remark. The reader should take note of the fact that in the above isomorphism, any reference to the weights on the graphs Λ_1 and Λ_2, i.e., the lengths of the corresponding paths in space, is lost, despite their importance in the construction of the classical functor F_C !

It is easy to see that in the case in which the refinement $(\Lambda_1, \Lambda_2, \iota)$ corresponds to addition of a single new edge instead of subdivision of an edge in Λ_1, then the map

$$
\mathsf{F}_Q(\Lambda_1, \Lambda_2, \iota) \colon A_{\hbar,\Lambda_1} \hookrightarrow A_{\hbar,\Lambda_2},
$$

induces a map between the gauge invariant parts of these algebras.

We thus arrive at the following theorem:

8.30 Theorem. *The following assignment defines a covariant functor* $\mathsf{F}_Q^{\mathrm{red}} \colon \mathsf{wtRefine} \to \mathsf{Quantum}$*:*

- *An object* (Λ, ℓ) *is mapped to the algebra* $A^{\mathrm{red}}_{\hbar,\Lambda}$*;*
- *A morphism* $(\Lambda_1, \Lambda_2, \iota)$ *from an object* (Λ_1, ℓ_1) *to an object* (Λ_2, ℓ_2) *is mapped to the injective* **-homomorphism*

$$A^{\mathrm{red}}_{\hbar,\Lambda_1} \hookrightarrow A^{\mathrm{red}}_{\hbar,\Lambda_2},$$

that is obtained by writing $(\Lambda_1, \Lambda_2, \iota)$ *as a composition of elementary refinements as in chapter 4, and composing their associated injective* **-homomorphisms described above.*

Furthermore, consider the map that assigns to each (Λ, ℓ) *the map*

$$\mathcal{Q}^{W,\mathrm{red}}_{\hbar} \colon \mathcal{A}^{\mathrm{red}}_{0,\Lambda} \to A^{\mathrm{red}}_{\hbar,\Lambda_2}, \quad f \mapsto p_\Lambda \mathcal{Q}^{W}_{\hbar}(f) p^*_\Lambda,$$

where p_Λ *denotes the orthogonal projection of* $\mathcal{H}_\Lambda$ *onto the subspace of gauge invariant vectors* $\mathcal{H}^{\mathcal{G}_\Lambda}_\Lambda$*. This map is a natural transformation from* $\mathsf{F}^{\mathrm{red}}_C$ *to* $\mathsf{F}^{\mathrm{red}}_Q$*.*

8.31 Remark. It should be understood that if $\Lambda_1 = \Lambda_2$, and ι is the identity functor on the free category of Λ^1, then $\mathsf{F}^{\mathrm{red}}_C(\Lambda_1, \Lambda_2, \iota)$ is defined to be the identity map on $\mathcal{A}^{\mathrm{red}}_{\hbar,\Lambda}$. Thus $\mathsf{F}^{\mathrm{red}}_C$ maps identity morphisms to identity morphisms.

Proof. From our discussion in this section, it is clear that if $(\Lambda_1, \Lambda_2, \iota)$ is an elementary refinement, or the trivial refinement from the previous remark, then the following diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{red}}_{0,\Lambda_2} \arrow[r, "\mathcal{Q}^{W,\mathrm{red}}_{\hbar,(\Lambda_2,\ell_2)}"] & A^{\mathrm{red}}_{\hbar,\Lambda_2} \\
\mathcal{A}^{\mathrm{red}}_{0,\Lambda_1} \arrow[u, "\mathsf{F}^{\mathrm{red}}_C(\Lambda_1,\Lambda_2,\iota)"] \arrow[r, "\mathcal{Q}^{W,\mathrm{red}}_{\hbar,(\Lambda_1,\ell_1)}"] & A^{\mathrm{red}}_{\hbar,\Lambda_1} \arrow[u, "\mathsf{F}^{\mathrm{red}}_Q(\Lambda_1,\Lambda_2,\iota)"']
\end{tikzcd}$$

is commutative. Let us sketch how to extend this result to general refinements $(\Lambda_1, \Lambda_2, \iota)$, and to show that $\mathsf{F}^{\mathrm{red}}_Q(\Lambda_1, \Lambda_2, \iota)$ is well-defined. It is convenient to replace the indices 1 and 2 by other ones, such as i and j, respectively. The problem here is that there may be more than one way in which $(\Lambda_i, \Lambda_j, \iota_{i,j})$ can be written as a composition of elementary refinements. Suppose that there exist two sequences of elementary refinements

$$(\Lambda_{k-1}, \Lambda_k, \iota_{k-1,k})_{k=1}^{m}, \quad (\Lambda'_{k-1}, \Lambda'_k, \iota'_{k-1,k})_{k=1}^{m'},$$

such that $\Lambda_0 = \Lambda_i = \Lambda'_0$, and $\Lambda_m = \Lambda_j = \Lambda'_{m'}$, and we have

$$\iota_{m-1,m} \circ \cdots \circ \iota_{0,1} = \iota_{i,j} = \iota'_{m'-1,m'} \circ \cdots \circ \iota'_{0,1}.$$

Then we immediately see that

$$m = |\Lambda_j^1| - |\Lambda_i^1| = m'.$$

Furthermore, it can be shown that there exists a finite sequence

$$\left((\Lambda_{k-1}^{(l)}, \Lambda_k^{(l)}, \iota_{k-1,k}^{(l)})_{k=1}^{m} \right)_{l=0}^{L},$$

of sequences of elementary refinements of which the composition is the refinement $(\Lambda_i, \Lambda_j, \iota_{i,j})$, such that the sequences corresponding to $l = 0$ and $l = L$ are $(\Lambda_{k-1}, \Lambda_k, \iota_{k-1,k})_{k=1}^{m}$ and $(\Lambda'_{k-1}, \Lambda'_k, \iota'_{k-1,k})_{k=1}^{m}$, respectively, and such that two consecutive sequences are equal, except for the entries corresponding to $k-1$ and k for some $k \in \{1, \ldots, m\}$. This fact can be used to reduce the problem to the case $m = 2$, for which there exists an exhaustive (and short) list of possible ways to write a given refinement $(\Lambda_i, \Lambda_j, \iota_{i,j})$ as a product of two elementary refinements, and by studying these cases and comparing the corresponding maps $\mathsf{F}_C^{\mathrm{red}}(\Lambda_i, \Lambda_j, \iota)$, one sees that this map is well-defined, and that the above diagram commutes. ■

8.32 Definition. We call the functor $\mathsf{F}_Q^{\mathrm{red}}$ the *reduced quantum functor.*

Note that $\mathcal{Q}_\hbar^{W,\mathrm{red}}$ is not just an approximate natural transformation as we defined it implicitly at the beginning of this section, but an actual natural transformation. However, we view this as a happy coincidence, and do not expect this to be a general feature of quantisation maps.

8.4 Discussion and outlook

We have defined a classical and a quantum category, as well as functors from $\mathsf{wtRefine}$ to both of these categories. On the classical side, before reduction to the gauge group, the classical functor F_C, as well as the functors from $\mathsf{wtRefine}$ to $\mathsf{RiemannSub}$ and $\mathsf{SympSub}$ are well-motivated, both from a mathematical and a physical point of view, and the particular form of images of refinements under F_C demonstrate the necessity of working with (classical analogues of) field algebras that are strictly larger than $C_0(T^*\mathbb{T}^n)$, which was our reason for defining a resolvent algebra on $T^*\mathbb{T}^n$ in the first place. In addition, the picture presented in this chapter is consistent with the idea that, when forming composite quantum systems, the embeddings of the observable algebras of a constituent in the observable algebra of the composite system is given by the map that sends an operator to its tensor product with the identity operator on the tensor product of the Hilbert spaces corresponding to all of the other constituents, as discussed in section 5.1. However, the framework is not free of shortcomings, which manifest themselves upon consideration of the reduced versions of the systems.

First of all, the notation $A^{\mathrm{red}}_{0,(\Lambda,\ell)}$ (or $\mathcal{A}^{\mathrm{red}}_{0,(\Lambda,\ell)}$) and $A^{\mathrm{red}}_{\hbar,(\Lambda,\ell)}$ suggests that the latter algebra is the quantum mechanical counterpart of the former, but this is not really the case; the classical analogue of $A^{\mathrm{red}}_{\hbar,(\Lambda,\ell)}$ should be a space of functions on a Marsden–Weinstein quotient of $T^*(\mathbb{T}^n)^{\Lambda^1}$ by the action of the gauge group $\mathbb{T}^{\Lambda^0}$ [64] (cf. [19] (with more details in [18, chapters 1–4]) and [51] for a discussion of quantisation and reduction in a simplified version of the present setting in the context of the Guillemin–Sternberg conjecture focussing on the Hilbert spaces). Note that we say "a" rather than "the" Marsden–Weinstein quotient, since its construction involves the choice of an orbit of the coadjoint action of the symmetry group - in the case at hand the gauge group - on the dual of its corresponding Lie algebra $(\mathfrak{t}^n)^{\Lambda^0}$. To construct the quotient, one considers the preimage of this orbit by the moment map $J\colon T^*(\mathbb{T}^n)^{\Lambda^1} \to ((\mathfrak{t}^n)^{\Lambda^0})^*$. (There is a canonical moment map due to the fact that the action of the gauge group on phase space is induced by an action on the configuration space; see [103, Proposition 10.1.20] and the subsequent discussion.) Given a direct system in $\mathsf{wtRefine}$, the only (*a priori*) consistent choice of a family of coadjoint orbits seems to be the family of orbits consisting of the singletons $\{0\}$, and this corresponds to the absence of charge.

Furthermore, a crucial piece of information in the construction of our classical functor F_C and its reduced counterpart $\mathsf{F}^{\mathrm{red}}_C$ is the function

$\ell\colon \Lambda^1 \to (0,\infty)$ that encodes the lengths of the paths corresponding to the elements of Λ^1; this is necessary to define the map between phase spaces, i.e., the image of a refinement under F_C, and is natural both from a mathematical and a physical point of view, as already noted in Example 8.14. By contrast, in our discussion in the previous section on the modification of the idea that $\mathcal{Q}_\hbar^W$ can be viewed as a natural transformation so as to include the case of subdivision of edges, in passing from operators on the unreduced Hilbert spaces $\mathcal{H}$ to their reduced counterparts $\mathcal{H}^\mathcal{G}$, every explicit reference to the lengths ℓ was lost. For this reason, we expect Marsden–Weinstein quotients to be similarly incapable of encoding this type of geometric information.

For these reasons, it is desirable to have a more flexible framework. We believe that in the case of $T^*\mathbb{T}^n$, the deformations constructed by Rieffel [98] which we already mentioned at the beginning of this chapter will offer this flexibility, however we do not know what the deformation of the commutative resolvent algebra $C_\mathcal{R}(T^*\mathbb{T}^n)$ looks like, let alone the direct limit of a direct system of such deformations. Moreover, Propositions 1.11 and 2.10 and Theorem 5.7 in the aforementioned reference will ensure that one can form direct systems of deformations, and that quantisation is a natural transformation between functors.

According to the discussions in [65, section III.3.7] and [70, section 7.7], the deformations of $C_0(T^*\mathbb{T}^n)$ are isomorphic to the groupoid C*-algebra $C^*(\mathbb{R}^n \times_{\mathbb{Z}^n} \mathbb{R}^n)$ of the gauge groupoid $\mathbb{R}^n \times_{\mathbb{Z}^n} \mathbb{R}^n$. It would be interesting to know whether this deformation admits a natural faithful representation on a Hilbert space with the property that for each equivalence class of its irreducible subrepresentations (which correspond to the irreducible group representations of $\mathbb{Z}^n$), there is a subrepresentation in that equivalence class that can be extended to a representation of each of Rieffel's deformations. Note that we cannot simply apply [65, Theorem III.3.7.1] here to obtain a representation on $L^2(\mathbb{R}^n)$, since $\mathbb{Z}^n$ is not compact. If there is such a Hilbert space, then we could try to see whether a formula similar to the one in part (3) of Proposition 7.1 holds. If this turns out to be the case as well, then, using representation theory of Lie algebras, it could probably be generalised to arbitrary compact connected reductive Lie groups, thereby offering a more elementary quantisation procedure than, or more elementary characterisation of, the Weyl quantisation formula in [65, section II.3.4], in that it does not explicitly refer to the exponential map. Needless to say, this is mostly speculation, and is left as future work.

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Publiekssamenvatting

Naast het wetenschappelijke geweld dat het leeuwendeel van de tekst vormt en voornamelijk dan wel uitsluitend door experts gelezen wordt, dient een proefschrift ook altijd een samenvatting van het werk te bevatten. Het is mijns inziens een goede gewoonte van de promovendi van de afdeling wiskunde van de Radboud Universiteit dat zij hierbij dikwijls ervoor kiezen zich te richten tot een algemener publiek. Een droge opsomming van de inhoud van ieder hoofdstuk leidt immers slechts tot glazige blikken bij familie en vrienden, waarvan de kandidaat er al meer dan genoeg ontvangen heeft gedurende het promotietraject en de daaraan voorafgaande studie. De gevorderde lezer wiens honger naar kennis na het lezen van deze samenvatting nog niet gestild is, kan zich desgewenst tot de inleiding van dit proefschrift wenden. Een bijkomend voordeel van deze aanpak is dat het mij in staat stelt het belang van mijn onderzoek te schetsen, of - met andere woorden - antwoord te geven op de vraag: "Zijn mijn zuurverdiende belastingcenten wel goed besteed?"

De vertaling van de titel van mijn proefschrift luidt *kwantisatie versus roosterijktheorie*. Voordat ik kan uitleggen wat ik tijdens mijn promotie onderzocht heb, is het noodzakelijk te weten wat het eerste en het laatste woord in de vertaling betekenen op een basaler niveau dan de uiteenzetting in hoofdstuk 1.

Kwantisatie

Kort gezegd is *kwantisatie* niets anders dan het vertalen van het formalisme van de klassieke mechanica naar het formalisme van de kwantummechanica. De *klassieke mechanica* is de natuurkundige theorie die beschrijft hoe krachten de beweging beïnvloeden van macroscopische objecten[1] die bewegen met een snelheid waarvan de grootte verwaarloosbaar is ten opzichte van de lichtsnelheid.

De bekendste naam die met deze theorie geassocieerd is, is zonder twijfel die van de Engelse wis- en natuurkundige Isaac Newton (1643-1727). De *tweede wet van Newton* beschrijft precies de relatie tussen kracht en versnelling: de kracht $\vec{F}$ die een object ondervindt is gelijk aan de versnelling $\vec{a}$ van dat object ten gevolge van de kracht maal de massa m van het object, oftewel $\vec{F} = m\vec{a}$. Als men op een gegeven tijdstip t_0 de krachten op alsmede de plaats $q(t_0)$ en de snelheid $\dot{q}(t_0)$ of equivalent de impuls $p(t_0) = m\dot{q}(t_0)$ van het object kent, kan men met behulp van deze wet in principe de plaats $q(t)$ en impuls $p(t)$ op elk ander tijdstip t berekenen. Het paar $(q(t_0), p(t_0))$ bepaalt op deze manier de toestand van het bewegende object. De verzameling van alle mogelijke waarden van q noemt men de *configuratieruimte* van het systeem. De verzameling van alle mogelijke waarden van het paar (q, p) noemt men de *faseruimte* van het systeem.

De (niet-relativistische) *kwantummechanica* is de natuurkundige theorie die beschrijft hoe de beweging van objecten in de aanwezigheid van een potentiaal V beïnvloed wordt als de objecten bewegen met een snelheid waarvan de grootte verwaarloosbaar is ten opzichte van de lichtsnelheid, ook wanneer deze objecten van microscopisch formaat zijn. Het begrip potentiaal is nauw verwant aan het begrip kracht, maar niet equivalent. De noodzaak om met de potentiaal te werken is slechts één van de verschillen tussen de formalismen van de klassieke mechanica en de kwantummechanica. Een ander, belangrijker verschil is dat de toestand van het object op een tijdstip t niet gegeven wordt door een paar $(q(t), p(t))$, maar door een functie ψ op de configuratieruimte Q, de zogeheten *golffunctie*, die aan bepaalde voorwaarden voldoet.[2] Om te benadrukken dat men met tijdsafhankelijke golffuncties werkt, schrijft men gewoonlijk Ψ, waarbij het verband met ψ is gegeven door $\psi(q) = \Psi(q, t)$ voor elk element q in

[1] Objecten worden macroscopisch genoemd wanneer hun grootte voor ons geen belemmering vormt om ze met het blote oog te kunnen zien.

[2] Een functie op een verzameling (in dit geval Q) is een wiskundig object dat aan elk element van die verzameling een (in dit geval complex) getal toekent. Overigens, strikt genomen is ψ geen functie, maar een verzameling van functies die (in wiskundige zin) bijna overal aan elkaar gelijk zijn.

de configuratieruimte.

De *Schrödingervergelijking*

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi,$$

is voor de kwantummechanica wat de tweede wet van Newton is voor de klassieke mechanica. Het enige wat in deze vergelijking van belang is voor de rest van dit stuk, is dat $\hbar = h/2\pi$ de gereduceerde constante van Planck is.[3] De vergelijking is vernoemd naar zijn bedenker Erwin Schrödinger (1887-1961), die samen met Werner Heisenberg (1901-1976) het tweetal vormt van natuurkundigen die aan de kwantummechanica gewerkt hebben dat het bekendst is bij het grote publiek.[4]

De verzameling van mogelijke golffuncties vormt een verzameling met extra structuur die het tot een zogenaamde *Hilbertruimte* maakt. Een van de vraagstukken binnen het onderwerp kwantisatie luidt: *hoe kan men uit de gegeven faseruimte van het klassieke systeem de bijbehorende Hilbertruimte van het kwantummechanische systeem construeren?* Het probleem hier is dat, in tegenstelling tot wat in het bovenstaande gesuggereerd wordt, de faseruimte van een klassiek systeem niet altijd een bijbehorende configuratieruimte heeft en men dus niet automatisch de Hilbertruimte kan definiëren als een verzameling van functies op de configuratieruimte. Het gebied dat bekend staat als *meetkundige kwantisatie* (Engels: *geometric quantisation*) geeft in bepaalde situaties een constructie voor de Hilbertruimte.

Dit proefschrift gaat echter over een andere vorm van kwantisatie, namelijk *deformatiekwantisatie*. Deze vorm van kwantisatie richt zich op de *observabelen* van een systeem. Verderop zal ik dieper ingaan op het begrip observabele, maar voor nu is het voldoende om te weten dat we zowel in de klassieke wereld als in de kwantumwereld verzamelingen van observabelen beschouwen die een zogeheten *C^*-algebra*[5] vormen die we

[3] De constante van Planck is de natuurconstante $h = 6.62607015 \cdot 10^{-34}\,\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. De constante $\hbar$ (spreek uit: "h streep") wordt ook wel de *constante van Dirac* genoemd.

[4] Naast zijn vergelijking is Schrödinger natuurlijk bekend van zijn beroemde gedachtenexperiment dat bekend staat als *Schrödingers kat*. Heisenberg is in eerste instantie voornamelijk bekend van de *onzekerheidsrelatie van Heisenberg*, maar sinds 2008 zijn er waarschijnlijk meer mensen die zijn naam associëren met het alter ego van de hoofdpersoon Walter White in de televisieserie *Breaking Bad*.

[5] Spreek uit: "C ster algebra."

de *observabelenalgebra* noemen; dat betekent onder andere dat men twee observabelen - zeg a en b - kan optellen en vermenigvuldigen, waarbij het resultaat ($a+b$ en $a \cdot b$ respectievelijk) weer een observabele is. Het cruciale verschil is dat hoewel de klassieke observabelenalgebra altijd *commutatief* is, dat wil zeggen $a \cdot b = b \cdot a$, of anders geschreven:

$$a \cdot b - b \cdot a = 0,$$

voor alle a en b in de algebra, dit niet het geval is voor de kwantumobservabelenalgebra. Integendeel, niet-triviale commutatierelaties zoals

$$\hat{q} \cdot \hat{p} - \hat{p} \cdot \hat{q} = i\hbar\mathbf{1}, \tag{1}$$

vormen het hart van de kwantummechanica; in het bijzonder ligt de bovenstaande relatie ten grondslag aan de onzekerheidsrelatie van Heisenberg, die tot gevolg heeft dat men niet tegelijkertijd de plaats $\hat{q}$ en de impuls $\hat{p}$ kan bepalen. Vergelijking (1) is het standaardvoorbeeld van een *canonieke commutatierelatie*.[6]

Beschouw in vergelijking (1) de constante $\hbar$ nu als een parameter, en stel deze parameter gelijk aan 0. Dan is de rechterzijde van die vergelijking gelijk aan 0, waardoor geldt dat $\hat{q} \cdot \hat{p} = \hat{p} \cdot \hat{q}$. In dit opzicht gedragen de observabelen $\hat{q}$ en $\hat{p}$ zich dan dus als klassieke observabelen. Het gelijkstellen van $\hbar$ aan 0 of het doen van de aanname dat $\hbar$ zeer dicht bij 0 ligt, wat bekend staat als het nemen van *de klassieke limiet*, is in algemenere zin een methode om het gedrag te simuleren van een systeem dat in een bepaald opzicht lijkt op een macroscopisch systeem. Doordat de klassieke mechanica een goede beschrijving vormt voor het gedrag van macroscopische systemen, kan men door de klassieke limiet te nemen de klassieke mechanica verklaren vanuit de kwantummechanica. Dit wordt ook wel het *correspondentieprincipe* genoemd en het verklaart de naam 'klassieke limiet'.

Bij het nemen van de klassieke limiet begint men dus met een kwantummechanische beschrijving van een systeem en verkrijgt men door het nemen van de limiet $\hbar \to 0$ een klassieke beschrijving. Bij deformatiekwantisatie bewandelt men de omgekeerde weg: uit een gegeven fysisch

[6] Hier is de configuratieruimte een lijn, $\hat{q}$ en $\hat{p}$ (en dus ook $i\hbar\mathbf{1}$) zijn observabelen, en $\mathbf{1}$ het unieke element van de observabelenalgebra is met de eigenschap dat $a \cdot \mathbf{1} = a = \mathbf{1} \cdot a$ voor elke observabele a.

systeem met een klassieke beschrijving wil men een kwantummechanische beschrijving van het systeem verkrijgen door op een bepaalde manier een niet-triviale commutatierelatie te introduceren zoals in vergelijking (1). Er zijn twee manieren om dit te bereiken:

- Door middel van *deformatie van het product*: gegeven een commutatieve C*-algebra A_0 met operatie van vermenigvuldiging $\cdot$ tracht men een familie van soortgelijke operaties $(\star_\hbar)_{\hbar>0}$ te definiëren. Door A_0 voorzien van de operatie $\star_\hbar$ weer tot een C*-algebra te maken, krijgt men een nieuwe algebra $A_\hbar$ die niet commutatief is op een manier die het tot een kandidaat voor een kwantumobservabelenalgebra maakt.
- Door middel van *kwantisatieafbeeldingen*: naast een commutatieve C*-algebra A_0 nemen we aan dat we beschikken over een familie van niet-commutatieve C*-algebra's $(A_\hbar)_{\hbar>0}$, en gaan we op zoek naar een familie van afbeeldingen $(\mathcal{Q}_\hbar)_{\hbar>0}$, waar

  $$\mathcal{Q}_\hbar \colon A_0 \to A_\hbar,$$

  aan elke klassieke observabele in A_0 een kwantumobservabele in $A_\hbar$ toekent.[7]

In het eerste geval wordt het product, oftewel de operatie van vermenigvuldiging, van de algebra A_0 gedeformeerd, terwijl in het tweede geval de observabelen zelf, oftewel de elementen van A_0, gedeformeerd worden. In beide gevallen moeten de gedeformeerde objecten na het kwantiseren bij het nemen van de klassieke limiet op een gecontroleerde manier met de oorspronkelijke klassieke objecten corresponderen. In dit proefschrift voeren kwantisatieafbeeldingen de boventoon, waarbij bovendien alle elementen van de familie $(A_\hbar)_{\hbar>0}$ van C*-algebra's één en dezelfde algebra zijn.

IJktheorie

Om het andere hoofdonderwerp van mijn proefschrift, roosterijktheorie, te motiveren, zal ik eerst een korte inleiding tot ijktheorie geven. IJkthe-

[7]Eigenlijk kwantiseert men niet alle elementen van A_0, maar alleen die elementen die in een bepaalde 'voldoende grote' deelverzameling $\mathcal{A}_0$ van A_0 liggen.

orieën[8] vormen de basis van het *standaardmodel van de deeltjesfysica*, dat de bouwstenen van materie en hun interactie beschrijft. In het bijzonder worden deze theorieën in het standaardmodel gebruikt om drie van de vier fundamentele natuurkrachten te beschrijven:

- Zoals de naam al doet vermoeden, is het *elektromagnetisme* de kracht die verantwoordelijk is voor elektriciteit en magnetisme;
- De *zwakke kernkracht* is de kracht die verantwoordelijk is voor radioactief verval van deeltjes en kernsplijting mogelijk maakt;
- De *sterke kernkracht* is de kracht die atoomkernen bijeenhoudt. Doordat atoomkernen uit elektrisch positief geladen protonen en elektrisch neutrale neutronen bestaan, zouden deze zonder de sterke kernkracht uiteenvallen. Protonen en neutronen bestaan op hun beurt elk weer uit drie zogeheten *quarks*, die ook door de sterke kernkracht bijeengehouden worden. Verder is de sterke kernkracht verantwoordelijk voor kernfusie.

De vierde kracht is de zwaartekracht en wordt beschreven door de algemene relativiteitstheorie. Op dit moment is er echter nog geen bevredigende manier om deze theorie te verenigen met het standaardmodel.

Het centrale object in elk van de ijktheorieën die de bovenstaande krachten beschrijven, is het *ijkveld*, dat in de literatuur aangeduid wordt met A (niet te verwarren met de notatie voor de observabelenalgebra's!). Een *veld* is een wiskundig object dat aan ieder punt van een bepaalde ruimte - in ons geval de vierdimensionale ruimtetijd of de driedimensionale ruimte - een element uit een verzameling toekent waarop men een notie van optelling en schaling heeft.[9] Het veld A is echter niet uniek bepaald, in de zin dat er een ander veld A' bestaat[10] zodanig dat wanneer men in alle natuurwetten die het systeem beschrijven het veld A vervangt door A' (en bepaalde andere velden op een overeenkomstige manier wijzigt), het voorspelde gedrag van het systeem niet verandert. Het proces

[8] Het woord *ijktheorie* (Engels: *gauge theory*) wordt gebruikt om zowel een specifiek voorbeeld van een bepaalde wiskundige constructie aan te duiden, als de tak van de theoretische en mathematische fysica die de deze voorbeelden en de algemene constructie bestudeert.

[9] Wiskundigen zullen begrijpen dat hier een vectorruimte bedoeld wordt.

[10] Er bestaan er zelfs oneindig veel.

van het vervangen van A door A' wordt een *ijktransformatie* genoemd; de mogelijkheid om een ijktransformatie uit te voeren wordt *ijkvrijheid* genoemd. Het kiezen van één veld uit een veelheid van mogelijke velden wordt het kiezen van een *ijk* genoemd, naar analogie met het ijken van een meetinstrument.

Denk bijvoorbeeld aan het kiezen van een schaalverdeling op een thermometer: het maakt niet uit of we de temperatuur weergeven in graden Celsius, graden Fahrenheit of Kelvin, omdat we de temperatuur uitgedrukt in elk van deze schalen kunnen omrekenen naar elke andere schaal. Bij het kiezen van een schaal kunnen bepaalde schalen daarentegen wel handiger zijn om mee te rekenen dan andere. Zo zal een natuurkundige geneigd zijn haar temperaturen in Kelvin uit te drukken, omdat deze van de drie genoemde schalen de enige is waarbij het nulpunt op de schaal correspondeert met het absolute nulpunt. In de context van ijktheorie kan een keuze voor een bepaalde ijk sommige vergelijkingen vereenvoudigen, waardoor ze makkelijker op te lossen zijn.[11]

Roosterijktheorie

Niet alleen objecten die krachten ondervinden, hebben klassieke beschrijvingen die men kan (proberen te) kwantiseren, maar ook de ijkvelden die deze krachten uitoefenen: men kan op zoek gaan naar een kwantumtheorie van velden, oftewel een *kwantumveldentheorie*. Dit blijkt een enorme uitdaging, zeker wanneer men eist dat dit op een wiskundig rigoureuze manier gebeurt. Het voornaamste obstakel is het feit dat velden oneindig veel *vrijheidsgraden* hebben: men kan de waarde van een veld in elk punt in de ruimte waarop dat veld gedefinieerd is, uitdrukken in eindig veel getallen/parameters, en aangezien de meeste ruimten waarin men geïnteresseerd is oneindig veel punten hebben, heeft men oneindig veel parameters nodig om het veld in zijn geheel te beschrijven. Wanneer men ijkvelden wil kwantiseren, zorgt de ijkvrijheid bovendien voor extra complicaties.

Roosterijktheorie probeert het eerste probleem op te lossen door het aantal vrijheidsgraden terug te brengen van een oneindig aantal naar een

[11] Het is overigens van belang op te merken dat het kiezen van een ijk wel iets wezenlijk anders is dan het kiezen van de eenheid waarin we de waarde van het ijkveld uitdrukken.

eindig aantal door een eindig rooster[12] te introduceren, waarbij elk punt in het rooster correspondeert men een punt in de ruimte waarop het veld gedefinieerd is, en elk lijnstuk tussen twee roosterpunten met een pad tussen de bijbehorende punten in de ruimte. Men kan nu het ijkveld A associëren met een bepaalde afbeelding op de verzameling van lijnstukken van het rooster, die daarmee in zekere zin een benadering is van A. Deze benadering heeft nog slechts eindig veel vrijheidsgraden, waardoor het probleem van kwantisatie een stuk eenvoudiger wordt.

Om nu A te reconstrueren dan wel volledig te karakteriseren heeft men niet voldoende aan slechts één benadering; men dient een hele familie van roosters te beschouwen die op een bepaalde manier in elkaar worden ingebed zoals weergegeven in sectie 2.1. Een dergelijke inbedding geeft een notie van een 'grof' en een 'fijn' rooster. Uit de benadering van A voor het fijne rooster kan nu de benadering van A voor het grove rooster gevonden worden.

Ter illustratie van dit laatste punt kan men denken aan een plaatje dat men op een beeldscherm wil weergeven. Een beeldscherm bevat heel veel kleine beeldelementen, die in de volksmond ook wel *pixels* genoemd worden. Hoewel het deel van het plaatje dat door een bepaalde pixel gerepresenteerd wordt heel veel punten bevat en daardoor even veel kleuren kan bevatten, kan een pixel op één moment slechts één kleur aannemen, waardoor het scherm een beperkt oplossend vermogen of resolutie heeft. Wanneer (een deel van) het plaatje op een beeldscherm wordt weergegeven, is de kleur die een bepaalde pixel aanneemt slechts een soort gemiddelde van de kleuren die te vinden zijn in het deel van het plaatje dat door de pixel vertegenwoordigd wordt.

Laten we gemakshalve aannemen dat we met grijstinten werken, en laten we aannemen dat elke grijstint correspondeert met een getal tussen 0 en 1, de zogeheten *grijswaarde*, waarbij een hoger getal correspondeert met een lichtere grijstint; in het bijzonder correspondeert 0 met zwart en 1 met wit.[13] Er zijn in essentie twee manieren waarop een beeldscherm

[12] Wiskundig gezien is het netter om dit een *graaf* te noemen.

[13] In moderne beeldschermen is de door het oog waargenomen kleur een combinatie van de kleuren van drie subpixels die elk één van de kleuren rood, groen en blauw aannemen. De lezer wordt aangemoedigd om na te denken over de vraag welke aanpassingen men in het verhaal moet aanbrengen om deze situatie te beschrijven.

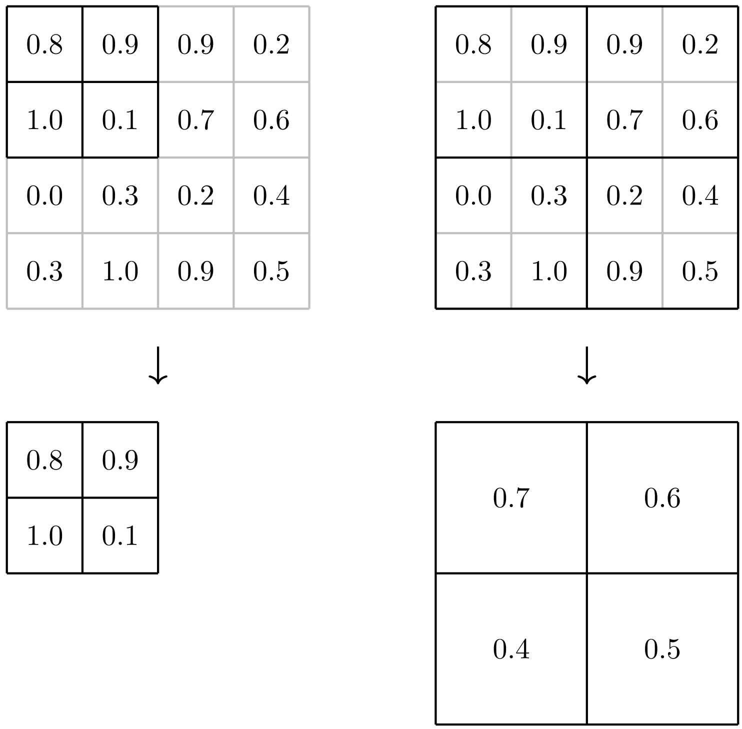
'fijner' kan zijn dan een ander beeldscherm:

- Het ene beeldscherm kan meer pixels hebben dan het andere, waardoor het een groter deel van het plaatje kan weergeven. Zo past er op een televisiescherm meer informatie dan op het scherm van een smartphone. In dit geval kan men de grijswaarden van de extra pixels van het fijne scherm 'negeren' of 'vergeten' om de waarden voor de pixels van het grove scherm te verkrijgen, zoals in Figuur 1a.

- Het ene beeldscherm kan een hogere resolutie hebben dan het andere, op de volgende specifieke wijze: elk deel van het plaatje dat door één pixel van het grove scherm gerepresenteerd wordt, wordt precies door een eindige verzameling van pixels van het fijne scherm gerepresenteerd. Bijvoorbeeld: als het grove scherm en het fijne scherm een resolutie hebben van 640×360 en 1920×1080 respectievelijk, en ze geven hetzelfde deel van het plaatje weer, dan correspondeert elke pixel van het grove scherm met een 3×3-blok van pixels van het fijne scherm en moet men de grijswaarden van deze pixels middelen om die van de bijbehorende pixel van het grove scherm te verkrijgen. Een ander voorbeeld met een aanzienlijk kleiner aantal pixels is weergegeven in Figuur 1b.

Uiteraard kan er ook een combinatie van deze twee situaties optreden. Bovendien is het niet eens nodig om aan te nemen dat alle pixels even groot zijn of dezelfde vorm hebben. In die algemenere versie moet men daar dan wel rekening mee houden in de tweede situatie door een gewogen gemiddelde van de grijswaarden van de kleinere pixels te nemen, waarbij de weegfactoren gegeven worden door de oppervlakten van de pixels van het fijne scherm.

We observeren nu het volgende: veronderstel dat we drie schermen S_1, S_2 en S_3 hebben, en dat de laatste twee schermen fijner zijn dan hun voorganger. Dan is S_3 fijner dan S_1. Bovendien zijn er nu twee manieren om de grijswaarden van een pixel van S_1 te bepalen: enerzijds kan men direct de grijswaarden van de bijbehorende pixels in S_3 middelen. Anderzijds kan men opmerken dat de grijswaarde van de pixel gevonden kan worden door de grijswaarden van de bijbehorende pixels in S_2 te middelen, die op hun beurt weer gemiddelden van grijswaarden van pixels in S_3 zijn.



(a) 'Vergeten' van grijswaarden. (b) 'Middelen' van grijswaarden.

Figuur 1: Transformeren van grijswaarden van een fijn scherm naar grijswaarden van een grof scherm.

Het maakt echter niet uit welke van deze twee methoden we gebruiken om de grijswaarden van de pixels in S_1 te vinden: beiden geven namelijk hetzelfde resultaat.

Veronderstel nu dat we een willekeurige familie van schermen hebben waarbij er voor bepaalde paren van schermen een notie van 'fijn' en 'grof' is, en waarbij er voor elk paar schermen S en S' een derde scherm S'' is zodanig dat S'' fijner is dan zowel S als S'. Deze eis vormt samen met de observatie in de vorige alinea de twee belangrijkste eigenschappen van de familie (geïndexeerd door de verzameling van schermen) van verzamelingen van mogelijke grijswaarden (van een gegeven scherm) die het tot een zogeheten *invers systeem* maken.

Uit dit inverse systeem kan men de *inverse limiet* vormen: dit is een verzameling die elementen bevat die in zekere zin de maximale hoeveelheid informatie bevatten over de grijswaarden van de schermen. Daarmee wordt bedoeld dat een element in deze limiet voor elk scherm voor elk pixel in dat scherm een bijbehorende grijswaarde geeft op een manier die consistent is met het middelen van de grijswaarden van pixels van een gegeven scherm om de grijswaarde van een pixel van een grover scherm te verkrijgen. Als er voor elk paar van verschillende punten in het plaatje een scherm is waarvan de twee punten in twee verschillende pixels vallen, en als bovendien de grijstinten in het plaatje op een geleidelijke manier van punt tot punt veranderen, dan kan men uit het element van de inverse limiet het plaatje reconstrueren.

Het idee achter roosterijktheorie is in essentie hetzelfde: men kan de roosters zodanig kiezen dat de ruimten van benaderingen van het ijkveld die corresponderen met verschillende roosters op een voor de hand liggende manier een invers systeem vormen, waarvan de inverse limiet de eigenschap heeft dat men het oorspronkelijke ijkveld A kan reconstrueren uit een element van deze limiet.

Inbedding van observabelen

Eerder is het begrip *observabelenalgebra* al besproken, maar daarbij is de definitie van het begrip *observabele* achterwege gelaten. In feite is een observabele van een (natuurkundig) systeem een vraag over dat systeem die men in principe door middel van metingen aan het systeem kan beantwoorden. Voor een systeem dat bestaat uit een bewegend object kan men vragen stellen als:

- Waar bevindt het object zich?
- Wat is de snelheid van het object?
- Hoeveel energie vertegenwoordigt de beweging die het object uitvoert, oftewel wat is de kinetische energie van het object?

Merk op dat in de klassieke mechanica al deze vragen beantwoord kunnen worden in termen van een getal of een verzameling[14] van getallen die

[14]Een geordende verzameling, oftewel een tupel, welteverstaan.

kunnen worden uitgedrukt in termen van de plaats en impuls van het systeem. In de klassieke mechanica is een observabele dan ook niets anders dan een functie op de faseruimte van het systeem.

In de context van de beeldschermmetafoor in de vorige sectie zou een observabele die correspondeert met een bepaald rechthoekig scherm kunnen zijn: "Wat is de grijswaarde van de pixel in de linkerbovenhoek van het scherm?" Als we nu twee van zulke schermen hebben, waarbij de een fijner is dan de ander, dan vertaalt de bovenstaande vraag over het grove scherm zich naar een vraag over het fijne scherm. In het voorbeeld van de beeldschermen met de resoluties 640×360 en 1920×1080 wordt de vraag over het tweede scherm: "Wat is de gemiddelde grijswaarde van het 3×3-blok van pixels in de linkerbovenhoek van het scherm?"

Dit werkt niet alleen voor de gegeven vraag, maar voor alle observabelen. We hebben dus een manier gevonden om de observabelen van het grove systeem te laten corresponderen met een deel van de observabelen van het fijne systeem; er is een *inbedding* van de observabelenalgebra van het grove systeem in die van het fijne systeem. Merk op dat ten opzichte van het vinden van de grijswaarden van pixels, dat van fijn naar grof gaat, dit proces in de omgekeerde richting verloopt. Dit heeft tot gevolg dat de observabelenalgebra's een *direct systeem* in plaats van een invers systeem vormen wanneer men een familie van schermen beschouwt zoals in de vorige sectie. Het directe systeem heeft een *directe limiet*, die men kan beschouwen als een verzameling vragen over de grijswaarden van de pixels van elk scherm in de familie ongeacht grootte of resolutie.

Overzicht van de inhoud

In dit proefschrift wordt geprobeerd directe systemen van observabelenalgebra's van roosterijktheorieën te kwantiseren. In de context van ijktheorie worden bepaalde *veldenalgebra's* geïdentificeerd: dat zijn C*-algebra's waaruit men de observabelenalgebra's kan halen, maar waarvan de elementen nog wel afhankelijk zijn van de gekozen ijk. Door eerst directe systemen van veldenalgebra's te identificeren, waarbij de inbeddingen zich op een goede manier gedragen ten opzichte van de ijktransformaties, en vervolgens op systematische wijze de ijkvrijheid te verwijderen, proberen we te komen tot de gewenste directe systemen.

In deel I van dit proefschrift vertrekken we vanuit bestaande theorie van kwantisatie gestoeld op de theorie van *groepoïden*: dat zijn meetkundige objecten waaruit men zowel klassieke als kwantummechanische velden- en observabelenalgebra's kan construeren, die men bovendien op een meetkundige manier aan elkaar kan relateren.

Na de algemene inleiding in hoofdstuk 1 en de inleiding tot roosterijktheorie in hoofdstuk 2, worden in hoofdstuk 3 twee verschillende manieren beschreven om uit een veldalgebra de bijbehorende observabelenalgebra te vinden, namelijk door *Rieffelinductie* toe te passen, of door op een bepaalde manier de *wet van Gauss* te implementeren. Vervolgens laten we zien dat deze twee procedures hetzelfde resultaat geven. Hier spelen groepoïden nog geen expliciete rol, maar de keuze voor de velden- en observabelenalgebra's is wel gemotiveerd vanuit het groepoïdenformalisme.

In hoofdstuk 4 laten we zien dat uit een invers systeem van groepoïden directe systemen van kwantummechanische velden- en observabelenalgebra's kunnen worden gehaald. Van elk van deze systemen karakteriseren we bovendien de limiet, en laten we zien dat de correspondentie tussen groepoïden en algebra's nog steeds geldig is in de limiet. Hoewel vanuit wiskundig oogpunt de constructie goed gedefinieerd en zelfs natuurlijk is, zijn er meerdere aanwijzingen dat het directe systeem van de algebra's evenals hun limiet niet op de juiste manier corresponderen met de ijktheorieën die zij dienen te beschrijven. Het eerste probleem komt voort uit een bepaalde klasse van inbeddingen, namelijk die inbeddingen die in de beeldschermmetafoor corresponderen met het 'vergeten' van grijswaarden. De inbeddingen van kwantummechanische velden- en observabelenalgebra's die men op basis van het groepoïdenformalisme vindt, zijn niet de inbeddingen die men op basis van de natuurkunde verwacht. Als men echter met de natuurkundige inbeddingen werkt, blijken de velden- en observabelenalgebra's in zekere zin 'te klein'.

De problemen met de constructie in hoofdstuk 4 motiveren deel II van dit proefschrift, waarin geprobeerd wordt een bepaald soort C*-algebra, namelijk de *resolventenalgebra*, aan te passen aan een situatie die van belang is voor bepaalde soorten roosterijktheorieën. In tegenstelling tot de algebra's die men uit groepoïden construeert, zijn families van resolventenalgebra's wel groot genoeg voor de inbeddingen van de kwantumobservabelen. In hoofdstuk 5 definiëren en bestuderen we de klassieke versie van onze resolventenalgebra, die we vervolgens in hoofdstuk 7 kwantiseren met behulp van een familie kwantisatieafbeeldingen die samen bekend staan als *Weylkwantisatie*. In hoofdstuk 6 slaan we een zijweg in door het *Gelfandspectrum* van de klassieke resolventenalgebra te bestuderen: dat is een bepaald meetkundig[15] object dat met een commutatieve C*-algebra correspondeert.

In hoofdstuk 8 gaan we net als in hoofdstuk 4 op zoek naar inbeddingen van velden- en observabelenalgebra's, maar laten we ons niet leiden door het groepoïdenformalisme. In plaats daarvan zorgen we eerst dat we een goed begrip hebben van de inbeddingen aan de klassieke kant. Het lastigste punt hierbij is dat roosterijktheorie wel voorschrijft hoe men uit een configuratie op een fijn rooster een configuratie op een grof rooster verkrijgt, oftewel wat de afbeelding tussen configuratieruimten is, maar niet wat de afbeelding tussen faseruimten is. We ontwikkelen wiskundige theorie om met behulp van meetkundige informatie van het rooster op systematische wijze de laatstgenoemde afbeelding uit de eerstgenoemde afbeelding te construeren. Vervolgens maken we gebruik van bestaande theorie om de bijbehorende inbeddingen van klassieke observabelenalgebra's te verkrijgen.

Om de inbeddingen tussen kwantummechanische velden- en observabelenalgebra's te vinden, nemen we aan dat voor elke klassieke observabele die correspondeert met het grove systeem, de kwantisatie van zijn klassieke inbedding ongeveer hetzelfde moet zijn als de kwantummechanische inbedding van zijn kwantisatie. Dit geeft de juiste inbeddingen, dat wil zeggen de inbeddingen die men op basis van de natuurkunde verwacht, in het geval dat duidelijk onjuist was in hoofdstuk 4. De in hoofdstuk 8 gevonden inbeddingen in het andere geval, dat in de beeldschermmetafoor correspondeert met het 'middelen' van grijswaarden, werkt tot op zekere hoogte, maar is nog niet geheel bevredigend. Er lijkt hier namelijk geen duidelijke kandidaat te zijn voor een inbedding van de kwantummechanische veldenalgebra's. Die is er echter wel in het geval van de observabelenalgebra's, maar die inbedding bevat niet de meetkundige informatie van het rooster die men gebruikt om de afbeeldingen tussen de (klassieke) faseruimten te construeren.

[15]Voor de wiskundigen: topologisch.

Het lukt dus nog niet om de twee gevallen, namelijk het 'vergeten' en het 'middelen' van pixels, op gelijke voet te behandelen, wat we wel graag zouden willen. Hiertoe stellen we voor om de C*-algebra's die we voor de kwantummechanische veldenalgebra's gebruiken, te vervangen door bepaalde C*-algebra's die, in tegenstelling tot de beschikbare literatuur over roosterijktheorie, niet met een specifieke Hilbertruimte corresponderen.

Wat is het belang van dit onderzoek?

Na de bovenstaande samenvatting zal het duidelijk zijn dat het onderzoek in dit proefschrift in beginsel fundamenteel van aard is; het levert noch een geneesmiddel tegen kanker, noch een schone energiebron, noch een supersnelle computer op.

Dit onderzoek heeft in eerste instantie betrekking op de *hoge-energiefysica*[16]. Roosterijktheorie is bedacht door de Amerikaanse natuurkundige Ken Wilson (1936-2013), met als doel het fenomeen dat bekend staat als *quark confinement* te beschrijven. Quark confinement is de naam van het verschijnsel dat quarks nooit alleen voorkomen, maar onder invloed van de sterke kernkracht altijd in een *gebonden toestand* met één of meer andere quarks. Naast een theoretisch begrip van dit verschijnsel maakt roosterijktheorie ook numerieke simulaties mogelijk die dit gedrag beschrijven.

Dit proefschrift valt daarentegen in een van vele onderzoekslijnen die als doel hebben om een wiskundig rigoureuze beschrijving van een kwantumveldentheorie (op een vierdimensionale ruimtetijd) te geven. Het probleem met dit soort theorieën is dat er tot op heden[17] nog niet een dergelijke beschrijving is, ondanks dat kwantumveldentheorie al sinds 1950 door natuurkundigen gebruikt worden om berekeningen uit te voeren waarvan de uitkomsten met zeer hoge nauwkeurigheid experimenteel geverifieerd zijn. Er is dus sprake van 'achterstallig onderhoud'.

Een zo mogelijk nog ambitieuzer project is het zoeken naar een theorie van *kwantumzwaartekracht*: een theorie die de kwantummechanica verenigt met de algemene relativiteitstheorie. Een van de pogingen om tot een dergelijke theorie te komen is *luskwantumzwaartekracht* (Engels: *loop*

[16] Bij het grote publiek beter bekend als de *deeltjesfysica*.

[17] Ook na het schrijven van dit proefschrift.

quantum gravity), waarvan het formalisme veel overeenkomsten vertoont met roosterijktheorie. Een beter begrip van roosterijktheorie kan dus leiden tot een beter begrip van luskwantumzwaartekracht, en vice versa.

Ten slotte zullen de meeste mensen die het nieuws volgen wel gehoord hebben van het *Higgsdeeltje*. Het bestaan daarvan is in 1964 door Robert Brout, François Englert en onafhankelijk door Peter Higgs voorspeld en is in 2012 door (verschillende detectoren in) de *Large Hadron Collider* (LHC), de grote deeltjesversneller onder Genève, geregistreerd. De laatste twee onderzoekers mochten een jaar later voor hun ontdekking de Nobelprijs voor de natuurkunde in ontvangst nemen (Brout was in 2011 al overleden). Er wordt nu nagedacht over een mogelijke opvolger van de LHC, namelijk de *Future Circular Collider* (FCC), die nieuwe deeltjes moet gaan detecteren. Critici van de FCC stellen dat andere ontdekkingen van deeltjes die men voorspeld had en waarvan men had verwacht dat ze door de LHC zouden worden gedetecteerd, uitgebleven zijn, en dat er bovendien geen redelijke indicatie is dat de FCC die wel zou kunnen detecteren [28]. Op dit moment is geld dat geïnvesteerd wordt in wiskundig onderzoek naar hoge-energiefysica daarom waarschijnlijk beter besteed dan geld dat gestoken gaat worden in de bouw van deze peperdure[18] nieuwe deeltjesversneller.

[18]Voorstellen voor de FCC bevatten kostenramingen die uiteenlopen van 9 miljard tot 21 miljard euro, zie de bovengenoemde bron.

Curriculum Vitae

Ruben Stienstra was born on June 30, 1991 in Oirschot, the Netherlands. After finishing his secondary education at Christiaan Huygens College in Eindhoven in 2009, he went on to study mathematics and physics at the Radboud University Nijmegen, obtaining bachelor's degrees in both disciplines in 2012. Emphasising the former without completely abandoning the latter, he continued his studies at the same university by pursuing a master's degree in mathematics, specialising in mathematical physics. He wrote his thesis "Complete motion in classical and quantum mechanics" under the supervision of prof. dr. N.P. Landsman, concluding his studies in 2014. Still at the same university, he commenced working on his PhD project immediately afterwards under the supervision of dr. W.D. van Suijlekom, of which this thesis is the end result.