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BACHELOR'S THESIS MATHEMATICS

# Relating Division Algebras to Hopf Fibrations using K-theory

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# Introduction

In 1898 Adolf Hurwitz showed in [Hur98] that there are exactly four division algebras: The well-known real numbers ($\mathbb{R}$) and the complex numbers ($\mathbb{C}$), and the lesser-known the quaternions ($\mathbb{H}$) and the octonions ($\mathbb{O}$). Moreover, there are exactly four Hopf fibrations: $S^0 \hookrightarrow S^1 \to S^1$; $S^1 \hookrightarrow S^3 \to S^2$; $S^3 \hookrightarrow S^7 \to S^4$; $S^7 \hookrightarrow S^{15} \to S^8$.

A natural question to ask is whether this is a coincidence. To answer this question we go back in time:

The first Hopf fibration was studied and discovered by Heinz Hopf, who published it in [Hop31] in 1931. In this article, he studied the continuous fibration $S^3 \to S^2$ with fibre $S^1$. Also, this is one of the first examples of a non-trivial principal $S^1$-bundle. After this discovery, Hopf continued the search to find other similar fibrations in which the only spaces are spheres. In 1935, he published in [Hop35] that he had also discovered three other fibrations.

The Hopf fibrations are a great discovery for the field of K-theory, which became clear in the 1960s. Moreover, Hopf fibrations are an important subject in theoretical physics. They have applications in, for instance, the theory of magnetic monopoles [Nak90] and quantum theory [MD01].

Several years after the discovery of Hopf, in 1955, Jean-Pierre Serre showed that there is a one-to-one correspondence between algebraic vector bundles and finite projective modules in [Ser55]. A similar result was proven in 1962 by Richard Gordan Swan in [Swa62]. He took a more analytic approach. In this article, Swan showed that there is an equivalence between topological vector bundles over compact Hausdorff spaces and finite projective modules. As a result, $C^*$-algebras could be linked to isomorphism classes of vector bundles.

A few years after Serre published his work, Raoul Bott discovered a certain periodicity within homotopy groups. This resulted in the so-called Bott Periodicity Theorem, which was published in [Bot59] and had a major impact on K-theory.

About 25 years after the first publication of Hopf about the first Hopf fibration, the connection between the Hopf fibrations and the division algebras was made by Bott and John Willard Milnor in 1958 in [BM58]. Bott and Milnor suspected that the morphisms $S^n \hookrightarrow S^{2n+1} \to S^{n+1}$ are in correspondence with $\mathbb{R}^{n+1}$ for $n = 0, 1, 3$ and 7. It was proven using the Hopf invariant by John Frank Adams, which was published in [Ada60]. A much more concise proof was given by Michael Atiyah and Adams in 1964, which was published in [AA66]. We refer to [Ada60] for a more detailed description of each mathematician that contributed to these results.

So it is not a coincidence that there are precisely four division algebras and four Hopf fibrations, there is a one-to-one correspondence between the two:

$$\begin{array}{ccc} S^0 \hookrightarrow S^1 \to S^1 & & \mathbb{R} \\ S^1 \hookrightarrow S^3 \to S^2 & \xleftrightarrow{1-1} & \mathbb{C} \\ S^3 \hookrightarrow S^7 \to S^4 & & \mathbb{H} \\ S^7 \hookrightarrow S^{15} \to S^8 & & \mathbb{O} \end{array}$$

In this thesis, we look closely at how the Hopf fibrations are defined. Furthermore, we will mainly follow [Hus66] and [Sty13] to prove that for $n = 0, 1, 2$ and 4 there exists a morphism $S^{2n-1} \to S^n$ of Hopf invariant $\pm 1$.

During this thesis, we will assume that all manifolds are smooth. Furthermore, we make the standing assumption that the base space of every fibre bundle is compact and Hausdorff. Notice that this holds for $S^n$, our main space of interest.

This thesis is divided into five chapters. In Chapter 1, we will cover the definitions of fibre bundles and principal $G$-bundles. This is followed by a description of K-theory. Here, we will treat two different versions: Topological K-theory (Section 2.1) and K-theory on $C^*$-algebras (Section 2.2). Then, we will zoom in on the Hopf fibrations in Chapter 3. We will construct the projections that correspond to the different Hopf fibrations, which makes it possible to state the Bott Periodicity Theorem in terms of this projection. To prove the Bott Periodicity Theorem, we will introduce the notion of exact sequences and different operations on topological spaces in Chapter 4. Finally, in Chapter 5 we start a discussion about H-space structures and the Hopf invariant. This chapter concludes with one side of the one-to-one correspondence: For $n = 0, 1, 2$ or 4 there exist a morphism $S^{2n-1} \to S^n$ of Hopf invariant $\pm 1$.

# 1 Fibre Bundles

In this chapter, we will follow [Hus66], [Lee13], [Lee97], [Cra15] and [Ste51] to cover the definition of a fibre bundle, vector bundle and a principal $G$-bundle and some important properties of these bundles.

*For convenience, we use the notation $\mathbb{K}$ for the division algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We will treat $\mathbb{O}$ separately.*

## 1.1 Vector Bundles

**Definition 1.1.1.** A continuous *fibre bundle* over a topological space $M$ *(the base)* consists of topological spaces $E$ *(total space)* and $\mathcal{F}$, and a surjective continuous mapping $\pi : E \to M$, such that

(i) for each $p \in M$ the *fibre* $\pi^{-1}(p) =: E_p$ is homeomorphic to $\mathcal{F}$,

(ii) for each $p \in M$ there is an open neighbourhood $U$ of $p$ satisfying *the local triviality condition*: There is a homeomorphism $\phi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathcal{F}$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times \mathcal{F} \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes[1].

**Example 1.1.2.** [Lee13, Example 10.3][Mun14, §74] The Möbius band (Figure 1.1), is an example of a non-trivial fibre bundle. Denote $I$ for the interval $[0,1]$, then the Möbius band is given by $M := I \times I / \sim$, where $(x, y) \sim (x', y')$ iff either $(x', y') = (x, y)$ or $x + x' = 1$ and $\{y, y'\} = \{0, 1\}$. Write $S^1 = \left\{e^{2\pi i x} : x \in I\right\}$, and let $\pi : M \to S^1$ be the continuous function given by $\pi : (x, y) \mapsto e^{2\pi i x}$. The map $\pi$ is surjective and continuous, and forms a fibre bundle over $S^1$ with fibre $I$. Namely, the fibre of a point $e^{2\pi i x} \in S^1$ is given by $\pi^{-1}(e^{2\pi i x}) \cong \{x\} \times I \cong I$. Also, the local triviality condition is satisfied. Namely, for $\alpha \in S^1$ and $U_\alpha$ an open neighbourhood of $\alpha$ we have that $\pi^{-1}(U_\alpha) \cong U_\alpha \times I$. △

[1] We denote $\mathrm{pr}_i$ for the projection onto the $i^{th}$ coordinate.

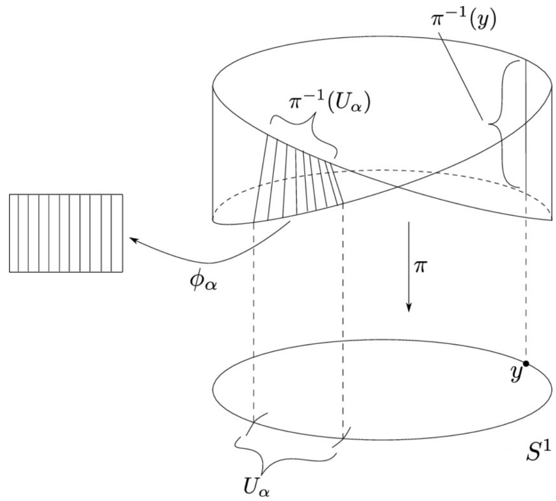



**Figure 1.1:** Möbius band viewed as a fibre bundle. (Adapted from reference [Mal18]).

**Definition 1.1.3.** A fibre bundle $E \xrightarrow{\pi} M$ is called a *subbundle* of the fibre bundle $E' \xrightarrow{\pi'} M'$ whenever $E$ is a subspace of $E'$ and $M$ is a subspace of $M'$, and $\pi = \pi'|_E \colon E \to M$.

**Definition 1.1.4.** A continuous (real) *k-dimensional vector bundle* over a topological space $M$ is a fibre bundle where

(i) the fibre is a k-dimensional vector space, $\mathcal{F} = \mathbb{R}^k$
(i.e. for each $p \in M$ the *fibre* $E_p = \pi^{-1}(p)$ is a k-dimensional vector space);

(ii) the morphism $E_p \xrightarrow{\sim} \{p\} \times \mathcal{F}$ is a linear isomorphism.

We will also call $k$ the *rank* of $E \xrightarrow{\pi} M$.

**Remark 1.1.5.** A one-dimensional vector bundle is often called a *line bundle*.

**Example 1.1.6.** A trivial $k$-dimensional vector bundle is a vector bundle $E \xrightarrow{\pi} M$, where the total space is given by $E = M \times \mathbb{R}^k$ and $\pi = \mathrm{pr}_1$. △

We will denote the trivial $k$-dimensional vector bundle over $M$ by $\varepsilon^k \to M$.

**Example 1.1.7.** A non-trivial example of a vector bundle is the *tautological line bundle*, denoted by $\mathcal{O}(-1)$ in algebraic geometry, over the projective space $\mathbb{PK}$. Another notation for the tautological line bundle is given by $H$. Take $\mathbb{K} = \mathbb{C}$, then we have that

$$\mathcal{O}(-1) \overset{d}{=} \left\{(\ell, y) \middle| l \in \mathbb{PC}, y \in \ell\right\} \subset \mathbb{PC} \times \mathbb{C}^2.$$

The projection of the tautological line bundle is given by

$$\pi_{\mathcal{O}} \colon \mathcal{O}(-1) \to \mathbb{PC}; \qquad \pi_{\mathcal{O}}(\ell, y) = \ell.$$

The preimage of $\ell \in \mathbb{PC}$ corresponds to all the points in the line $\ell$ through the origin of $\mathbb{C}^2$. We note that the space $\mathbb{PK}$ is also a manifold. Hence, the local trivialisations of the tautological line bundle can be given with respect to the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$. This atlas is given by

$$U_1 = \{[1, \alpha] | \alpha \in \mathbb{C}\}; \qquad \varphi_1([1, \alpha]) = \alpha$$
$$U_2 = \{[\alpha, 1] | \alpha \in \mathbb{C}\}; \qquad \varphi_2([\alpha, 1]) = \alpha.$$

Then the local trivialisation $\phi\colon U_1 \times \mathbb{C} \to \mathcal{O}(-1)\big|_{U_1}$ is given by $\phi\colon ([1, \alpha], c) \mapsto ([1, \alpha], (c, c\alpha))$ and $\phi\colon U_2 \times \mathbb{C} \to \mathcal{O}(-1)\big|_{U_2}$ is given by $\phi\colon ([\alpha, 1], c) \mapsto ([\alpha, 1], (c\alpha, c))$. △

**Definition 1.1.8.** A *continuous section* of a vector bundle $\pi\colon E \to M$ is a continuous function $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The vector space of all continuous section of $\pi$ is denoted $\Gamma(E, M)$.

Definition 1.1.4 of a vector bundle is not often used to construct vector bundles. A more convenient way to construct a vector bundle is by giving its local trivialisations. These are the homeomorphisms $\phi\colon \pi^{-1}(U) \to U \times \mathbb{K}^n$ of Definition 1.1.1 where $\mathcal{F} = \mathbb{K}^n$. This construction of a vector bundle is given in the following lemma.

**Lemma 1.1.9.** [Lee97, Lemma 2.2] *Let $\pi\colon E \to M$ be a surjection, where $M$ is a topological space and $E$ is a set, and let $\{U_\alpha\}$ be an open cover of $M$. Assume that for each $U_\alpha$ there is a bijective map $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that* $\mathrm{pr}_1 \circ \phi_\alpha = \pi$ *and satisfies for $U_\alpha \cap U_\beta \neq \emptyset$ that the map*

$$\phi_\alpha \circ \phi_\beta^{-1}\colon U_\alpha \cap U_\beta \times \mathbb{R}^k \to U_\alpha \cap U_\beta \times \mathbb{R}^k$$

*is given by $\phi_\alpha \circ \phi_\beta^{-1}(p, V) = (p, \tau_{\alpha\beta}(p)V)$, for a continuous map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$. Then $E$ has a unique vector bundle structure over $M$, with $\phi_\alpha$ the local trivialisations.*

We call the mapping $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, the *transition function* between the local trivialisations $\phi_\alpha$ and $\phi_\beta$. Also, the converse of Lemma 1.1.9 holds, for which we refer to [Lee13, Lemma 10.5].

Given two vector bundles one can define a so-called bundle morphism between the vector bundles as a continuous function which respects the vector space structure on the fibres.

**Definition 1.1.10.** Let $E_1 \xrightarrow{\pi_1} M_1$ and $E_2 \xrightarrow{\pi_2} M_2$ be vector bundles. A *bundle morphism* is a pair $(f, f')\colon (E_1, \pi_1, M_1) \to (E_2, \pi_2, M_2)$ of continuous morphisms such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[f']{} & M_2 \end{array}$$

commutes, and for all $p \in M$ the map $\pi_1^{-1}(p) \to \pi_2^{-1}\left(f'(p)\right)$ is linear. We call $E_1 \xrightarrow{\pi_1} M_1$ and $E_2 \xrightarrow{\pi_2} M_2$ *isomorphic* if there exists a bundle morphism as above such that $f'$ is a homeomorphism.

One can also construct new vector bundles from the pull-back, direct sum or the tensor product of vector bundles.

**Definition 1.1.11.** Let $\pi\colon E \to M$ a vector bundle, and $f\colon N \to M$ a continuous morphism. Define the *pull-back vector bundle* as $f^*E\colon = \{(n,e) \in N \times E \big| f(n) = \pi(e)\}$, with the projection map $\pi'\colon f^*E \to N$ as the projection onto the first coordinate.

With the above definition we also obtain a morphism $\mathrm{pr}_2\colon f^*E \to E$ such that the diagram

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

commutes.

**Example 1.1.12.** Let $\pi\colon E \to M$ be a vector bundle and $\mathrm{pr}_1\colon M \times N \to M$ be the projection on the first coordinate. Then the pull-back vector bundle is given by

$$\mathrm{pr}_1^* E = \{(m,n,e) \in M \times N \times E\colon\ m = \pi(e)\},$$

with the projection map $\pi'\colon\ \mathrm{pr}_1^* E \to M \times N$, where $\pi'(m,n,e) = (\pi(e), n)$ for all $n \in N$. △

**Definition 1.1.13.** Let $E \xrightarrow{\pi_1} M$ and $F \xrightarrow{\pi_2} M$ be vector bundles of ranks $k$ and $l$, respectively. We can define the *direct sum* (also known as the Whitney sum) of the vector bundles by the projection onto $M$ of the total space

$$E \oplus F\colon = \bigsqcup_{p \in M} \left(E_p \oplus F_p\right).$$

Furthermore, we have local trivialisations $\{U_\alpha, \phi_\alpha\}$ and $\{U_\beta, \phi_\beta\}$ of $E$ and $F$ with the corresponding transition functions $\tau_{\alpha_i\alpha_j}\colon U_{\alpha_i} \cap U_{\alpha_j} \to GL(k,\mathbb{R})$ and $\tau_{\beta_i\beta_j}\colon U_{\beta_i} \cap U_{\beta_j} \to GL(l,\mathbb{R})$. Then, we define the transition function for $E \oplus F$ by

$$\tau_{\alpha_i\alpha_j} \oplus \tau_{\beta_i\beta_j}\colon \left(U_{\alpha_i} \cap U_{\alpha_j}\right) \cap \left(U_{\beta_i} \cap U_{\beta_j}\right) \longrightarrow GL(k+l,\mathbb{R}); \qquad x \mapsto \begin{pmatrix} \tau_{\alpha_i\alpha_j}(x) & 0 \\ 0 & \tau_{\beta_i\beta_j}(x) \end{pmatrix}.$$

Hence, we have that $(\pi_1 \oplus \pi_2)^{-1}(U) = U \times \mathbb{R}^{k+l}$, where $U = \left(U_{\alpha_i} \cap U_{\alpha_j}\right) \cap \left(U_{\beta_i} \cap U_{\beta_j}\right)$.

**Lemma 1.1.14.** *For each vector bundle $P \xrightarrow{\pi} M$, where $M$ is compact and Hausdorff, there exists a vector bundle $Q \xrightarrow{\overline{\pi}} M$ such that $P \oplus Q$ is trivial.*

*Proof.* We will only give a sketch of the proof following [Hat17, Prop. 1.4];
For each $m \in M$ there is an open neighbourhood $U_m$ such that the $\pi|_{U_m}\colon P|_{U_m} \longrightarrow U_m$ is trivial. We use Urysohn's lemma ([Mun14, Thm. 33.1])to construct a map $\varphi_m\colon M \to [0,1]$, where for $x \in M \setminus U_m$ we have that $\varphi_m(x) = 0$. The set $\{\varphi_m^{-1}((0,1])\}_{m \in M}$ is an open cover of $M$. The base $M$ is compact, hence there exist a finite subcover of $M$, $\{\varphi_i^{-1}((0,1])\}_i$ with corresponding open neighbourhoods $U_i$. Define the map

$$g_i\colon P \to \mathbb{R}^n; \qquad g_i(x) = \begin{cases} \varphi_i(\pi(x))(\mathrm{pr}_i\, h_i(x)), & \text{for } x \in \pi^{-1}(U_i) \\ 0 & \text{for } x \notin \pi^{-1}(U_i), \end{cases}$$

for $h_i$ the local trivialisation $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$ and $\mathrm{pr}_i$ the projection of $U_i \times \mathbb{R}^n$ to $\mathbb{R}^n$. Now $g_i$ is a linear injection on each fibre. Also, note that because $M$ is compact Hausdorff, we have that $M$ is normal [Mun14, Thm. 32.3]. Therefore, we have that $\mathrm{supp}(\varphi_i) \subset U_i$. This makes $g_i$ a continuous morphism on $P$.

One can construct the following map

$$f\colon P \to M \times \mathbb{R}^n; \qquad f(x) = (\pi(x), g(x)).$$

Now $P$ is isomorphic to a subbundle of $M \times \mathbb{R}^n$. Using [Hat17, Prop. 1.3] one finds a complementary subbundle $\overline{P}$ such that $P \oplus \overline{P}$ is the trivial bundle. □

**Definition 1.1.15.** In the same setting as in the definition of the direct sum we define the *tensor product* of two (real) vector bundles, $E \xrightarrow{\pi_1} M$ and $F \xrightarrow{\pi_2} M$ as the projection onto $M$ of the total space

$$E \otimes F := \bigsqcup_{p \in M} \left(E_p \otimes F_p\right).$$

Then the transition function for $E \otimes F$ are given by the pointwise tensor product of vector spaces.

$$\tau_{\alpha_i\alpha_j} \otimes \tau_{\beta_i\beta_j} \colon \left(U_{\alpha_i} \cap U_{\alpha_j}\right) \cap \left(U_{\beta_i} \cap U_{\beta_j}\right) \to GL(k \cdot l, \mathbb{R}); \qquad x \mapsto \tau_{\alpha_i\alpha_j} \otimes \tau_{\beta_i\beta_j}(x, x),$$

which equips $E \otimes F$ with a vector bundle structure. Then the local trivialisations are given by $(\pi_1 \otimes \pi_2)^{-1}(U) \cong U \times \mathbb{R}^{kl}$, where $U = \left(U_{\alpha_i} \cap U_{\alpha_j}\right) \cap \left(U_{\beta_i} \cap U_{\beta_j}\right)$.

## 1.2 Principal Bundles

**Definition 1.2.1.** A *principal $G$-bundle* over $M$ consists of a topological space $P$ and a topological group $G$ with a right action of $G$ on $P$ and a surjective map $\pi\colon P \to M$ which is $G$-invariant (i.e. we have that $\pi_P(pg) = \pi_P(p)$ for all $p \in P$ and $g \in G$) and satisfies the local triviality condition: There is a homeomorphism $\phi : \pi^{-1}(U) \xrightarrow{\sim} U \times G$ which is $G$-invariant, where the action of $G$ on $U \times G$ is given by $(u, a)g = (u, ag)$.

We often write $G \hookrightarrow P \xrightarrow{\pi} M$ for a principal $G$-bundle.

Before we can state what it means for two principal bundles to be isomorphic, we need the definition of $G$-equivariance:

**Definition 1.2.2.** Let $X$ and $Y$ be sets on which a right group action of a group $G$ is defined. A function $f\colon X \to Y$ is called *equivariant* if $f(xg) = f(x) \cdot g$ for all $x \in X$ and $g \in G$.

**Definition 1.2.3.** Let $P_1 \xrightarrow{\pi_1} M$ and $P_2 \xrightarrow{\pi_2} M$ be principal $G$-bundles over the same base. An *isomorphism of principal $G$-bundles* is a homeomorphism $F\colon P_1 \to P_2$ such that $\pi_2 \circ F = \pi_1$ and $F$ is $G$-equivariant.

To a principal $G$-bundle we can associate a vector bundle over the same base space. For this we need a so-called *representation*:

**Definition 1.2.4.** A vector space $V$ is called a *representation space* of $G$ if there exist a *representation* $A\colon G \to \mathrm{GL}(V)$, which is a morphism of topological groups.

Then for each $g \in G$ we have a linear invertible morphism $A_g := A(g)\colon V \to V$, with property that $A_{g\circ h} = A_g \circ A_h$.

With the representation space of Definition 1.2.4 one can construct a vector bundle associated to a principal $G$-bundle.

For a principal $G$-bundle $\pi\colon P \to M$ with a representation $A\colon G \to \mathrm{GL}(V)$ where $A(g) = g\cdot(\_)$, we denote the total space for the associated vector bundle by $E(P,V) = (P\times V)/G$. The representation $A$ defines a linear action of $G$ onto $V$, which in combination of the action of $G$ on $P$ gives that $G$ can act on the product $P\times V$ by $(p,v)\cdot g = (pg, A_g v)$. Hence the equivalence classes are determined by the representation $A$. This yields equivalences classes, which for each $g \in G$ are given by

$$\left[pg, v\right] = \left[p, A_g(v)\right].$$

**Definition 1.2.5.** The *associated vector bundle* for the principal $G$-bundle $P \to M$ is given by

$$\pi_{as}: E(P,V) \to M; \qquad \pi_{as}(p,v) = \pi(p).$$

The fibres $E(P,V)_p$, of the associated vector bundle, are given by the set of equivalence classes $\left\{[x,v]\colon x \in P_p, v \in V\right\}$. Denote $\pi(p) = x$. Then the morphism $E(P,V)_x \to V$ with $[p,v] \mapsto v$ forms an isomorphism of vector spaces. Moreover, the local trivialisations for the principal $G$-bundle: $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U\times G$, induce the local trivialisations

$$\phi : \pi_{as}^{-1}(U) \xrightarrow{\sim} U\times V; \qquad p \mapsto \left(\pi(p), (\mathrm{pr}_2 \circ\varphi)(p)v\right)$$

for the associated vector bundle. Consequently, we find that the associated vector bundle is a vector bundle of rank $\dim V$ over $M$ [Cra15, §2.2].

# 2 K-theory

There are various equivalent ways of defining K-theory, in this section we look at two of those definitions. In both of these settings, we will restrict the discussion to complex K-theory on finite-dimensional (complex) vector bundles. Furthermore, we will assume that the base space is a compact Hausdorff space because this holds for $S^n$, our main space of interest.

## 2.1 Topological K-theory

In this section we will follow [Hat17] and [Hus66] for the description of $K(M)$ (called the *K-ring*) and $\widetilde{K}(M)$ (called the *reduced K-ring*).

### 2.1.1 Construction of $K(M)$

Before we can define the K-ring over a topological space $M$, we need some definitions.

**Definition 2.1.1.** The set of isomorphism classes vector bundles over the same base $M$, which is compact and Hausdorff, is given by $\mathrm{Vect}(M)$.

The set $\mathrm{Vect}(M)$ with $\oplus$ has a semigroup structure, i.e. it has no additive inverses. In order to give $\mathrm{Vect}(M)$ a group structure we use the construction of the Grothendieck group.

**Definition 2.1.2.** Let $(S,+)$ be an abelian semigroup. Then, the *Grothendieck group* is defined as $Gr(S) = (S \times S)/\sim$, where $(x,y) \sim (x',y')$ if and only if there is $c \in S$ such that $x + y' + c = x' + y + c$. The addition given by

$$Gr(S) \times Gr(S) \to Gr(S) \qquad \text{where } ([x,x'],[y,y']) \mapsto [x+y, x'+y'],$$

is well-defined and makes $(Gr(S),+)$ into an abelian group. Here we have that $[x,x] = 0$ and $-[x,y] = [y,x]$ for all $x, y \in K(X)$.

The semigroup $(S,+)$ and $Gr(S)$ are linked via the corresponding *Grothendieck map* $\gamma_S : S \to Gr(S)$ given by $x \mapsto [x+y, y]$ for all $y \in S$.

The Grothendieck construction has the following universal property:

**Proposition 2.1.3.** *Let $A$ be an abelian group, and suppose $f: S \to A$ is an additive map, then there is a unique group homomorphism $g: Gr(S) \to A$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\gamma_S"'] & A \\ Gr(S) \arrow[ur, "\exists! g"'] & \end{tikzcd}$$

*commutes.*

*Proof.* [RLL00, 3.1.1(i)] Define $g$ by $g(x,y) = f(x)-f(y)$. Then $(g\circ\gamma_S)(x) = g((x+y,y)) = f(x)+f(y)-f(y) = f(x)$. Hence, $g$ makes the diagram commute and because $g(x,y)+g(x',y') = f(x)-f(y)+f(x')-f(y') = f(x+x')-f(y+y') = g(x+x',y+y')$, $g$ is also additive. For $x,x',y,y' \in S$, and suppose that $(x,y)\sim(x',y')$. Then there is a $c \in S$ such that $x+y'+c = x'+y+c$. Then because $f$ is additive we have that $f(x)+f(y')+f(c) = f(x')+f(y)+f(c)$ and thus that $f(x)-f(y) = f(x')-f(y')$. Hence, $g$ is well-defined. Left to show is the uniqueness of $g$. First note that each element in $(x,y)\in Gr(S)$ can be written as $\gamma_S(x)-\gamma_S(y)$. This means that $g(\gamma_S(x)-\gamma_S(y)) = f(x)-f(y)$, which shows that $g$ is unique. □

**Example 2.1.4.** We construct the Grothendieck group of the natural numbers. Using this construction we add additive inverse of all non-zero elements. The equivalence relation becomes $(n,m)\sim(\tilde{n},\tilde{m})$ if and only if $n+\tilde{m}+c = m+\tilde{n}+c$, where $(n,m)\hat{=}n-m$. Now, defining $n=(n,0)$ and $-n=(0,n)$ gives the integers. Hence, we have that $Gr(\mathbb{N}) = \mathbb{Z}$. △

Now, we have all the tools to construct the K-group of $M$;

**Definition 2.1.5.** The K-ring of $M$, $K(M)$ is defined as the Grothendieck group of $\mathrm{Vect}(M)$.

**Remark 2.1.6.** In order to make $K(M)$ an abelian group, the topological space $M$ needs to be a compact space.

The tensor product defined on $\mathrm{Vect}(M)$ gives $K(M)$ a ring structure. Taking the tensor product through the Grothendieck group yields the multiplication given by

$$(E_1-E_1')(E_2-E_2') = E_1\otimes E_2 - E_1\otimes E_2' - E_1'\otimes E_2 + E_1'\otimes E_2'. \tag{2.1}$$

**Lemma 2.1.7.** *Let $E_i, E_i'$ be vector bundles over $M$, for $n = 1,2$. Then the multiplication of* (2.1) *and the addition given by*

$$(E_1-E_1')+(E_2-E_2') = (E_1\oplus E_2)-(E_1'\oplus E_2')$$

*give $K(M)$ a commutative ring structure.*

*Proof.* From the Grothendieck construction it follows that $(K(M),+,\varepsilon^0)$ has a group structure, where the inverse of $(E-E')$ is $(E'-E)$. The associativity of the product is obtained due to the associativity of the tensor product. We note that the multiplication is commutative and distributive, due to the fact that there is an isomorphism of vector bundles: $E_1\otimes E_2 \cong E_2\otimes E_1$. For the identity element take $\mathbb{I} = (\varepsilon^1-\varepsilon^0)$ then $\mathbb{I}(E_1-E_1') = (E_1-E_1') = (E_1-E_1')\mathbb{I}$. Therefore $K(M)$ has the structure of a commutative ring. □

**Example 2.1.8.** Take $M = \{*\}$, a point. Then the only vector bundles over $\{*\}$ are the trivial vector bundles, i.e. $\{*\} \times \mathbb{C}^n \xrightarrow{\pi_n} \{*\}$ for $n \in \mathbb{N}$. Now, we can make an isomorphism $p\colon \mathrm{Vect}(\{*\}) \to \mathbb{N}$ where $\{*\} \times \mathbb{C}^n \mapsto n$. Then, by Example 2.1.4 we find that $K(\{*\}) = \mathbb{Z}$. △

Let $M, N$ and $P$ be topological spaces. From the definition of the pull-back (Definition 1.1.11) one can construct a morphism between $K(M)$ and $K(N)$. Let $f\colon N \to M$ be a continuous morphism. Then the morphism $K(f)\colon K(M) \to K(N)$ is given by making the diagram

$$\begin{tikzcd} \mathrm{Vect}(M) \arrow[r, "\gamma_M"] \arrow[d, "f^*"'] & K(M) \arrow[d, "K(f)"] \\ \mathrm{Vect}(N) \arrow[r, "\gamma_N"'] & K(N) \end{tikzcd}$$

commute. Suppose we have another morphism $g\colon P \to M$, then the composition of $f \circ g$ yields the composition $K(f \circ g) = K(g) \circ K(f)$. Moreover we have that $K(\mathrm{id}_M) = \mathrm{id}_{K(M)}$ and for the difference $E_1 - E_2 \in K(M)$ we have that $K(f)(E_1 - E_2) = f^*(E_1) - f^*(E_2)$.

### 2.1.2 Construction of $\widetilde{K}(M)$

The construction of $\widetilde{K}(M)$ is based on the map rank: $\mathrm{Vect}(M) \to \mathbb{Z}$, which sends a vector bundle to the dimension of its fibre.

Using the universal property of the Grothendieck construction of Proposition 2.1.3, with the abelian group $\mathbb{Z}$, one finds the following commutative diagram

$$\begin{tikzcd} \mathrm{Vect}(M) \arrow[r, "\mathrm{rank}"] \arrow[d] & \mathbb{Z}\ . \\ K(M) \arrow[ur, "\mathrm{rk}"'] & \end{tikzcd} \tag{2.2}$$

**Definition 2.1.9.** The reduced K-ring is the kernel of the map induced by the inclusion $\iota : \{*\} \hookrightarrow M$;

$$\widetilde{K}(M) := \ker\big(\mathrm{rk} : K(M) \to K(\{*\})\big),$$

where the kernel is taken with respect to 1, i.e. the ring structure.

In order to see how this definition is related to the map rank : $\mathrm{Vect}(M) \to \mathbb{Z}$ we consider the following diagram

$$\begin{tikzcd} \mathrm{Vect}(M) \arrow[r, "\gamma_M"] \arrow[d, "\iota^*"'] & K(M) \arrow[d, "\mathrm{rk}"] \\ \mathrm{Vect}(\{*\}) \arrow[r, "\gamma_{\{*\}}"'] & K(\{*\}). \end{tikzcd}$$

The pull-back morphism $\iota^*$ sends vector bundles of rank $k$ over $M$ to the trivial vector bundle of rank $k$. In Example 2.1.8, we showed that $\mathrm{Vect}(\{*\}) \cong \mathbb{N}$. Hence, the map rank is induced by the inclusion

$\iota$, where each vector bundle is mapped to its rank. Furthermore, the map rk is the morphism $K(\iota)$ which is given by making the diagram commute.

Note that in Example 2.1.8, we found that $K(\{*\}) = \mathbb{Z}$. Hence, we can also write

$$\widetilde{K}(M) = \ker(\mathrm{rk}: K(M) \to \mathbb{Z}).$$

**Example 2.1.10.** Using Example 2.1.8, we find that $\widetilde{K}(\{*\}) = \ker(\mathrm{rk}\colon K(\{*\}) \to K(\{*\})) = 0$. △

Another, more direct way of looking at $\widetilde{K}(M)$, is to see $\widetilde{K}(M)$ as the set of s-equivalence classes, w.r.t. $\oplus$.

**Definition 2.1.11.** Let $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ be vector bundles over the same base $M$. Then we call $E_1$ and $E_2$ over $M$ *stably isomorphic* or *s-isomorphic*, denoted $E_1 \approx_s E_2$, if $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$ for some $m, n \in \mathbb{N}$ (note that $\cong$ is the isomorphism of vector bundles).

**Lemma 2.1.12.** *Stable equivalence or s-equivalence of vector bundles defines an equivalence relation on the set of vector bundles over $M$.*

*Proof.* Let $E_1 \xrightarrow{\pi_1} M, E_2 \xrightarrow{\pi_2} M$ and $E_3 \xrightarrow{\pi_3} M$ be vector bundles over the base $M$. Then we have that $E_1 \oplus \varepsilon^k = E_1 \oplus \varepsilon^k$ for all $k$, hence $E_1 \approx_s E_1$. Furthermore, for $E_1 \approx_s E_2$ we have that there are $n, m \in \mathbb{N}$ such that $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$. This property is symmetric. So we have that $E_2 \oplus \varepsilon^m \cong E_1 \oplus \varepsilon^n$ and thus that $E_2 \approx_s E_1$. Finally, we need to show that this relation is transitive. Suppose $E_1 \approx_s E_2$ and $E_2 \approx_s E_3$. Then there are $n, m, o, p \in \mathbb{N}$ such that $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$ and $E_2 \oplus \varepsilon^o \cong E_3 \oplus \varepsilon^p$. Then we have that $E_1 \oplus \varepsilon^n \oplus \varepsilon^o \cong\ E_2 \oplus \varepsilon^m \oplus \varepsilon^o$ and $E_2 \oplus \varepsilon^o \oplus \varepsilon^m \cong E_3 \oplus \varepsilon^p \oplus \varepsilon^m$. This can be rewritten to $E_1 \oplus \varepsilon^{n+o} \cong\ E_2 \oplus \varepsilon^{m+o} = E_2 \oplus \varepsilon^{m+o} \cong E_3 \oplus \varepsilon^{m+p}$ yielding the s-equivalence: $E_1 \approx_s E_3$. Therefore the s-isomorphism of vector bundles defines an equivalence relation. □

The set of s-isomophism classes of vector bundles over $M$ together with the direct sum forms an abelian group with zero element $\varepsilon^0$. Let $E$ be a vector bundle over $M$. Due to Lemma 1.1.14, there exists a vector bundle such that $E \oplus E' \cong \varepsilon^0$. Hence, the inverse of $E$ is given by $E'$.

**Lemma 2.1.13.** *The reduced K-ring of M defined in Definition 2.1.9 is equivalent to the set of s-equivalence classes of vector bundles over M.*

*Proof.* We sketch the proof of [Hus66, Thm. 8.3.8]: Define the function $\alpha\colon \mathrm{Vect}(X) \to \widetilde{K}(X)$, where $\alpha(x) = x - \mathrm{rk}(x)$. For $x - y \in \widetilde{K}(X)$, with $\mathrm{rk}(x) = \mathrm{rk}(y)$. Then, from the additive structure of the map rk of (2.2) it follows that the map $\alpha$ is surjective. Subsequently, one finds that $\alpha(x) = \alpha(y)$ if and only if the vector bundle $x$ and $y$ are s-equivalent. □

With the alternative definition of $\widetilde{K}(M)$ as the set of $s$-equivalence classes one can write $K(M)$ as the direct sum of $\widetilde{K}(M)$ and $\mathbb{Z}$. For this, we note that every element in $K(M)$ can be written as the formal difference of over vector bundles, $E - E'$, over base $M$, with an equivalence relation given by $E - E' = F - F'$ if and only if $E \oplus F' \approx_s F \oplus E'$. Which also means that we can write every element of $K(M)$ as the difference $E - \varepsilon^n$ for suitable $n$.

**Lemma 2.1.14.** *The reduced K-ring is given by $K(M) = \widetilde{K}(M) \oplus \mathbb{Z}$.*

*Proof.* We follow the proof of [Hat17, p. 40].
We have constructed $\widetilde{K}(M)$ as the kernel of the map rk, hence we have that there is a inclusion $K(M) \hookrightarrow \widetilde{K}(M)$, where $E - \varepsilon^n$ maps to the equivalence class of $E$ in $\widetilde{K}(M)$. Then for $E - \varepsilon^n = E' - \varepsilon^m$ in $K(M)$, we have that $E \oplus \varepsilon^n \approx_s E' \oplus \varepsilon^m$. Therefore we also have that $E \cong E'$. Hence, the map is well-defined. The kernel of the inclusion are all the elements in $K(M)$ that are mapped to $\varepsilon^0$. Hence, all elements in $\ker\big(K(M) \hookrightarrow \widetilde{K}(M)\big)$ are of the form $E - \varepsilon^n$ with $E \cong \varepsilon^0$. Moreover, we also have that $E \approx_s \varepsilon^m$ for a suited $m \in \mathbb{N}_{\geq 0}$. Thus we have that $\ker\big(K(M) \hookrightarrow \widetilde{K}(M)\big) = \{\varepsilon^m - \varepsilon^n | m, n \in \mathbb{N}_{\geq 0}\} \cong \mathbb{Z}$. Furthermore, we note that for the morphism $K(M) \to K(\{*\}) \cong \mathbb{Z}$ all elements are mapped to elements of the form $\varepsilon^m - \varepsilon^n$. This gives rise to the splitting $\widetilde{K}(M) = K(M) \oplus \mathbb{Z}$. □

## 2.2 K-theory on $C^*$-algebras

In this section we will follow [RLL00] and [Bla98] for the description of $K(M)$ and $\widetilde{K}(M)$.

At first sight, it might not be evident why one would include $C^*$-algebras in a discussion about vector bundles. Hence, the first part of the section is spent on the relation between the two. This will be followed by the construction of the K-ring and the reduced K-ring.

Denote $X$ for the base space, and denote $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$ for the vector space of continuous functions on $X$.

**Lemma 2.2.1.** *Let $X$ be a compact Hausdorff space. Then the vector space $C(X)$, with the supremum norm, pointwise multiplication and addition and scalar multiplication forms a unital $C^*$-algebra.*

*Proof.* It is clear that $C(X)$ is a normed linear space. First we show that $C(X)$ is complete with respect to the supremum norm, then we will show that it is a $C^*$-algebra.
Let $f_n$ be a Cauchy sequence in $C(X)$. For a fixed $x \in X$ we have that $f_n(x)$ is a Cauchy sequence in $\mathbb{C}$. Because $\mathbb{C}$ is complete, we have that $\lim_{n\to\infty} f_n(x) \in \mathbb{C}$ exists. Define $f(x) := \lim_{n\to\infty} f_n(x)$ as the pointwise limit. Then there exists $M > 0$ such that $||f_n||_\infty < M$, and therefore $|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq M$ for all $x \in X$. Hence, for all $x \in X$ and $n > N$ we have that $|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| < \varepsilon$. Thus, for $n > N$ we have that $||f_n - f||_\infty < \varepsilon$, making $C(X)$ into a Banach space.

The multiplication is given by $(f \cdot g)(x) = f(x)g(x)$ for all $f, g \in C(X)$. Taking the supremum norm of $f \cdot g$, yields

$$\begin{aligned}
||f \cdot g||_\infty &= \sup\{|f(x)g(x)| : \text{ for } x \in X\} \\
&\leq \sup\big\{|f(x)| \cdot |g(x)| : \text{ for } x \in X\big\} \\
&= ||f||_\infty \cdot ||g||_\infty .
\end{aligned}$$

Thus, $C(X)$ is a Banach algebra. In order to show that $C(X)$ is a $C^*$-algebra, define the involution mapping as the complex conjugate. Then,

$$||ff^*||_\infty = \sup_{x\in X}\{|f(x)\overline{f(x)}|\} = \sup_{x\in X}\{|f(x)|^2\} = \sup_{x\in X}\{|f(x)|\}^2 = ||f||_\infty^2 .$$

As a result we find that $C(X)$ is a $C^*$-algebra, because the $C^*$-identity is satisfied. Finally, the map $1 \colon x \mapsto 1 \in \mathbb{C}$ is a continuous map. Thus the space of continuous functions $C(X)$ is a unital $C^*$-algebra. □

Recall that an element $p$ in a $C^*$-algebra is called a *projection* whenever $p = p^2 = p^*$. We denote the set of all projection of the $C^*$-algebra $A$ as $\mathcal{P}(A)$.

**Definition 2.2.2.** Let $A$ be a $C^*$-algebra. We define the *semigroup of projections* by

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}(M_n(A)),$$

where $M_n(A)$ is the set of $n \times n$ matrices over $A$. The semigroup operation on $\mathcal{P}_\infty(A)$ is given by $p \oplus q = \operatorname{diag}(p, q)$. Then we have that $p \oplus q \in \mathcal{P}(M_{n+m}(A))$ for $p \in \mathcal{P}(M_n(A))$ and $q \in \mathcal{P}(M_m(A))$.

We can also define a equivalence relation on $\mathcal{P}_\infty(A)$, denoted by $\sim$, analogous to the so-called *Murray-von Neumann equivalence* (see [RLL00, Def. 2.2.1]). Two projections $p \in \mathcal{P}(M_n(A))$ and $q \in \mathcal{P}(M_m(A))$ are called equivalent, denoted $p \sim q$, whenever there is an element $v \in M_{n,m}(A)$ with $p = v^* v$ and $q = v v^*$.

**Remark 2.2.3.** For each $C^*$-algebra, we have that $M_n(A)$ is a $C^*$-algebra with a given norm (for more details on this norm we refer to [RLL00, §1.3]). Then $\mathcal{P}(M_n(A))$ forms a closed subspace with respect to this norm by the induced topology.

An important notion, which we will employ later on in this thesis, is that we can relate the equivalence relation of Definition 2.2.2 to homotopy equivalence in $\mathcal{P}(M_n(A))$.

Recall from general topology, that for a topological space $X$ two elements $x, y \in X$ are called *homotopic*, i.e. $x \sim_h y$, whenever there exists a continuous function $f \colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

To show a relation between homotopy equivalence and the equivalence relation of Definition 2.2.2 we need the notion of polar decomposition:

**Definition 2.2.4.** For each invertible element $x$ in a unital $C^*$-algebra $A$ we can write the *polar decomposition* $x = u|x|$ for a unique $u \in A$ where $|x|$ is positive definite.

**Proposition 2.2.5.** *For projections $p, q \in \mathcal{P}(M_n(A))$, if $p \sim_h q$ then $p \sim q$.*

*Proof.* We follow the proof of [RLL00, Prop. 2.2.7, Prop 2.2.6].
Assume that $A$ is a unital $C^*$-algebra and suppose there are projections $p, q \in \mathcal{P}(M_n(A))$ such that $p \sim_h q$. Let $p_t \colon [0,1] \to \mathcal{P}(M_n(A))$ be the homotopy for $t \in [0,1]$ such that $p_{t=0} = p$ and $p_{t=1} = q$. Subsequently, we can find $p_0, p_1, \ldots, p_n$ where $p_{t=0} = p_0 = p$ and $p_n = p_{t=1} = q$ such that $||p_i - p_j|| < \frac{1}{2}$ (using that $[0,1]$ is compact). Hence, it is enough to show for $p_0 = p$ and $p_1 = q$ with $||p - q|| < \frac{1}{2}$ that $p \sim q$. Let

$$z = pq + (1-p)(1-q).$$

Because $p$ and $q$ are projections we have that

$$pz = p^2 q + (p - p^2)(1-q) = pq = pq^2 + (1-p)(q - q^2) = zq. \tag{2.3}$$

Furthermore, we have that

$$||z - 1|| = ||pq + (1-p)(1-q) - 1|| = ||p(q-p) + (1-p)((1-q) - (1-p))|| \leq 2||q-p|| < 1.$$

Hence, we know from functional analysis (see for instance [Mac09, Lemma 5.8.]) that $z$ is invertible in $\mathcal{P}(M_n(A))$. Define

$$c_t = (1-t)z + 1t$$

then $||1 - z^{-1}c_t|| = ||z^{-1}(z - c_t)|| = ||z^{-1}|| \cdot ||z - c_t|| < 1$ (because we have that $||z - c_t|| = t||z-1|| < ||z^{-1}||^{-1}$). So, $z^{-1}c_t$ is also invertible. Consequently, $c_t$ is invertible. Hence $c_t$ gives a homotopy between $z$ and 1.

Because $z$ is invertible, it has a polar decomposition, which we will denote by $z = u|z|$ for some unitary $u$. From equation (2.3) it follows that $pz = zq$, and because $p$ and $q$ are projections we have that $z^*p = qz^*$. Combining these results gives that

$$|z|^2 q = (z^*z)q = z^*pz = q(z^*z) = q|z|^2.$$

Hence, $q$ commutes with all elements of the form $|z|^2$. Therefore, $q$ also commutes with $|z|^{-1}$. Thus we have that $uqu^* = z|z|^{-1}qu^* = zq|z|^{-1}u^* = pz|z|^{-1}u^* = puu^* = p$. Let $v = uq$, then $vv^* = p$ and $v^*v = q$. As a result we find that $p \sim q$. □

With this in mind, we go back by using the equivalence relation of Definition 2.2.2 to construct an abelian group of projections.

**Definition 2.2.6.** Let $A$ be a unital $C^*$-algebra. The abelian semigroup $(\mathcal{D}(A), +)$ is given by

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim$$

where the addition is given by $[p] + [q] = [p \oplus q]$.

Using the Grothendieck construction of Definition 2.1.2, one can make $(\mathcal{D}(A), +)$ into a abelian group.

**Definition 2.2.7.** Let $A$ be a unital $C^*$-algebra. Define the K-group by the Grothendieck group of $(\mathcal{D}(A), +)$, i.e.

$$K(A) := Gr(\mathcal{D}(A)).$$

We can relate $\mathcal{P}_\infty(A)$ with $K(A)$, with the help of the Grothendieck map $\gamma$, by the morphism

$$[-] : \mathcal{P}_\infty(A) \to K(A) \qquad \text{where } p \mapsto \gamma\left([p]_{\mathcal{D}}\right).$$

A similar definition of stable equivalence of vector bundles, Definition 2.1.11, can be given on the semigroup of projections.

**Definition 2.2.8.** Let $p, q \in \mathcal{P}_\infty(A)$ then we call $p$ and $q$ *stable equivalent*, $p \sim_s q$ if and only if $p \oplus r \sim_s q \oplus r$ for some $r \in \mathcal{P}_\infty(A)$.

The above definition suggests that there might be some relation between $\mathcal{P}_\infty(X)$ and Vect$(X)$. In fact we have that $K(C(X))$ and the topological $K(X)$ are isomorphic as abelian groups. But there is an even stronger relation. This relation is known as the *Serre-Swan Theorem*, which gives an important correspondence between geometry and algebra. The first proof was given by Serre in 1955, which was published in [Ser55]. Serre showed that there is a one-to-one correspondence between algebraic vector bundles and finite projective modules. Several years later, in 1962, Swan proved a similar statement, which states that there is an equivalence between topological vector bundles over compact Hausdorff spaces and finite projective modules [Swa62].

The *Serre-Swan Theorem* states a relation with finite projective modules, hence the following definition:

**Definition 2.2.9.** A (right) $\mathcal{A}$-module $\mathcal{E}$ is called *finite projective* (i.e. finitely generated projective) if there exists a finitely generated free module[2] $\mathcal{F}$ and a module $\mathcal{E}'$ such that $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$.

**Lemma 2.2.10.** *Let $\mathcal{E}$ be a (right) $\mathcal{A}$-module then $\mathcal{E}$ is finite projective if and only if*

*(i) There is a surjective homomorphism $\rho\colon \mathcal{A}^N \to \mathcal{E}$ and a lift $\lambda\colon \mathcal{E} \to \mathcal{A}^N$ such that $\rho\circ\lambda = \mathrm{id}_{\mathcal{E}}$, for a suited $N \in \mathbb{N}_{>0}$.*

*(ii) There exists a projection $p = \lambda\circ\rho$ such that $\mathcal{E} \cong p\mathcal{A}^N$.*

For the proof of Lemma 2.2.10 we refer to [Lan02, §4.2].

**Theorem 2.2.11** (Serre-Swan)**.** *Let $X$ be a compact finite-dimensional Hausdorff space. A $C(X)$-module $\mathcal{E}$ is isomorphic to a module $\Gamma(E,X)$ of continuous sections of a bundle $E \to X$ if and only if it is finite projective.*

*Proof.* We first prove that if a $C(X)$-module $E$ is isomorphic to a module $\Gamma(E,X)$ of continuous sections of a vector bundle $E \to X$ then it is finite projective.

Define $\mathcal{A} := C(X)$. We note that for the trivial bundle $\varepsilon^N \to X$, the set of sections is given by

$$\Gamma(\mathbb{C}^N, X) = \{s : X \to \mathbb{C}^N | s \text{ is a continuous}\} \cong \{(s_1,\ldots,s_N)|\ s_i : X \to \mathbb{C} \text{ is continuous}\} \cong \mathcal{A}^N,$$

which is an $\mathcal{A}$-module. Hence we have that $\Gamma(\varepsilon^N, X)$ is finite projective. Denote $\{e_i\}_{i=1,\cdots N}$ for the standard basis of $\mathbb{C}^N$, and $\phi$ for the homeomorphism $\varepsilon^N \cong X\times\mathbb{C}^N$. Subsequently, we define $\sigma_i(x) := \phi^{-1}(x,e_i) \in \varepsilon^N_x$. The vectors $\sigma_i(x)$ form a basis for $\Gamma(\varepsilon^N, X)$, because $\phi$ is a linear isomorphism on the fibres. Consequently we find that $\Gamma(\varepsilon^N, X)$ is a finite projective free module.

For the vector bundle $E \to X$, we know by Lemma 1.1.14, that there exists a vector bundle, $Q \to X$, such that $E\oplus Q \cong \varepsilon^n$ for a suitable $n$. Moreover, we have that $\Gamma(\varepsilon^n, X) = \Gamma(E\oplus Q, X) = \Gamma(E,X)\oplus\Gamma(Q,X)$. Then by Definition 2.2.9 it follows that $\Gamma(E,X)$ is a finite projective module.

For the other direction, we follow the proof of [Lan02, Prop. 21].
Assume that $\mathcal{E}$ is a finite projective $\mathcal{A}$-module. Then we can find morphisms $\rho$ and $\lambda$ with a suitable $N$, such that $\rho\circ\lambda = \mathrm{id}_{\mathcal{E}}$. Define $p = \lambda\circ\rho \in M_N(\mathcal{A})$. Note that $p^2 = \rho\circ\lambda\circ\rho\circ\lambda = \rho\circ\lambda = p$. Hence, $p$ is an idempotent. This allows to write $\mathcal{A}^n$ in terms of p: $\mathcal{A}^N = p\mathcal{A}^n \oplus (1-p)\mathcal{A}^N$. Consequently, $\rho$ and $\lambda$ are isomorphisms between $\mathcal{E}$ and $p\mathcal{A}^N$.

For $f \in \mathcal{A}$ and $s \in \mathcal{A}^N$ one has that $p(sf) = p(s)f$, since $p$ is a module homomorphism which is $\mathcal{A}$-linear. For $x \in X$ we define the ideal $\mathcal{I}_x = \{f \in \mathcal{A} | f(x) = 0\}$. Notice that $p\mathcal{A}^N\mathcal{I}_x = \mathcal{A}^N\mathcal{I}_x$. Therefore, one finds that $p(s)(x) \in (X\times\mathbb{C}^N)_x$ for all $s \in \mathcal{A}^N$. We define the bundle morphism $\pi$ such that the

[2] Recall that a free module is module with a basis [Haz89, p. 110].

diagram

$$\begin{tikzcd}
X \times \mathbb{C}^N \arrow[rr, "\pi"] \arrow[dr] & & X \times \mathbb{C}^N \arrow[dl] \\
& X \arrow[ul, dashed, bend left, "s"] \arrow[ur, dashed, bend right, "p(s)"'] &
\end{tikzcd}$$

commutes, i.e. we have that $s(x) = p(s)(x)$. The defined map satisfies $\pi \circ s = p(s)$. Furthermore, we have that $p^2 = p$ and thus we have that $\pi^2 = \pi$. Let $\dim \pi\left((X \times \mathbb{C}^N)_x\right) = k$, then there are $k$ linearly independent continuous local sections $s_i \in \mathcal{A}^N$ such that $(\pi \circ s_i)(x) = s_i(x)$ in a small neighbourhood of $x \in X$. Consequently, $\pi \circ s_i$ are $k$ linearly independent morphisms in a neighbourhood $U$ of $x \in X$. Thus for each $y \in U$ we have that $\dim \pi\left((X \times \mathbb{C}^N)_y\right) \geq k$. Analogously, for the bundle morphism $(1-\pi)$ we have for any $y \in X$ that $(1-\pi)\left((X \times \mathbb{C}^N)_y\right) \geq N - k$. Because, $X \times \mathbb{C}^N \cong \pi\left(X \times \mathbb{C}^N\right) \oplus (1-\pi)\left(X \times \mathbb{C}^N\right)$, we have that

$$\dim\left(X \times \mathbb{C}^N\right) = \dim \pi\left(X \times \mathbb{C}^N\right) + \dim(1-\pi)\left(X \times \mathbb{C}^N\right),$$

and thus $\dim \pi\left((X \times \mathbb{C}^N)_y\right)$ is locally constant. Consequently, there is a subvector bundle $E \to X$ such that $\pi(X \times \mathbb{C}^N) = E$. Furthermore, we have that $X \times \mathbb{C}^N = E \oplus \ker \pi$. Finally, we note that

$$\Gamma(E, X) = \left\{\pi \circ s | s \in \Gamma(X \times \mathbb{C}^N, X)\right\} = \left\{p(s) | s \in \mathcal{A}^N\right\} = p\mathcal{A}^N = \mathcal{E},$$

which concludes the proof. □

For a vector bundle $E \to X$ we now have that $\Gamma(E, X) \subset \mathcal{D}(C(X))$. Furthermore, for $p \in \mathcal{D}(C(X))$ we have that $p$ is a $C(X)$-module and therefore is isomorphic to a module of continuous sections of a bundle $E \to X$.

**Corollary 2.2.12.** *The semigroup $\mathcal{D}(C(X))$ is isomorphic to* $\mathrm{Vect}(X)$*, for $X$ a compact Hausdorff space.*

With this notion, one can relate the isomorphism classes of complex vector bundles over $X$, to projections in $M_n(C(X))$. Consequently, we find that:

**Corollary 2.2.13.** *For $X$ a compact Hausdorff space we have that $K(C(X)) \cong K(X)$.*

### 2.2.1 Realisation of Projections

For a principal $G$-bundle $P \xrightarrow{\pi} M$, one can construct such a projection by the using a finite-dimensional representation space (see Definition 1.2.4). Let $\tilde{\pi}\colon E(P,V) \to M$ be the associated vector bundle over $M$ where $\tilde{\pi}\left([p,v]\right) = \pi(p)$. Take $N$ continuous vector valued functions $v_i\colon P \to V$ such that the equivariance condition $v_i(pg^{-1}) = A_g v_i(p)$ holds.

We note that the representation space is a vector space. Therefore, we can introduce the notion of an inner product on the representation space $V$. Let $\langle v, w\rangle$ be the inner product which is linear in the second component. Furthermore, denote $|v_1\rangle\langle v_2|$ for the linear operator given by $|v_1\rangle\langle v_2|(w) = v_1\langle v_2, w\rangle$ on which $G$ may act unitary.

**Lemma 2.2.14.** *If $\sum_{i=1}^N |v_i\rangle\langle v_i| = \mathrm{id}_{E(P,V)}$, then $p = (p_{ij}) = \langle v_i, v_j\rangle$ is a projection in $M_n(C(M))$.*

*Proof.* Take $p = \left(p_{ij}\right) = \langle v_i, v_j\rangle \in M_N(C(P))$. Then we have the following properties:

$$\sum_j \langle v_i, v_j\rangle\langle v_j, v_k\rangle = \sum_j \langle v_i, v_j\langle v_j, v_k\rangle\rangle = \sum_j \langle v_i, (|v_j\rangle\langle v_j|(v_k))\rangle = \langle v_i, v_k\rangle = p_{ik}$$
$$(p_{ij})^* = p^*_{ji} = \langle v_j, v_i\rangle^* = \langle v_i, v_j\rangle = p_{ij}.$$

Hence, we find that $p = p^2 = p^*$. Thus $p$ is a projection.
Define $\varphi\colon C(M) \to C(P)$ by $\varphi(f) = f \circ \pi$. Note that $\varphi$ is an injective map. Hence, we have that $C(M) \subset C(P)$. The space of $G$-invariant elements (by definition of the principal $G$-bundle) of the group action $\alpha_g\colon C(P) \to C(P)$, $\alpha_g(f)(p) = f(pg^{-1})$ is given by $C(M)$. Consequently, we find that

$$\begin{aligned}\alpha_g(p_{ij}) &= \alpha_g(\langle v_i, v_j\rangle\\ &= \langle A_{g^{-1}} v_i, A_{g^{-1}} v_j\rangle\\ &= A_{g^{-1}}\overline{A_{g^{-1}}}\langle v_i, v_j\rangle\\ &= \langle v_i, v_j\rangle = p_{ij}.\end{aligned}$$

Hence, $p_{ij} \in C(M)$. □

The projection obtained by applying Lemma 2.2.14 corresponds to the associated vector bundle of the given principal bundle:

**Proposition 2.2.15.** *For a principal $G$-bundle $P \xrightarrow{\pi} M$ there is an isomorphism $E(P,V) \cong pC(X)^N$, where $N$ equals the number of continuous vector valued functions.*

# 3 The Hopf Fibrations

In this chapter we follow [Jac85] and [Lee13] for the description of division algebras and the stereographic projection. We show that the spheres $S^1, S^2, S^4$ and $S^8$ are diffeomorphic (and thus homeomorphic) to projective lines. These diffeomorphism we will use to construction of the Hopf fibrations. Subsequently, we will prove that the Hopf fibration are principal $G$-bundles. Finally, we follow [Lan00] in order to realise the corresponding matrix-valued projections describing an associated vector bundle.

## 3.1 Division Algebras

**Definition 3.1.1.** A *division algebra* $A$ over $\mathbb{R}$ is an algebra over $\mathbb{R}$, which is also a division ring. i.e. for all $x, y, z \in A$ and $a, b \in \mathbb{R}$ holds that $(x+y)z = xz + yz$; $z(x+y) = xz + yz$; $(ax)(by) = (ab)(xy)$; $xy = 0 \Rightarrow x = 0$ or $y = 0$.

There are four classical examples of $\mathbb{R}$-division algebras;

1. *Real numbers*: $\mathbb{R}$, with the standard addition and multiplication.
2. *Complex numbers*: $\mathbb{C} \cong \mathbb{R}^2$, with the standard multiplication of the complex numbers.
3. *Quaternions*: $\mathbb{H} \cong \mathbb{R}^4 = \{a + bi + cj + dk \colon a, b, c, d \in \mathbb{R}\}$ where $i^2 = j^2 = k^2 = ijk = -1$ and and $k = ij = -ji, j = ki = -ik, i = jk = -kj$
4. *Octonions*[3]: $\mathbb{O} \cong \mathbb{R}^8 = \{a + bi + cj + dk + el + fm + gn + ho \colon a, b, c, d, e, f, g, h \in \mathbb{R}\}$, where $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1$ and the multiplication is given by Figure 3.1.

## 3.2 Projective Spaces and Spheres

In this section, we define the projective space over different division algebras, and discover the relation between projective lines and spheres.

[3] This is also known as the Cayley-Graves algebra.

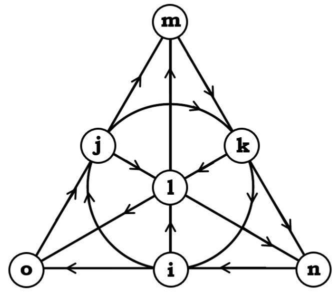



**Figure 3.1:** The multiplication rules for the quaternions and octonions. For example multiplication with the arrow direction: $m \cdot k = n$, and with the anti-arrow direction: $i \cdot n = -o$.

**Definition 3.2.1.** The *n-dimensional projective space*, $\mathbb{P}^n\mathbb{K}$, is given by the quotient space

$$\left(\mathbb{K}^{n+1} - \{0\}\right) / \sim,$$

where two nonzero vectors $x$ and $y$ are called equivalent if and only if $x = \lambda y$ for $\lambda \in \mathbb{K}_{\neq 0}$.

For now, we restrict the discussion the the one-dimensional projective spaces (*projective lines*), denoted by $\mathbb{P}\mathbb{K}$. Such a projective line, can also be seen as the collection of one dimensional lines through the origin of $\mathbb{K}$, i.e. we see the projective line as the set of ratios

$$\mathbb{P}\mathbb{K} = \left\{ \left[x^1, x^2\right] \middle| x^i \in \mathbb{K} \text{ and } (x^1, x^2) \neq 0 \right\},$$

where $= [x^1, x^2] = [\lambda x^1, \lambda x^2]$ for all $\lambda \neq 0$.

**Definition 3.2.2.** The *n-dimensional sphere* is defined as

$$S^n : = \left\{ (x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} \middle| \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}.$$

**Definition 3.2.3.** Let $N$ denote the north pole, i.e. $(0, \cdots, 0, 1) \in S^n$. Then, the *stereographic projection* $\sigma : S^n \backslash N \to \mathbb{R}^n$ is a bijective morphism given by

$$\sigma(x^1, ..., x^{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}};$$
$$\sigma^{-1}(u^1, ..., u^n) = \frac{(2u^1, ..., 2u^n, |u|^2 - 1)}{|u|^2 + 1},$$

for $(x^1, ..., x^{n+1}) \in S^n$ and $(u^1, ..., u^n) \in \mathbb{R}^n$. For $S^n \backslash S$ the stereographic projection is given by $\widetilde{\sigma}(x) = -\sigma(-x)$.

From the definitions 3.2.1,3.2.2 and 3.2.3, we can find a relation between spheres and projective lines.

**Proposition 3.2.4.** *The projective lines are diffeomorphic to spheres,* *(i)* $S^1 \cong \mathbb{PR}$ *(ii)* $S^2 \cong \mathbb{PC}$ *(iii)* $S^4 \cong \mathbb{PH}$ *(iv)* $S^8 \cong \mathbb{PO}$.

*Proof.* We first proof *(iii)*: Denote $N := (0,0,0,0,1)$ for the north pole and $S := (0,0,0,0,-1)$ for the south pole. Take the atlas $\{(U_1,\varphi_1),(U_2,\varphi_2)\}$ on $S^4$, where

$$U_1 = S^4 \setminus N; \qquad \text{with } \varphi_1 \colon (x^1,x^2,x^3,x^4,x^5) \mapsto \frac{(x^1,x^2,x^3,x^4)}{1-x^5},$$
$$U_2 = S^4 \setminus S; \qquad \text{with } \varphi_2 \colon (x^1,x^2,x^3,x^4,x^5) \mapsto \frac{(x^1,x^2,x^3,x^4)}{1+x^5}.$$

Define the atlas on $\mathbb{PH}$ as: $\{(V_1,\phi_1),(V_2,\phi_2)\}$, where

$$V_1 = \{[q^1,q^2] \in \mathbb{PH} \colon q^2 \neq 0\}; \qquad \text{with } \phi_1 \colon [q^1,q^2] \mapsto \frac{q^1}{q^2}$$
$$V_2 = \{[q^1,q^2] \in \mathbb{PH} \colon q^1 \neq 0\}; \qquad \text{with } \phi_2 \colon [q^1,q^2] \mapsto \frac{q^2}{q^1}.$$

We will construct a morphism between $S^4$ and $\mathbb{PH}$. For $(x^1,x^2,x^3,x^4,x^5) \in S^4 \setminus N$ and for $(x^1,x^2,x^3,x^4,x^5) \in S^4 \setminus S$, we define the morphisms

$$\begin{aligned} F : S^4\setminus N \to \mathbb{PH}; \qquad F(x^1,x^2,x^3,x^4,x^5) &= [\sigma(x^1,x^2,x^3,x^4,x^5),1] \\ &= [x^1 + ix^2 + jx^3 + kx^4, 1-x^5]; \\ F : S^4\setminus S \to \mathbb{PH}; \qquad F(x^1,x^2,x^3,x^4,x^5) &= [1, \overline{(\widetilde{\sigma}(x^1,x^2,x^3,x^4,x^5))}] \\ &= [1+x^5, x^1 - ix^2 - jx^3 - kx^4]. \end{aligned} \tag{3.1}$$

Note that on the intersection $U_1 \cap U_2$, the image of $F$ yields the same line in $\mathbb{PH}$ for both morphisms $F$. The inverse of the map $F$ can be constructed using the inverse of the stereographic projection. For $[q,1] \in V_1$ define

$$F^{-1}([q,1]) = \sigma^{-1}(q) = \frac{(|2q|, |q|^2 - 1)}{|q|^2+1},$$

and for $[1,q] \in V_2$ define the inverse by

$$F^{-1}([1,q]) = \sigma^{-1}(q) = \frac{(|2q|, 1-|q|^2)}{|q|^2+1},$$

where $|q| = \sqrt{a^2+b^2+c^2+c^2}$ for $q = a+bi+jc+kd \in \mathbb{H}$. From Definition 3.2.3 it follows that $F^{-1}$ is the inverse of $F$.

Left to show is that the morphisms $F$ and $F^{-1}$ are smooth. We write $F$ and in local coordinates, i.e. we look at $\hat{F}$.

$$\begin{tikzcd} S^4 \arrow[r, "F"] \arrow[d, "\varphi_1{,}\varphi_2"'] & \mathbb{PH} \arrow[d, "\phi_1{,}\phi_1"] \\ \mathbb{R}^4 \arrow[r, "\hat{F}"] & \mathbb{R}^4 \end{tikzcd}$$

Because, we are working over two charts on each space we have four possible combinations of charts. For convenience we denote $(x^1,x^2,x^3,x^4) = x$ and $x^5 = y$, hence $(x^1,x^2,x^3,x^4,x^5) \in S^4$ becomes $(x,y)$.

- For $(x, y) \in U_1$ and $F(x, y) \in V_1$, we have that $\varphi_1(x, y) = \frac{x}{1-y}$ and $(\phi_1 \circ F)(x, y) = \frac{x}{1-y}$. Thus $\hat{F}\left(\frac{x}{1-y}\right) = \frac{x}{1-y}$ which is smooth because it is the identity on $\mathbb{R}^4$.
- For $(x, y) \in U_1$ and $F(x, y) \in V_2$, we have that $\varphi_1(x, y) = \frac{x}{1-y}$ and $(\phi_2 \circ F)(x, y) = \frac{1-y}{x}$. Thus $\hat{F}\left(\frac{x}{1-y}\right) = \frac{1-y}{x}$ which is smooth iff $x \neq 0$. Note that we have that $x \neq 0$ because $F(x, y) \in V_2$
- For $(x, y) \in U_2$ and $F(x, y) \in V_1$, we have that $\varphi_2(x, y) = \frac{x}{1+y}$ and $(\phi_1 \circ F)(x, y) = \frac{1+y}{\overline{x}}$. Thus $\hat{F}\left(\frac{x}{1+y}\right) = \frac{1+y}{\overline{x}}$ which is smooth iff $\overline{x} \neq 0$. Note that we have that $\overline{x} \neq 0$ because $F(x, y) \in V_1$.
- For $(x, y) \in U_2$ and $F(x, y) \in V_2$, we have that $\varphi_2(x, y) = \frac{x}{1+y}$ and $(\phi_2 \circ F)(x, y) = \frac{\overline{x}}{1+y}$. Thus $\hat{F}\left(\frac{x}{1+y}\right) = \frac{\overline{x}}{1+y}$ which is smooth because it is conjugation on $\mathbb{R}^4$.

Hence, we find that the coordinate representation is smooth. So, $F$ is smooth. Analogously, we find that $F^{-1}$ is smooth. Consequently, the morphism $F\colon S^4 \to \mathbb{PH}$, is a diffeomorphism.

To prove *(i)* and *(ii)*, we use the morphism of (3.1). For *(i)* let $x^2 = x^3 = x^4 = 0$ and let $y^1 = x^1$ and $y^3 = x^5$, and for *(ii)* let $x^3 = x^4 = 0$ and let $y^1 = x^1$, $y^2 = x^2$ and $y^3 = x^5$. Then we find the following morphisms

$$F\colon S^1 \to \mathbb{PR}; \qquad F\colon \quad (y^1, y^2) \quad = (x^1, 0\,, 0\,, 0\,, x^5) \mapsto \begin{cases} [x^1, 1 - x^5] & \text{if } (y^1, y^2) \in S^1 \backslash N \\ [1 + x^5, x^1] & \text{if } (y^1, y^2) \in S^1 \backslash S, \end{cases}$$

$$F\colon S^2 \to \mathbb{PC}; \qquad F\colon (y^1, y^2, y^3) = (x^1, x^2, 0, 0, x^5) \mapsto \begin{cases} [x^1 + i x^2, 1 - x^5] & \text{if } (y^1, y^2, y^3) \in S^2 \backslash N \\ [1 + x^5, x^1 - i x^2] & \text{if } (y^1, y^2, y^3) \in S^2 \backslash S. \end{cases}$$

The proof of $(iv)$ is much more subtle, due to the fact that $\mathbb{O}$ is noncommutative and nonassociative. For details of this proof we refer to [Bae02]. □

Note that diffeomorphic implicates homeomorphic, hence the spheres are also homeomorphic to the projective lines.

## 3.3 Construction of the Hopf Fibrations

The Hopf fibrations are specific cases of fibre bundles, where the total space, the base space and the fibres are spheres. There are four Hopf fibrations:

$$\begin{aligned} S^0 \hookrightarrow S^1 &\xrightarrow{\pi_0} S^1 \cong \mathbb{PR} \\ S^1 \hookrightarrow S^3 &\xrightarrow{\pi_1} S^2 \cong \mathbb{PC} \\ S^3 \hookrightarrow S^7 &\xrightarrow{\pi_3} S^4 \cong \mathbb{PH} \\ S^7 \hookrightarrow S^{15} &\xrightarrow{\pi_7} S^8 \cong \mathbb{PO} \end{aligned} \tag{3.2}$$

Here, $\pi_i$ denotes the surjective map and $S^i$ the fibre of each point. With the identification of Proposition 3.2.4, the Hopf fibrations are given by $\widetilde{\pi_i}\colon S^i \to \mathbb{PK}$ given by $\widetilde{\pi_i}(x, y) = [x, y] \in \mathbb{PK}$.

**Remark 3.3.1.** For $i = 1,3,7,15$ we have that $S^i = S(\mathbb{K}^2) = \{(k,l) \in \mathbb{K}^2 \colon (k)^2 + (l)^2 = 1\}$.

The Hopf fibrations (of (3.2)) are early examples of non-trivial fibre bundles. These fibrations can be written in a general form:

$$S^n \hookrightarrow S^{2n+1} \xrightarrow{\pi_n} S^{n+1} \cong \mathbb{PK},$$

where $n = 0,1,3,7$.

With this general description, we can show that the Hopf fibrations are fibre bundles. For this, we use that $S^n$ is also a manifold, such that we may work with atlases.

**Proposition 3.3.2.** *The Hopf fibrations are fibre bundles.*

*Proof.* It is clear that $\pi_n$ is surjective. Then we can look at the fibres of the points $x = [x^1, x^2] \in \mathbb{PK}$. We have that

$$\pi_n^{-1}(x) = \{y \in S^{2n+1} \,|\, \text{where } (\lambda x^1, \lambda x^2) = y \text{ and } |\lambda x^1|^2 + |\lambda x^2|^2 = 1 \text{ for } \lambda \in \mathbb{K}_{\neq 0}\}.$$

Without loss of generality, we may assume that $|x^1|^2 + |x^2|^2 = 1$. Hence, we find that the fibres are all spheres; $\pi_n^{-1}(x) \cong \{\lambda \in \mathbb{K}_{\neq 0} \,|\, |\lambda|^2 = 1\} \cong S^n$. We also have the following local triviality condition for any $x \in S^{n+1}$ we have that $x \in U_i$, where $i = 1,2$ which are the opens of the chosen atlas. Then for $U_i \subset S^{n+1}$ we find the diagram

$$\begin{tikzcd} \pi_n^{-1}(U_i) \arrow[rr, "\phi"] \arrow[dr, "\pi_n"'] & & U_i \times S^n \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

where $\pi_n^{-1}(U_i) = \bigcup_{u \in U_i} \pi_n^{-1}(u) \cong \bigsqcup_{u \in U_i} S^n$. Hence, we have the diffeomorphism $\phi \colon \pi_n^{-1}(U_i) \to U_i \times S^n$. □

A more convenient way of representing these Hopf fibrations is in the coordinates of the domain. This is shown in the following examples.

**Example 3.3.3.** The first Hopf fibration $S^1 \xrightarrow{\pi_0} S^1 \cong \mathbb{PR}$, can be explicitly given, by using Proposition 3.2.4. We have that $\pi_0(x,y) = F^{-1}(\widetilde{\pi_0}(x,y)) = F^{-1}([x,y])$. Consequently, we find that the first Hopf fibration is given by

$$\pi_0 \colon S^1 \to S^1; \qquad \pi_0(x,y) = \frac{1}{\frac{y^2}{x^2}+1}\left(2\frac{y}{x}, \frac{y^2}{x^2} - 1\right) = \left(2xy, |y|^2 - |x|^2\right).$$

△

**Example 3.3.4.** The second Hopf fibration $S^3 \xrightarrow{\pi_1} S^2 \cong \mathbb{PC}$ is given by the map $\pi_1(\alpha,\beta) = F^{-1}(\widetilde{\pi_1}(\alpha,\beta)) = F^{-1}([\alpha,\beta])$ and of Proposition 3.2.4. Using the construction in the proof of Proposition 3.2.4 we find the Hopf fibration

$$\pi_1 \colon S^3 \to S^2; \qquad \pi_1(\alpha,\beta) = \frac{1}{\left|\frac{\beta}{\alpha}\right|^2 + 1}\left(2\Re\left(\frac{\beta}{\alpha}\right), 2\Im\left(\frac{\beta}{\alpha}\right), \left|\frac{\beta}{\alpha}\right|^2 - 1\right) = \left(2\alpha\overline{\beta}, |\beta|^2 - |\alpha|^2\right).$$

If we now set $\pi_1(\alpha,\beta) = (x,y,z)$, we find that $x = \alpha\overline{\beta} + \beta\overline{\alpha}$; $y = i\alpha\overline{\beta} - i\beta\overline{\alpha}$ and $z = |\beta|^2 - |\alpha|^2$. △

**Example 3.3.5.** The third Hopf fibration $S^7 \xrightarrow{\pi_3} S^4 \cong \mathbb{PH}$, is constructed in a similar manner as in Example 3.3.4. Using Proposition 3.2.4, we find that $\pi_3(p,q) = F^{-1}(\widetilde{\pi_3}(p,q)) = F^{-1}([p,q])$ Then, one finds that the map is explicitly given by

$$\pi_3\colon S^7 \xrightarrow{\pi_3} S^4; \qquad \pi_3(p,q) = \frac{1}{\left|\frac{q}{p}\right|^2+1}\left(2\left(\frac{q}{p}\right), \left|\frac{q}{p}\right|^2 - 1\right) = \left(2p\overline{q}, |q|^2 - |p|^2\right).$$

Where $|q| = \sqrt{a^2+b^2+c^2+c^2}$ and $\overline{q} = a - bi - cj - dk$ for $q = a + bi + cj + dk \in \mathbb{H}$. If we set $\pi_3(p,q) = (x^1, x^2, x^3, x^4, x^5)$, we find that

$$\begin{aligned} x^1 &= p\overline{q} + q\overline{p} \\ ix^2 + jx^3 + kx^4 &= p\overline{q} - q\overline{p} \\ x^5 &= |q|^2 - |p|^2. \end{aligned}$$

△

With the description of the Hopf projections as given above we can prove that the first three of these fibrations are also principal $G$-bundles.

**Lemma 3.3.6.** *The first three Hopf fibrations of* (3.2)*, are principal $G$-bundles, where the fibres are Lie groups:*

- $S^0 \cong U(\mathbb{R}) \cong \mathbb{Z}_2$,
- $S^1 \cong U(\mathbb{C}) \cong U(1)$,
- $S^3 \cong U(\mathbb{H}) \cong SU(2)$.

*Proof.* We only need to show that the surjective morphism $\pi_i$ and the diffeomorphism

$$\phi\colon \pi_i^{-1}(U) \xrightarrow{\sim} U \times S^i,$$

where $U \subseteq S^{i+1}$, are $G$-invariant, for $i = 0, 1, 3$.

Let $g \in S^i$ and $(p,q) \in S^{2i+1}$. Then

$$\begin{aligned} \pi_i(g(p,q)) &= \pi(gp, gq) \\ &= \left(2gp\overline{gq}, |gq|^2 - |gp|^2\right) \\ &= \left(|g|^2 p\overline{q}, |g|^2(|q|^2 - |p|^2)\right) \\ &= \left(p\overline{q}, |q|^2 - |p|^2\right) \\ &= \pi_i(p,q), \end{aligned}$$

where we used that $g\overline{g} = |g|^2 = 1$. Hence, the surjective map $\pi_i$ is $G$-invariant.

To show that the diffeomorphism is $G$-invariant we use the atlas of Example 3.3.2. Then, by the above we already have the inclusion $\pi_i^{-1}(U_j) \supset U_j \times S^i$ for $j = 1, 2$.

For the other inclusion, assume that $\pi_i(p,q) = \pi_i(p',q')$. Then we have that

$$p\overline{q} = p'\overline{q'}, \tag{3.3}$$

$$|q|^2 - |p|^2 = |q'|^2 - |p|^2, \tag{3.4}$$

$$|p|^2 + |q|^2 = |p'|^2 + |q'|^2 = 1. \tag{3.5}$$

Hence, by (3.4) and (3.5) we find that $|q|^2 = |q'|^2$. Furthermore, we may assume without loss of generality that $p \neq 0$. Define $g := (p)^{-1}q$, then (3.3) translates to $q = \overline{g}q'$. Taking the norm gives $|q|^2 = |\overline{g}q'|^2 = |g|^2|q'|^2$. Now, there are two options:

1. Suppose $q = 0$, then $q^2 = 0$ and $|p| = |p'| = 1$ and thus $|g| = 1$.
2. Suppose $q \neq 0$, then $|g| = 1$.

Conclusively, we have that $g^{-1} = \overline{g}$ thus $g \in S^i$ and $g(p,q) = (p', q')$. Hence, we have that $\pi_i^{-1}(U_j) \subset U_j \times S^i$. Hence, both inclusions are $G$-invariant. □

Note that Lemma 3.3.6 also implicates that the Hopf fibrations are principal $G$-bundles in a topological setting.

**Remark 3.3.7.** The sphere $S^7$ is not a topological group (see [JS88]), hence the fourth Hopf fibration is not a principal $G$-bundle, in the sense of definition 1.2.1. However, we note that $S^7$ hast the structure of a H-space, which we will cover in Chapter 5.

## 3.4 Projections from the Hopf Fibrations

Following Section 1.2, we can form an associated vector bundle to a principal $G$-bundle.

**Example 3.4.1.** We construct the vector bundle associated to the second Hopf fibration: $S^1 \hookrightarrow S^3 \xrightarrow{\pi_1} S^2$. The representations space of $S^1$ is the vector space $C$, where we have the representation

$$A\colon S^1 \to GL(\mathbb{C}); \qquad A\colon \cos x + i\sin x \mapsto e^{-ix}. \tag{3.6}$$

This means that the associated vector bundle is given by $(S^3 \times \mathbb{C})/S^1 = E \xrightarrow{\pi_{as}} S^2$, where $\pi_{as}([p,v]) = \pi_1(p)$. For each $p \in S^2$ we have that the fibre is given by $E_p = \left\{[u,v] \middle| u \in S^3_p \text{ and } v \in \mathbb{C}\right\} \cong \mathbb{C}$. Hence, the associated vector bundle is a vector bundle of rank $\dim \mathbb{C} = 1$. △

The associated vector bundle of the second Hopf fibration is a well-known vector bundle:

**Proposition 3.4.2.** *The associated vector bundle of the Hopf fibration $\pi_1$ is in correspondence with the tautological line bundle (Example 1.1.7).*

*Proof.* To prove this statement, we use the identification $S^2 \cong \mathbb{PC}$ and the fact that $S^2$ is a manifold. Then the associated vector bundle given by $(S^3 \times \mathbb{C})/S^1 = E \xrightarrow{\pi_{as}} \mathbb{PC}$. It is enough to show that the diagram

$$\begin{tikzcd}
E = (S^3 \times \mathbb{C})/S^1 \arrow[rr, "F"] \arrow[dr, "\pi_{as}"'] & & \left\{(\ell, v) \middle| \ell \in \mathbb{PC}, v \in \ell\right\} = \mathcal{O}(-1) \arrow[dl, "\pi_{\mathcal{O}}"] \\
& \mathbb{PC} &
\end{tikzcd}$$

commutes for $F$ a diffeomorphism given by $F(\alpha,\beta,\lambda) = \left(\pi_1(\alpha,\beta), \lambda(\alpha,\beta)\right) = \left([\alpha,\beta], \lambda(\alpha,\beta)\right)$, and that $\pi_{as}^{-1}(p) \to \pi_{\mathcal{O}}^{-1}(p)$ is linear for all $p \in \mathbb{PC}$.

Pick $(\alpha,\beta,\lambda)\in(S^3\times\mathbb{C})/S^1$ then $\pi_{as}(\alpha,\beta,\lambda)=\pi_1(\alpha,\beta)=[\alpha,\beta]$ and $(\pi_{\mathcal{O}}\circ F)(\alpha,\beta,\lambda)=\pi_{\mathcal{O}}(\pi_1(\alpha,\beta),\lambda)=[\alpha,\beta]$. Hence, the diagram commutes. Furthermore, we have that $\pi_{as}^{-1}\left([\alpha,\beta]\right)=\pi_1^{-1}\left([\alpha,\beta]\right)\times\mathbb{C}$ and $\pi_{\mathcal{O}}^{-1}\left([\alpha,\beta]\right)=\left\{([\alpha,\beta],v)\,|\text{ for } v=\lambda(\alpha,\beta),\lambda\in\mathbb{C}\right\}$. Hence, the mapping $\pi_{as}^{-1}\left([\alpha,\beta]\right)\to\pi_{\mathcal{O}}^{-1}\left([\alpha,\beta]\right)$ is given by $\left([\alpha,\beta],\lambda\right)\mapsto\left([\alpha,\beta],\lambda[\alpha,\beta]\right)$, which is linear.

Notice, that the morphism $F$ is $S^1$-invariant: Let $(\cos x+i\sin x)\in S^1$, then we have that

$$F\big((\alpha,\beta,\lambda)(e^{ix})\big)=F\Big(\alpha e^{ix},\beta e^{ix},\lambda e^{-ix}\Big)=\big([\alpha,\beta],\lambda[\alpha,\beta]\big)$$

Left to check is that $F$ is a diffeomorphism. First, we notice that we can write each line $\ell\in\mathbb{PC}$ as $\ell=[1,\alpha]$ uniquely, for all lines not parallel with the y-axis. Then the inverse of $F$ for a point in $\mathcal{O}(-1)$ is given by: $F^{-1}\left([1,\alpha],z^1,z^2\right)=\left(\pi_1^{-1}\left([1,\alpha]\right),z^1\right)$. In a similar manner if $\ell$ is parallel with the y-axis we find that the inverse is given by $F^{-1}\left([0,1],z^1,z^2\right)=\left(\pi_1^{-1}([0,1]),z^2\right)$. Because, $F\circ F^{-1}=\mathrm{id}$ and $F^{-1}\circ F=\mathrm{id}$ the morphism $F$ is indeed a bijection on this set.

Secondly, we recall that the mapping $\mathrm{pr}:S^3\times\mathbb{C}\to(S^3\times\mathbb{C})/S^1$ is an open projection. Hence, the atlas on $(S^3\times\mathbb{C})/S^1$ can be constructed from an atlas on $S^3\times\mathbb{C}$. In order to construct the atlas we express, all coordinates in $\mathbb{R}^n$ for a suited $n\in\mathbb{N}$. Take the atlas $\{U_1,\varphi_1).(U_2,\varphi_2)\}$ on $S^3\times\mathbb{C}$, where

$$\begin{aligned}
&U_1=(S^3\backslash(0,0,0,1))\times\mathbb{C}; &&\text{with } \varphi_1:(x^1,x^2,x^3,x^4,x^5,x^6)\mapsto\left(\tfrac{x^1}{1-x^4},\tfrac{x^2}{1-x^4},\tfrac{x^3}{1-x^4},x^5,x^6\right)\\
&U_2=(S^3\backslash(0,0,0,-1))\times\mathbb{C}; &&\text{with } \varphi_2:(x^1,x^2,x^3,x^4,x^5,x^6)\mapsto\left(\tfrac{x^1}{1+x^4},\tfrac{x^2}{1+x^4},\tfrac{x^3}{1+x^4},x^5,x^6\right).
\end{aligned}$$

Then, the atlas on $(S^3\times\mathbb{C})/S^1$ is just the restriction to the projection. For the atlas on $\mathcal{O}(-1)$, we will use a similar construction. First we define an atlas on $\mathbb{PC}\times\mathbb{C}^2$, where

$$\begin{aligned}
&V_1=\left\{[\alpha^1,\alpha^2]\in\mathbb{PC}:\,\alpha^2\neq0\right\}\times\mathbb{C}^2; &&\text{with } \phi_1:([\alpha^1,\alpha^2],\alpha^3,\alpha^4)\mapsto\left(\frac{\alpha^1}{\alpha^2},\alpha^3,\alpha^4\right)\\
&V_2=\left\{[\alpha^1,\alpha^2]\in\mathbb{PC}:\,\alpha^1\neq0\right\}\times\mathbb{C}^2; &&\text{with } \phi_2:([\alpha^1,\alpha^2],\alpha^3,\alpha^4)\mapsto\left(\frac{\alpha^2}{\alpha^1},\alpha^3,\alpha^4\right).
\end{aligned}$$

for $i=1,2,3,4$. With these atlases we can give the coordinate representation of $F$ for the different charts: Take $x=(x^1,...,x^6)\in(S^3\times\mathbb{C})/S^1$, then If $x\in U_1$ and $F(x)\in V_1$. Then for $(u,v)\in\varphi_1(U_1)$ holds that

$$\begin{aligned}
\hat{F}(u,v)&=(\psi_1\circ F\circ\varphi_1)(u^1,u^2,u^3,v^4,v^5)\\
&=\big(\psi_1\circ F\big)\Big(\tfrac{2u^1}{|u|^2+1},\tfrac{2u^2}{|u|^2+1},\tfrac{2u^3}{|u|^2+1},|u|^2-1,v^4,v^5\Big)\\
&=\psi_1\Big(\big[2\tfrac{u^1+iu^2}{|u|^2+1},\tfrac{2u^3}{|u|^2+1}+i(|u|^2-1)\big],\big(v^4+iv^5\big)\big(2\tfrac{u^1+iu^2}{|u|^2+1},\tfrac{2u^3}{|u|^2+1}+i(|u|^2-1)\big)\Big)\\
&=\Big(\frac{2u^1+iu^2}{2u^3+i(|u|^4-1)},\big(v^4+iv^5\big)\big(2\tfrac{u^1+iu^2}{|u|^2+1},\tfrac{2u^3}{|u|^2+1}+i|u|^2-1\big)\Big).
\end{aligned}$$

Notice, that $2u^3+i(|u|^4-1)\neq0$. Hence, the coordinate representation is smooth and thus continuous. The calculation of the other coordinate representations of $F$ and $F^{-1}$ are found similarly. □

For the Hopf fibrations, which are principal bundles, we can construct projections as described in Section 2.2.1. Here we construct the projection for the second and third Hopf fibration.

**Example 3.4.3.** We construct a projection from the second Hopf fibration: $S^1 \hookrightarrow S^3 \xrightarrow{\pi_1} S^2 \cong \mathbb{P}\mathbb{C}$. A representation space of $G = S^1$ is the vector space (with inner product) $V = \mathbb{C}$. Now we construct the continuous vector valued functions such that the equivariance condition is satisfied (i.e. $G$-invariant) and $\sum_{i=1}^{N} |v_i\rangle\langle v_i| = \mathrm{id}_V$. Note that we have the bijection of groups: $S^1 \cong U(\mathbb{C})$. Hence, the representation $A\colon U(\mathbb{C}) \to \mathrm{GL}(\mathbb{C})$ is the left multiplication by $x \in U(1)$ [Lan00]. Define the maps

$$v_1\colon S^3 \to \mathbb{C}; \qquad (\alpha, \beta) \mapsto \alpha$$
$$v_2\colon S^3 \to \mathbb{C}; \qquad (\alpha, \beta) \mapsto \beta,$$

then $v = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is a map to $\mathbb{C}^2$. With this definition the equivariance condition and $\sum_{i=1}^{2} |v_i\rangle\langle v_i| = \mathrm{id}_V$ are satisfied. Furthermore, we have that

$$\mathbf{p} = (\langle v_i, v_j\rangle) = \begin{pmatrix} v_1 v_1^* & v_1 v_2^* \\ v_2 v_1^* & v_2 v_1^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha\overline{\alpha} & \alpha\overline{\beta} \\ \beta\overline{\alpha} & \beta\overline{\beta} \end{pmatrix},$$

which means that the projection $\mathbf{p}$ forms a projection in $\mathbb{C}^2$ on the linear one-dimensional subspace spanned by the vector $v$.

Using $\pi_1$ of Example 3.3.4 we find that

$$\alpha\overline{\alpha} = |\alpha|^2 = \tfrac{1}{2}(1+z); \qquad \beta\overline{\beta} = |\beta|^2 = \tfrac{1}{2}(1-z);$$
$$\alpha\overline{\beta} = \tfrac{1}{2}(x - iy); \qquad \beta\overline{\alpha} = \tfrac{1}{2}(x+iy).$$

Hence, one can write $p$ in terms of coordinates in $S^2$:

$$\mathbf{p} = \frac{1}{2}\begin{pmatrix} \mathbf{1}+z & x-iy \\ x+iy & \mathbf{1}-z \end{pmatrix} = \tfrac{1}{2}(1 + x\sigma_x + y\sigma_y + z\sigma_z), \tag{3.7}$$

where $\sigma_{x,y,z}$ denote the Pauli matrices:

$$\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right); \qquad \sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right); \qquad \sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

Note that we also have the relations $\mathbf{p} = \mathbf{p}^2 = \mathbf{p}^*$ and $\mathrm{trace}(\mathbf{p}) = 1$. Hence, we have constructed a projection $\mathbf{p} \in M_2(C(S^2))$ of rank 1 over $\mathbb{C}$.

Furthermore, we note that $\mathbf{p}$ maps $S^2$ to $\ell \in \mathbb{C}^2$, determined by the point of $S^2 \cong \mathbb{P}\mathbb{C}$. We observe that for some basis vectors $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ we have that

$$\mathbf{p}e_1 = \frac{1}{2}\begin{pmatrix} \mathbf{1}+z & x-iy \\ x+iy & \mathbf{1}-z \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+z \\ x+iy \end{pmatrix}; \qquad \mathbf{p}e_2 = \frac{1}{2}\begin{pmatrix} \mathbf{1}+z & x-iy \\ x+iy & \mathbf{1}-z \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix} x-iy \\ 1-z \end{pmatrix}$$

Because, $x, y, z \in S^2$, we have that the following equality

$$(1+z)(1-z) = 1 - z^2 = x^2 + y^2 = (x+iy)(x-iy) \tag{3.8}$$

holds. Dividing equation (3.8) by $(1+z)(x-iy)$ we find that

$$\frac{1-z}{x-iy} = \frac{x+iy}{1+z}.$$

From this we can conclude that $\mathbf{p}e_1$ and $\mathbf{p}e_2$ lie on the same line in $\mathbb{C}^2$. Denote this line by

$$\ell_{\mathbf{p}} :\ = [x + iy, 1 + z].$$

Consequently, we find that the range of the projection $p$ is a linear combination $\mathbf{p}e_1$ and $\mathbf{p}e_2$, which is the same as stating range($\mathbf{p}$) = $\ell_{\mathbf{p}}$. △

From this, we may conclude that the projection $\mathbf{p}$ coincides with the tautological line bundle. Note that this is compatible with Proposition 3.4.2. Thus the constructed projection $\mathbf{p}$ uniquely determines the sections of the tautological line bundle as a result of the *Serre-Swan Theorem* (Theorem 2.2.11).

**Example 3.4.4.** We construct a projection in $M_4(C(S^4))$, from the third Hopf fibration: $S^3 \hookrightarrow S^7 \xrightarrow{\pi_3} S^4 \cong \mathbb{P}\mathbb{H}$. A representation space of $G = S^3$ is the vector space (with inner product) $V = \mathbb{H}$. Furthermore, we have that $S^3 \cong U(\mathbb{H}) \cong SU(2)$. Hence, the representation $A\colon SU(2) \to \mathrm{GL}(\mathbb{H})$ is the left multiplication by $x \in SU(2)$ [Lan00]. Define the maps

$$\begin{aligned} w_1 &: S^7 \to \mathbb{H}; \qquad && (p,q) \mapsto p \\ w_2 &: S^7 \to \mathbb{H}; \qquad && (p,q) \mapsto q, \end{aligned}$$

which satisfy the equivariance condition and $\sum_{i=1}^{2} |w_i\rangle\langle w_i| = \mathrm{id}$. In a similar manner as in Example 3.4.3, we find that

$$\mathbf{q} = (\langle w_i, w_j \rangle) = \begin{pmatrix} p\overline{p} & p\overline{q} \\ q\overline{p} & q\overline{q} \end{pmatrix}.$$

Now we can use Example 3.3.5 to give $\mathbf{q}$ in coordinates in $S^4$

$$\begin{aligned} p\overline{p} &= |p|^2 = \tfrac{1}{2}(1 + x^5) \\ p\overline{q} &= \tfrac{1}{2}(x^1 + ix^2 + jx^3 + kx^4) = \tfrac{1}{2}\begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -(x^3 - ix^4) & x^1 - ix^2 \end{pmatrix} \\ q\overline{q} &= |q|^2 = \tfrac{1}{2}(1 - x^5) \\ q\overline{p} &= \tfrac{1}{2}(x^1 - ix^2 - jx^3 - kx^4) = \tfrac{1}{2}\begin{pmatrix} x^1 - ix^2 & -(x^3 + ix^4) \\ x^3 - ix^4 & x^1 + ix^2 \end{pmatrix} \end{aligned}$$

Using the expressions for $p\overline{p}$, $p\overline{q}$, $q\overline{q}$ and $q\overline{p}$ we can write $\mathbf{q}$ in terms of the coordinates $(x^1, x^2, x^3, x^4, x^5)$:

$$\mathbf{q} = \frac{1}{2}\begin{pmatrix} 1 + x^5 & 0 & x^1 + ix^2 & x^3 + ix^4 \\ 0 & 1 + x^5 & -(x^3 - ix^4) & x^1 - ix^2 \\ x^1 - ix^2 & -(x^3 + ix^4) & 1 - x^5 & 0 \\ x^3 - ix^4 & x^1 + ix^2 & 0 & 1 - x^5 \end{pmatrix}.$$

Note that $\mathbf{q}$ satisfies $\mathbf{q} = \mathbf{q}^2 = \mathbf{q}^*$. Hence, $\mathbf{q}$ is a projection in $M_4(C(S^4))$. △

**Remark 3.4.5.** The projection constructed in Example 3.4.4, is constructed via the left multiplication. However, $\mathbb{H}$ is not commutative. Hence, one can also construct the projection via the right multiplication, which gives rise to a similar, but different, projection.

# 4 Bott Periodicity

Bott Periodicity was discovered by R. Bott in 1959 [Bot59], in 1968 the connection with K-theory was made by M.F. Atiyah [Ati68]. In K-theory, the Bott Periodicty plays an important role, especially when considering spheres. In this chapter we prove the Bott Periodicity Theorem following [Sty13, Ch. 1],[RLL00] and [Hat17, §2.1-2.2] with respect to the projection of Example 3.7.

## 4.1 The Product Theorem and the Bott Generator

We observe in Example 3.7 the following relation:

**Lemma 4.1.1.** *For* $\mathbf{p}$ *as defined in* (3.7)*, in* $K(C(S^2))$ *we have that*

$$\mathbf{p} \oplus \mathbf{p} \sim_h \mathbf{p} \otimes \mathbf{p} \oplus \mathbf{1}.$$

In order to prove Lemma 4.1.1, we introduce the notion of the *Khatri-Rao product* [LT08]. Let $A$ and $B$ be matrices. Denote the partitions of the matrices as $A = (A_{ij})$ and $B = (B_{ij})$, where $A_{ij}$ and $B_{ij}$ are blocks of size $n_i \times m_j$. Then the Khatri-Rao product of matrices $A$ and $B$ is given by

$$A \odot B = \left(A_{ij} \otimes B_{ij}\right)_{ij},$$

of size $\left(\sum (n_i)^2\right)\left(\sum (m_j)^2\right)$. This product can be seen as the block-wise tensor product of matrices.

**Example 4.1.2.** The Khatri-Rao product of the $4 \times 4$ matrix $A = (a_{ij})$ with the $4 \times 4$ matrix $B = (b_{ij})$ for blocks of size $2 \times 2$ is given by

$$\left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right) \odot \left(\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right) = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \otimes \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & a_{24} \end{pmatrix} \\ \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \otimes \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & a_{42} \end{pmatrix} & \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix} \otimes \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & a_{44} \end{pmatrix} \end{pmatrix}.$$

△

*Proof of Lemma 4.1.1.* We introduce a family of invertible matrices $A_t$ that forms a continuous path in $\mathrm{GL}_4(\mathbb{C})$, which is given by

$$A_t = \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2}t & 0 & -\sin\frac{\pi}{2}t & 0 \\ 0 & \cos\frac{\pi}{2}t & 0 & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & 0 & \cos\frac{\pi}{2}t & 0 \\ 0 & \sin\frac{\pi}{2}t & 0 & \cos\frac{\pi}{2}t \end{pmatrix},$$

for $t \in [0,1]$. Then for $t = 0$, we find that $A_0 = \mathbb{1}_4$ and for $t = 1$ we find that $A_1 = \left(\begin{smallmatrix} 0_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0_2 \end{smallmatrix}\right)$. Then the mapping

$$P_t = \begin{pmatrix} \mathbf{p} & 0_2 \\ 0_2 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \odot A_t \begin{pmatrix} \mathbf{p} & 0_2 \\ 0_2 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} A_t^{-1},$$

forms a homotopy with $\odot$ the Khatri-Rao product for 2 by 2 matrices. We note that this is indeed a continuous function, because it is the composition of continuous functions. Subsequently, we find that

$$\begin{aligned} P_0 &= \begin{pmatrix} \mathbf{p}\otimes\mathbf{p} & 0_4 \\ 0_4 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} = \mathbf{p}\otimes\mathbf{p}\oplus 1 \oplus 0_3 \sim \mathbf{p}\otimes\mathbf{p}\oplus 1 \\ P_1 &= \begin{pmatrix} \mathbf{p}\otimes\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & 0_4 \\ 0_4 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes \mathbf{p} \end{pmatrix} = \mathbf{p}\oplus 0_2 \oplus \mathbf{p} \oplus 0_2 \sim \mathbf{p}\oplus\mathbf{p}. \end{aligned}$$ □

As a result we find that $\operatorname{rk}([\mathbf{p}\otimes\ \mathbf{p}\oplus\mathbf{1}] - [\mathbf{p}\oplus\mathbf{p}]) = 0$, where rk is defined in Definition 2.1.9. Consequently, we have that $[\mathbf{p}\otimes\ \mathbf{p}\oplus\mathbf{1}] - [\mathbf{p}\oplus\mathbf{p}] \in \widetilde{K}(C(S^2))$. We note that $K(C(S^2))$ is an abelian group, due to the Grothendieck construction. Hence, there is and injection $\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2 \hookrightarrow K(S^2)$.

**Definition 4.1.3.** Multiplication of K-rings is called the *external product*

$$\mu\colon K(X)\otimes K(Y) \to K(X\times Y),$$

where the unique morphism on pure tensors is given by $\mu(x\otimes y) = \operatorname{pr}_1^*(x)\operatorname{pr}_2^*(y)$ for $\operatorname{pr}_1$ and $\operatorname{pr}_2$ projections of $X\times Y$ onto $X$ and $Y$ respectively, which can be extended bilinearly.

From Definition 4.1.3 the following theorem is deduced.

**Theorem 4.1.4.** *(Product Theorem) The homomorphism* $\mu\colon K(X)\otimes\mathbb{Z}[H]/(H-1)^2 \to K(X\times S^2)$ *is an isomorphism of rings for all compact Hausdorff spaces X.*

The proof of the product theorem is out of the scope of this thesis. However, for a proof we refer to [Hat17, Thm. 2.2].

In Example 3.4.3 we found that the projection $\mathbf{p}$ coincides with the tautological line bundle. Thus we have that $[\mathbf{p}] = H$ (We will often omit the brackets when it is not likely to cause confusion). For $H$ we have the relation that $(H-1)^2 = ([\mathbf{p}]-1)^2 = [\mathbf{p}\otimes\mathbf{p}] - [\mathbf{p}\oplus\mathbf{p}] + 1 = [\mathbf{p}\otimes\mathbf{p}\oplus 1] - [\mathbf{p}\oplus\mathbf{p}] = 0$, because we have that $\mathbf{p}\oplus\mathbf{p} \sim_h \mathbf{p}\otimes\mathbf{p}\oplus\mathbf{1}$ and thus represent the same equivalence class. Consequently, we can rewrite the Product Theorem in terms of the projection $\mathbf{p}$:

$$\mu\colon K(C(X))\otimes\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2 \to K(C(X)\times C(S^2)) \cong K(C(X\times S^2))$$

As a consequence of the Product Theorem we can explicitly describe the ring $K(C(S^2))$:

**Corollary 4.1.5.** *The rings* $K(C(S^2))$ *and* $\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2$ *are isomorphic.*

*Proof.* Take $X = \{*\}$, according to the product theorem we obtain that

$$\begin{gathered} K(C(\{*\}))\otimes\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2 \cong K(C(\{*\}\times S^2)) \\ \iff \\ \mathbb{Z}\otimes\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2 \cong K(C(S^2)) \end{gathered} \tag{4.1}$$

Note that in Example 2.1.8 we found that $K(\{*\}) = \mathbb{Z}$, and we have that $K(C(\{*\})) \cong K(\{*\})$. Furthermore, we note that the tensor product in (4.1) is a tensor product over $\mathbb{Z}-$modules. Hence, we find that

$$\mathbb{Z}[\mathbf{p}]/(\mathbf{p}-1)^2 \cong K(C(S^2)).$$ □

As a result of Corollary 4.1.5, we find that the group $K(C(S^2))$ is generated by $[1,\mathbf{p}]$. Furthermore, we also observe from Definition 2.1.9 that $\widetilde{K}(C(S^2))$ is generated by $\mathcal{B} = (\mathbf{p}-1)$. Moreover, we have the relation $\mathcal{B}^2 = 0$. Thus the multiplication in $\widetilde{K}(C(S^2))$ is trivial, i.e. the multiplication of any two elements in $\widetilde{K}(C(S^2))$ is zero.

## 4.2 Exact sequences

In this section we recall the concept of exact sequences. For more in depth information on exact sequences we refer to [Rot98].

**Definition 4.2.1.** An infinite sequence of (abelian) groups and homomorphisms

$$\ldots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \ldots$$

is called a *long exact sequence* if for all $i \in \mathbb{Z}$, $\mathrm{Im}(d_{i+1}) = \ker(d_i)$.

**Definition 4.2.2.** An exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

is called a *short exact sequence*.

**Proposition 4.2.3.** *For $X$ a compact Hausdorff space, and a closed subset $U \subseteq X$, the sequence*

$$U \xrightarrow{i} X \xrightarrow{p} X/U,$$

*where $p$ is the quotient map and $i$ the inclusion, induces the exact sequence*

$$\widetilde{K}(U) \xleftarrow{i^*} \widetilde{K}(X) \xleftarrow{p^*} \widetilde{K}(X/U).$$

*Proof.* We will follow the proof of [Hat17, Prop. 2.9]:
We first proof the inclusion $Im(p^*) \subseteq \ker(i^*)$: The inclusion is the same as stating $i^*p^* \equiv 0$. We have that $(pi)^* = i^*p^*$, and we can form the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ U/U \arrow[r] & X/U. \end{tikzcd}$$

This diagram shows that all $a \in U$ are mapped to one point $\{*\} \in X/U$. We also know that $U/U \cong \{*\}$, and $\widetilde{K}(\{*\}) = 0$. Hence, we find that $i^*p^* \equiv 0$.

Subsequently, we poof the inclusion $Im(p^*) \supseteq \ker(i^*)$:
Let $p\colon E \to X$ be a vector bundle, and assume that the restriction vector bundle $E|_U \to U$ is s-isomorphic to the trivial vector bundle, and that $E$ itself is trivial over $U$. Let $h\colon p^{-1}(U) \to U \times \mathbb{C}^n$ be a trivialisation of $U$. Furthermore, let $E/h$ be the quotient space where for each $x, y \in U$ we have that $h^{-1}(x, v) \sim h^{-1}(y, v)$. With the use of the trivialisation one can construct a projection from the given vector bundle $p$:

$$p_h\colon E/h \to X/U.$$

Subsequently, we show that the projection $p_h$ is a vector bundle.
Using the trivialisations $h$, one obtains sections of $E \to U$. The sections can be given with respect to a basis on $\mathbb{C}^n$. Suppose that $\{e_i\}_{i\in[1,n]}$ forms a basis for $\mathbb{C}^n$, then the sections $s_i := h^{-1}(u, e_i)$ form a basis in each of the fibres over $U$. Subsequently, choose a finite open cover $\mathscr{U}$ of $U$ such that the bundle $E \to U_j$ is trivial for an open $U_j \in \mathscr{U}$ (this open cover can be constructed because $U$ is a compact set). The restriction of the sections $s_{ij} : U \cap U_j \to E$ form sections to a single fibre. Then applying the *Tietze Extension Theorem* (See [Mun14, Thm. 35.1]) from general topology, one can extend the sections to a continuous map on all of $U_j$ for each $U_j \in \mathscr{U}$. Let $\mathscr{X} = \{\mathscr{U}, X\backslash U\}$ be an open cover of $X$. Then, there exists a *partition of unity* subordinate to $\mathscr{X}$ (see [Lee13, Thm. 2.23]) given by the set indexed set of continuous functions $\{\psi_j, \psi\}$. Then the continuous function

$$\sum \psi_j s_{ij},$$

defines an extension of the sections $s_{ij}$ on $X$. We note that the sections where constructed to form a basis in each of the fibres. Hence, they also form a basis in all surrounding fibres. Consequently, the trivialisation $h$ of $E$ gives a trivialisation of $E/h$, and thus is $p_h$ a vector bundle. Finally, we need to check that $E \cong q^*(E/h)$. In order to prove the final statement we use the following commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E/h \arrow[d, "p_h"] \\ X \arrow[r, "q"'] & X/U, \end{tikzcd}$$

where $q$ is the projection of $X$ of its quotient with $U$. Due to the construction of the trivialisations with $h^{-1}(x, v) \sim h^{-1}(y, v)$, we have a isomorphism on the fibres. Thus, by the pull-back construction we find that $E \cong q^*(E/h)$, which concludes the proof. □

Notice that as a result of the *Serre-Swan Theorem* (Theorem 2.2.11) there is an isomorphism between $K(X)$ and $K(C(X))$ for $X$ a compact Hausdorff space. Consequently, there also is an isomorphism between $\widetilde{K}(X)$ and $\widetilde{K}(C(X))$. With this isomorphism Proposition 4.2.3 can be translated to the reduced K-rings over $C^*$-algebras.

**Corollary 4.2.4.** *For $X$ a compact Hausdorff space, and a closed subset $U \subseteq X$, the sequence*

$$U \xrightarrow{i} X \xrightarrow{p} X/U$$

*induces the exact sequence*

$$\widetilde{K}(C(U)) \xleftarrow{i^*} \widetilde{K}(C(X)) \xleftarrow{p^*} \widetilde{K}(C(X/U)).$$

**Remark 4.2.5.** Because, $U \subseteq X$ is a closed subset of X, $U$ is a compact Hausdorff space. Hence, $C(U)$ is also a $C^*$-algebra.

## 4.3 Operations on Topological Spaces

In this section we introduce new topological definitions, with the aim to extend the sequence $U \to X \to X/U$ of Corollary 4.2.4.

**Definition 4.3.1.** The *cone* of $X$, denote by $CX$ is defined as $(X \times I)/(X \times \{1\})$.

**Definition 4.3.2.** The *join* of $X$ and $Y$, denoted $X * Y$, is defined by the quotient space $(X \times I \times Y)/R$. Here $R$ is the equivalence relation generated by:

$$\langle x, 0, y\rangle \cong \langle x', 0, y\rangle \qquad \text{and} \qquad \langle x, 1, y\rangle \cong \langle x, 1, y'\rangle,$$

for $x, x' \in X$ and $y, y' \in Y$.

**Example 4.3.3.** For spheres we find that $S^n * S^m \cong S^{n+m+1}$
This statement can be proved by induction to the degree of the spheres. Here we will only treat the base case: $n = m = 1$.

We have that $S^1 * S^1 = (S^1 \times I \times S^1)/R$. Consider the map

$$\begin{aligned} f\colon S^1 \times I \times S^1 &\to S^3 \subset \mathbb{R}^2 \times \mathbb{R}^2 \\ (x, t, y) &\mapsto \big(x \sin(\tfrac{\pi t}{2}), y \sin(\tfrac{\pi(1-t)}{2})\big). \end{aligned}$$

The map $f$ is continuous and satisfies the equivalence relation $R$. Concequently, we find that $f$ defines a homeomorphism between $S^1 * S^1$ and $S^3$:

$$\begin{aligned} f\big([(x, t, y)]\big) &= \big(x \sin(\tfrac{\pi t}{2}), y \sin(\tfrac{\pi(1-t)}{2})\big) \\ f^{-1}(u, v) &= \big[\big(\tfrac{u}{|u|}, \tfrac{2sin^{-1}(|u|)}{\pi}, \tfrac{v}{|v|}\big)\big]. \end{aligned}$$

So we have that $S^1 * S^1 \cong S^3$. △

**Definition 4.3.4.** The *suspension* of $X$, denoted by $SX$ is defined as $(X \times I)/(X \times \{0\} \sqcup X \times \{1\})$, which is equivalent to the join of $X$ and $S^0$.

**Definition 4.3.5.** The *reduced suspension* of a pointed space $X$, denoted by $\Sigma X$ for $x \in X$ is defined as the quotient $SX/(\{x\} \times I)$, which is the same as

$$(X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x\} \times I).$$

**Remark 4.3.6.** The suspension can be seen as the union of two cones: $CX \cup CX$, where they are glued along the body $X$.

**Example 4.3.7.** For spheres the suspension is given by $S(S^n) = S^{n+1}$. △

**Example 4.3.8.** [RLL00, Ex. 4.1.5.] The definition of the cone and the suspension can also be given in terms of $C^*$-algebras. Let $A$ be a $C^*$-algebras then the definitions are given by

$$\begin{aligned} CA &= \big\{f \in C([0,1], A)\colon f(0) = 0\big\} \\ SA &= \big\{f \in C([0,1], A)\colon f(0) = f(1) = 0\big\} \end{aligned}$$

△

With the definition of the cone and the suspension we find the following sequence:

$$\begin{tikzcd} U \arrow[r] & X \arrow[r] & X\cup CU \arrow[r] \arrow[d] & (X\cup CU)\cup CX \arrow[r] \arrow[d] & ((X\cup CU)\cup CX)\cup C(X\cup CU) \arrow[d] \\ & & X/U & SU & SX \end{tikzcd} \tag{4.2}$$

where the vertical arrows are projections onto the quotient with the most recently attached cone. Notice, that there is a repeated structure in the sequence. For each term the union of the previous term with the cone of the term two steps back is taken.

**Definition 4.3.9.** A topological space is *contractible* if the identity morphism is homotopic to a constant morphism.

**Lemma 4.3.10.** *The cone of any topological space is contractible.*

*Proof.* We follow the proof of [Rot98, Thm. 1.11].
Suppose $X$ is a topological space, and let $CX$ be the cone of $X$. We construct a homotopy by first defining the map $F: CX\times I \to CX$, where $F([x,t],s) = [x,(1-s)t+s]$. This map is clearly continuous. Moreover, for $s=0$ we have the identity mapping on $CX$, and for $s=1$ one has that for all $x,y\in X$ $[x,s]\sim[y,s]$, which is the constant map. Therefore the cone $CX$ is contractible. □

With Lemma 4.3.10 in mind we can use the following lemma in order find the exact sequence derived from (4.2).

**Lemma 4.3.11.** *Suppose $U\subset X$ is contractible, then the projection onto the quotient* $\mathrm{pr}\colon X\to X/U$ *induces a bijection* $\mathrm{pr}^*\colon \mathrm{Vect}^n(X/U)\to\mathrm{Vect}^n(X)$ *for all $n$, where* $\mathrm{Vect}^n$ *denotes all vector bundles of rank $n$.*

For the proof of Lemma 4.3.11 we refer to [Hat17, Lemma 2.10].

As a result of Lemma 4.3.11 we find that there is an isomorphism $\widetilde{K}(X\cup CU)\cong\widetilde{K}(X/U)$. Then, it also holds for the following terms in the sequence of (4.2).

**Lemma 4.3.12.** *There is an homeomorphism between $(X\cup CU)/X$ and $SU$.*

*Proof.* Such a homeomorphism can be easily obtained by considering the visualisation in Figure 4.1. Note that $X\cup CU$, is a cone on a subspace $U$ of $X$. A cone can be seen as copies of $U$ on the unit interval where at $1\in I$ there is only a point. If one now takes the union with $X$ one obtains the left side of Figure 4.1.

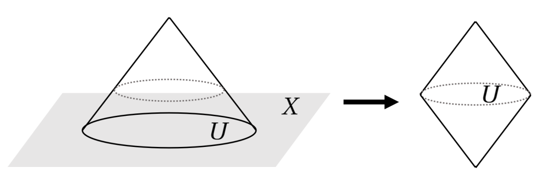



**Figure 4.1:** The quotent of $X\cup CU$ with $X$.

Subsequently, we take the quotient with $X$, and obtain the right side of Figure 4.1. Again, one realises that this coincides with the suspension of $U$. □

Lemma 4.3.12 can be extended to the components of (4.2). We note that the cone and the suspension of a closed set are again closed. Hence, the repeated application of Corollary 4.2.4 and Lemma 4.3.11 to the sequence in (4.2) gives the exact sequence:

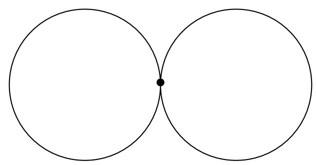
$$\cdots \to \widetilde{K}(SX) \to \widetilde{K}(SU) \to \widetilde{K}(X/U) \to \widetilde{K}(X) \to \widetilde{K}(U). \tag{4.3}$$

In order to obtain a reduced version of the external product we introduce three new definitions (See [Rot98]):

**Definition 4.3.13.** The *wedge sum* of two topological spaces $X$ and $Y$, with chosen points $x_0 \in X$ and $y_0 \in Y$, is defined as

$$X \vee Y := X \sqcup Y/(x_0 \sim y_0).$$

**Example 4.3.14.** A widely used example of the wedge sum, is the wedge of two spheres. For instance take twice the sphere $S^1$ with basepoints $x, y \in S^1$. Then, the wedge $S^1 \vee S^1$ is visually given by:



where the black dot represents that there is an equivalence between the points $x$ and $y$. △

**Lemma 4.3.15.** *For* $(X, x)$ *and* $(Y, y)$ *pointed topological spaces*

$$\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y.$$

*Proof.* First we write out the definition of the wedge sum:

$$X \vee Y \times I = (X \times I \sqcup Y \times I)/(\{x\} \times I \sim \{y\} \times I).$$

Subsequently, we take the quotient

$$\begin{aligned}
&(X \vee Y \times I)/(X \vee Y \times \{0\} \sqcup X \vee Y \times \{1\} \sqcup (x, y) \times I) \\
&= (X \times I \sqcup Y \times I)/(\{x\} \times I \sim \{y\} \times I)(X \vee Y \times \{0\} \sqcup X \vee Y \times \{1\} \sqcup (x, y) \times I) \\
&= ((X \times I)/(X \times \{0\} \sqcup X \times \{1\} \sqcup x \times I) \sqcup (Y \times I)/(Y \times \{0\} \sqcup Y \times \{1\} \sqcup y \times I))/(\{x\} \times I \sim \{y\} \times I) \\
&= (X \times I)/(X \times \{0\} \sqcup X \times \{1\} \sqcup x \times I) \vee (Y \times I)/(Y \times \{0\} \sqcup Y \times \{1\} \sqcup y \times I) \\
&= \Sigma X \vee \Sigma Y,
\end{aligned}$$

which concludes the proof. □

**Definition 4.3.16.** The *smash product* of two topological spaces $X$ and $Y$, with chosen points $x_0 \in X$ and $y_0 \in Y$, is defined as

$$X \wedge Y := X \times Y/X \vee Y.$$

Recall from general topology that the *one-point compactification* of a Hausdorff space $X$ is given by a compact Hausdorff space $Y$ which contains $X$ as a dense subset [Mun14, §29]. Denote $X^*$ for the one-point compactification of $X$. The one-point compactification of compact Hausdorff spaces has the following property:

**Lemma 4.3.17.** *Let $X$ and $Y$ be compact Hausdorff spaces, then*

$$X^* \wedge Y^* \cong (X \times Y)^*,$$

*where $\{*\}$ denotes the basepoint of the one-point compactification.*

For the proof of Lemma 4.3.17 we refer to [Rot98, Lemma 11.15].

**Example 4.3.18.** We compute the smash product of the sphere $S^n$ and $S^m$. Recall that the one-point compactification of $\mathbb{R}^n$ is $S^n$ [Mun14, §29]. As a result of Lemma 4.3.17 we find that

$$S^n \wedge S^m = (\mathbb{R}^n)^* \wedge (\mathbb{R}^m)^* \cong (\mathbb{R}^n \times \mathbb{R}^m)^* = (\mathbb{R}^{n+m})^* = S^{n+m}. \qquad \triangle$$

**Lemma 4.3.19.** *If $(X, x)$ is a compact Hausdorff space then $\Sigma X \cong X \wedge S^1$.*

*Proof.* Note that we can write $S^1$ in terms of an equivalence on the unit interval, i.e. $S^1 = [0,1]/(\{0\} \sim \{1\})$. Subsequently, the smash product of $X$ with $S^1$ becomes

$$\begin{aligned} X \wedge S^1 &= X \times (I/(\{0\} \sim \{1\})/(X \vee S^1) \\ &= X \times I/(X \times (\{0\} \sim \{1\}) \sqcup (X \vee S^1)) \\ &= (X \times I)/(X \times \{0\} \sqcup X \times \{1\} \sqcup \{x\} \times I) \\ &= \Sigma X, \end{aligned}$$

which concludes the proof. $\square$

**Definition 4.3.20.** A short exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0$ is called *split exact* if it is isomorphic to a sequence of the form $0 \to A \to A \oplus C \to C \to 0$. In other words, there exists a homomorphism $s : C \to B$ with $h \circ s = \mathrm{id}_C$.

**Example 4.3.21.** Let $X = A \vee B$, then we have that $X/A = B$. Then, one can express the sequence $A \xrightarrow{i} X \xrightarrow{p} X/A$ as $A \to X \to B$. Subsequently, we can apply Proposition 4.2.3, to obtain the exact sequence

$$\widetilde{K}(B) \xrightarrow{p^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A).$$

Suppose, there is another morphism $q : X \to A$, which is similarly defined as $p$, i.e. the contraction of $B$ to a point. Then we have the sequence $A \xrightarrow{i} X \xrightarrow{q} A$. This yields the composition $q \circ i = \mathrm{id}_A$, which yields the composition $i^* \circ q^* = \mathrm{id}^*$ on $\widetilde{K}(A)$. So, the given exact sequence is also split exact. Hence, we have an isomorphism

$$\widetilde{K}(X) \to \widetilde{K}(A) \oplus \widetilde{K}(B). \qquad \triangle$$

**Example 4.3.22.** Let $(X, x)$ be a pointed topological space. Using the definition of the reduced suspension and the suspension we obtain the quotient morphism

$$SX \to SX/(\{x\} \times I) = \Sigma X.$$

Subsequently applying Lemma 4.3.11 yields an isomorphism

$$\widetilde{K}(SX) \cong \widetilde{K}(\Sigma X). \tag{4.4}$$

Now, we use the property of Lemma 4.3.15 in combination with Example 4.3.21 and (4.4). Then, we find that

$$\widetilde{K}(S(X \vee Y) \cong \widetilde{K}(\Sigma(X \vee Y) \cong \widetilde{K}(\Sigma X \vee \Sigma Y) \cong \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \cong \widetilde{K}(SX) \oplus \widetilde{K}(SY). \qquad \triangle$$

Substituting $X \times Y$ for $X$ and $X \vee Y$ for $U$ in (4.3), one obtains the diagram

$$\begin{tikzcd} \widetilde{K}(S(X\times Y)) \arrow[r] & \widetilde{K}(S(X\vee Y)) \arrow[r] \arrow[d, "\sim"] & \widetilde{K}(X\wedge Y) \arrow[r] & \widetilde{K}(X\times Y) \arrow[r] & \widetilde{K}(X\vee Y) \arrow[d, "\sim"] \\ & \widetilde{K}(SX)\oplus\widetilde{K}(SY) & & & \widetilde{K}(X)\oplus\widetilde{K}(Y), \end{tikzcd} \tag{4.5}$$

where the first vertical isomorphism follows from Example 4.3.22 and the second vertical isomorphism follows form Example 4.3.21. Furthermore, we notice that the last part of the sequence in (4.5): $\widetilde{K}(X\wedge Y) \to \widetilde{K}(X\times Y) \to \widetilde{K}(X\vee Y) \cong \widetilde{K}(X)\oplus\widetilde{K}(Y)$, forms a split exact sequence. This can be seen from the surjection

$$\widetilde{K}(X)\oplus\widetilde{K}(Y) \to \widetilde{K}(X\times Y); \qquad \text{where } (x,y)\mapsto p_1^*(x)\oplus p_2^*(y),$$

where $p_1$ and $p_2$ are projection of $X\times Y$ on $X$ and $Y$ respectively. Consequently, we have the splitting

$$\widetilde{K}(X\times Y) \cong \widetilde{K}(X\wedge Y)\oplus\widetilde{K}(X)\oplus\widetilde{K}(Y).$$

We use the external product (Definition 4.1.3) and the definition of the reduced K-ring (Definition 2.1.9), to compute a reduced version of the external product. Due to the morphism rk, we concluded that we have an isomorphism $K(X) \cong \widetilde{K}(X)\oplus\mathbb{Z}$. Hence, the external product gives

$$\begin{array}{ccccccccc} K(X)\otimes K(Y) & \cong & (\widetilde{K}(X)\otimes\widetilde{K}(Y)) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus & \mathbb{Z} \\ \downarrow & & \downarrow & & \| & & \| & & \| \\ K(X\times Y) & \cong & \widetilde{K}(X\wedge Y) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus & \mathbb{Z}. \end{array} \tag{4.6}$$

Note, that we again used that the tensor product is taken over $\mathbb{Z}$-modules. From this we can deduce the *reduced external product.*

**Definition 4.3.23.** Multiplication of $\widetilde{K}$-rings is called the *reduced external product*

$$\widetilde{\mu}: \widetilde{K}(X)\otimes\widetilde{K}(Y) \to \widetilde{K}(X\wedge Y),$$

where the unique morphism on pure tensors is given by $\mu(x\otimes y) = \mathrm{pr}_1^*(x)\,\mathrm{pr}_2^*(y)$ for $\mathrm{pr}_1$ and $\mathrm{pr}_2$ projections of $X\times Y$ onto $X$ and $Y$ respectively, which can be extended bilinearly.

**Notation:** We denote $x * y := \mathrm{pr}_1^*(x)\,\mathrm{pr}_2^*(y) = \mu(x\otimes y)$ for the reduced external product.

Now we have constructed all the tools to state the Bott Periodicity Theorem.

## 4.4 The Bott Periodicity Theorem

In section 4.2 and 4.3 we have given the machinery to state and proof the Bott Periodicity Theorem. In this section we will proof the Bott Periodicity Theorem and look into some of its consequences.

**Theorem 4.4.1** (Bott Periodicity)**.** *The homomorphism*

$$\beta\colon \widetilde{K}(X) \to \widetilde{K}(S^2X); \qquad \textit{where } \beta(a) = (H-1)*a,$$

*is an isomorphism for $X$ a compact Hausdorff space and $H$ the tautological line bundle over* $\mathbb{PC}$.

*Proof.* Note that we may write $\beta$ as the composition of an inclusion and the reduced external product:

$$\widetilde{K}(X) \to \widetilde{K}(S^2) \otimes \widetilde{K}(X) \xrightarrow{\text{reduced external product}} \widetilde{K}(S^2 \wedge X) \cong \widetilde{K}(S^2 X),$$

where we have used (4.4) in combination with Lemma 4.3.19 for the last equivalence. As a result of Corollary 4.1.5 we have that $\widetilde{K}(C(S^2)) \cong \widetilde{K}(S^2) \cong \mathbb{Z}$, with generator $(H-1)$ and trivial multiplication. Furthermore, we have a tensor product over $\mathbb{Z}$-modules, hence $\widetilde{K}(X) \cong \mathbb{Z} \otimes \widetilde{K}(X) \cong \widetilde{K}(S^2) \otimes \widetilde{K}(X)$. Moreover, we have that the reduced external product is an isomorphism due to the product theorem. Hence, we find that $\beta$ is an isomorphism. Furthermore, we note that the first morphism $\widetilde{K}(X) \to \widetilde{K}(S^2) \otimes \widetilde{K}(X)$ is given by $a \mapsto (H-1) \otimes a$, and the reduced external product is given by $(H-1) \otimes a \mapsto (H-1) * a$, Hence, the composition $\beta$ is given by $a \mapsto (H-1) * a$. □

The Bott Periodicity Theorem as given in Theorem 4.4.1, is stated for topological groups. However, we are interested in the Bott Periodicity Theorem in terms of the projection **p**, i.e. $C^*$-algebras. We use that there is an isomorphism between $\widetilde{K}(X)$ and $\widetilde{K}(C(X))$ for $X$ a compact Hausdorff space. Then the Bott Periodicity Theorem becomes for $C^*$-algebras becomes:

**Corollary 4.4.2** (Bott Periodicity Theorem for **p**)**.** *The homomorphism*

$$\beta\colon \widetilde{K}(C(X)) \to \widetilde{K}(C(S^2 X)); \qquad \text{where } \beta(a) = (\boldsymbol{p}-1) * a,$$

*is an isomorphism for $X$ a compact Hausdorff space and* $\boldsymbol{p} = \frac{1}{2}\begin{pmatrix} \boldsymbol{1}+z & x-iy \\ x+iy & \boldsymbol{1}-z \end{pmatrix}$.

Furthermore, we note the in Corollary 4.4.2 **p** and $a$ are continuous functions. Hence, the pull-back with the projections can be seen as the composition of the functions:

$$(\mathbf{p}-1) * a = p_1^*(\mathbf{p}-1) p_2^*(a) = ((\mathbf{p}-1) \circ p_1)(a \circ p_2).$$

A keen property of the Bott Periodicity Theorem is that we can compute the $\widetilde{K}$-groups for all spheres. Recall that $S^2 S^n = S^{n+2}$, $\widetilde{K}(S^1) = 0$, and $\widetilde{K}(S^2) = \mathbb{Z}$, from which immediately follows that:

**Corollary 4.4.3.**

$$\widetilde{K}(S^n) \cong \begin{cases} \mathbb{Z}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

In Section 4.3 we have stated that $\widetilde{K}(C(S^2))$ is generated by the Bott element $\mathcal{B}$, i.e. $\widetilde{K}(C(S^2)) \cong \langle \mathcal{B} \rangle$. Applying the Bott Periodicity Theorem gives that $\widetilde{K}(C(S^{2n})) \cong \widetilde{K}(C(S^2))^{\otimes n}$. Hence, the generator of $\widetilde{K}(C(S^{2n}))$ is given by $\beta(\mathcal{B} \otimes \cdots \otimes \mathcal{B}) =: \mathcal{B}_n$.

**Lemma 4.4.4.** *For $n \in \mathbb{Z}$ we have that $K(C(S^{2n})) \cong \mathbb{Z}[\mathcal{B}]/(\mathcal{B}^2)$.*

*Proof.* In Corollary 4.4.3, we concluded that $\widetilde{K}(S^m) \cong \mathbb{Z}$ for $m$ an even integer. Hence for all $n \in \mathbb{Z}$ we have that $\widetilde{K}(C(S^{2n})) \cong \mathbb{Z} \cong \langle \mathcal{B} \rangle$, with $\mathcal{B}^2 = 0$. Moreover, we have that

$$K(C(S^{2n})) = \widetilde{K}(C(S^{2n})) \oplus \mathbb{Z} \cong \langle \mathcal{B} \rangle \oplus \mathbb{Z},$$

which is the same a stating $K(C(S^{2n})) \cong \mathbb{Z}[\mathcal{B}]/(\mathcal{B}^2)$. □

One can also generalise the Bott Periodicity Theorem for a topological space $X$ instead of a sphere by repeatedly applying the Bott Periodicity Theorem.

**Corollary 4.4.5.** *The (reduced) external product*

$$\widetilde{K}(S^{2n}) \otimes \widetilde{K}(X) \cong \widetilde{K}(S^{2n} \wedge X) \tag{4.7}$$

*is an isomorphism.*

*Proof.* The external product of $\widetilde{K}(S^2) \otimes \widetilde{K}(S^2)$, is equivalent to $\widetilde{K}(S^4)$ according to the Bott Periodicity Theorem. By induction to the tensor power we find that $\widetilde{K}(S^2)^{\otimes n} \otimes \widetilde{K}(X) \cong \widetilde{K}(S^{2n}) \otimes \widetilde{K}(X)$. Moreover, we have that

$$\widetilde{K}(S^2)^{\otimes(n-1)} \otimes \widetilde{K}(S^2) \otimes \widetilde{K}(X) \cong \widetilde{K}(S^2)^{\otimes(n-1)} \otimes \widetilde{K}(S^2 X) \cong \cdots \cong \widetilde{K}(S^2) \otimes \widetilde{K}(S^{2(n-1)} X) \cong \widetilde{K}(S^{2n} X)$$

which concludes the proof. □

From Corollary 4.4.5, and (4.6) the isomorphism

$$K(S^{2n}) \otimes K(X) \cong K(S^{2n} \times X),$$

follows immediately.

**Remark 4.4.6.** In Example 3.4.4 we constructed a projection $\mathbf{q} \in M_4(C(S^4))$ from the third Hopf fibration. For this projection there is not such a notion as the Bott Projection.

# 5 The Hopf Invariant One Problem

In this chapter we build the tools to define the Hopf Invariant One Problem. This theorem allows linking the division algebras to the Hopf fibrations. The original proof of this major theorem was given by F. Adams in 1960 [Ada60]. This theorem was also proved in the context of K-theory by F. Adams and M. Atiyah in 1966 [AA66]. In this chapter we will follow the proof given by [Sty13], which is based on the proof by Adams and Atiyah.

This chapter is divided in two subjects. First we will examine the concept of H-spaces. Then we conclude the chapter with the Hopf Invariant One Problem.

## 5.1 H-space structures

In this section we cover the definition of an H-space structure as well as H-space structures in combination with spheres and division algebras. In this section we will mainly follow the discussion of [Sty13, Ch. 2]. Furthermore, we will follow [Hat17, §2.3].

**Definition 5.1.1.** A topological space $M$ with basepoint $m \in M$ with a continuous map $\mu\colon M \times M \to M$ satisfying

1. $\mu(e,e) = e$,
2. the map $M \to M$ given by $m \mapsto \mu(m,e)$ and the map $M \to M$ given by $m \mapsto \mu(e,m)$ are homotopic to the identity map on $M$,

is called a *H-space.*

**Remark 5.1.2.** If $X$ is a topological group then it is also a H-space, with $\mu$ given by the group multiplication.

In order to relate spheres to H-spaces we need to recall the definition of a parallelisable space.

**Definition 5.1.3.** We call a manifold $M$ *parallelisable* if it admits a global frame, i.e. we have an ordered n-tuple of linear independent vector fields $(E_1,...,E_n)$ such that for each $p \in M$ the vectors $(E_1|_P,...,E_n|_p)$ form a basis for $T_pM$.

**Remark 5.1.4.** The sphere $S^n$ has the structure of a manifold.

**Lemma 5.1.5.** *If $\mathbb{R}^n$ has the structure of a division algebra, then the sphere $S^{n-1}$ is parallelisable. Moreover, if $S^{n-1}$ is parallelisable, then it is a H-space.*

The proof of Lemma 5.1.5 follows from some computations in linear algebra; for a proof we refer to [Sty13, Prop. 2.1.1 and 2.1.2].

**Example 5.1.6.** We already know that spheres $S^0, S^1$ and $S^3$ are H-spaces because they are Lie groups. Furthermore, we have that $\mathbb{R}^8 \cong \mathbb{O}$ has the structure of a division algebra. Hence, by Lemma 5.1.5 we find that $S^7$ is also a H-space. △

**Lemma 5.1.7.** *The sphere $S^{n-1}$ cannot be a H-space when $n \neq 0$ is odd.*

*Proof.* We follow the proof of [Sty13, Prop. 2.3.1.]
Let $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$, be the morphism giving $S^{n-1}$ its H-space structure. Suppose that $n > 1$ is odd, then $n-1$ is even. Hence, $K(S^{n-1}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$ by lemma 4.4.4 and the fact that $K(X) \cong K(C(X))$. By the reduced external product applied to $X = S^{n-1}$, we obtain a homomorphism $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$ (Note that we have used the tensor product of $\mathbb{Z}$-modules). Then the image of $\gamma$ under $\mu^*$ is a linear combination of the basis elements of $\mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$, i.e. $\{1,\alpha,\beta,\alpha\beta\}$. Write $\mu^*(\gamma) = m + n\alpha + o\beta + p\alpha\beta$, for $m,n,o,p \in \mathbb{Z}$. Since, $\mu^*$ is a ringhomomorphism, we have that $\mu^*(\gamma)^2 = \mu^*(\gamma^2) = 0$. Thus, we have that

$$m^2 + 2mn\alpha + 2mo\beta + 2(mp + no)\alpha\beta = 0. \tag{5.1}$$

Then $m = 0$ and $no = 0$ in order to satisfy (5.1).
Next, we take a look at the inclusions $\iota_{1,2} : S^{n-1} \hookrightarrow S^{n-1} \times S^{n-1}$, to the first and second component. We note that the composition $\iota_{1,2}^* \circ \mu^* = \mathrm{id}_{S^{n-1}}^*$, due to the H-space structure on $S^{n-1}$. Then $(\iota_1 \circ \mu)^*(\gamma)$ gives that $n = 1$, and $(\iota_2 \circ \mu)^*(\gamma)$ gives that $o = 1$. Consequently, we have that $n = o = 1$, which is in contradiction with (5.1). □

## 5.2 The Hopf Invariant

In this section, we will use a different approach than the approach of Adams to show a similar result. To show this we will follow the approaches used by [Hus66] and [Sty13].

First, we will define the map known as the *Hopf Construction*. Then, we will examine what it means to be of *Hopf Invariant One*. Finally, we can show how the division algebras correspond to the Hopf fibrations.

### 5.2.1 The Hopf Construction

**Definition 5.2.1.** Let $f : X \to Y$ be a continuous map. Then the *mapping cylinder of $f$* is defined as

$$Z_f = ((X \times I) \sqcup Y) / \big((x,0) \sim f(x)\big).$$

**Definition 5.2.2.** Let $f\colon X \to Y$ be a continuous map. Then the *mapping cone of $f$* is defined as

$$C_f = ((X \times I) \sqcup Y) / \big((x,1) \sim (x',1) \sqcup (x,0) \sim f(x)\big)$$

for all $x, x' \in X$.

An important construction used to relate the division algebras to the Hopf fibrations is the so-called *Hopf construction*:

**Definition 5.2.3.** For a continuous map $f\colon X \times Y \to Z$ the *Hopf construction* is given by the map

$$H(f)\colon X * Y \to \Sigma Z; \qquad H(f)(x,t,y) = (f(x,y),t).$$

### 5.2.2 Hopf Invariant One

From now on we will focus on spheres $S^{2n-1}$ which admit a H-space structure. Hence, by Lemma 5.1.7, we take $2n$ for $n$ because it has to be even.

Let $g\colon S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ be a continuous map. Then from Example 4.3.3 it follows that the join is given by $S^{2n-1} * S^{2n-1} = S^{2n-1+2n-1+1} = S^{4n-1}$. Moreover, from Example 4.3.18 and Lemma 4.3.19 it follows that $\Sigma S^{2n-1} = S^{2n-1} \wedge S^1 = S^{2n}$. Hence, the Hopf Construction of $g$ is the map

$$H(g)\colon S^{4n-1} \to S^{2n}.$$

For the Hopf construction and the mapping cylinder and mapping cone one can compute the following diagram [Hus66, S. 9(2.4)]

$$\begin{tikzcd}
& S^{2n} \arrow[d, "v"] \arrow[dr, "a"] & & & \\
S^{4n-1} \arrow[ur, "H(g)"] \arrow[r, "u"] & Z_{H(g)} \arrow[r, "w"] & C_{H(g)} \arrow[r, "b"] & SS^{4n-1} \arrow[r, "S(H(g))"] & SS^{2n},
\end{tikzcd} \tag{5.2}$$

where $u(x)$ is the class of $(x,1) \in Z_{H(g)}$, $v(y)$ is the class of $y \in Z_{H(g)}$, $w$ is the projection given by $Z_{H(g)} \to Z_{H(g)}/u(X)$, $a(y)$ is the class of $y \in C_{H(g)}$ and for $b$ we have that $b(y) = \left[S^{4n-1} \times \{1\}\right] \cong \{*\}$ and $b(x,t) = [x,t]$. In order to clarify the morphism $b$, a visual representation is given in Figure 5.1.

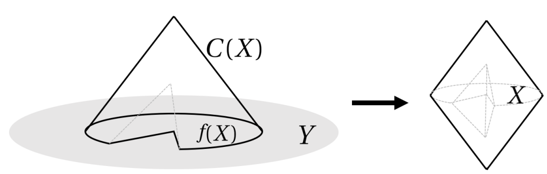



**Figure 5.1:** The morphism $b\colon C_f \to SX$ for $f\colon X \to Y$, where we take the quotient with the space $Y$.

**Lemma 5.2.4.** *The sequence*

$$\widetilde{K}(S^{2n}) \leftarrow \widetilde{K}(C_{H(g)}) \leftarrow \widetilde{K}(SS^{4n-1})$$

*is an exact sequence.*

*Proof.* We follow the proof of [Hus66, §9(2)].
We notice that $S^{4n-1}$ is a closed subset of $Z_{H(g)}$. Then, as a result of Proposition 4.2.3, we have that the sequence $S^{4n-1} \to Z_{H(g)} \to C_{H(g)}$ induced the exact sequence

$$\widetilde{K}(C_{H(g)}) \to \widetilde{K}(Z_{H(g)}) \to \widetilde{K}(S^{4n-1}).$$

Moreover, we have that $a$ is an inclusion, and the map $b$ is a projection $C_{H(g)} \to C_{H(g)}/a(Y)$. Therefore, we can again apply Proposition 4.2.3 and find that the sequence

$$\widetilde{K}(SS^{4n-1}) \to \widetilde{K}(C_{H(g)}) \to \widetilde{K}(S^{2n})$$

is exact. In [Hus66, Prop. 9(2.5)] it is proven that the map $v$, of (5.2), is a homotopy equivalence, from which it follows that $\widetilde{K}(S^{2n}) \cong \widetilde{K}(Z_{H(g)})$. Combining the exact sequences yields the following diagram

$$\begin{tikzcd}
 & & \widetilde{K}(S^{2n}) \arrow[dr, "u^*\circ(v^*)^{-1}"] & \\
\widetilde{K}(SS^{4n-1}) \arrow[r, "b^*"] & \widetilde{K}(C_{H(g)}) \arrow[ur, "a^*"] \arrow[r, "w^*"] & \widetilde{K}(Z_{H(g)}) \arrow[u, "v^*"] \arrow[r, "u^*"] & \widetilde{K}(S^{4n-1}),
\end{tikzcd}$$

where $\mathrm{Im}(b^*) = \ker(a^*)$ and $\mathrm{Im}(w^*) = \ker(u^*)$. Then, we also have that

$$\begin{aligned}
\mathrm{Im}(a^*) &= \mathrm{Im}(v^* \circ w^*) \\
&= v^*(\mathrm{Im}(w^*)) \\
&= v^*(\ker(u^*)) \\
&= \ker(u^* \circ (v^*)^{-1}).
\end{aligned}$$

Therefore, the sequence

$$\widetilde{K}(SS^{4n-1}) \xrightarrow{b^*} \widetilde{K}(C_{H(g)}) \xrightarrow{a^*} \widetilde{K}(S^{2n}) \xrightarrow{u^*\circ(v^*)^{-1}} \widetilde{K}(S^{4n-1}), \tag{5.3}$$

is exact. □

**Remark 5.2.5.** As a result of Corollary 4.4.3 we have that $\widetilde{K}(S^{4n-1}) = 0$. Thus the sequence (5.3) becomes

$$\widetilde{K}(SS^{4n-1}) \xrightarrow{\psi} \widetilde{K}(C_{H(g)}) \xrightarrow{\phi} \widetilde{K}(S^{2n}) \to 0. \tag{5.4}$$

Suppose that $\mathcal{B}_{4n}$ is the generator of $\widetilde{K}(S^{4n}) = \widetilde{K}(SS^{4n-1})$, then we denote $b_{H(g)}$ for $\psi(\mathcal{B}_{4n})$. Subsequently, denote $a_{H(g)} \in \widetilde{K}(C_{H(g)})$ for an element such that $\phi(a_{H(g)}) = \mathcal{B}_{2n}$ . From this two conclusions about the relations of $a_{H(g)}$ and $b_{H(g)}$ can be made.

**Lemma 5.2.6.** *For the introduced elements $a_{H(g)}$ and $b_{H(g)}$ we have that:*

1. *$a^2_{H(g)}$ is a multiple of $b_{H(g)}$.*
2. *$a_{H(g)}b_{H(g)} = 0$*

*Proof.* First, we prove the first statement: Note that $\psi$ and $\phi$ are ringhomomorphisms and that $\mathcal{B}^2_{2n} = 0$. We have that $\phi(a_{H(g)}) = \mathcal{B}_{2n}$ and that $\mathcal{B}^2_{2n} = 0$. This implies that $a^2_{H(g)} \in \ker(\phi) = \mathrm{im}(\psi) = \langle b_{H(g)} \rangle$. Consequently, there exists $\lambda \in \mathbb{Z}$ such that $a^2_{H(g)} = \lambda b_{H(g)}$.

Next, we prove the second statement: We have that $\phi(a_{H(g)}b_{H(g)}) = \phi(a_{H(g)})\phi(b_{H(g)}) = 0$, because $b_{H(g)} \in \ker(\phi)$ due to the exactness of sequence (5.4). Then there exists $m \in \mathbb{Z}$ such that $a_{H(g)}b_{H(g)} = mb_{H(g)}$. Hence we find that

$$ma_{H(g)}b_{H(g)} = a_{H(g)}(a_{H(g)}b_{H(g)}) = a^2_{H(g)}b_{H(g)} = \lambda b^2_{H(g)} = 0.$$

So, either $m = 0$ or $a_{H(g)}b_{H(g)} = 0$. Note that if $m = 0$, then by the relation $a_{H(g)}b_{H(g)} = mb_{H(g)}$ we also find that $a_{H(g)}b_{H(g)} = 0$. Thus $a_{H(g)}b_{H(g)} = 0$. □

With the definitions of $a_{H(g)}$ and $b_{H(g)}$ we can define the Hopf Invariant.

**Definition 5.2.7.** The *Hopf Invariant* of a morphism $H(g) : S^{4n-1} \to S^{2n}$ is the integer $h_{H(g)}$ such that $a^2_{H(g)} = h_{H(g)}b_{H(g)}$ in $\widetilde{K}(C_{H(g)})$.

**Lemma 5.2.8.** *The Hopf Invariant is well-defined.*

*Proof.* Let $c \in \widetilde{K}(C_{H(g)})$ such that $\phi(c) = \phi(a_{H(g)}) = \mathcal{B}_{2n}$. Then there exists $n \in \mathbb{Z}$ such that $c = a_{H(g)} + nb_{H(g)}$. Then we also have that $c^2 = a^2_{H(g)} + 2n(a_{H(g)}b_{H(g)} + n^2 b^2_{H(g)} = a^2_{H(g)}$. Thus, we have that $c^2 = a^2_{H(g)} = h_{H(g)}b_{H(g)}$. Hence, we find that $h_{H(g)}$ is independent of the choice of $a_{H(g)}$. □

**Proposition 5.2.9.** *If $S^{2n-1}$ admits a H-space structure $\mu$, then the Hopf Construction $H(\mu)$ has Hopf Invariant $\pm 1$.*

We say that the Hopf construction of a map $g$ is of Hopf invariant one, if $h_{H(g)} = \pm 1$.

To prove Proposition 5.2.9 we need some machinery.

**Lemma 5.2.10.** *For $A, B \subseteq X$ topological spaces we have that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$ , where $\partial$ denotes the boundary.*

*Proof.* The proof consists of some basic topological properties for which we refer to [Mun14]. Then, we can decompose $\partial(A \times B)$ as follows:

$$\begin{aligned}
\partial(A \times B) &= \overline{A \times B} - (A \times B)^\circ \\
&= \overline{A} \times \overline{B} - A^\circ \times B^\circ \\
&= (\overline{A} \times \overline{B}) \cap (A^\circ \times B^\circ)^c \\
&= \big((\overline{A} \times \overline{B}) \cap ((A^\circ)^c \times X)\big) \cup \big((\overline{A} \times \overline{B}) \cap ((X \times (B^\circ)^c\big) \\
&= \big((\overline{A} \cap (A^\circ)^c) \times (\overline{B} \cap X\big) \cup \big(\overline{A} \cap X) \times (\overline{B} \times (B^\circ)^c)\big) \\
&= \big((\overline{A} - A^\circ) \times \overline{B}\big) \cup \big(\overline{A} \times (\overline{B} - B^\circ)\big) \\
&= (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B),
\end{aligned}$$

which concludes the proof. □

Denote $D^n := \{x \in \mathbb{R}^n : ||x|| \le 1\}$, then we can decompose the sphere $S^{n-1}$:

$$S^{2n-1} = \partial(D^{2n}) = \partial(D^n \times D^n) = S^{n-1} \times D^n \cup D^n \times S^{n-1},$$

where for the last equality we used Lemma 5.2.10.

*Proof of Proposition 5.2.9.* We follow the proof of [Hus66, Lemma 2.18.] and [Sty13, Prop. 2.1.5.] We have that $\mu : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ and that the Hopf construction is give by $H(\mu) : S^{4n-1} \to S^{2n}$. Subsequently, we have the following commutative diagram:

$$\begin{tikzcd}
S^{4n-1} \arrow[r, hook, "\iota"] \arrow[d, "H(\mu)"'] & D^{4n} \arrow[d] \\
S^{2n} \arrow[r, hook, "\iota"] & C_{H(\mu)},
\end{tikzcd}$$

where $\iota$ denotes the inclusion morphism. Hence, we obtain a morphism $\phi : D^{4n}/S^{4n-1} \to C_{H(\mu)}/S^{2n}$. Using the boundary and $D^n \times D^n = D^{2n}$ we can rewrite $\phi$ as:

$$\Phi : (D^{2n} \times D^{2n}/\partial(D^{2n} \times D^{2n})) \longrightarrow (C_{H(\mu)}/S^{2n}).$$

Moreover, note that if we denote $D^n_+$ for the upper hemisphere and $D^n_-$ for the lower hemisphere, then we may write $S^n = D^n_+ \cup D^n_-$.

Next, we consider the following diagram:

$$\begin{tikzcd}
\widetilde{K}\big(C_{H(\mu)}\big) \otimes \widetilde{K}\big(C_{H(\mu)}\big) \arrow[r] & \widetilde{K}\big(C_{H(\mu)}\big) \\
\widetilde{K}\big(C_{H(\mu)}/D^{2n}_+\big) \otimes \widetilde{K}\big(C_{H(\mu)}/D^{2n}_-\big) \arrow[r] \arrow[u, "(\text{Lemma 4.3.11}) \cong"'] \arrow[d, "\Phi^* \otimes \Phi^*"'] & \widetilde{K}\big(C_{H(\mu)}/S^{2n}\big) \arrow[u, "b^*"'] \arrow[d, "\cong"] \\
\widetilde{K}\big(D^{2n} \times D^{2n}/S^{2n-1} \times D^{2n}\big) \otimes \widetilde{K}\big(D^{2n} \times D^{2n}/D^{2n} \times S^{2n-1}\big) \arrow[r] \arrow[d, "\cong"'] & \widetilde{K}\big(D^{2n} \times D^{2n}/\partial(D^{2n} \times D^{2n})\big) \\
\widetilde{K}\big(D^{2n} \times \{*\}/S^{2n-1} \times \{*\}\big) \otimes \widetilde{K}\big(\{*\} \times D^{2n}/\{*\} \times S^{2n-1}\big) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The diagonal isomorphism is equivalent to $\widetilde{K}(S^{2n}) \otimes \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{4n})$, which is an isomorphism due to Corollary 4.4.5. The horizontal morphisms follow from the product, and the morphism $b^*$ is stems form the exact sequence in (5.3).

Now it all comes down to diagram chasing. Take the generator $\mathcal{B}^2 = \mathcal{B} \otimes \mathcal{B} \in \widetilde{K}(C_{H(\mu)}) \otimes \widetilde{K}(C_{H(\mu)})$, following the diagram this generator maps to the generator of the ring in the bottom of the diagram. If we now continue upwards to the right of the diagram via $p^*$ to the generator of $\widetilde{K}(C_{H(\mu)})$ which equals $\pm a_{H(\mu)}$. Hence, by the commutativity of the diagram $\mathcal{B} = \pm a_{H(\mu)}$. Consquently, the Hopf invariant of $H(\mu)$, $h(H(\mu))$, equals $\pm 1$. $\square$

The British mathematician John Frank Adams proved in [Ada60] that one can make the statement of Proposition 5.2.9 much stronger, which known as the *Adams' Theorem*:

**Theorem 5.2.11** (Adams' Theorem)**.** *There exist a map $f : S^{4n-1} \to S^{2n}$ of Hopf Invariant $\pm 1$, only when $n = 1, 2$ or $4$.*

The proof of Theorem 5.2.11 uses the so-called *Adams operations* (see [Sty13, Thm. 2.2.1]) and the *Splitting Principle* (see [Sty13, Thm. 2.2.2]), which goes beyond the scope of this thesis. Hence for a proof we refer to [Sty13, §2.2.].

We conclude this thesis by a major result, which is the correspondence between the division algebras and the Hopf fibrations. This result is a consequence of *Adams' Theorem* and was already known by Bott and Milnor in 1958 [BM58]:

$$\begin{array}{ccc} S^0 \hookrightarrow S^1 \xrightarrow{\pi_0} S^1 \cong \mathbb{PR} & & \mathbb{R} \\ S^1 \hookrightarrow S^3 \xrightarrow{\pi_1} S^2 \cong \mathbb{PC} & \xleftrightarrow{1-1} & \mathbb{C} \\ S^3 \hookrightarrow S^7 \xrightarrow{\pi_3} S^4 \cong \mathbb{PH} & & \mathbb{H} \\ S^7 \hookrightarrow S^{15} \xrightarrow{\pi_7} S^8 \cong \mathbb{PO} & & \mathbb{O} \end{array}$$

The correspondence between the Hopf fibration and the division algebras yields from the map given by the Hopf construction (Definition 5.2.3) of Hopf invariant one. Given the division algebras one can form spheres which admit a H-space structure. Subsequently, the Hopf construction can be applied to the continuous morphism that is included in the definition of a H-space. This Hopf construction has Hopf invariant one and corresponds with the known Hopf fibrations. Subsequently, Adams' Theorem gives that the four Hopf fibrations are the only maps of the form $f : S^{4n-1} \to S^{2n}$ that are of Hopf invariant one.

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