FINITE APPROXIMATION AND CONTINUOUS CHANGE IN SPECTRAL GEOMETRY

BY

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Finite approximation and continuous change in spectral geometry

Proefschrift

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Summary

This dissertation explores two aspects of spectral geometry: that of *finite approximation* and that of *continuous change*.

Finite approximation

The main theme of Part I is that the spectral description of geometry, like the point-set one, should include a natural way to represent *partial knowledge*. This would both be very desirable from the side of applications (such as to quantum field theory and computer science) and be worth seeking as a further enrichment of the dictionary between noncommutative and differential geometry. The proposed candidate is that of *truncations* of *Dirac-type spectral triples*, selected both for its physical intuitiveness and for its mathematical simplicity.

The body of this part comprises three chapters, each a freestanding scientific article. The first, Chapter 2, addresses the relation between purely spectral *asymptotic* invariants (that is, geometric invariants determined uniquely by the asymptotics of the spectrum of any Laplace-type operator) and the asymptotics of functionals of the *truncated* spectrum. The requisite technical innovation is a careful balance between the asymptotic short-time behaviour of the heat trace and its exponential decay in the operator spectrum.

Chapter 3 addresses the issue of reconstructing a Riemannian manifold from successively larger truncations of any associated Dirac-type spectral triple. A new notion of quantitatitve *dispersion* (roughly, under the Riesz representation theorem, corresponding to a statistical variance) is introduced to present a further balancing consideration, that of *localization* of states. This is then combined with time-spectrum balance of the previous chapter to show that from these truncations one can reconstruct a metric space that increasingly approximates the manifold itself. Computability being a guiding

principle, a conjectural assumption about the equivalence of certain metrics on the localized states is applied to propose a computer algorithm to isometrically visualise the approximating metric spaces.

Finally, given the vast generality of the notion of spectral triples and the difficulty in recognizing them from any given truncation, Chapter 4 asks whether we can recognize whether a given operator system spectral triple is the truncation of a Dirac-type spectral triple, as a truncated variant of Connes' reconstruction theorem. The higher Heisenberg relation in noncommutative geometry is used to present an asymptotic algebraic constraint on the truncated operator. A numerical investigation of this constraint in the case of the sphere then results in the analytic discovery of a pseudodifferential operator on the sphere that, surprisingly, *better* exhibits the properties of a commutative spectral triple in its truncations. The results of the previous articles are then applied to compare these, no longer Dirac-type but still approximately commutative, truncations of spectral triples.

Continuous change

Part II extends the concept of *Schatten-class operators* to (countably generated) Hilbert modules over commutative C^*-algebras. Two main ingredients guide the construction.

First, we realize that there are *two* extremal generalizations of the Schatten-class criterion to operators on the standard module, namely, either that a family be continuous (and vanishes at infinity) in Schatten norm or that its pointwise Schatten norm be continuous (and vanish at infinity). Each has desirable properties, but *a priori* neither is obviously the correct choice. A refined understanding of the Schatten-norm convergence of finite matrix truncations of Schatten-class operators on the standard Hilbert space is then developed to show that our considered generalizations do in fact coincide. As a corollary the main, and very desirable, properties of the Schatten classes on Hilbert spaces are extended to those on the standard Hilbert module over a commutative C^*-algebra.

Second, Kasparov's stabilization theorem allows us to view all countably generated Hilbert modules over a C^*-algebra as complementable submodules of its standard module. The pullbacks of the Schatten classes on the standard module under this identification form two-sided ideals and are therefore invariant under the choice of isomorphic submodules. Moreover, because the Schatten norms are unitarily invariant, the two-sided ideal thus obtained canonically inherits a Banach space structure and is identified with the set of adjointable operators whose characterwise Schatten norms lie in the base algebra.

As direct applications, the Fredholm determinant and the operator zeta function obtain their generalization to the setting of countably generated Hilbert modules over commutative C*-algebras, while retaining essential properties such as multiplicativity (of the former) and holomorphicity.

The relation between traces and geometric invariants, already central to Chapter 2 of the thesis, should generalize to *geometric correspondences*, that is, Hilbert bimodules over commutative C*-algebras equipped with a suitable selfadjoint regular operator. An important step in that direction is a definition of the relevant class of *finitely summable cycles*, so as to give rise to at least a spectral notion of (fiber) dimension and allow application of the preliminary theory of operator zeta functions on Hilbert modules developed earlier in the thesis. This leads to the conjecture that the new notion of summability be *additive* under the *unbounded Kasparov product*, followed by the presentation of some important supporting examples.

Operators on Hilbert modules over *non*commutative C*-algebras exhibit deeply contrasting properties with respect to the trace. Chapter 8 presents the difficulty in the form of a highly problematic and basic example. As a possible approach to a solution, it goes on to discuss the idea that one should not work in the space of all adjointable operators, but rather with a subalgebra. The natural setting would then become that of inner-product *bi*modules.

Part I

Finite spectral geometry

Chapter 1

Introduction

Now we see as through a glass, darkly ...
Now what I know is incomplete.

1 Corinthians 13:12

The spectral picture of geometry begets its naturality from deep ties to physics, both classical – through the language of vibrations and heat flow – and quantum – through its natural accomodation of noncommutativity. Both aspects call for a coherent approach to *approximate geometry*, to accomodate the limited physical information accessible by ourselves as well as the viewpoints of renormalization and effective field theory. The present work engages the issue through a study of *truncations* of *Dirac-type spectral triples*.

Sketches and building blocks

Both *construction* and *observation* of shapes in the physical world proceed successively, in stages of increasing refinement. The knowledge of the final shape that each intermediate stage represents is finite and incomplete. It is, however, nontrivial. The theory of spectral geometry, in contrast to that of metric spaces, reflects this notion only rather opaquely. It is this opacity that we aim to alleviate by introducing the spectral analogon of partial geometric knowledge.

Spectral truncations

The basic objects of noncommutative geometry are the spectral triples (A, H, D), which represent a Riemannian manifold whenever the triple is of Dirac type, that is, when A is the algebra of its smooth functions and D is a Dirac-type operator on a Hermitian bundle, both represented on the Hilbert space H of its sections. Spectral triples come with a natural momentum cutoff, the projection P_Λ onto the $[-\Lambda, \Lambda]$-eigenspaces of D that yields the *truncation* $(P_\Lambda A P_\Lambda, P_\Lambda H, P_\Lambda D)$ [AB84; DLM14b; CS20]. Because P_Λ commutes with D, this truncation preserves all symmetries – including the isometries of an underlying Riemannian manifold.

Chapters 2 and 3 investigate a natural framework for incremental observation in spectral geometry, by connecting metric geometry and its invariants to truncations of Dirac-type spectra. Chapter 4 explores the issue of recognizing those objects that are truncations of Dirac-type spectral triples and thereby takes a first step towards a spectral account of cumulative construction.

Nonperturbative regularization in quantum gravity

A common starting point for several candidate theories of quantum gravity is the path integral over the space of all Lorentzian metrics (on a given smooth background manifold, say), with respect to some gravitational action. To obtain effective field theories from this paradigm, one introduces some type of regularization, restricting the integral to metrics constrained by some cutoff parameter Λ – ideally one that preserves the appropriate symmetries. These remaining metrics are then the building blocks that constitute the geometries considered at given scale. One salient example is that of Causal Dynamical Triangulations, where the Regge calculus is used to regularize the space of Lorentzian metrics in terms of flat simplicial manifolds.

In noncommutative geometry, the *spectral action principle* guides an approach to Euclidean quantum gravity that is naturally regularized by spectral truncation [CL90; Con96; Bar07; CCM07]. The construction of observables for such a theory then requires a metric understanding of truncated spectral triples (again, see Chapters 2, 3), and the parametrization of a suitable domain for the path integral requires a demarcation of the truncations of Dirac-type (that is, commutative) spectral triples (see Chapter 4) within the broader set of operator system spectral triples.

Digital representation of geometric objects

In computer science, the discrete and finite nature of our machines necessitate approximation and precision management when dealing with geometric objects. This often leads to discrete representations thereof through e.g. polygonal meshes, which exemplifies the metric-spaces approach to partial geometric knowledge. In contrast, the philosophy of spectral geometry in its most basic form, the Fourier transform, underpins many important applications such as the lossy compression methods of the JPEG and MP3 standards.

Laplacian-based approaches to shape recognition and manipulation have been the subject of persistent interest in the fields of computer graphics (following e.g. [Lev06]) and manifold learning (such as [BN02]). A true understanding of truncated spectral geometry would provide a consistent framework for methods in this direction. As a first step, the method of [BN07] to recapture part of the Laplacian spectrum and eigenfunctions from a sufficiently dense finite metric subspace finds its dual counterpart in the algorithm of Chapter 3 that reconstructs such metric subspaces from finite spectral data.

Spectral timescale and localization

An important ingredient in the relation between truncated Dirac-type spectral triples (and, in particular, the truncated spectrum of generalized Laplacians $\varDelta$) and Riemannian geometry is the *short-time asymptotic behaviour* of the *heat kernel*, see Section 1.1. It is well-known that the heat equation $\partial_t u(x,t) = -\varDelta u$ has a fundamental solution $K(t,x,y)$ which satisfies $\int_M K(t,\cdot,y)u(y)dy = e^{-t\varDelta}u$, and the behaviour of $K(t,x,y)$ as $t \to 0$ captures much (indeed, all) of the metric geometry of a Riemannian manifold M. In order to connect to *partial* spectra, we need to balance the exponential decay, in the spectrum of $\varDelta$, of K with its short-time asymptotics. The timescale at which this balance is preserved, as first seen in Chapter 2, is that of $t_\varLambda = O(\varLambda^{-1}\log\varLambda)$ – where $\varLambda$ is the relevant cut-off of the spectrum of $\varDelta$. Interestingly, the principle of heat flow on an appropriate timescale underlies not only our Chapters 2, 3 – as a bridge from spectral to metric data – but also – in the other direction – many of the applications of the Laplacian in computer science discussed above.

At the spectral timescale $t_\varLambda$, the heat kernel is not only well approximated by that of the truncated Laplace-type operator $P_\varLambda\varDelta$, but is also optimally *localized* given that condition. In Section 3.3 we show that the probability measure on M associated to the fiberwise square norm of the vector $P_\varLambda K(t_\varLambda,x,\cdot)$ has second Wasserstein distance

$\widetilde{O}\left(\Lambda^{-2}\right)$ to $x \in M$, providing an asymptotic counterpart to the points of M in terms of truncations of its Dirac-type spectral triple.

The continuum nature of Connes' reconstruction theorem

Somewhat orthogonal to the issue of connecting metric and spectral geometry is that of demarcating spectral geometry inside the larger realm of *noncommutative* geometry. Connes' reconstruction theorem [Con13] (cf. pp. 20 below) provides algebraic criteria to determine when a spectral triple is *commutative*, that is, of Dirac type. These criteria, however, do not coexist well with truncation. It is very much an open problem to find conditions that, if only asymptotically, ensure that a given operator system spectral triple corresponds to the truncations of a Dirac-type spectral triple. Chapter 4 investigates a first approach, employing the *higher Heisenberg relation* of [CCM14] as a version of the crucial orientability axiom.

Both the higher Heisenberg relation - as an index constraint - and the results of Chapter 2 relate truncations of spectral triples to K-theory invariants. The fascinating result of [LS17] shines a different light on that relation, and combining the two presents an interesting avenue for further work.

1.1 Mathematical preliminaries

The following chapters are concerned with the geometric properties of *truncated*, *Dirac-type spectral triples*. The basic philosophy is that the geometric information about M that is present in such a spectral triple should be recoverable, asymptotically, from its truncations. The most important technique that connects a Dirac-type spectral triple to its truncations is the asymptotic expansion of the heat kernel of generalized Laplacians. The interplay between that expansion and the (metric and topological parts of) Connes' reconstruction theorem constitutes the central theme of Chapters 2 and 3.

Dirac-type spectral triples

This section introduces the topic of Dirac-type operators on Hermitian bundles over smooth Riemannian manifolds. For further reading and context, see [BGV04; GVF01].

Definition 1.1.1. A second-order differential operator Δ on a vector bundle E over a smooth Riemannian manifold M is a *generalized Laplacian* if its principal symbol equals the metric. That is, for all $f \in C^\infty(M)$, the section $\sigma_2(\Delta, df)$ of $\operatorname{End} E$ is given by

$$\sigma_2(\Delta, df) \stackrel{\text{def}}{=} -\frac{1}{2}[[\Delta, f^*], f] = (df, df),$$

where (df, df) is the element of $C^\infty(M)$ given by $x \mapsto (df_x, df_x)_{T_x^*M}$.

The definition sets up a strong expectation that the Riemannian structure of M is fully captured by such an (elliptic) operator Δ and the action of $C^\infty(M)$ on $\Gamma^\infty(E)$. This (and more) is indeed the case. The precise statement will be part of Connes' reconstruction theorem, below.

Definition 1.1.2. A *Dirac-type operator* D on a vector bundle E over a smooth manifold M is a first-order differential operator on E such that D^2 is a generalized Laplacian.

Proposition 1.1.3. The principal symbol of a Dirac-type operator induces a Clifford module structure on E. That is to say, for $f \in C^\infty(M)$, the section

$$c(df) \stackrel{\text{def}}{=} i[D, f]$$

of $\operatorname{End} E$ satisfies

$$c(df)c(dg) + c(dg)c(df) = 2(df, dg).$$

Proof. Because D is of order 1, the commutator $[D, f]$ is an endomorphism of E and so commutes with $C^\infty(M)$. Therefore, we have $Df[D,g] + [D,g]Df = D[D,g]f + [D,g]Df = [D^2,g]f$ and similarly $fD[D,g] + [D,g]fD = fD[D,g] + f[D,g]D = f[D^2,g]$ so that $[D,f][D,g] + [D,g][D,f] = [[D^2,g],f]$, which equals $-2(df,dg)$ by polar decomposition of $\sigma_2(\Delta, \cdot)$. ☐

Corollary 1.1.4. For $f \in C^\infty(M)$, one has $\sup_{x\in M} \|df_x\|_{T_x^*M} = \|[D,f]\|_{\Gamma(\operatorname{End} E)}$ which is in turn, by smooth extension of extremizing elements of E, equal to $\|[D,f]\|_{B(L^2(E))}$.

The principal symbol of D is therefore nothing more or less than a Clifford module structure. In fact, there exists a canonical connection ∇ on E associated to D^2 such that $D = -ic \circ \nabla$ up to a bundle endomorphism, as follows.

Proposition 1.1.5. Let Δ be a generalized Laplacian and Δ_0 be the Laplacian on functions. Then, the map $\nabla\colon \Gamma(E) \times \Gamma(TM) \to \Gamma(E)$ defined by $\nabla_{(df)^\#} s = \frac{1}{2}(\Delta_0(f) - [\Delta, f])s$ is an affine connection on E. Moreover, up to a bundle endomorphism of E, the associated connection Laplacian on E is equal to Δ.

Corollary 1.1.6. The operator $D' = -ic \circ \nabla$ is a first-order Dirac-type differential operator on E. In particular, $D - D'$ is a bundle endomorphism of E.

Remark 1.1.7. Any compact, orientable Riemannian manifold carries a Dirac-type operator on some Hermitian vector bundle. For spin^c manifolds the most natural choice of Dirac-type operator is given by the Dirac operator $D = -i(1 \otimes c) \circ \nabla$, where ∇ is a Clifford connection on the chosen spinor bundle – if the manifold is in fact spin, then the connection can be determined uniquely if one demands that it commute with the charge conjugation operator.

On spin^c manifolds equipped with a spinor bundle S, each bundle E carrying a Dirac-type operator D_E is of the form $S \otimes F$, where F is a Hermitian vector bundle carrying a Hermitian connection ∇^F, and D_E is of the form $D_S \otimes 1 - ic \otimes \nabla^F$, plus possibly an endomorphism of E. That is to say, the rank of S is minimal among those carrying Dirac-type operators, and each Dirac-type operator is just a twist of D_S - see [Ply86].

The interplay between the algebra $C^\infty(M)$ and a Dirac-type operator D is realized on the Hilbert space $L^2(E; M)$ given by the completion of $\Gamma(E)$ with respect to the inner product induced by the Hermitian structure on E and the volume form on M. This triple is in fact sufficient to recover the geometric data used to define it, such as the smooth, Riemannian manifold M, the bundle $E \to M$ with its Hermitian structure, and of course the differential operator D.

Definition 1.1.8 (Dirac-type spectral triples). A *Dirac-type spectral triple* (associated to the compact, Riemannian manifold M) consists of the algebra $C^\infty(M)$ and the Dirac-type operator D on a Hermitian vector bundle E over M, both viewed in their representation on the Hilbert space $L^2(E; M)$.

By Proposition 1.1.3, Corollary 1.1.14 (below) and the discussion preceding Proposition 1.1.13 (below), a Dirac-type spectral triple is a special case of the following definition.

Definition 1.1.9 (Spectral triples). Let A be a unital C*-algebra represented on a Hilbert space H, and let D be a selfadjoint, possibly unbounded, operator on H. Let $\mathcal{A} \subset A$ be a dense $*$-subalgebra. Then, $(\mathcal{A}, H, D)$ is said to be a *spectral triple* whenever

- The resolvent $(D + i)^{-1}$ is compact.
- $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Connes' reconstruction theorem (Theorem 1.1.39, below) shows that each *commutative* spectral triple is (unitarily equivalent to) a Dirac-type spectral triple. That is to say, given a spectral triple $(\mathcal{A}, H, D)$, there is a known list of algebraic conditions on $\mathcal{A}, D$ that allow one to determine whether it is unitarily equivalent to a Dirac-type spectral triple. As such, the Dirac-type spectral triples form a reasonably general and well-understood spectral picture of Riemannian geometry.

Truncated spectral triples

In quantum field theory, the scaling behaviour of classical field theories and the postulated infinite granularity of space(time) often conspire to introduce (ultraviolet) divergences. This necessitates a careful balancing act, *regularizing* the scaling behaviour, that is then combined with experimental evidence to produce a *renormalized* theory with physical predictions. A common way to regularize the theory is to introduce a *momentum cutoff*, restricting the considered interactions to those whose momentum is bounded by a cutoff parameter Λ, and then investigating the scaling with Λ.

In the fermionic sector[1] of gauge theories, this was expressed in [AB84] as a truncation $D \mapsto P_\Lambda D P_\Lambda$ of the relevant Dirac operator, where $P_\Lambda = \mathbb{1}_{[-\Lambda,\Lambda]}(D)$ is the spectral projection onto the eigenspaces of eigenvalue $|\lambda| \leq \Lambda$ of D.

Such a truncation is very natural: one does not need to make additional choices (unlike when discretizing spacetime[2], for instance), the high-momentum interactions that are neglected are naturally difficult to observe, and all symmetries of the original Dirac operator (including, notably, those induced by isometries) are preserved because they commute with P_Λ.

Thirty years later, after the birth of noncommutative geometry, [DLM14b] investigated what the geometric consequences of such a truncation would be, when interpreted in the context of that discipline. Viewed from the perspective of comparison to the non-truncated geometry, the relevant objects are of the following form.

Definition 1.1.10. Let $(\mathcal{A}, H, D)$ be a spectral triple, let $\Lambda \in \mathbb{R}_+$ and let $P_\Lambda = \mathbb{1}_{[-\Lambda,\Lambda]}(D) \in B(H)$. Then, $(P_\Lambda \mathcal{A} P_\Lambda, P_\Lambda H, P_\Lambda D P_\Lambda)$ is said to be a *truncated* spectral triple and a *spectral truncation* of $(\mathcal{A}, H, D)$.

[1] And in far greater generality: see Section 1.1.

[2] In the sense that the only parameter here is Λ, whereas discretizations require a choice of triangulation or other decomposition into building blocks. It should be noted, however, that such geometric choices may exhibit a degree of universality, making them less relevant from a physical point of view (cf. e.g. [JL13; CS12]).

It is important to realize that $P_\Lambda \mathscr{A} P_\Lambda$ is no longer naturally an algebra, because P_Λ is usually far from commuting with $\mathscr{A}$ (compare Proposition 1.1.3). This puts us firmly outside the realm of spectral triples *per se*, and into the realm of *operator system spectral triples* as introduced in [CS20]. The general theory developed there is very much in line with the physics angle discussed here, and aims in particular to bring its increased[3] flexibility to bear on quantum gravity.

The central objects of study of the following chapters are the spectral truncations of *Dirac-type* spectral triples. Chapters 2 and 3 explore how the geometry of M relates to truncations of Dirac-type spectral triples $(C^\infty(M), L^2(E;M), D)$. Chapter 4, in contrast, probes whether one can recognize spectral truncations of Dirac-type spectral triples algebraically.

The spectral function of a generalized Laplacian

The interplay between the geometric properties encoded by the principal symbol of a generalized Laplacian and the spectral properties of its closure Δ in $L^2(E;M)$ is largely mediated through the *heat kernel* and its asymptotic expansion. At the heart of Chapter 2 lies a balance between the exponential decay of the (trace of) the associated heat operator in the spectrum of Δ and its well-known short-time divergence. Additional control over the exponential off-diagonal decay of the heat kernel, balanced with both, then allows for technical control over the tension between the local geometry of M and the global invariants provided by the spectrum of Δ, as figures prominently in Chapter 3.

Definition 1.1.11 (Heat kernel). Let M be a compact, orientable Riemannian manifold and let Δ be a generalized Laplacian acting on a smooth bundle $E \to M$. A *heat kernel* of Δ is a smooth section $p_t(x,y)$ of the bundle $E \boxtimes E^*$ over $M \times M \times \mathbb{R}_{>0}$, such that

- $p_t(x,y)$ satisfies the *heat equation*
$$(\partial_t + \Delta_x) p_t(x,y) = 0.$$
- For all smooth sections s of E, one has
$$\lim_{t \to 0} \int_M p_t(x,y) s(y) d\,\mathrm{vol}(y) = s(x),$$

[3] As compared to Riemannian geometry.

uniformly in $x \in M$.

The second condition guarantees uniqueness, if a heat kernel exists.

Note that $p_t(x,y)$ is the integral kernel of the operator $e^{-t\Delta}$.

Let ρ be the injectivity radius of M and let χ be a smooth, monotone decreasing function on $\mathbb{R}_+$ such that $\chi(s^2) = 1$ for $0 \leq s < \rho/2$ and $\chi(s^2) = 0$ for $s > \rho$.

Theorem 1.1.12 (Asymptotics of the heat kernel)**.** Let M be a compact, orientable Riemannian manifold and let Δ be a generalized Laplacian acting on a smooth bundle $E \to M$. Then, there exists a heat kernel $p_t(x,y)$ of Δ.

Moreover, there exist unique smooth sections $\Phi_k \in \Gamma^\infty(M \times M, E \boxtimes E^*)$ such that asymptotically

$$p_t(x,y) \sim (4\pi t)^{-m/2} e^{-d(x,y)^2/4t} \chi(d(x,y)^2) \sum_{k=0}^{\infty} t^k \Phi_k(x,y),$$

in the sense that

$$\sup_{x,y \in \mathcal{J}} \left\| \partial_t^l \left(p_t(x,y) - (4\pi t)^{-m/2} e^{-d(x,y)^2/4t} \chi(d(x,y))^2 \sum_{k=0}^{N} t^k \Phi_k(x,y) \right) \right\| = O(t^{N-m/2-l}),$$

for all $N > m/2$ and all $l \geq 0$. Here, Φ_0 is the parallel transport map associated to the connection determined by Δ.

Proof sketch, as in [BGV04, Theorem 2.30]. Construct a formal solution to the heat equation by termwise solving $(\partial_t + \Delta_x)(4\pi t)^{-m/2} e^{-d(x,y)^2/4t} \chi(d(x,y))^2 \sum_{k=0}^{\infty} t^k \Phi_k(x,y) = 0$ in order to obtain the Φ_k. Then show that the partial sum

$$k_t^N(x,y) \stackrel{\text{def}}{=} (4\pi t)^{-m/2} e^{-d(x,y)^2/4t} \chi(d(x,y))^2 \sum_{k=0}^{N} t^k \Phi_k(x,y)$$

constitutes a parametrix of the heat equation, meaning that $r_t^N(x,y) \stackrel{\text{def}}{=} (\partial_t + \Delta_x) k_t^N(x,y)$ lies in $\Gamma^l(\mathbb{R}_{\geq 0} \times M \times M, E \boxtimes E^*)$ and is $O(t^{N-m/2-l/2})$ in C^l norm, for $N > l + m/2$, and that

$$\lim_{t \to 0} \int_M k_t^N(x,y) s(y)\, d\,\mathrm{vol}(y) = s(x)$$

for sections s of E. Fix $N > m/2 + 1$ and define the iterated convolution products $q_t^l \stackrel{\text{def}}{=} k_t^N * (r_t^N)^{*l-1}$ in the variable t. Then, the series $\sum_{l=0}^{\infty} (-1)^l q_t^l(x,y)$ will converge in C^2 norm, its limit $p_t(x,y)$ will be a heat kernel, and the remainder $\partial_t^p(p_t(x,y) - k_t^N(x,y)) = \partial_t^p \sum_{l=1}^{\infty}(-1)^l q_t^l$ will be $O(t^{N-m/2-p-q/2+1})$ in C^q norm. □

Let P_t be the operator defined by the integral kernel $p_t(x,y)$. One can use the facts that, due to the heat equation, P_t commutes with Δ and that $\lim_{t\to 0} P_t s \to s$ to show that Δ is essentially selfadjoint whenever it is symmetric, with respect to a Hermitian structure on E that defines the Hilbert space $L^2(E;M)$. For brevity, we will denote its closure by Δ as well.

Proposition 1.1.13. Let Δ be the selfadjoint closure of a symmetric generalized Laplacian on $L^2(E;M)$ with heat kernel $p_t(x,y)$. Then, we have $P_t = e^{-t\Delta}$. In particular, if E_λ is the projection on the λ-eigenspace of Δ, we have $P_t = \sum_\lambda e^{-t\lambda}E_\lambda$ in norm.

Proof. Note that the operators P_t form a semigroup by uniqueness of the heat kernel. Thus, there is an increasing sequence of real numbers λ_i such that the eigenvalues of the positive operators $P_t = P_{t/2}^2$ are of the form $e^{-\lambda_i t}$. Now, for eigenspinors ψ of P_t with eigenvalue $e^{-\lambda t}$, we have $-\lambda\psi = \partial_t P_t \psi|_{t=0} = -\Delta\psi$, so that in fact $P_t = e^{-t\Delta}$. □

This exponentially decaying series expression provides an essential link between the spectral truncations of Δ and the geometric properties of M encoded in its symbol.

Corollary 1.1.14 (Asymptotic trace of the heat kernel). Let E be equipped with a Hermitian structure such that the generalized Laplacian Δ is symmetric. Then, the operator $P_t = e^{-t\Delta}$ is of trace class for all $t > 0$, and its trace,

$$\operatorname{tr} P_t = \int_M \operatorname{tr}_{E_x} p_t(x,x)\,dx,$$

satisfies

$$\operatorname{tr} P_t \sim (4\pi t)^{-m/2} \sum_{k=0}^{\infty} t^k \int_M \operatorname{tr}_{E_x} \Phi_k(x,x)\,d\operatorname{vol}(x).$$

Proof sketch. That P_t is of trace class follows from the fact that the kernel of its square root, $P_{t/2}$, is square-integrable. The asymptotic expansion of the trace follows from the restriction of the expansion of $p_t(x,y)$ to the diagonal. □

Corollary 1.1.15. The operator Δ has compact resolvent.

Proof. Because $e^{-t\varDelta}$ is in the trace class, the spectrum of $\varDelta$ must be discrete, with finite multiplicity, and accumulate only at ∞. $\square$

Corollary 1.1.16 (The zeta function). Let E_λ be the projections onto the λ eigenspaces of $\varDelta$. Then, define the one-parameter semigroups $\varDelta^{-s}$ by $\varDelta^{-s} \overset{\text{def}}{=} \bigoplus_{\lambda>0} \lambda^{-s} E_\lambda$, which are clearly bounded (and compact) for $\Re s > 0$.

Then, for $\Re s > m/2$, the operator $\varDelta^{-s}$ is of trace class. Moreover, the *zeta function*, defined for $\Re s > m/2$ as

$$\zeta(\varDelta, s) \overset{\text{def}}{=} \operatorname{tr} \varDelta^{-s}$$

extends meromorphically to $\mathbb{C}$, with simple poles at $\{m/2, m/2-1, \dots, 1\}$ whenever m is even, and at $\{m/2, m/2-1, \dots\}$ whenever m is odd, and with residues

$$\operatorname{res}_{s=m/2-k} \zeta(\varDelta, s) = \frac{\int_M \varPhi_k(x,x)\, d\operatorname{vol}(x)}{(4\pi)^{m/2} \varGamma(m/2-k)}$$

whenever $m/2-k$ is indeed a pole.

Proof. By the Jensen-Cahen theorem [HR64, Theorem 2], the series $\sum_{\lambda>0} \operatorname{rk} E_\varDelta \lambda^{-s}$ is uniformly convergent on angular regions in the half-plane that forms its domain, so that it is holomorphic there.

Let $g(t) \overset{\text{def}}{=} \sum_{\lambda>0} e^{-t\lambda}$, so that $g(t) = \operatorname{tr} P_t - \operatorname{rk} E_0$. Consider the Mellin transform $f(s) \overset{\text{def}}{=} \int_0^\infty t^{s-1} g(t)$, which is well-defined for $\Re s > m/2$ due to the small-time asymptotics of $g(t)$, its positivity and its monotone decay. Then, as in [FGD95, Theorem 3], $f(s)$ has the singular expansion $f(s) = \sum_k \frac{c_k}{s-s_k}$, where $s_k = m/2-k$ and c_k is the coefficient of $t^{k-m/2}$ in the asymptotic expansion of the heat trace. Moreover, by the Fubini-Tonelli theorem we may write $f(s) = \sum_{\lambda>0} \operatorname{rk} E_\lambda \int_0^\infty e^{-\lambda t} t^{s-1} dt = \varGamma(s) \sum_{\lambda>0} \lambda^{-s} \operatorname{rk} E_\lambda = \varGamma(s) \operatorname{tr} \varDelta^{-s}$. We must therefore conclude that $\zeta(\varDelta, s) = f(s)/\varGamma(s)$ continues analytically to a meromorphic function on $\mathbb{C}$. Moreover, by the knowledge of the poles of $f(s)$ and $\varGamma(s)$, the poles of $\zeta(\varDelta, s)$ must all be simple and must be located at $s_k = m/2-k$, for $k \in \mathbb{Z}_{\geq 0}$, unless s_k is a nonpositive integer. $\square$

Proposition 1.1.17 (Identification of the first asymptotic heat coefficients). We have

$$\operatorname{tr}_{E_x} \varPhi_0(x,x) = \operatorname{rk} E, \qquad \operatorname{tr}_{E_x} \varPhi_1(x,x) = \operatorname{rk} E \cdot R(x)/6,$$

where R is the scalar curvature of the Riemannian metric $\sigma(\varDelta)$ on M.

Proof sketch, cf. [Ros97]. Clearly $\varPhi_0(x,x) = \mathrm{id}_{E_x}$. If we locally trivialize the bundle E in Riemannian normal coordinates around x, then we have $\varPhi_0(\exp(0), \exp(y)) = \sqrt{\det g_{\exp y}}$ for sufficiently small $y \in \mathbb{R}^m$, so that $\varPhi_0$ is expressible as a Taylor series in universal polynomials of the components of the Riemann tensor and its covariant derivatives by a theorem due to Cartan, cf. [Ber03, Proposition 67]. By construction, then, all $\varPhi_k$ are expressible in that fashion, so that $\operatorname{tr}\varPhi_1(x,x)$ is expressible as a universal polynomial in the Riemann tensor. Now, note that the endomorphisms $\varPhi_k^{\alpha^2}$ associated to the generalized Laplacian $\alpha^2\varDelta$ (with respect to the rescaled metric) must equal $\alpha^{-2k}\varPhi_k$, whereas the kth iterated covariant derivatives of the Riemann tensor scale as α^{-2-k}. This allows us to conclude that $\operatorname{tr}\varPhi_1$ cannot contain constant or derivative terms, so that it is a universal linear polynomial in the Riemann tensor. It must, therefore, be proportional to the scalar curvature. Explicit calculation on e.g. n-spheres completes the proof. □

Chapter 2 discusses how to approximate these coefficients of the heat trace asymptotics using only a *finite* part of the spectrum of $\varDelta$.

Corollary 1.1.18 (Weyl's law). The counting function $N(\varLambda) \stackrel{\text{def}}{=} \#\{\sum_{\sigma(\varDelta)\ni\lambda<\varLambda} \operatorname{tr} E_\lambda\}$ satisfies

$$N(\varLambda) \sim \varLambda^{m/2} \frac{\operatorname{vol}(M)\operatorname{rk}(E)}{(4\pi)^{m/2}\varGamma(m/2+1)}$$

as $\varLambda \to \infty$.

Proof. The Wiener-Ikehara theorem [Wie88, Theorem 19.16] shows that, whenever the function g is positive and the integral $f(s) = \int_{\mathbb{R}_+} x^{-s}\,dg(x)$ converges for $\Re s > 1$, we have $g(x) \sim x \cdot \lim_{s\to 1}(s-1)f(s)$. Now write $g(x) = N(x^{m/2})$, so that $\operatorname{res}_{s=1} f(s) = \operatorname{res}_{s=1}\zeta(\varDelta, \frac{ms}{2})$. □

States and Gelfand duality

Let $\mathrm{Top}_{\mathrm{HCpt}}$ be the category of compact Hausdorff spaces and continuous maps, and let $C^*\,\mathrm{Alg}_{\mathrm{Com}}$ be the category of unital, abelian C*-algebras and $*$-homomorphisms.

Abstractly speaking, *Gelfand duality* is the statement that a specific pair $C, \varGamma$ of functors implements an equivalence between these categories. For further details on the topic of this section, see [GVF01, Chapter 1] and the references therein.

Definition 1.1.19. The (contravariant) functor $C\colon \mathrm{Top}_{\mathrm{HCpt}} \to C^* \,\mathrm{Alg}_{\mathrm{Com}}$ sends a compact topological space X to the unital, abelian C*-algebra $C(X)$ of $\mathbb{C}$-valued continuous functions.

On morphisms, C sends a continuous map $\phi\colon X \to Y$ to the $*$-homomorphism ϕ^* from $C(Y)$ to $C(X)$ given by $f \mapsto f \circ \phi$.

Now, points $x \in X$ correspond to elements of $\mathrm{Hom}(*, X)$, and so under the functor C correspond to evaluation morphisms $\mathrm{ev}_x\colon C(X) \to C(*) = \mathbb{C}$. The construction of the functor Γ proceeds from the realization that such $\mathbb{C}$-valued homomorphisms (*characters*), endowed with the weak $*$-topology, correctly capture the notion of *point* in the setting of $C^* \,\mathrm{Alg}_{\mathrm{Com}}$.

For the purposes of Chapter 3, we will first work with the larger subspace of the topological dual A^* consisting of *states* (instead of just the characters), so as to gain a more flexible picture of the dictionary between $\mathrm{Top}_{\mathrm{HCpt}}$ and $C^* \,\mathrm{Alg}_{\mathrm{Com}}$.

Definition 1.1.20. Let A be a C*-algebra. A bounded linear map $\phi\colon A \to \mathbb{C}$ is said to be a *state* of A if $\phi(1) = 1$ and $\phi(a^* a) \geq 0$ for all $a \in A$. The convex set of all such states is denoted by $S(A)$.

Indeed, all states of $C(X)$ are convex combinations of evaluation maps, in the following sense.

Theorem 1.1.21 (Riesz representation theorem). Let ϕ be a state of $C(X)$, for X a compact Hausdorff space. Then there exists a regular Borel probability measure μ_ϕ on X such that

$$\phi(f) = \int_X f \, d\mu_\phi$$

Proof. See [Con90, Appendix C]. □

Corollary 1.1.22. The extreme points of the convex set $S(C(X))$ are evaluation functionals $\{\mathrm{ev}_x \mid x \in X\}$.

Proof. Let μ be the measure corresponding to an extreme point of the convex set of states.

First, assume $K \hookrightarrow X$ has measure $0 < \mu(K) < 1$. Then, let $\mu_0(Y) \overset{\mathrm{def}}{=} \mu(Y \cap X)/\mu(K)$ and $\mu_1(Y) \overset{\mathrm{def}}{=} \mu(Y \setminus K)/\mu(X \setminus K)$ and observe that $\mu = \mu(K) \cdot \mu_0 + \mu(X \setminus K) \cdot \mu_1$. Thus, μ takes values in $\{0, 1\}$ whenever it is an extreme point.

Let now x, y be both in the support of μ and let K_x, K_y be disjoint open sets containing x, y respectively. Now, by additivity and positivity of μ, either $\mu(K_x) = 0$ or $\mu(K_y) = 0$, contradicting the assumption. We conclude that $\operatorname{supp}(\mu)$ is a single point $x \in X$, so that $\int_X f d\mu = f(x) = \mathrm{ev}_x(f)$. □

That is to say, we can recognize the points of X purely in terms of the C*-algebra $C(X)$. What about its topology?

Definition 1.1.23. Let A be a unital, abelian C*-algebra. Its *Gelfand spectrum* $\widehat{A}$ is the set of extreme points of $S(A)$, equipped with the weak $*$-topology.

Theorem 1.1.24. Let A be a unital, abelian C*-algebra. Then $\widehat{A}$ is compact and Hausdorff.

Proof. First, note that the weak $*$-topology on A^* is completely Hausdorff: after all, the evaluation functionals $\{\mathrm{ev}_v \mid v \in \mathcal{X}\}$ are continuous and separate points. Moreover, by Banach-Alaoglu (see [Con90, p. V.3.1]) its unit ball is compact.

Now let us show that $\widehat{A}$ is a closed subspace (which is contained in that unit ball by Cauchy-Schwartz). To that end, we will show that $\widehat{A}$ consists precisely of the multiplicative elements of A^*.

Assume $\phi \in \widehat{A}$ is multiplicative and that there exist $\phi_1, \phi_2, \lambda_1, \lambda_2$ with $\phi = \lambda_1\phi_1 + \lambda_2\phi_2$. In particular, $\lambda_1\phi_1 \leq \phi$. By Cauchy-Schwartz for the functional $(a, b) \to \phi_1(a^*b)$, then, we see that $\ker\phi \subset \ker\lambda_1\phi_1$, so that we must have $\phi_1 = \phi$. That is to say, ϕ must be an extreme point.

Conversely, suppose $\phi \in A^*$ is not multiplicative. Then the (GNS) representation π_ϕ of A by multiplication on $A/\ker\phi$, with the inner product induced by ϕ, cannot be one-dimensional: there exist a, b in A such that $\left\langle \pi_\phi(a)(1_A), \phi_\phi(b)(1_A)\right\rangle = \phi(a^*b)$, which by assumption on a, b does not equal $\phi(a^*)\phi(b) = \left\langle \phi_\phi(a)(1_A), 1_A\right\rangle \left\langle 1_A, \pi_\phi(b)(1_A)\right\rangle$, so that 1_A cannot be an orthonormal basis even though it is normalized. Thus, by Schur's lemma there exist a projection P commuting with π_ϕ. Then, the linear functionals $a \mapsto \left\langle 1_A, P\phi_\phi(a)1_A\right\rangle$ and $a \mapsto \left\langle 1_A, (1-P)\pi_\phi(a)1_A\right\rangle$ are positive and sum to ϕ, so that ϕ cannot be an extreme point of the state space.

Now, the weak $*$-limit of multiplicative states is multiplicative, so that the set of extremal states is a closed subset of a compact Hausdorff space. □

Corollary 1.1.25. Let A be a unital, abelian C*-algebra and let $\widehat{A}$ be its space of extremal states with the weak $*$-topology. Then the map $\mathrm{ev}\colon A \to C(\widehat{A})$ is an isomorphism of C*-algebras.

Proof. As $\widehat{A}$ consists of multiplicative $*$-functionals, the map ev is itself a $*$-homomorphism. It is injective because the states are the convex hull of the extremal states by Krein-Milman and separate A. Thus, $\mathrm{ev}(A)$ is a closed subalgebra of $C(\widehat{A})$ that separates points. By Stone-Weierstrass (the proof of which, as in [Con90, p. V.8.1], is similar to the arguments above: roughly, the unit ball of the set of functionals vanishing on such a closed subalgebra must be the convex hull of its extreme elements, which must then correspond to points of $\widehat{A}$) we conclude that ev is surjective. □

Corollary 1.1.26. Let X be a compact Hausdorff space. Then the map $\mathrm{ev}\colon X \to \widehat{C(X)}$ is a homeomorphism.

Proof. That ev is surjective was already shown, and it is injective by Urysohn's lemma. If $x_n \to x$, then $\mathrm{ev}_{x_n} \to \mathrm{ev}_x$ by definition of the weak $*$-topology. Thus, ev is a continuous bijection of compact Hausdorff spaces and is therefore a homeomorphism. □

Definition 1.1.27. The *Gelfand spectrum* is the covariant functor $\Gamma\colon C^*\,\mathrm{Alg}_{\mathrm{Com}} \to \mathrm{Top}_{\mathrm{HCpt}}$ that maps objects $A \in C^*\,\mathrm{Alg}_{\mathrm{Com}}$ to their extremal state space $\widehat{A}$ and morphisms $\psi\colon A \to B$ to the pullback $\psi^*\colon \widehat{B} \to \widehat{A}, \chi \mapsto \chi \circ \psi$.

Gelfand duality now follows from Corollary 1.1.25 and 1.1.26:

Theorem 1.1.28 (Gelfand duality). The functors C, Γ form an equivalence of categories.

This is, roughly, the *topological* statement at the heart of spectral geometry. For the metric part, we need to introduce the Connes metric on $S(C(X))$.

The Connes metric

Definition 1.1.29 (Connes metric). Let $(C^\infty(M), L^2(E;M), D)$ be a Dirac-type spectral triple. Then, the *Connes metric* on the space $S(C(M))$ of states of $C(M)$ is given by

$$d(\phi,\psi) \stackrel{\mathrm{def}}{=} \sup\left\{ |\phi(f) - \psi(f)| \,\middle|\, f \in C^\infty(M), \|[D,f]\| \le 1 \right\}.$$

Definition 1.1.30 (Wasserstein[4] distance). Let M be a compact Riemannian manifold and let μ, ν be probability measures. Then the p-th *Wasserstein distance* between μ and ν is

$$W_p(\mu,\nu) \stackrel{\text{def}}{=} \left(\inf_{\rho\in\Pi(\mu,\nu)} \int_{M\times M} d(x,y)^p \, d\rho(x,y) \right)^{1/p},$$

where $\Pi(\mu,\nu)$ is the set of probability measures ρ on $M\times M$ such that $\rho[A\times M] = \mu[A]$, $\rho[M\times A] = \nu[A]$, for all measurable $A \subset M$.

As, for atomic measures δ_x, δ_y supported at x, y in M, we have $\Pi(\delta_x, \delta_y) = \{\delta_{(x,y)}\}$, the following proposition follows immediately from the definition.

Proposition 1.1.31. With respect to the metric W_1, the embedding $x \to \delta_x$ of M into its space of probability measures is an isometry.

The relation between the Connes metric and the first Wasserstein distance is cemented by the following important theorem.

Theorem 1.1.32 (Kantorovich-Rubinstein duality). Let μ, ν be probability measures on M. Then,

$$W_1(\mu,\nu) = \sup\left\{ \int_M f \, d(\mu-\nu) \,\middle|\, f \in C(M), \operatorname{Lip} f \leq 1 \right\},$$

where $0 \leq \operatorname{Lip} f \leq \infty$ denotes the Lipschitz constant $\sup_{x,y\in M} \frac{|f(x)-f(y)|}{d(x,y)}$.

Proof. See [Vil09, Theorem 5.10]. □

In particular, by density of the inclusion $C^\infty(M) \subset C(M)$, we now see that Dirac-type operators carry not just the Riemannian metric in their symbol but, rather elegantly, the geodesic distance as well:

Corollary 1.1.33 (Connes metric, commutative case). For states ϕ, ψ of the algebra $C(M)$ of continuous functions on a compact Riemannian manifold M with associated probability measures μ_ϕ, μ_ψ, the Connes metric is equal to the first Wasserstein distance:

$$d(\phi,\psi) = W_1(\mu_\phi, \nu_\psi).$$

[4]This metric is known under various other names, especially for the case $p = 1$, including '1-Wasserstein', 'Monge-Kantorovich' and 'Earth Mover's'. See e.g.[Vil09, Chapter 6], where it is argued that the 'Wasserstein' nomenclature is unfortunate; it is, however, the most common.

Corollary 1.1.34. Let $(C^\infty(M), L^2(E;M), D)$ be a Dirac-type spectral triple. Then the metric space M is isometric to the Gelfand dual $\widehat{C(M)}$ when the latter is equipped with the Connes metric.

The Riemannian metric on M is given by the map $(f,g) \mapsto (df, dg)_{T^*M} = [D,f]^*[D,g]$. We now see that it is in fact already fully determined by the seminorm $f \mapsto \|[D,f]\|$ on $C^\infty(M)$.

Corollary 1.1.35. Let D_1, D_2 be Dirac-type operators on Hermitian bundles E_1, E_2 over Riemannian manifolds M_1, M_2. Then M_1, M_2 are isometric if and only if there exists an isomorphism $\phi\colon C(M_1) \to C(M_2)$ such that $\|[D_1, f]\| = \|[D_2, \phi(f)]\|$ for all $f \in C(M_1)$.

Proof. The 'if' part is an immediate corollary, and the 'only if' part follows from the uniqueness of the Levi-Civita connection and the fact that $\|[D,f]\|^2 = \|\nabla f\|^2$. □

Chapter 3 is occupied with approximating points (as elements of $\widehat{C(M)} \subset S(C(M))$) and the Connes distance between them in terms of truncations of Dirac-type spectral triples, so as to recover the metric space M.

The reconstruction theorem

Dirac-type spectral triples are the prime example of *commutative* spectral triples. Indeed, Connes' reconstruction theorem (Theorem 1.1.39 below) shows that they are the *only* examples thereof and are therefore uniquely characterized by the following properties.

Definition 1.1.36. A spectral triple $(\mathcal{A}, H, D)$ is said to be *commutative* whenever the $*$-algebra $\mathcal{A}$ is a commutative and there exists some integer p such that

1. *Dimension*: The n-th singular value, with multiplicity, of $(D+i)^{-1}$ is $O(n^{-1/p})$.

2. *First-order*: For all $f, g \in \mathcal{A}$, $[[D,f],g] = 0$.

3. *Regularity*: For $a \in \mathcal{A}$, all iterated commutators of a and $[D,a]$ with $|D|$ are bounded.

4. *Orientability*: There exists a Hochschild cycle $\sum_i c_i a_i^0 \otimes \cdots \otimes a_i^p$ in $Z_p(\mathscr{A},\mathscr{A})$ such that
$$\sum_i c_i a_i^0 [D, a_i^1] \cdots [D, a_i^p] = \gamma,$$
where $\gamma = 1$ if p is odd and γ is a grading on H that anticommutes with D otherwise.

5. *Finiteness and absolute continuity*: The $\mathscr{A}$-module $E = \bigcap_m \operatorname{dom} D^m$ is finite and projective, and there exists an $\mathscr{A}$-linear, $\mathscr{A}$-valued inner product on E such that
$$\langle \xi, a\eta \rangle_{\mathbb{C}} = \operatorname{res}_{s=0} \operatorname{tr} a \langle \xi, \eta \rangle_{\mathscr{A}} |D|^{-p-s},$$
for all $a \in \mathscr{A}$.

In Dirac-type spectral triples, the first axiom corresponds to Weyl's asymptotics (Corollary 1.1.18), the second to the fact that D is a first-order differential operator, the third to the fact that a and $[D,a]$ are smooth sections of the endomorphism bundle for all $a \in \mathscr{A}$, the fourth to the existence of a Riemannian volume form, and the fifth to the fact that H is the completion of the smooth sections $\Gamma^\infty(E;M)$ with respect to a Hermitian structure on E.

Definition 1.1.37. Two spectral triples $(\mathscr{A}_0, H_0, D_0)$, $(\mathscr{A}_1, H_1, D_1)$ are said to be *unitarily isomorphic* if there exists a unitary $U\colon H_0 \to H_1$ such that $U\mathscr{A}_0 U^* = \mathscr{A}_1$, $U \operatorname{dom} D_0 = \operatorname{dom} D_1$ and $UD_0U^* = D_1$.

Remark 1.1.38. When $C^\infty(M) \simeq \mathscr{A}_0 \simeq \mathscr{A}_1 \simeq C^\infty(N)$, the isomorphism with $\mathscr{A}_1$ given by $a \mapsto uau^*$ is necessarily given by the pullback through a diffeomorphism $\phi\colon N \to M$, as in [Mrč05]. If $(\mathscr{A}_0, H_0, D_0)$ and $(\mathscr{A}_1, H_1, D_1)$ are Dirac-type spectral triples, the operator $|da|^2 = [[D_0^2, a], a]$ lies in $\mathscr{A}_0$ for all $a \in \mathscr{A}_0$, so that $\phi^*(|da|_M^2) = u[[D_0^2, a], a])u^* = [[D_1^2, \phi^*(a)], \phi^*(a)] = |d\phi^*(a)|_N^2$ so that ϕ must be a Riemannian isometry. Moreover, using Gelfand duality and the Serre-Swan theorem, we can prove that u must in fact be the pullback by a bundle isomorphism that covers ϕ and intertwines the Dirac-type operators.

Theorem 1.1.39 (Connes' reconstruction theorem, [Con13][5]). If a spectral triple $(\mathscr{A}, H, D)$ is commutative (and only then), there exist a compact, oriented, smooth

[5] For reference, [Con13] contains the proof, initiated by Rennie and Varilly in [RV06], that $\mathscr{A}$ is of the form $C^\infty(M)$; for proof that under these assumptions D is then a Dirac-type operator, cf. [GVF01, Chapter 11].

manifold M and a Dirac-type spectral triple $(C^\infty(M), L^2(E;M), D_M)$ that is unitarily isomorphic to $(\mathcal{A}, H, D)$.

Note that the criteria of Definition 1.1.36 do not translate easily to the *truncations* of a commutative spectral triple. This leads to the question whether there is a version of these axioms that does, so as to detect whether a given operator system spectral triple is the truncation of a Dirac-type spectral triple. In Chapter 4 we investigate the cycle for the orientability axiom that is provided by the higher Heisenberg equation [CCM14] in a first step towards such detection.

The orientability axiom and the higher Heisenberg equation

The orientability axiom for commutative spectral triples, as in Definition 1.1.36 is essential to the proof of the reconstruction theorem, where it is used to construct a smooth atlas of M. It is so called because it expresses the grading (or the identity, in the odd case) as the Clifford action of a differential form of top degree.

In [CCM14], Chamseddine, Connes and Mukhanov prove that, in certain circumstances, there exist a very special representative of the orientability axiom. This relates to a specific choice of generators of the coordinate algebra $C^\infty(M)$, which satisfy a version of the orientability axiom that is analogous to the Heisenberg commutation relation $[x,p] = i\hbar$, as follows.

Theorem 1.1.40 (One-sided higher Heisenberg equation). If $(\mathcal{A}, H, D)$ is a Dirac-type spectral triple of dimension $m \in 2\mathbb{N}$, then there exists an element $Y \in \mathcal{A} \otimes \mathrm{Cl}^+_m$ with $Y^2 = 1, Y^* = Y$ such that

$$\mathrm{tr}_{B(H)}\, Y[D \otimes 1_{\mathrm{Cl}^+_m}, Y]^m = m!\gamma,$$

if and only if M is diffeomorphic to a disjoint sum of spheres, each of which has unit volume with respect to the Riemannian metric on M.

Here Cl^+_m is the real (Clifford) algebra generated by $1, e_1, \ldots, e_m$, with the relation $e_i e_j + e_j e_i = 2\delta_{ij}$ (which is, as a vector space, canonically isomorphic to $\bigwedge \mathbb{R}^m$), and the $B(H)$-valued trace $\mathrm{tr}_{B(H)} : B(H) \otimes \mathrm{Cl}^+_m \to B(H)$ gives m times the coefficient of $1_{\mathrm{Cl}^+_m}$.

The construction of a smooth atlas using the orientability axiom for commutative spectal triples is here illustrated by the fact that, by its algebraic properties, Y must be a covering of the sphere: its components, viewed as maps into $\mathbb{R}^{m+1}$, therefore locally form a coordinate system. The sphere being simply connected, this is also what determines the topology of M as a disjoint sum of spheres.

The reason that each disjoint summand must be of unit volume is that the Riemannian volume form on M is now equal to the pullback, under Y, of the volume form on S^m, so that the integral evaluates to $\deg Y$ times the volume of the standard m-sphere. More abstractly, the top zeta residue, $\operatorname{res}_{s=0} \operatorname{tr} D^{-m-s}$, which is proportional to the volume by Proposition 1.1.17, now equals $\operatorname{res}_{s=0} \operatorname{tr} \gamma \operatorname{tr}_{\mathcal{A}} Y[D \otimes 1, Y]^m / m!$, and the latter is proportional to the pairing between the Chern character of D and the Chern character of the idempotent $e = (Y+1)/2$, so that it is proportional to the integer $\operatorname{Index} e(D \otimes 1)e$.

The authors of [CCM14] go on to show that, for spectral triples associated to a *spin* Dirac operator, the statement can be slightly modified so as to hold in much greater generality, including for all 4-dimensional compact, oriented, connected spin manifolds.

The spectral action

The hypothetical (Wick rotated) path integral in pure Euclidean quantum gravity takes the form

$$\mathcal{Z} = \int \mathcal{D}g \exp(-S[g]),$$

where the Euclidean Einstein-Hilbert action $S[g]$ equals the total scalar curvature associated to g and the integral is over the space of four-dimensional, compact Riemannian manifolds without boundary – or perhaps of a fixed background manifold M.

In noncommutative geometry, the natural analogon of a compact Riemannian manifold (albeit with extra structure) is a Dirac-type spectral triple. The heat trace asymptotics of Corollary 1.1.14 allow one to express the Euclidean Einstein-Hilbert action in terms of the spectrum of the Dirac-type operator D. The *spectral action principle* of [CC97] now conjectures that the physical action (of any[6] reasonable field theory, including gravity) should likewise depend only on the spectrum of D and, in particular, that the natural (bosonic) physical action functional $S[D]$ should be of the form

$$S[D] = \operatorname{tr}(f(D/\Lambda)),$$

for some positive, even function f, whose moments determine the bare couplings. For an in-depth overview of this program, see [CM08, Chapters 11-19].

[6] Including the Standard Model. The surprisingly natural treatment of gauge theories in noncommutative geometry, through the inner automorphisms of almost-commutative finite spectral triples, is beyond the scope of this thesis: see [CM08, Chapter 11] and [Sui15].

The suppression of higher eigenvalues through the function f, parametrized by Λ, is a smoothed version of the finite-mode regularization discussed in Section 1.1 – this smoothed version is however *a priori* still strongly dependent on the higher eigenvalues of D. Computer simulations of the regularized path integral (see e.g. [BG16]) rather imply working with the finite-dimensional truncations $P_\Lambda D$. To bridge the gap, the results of Chapter 2 can be interpreted as the construction of admissible[7] functions f with *compact* support: see Theorem 2.2.3.

A further important question is that of *observables* of such a theory. To that end, the results of Chapter 2 can be applied immediately to transfer the spectral observables $\mathrm{res}_{s=m/2-k} \operatorname{tr} a\Delta^{-s}$, for $a \in C(M)$, to the truncated realm. Moreover, the algorithm of Chapter 3 allow one to interpret the integrands of the truncated path integral as metric spaces in their own right, so that we can translate observables from the purely metric realm (including e.g. the Gromow-Hausdorff distance to a fixed comparison space) to the setting of the spectral action.

When constructing the (truncated) spectral path integral we are faced with the choice of a domain of integration. For concreteness, let us say that a background manifold M and a energy scale Λ were fixed, so that we want to integrate $\exp(-S[D_\Lambda])$ over the space of 'truncated Dirac operators' $\{D_\Lambda\}$. Which finite-rank operators D_Λ should then be considered? The original integral over Riemannian metrics suggests we should restrict to truncations $D_\Lambda = P_\Lambda D$ of Dirac-type operators D. The question is then how to implement this restriction in the path integral, keeping in mind the difficulty of the corresponding recognition problem as discussed in Section 1.1. The approach laid out in [CM08, Chapter 18.4] is to implement the higher Heisenberg equation of [CCM14] by adding the corresponding term to the action as a dynamical constraint. To that end, Chapter 4 investigates the behaviour of such a dynamical constraint when $M = S^1$ or $M = S^2$.

[7] In the sense of producing the desired asymptotics of $\operatorname{tr}(f(D/\Lambda))$ as $\Lambda \to \infty$.

Chapter 2

Finite-rank approximation of spectral zeta residues[1]

We employ the asymptotic expansion of the heat trace to express all residues of spectral zeta functions as regularized sums over the spectrum. The method extends to those spectral zeta functions that are localized by a bounded operator.

2.1 Introduction

The spectral theory of elliptic operators presents a major connection between functional analysis and differential geometry. It provides a number of interesting relations between the spectrum with multiplicities (which is the complete unitary invariant of self-adjoint operators) and the symbol (which is directly tied to the local expression of pseudodifferential operators). Thereby, it shows how the local structure of such an operator influences its global properties, and vice versa. Of particular interest is the relation between the symbol of an elliptic operator and its spectral asymptotics that is conveyed by the spectral zeta residues.

This chapter is concerned with expressing the spectral zeta residues as a limit of partial sums of particular functions over the spectrum. Together with the well-known relations between the zeta residues, the symbol, and the heat expansion, this bridges a gap between the continuum setup of spectral geometry and the finite objects in combinatorial geometry and computer science. For instance, this method allows one to approximate

[1]This work was published as [Ste19b].

the scalar curvature on a compact Riemannian manifold using only a finite part of its Laplacian spectrum.

We will first provide some background material in Section 2.1 and then introduce the main topic and results of this chapter in Section 2.1.

Background: spectral zeta residues and Weyl's law

In 1949, Minakshisundaram and Pleijel showed [MP49] that the zeta function

$$\zeta(\Delta, s) \stackrel{\text{def}}{=} \operatorname{tr} \Delta^{-s}$$

of the Laplace operator on a compact Riemannian d-dimensional manifold can be meromorphically extended to the complex plane, with simple poles occurring in the points $d/2 - \mathbb{Z}_{\geq 0} \subset \mathbb{R}$. The residues at these poles (which are proportional to the so-called heat kernel coefficients) relate the spectrum of the Laplace operator, itself an isometry invariant, to other known isometry invariants of the manifold, such as its volume and scalar curvature.

More generally, an elliptic pseudodifferential operator defines a spectral zeta function, by the work of Seeley [See67], and its residues relate geometric information to the operator spectrum in the following way. If Δ is a positive elliptic classical pseudodifferential operator of order $m \in \mathbb{R}_{\geq 0}$ and k is any nonnegative integer, the residue at $s = (d-k)/m$ of $\zeta(\Delta, s)$ equals the Wodzicki residue

$$\frac{1}{m} \int_{S^*M} \operatorname{tr} a_{-n}^{(k-d)/m}(x, \xi) d^{n-1} \xi dx,$$

where $a_{-n}(x, \xi)$ is the homogenous term of order $-n$ in the decomposition of the symbol of Δ and $S^*M \subset T^*M$ is the cosphere bundle, cf. [Wod87; Gil95; Con94]. In the inversely Mellin transformed picture, the residues correspond to integrals of terms in the asymptotic expansion of the integral kernel of $e^{-t\Delta}$ along the diagonal, as $t \to 0^+$.

One wonders which conclusions about the operator spectrum can be drawn from the zeta residues, apart from the rather opaque one provided by their definition. The most well-known result in this direction is provided by the Wiener-Ikehara theorem, which relates the first residue of the zeta function to the asymptotics of the number $N(\Lambda)$ of eigenvalues smaller than Λ. If α is a monotone increasing function and the zeta function

$$\zeta(z) = \int_1^\infty \lambda^{-z} d\alpha(\lambda)$$

converges for $\Re z > 1$ and can be meromorphically extended to $\Re z \geq 1$ with a simple pole at $z = 1$, then the Wiener-Ikehara theorem states that $\alpha(\Lambda) \sim \Lambda \operatorname{res}_{z=1} \zeta(z)$ as $\Lambda \to \infty$. By Seeley's results on complex powers of elliptic operators, we may apply this theory to the spectral zeta function of a degree m elliptic pseudodifferential operator on a d-dimensional manifold. As in [MP49], this yields Weyl's famous law

$$N(\Lambda) = \operatorname{res}_{s=d/m} \zeta(\Delta, s) \Lambda^{d/m} + o(\Lambda^{d/m}).$$

Note that this restriction only uses the first residue of the zeta function: the zeta series $\operatorname{tr} \Delta^{-s}$ is absolutely convergent for $\Re s > d/m$.

The problem of improving on the accuracy in Weyl's law has attracted much attention over the last century. That is, one wonders whether we can obtain an asymptotic expansion of $N(\Lambda) - \operatorname{res}_{s=d/m} \zeta(\Delta, s) \Lambda^{d/m}$ for specific classes of operators. However, sharp bounds that depend only on the spectral zeta function of Δ have not yet been produced, even in the well-studied case of the Laplacian on flat tori. It would seem natural that the Wiener-Ikehara result could be extended to relate the asymptotic expansion of $N(\Lambda)$ to the location of the poles of the zeta function. However, this approach is limited by the difficulties of inverse Mellin and Laplace transforms, see e.g. [Ara96], [Ber03, p. 9.7.2]. The lower poles of the zeta function can therefore not yet be related to the asymptotics of $N(\Lambda)$. However, we will explain in the present chapter how to relate their residues to the asymptotics of other functionals of the operator spectrum.

Zeta residues as a resummation of the spectrum

In the converse direction to Weyl's law, we ask which conclusions one can draw about the residues, given access to increasingly large *finite* subsets of the operator spectrum. For the first residue, Weyl's law gives rise to the Dixmier trace[2] formula: if $\lambda_0 \leq \lambda_1 \leq \cdots$ are the eigenvalues of Δ, then

$$\operatorname{res}_{s=d/m} \zeta(\Delta, s) = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} \lambda_i^{-d/m}}{\log N} = \lim_{\Lambda \to \infty} \frac{\sum_{\lambda < \Lambda} \lambda^{-d/m}}{\frac{d}{m} \log \Lambda}.$$

[2] As restricted to the ideal on which this formula converges. For a treatment of the Dixmier trace on the larger ideal $L^{(1,\infty)}$, see [Car+07].

The Dixmier trace is singular: it vanishes if $\operatorname{tr}\Delta < \infty$. However, if Δ_j is a sequence of finite rank operators of operator norm $\|\Delta_j\| \stackrel{\text{def}}{=} \Lambda_j \to \infty$ such that

$$\operatorname{tr}\Delta_j^{-s_0} - \sum_{0<\lambda<\Lambda_j} \lambda^{-s_0} = o(\log \Lambda_j),$$

the Dixmier trace of Δ^{-s_0} clearly equals $\lim_{j\to\infty} \operatorname{tr}\Delta_j^{-s_0} / \log \Lambda_j$. It would seem plausible that this result can be extended to further poles, i.e. that there are also functionals c_k on the finite rank operators such that

$$\lim_{j\to\infty} c_k(\Delta_j) = \operatorname{res}_{s=s_k} \zeta(\Delta, s),$$

if $\zeta(\Delta, s)$ has poles at $s_0 > \cdots > s_k$ and the finite-rank Δ_j converge to Δ in some appropriate sense. The main question, then, is whether we can write these functionals down explicitly.

The existence of such asymptotic residue functionals is shown by our Theorem 2.2.3, and explicit expressions follow from the conditions of Propositions 2.2.7 and 2.2.8. Corollary 2.2.9 uses the theorem to improve on the convergence of the Dixmier trace formula, and Corollary 2.2.11 exhibits the resulting expression for the second pole. Finally, Theorem 2.3.1 is a simple extension of our treatment to the localized residue traces $\operatorname{res}_{s=s_k} \operatorname{tr} h\Delta^{-s}$, for any bounded operator h.

2.2 Zeta residues as normal functionals

We can ask how spectral zeta residues relate to finite subsets of the operator spectrum without referring to the original setting of differential geometry. We will henceforth consider the question in such generality, but we will need to restrict ourselves to operators whose zeta functions share an essential property with the Minakshisundaram-Pleijel zeta function.

Definition 2.2.1. A positive, invertible[3], self-adjoint, unbounded operator Δ with compact resolvent is said to be *spectrally elliptic* if the 'heat trace' $\operatorname{tr} e^{-t\Delta}$ admits an asymptotic expansion $\sum_{i=0}^{\infty} c_i t^{-s_i}$ as $t \to 0^+$, where the s_i are decreasingly ordered reals. We say that $\Pi = \{s_i\}$ is the *heat spectrum* of Δ.

[3] If not, just restrict to the complement of the kernel.

The definition implies that there is some $s_0 \in \mathbb{R}$ such that $\Gamma(s)\,\mathrm{tr}\,\Delta^{-s}$ converges for $\Re s > s_0$ and can be analytically continued to a meromorphic function $\Gamma(s)\zeta(\Delta, s)$ whose poles are all simple and located in $\Pi \subset (-\infty, s_0]$.

This particular terminology was chosen because the motivating example of such operators are the positive elliptic differential[4] operators, cf. [See67; DG75]. Indeed, for a positive elliptic differential operator of order m on a n-dimensional manifold, the heat spectrum is contained in $\frac{n}{m}, \frac{n-1}{m}, \ldots, \frac{1}{m}$.

Definition 2.2.1 suggests that we will in fact be technically concerned with the asymptotic heat trace coefficients c_i, and the reader who is more familiar with that terminology may rest assured that they are indeed what we are talking about. However, because zeta residues are mathematically the more general notion, the text will mainly refer to the coefficients by that name.

Our main theorem shows how the zeta residues of a spectrally elliptic operator are related to restrictions of its spectrum. We will provide the proof in Section 2.2, below.

The precise asymptotic formula for the residue $\mathrm{res}_{s=s_k} \zeta(\Delta, s)$ at some pole $s_k \in \Pi$ depends only on the location of the 'previous' poles, that is, on the set $\Pi \cap [s_k, \infty)$. The Dixmier trace, for instance, requires knowledge of s_0 and of the fact that no poles $s_{-1} > s_0$ exist. For a given finite set $\{s_i\}$ of such poles, all operators with zeta residues contained in this set can be considered simultaneously. Hence the following definition.

Definition 2.2.2. For any finite set $\{s_i\}_{i=0}^k$ of decreasingly ordered reals, let $\mathscr{D}(\{s_i\}_{i=0}^k)$ be the set of all spectrally elliptic operators Δ whose heat spectrum Π satisfies $\Pi \cap [s_k, \infty) \subset \{s_i\}$, i.e., whose heat trace admits an asymptotic expansion $\sum_{i=0}^k c_i t^{-s_i} + O(t^{-s_{k+1}})$ as $t \to 0^+$, for some $s_{k+1} < s_k$.

If $\sigma(\Delta)$ is the (necessarily point) spectrum of some spectrally elliptic operator Δ and F is any Borel measurable function, we will denote the summation of F over the strictly positive eigenvalues smaller than Λ by

$$\mathrm{tr}_\Lambda F(\Delta) \overset{\mathrm{def}}{=} \sum_{\Lambda > \lambda \in \sigma(\Delta)} F(\lambda).$$

[4]The classical elliptic *pseudo*differential operators may have logarithmic terms in the heat expansion, leading to double poles of $\Gamma(z)\zeta(z)$ (e.g. at negative integers), and are thus excluded in general. However, see remark 2.2.6.

Theorem 2.2.3. For any finite set $\{s_i\}_{i=0}^k$ of decreasingly ordered reals there exists a function F such that, for all $\Delta \in \mathscr{D}(\{s_i\}_{i=0}^k)$,

$$\operatorname{res}_{s=s_k} \zeta(\Delta, s)\Gamma(s) = \lim_{\Lambda\to\infty} \epsilon(\Lambda)^{s_k} \operatorname{tr}_\Lambda F(\Delta\epsilon(\Lambda)), \tag{2.1}$$

where $\epsilon(\Lambda) \stackrel{\text{def}}{=} m \log \Lambda / \Lambda$ for any $m > s_0 - s_k$.

The problem posed in the introduction is then trivially resolved by the following corollary, together with an explicit choice of F.

Corollary 2.2.4. Let $\{\Delta_j\}$ be any sequence of finite rank positive operators of norm $\|\Delta_j\| \stackrel{\text{def}}{=} \Lambda_j$ such that, as $j \to \infty$, $\Lambda_j \to \infty$ and

$$\operatorname{tr}_{\Lambda_j} F(\Delta\epsilon(\Lambda_j)) - \operatorname{tr} F(\Delta_j\epsilon(\Lambda_j)) = o((\epsilon(\Lambda_j)^{-s_k}).$$

Then,

$$\operatorname{res}_{s=s_k} \zeta(\Delta, s)\Gamma(s) = \lim_{j\to\infty} \epsilon(\Lambda_j)^{s_k} \operatorname{tr} F(\Delta_j\epsilon(\Lambda_j)).$$

Remark 2.2.5. A uniform bound over $\mathscr{D}(\{s_i\}_{i=0}^k)$ on the rate of convergence is too much to ask: it depends on the complete spectrum of Δ. However, the convergence rate will be shown to always be $O(\Lambda^{-1})$ if the next pole is at $s_k - 1$. For examples of arbitrary error for given Λ, see for instance [Col87; Loh96]. For comparison, see remark 2.4.1 on page 37.

Remark 2.2.6. The exclusion of logarithmic terms in the heat expansion from Definition 2.2.1 corresponds to an exclusion of higher order poles of $\Gamma(s)\zeta(\Delta, s)$. However, Theorem 2.2.3 is unchanged if higher order poles are allowed, as long as they lie below s_k. Moreover, the approach can probably be modified to accomodate such logarithmic terms, at the expense of brevity and asymptotic rate of convergence.

Asymptotic series for zeta function residues

For any spectrally elliptic operator Δ, the function $\zeta(\Delta, s)\Gamma(s)$ is the Mellin transform of $e^{-t\Delta}$. Therefore, the asymptotic expansion of $e^{-t\Delta}$

$$\operatorname{tr} e^{-t\Delta} \sim \sum_{i=0}^{k} c_i t^{-s_i} + o(t^{-s_k})$$

has coefficients

$$c_i = \operatorname{res}_{s=s_i} \zeta(\Delta, s)\Gamma(s).$$

The proof of Theorem 2.2.3 relies on the asymptotic expansion of a Mellin convolution integral with $\operatorname{tr} e^{-t\Delta}$. This strategy was inspired by the spectral action principle of [CM08, Thm 1.145] and the approach of [Ara96]. However, here we have specifically constructed the function we convolve with such that its first moments vanish, which allows us to recover the zeta residue in the leading term.

Proof of Theorem 2.2.3. The proof proceeds in two elementary steps. First, we show (Proposition 2.2.7, below) that for suitable functions f one has

$$\int_0^\infty \operatorname{tr} e^{-\epsilon t\Delta} f(t) = c_k \epsilon^{-s_k} + O(\epsilon^{-s_{k+1}})$$

as $\epsilon \to 0$. This part requires the asymptotic expansion of $\operatorname{tr} e^{-t\Delta}$, implying in particular that the poles of $\Gamma(s)\zeta(\Delta, s)$ must be simple and discrete (i.e. Δ has simple dimension spectrum disjoint with the negative integers) and lie on the real line.

Then, we use the Laplace transform F to rewrite the integral as a trace

$$\int_0^\infty \operatorname{tr} e^{-\epsilon t\Delta} f(t)\, dt = \operatorname{tr} F(\epsilon\Delta),$$

and we prove in Proposition 2.2.8 that there is a choice $\epsilon(\Lambda) = \Lambda^{-1}\log\Lambda^m$ such that F decays as

$$(\operatorname{tr} - \operatorname{tr}_\Lambda) F(\Delta\epsilon(\Lambda)) = O(\epsilon^{-s_{k+1}}),$$

so that we obtain the useful asymptotic behaviour

$$\operatorname{tr}_\Lambda F(\Delta\epsilon(\Lambda)) = c_k \epsilon(\Lambda)^{-s_k} + O(\epsilon^{-s_{k+1}}).$$

This second part relies on the finite summability of Δ, that is, on the fact that $\operatorname{tr} e^{-t\Delta} = O(t^{-s_0})$ as $t \to 0^+$. $\square$

The following asymptotic estimate is completely straightforward.

Proposition 2.2.7. Let f be a piecewise continuous function supported in $(1, \infty)$ that is $O(t^{-m})$ for all $m \in \mathbb{R}$ towards ∞. Let $\Delta \in \mathscr{D}(\{s_i\}_{i=0}^k)$. Then, the Mellin convolution integral of f with $\operatorname{tr} e^{-t\Delta}$ has an asymptotic expansion

$$\int_0^\infty \operatorname{tr} e^{-\epsilon t\Delta} f(t)\, dt = \sum_{i=0}^k c_i \epsilon^{-s_i} \int_0^\infty t^{-s_i} f(t)\, dt + o(t^{-s_k}).$$

Thus, if we simply choose f so that its moments $\int_0^\infty t^{-s_i} f(t)dt$ vanish for $i \neq k$ and normalize it so that $\int_0^\infty t^{-s_k} f(t)dt = 1$, we gain access to the coefficient c_k.

By asymptotic decay of such a function f towards ∞, it has absolutely convergent Laplace transform F. As $|f(t)\operatorname{tr} e^{-\epsilon t\Delta}|$ is absolutely integrable as well, we can apply the Fubini-Tonelli theorem to obtain

$$\begin{aligned}\int_0^\infty \operatorname{tr} e^{-\epsilon t\Delta} f(t)dt &= \sum_{\lambda\in\sigma(\Delta)} \int_0^\infty e^{-\lambda\epsilon t} f(t)dt \\ &= \operatorname{tr} F(\Delta\epsilon),\end{aligned}$$

where $\sigma(\Delta)$ is the spectrum (with multiplicities) of Δ. Therefore, Proposition 2.2.7 leads us to the conclusion that

$$\operatorname{tr} F(\epsilon\Delta) = c_k \epsilon^{-s_k} + O(\epsilon^{-s_{k+1}}). \tag{2.2}$$

The following proposition shows that we retain the same asymptotics if we replace the trace of $F(\epsilon\Delta)$ by a sum over a finite part of the spectrum, provided we scale ϵ accordingly.

Proposition 2.2.8. If f and Δ are as in Proposition 2.2.7 and additionally

$$\int_0^\infty t^{-s_i} f(t)dt = 0 \quad \text{(for all } i < k\text{)},$$

then the Laplace transform F satisfies

$$\sum_{\Lambda<\lambda\in\sigma(\Delta)} F(\lambda \log \Lambda^m/\Lambda) = o((\log \Lambda^m/\Lambda)^{-s_k})$$

for all $m > s_0 - s_k$

Proof. By the fact that $\operatorname{supp} f \subset [1,\infty)]$, its Laplace transform decays as $F(s) = O(s^{-1}e^{-s})$ towards ∞. As $\sum_{\lambda>\Lambda} e^{-\epsilon(\lambda-\Lambda)t}$ is monotone decreasing in t, we see that

$$\sum_{\lambda>\Lambda} F(\epsilon\lambda) = O\left(\sum_{\lambda>\Lambda} (\epsilon\Lambda)^{-1} e^{-\epsilon\lambda}\right) = O\left((\epsilon\Lambda)^{-1} e^{-\epsilon\Lambda} \sum_{\lambda>\Lambda} e^{-\epsilon(\lambda-\Lambda)t}\right)$$

for any $t \leq 1$, as $\Lambda\epsilon \to \infty$. This, in turn, is $O((\epsilon\Lambda)^{-1} e^{-\epsilon\Lambda(1-t)} \epsilon^{-s_0})$. With $\epsilon(\Lambda) = m\Lambda^{-1}\log\Lambda$ for any $m > (s_0 - s_k)$, the remainder $\sum_{\lambda>\Lambda} F(\epsilon\lambda)$ will be $o(\epsilon^{-s_k})$. □

This completes the proof of Theorem 2.2.3 that started on page 31.

Explicit formulas for the first two poles

One reason to look for series that converge to zeta residues is to obtain geometric information from finite-dimensional approximations of the Laplacian on a Riemannian manifold. For instance, if Δ is the Laplacian, the first two residues are proportional to the volume and the total scalar curvature, respectively.

The first pole is a classical object of study. Its residue is expressed by Dixmier's singular trace, and is used, for instance, for the zeta regularization of divergent series. It is connected to counting asymptotics by the Wiener-Ikehara theorem, and for a Laplace operator on a compact Riemannian manifold it is proportional to the volume.

Our Theorem 2.2.3 provides the following formula for the first residue.

Corollary 2.2.9. If Δ is a spectrally elliptic operator with heat spectrum bounded from above by $s_0 \in \mathbb{R}$, then if $\epsilon(\Lambda) = \log \Lambda / \Lambda$,

$$\Gamma(s_0)\,\mathrm{res}_{s=1}\,\zeta(\Delta^{s_0}, s) = \lim_{\Lambda\to\infty} \frac{\epsilon(\Lambda)^{s_0}}{e\Gamma(1-s_0,1)}\,\mathrm{tr}_\Lambda\,\frac{e^{-\Delta\epsilon(\Lambda)}}{1+\Delta\epsilon(\Lambda)}.$$

Proof. Use $f = [t \geq 1]e^{-t}$, with Laplace transform $F(s) = e^{-1-s}/(1+s)$, in Proposition 2.2.7. Then, divide by $\int_0^\infty t^{-s_0} f(t)\,dt$ to satisfy the conditions of Proposition 2.2.8. □

Remark 2.2.10. The present series converges faster in general than the logarithmic trace suggested by Weyl's asymptotic formula; the remainder is $O(\epsilon^{s_0-s_1}) = O((\Lambda/\log\Lambda)^{s_1-s_0})$, whereas for e.g. the Laplacian on the circle the logarithmic trace of $\Delta^{-1/2}$ convergences only as $\sum_{n=1}^N \frac{1}{n\log N} - 1 = \frac{\gamma + O(N^{-1})}{\log N}$.

Now, the second pole is of particular interest because it is the first pole for which no asymptotic residue formula was previously known and because it provides a way to calculate the total scalar curvature of a Riemannian manifold from partial spectra of the Laplacian.

Corollary 2.2.11. If Δ is a spectrally elliptic operator with heat spectrum $\{s_0, s_0-1, \ldots\}$, then if $\epsilon(\Lambda) = 2\log\Lambda/\Lambda$,

$$\mathrm{res}_{s=s_1}\,\zeta(\Delta,s)\Gamma(s) = \lim_{\Lambda\to\infty} \frac{\epsilon(\Lambda)^{s_0-1}}{e\Gamma(2-s_0,1)}\,\mathrm{tr}_\Lambda\left(\frac{2^{s_0}e^{-2\Delta\epsilon(\Lambda)}}{1+2\Delta\epsilon(\Lambda)} - \frac{e^{-\Delta\epsilon(\Lambda)}}{1+\Delta\epsilon(\Lambda)}\right).$$

Proof. Use $f = [t \geq 1]e^{-t} - 2^{s_0-1}[t \geq 2]e^{-t/2}$, which clearly satisfies the conditions of Proposition 2.2.7 and has vanishing moment $\int_0^\infty t^{-s_0} f(t)dt$. Therefore, if we divide by the factor $\int_0^\infty t^{1-s_0} f(t)dt$, the result will be as in Proposition 2.2.8. The Laplace transform of f is $F(s) = e^{-1-s}/(1+s) - 2^{s_0}e^{-1-2s}/(1+2s)$. □

Example 2.2.12 (The total scalar curvature of the sphere S^2)**.** Let F be the residue functional given by

$$F(s) := \left(\frac{\exp(-s)}{1+s} - \frac{2\exp(-2s)}{1+2s}\right),$$

let $q_n = n(n+1)$ be the nth eigenvalue of the Laplacian on the sphere and let $p_n = 2n+1$ be its multiplicity. As before, write $\epsilon_m = 2q_m^{-1}\log q_m$. We will use the Euler-Maclaurin formula to estimate the series from Corollary 2.2.11,

$$S(m) := \sum_{n=0}^{m} p_n F(\epsilon_m q_n).$$

Sufficient accuracy will turn out to require only the first nontrapezoidal correction term. The Euler-Maclaurin estimate reads

$$S(m) = \int_0^m p_n F(\epsilon_m q_n)dn + \left(\frac{1}{2} + \frac{B_2}{2}\frac{\partial}{\partial n}\right) p_n F(\epsilon_m q_n)\bigg|_0^m + R_3(m),$$

where $B_2 = \frac{1}{6}$, and we will employ the usual rough bound

$$|R_3(m)| \leq \frac{\zeta(3)}{(2\pi)^3} m \sup_{0\leq n\leq m} \left| \frac{\partial^3 \left(p_n F(\epsilon_m q_n)\right)}{\partial n^3} \right|.$$

The integral in the first term can be taken analytically,

$$\int_0^m p_n F(\epsilon_m q_n)dn = \int_0^{q_m} F(q, \epsilon_m)dq = e\epsilon_m^{-1}\left(E_1(1+2\epsilon_m q_m) - E_1(1+\epsilon_m q_m)\right),$$

and is $O(q_m^{-2}\log\left(1+q_m/(2\log q_m)\right)$.

Because $F(s) = O(s^{-1}e^{-s})$ as $s \to \infty$, the order zero boundary term $\frac{1}{2} + \frac{1}{2}p_M(F(\epsilon_m q_m))$ is $\frac{1}{2} + O(p_m q_m^{-2}/\log q_m)$.

The first correction term to the trapezoidal approximation equals

$$\frac{1}{12}\left(-2 + 2F(\epsilon_m q_m) + \epsilon_m p_m^2 F^{(1)}(\epsilon_m q_m)\right) = -\frac{1}{6} + O(q_m^{-2}\log q_m).$$

Finally, the third derivative of $p_n F(\epsilon_m q_n)$ with respect to n, whose supremum appears in the error estimate, is

$$12\epsilon_m F^{(1)}(\epsilon_m q_n) + 12\epsilon_m^2 p_n^2 F^{(2)}(\epsilon_m q_n) + \epsilon_m^3 p_n^4 F^{(3)}(\epsilon_m q_n).$$

All derivatives of F are bounded on $[0,\infty)$, so these terms are all $O(q_m^{-1}\log q_m)$ uniformly in n. Thus, $R_3(m) = O(m^{-1}\log q_m)$.

Combining these facts with Corollary 2.2.11, we see that the nonleading residue for the zeta function ζ_{S^2} is

$$\operatorname{res}_{s=0}\zeta_{S^2}^{-s} = \lim_{m\to\infty}\left(\frac{1}{2} - \frac{1}{6} + O(m^{-1}\log q_m)\right) = \frac{1}{3} = \frac{2\cdot 4\pi}{6\cdot(4\pi)^{d/2}}.$$

Of course, one could apply Euler-Maclaurin directly to the sum that defines the heat trace on S^2; since the corresponding limits $m\to\infty$ can termwise be identified as power series in t, one can reproduce the asymptotics as $t\to 0$ afterwards. In the present formulation, however, we need only take a single limit in m and have expressed the residue as a single universal series in the eigenvalues, with an asymptotic bound on the error over all of $\mathscr{D}(s_0, s_0 - 1)$.

2.3 Localization

If Δ is spectrally elliptic, Theorem 2.2.3 allows us to calculate its zeta residues as a series in its eigenvalues. However, the classical theory of elliptic pseudodifferential operators assigns to them not just the total zeta residues, but rather a set of zeta *densities*. To be precise, the somewhat simpler situation for elliptic differential operators is as follows.

Let M be a Riemannian manifold of dimension n equipped with a smooth Hermitian vector bundle E, and let Δ be a positive, self-adjoint, elliptic differential operator of order $m > 0$, acting inside the Hilbert space of square-integrable sections of E. Then, the following localized asymptotics are available.

1. The operator $e^{-t\Delta}$ is given by a smooth kernel $k(x,y,t) : E_y \to E_x$, and as $t \to 0^+$ its restriction to the diagonal admits an asymptotic expansion in smooth sections k_j of $\operatorname{End} E$,

$$k(x,x,t) \sim \sum_{j=0}^{\infty} k_j(x) t^{(j-n)/m}.$$

2. For any continuous section h of $\operatorname{End} E$ and any $j \geq 0$, the following residues exist and satisfy

$$\operatorname{res}_{s=(j-n)/m} \Gamma(s) \operatorname{tr} h\Delta^{-s} = \int_M \operatorname{tr}(h(x)k_j(x))\, d\operatorname{vol}_M(x).$$

We say that h *localizes* the residue trace, and we are interested in expressing the localized residues in a fashion similar to Theorem 2.2.3. We will solve this localization problem in a slightly more general setting, along the lines of the previous treatment of the zeta function residues.

Let Δ be a spectrally elliptic operator on a Hilbert space $\mathscr{H}$ and let $h \in \mathscr{B}(\mathscr{H})$ be bounded. Then, there exist $s_0 > \cdots > s_{k+1}$ such that, as $t \to 0^+$,

$$\operatorname{tr} he^{-t\Delta} - \sum_{i=0}^{k} t^{-s_k} c_i(h) = O(t^{-s_{k+1}}).$$

As before, let $\{s_i\}_{i=1}^k$ be a set of decreasingly ordered reals and let F be the Laplace transform of a piecewise continuous function f supported in $[1,\infty)$, which is $O(t^{-m})$ for all $m \in \mathbb{R}$ towards ∞, satisfies $\int_0^\infty t^{-s_i} f(t)\,dt = 0$ for all $i < k$, and is normalized to satisfy $\int_0^\infty t^{-s_k} f(t)\,dt = 1$.

Theorem 2.3.1. Let $\Delta \in \mathscr{D}(\{s_i\}_{i=1}^k)$ be a spectrally elliptic operator on a Hilbert space $\mathscr{H}$ and let $h \in \mathscr{B}(\mathscr{H})$ be bounded. Then, for any orthonormal basis $\{\phi_\lambda\}_{\lambda \in \sigma(\Delta)}$ of $\mathscr{H}$ diagonalizing Δ, we have

$$\operatorname{res}_{s=s_k} \Gamma(s) \operatorname{tr} h\Delta^{-s} = \lim_{\Lambda \to \infty} \epsilon(\Lambda)^{s_k} \sum_{\Lambda > \lambda \in \sigma(\Delta)} \langle h\phi_\lambda, \phi_\lambda \rangle F(\lambda \epsilon(\Lambda)),$$

where $\epsilon(\Lambda) \overset{\text{def}}{=} \Lambda^{-1} \log \Lambda^m$ for any $m > s_k - s_0$.

Proof. As before, Proposition 2.2.7 holds for any such function f. That is,

$$\int_0^\infty \operatorname{tr} he^{-\epsilon t\Delta} f(t)\,dt = c_k(h)\epsilon^{-s_k} + \|h\|_{\mathrm{op}} O(\epsilon^{-s_{k+1}}).$$

Moreover, $\sum_{\lambda \in \sigma(\Delta)} |f(t)e^{-t\lambda} \langle h\phi_\lambda, \phi_\lambda \rangle| \leq |f(t)|\,\|h\|_{\mathrm{op}} \operatorname{tr} e^{-t\Delta}$ is in $L^1(0,\infty)$. Therefore, by dominated convergence,

$$\int_0^\infty \operatorname{tr} he^{-\epsilon t\Delta} f(t)\,dt = \sum_{\lambda \in \sigma(\Delta)} \langle h\phi_\lambda, \phi_\lambda \rangle F(\epsilon\lambda)$$

and by Proposition 2.2.8, the rest term decays like

$$\sum_{\Lambda<\lambda\in\sigma(\Delta)} F(\lambda\epsilon(\Lambda))\langle h\phi_\lambda,\phi_\lambda\rangle = \|h\|_{\mathrm{op}} O((\epsilon(\Lambda))^{-s_{k+1}}).$$

We have $\operatorname{res}_{s=s_k}\Gamma(s)\operatorname{tr} h\Delta^{-s} = c_k(h)$ and the conclusion follows. □

2.4 Final remarks and suggestions

The original motivation behind this chapter was to confirm the point of view that finite-rank cutoffs of spectral triples can carry geometric information in noncommutative geometry. The following remarks all proceed in that direction.

Remark 2.4.1. Our Theorem 2.2.3 provides a partial counterweight to a classical result by Colin de Verdière [Col87]. On the one hand, he showed that a finite set of Laplace eigenvalues carries no information on the metric if no pointwise bounds on the sectional curvature are imposed. On the other hand, we now see that for each metric and each desired accuracy, there is a bound Λ such that the set of eigenvalues smaller than Λ, with multiplicity, allow computation of all zeta residues (and hence, of the associated global invariants of the metric) up to that accuracy. The local version, using Theorem 2.3.1, of this statement is that for each metric and each ϵ there is a finite-rank projection $P_\Lambda = [0,\Lambda](\Delta_g)$ such that the cutoff matrix $P_\Lambda\Delta_g P_\Lambda$ together with the cutoff $P_\Lambda C(M)P_\Lambda$ of the function algebra yield the residues of $\operatorname{tr} a\Delta^{-s}$ up to an error of $\|a\|\epsilon$.

Remark 2.4.2. The method of Theorem 2.3.1 may be applied to obtain a reasonable definition of scalar curvature in *finite-dimensional* noncommutative geometry. Let (A,H,D) be a spectral triple such that D^2 is spectrally elliptic with heat spectrum $\{(j-n)/2\}_{j\in\mathbb{Z}_{>0}}$. In [CM08] the scalar curvature functional of such a spectral triple was defined to be the map

$$a\mapsto R(a) \stackrel{\text{def}}{=} \operatorname{res}_{s=n-2}\Gamma(s)\operatorname{tr} aD^{-s}.$$

If H is finite-dimensional but a dimension spectrum of D is specified in advance (e.g. by modelling considerations), this residue always vanishes and Theorem 2.3.1 suggests it should perhaps be replaced by

$$a\mapsto R_\Lambda(a) \stackrel{\text{def}}{=} \sum_{\lambda\in\sigma(D^2)} \langle a\phi_\lambda,\phi_\lambda\rangle \frac{F(\lambda\log\Lambda^m/\Lambda)}{(\log\Lambda^m/\Lambda)^{n/2-1}},$$

where $\Lambda \stackrel{\text{def}}{=} \|D^2\|^\rho$ for any $0 < \rho < 1$, $\{\phi_\lambda \mid D^2\phi_\lambda = \lambda\phi_\lambda\}$ is an orthonormal basis of H and F is as in Corollary 2.2.11. Mutatis mutandis, the same applies to the volume and other spectral invariants.

Remark 2.4.3. The calculation of the residues of localized zeta functions, Theorem 2.3.1, can be combined with the local index formula of Connes and Moscovici [CM95] for the Chern character of the Fredholm module associated to a spectral triple (A, H, D), in order to estimate some KK-theoretic index pairings numerically. If the square of the operator D is spectrally elliptic, the index pairings can be expressed as a series in the spectrum of D^2 and the coefficients $\operatorname{tr} \pi_\lambda a_0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} \pi_\lambda$, where π_λ projects onto the eigenspace associated with the eigenvalue λ, the commutators are defined recursively as $[D, a]^{(k+1)} \stackrel{\text{def}}{=} [D^2, [D, a]^{(k)}]$ and $a_0, \ldots, a_n \in A$. A similar statement holds in the presence of a grading.

Chapter 3

Reconstructing manifolds from truncated spectral triples[1]

We explore the geometric implications of introducing a spectral cut-off on compact Riemannian manifolds. This is naturally phrased in the framework of non-commutative geometry, where we work with spectral triples that are *truncated* by spectral projections of Dirac-type operators. We associate a metric space of 'localized' states to each truncation. The Gromov-Hausdorff limit of these spaces is then shown to equal the underlying manifold one started with. This leads us to propose a computational algorithm that allows us to approximate these metric spaces from the finite-dimensional truncated spectral data. We subsequently develop a technique for embedding the resulting metric graphs in Euclidean space to asymptotically recover an isometric embedding of the limit. We test these algorithms on the truncated sphere and a recently investigated perturbation thereof.

3.1 Introduction

A natural notion of *scale* is a major asset to any geometric theory with ties to the physical world. After all, our geometric knowledge of objects appearing around us is finite, being limited by our observational power which fails us at high energy scales. Additionally, the appearance of divergences in e.g. quantum field theory, especially when combined with gravity, is closely tied to bridging the gap between finite and infinite, or discrete and

[1]This work was written jointly with Lisa Glaser and published as [GS21]

continuum, models. Moreover, computational representation and analysis of geometric models strongly relies on our ability to extract from our model what is relevant and computationally feasible.

The field of noncommutative geometry has had close ties to physics ever since its inception, and yet lacks a consistent treatment of scale in the sense imagined. The aim of this chapter is to ameliorate this situation, by constructing a natural metric counterpart to the finite objects that are here referred to as *truncations* of (commutative) *spectral triples*, and aiming to show that these do indeed carry enough information to describe their continuum limit in arbitrary detail. For a wide-ranging and systematic approach to such truncations, see [CS20].

Admittedly, finite-dimensional objects in noncommutative geometry have enjoyed enduring attention. General finite spectral triples have been classified [GP95; Ćać11; CC08] and parametrized [Bar15], and the Connes metric on these spaces has been studied in depth, *cf.* [IKM01]. However, this framework seems to lack simultaneous presence of 1) a natural link to the continuum in terms of metric spaces and 2) a natural link to the continuum in terms of spectral triples.

When representation-theoretic knowledge of the continuum is available, the framework of *fuzzy spaces* such as those of [GP95; Bal+02; DO03] seems to provide at least a natural link to the continuum in terms of spectral triples, and even some metric knowledge is available there [SS16]. However, it might be said that the construction of a 'fuzzy' version of a manifold is somewhat *ad hoc* from a Riemannian viewpoint, and at least the framework has not yet seen successful extension to a reasonably large class of manifolds.

Truncations of spectral triples (see Section 3.2 and beyond) provide the advantage of a natural scale parameter and a natural symmetry-preserving correspondence between different scales. The natural framework in which to study these truncations themselves as metric spaces is that of state spaces, as in [CS20], which can then be equipped either with the Connes metric associated to the truncation of a spectral triple itself or with the pullback of the Connes metric on the full spectral triple.

An early and interesting study of the topological and metric properties of such spaces can be found in [DLM14b]. More recently, [Sui20] investigates the question of Gromov-Hausdorff convergence of such truncated spaces under the truncated metric; see [Ber19] for the example of the torus. By contrast, here we are interested in *localized* states, in order to recover e.g. a manifold M, rather than its metric space of probability measures. Chapter 4 investigates the relevance of the higher Heisenberg equation of [CCM15] in the framework of truncated spectral triples (see Section 3.6).

Arguably the main mathematical result of this chapter is Corollary 3.3.10, which shows that the 'localized' states of Section 3.3, when equipped with the pullback metric, recover the compact Riemannian manifold one started with in the Gromov-Hausdorff limit.

In order to make the result more concrete and link back to the 'computational representation' alluded to above, Section 3.4 is devoted to the description of an algorithm to approximate (finite subsets of) these metric spaces from the raw datum of the truncation of a spectral triple. This allows us to test some of the results of Section 3.3 on the example of the sphere in Section 3.5.

Finite non-Euclidean metric spaces, as obtained by our algorithm, do not necessarily lend themselves to easy visualization or comparison by standard computational techniques. In order to gain traction in this direction, we propose to look for new (asymptotically, locally isometric) embedding techniques and present a candidate approach in Section 3.6. This allows us to visualize the metric results of Section 3.5 and – as originally inspired the technique – compare the truncation of the spectral triple for S^2 and its perturbation as in [GS20].

3.2 Truncated spectral geometries and point reconstruction

In noncommutative geometry one describes a compact Riemannian manifold M in terms of an associated spectral triple $(C^\infty(M), L^2(M,\mathcal{S}), D_\mathcal{S})$, where $\mathcal{S}$ is a Hermitian Clifford module bundle over M and $D_\mathcal{S}$ is a Dirac-type operator on $\mathcal{S}$.

Connes' reconstruction theorem [Con13] shows that this association is a bijection: one can fully reconstruct the underlying manifold M and the chosen Hermitian Clifford module bundle $\mathcal{S}$ from a triple (A,H,D) that satisfies the axioms of a *commutative* spectral triple.

Of particular interest for the present chapter is the way one can recover the Kantorovich-Rubinstein[2] distance between probability measures on M from the interplay of the algebra $C^\infty(M)$ and the Dirac operator $D = D_\mathcal{S}$, acting on the Hilbert space $L^2(M,\mathcal{S})$. Probability measures correspond to states on the C*-algebra $C(M) \supset C^\infty(M)$, which carries the topological, as opposed to differentiable, information about M. Moreover, a function $f \in C^\infty(M)$ has Lipschitz constant k if and only if $\|[D,f]\| \leq k$ (as operators

[2]This metric is known under various names including '1-Wasserstein', 'Monge-Kantorovich' and 'Earth Mover's'. See e.g.[Vil09, Chapter 6].

on $H = L^2(M, \mathcal{S}_M)$). Thus, Kantorovich-Rubinstein duality allows us to write

$$d(\omega_1, \omega_2) = \sup_{a \in C^\infty(M)} \{|\omega_1(a) - \omega_2(a)| \mid \|[D,a]\| \leq 1\} . \tag{3.1}$$

When the states ω of $C(M)$ are *pure*, they correspond to the atomic measures on single points, and we can thus recover M with its metric. This chapter answers the question as to how we can understand this recovery of M from the perspective of finite-dimensional parts of the representation of $\mathcal{A}$ and D on H. That is, we will construct natural counterparts to the ingredients above in the setting of *truncations* of spectral triples, and show that the metric space M can be recovered as an asymptotic limit thereof.

Truncated spectral geometries

From a mathematical perspective, it is desirable to be able to describe the infinite-dimensional datum of a spectral triple as a limit of finite-dimensional data of increasing precision, just like one can describe a compact Riemannian manifold as a Gromov-Hausdorff limit of finite metric spaces (by, for instance, equipping suitably dense finite subsets with the induced metric). From a physical perspective, the same desire results from the view that one should be able to gain at least *some* information about the geometry by probing it at finite energies.

A natural way to introduce such a 'finite-energy cutoff', that is, truncation, of the geometric data (A, H, D) is to pick a scale Λ, and define the corresponding spectral projection of D,

$$P_\Lambda \stackrel{\text{def}}{=} \chi_{[-\Lambda,\Lambda]}(D)$$

to generate the finite-dimensional data

$$(A_\Lambda, H_\Lambda, D_\Lambda) \stackrel{\text{def}}{=} (P_\Lambda A P_\Lambda, P_\Lambda H, P_\Lambda D) . \tag{3.2}$$

By compactness of the resolvent of D, one might as well write P_N for the projection onto the first N eigenspaces ordered in absolute value and work with P_N instead. However, this would require keeping track of the relation between N and Λ in the analysis to follow and thus complicate notation. A further advantage of using P_Λ is it is physically more intuitive to work with the energy scale rather than with the number of modes considered.

The study of truncations of spectral triples and their convergence was initiated in [DLM14b], while the structure of the truncations as operator systems was explored in depth by Connes and van Suijlekom in [CS20] and [Sui20].

Optimal transport in the framework of noncommutative geometry has seen widespread inquiry. See e.g. [BLS94; DM99] for the case of lattices, [IKM01] for finite spectral triples, [Mar06] for the context of gauge theory, [Wal12; ŻS01; DAn16; DLM14a] for several quantum mechanically inspired examples and deformations, [Sui20; DLM14b] for truncations of the circle, and notably [Rie99; Rie04] for generalization of the inducing Lipschitz seminorm.

This is the setting in which we wonder what counterpart to the duality $(A, H, D) \leftrightarrow M$, provided by the reconstruction theorem, can be found at the level of $(A_\Lambda, H_\Lambda, D_\Lambda)$.

Point reconstruction

We aim to refine the reconstruction of M from its spectral triple (A, H, D) through an understanding of the metric information contained in the truncations $(A_\Lambda, H_\Lambda, D_\Lambda)$. The full manifold should then emerge asymptotically, such as through a Gromov-Hausdorff limit of the objects corresponding to the truncations.

The vector states of $C(M)$ that are induced by elements of H_Λ appear naturally as vector states of A_Λ as well, so that we have access to probability measures on M directly in the truncated setting. This, together with the distance formula (3.1), is the main ingredient of our approach: we will identify states that correspond to points of M in a suitable (asymptotic) sense, such that (3.1) asymptotically recovers the corresponding geodesic distance.

Our 'proxy' approach to state localization, as discussed in section 3.3, was inspired by the notion of quasi-coherent states on fuzzy spaces defined in [SS16]. However, as we aim just for the induced metric geometry on M and view $(C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)$ rather as a finite observation of a spectral triple than as a quantization thereof, we will not construct coherent states in any quantum-mechanical sense but rather aim for localization only. Moreover, as discussed below, this 'proxy' approach is merely introduced to gain computational feasibility, at the expense of requiring identification of an embedding $\phi : M \to \mathbb{R}^n$. See also the end of Section 3.3.

After we define localized states, we prove existence of the desired objects: a sequence of metric spaces associated to the truncations $(A_\Lambda, H_\Lambda, D_\Lambda)$ that do indeed converge to M in the Gromov-Hausdorff sense. Then, Section 3.4 proposes an algorithm to construct these metric spaces computationally, in order to make actual examples amenable to computer simulation.

In Section 3.6 we propose a simple algorithm to obtain approximately locally isometric embedding of the resulting finite metric spaces into Euclidean space. These should asymptotically converge to an isometric embedding of M itself, and allow us to view the resulting finite metric spaces and investigate them more easily.

3.3 The metric space of localized states

Given a truncation of a commutative spectral triple $(C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)$, we aim to construct a finite metric space that describes M to the level of accuracy that the truncated spectrum will allow.

We will identify a *subset* of the (vector) states of $C^\infty(M)_\Lambda$, consisting of those states that are *localized* in a suitable sense and, therefore, correspond approximately to points of M. The Connes metric on these vector states will then turn this subset into a metric space.

The guiding demand for this construction will be that the resulting metric spaces should asymptotically (as $\Lambda \to \infty$) converge, in the Gromov-Hausdorff sense, to the metric space M.

Now, the pure states of $C(M)$ – that is, actual points in the metric space M we are approximating – do not necessarily extend to $C(M)_\Lambda$. We do, however, have access to vector states induced by $v \in H_\Lambda$, which can be applied to either because both $C(M)$ and $C(M)_\Lambda$ are subsets of $B(H)$.

Definition 3.3.1. $\mathbb{P}(H_\Lambda)$ is the projective space over H_Λ.

Each element v of $\mathbb{P}(H_\Lambda)$ corresponds to a positive linear functional of norm 1 on $C(M)$ given by $a \mapsto \langle v, av\rangle / \langle v, v\rangle$ (for any representative v of v). By the Riesz Representation theorem, such functionals correspond uniquely to probability measures on M.

Definition 3.3.2. For $\mathrm{v} \in \mathbb{P}(H_\Lambda)$, μ_v is the unique probability measure such that $\langle v, av\rangle = \int_M a(x) d\mu_v(x)$ for all $a \in C(M)$ and any unit norm representative v of v.

By this identification, the Kantorovich-Rubinstein metric on the probability measures of M induces a metric d_Λ on $\mathbb{P}(H_\Lambda)$. It is an open conjecture that this metric can be (asymptotically) computed using the data $(C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)$: see Section 3.3 for a discussion.

We will say that v is localized when μ_v is sufficiently concentrated near a single point in M. In order to quantify this notion, we will introduce the *dispersion* functional η on $\mathbb{P}(H_\Lambda)$, below.

Let d_Λ and d_M denote the metrics on $\mathbb{P}(H_\Lambda)$ and M, respectively, so that we may regard both as subsets of the space of probability measures on M equipped with the Kantorovich-Rubinstein metric. We now wish to construct a subspace of $(\mathbb{P}(H_\Lambda), d_\Lambda)$ that is (Gromov-)Hausdoff close to (M, d_M).

Proposition 3.3.6 will show that there is a map $b\colon \mathbb{P}(H_\Lambda) \to M$ such that $|d_\Lambda(\mathrm{v}_1, \mathrm{v}_2) - d_M(b(\mathrm{v}_1), b(\mathrm{v}_2))| = O(\sqrt{\eta(\mathrm{v}_1)} + \sqrt{\eta(\mathrm{v}_2)})$; that is, our localized states can be identified almost isometrically with points of M. Proposition 3.3.8 will then show that there is a corresponding asymptotically inverse map $\Phi_\Lambda\colon M \to \mathbb{P}(H_\Lambda)$ such that $d_M(x, b(\Phi_\Lambda(x))) = \widetilde{O}\left(\Lambda^{-1}\right)$ and $\eta(\Phi_\Lambda(x)) = \widetilde{O}\left(\Lambda^{-2}\right)$ uniformly in x. This leads to Corollary 3.3.10, which shows that there exist a subspace $\mathbb{P}(H_\Lambda)_{\epsilon^2}$ of $\mathbb{P}(H_\Lambda)$ that is ϵ-close to M in Gromov-Hausdorff distance, where $\epsilon = \widetilde{O}\left(\Lambda^{-1}\right)$.

Finally, Section 3.3 discusses how these notions connect to the setting of truncated spectral triples $(C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)$.

Localization: ϕ and the dispersion functional

Since elements of $\mathbb{P}(H_\Lambda)$ correspond uniquely to probability measures on M, a natural way to measure the localization of such an element would be to take e.g. the variance of (isometrically embedded) position under this measure; that is, one would naturally define the dispersion of $\mathrm{v} \in \mathbb{P}(H_\Lambda)$ to be $\inf_{x\in M} E_{\mu_v}\left[d(x,\cdot)^2\right]$, where E_{μ_v} denotes expectation values under μ_v. In terms of the algebraic data, this quantity can be estimated as $\sup_{a\in C^\infty_\Lambda(M)}\{\langle v, a^* a v\rangle - |\langle v, a v\rangle|^2 \mid \|[D,a]\| \le 1\}$. However, the relevant non-convex double optimization problem – to find minima v of this dispersion in high-dimensional H – is computationally extremely challenging except in the simplest cases.

Therefore, we will require a proxy, $\phi\colon M \to \mathbb{R}^n$, for the extremizing element a above, in the sense that the Euclidean distance $d_{\mathbb{R}^n}\left(\phi(x), \phi(y)\right)$ on M is bi-Lipschitz equivalent to the distance $d_M(x,y)$ appearing in the variance. Thus, let $\phi : M \to \mathbb{R}^n$ be a (not necessarily Riemannian) embedding.

In other words, we take M to be a compact embedded submanifold of $\mathbb{R}^n$, with a Riemannian metric not necessarily the one induced by the embedding. However, we will keep the notation ϕ to emphasize the arbitrary nature of the embedding, especially in connection with Section 3.4.

Definition 3.3.3. Let μ be a probability measure on M. Then, the dispersion $\eta(\mu)$ equals

$$\eta(\mu) \stackrel{\text{def}}{=} \int_M d_{\mathbb{R}^n}\left(\phi(x), E_\mu[\phi]\right)^2 d\mu(x)$$

In probabilistic terms, $\eta(\mu)$ is just the trace of the covariance matrix of the vector-valued random variable ϕ, under the probability measure μ.

The ϕ-barycenter of a localized state

An element v of $\mathbb{P}(H_\Lambda)$ that is considered to be localized should be localized *somewhere*, that is, around some 'barycenter' $x_v \in M$. In order to control the localization of μ_v around the point x_v by the dispersion $\eta(\mu_v)$, it is important that $\phi(x_v)$ be close to $E_{\mu_v}[\phi]$ in $\mathbb{R}^n$. Hence,

Definition 3.3.4. Let μ be a probability measure on M. Then a *ϕ-barycenter* of μ is any point $x \in M$ that minimizes $d_{\mathbb{R}^n}\left(\phi(x), E_\mu[\phi]\right)$.

By compactness of M and continuity of $d_{\mathbb{R}^n}\left(\phi(\cdot), E_\mu[\phi]\right)$, there always exists a ϕ-barycenter.

Localized states are indeed concentrated near their ϕ-barycenters, as the following lemma shows. That is, the dispersion $\eta(\mu)$ is a good proxy for the squared second Wasserstein distance[3] $W_2(\mu, \delta_x)^2$ between the measure μ and any given ϕ-barycenter x thereof.

Lemma 3.3.5. Any ϕ-barycenter x of a probability measure μ satisfies

$$W_2(\mu, \delta_x)^2 \stackrel{\text{def}}{=} \int_M d_M(z, x)^2 d\mu(z) = O(\eta(\mu)),$$

uniformly[4] in μ, where δ_x denotes the Dirac measure centered on x. Moreover, any two ϕ-barycenters of μ are within distance $O(\sqrt{\eta(\mu)})$, uniformly in μ, of each other.

[3] See e.g. [Vil03] for an introduction to the measure-theoretic notions that are applied (without any hint of sophistication) in this section.

[4] That is, the relevant constant depends only on ϕ and M, not on μ.

Proof. By Chebyshev's inequality, $\mu(\{x \in M \mid \|\phi(x) - E_\mu[\phi]\| \geq t\})$ must be bounded by $t^{-2} E_\mu\left[\|\phi - E_\mu[\phi]\|^2\right] = t^{-2}\eta(\mu)$. Therefore, if $d_{\mathbb{R}^n}(\phi(\cdot), E_\mu[\phi])^2 \geq t$ on the support of μ, we see that $1 \leq t^{-2}\eta(\mu)$. Therefore, we conclude that $\inf_{x \in \operatorname{supp}\mu} d_{\mathbb{R}^n}(\phi(x), E_\mu[\phi]) \leq \sqrt{\eta(\mu)}$. Any ϕ-barycenter of μ must therefore, as a minimizer of $d_{\mathbb{R}^n}(\phi(\cdot), E_\mu[\phi])$ in $M \supset \operatorname{supp}\mu$, also satify this inequality.

Now, as a smooth embedding, ϕ is automatically bi-lipschitz. In particular, there exists β such that $d_M(x,y) \leq \beta d_{\mathbb{R}^n}(\phi(x), \phi(y))$ uniformly in x, y. We see, therefore, that any two ϕ-barycenters of μ are at a distance at most $2\beta\sqrt{\eta(\mu)}$.

Moreover, we conclude that $\int_M d_M(z,x)^2 d\mu(z) \leq \beta^2 \int_M d_{\mathbb{R}^n}(\phi(z), \phi(x))^2 d\mu(z)$ for all $x \in M$. As $|\int_M f^2 - g^2 d\mu| \leq \int_M (2|g| + |f-g|)|f-g| d\mu$ and all ϕ-barycenters x of μ satisfy $\left|d_{\mathbb{R}^n}(\phi(z), \phi(x)) - d_{\mathbb{R}^n}(\phi(z), E_\mu[\phi])\right| \leq \sqrt{\eta(\mu)}$, we can estimate the multivariate variance $\int d_{\mathbb{R}^n}(\phi(z), \phi(x))^2 d\mu(z)$ by $\eta(\mu)$, up to an error that is bounded by $\int_M (2 d_{\mathbb{R}^n}(\phi(z), E_\mu[\phi]) + \sqrt{\eta(\mu)})\sqrt{\eta(\mu)} d\mu(z)$ which is, by the classical Jensen inequality and the definition of $\eta(\mu)$, bounded by $3\eta(\mu)$. The Lemma follows. □

We now consider the implications of the above for the barycenters of probability measures μ_v, for $\mathrm{v} \in \mathbb{P}(H_\Lambda)$.

Proposition 3.3.6. There exists a map $b\colon \mathbb{P}(H_\Lambda) \to M$ such that

$$|d_\Lambda(\mathrm{v}, \mathrm{w}) - d_M(b(\mathrm{v}), b(\mathrm{w}))| = O(\sqrt{\eta(\mu_v)} + \sqrt{\eta(\mu_w)})$$

as $\eta(\mu_v), \eta(\mu_w) \to 0$, uniformly in v, w.

Proof. Let b assign a choice of ϕ-barycenter to each μ_v, $\mathrm{v} \in \mathbb{P}(H_\Lambda)$.

Now let δ_v, δ_w be the Dirac measures centered on $b(\mathrm{v})$, $b(\mathrm{w})$. Recall that the distance $d_\Lambda(\mathrm{v}, \mathrm{w})$ is the Kantorovich-Rubinstein distance $W_1(\mu_v, \mu_w)$ between μ_v and μ_w, and similarly $d_M(b(\mathrm{v}), b(\mathrm{w})) = W_1(\delta_v, \delta_w)$.

Then, by the triangle inequality for the metric W_1, we have $\left|W_1(\mu_v, \mu_w) - W_1(\delta_v, \delta_w)\right| \leq W_1(\mu_v, \delta_v) + W_1(\mu_w, \delta_w)$.

Now, by the classical Jensen inequality $\int_M |f|d\mu \le \sqrt{\int_M |f|^2 d\mu}$ we have $W_1(\cdot,\cdot) \le W_2(\cdot,\cdot)$ and we conclude that

$$\begin{aligned} |d_\Lambda(\mathrm{v},\mathrm{w}) - d_M(b(\mathrm{v}),b(\mathrm{w}))| &= \left|W_1(\mu_v,\mu_w) - W_1(\delta_v,\delta_w)\right| \\ &\le W_1(\mu_v,\delta_v) + W_1(\mu_w,\delta_w) \\ &\le W_2(\mu_v,\delta_v) + W_2(\mu_w,\delta_w) \\ &= O(\sqrt{\eta(\mu_v)} + \sqrt{\eta(\mu_w)}), \end{aligned}$$

where the last line is Lemma 3.3.5. □

Existence of localized states near any point

Proposition 3.3.6 tells us that probability measures μ on M of sufficiently small dispersion correspond well to their ϕ-barycenters. This holds in particular for the probability measures μ_v associated to $\mathrm{v} \in \mathbb{P}(H_\Lambda)$. We would now like to estimate the converse, i.e. to show that each point x corresponds to an element $\mathrm{v} \in \mathbb{P}(H_\Lambda)$ whose probability measure is of small dispersion and such that $b(\mathrm{v})$ is close to x. When we thus construct an asymptotically isometric embedding $M \hookrightarrow \mathbb{P}(H_\Lambda)$ that is asymptotically inverted by b, we would rightly be able to say there is a picture of M inside $\mathbb{P}(H_\Lambda)$.

To simplify the asymptotic estimates we will introduce the notation $\widetilde{O}\,()$ common in computer science:

Definition 3.3.7. Let X be a set and consider functions $f\colon X \times \mathbb{R}_+ \to \mathbb{C}$, $g\colon X \times \mathbb{R}_+ \to \mathbb{R}_+$. We say that $f = O\left(g\right)$ uniformly when there exist finite $C, r_0 > 0$ such that $|f(x,r)| \le Cg(x,r)$ for all $r > r_0$ and all $x \in X$. We say that $f = \widetilde{O}\left(g\right)$ uniformly when $f = O\left(g|\log g|^s\right)$ uniformly for some $s \ge 0$.

Proposition 3.3.8. Let M be equipped with a Dirac-type operator D on a Hermitian vector bundle $\pi\colon \mathcal{S} \to M$, and let $\tilde{\pi}\colon \mathbb{P}(\mathcal{S}) \to M$ be its projectivized bundle. Then, there exists a family $\{\Phi_\Lambda\}_\Lambda$ of maps $\Phi_\Lambda : \mathbb{P}(\mathcal{S}) \to \mathbb{P}(H_\Lambda)$ such that for all $\epsilon > 0$,

- $d_\Lambda(\Phi_\Lambda(v), \Phi_\Lambda(w)) = d_M(\tilde{\pi}(v), \tilde{\pi}(w)) + \widetilde{O}\left(\Lambda^{-1}\right)$ uniformly.
- The dispersion $\eta(\mu)$ of the measure μ associated to $\Phi_\Lambda(v)$ is $\widetilde{O}\left(\Lambda^{-2}\right)$ uniformly.
- The maps Φ_Λ asymptotically invert b, in the sense that uniformly

  $$d_M(\tilde{\pi}(v), b(\Phi_\Lambda(v))) = \widetilde{O}\left(\Lambda^{-1}\right), \qquad d_\Lambda(\Phi_\Lambda(v)), \mathrm{v}) = \widetilde{O}\left(\sqrt{\eta(\mu_v)} + \Lambda^{-2}\right)$$

  whenever $b(\mathrm{v}) = \tilde{\pi}(v)$.

The proof depends on a balanced rescaling of the truncated heat flow and will be presented at the end of this section.

To discuss the geometric consequences of Proposition 3.3.8, we will introduce notation for the small-dispersion subset into which Φ_Λ maps.

Definition 3.3.9. Let $\epsilon > 0$. Then, $\mathbb{P}(H_\Lambda)_\epsilon \subset \mathbb{P}(H_\Lambda)$ consists of those $\mathrm{v} \in \mathbb{P}(H_\Lambda)$ for which $\eta(\mu_v) < \epsilon$.

As $\mathbb{P}(H_\Lambda)$ is the projectivization of a finite-dimensional complex vector space – in particular a compact, smooth (real) manifold – and η is a smooth real function, presenting as a quartic polynomial on the standard real charts, each sublevel set $\mathbb{P}(H_\Lambda)_\epsilon$ is itself the interior of a smooth compact manifold with boundary.

Corollary 3.3.10. As $\Lambda \to \infty$, there exists $\epsilon = \widetilde{O}\left(\Lambda^{-1}\right)$ such that the Gromov-Hausdorff distance between M and the space $\mathbb{P}(H_\Lambda)_{\epsilon^2}$, equipped with the metric d_Λ, is $O(\epsilon)$.

Proof. Let $\epsilon^2 = \sup_{x \in M} \eta(\mu_{\phi_\Lambda(x)})$. By the second part of Proposition 3.3.8, $\epsilon^2 = \widetilde{O}\left(\Lambda^{-2}\right)$.

Now, the map $\mathrm{v} \to \mu_v$ sends $\mathbb{P}(H_\Lambda)_{\epsilon^2}$ isometrically into the space of probability measures on M with the Kantorovich-Rubinstein metric W_1, and the map $x \mapsto \delta_x$ sends M isometrically into the same space.

For $\mathrm{v} \in \mathbb{P}(H_\Lambda)_{\epsilon^2}$, there is a point $x = b(\mathrm{v})$ in M such that $W_1(\mu_v, \delta_x) = O(\epsilon)$ by Proposition 3.3.6.

For $x \in M$, there is an element $\mathrm{v} = \Phi_\Lambda(x)$ of $\mathbb{P}(H_\Lambda)_{\epsilon^2}$ that satisfies $d_M(x, b(\mathrm{v})) = \widetilde{O}\left(\Lambda^{-1}\right)$ $= O(\epsilon)$ by the third part of Proposition 3.3.8. Let δ_v be the Dirac measure centered at $b(\mathrm{v})$, so that $W_1(\delta_v, \delta_x) = d_M(x, b(\mathrm{v}))$. Now, $W_1(\mu_v, \delta_x) \leq W_1(\mu_v, \delta_v) + W_1(\delta_v, \delta_x)$, and moreover $W_1(\mu_v, \delta_v) = O(\sqrt{\eta(\mu_v)}) = O(\epsilon)$ by Proposition 3.3.6, so that indeed $W_1(\mu_v, \delta_x) = O(\epsilon)$. We conclude that the Hausdorff distance between M and $\mathbb{P}(H_\Lambda)_{\epsilon^2}$, as subsets of the space of probability measures on M, is $O(\epsilon)$. ☐

Proof of Proposition 3.3.8

Recall the spectral triple $(C^\infty(M), L^2(M, \mathcal{S}), D)$ associated to M, where $\mathcal{S}$ is a Hermitian Clifford bundle over M and D is a Dirac-type operator on $\mathcal{S}$.

We will define a localization map $F_\Lambda \colon \mathcal{S} \to P_\Lambda H$ such that

$$\langle F_\Lambda(v_x), aF_\Lambda(w_x)\rangle = (v_x, a_x w_x)_{\mathcal{S}} + \|v_x\| \, \|w_x\| \, O\left(\|a\| \Lambda^{-2} + \mathrm{Lip}_x^{(k)}(a)\Lambda^{-k}\right)$$

for all $\epsilon > 0$, whenever $v_x, w_x \in \mathcal{S}_x$ and $a \in \Gamma(\operatorname{End}\mathcal{S})$. If Ψ_{xy} is the parallel transport map from $\mathcal{S}_y$ to $\mathcal{S}_x$, define the constant $\operatorname{Lip}_x^{(k)}(a) \stackrel{\text{def}}{=} \sup_{y:\, d(x,y)\le\rho} \left\| a_x - \Psi_{xy}^* a_y \right\| / d(x,y)^k$.

To do so, we will take the element $v_x \in \mathcal{S}_x$ and use the short-time heat flow associated to the Laplace-type operator D^2 to obtain a smooth section $y \mapsto p_t(x,y)(v_x)$ of $\mathcal{S}$, which then corresponds to an element of H. The known estimates on heat asymptotics will allow us to bound the dispersion of $p_t(x,y)(v_x)$ for small t. Then, the fact that p_t is the heat kernel associated to D^2 whereas P_Λ is an associated projection, will allow us to control the behaviour of $(1 - P_\Lambda)p_t(x,y)(v_x)$.

Definition 3.3.11. Let $v_x \in \mathcal{S}_x$, for $x \in M$. Then $p_t(v_x)$ is the section $y \mapsto p_t(x,y)(v_x)$ of $\mathcal{S}$, where p_t is the integral kernel associated to the operator e^{-tD^2}.

The following Lemma allows us to control the leading term in the short-time behaviour of the heat flow $p_t(v_x)$. To that end, let $h_t(x,y)$ equal the scalar coefficient $e^{-d_M(x,y)^2/4t}(4\pi t)^{-m/2}$ of the leading term in the asymptotics of the heat kernel. For $x \in M$ and $s \in \mathbb{R}$, let $B_s(x) \subset M$ be the metric ball of radius s around x.

Lemma 3.3.12. Let $a \in \Gamma(\operatorname{End}\mathcal{S})$. Then, we have for all s smaller than the injectivity radius of M, and all $v, w \in \Gamma(\mathcal{S})$,

$$\begin{aligned} \int_{B_s(x)} h_t(x,y) \left(v_x, \Psi_{xy}^* a_y w_x \right)_{\mathcal{S}} dy = (v_x, a_x w_x)_{\mathcal{S}} \int_{B_s(x)} h_t(x,y) dy + \\ + \|v\| \, \|w\| \operatorname{Lip}_x^{(k)}(a) O(t^{k/2} + s^{-2} t^{(k+2)/2}), \\ \int_{B_s(x)} h_t(x,y) dy = 1 + O(t + s^{-4} t^2), \end{aligned}$$

uniformly in $v_x, a, x \in M$.

Proof. For $k \geq 0$, consider the integral $m_{t,s,k}(x) \stackrel{\text{def}}{=} \int_{B_s(x)} h_t(x,y) d^k(x,y) dy$. Let $m'_{t,s,k} \stackrel{\text{def}}{=} (4\pi t)^{-m/2} \int_{\|y\| \le s} e^{-\|y\|^2/4t} \left\| y \right\|^k dy$. There exists a global constant C such that the pullback of the volume form on M is bounded by $C \left\| y \right\|^2$ times the Euclidean volume form. Thus, pulling back our integral through the exponential map at x, we have $|m_{t,s,k}(x) - m'_{t,s,k}| \le C m'_{t,s,k+2}$.

By Chebyshev's inequality, we have

$$m'_{t,s,2k} = (4\pi t)^{-m/2} \int e^{-\|y\|^2/4t} \left\| y \right\|^{2k} dy + O(t^{-m/2} s^{-4} \int_{\complement B_s(0)} e^{-\|y\|^2/4t} \left\| y \right\|^{2k+4} dy),$$

where $\complement$ denotes the complement. With Isserlis' theorem to calculate the full Gaussian integrals, we see that $m'_{t,s,2k} = c_k t^k + O(s^{-4}t^{k+2})$ for all k. Thus, $m_{t,s,2k}(x) = c_k t^k + O(t^{k+1}) + O(s^{-4}t^{k+2})$.

Now, estimate $\left|\left(v_x, \Psi^*_{xy} a_y w_x\right)_{\mathcal{S}} - (v_x, a_x v_x)_{\mathcal{S}}\right| \leq \mathrm{Lip}^{(k)}_x(a) d(x,y)^k \|v_x\| \|w_x\|$.

As $m^2_{t,s,k} \leq m_{t,s,0} m_{t,s,2k}$ by the classical Jensen's inequality, we can use the simple estimate $\sqrt{m_{t,s,0}(x) m_{t,s,k}(x)} = O(t^{k/2} + s^{-2}t^{k/2})$ to conclude that the remaining error is itself

$$O(\|v_x\| \|w_x\| \mathrm{Lip}^{(k)}_x(a)(t^{k/2} + s^{-2}t^{(k+2)/2}))$$

uniformly. □

We are now in a position to show that the rescaled heat flow $(2\pi t)^{m/4} p_t \colon \mathcal{S}_x \to H$ is asymptotically isometric, in the following sense:

Lemma 3.3.13. For $a \in \Gamma(\operatorname{End}\mathcal{S})$ and $v, w \in \Gamma(s)$, we have uniformly

$$\begin{aligned} \langle p_t(v_x), a p_t(w_x)\rangle = &(2\pi t)^{-m/2} (v_x, a_x w_x)_{\mathcal{S}} + \\ &+ \|v_x\| \|w_x\| O(\mathrm{Lip}^{(k)}_x(a) t^{(k-m)/2} + \|a\| t^{(2-m)/2}) \end{aligned}$$

Proof. It is well-known, see e.g. [BGV04, Theorem 2.30], that there exist a nonzero radius s around x such that for $d_M(x,y) < s$, one has $p_t(x,y)(v_x) = h_t(x,y)(\Psi_{xy}(v_x) + O(t))$ as $t \to 0$, where Ψ is the parallel transport along the Clifford connection. Therefore,

$$\begin{aligned} \left(p_t(x,y)(v_x), a_y p_t(x,y)(w_x)\right)_{\mathcal{S}} &= (h_t(x,y))^2 \left(v_x, \Psi^*_{xy} a_y w_x\right)_{\mathcal{S}} \\ &\quad + O(t(h_t(x,y))^2 \|a\| \|v_x\| \|w_x\|), \end{aligned}$$

uniformly in x. Moreover, for $d_M(x,y) > s$, one has

$$\left(p_t(x,y)(v_x), a_y p_t(x,y)(w_x)\right)_{\mathcal{S}} = O((h_t(x,y))^2 \|a\| \|v_x\| \|w_x\|).$$

Now, outside an s-ball around x, we have

$$\int_{\complement B_s(x)} \left(p_t(x,y)(v_x), a_y p_t(x,y)(w_x)\right)_{\mathcal{S}} dy = O(e^{s^2/2t} t^{-m} \|a\| \|v_x\| \|w_x\|),$$

as $t \to 0$. Now set $s \stackrel{\text{def}}{=} t^{1/4}$ and note that $(h_t(x,y))^2 = (2\pi t)^{-m/2} h_{2t}(x,y)$. The estimate of Lemma 3.3.12 on the integral over $B_s(x)$ then completes the proof. □

In order to estimate the scaling of the truncated heat flow $P_\Lambda p_t(v_x)$ with Λ, we will relate $p_t(v_x)$ to the spectral resolution of D^2, as follows.

Definition 3.3.14. Let P_λ be the projection onto the λ-eigenspace of the first-order elliptic differential operator D and let $\widetilde{E}_\lambda$ be its integral kernel, so that for all sections v of $\mathcal{S}$ we have

$$P_\lambda(v)(y) = \int_M \widetilde{E}_\lambda(x,y)(v_x)dx.$$

Then, $E_\lambda : \mathcal{S} \to \Gamma(\mathcal{S})$ is the associated lifting $E_\lambda(v_x) : y \mapsto \widetilde{E}_\lambda(x,y)(v_x)$.

In particular, we have $p_t = \sum_\lambda e^{-t\lambda^2} E_\lambda$ weakly. To estimate the L^2-norm of $E_\lambda(v_x)$, we will need the following classical result by Hörmander[Hör68].

Theorem 3.3.15 ([Hör68, Theorem 4.4]). There exists a constant C such that

$$\sup_{x,y\in M} \left\|\widetilde{E}_\lambda(x,y)\right\| \leq C(1+|\lambda|)^{\dim M-1}$$

uniformly in x,y,λ. In particular, there exists a constant c such that

$$\|E_\lambda(v_x)\|_H \leq c(1+|\lambda|)^{\dim M-1} \|v_x\|_{\mathcal{S}_x},$$

for all $v_x \in \mathcal{S}$, all $x \in M$ and all $\lambda \in \sigma(D) \subset R$.

Due to the polynomial scaling of the fiberwise inner product, we can now show that the exponential dependence of $(1-P_\Lambda)p_t(v_x)$ on Λ implies that we retain the asymptotic properties of the heat flow when we truncate such that $\Lambda^2 t = c\log\Lambda$ for sufficiently large c.

Lemma 3.3.16. We have for all $a \in B(H)$,

$$\left|\left\langle p_t(v_x), (a - P_\Lambda a P_\Lambda)p_t(w_x)\right\rangle_H\right| = O(\|v_x\|\,\|w_x\|\,\|a\|\,t^{1-2m}e^{-tz\Lambda^2}),$$

for all fixed $0 \leq z < 1$, uniformly in $v_x, w_x \in \mathcal{S}_x, x \in M$, as $\Lambda \to \infty$.

Proof. Recall that the integral transform associated to the kernel $p_t(x,y)$ equals the bounded linear operator $w \mapsto e^{-tD^2}w$ on H, so that $p_t = \sum_\lambda e^{-t\lambda^2}E_\lambda$ weakly. Thus, for $w \in H$, we have $\langle P_\Lambda p_t(v_x), w\rangle = \sum_{|\lambda|<\Lambda\in\sigma(D)} e^{-t\lambda^2}\langle E_\lambda(v_x), w\rangle$.

The difference to be estimated then consists of the sum of the missing terms, which equals $\sum_{\lambda_1,\lambda_2\notin[-\Lambda,\Lambda]^2} e^{-t(\lambda_1^2+\lambda_2^2)} \left\langle E_{\lambda_1}(v_x), aE_{\lambda_2}(w_x)\right\rangle$.

First note that Theorem 3.3.15 provides a global constant c such that

$$|\left\langle E_{\lambda_1}(v_x), aE_{\lambda_2}(w_x)\right\rangle| \leq \|v_x\|\,\|w_x\|\,\|a\|\,c^2\left((1+\lambda_1^2)(1+\lambda_2^2)\right)^{m-1}.$$

Now, $\sum_{|\lambda|>\Lambda} e^{-t\lambda^2}(1+\lambda^2)^{m-1}$ is, for $0<\epsilon\leq 1$, bounded by $e^{-(1-\epsilon)t\Lambda^2}$ times the shifted sum $\sum_{|\lambda|>\Lambda} e^{-t\epsilon\lambda^2}(1+\lambda^2)^{m-1}$. Moreover, the entire sum $\sum_{\lambda} e^{-t\lambda^2}(1+\lambda^2)^{m-1}$ is, by the heat asymptotics for the Laplace-type operator D^2, bounded by $O(t^{\frac{1}{2}-m})$.

Thus, we obtain a bound of $O\left(c^2\,\|v_x\|\,\|w_x\|\,\|a\|\,t^{1-2m}e^{-t(1-\epsilon)\Lambda^2}\right)$. □

Our localization map is thus given by a truncated, rescaled heat flow, as follows.

Definition 3.3.17. Let $t_\Lambda \overset{\text{def}}{=} 2m\Lambda^{-2}\log\Lambda$. The map $F_\Lambda\colon \mathcal{S}\to P_\Lambda H$ is given by

$$v_x \mapsto (2\pi t_\Lambda)^{m/4}\sum_{|\lambda|\leq\Lambda} e^{-t_\Lambda\lambda^2}E_\lambda(v_x).$$

There exists finite Λ such that F_Λ is injective, by Lemma 3.3.13 and injectivity of the heat flow $p_t(v_x)$.

Now we are in a position to connect F_Λ to the localization question of Proposition 3.3.8.

Proposition 3.3.18. Consider the map $\Phi_\Lambda\colon \mathbb{P}\mathcal{S}\to\mathbb{P}(H_\Lambda)$ given by

$$\Phi_\Lambda([v_x]) \overset{\text{def}}{=} \left[F_\Lambda(v_x)\right]\in\mathbb{P}(H_\Lambda),$$

for Λ sufficiently large that F_Λ is injective. Then, $\Phi_\Lambda([v_x])$ is localized near x in the sense that

$$\eta(\mu_{\Phi_\Lambda([v_x])}) = \widetilde{O}\left(\Lambda^{-2}\right), \qquad W_2(\mu_{\Phi_\Lambda([v_x])},\delta_x)^2 = \widetilde{O}\left(\Lambda^{-2}\right).$$

Proof. Note that for any $\epsilon>0$ we may pick z such that $t_\Lambda^{1-2m}e^{-t_\Lambda z\Lambda^2} = \widetilde{O}\left(\Lambda^{-2}\right)$. Thus, for $a\in\Gamma(\operatorname{End}\mathcal{S})$ we have

$$\begin{aligned}\langle F_\Lambda(v_x), aF_\Lambda(v_x)\rangle &= (2\pi t)^{m/2}\left\langle p_t(v_x), ap_t(v_x)\right\rangle + \|v_x\|^2\,\|a\|\,\widetilde{O}\left(\Lambda^{-2}\right)\\ &= (v_x, a_x v_x) + \|v_x\|^2\,\widetilde{O}\left(\operatorname{Lip}_x^{(k)}(a)\Lambda^{-k} + \|a\|\,\Lambda^{-2}\right)\end{aligned}$$

so that in particular $\|E_\Lambda(v_x)\|^2 = \|v_x\|^2 (1 + \widetilde{O}(\Lambda^{-2}))$.

With $f_x(y) \stackrel{\text{def}}{=} d(x,y)^2$ and $\|v_x\|^2 = 1$, we have $W_2(\mu_{\Phi_\Lambda([v_x])}, \delta_x)^2 = \Phi_\Lambda([v_x])(f_x)$ which is $\widetilde{O}(\Lambda^{-2})$, and with $g(y)_i \stackrel{\text{def}}{=} \phi(y)_i - \phi(x)_i$ we see that $\Phi_\Lambda([v_x])(g_i)$ is $\widetilde{O}(\Lambda^{-1})$. The dispersion of the associated measure is therefore $\widetilde{O}(\Lambda^{-2})$. □

Proof of Proposition 3.3.8. Let μ_1, μ_2 be the measures associated to $\Phi_\Lambda(v_x)$ and $\Phi_\Lambda(w_y)$ respectively. Then, $\eta(\mu_i) = \widetilde{O}(\Lambda^{-2})$ by Proposition 3.3.18 and we have

$$\left| d_M(x,y) - d(\mu_1,\mu_2) \right| \le W_2(\delta_x,\mu_1) + W_2(\mu_2,\delta_y) = \widetilde{O}(\Lambda^{-1}).$$

Let $p \stackrel{\text{def}}{=} E_{\mu_{\Phi_\Lambda(v_x)}}[\phi]$. Then, $d_{\mathbb{R}^n}(\phi(x),p) \le W_2(\phi(x),\phi_*\mu_1) + \sqrt{\eta(\mu)}$ and the first term is, by bi-Lipschitz equivalence, $O(W_2(x,\mu_1))$ so that $d_{\mathbb{R}^n}(\phi(x),p) = \widetilde{O}(\Lambda^{-1})$. Therefore, one has $d(x, b(\Phi_\Lambda(v_x))) = \widetilde{O}(\Lambda^{-1})$ as well.

Finally, for probability measures ν, $x = b(\nu)$ and $0 \ne v \in \mathcal{S}_x$, we have $d(\Phi_\Lambda(v),\nu) \le W_2(\delta_x,\nu) + W_2(\Phi_\Lambda(v),\delta_x)$, and Lemma 3.3.5 leads to bounds of $O(\sqrt{\eta(\nu)})$ and $\widetilde{O}(\Lambda^{-1})$ (when combined with Proposition 3.3.18), respectively, on the latter. □

The space $\mathbb{P}(H_\Lambda)$ in terms of the truncation of a spectral triple

By Corollary 3.3.10, there exists $\epsilon = \widetilde{O}(\Lambda^{-1})$ such that the space $\mathbb{P}(H_\Lambda)_{\epsilon^2}$ is ϵ-close to M, when equipped with the Kantorovich-Rubinstein metric. By Kantorovich-Rubinstein duality, that metric can be computed by Connes' formula (3.1):

$$d_\Lambda(\mathrm{v},\mathrm{w}) = \sup_{f \in C^\infty(M)} \left\{ \left| \int_M f \, d\mu_v - \int_M f \, d\mu_w \right| \mid \left\| [D,f] \right\| \le 1 \right\}$$

Now, each element v of $\mathbb{P}(H_\Lambda)$ induces a state ω_v of $C^\infty(M)_\Lambda$, which corresponds to representatives $v \in H_\Lambda$ of v as

$$\begin{aligned}\omega_v(P_\Lambda f P_\Lambda) &= \langle v, P_\Lambda f P_\Lambda v\rangle / \|v\|_H^2 = \langle v, f v\rangle / \|v\|_H^2 \\ &= \int_M f(x) \, d\mu_v(x).\end{aligned}$$

With this identification, we have

$$d_\Lambda(\mathrm{v},\mathrm{w}) = \sup_{f \in C^\infty(M)} \left\{ \left| \omega_v(P_\Lambda f P_\Lambda) - \omega_w(P_\Lambda f P_\Lambda) \right| \mid \left\| [D,f] \right\| \le 1 \right\}. \tag{3.3}$$

It is an open question whether, in the limit $\Lambda \to \infty$, this metric can be approximated arbitrarily well in Gromov-Hausdorff distance by the functional on the *truncation* of the spectral triple given by

$$\widetilde{d_\Lambda}(\mathrm{v},\mathrm{w}) = \sup_{f \in C^\infty_\Lambda(M)} \left\{ \left| \omega_v(f) - \omega_w(f) \right| \mid \left\| [D_\Lambda, f] \right\| \le 1 \right\}. \tag{3.4}$$

Although we clearly have $\| [D_\Lambda, P_\Lambda f P_\Lambda] \| \le \| [D, f] \|$ so that $d_\Lambda \le \widetilde{d_\Lambda}$, it is a highly non-trivial undertaking to obtain a bound in the *opposite* direction. See e.g. [DLM14b; Sui20] for further perspective on the problem.

Importantly, the distance $\widetilde{d_\Lambda}$ is directly amenable to *algorithmic* computation; see Section 3.4. Therefore, that is the metric we propose to use in the PointForge algorithm introduced below. The question of correctness of that algorithm, in the sense of Gromow-Hausdorff convergence of its result to the original compact Riemannian manifold, then at least partly hinges on the relation between the metrics d_Λ and $\widetilde{d_\Lambda}$. Attacking this difficult question is however beyond the scope of the present chapter.

3.4 The PointForge algorithm: associating a finite metric space

Once a set of localized vector states is found, the Connes (Kantorovich-Rubinstein) distance between them will serve as an estimate for the geodesic distance between the points in M near which they are concentrated. Keeping in mind the discussion of Section 3.3, we will regard the truncated metric of Equation (3.4) as the natural metric on $\mathbb{P}(H_\Lambda)$.

Localized vector states can be found by minimizing the dispersion functional in H. Apart from the comparison of the metrics (3.3) and (3.4), nonzero dispersion induces a distortion of estimated distances (see section 3.3, below). Therefore, the dispersion supplies a lower bound on the Gromov-Hausdorff distance between any graph of localized states and the manifold M. Computationally speaking, then, it would be desirable to minimize the number of states (and, hence, computational resources) required to approach this bound.

The main other factor, besides localization, that influences the Gromov–Hausdorff distance is the density (in the Hausdorff sense) of our set of points inside M. Optimally, therefore, the states would be equidistributed on M.

In order to construct a potential whose minima are both localized and roughly (that is, under the map ϕ) equidistributed, we add an electrostatic repulsion term to the dispersion. Given a set V of states, the next state is then generated as the minimum of the energy functional

$$e(v;V) \stackrel{\text{def}}{=} -\eta(v)^{-1} + g_e \sum_{w \in V} \left(\sum_i \left(\langle v, \phi_i v \rangle - \langle w, \phi_i w \rangle \right)^2 \right)^{-1}. \tag{3.5}$$

The value of the coupling constant g_e should ideally be sufficiently large to overcome local variation in minimal dispersion but is otherwise not expected to influence the generated states much – this is consistent with our observations for $M = S^1, S^2$.

The PointForge algorithm

Using the functional (3.5) we propose the following algorithm to construct states and thus a finite metric space M_Λ that models the metric information about M contained at cutoff Λ.

As preparation, we must estimate the number N of states to generate.

- Estimate $\operatorname{vol} M_\Lambda$ and $\dim M_\Lambda$, e.g. using the asymptotic formulas of [Ste19b].
- Estimate the Euclidean dispersion $\eta_0 = E_\nu\left[\|X\|^2\right]$ under the multivariate normal distribution ν of covariance matrix $2\Lambda^{-2}\log\Lambda\,\mathrm{id}$ on $\mathbb{R}^{\dim M_\Lambda}$.
- Set $N = \operatorname{vol} M_\Lambda / (\operatorname{vol}(B_{\dim M_\Lambda})\eta_0^{\dim M_\Lambda/2})$, where $B_{\dim M_\Lambda}$ is the Euclidean unit ball of dimension $\dim M_\Lambda$.

For cases where ϕ is a Riemannian embedding of M, any g_e will suffice to lead to equidistributed states in M, while for $g_e = 0$ the states generated numerically would mostly lie very close together. However in the cases where ϕ is far from Riemannian, we need to chose g_e to be sufficiently large to overcome local variations in minimal dispersion, and assume this to mean that $-\alpha^2\eta_0^{-1} + g_e\alpha^2\eta_0^{-1} \geq -\beta^2\eta_0^{-1}$, where α and β are the optimal local Lipschitz constants of ϕ and ϕ^{-1}, respectively. This ensures that states in regions of M where the dispersion is over-reported (due to stretching by ϕ) will be generated once the regions where the dispersion is under-reported are saturated with states, instead of being skipped.

Then, simply generate N states by minimizing the iterative energy functional and calculate the Connes distance between them:

```
while |V| ≤ N do
    Find a vector w (locally) minimizing e(w;V).
    Append w to V.
    for v ∈ V, do
        Set d(v,w) = max{|⟨v,av⟩ − ⟨w,aw⟩| : ‖[D,a]‖ ≤ 1}.
    end for
end while
```

The algorithm, including the distance calculation and the examples S^1 and S^2, has been implemented in Python and is publicly available at [SG19].

Implementation: calculating the metric on $\mathbb{P}(H_\Lambda)$

When $v, w \in H_\Lambda$, the distance between the vector states $\langle v|\cdot|v\rangle$ and $\langle w|\cdot|w\rangle$ of the algebra $A = C^\infty(M)$ equals

$$\max_{a\in A_\Lambda}\{|\langle v,av\rangle - \langle w,aw\rangle| : \ \|[D_\Lambda,a]\| \le 1\},$$

as in discussed in section 3.3.

The functional $a \mapsto \langle v|\,a\,|v\rangle - \langle w|\,a\,|w\rangle$ is linear and the space $\{a \in A_\Lambda \mid \|[D_\Lambda,a]\| \le 1\}$ is convex, which ensures that computing the minimum is computationally feasible.

Indeed, if $a_0,\dots,a_n$ is a basis for $(A_\Lambda)_{\mathrm{sa}}$, we can reformulate the problem as:

Problem 1. Minimize $\sum_i c_i\,(\langle v,a_iv\rangle - \langle w,a_iw\rangle)$ over $c\in\mathbb{R}^{n+1}$, subject to the constraint

$$\begin{bmatrix} I & \sum_i c_i[D_\Lambda,a_i] \\ \sum_i c_i[D_\Lambda,a_i]^* & I \end{bmatrix} > 0$$

With the constraints formulated as a linear matrix inequality, we have put the problem in a form directly amenable to techniques from semi-definite programming. A reasonably effective algorithm, given the scale of the problem, is then provided by the Splitting Cone Solver of [ODo+16].

Complexity and the dimension of $C^\infty(M)_\Lambda$

Step 2 of the PointForge algorithm amounts to finding a local minimum of a quadratic function under quadratic constraints in a vector space of dimension $\dim H_\Lambda$, which can be done in $O(\dim H_\Lambda)^2$, e.g. with the BFGS algorithm.

The problem in step 5 is convex, of dimension $\dim C_\Lambda(M)$. This factor is what limits the computational feasibility of high Λ in our experiments, so it would be informative to analyze the scaling of $\dim C_\Lambda(M)$ with Λ.

As a simple example, one can represent the generator $e^{i\theta}$ of $C^\infty(S^1)$ as the shift operator on $H = l^2$, with basis indexed by $\mathbb{Z}$, where the corresponding Dirac operator acts diagonally as $De_n = ne_n$. It is then easy to see that the dimension of $C^\infty(S^1)_\Lambda$ is equal to $\dim H_\Lambda = 2\lfloor\Lambda\rfloor + 1$.

For $M = S^2$, if we choose an orthonormal basis e_{lm} of eigenvectors of D and introduce the spherical harmonics ${}_0Y_{lm}$ then we can express $\langle(e_{l_1m_1}\cdot e_{l_2,m_2}), {}_0Y_{l_3m_3}\rangle_{L^2(M)}$ in terms of $3j$-symbols. In particular, these vanish unless $||l_1| - |l_2|| \le l_3 \le |l_1| + |l_2|$, which tells us that $C^\infty(S^2)_\Lambda$ is spanned by $({}_0Y_{lm})_\Lambda$ for $l \le 2\Lambda$ and is thus of dimension bounded by $(2\Lambda+1)^2$.

The general situation is not entirely clear. However, our Proposition 3.3.8, as noted there, provides a lower bound of $\Theta(\Lambda^{\dim M})$ on the scaling of $\dim C^\infty(M)_\Lambda$ with Λ.

3.5 Example: S^2

The simplest interesting example of a commutative spectral triple that allows for an isometric embedding in $\mathbb{R}^3$ is probably the sphere S^2. This section will cover the application of the PointForge algorithm to truncations of $(C^\infty(S^2), L^2(S^2, \mathcal{S}_{S^2}), D_{S^2})$, and thereby illustrate (and test the optimality of) the analytic results of Section 3.3.

Implementation

The main ingredients are the vector space $C^\infty_\Lambda(S^2)$, the spectrum of D_{S^2}, the element ϕ, and their representation on $L^2(S^2, \mathcal{S}_{S^2})_\Lambda$.

The vector space $C^\infty_\Lambda(S^2)$ is spanned by the spherical harmonic functions Y_{lm} up to $l = 2\Lambda$, as in section 3.4. An eigenbasis e_{lm} of D can be expressed in terms of the spin-weighted spherical harmonics ${}_sY_{lm}$, with $s = \pm\frac{1}{2}$, as discussed e.g. in [GVF01], section 9.A. The matrix coefficients of the representation of $C^\infty_\Lambda(S^2)$ can then be expressed in terms of triple integrals of spin-weighted spherical harmonics. Note that a brute-force approach of calculating the inner products $\langle e_{lm}, Y_{l'm'}e_{l''m''}\rangle$ in order to obviate knowledge of the representation-theoretic machinery attached to S^2 would have been possible, however it would have introduced the additional complexity of calculating $(\dim H_\Lambda)^2 \cdot \dim C^\infty_\Lambda(S^2) \cdot \operatorname{rk}\mathcal{S}$ integrals numerically.

The element ϕ is just the idempotent associated to the Bott projection $\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$, where x, y, z are the standard coordinates on the embedding $S^2 \hookrightarrow \mathbb{R}^3$. Note that this embedding ϕ is isometric, although that is *not* necessary for the algorithm or the theory in Section 3.3 to work.

The source code to this implementation is publicly available as part of the full Python implementation of the algorithm at [SG19].

The localized state densities

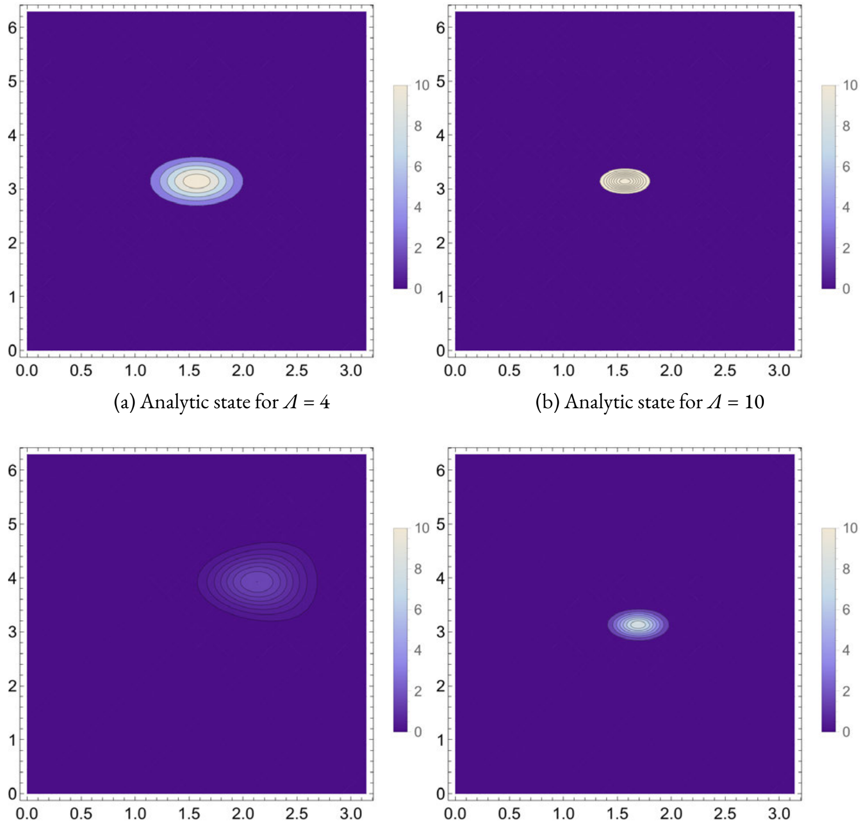
Because the measures associated to states in $\mathbb{P}(H_\Lambda)$ are of the form $(v, v)\,\mathrm{vol}_M$, with v in the finite-dimensional vector space $P_\Lambda H$, one can easily plot the corresponding function (v, v) on M. This allows us to test them, by simply plotting the corresponding fiberwise inner product of the spinor spherical harmonics in the continuum. We can then compare these with the numerical states generated through the PointForge algorithm for different Λ. The expectation is that the numerical states will be comparable to the states obtained through Φ_Λ but will be slightly less localized.

Figure 3.1 shows plots for $\Phi_\Lambda(v_x)$, for fixed $v_x \in \mathcal{S}$ is fixed, and plots of numerical states for $\Lambda = 4, 10$. It is evident that the states are indeed peaked neatly near x, in both cases. We thus find that the states are well localized and become more localized the larger the cutoff is.

Other than this qualitative comparison we also have analytic control. Proposition 3.3.6 gives the functional form of the dispersion as a function of the cut-off Λ as $\log \Lambda / \Lambda^2$. We can check this relation explicitly by plotting the size of the dispersion against the cutoff value, as done in Figure 3.2 for the cutoff up to $\Lambda = 16$.

Distribution of states over the sphere

Plotting several states simultaneously allows us to show how the repulsion term distributes them over the sphere. Figure 3.3 shows 17 states for $\Lambda = 11$. The distribution of states in Figure 3.3 has some inhomogeneities, some gaps between states are very large. This is because we only generated 17 states instead of the 110 we would expect to generate in the PointForge algorithm. Restricting the number of states reduced computation time, and allowed for a clearer visualization of the independent states. In the right hand Figure we see the states as densities on the sphere, while the left hand plot shows the densities in the θ, ϕ plane.



(a) Analytic state for $\Lambda = 4$

(b) Analytic state for $\Lambda = 10$

(c) Numerical state for $\Lambda = 4$

(d) Numerical state for $\Lambda = 10$

Figure 3.1: Plot of analytic and truncated localized states.

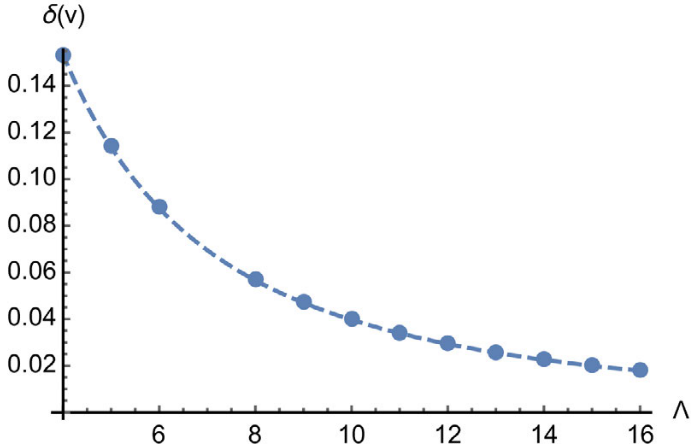


Figure 3.2: Plot of the dispersion of states versus the value of the cutoff for the states. The dashed line is a fit of the analytic result that the dispersion should scale like $\log \Lambda / \Lambda^2$.

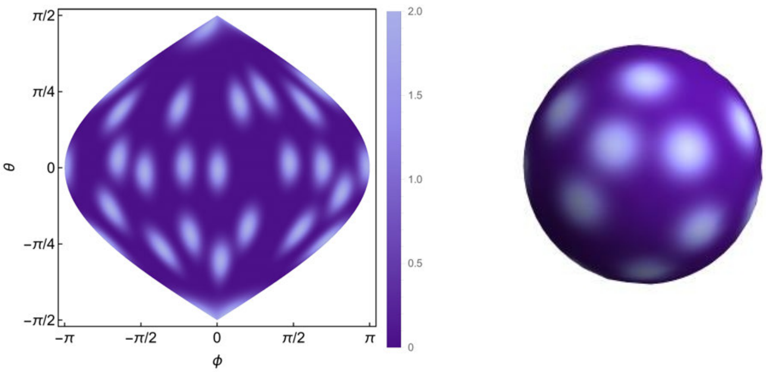
To test how the repulsive potential acts we can generate states on the sphere and just plot the coordinates for their center of mass associated with the embedding maps ϕ_i. We show this in Figure 3.4 for a maximal eigenvalue of $\Lambda = 10$, it is clear that without potential all states generated cluster at one point, while even a weak repulsive potential leads to points that are evenly distributed.

Error analysis

If a measure μ on S^2 is reasonably localized, so that $x \stackrel{\text{def}}{=} E_\mu\left[\phi(X)\right] \neq 0$, then it possesses a unique ϕ-barycenter p given by the projection of x onto the sphere. The Euclidean distance between x and $\phi(p)$ is then given by $\sqrt{1 - \|x\|^2}$.

Figure 3.5(a) shows how closely the geodesic distance between barycenters is approximated by the truncated Connes distance $\widetilde{d_\Lambda}$. The monotone scaling of the error reflects the fact that antipodal, imperfectly localized measures are significantly closer in Wasserstein distance than their barycenters are, due to the presence of the cut locus.

Interestingly, the error is strictly positive, so that $\widetilde{d_\Lambda}(\mu_1, \mu_2)$ turns out to be – for the



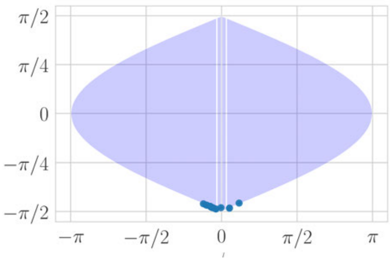
(a) States in $\theta - \phi$ plane

(b) States on the sphere

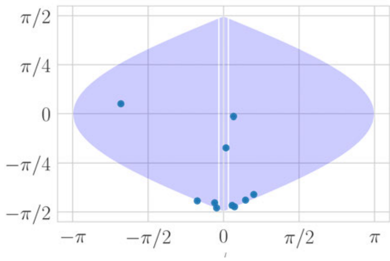
Figure 3.3: Localized states on the sphere, the left hand image shows the states projected on the two dimensional plane using a sinusoidal projection, while the right hand image shows the states on the sphere.

states considered – a *better* approximation to $d_M(p_1, p_2)$ than $W_1(\mu_1, \mu_2) = d_A(\mu_1, \mu_2)$ itself. In particular, as long as the error is positive, the convergence of $d_A(\mu_1, \mu_2)$ to $d_M(p_1, p_2)$ as the dispersions fall implies convergence of $\widetilde{d_A}$ to d_A as well. Whether this points to special behaviour of the (truncated) Connes distance between *localized* elements of $\mathbb{P}(H_A)$ remains to be seen.

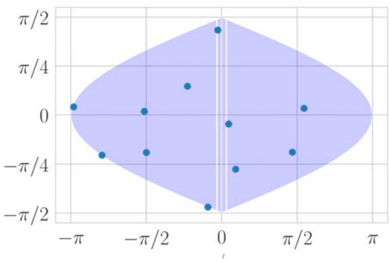
For measures μ_1, μ_2 on S^2, the analysis of Section 3.3 shows that the error $|d_M(p_1, p_2) - W_1(\mu_1, \mu_2)|$ is bounded by $W_2(p_1, \mu_1) + W_2(p_2, \mu_2)$, and moreover that we have $W_2(p, \mu)^2 \leq \beta^2 E_\mu \left[\|\phi(p) - \phi(X)\|^2 \right]$, where $\beta = \pi/2$ is the Lipschitz constant of ϕ. Thus, we have a bound of $\frac{\pi^2}{4} \left(\sum_i \eta(\mu_i) + (1 - \|x_i\|^2) \right)$ on the squared error $|d_M(p_1, p_2) - W_1(\mu_1, \mu_2)|^2$. Since the other terms in this inequality can readily be calculated, it provides us with a theoretical lower (upper) bound on $W_1(\mu_1, \mu_2)$. Figure 3.5(b) shows this lower bound, and the upper bound provided by $\widetilde{d_A}$, for $W_1(\mu_1, \mu_2)$, with $d_M(p_1, p_2)$ shown for reference.



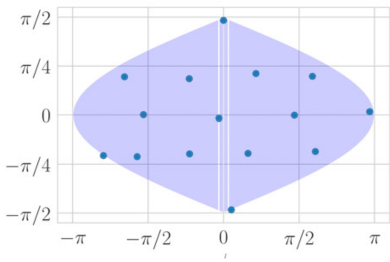
(a) No repulsive potential



(b) Repulsive potential with coupling 0.001

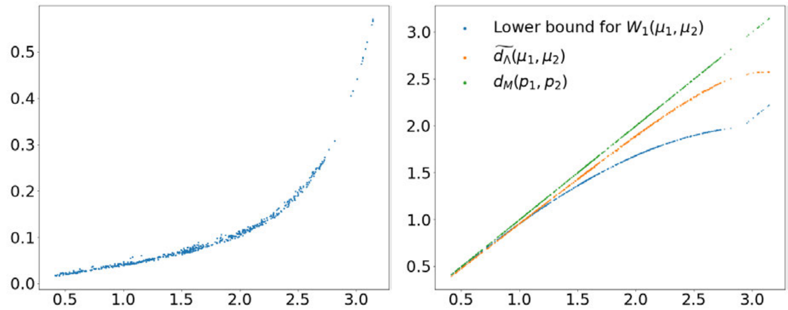


(c) Repulsive potential with coupling 0.1



(d) Repulsive potential with coupling 100

Figure 3.4: This shows how the states are distributed dependent on the repulsive potential. We can see that even a weak repulsive potential suffices to lead to well distributed points. This figure shows the point distribution in the Sinusoidal projection, flattening the sphere onto the plane.



(a) Absolute error

(b) Bounds for $d_\Lambda(\mu_1, \mu_2)$

Figure 3.5: Distance errors and bounds for pairs of localized states at $\Lambda = 5$.

3.6 Embedding a distance graph in $\mathbb{R}^n$

Let M be a compact Riemannian manifold, and assume that M embeds isometrically into $\mathbb{R}^n$. Now take a finite set V of points in M and their geodesic distances $d(\cdot,\cdot)|_{V\times V}$, e.g. by generating states as above and calculating their distances using (3.1). Optimally, we would ask for a way to embed V in $\mathbb{R}^n$ such that its image under this embedding equals its image under some Riemannian isometry $M \to \mathbb{R}^n$.

Of course, without knowledge of M such a problem is unsolvable for any given V. Instead, we hope for our embedding procedure to satisfy such a property asymptotically, i.e. that for sequences of V of increasing density their embeddings converge to an embedding of M, under some suitable notions of density and convergence. This is an open problem, and since our primary purpose at this point is one of visualization, we will only take it as a guiding principle.

Stress and local isometry of embeddings

The field of optimal graph embedding is well-established and provides many approaches to questions similar to the above. A particular model of interest is *metric multidimensional scaling*[5], where one looks for an embedding $X : V \to \mathbb{R}^n$ that minimizes the

[5] See e.g. [BG05]

stress function,

$$\sigma(X) = \frac{\sum_{p \neq q \in V} w(p,q)(d(p,q) - ||X(p) - X(q)||)^2}{\sum_{p \neq q \in V} w(p,q)},$$

where w is a positive weight function: this is just a weighted version of the second Gromov-Wasserstein distance between $V \subset M$ and $X(V) \subset \mathbb{R}^n$.

Because our M is not assumed to be Euclidean, the usual choice $w = 1$ would be quite unnatural here. In particular, an isometric embedding of M in the Riemannian sense would not necessarily have minimal stress, because the model instead asks for isometry in the sense of maps of *metric spaces*, not Riemannian manifolds. Since all tangent space information is lost when discretizing like this, the Riemannian notion of isometry does not translate immediately and we must replace it using a measure of locality.

By the smoothness of an isometric embedding ϕ of M, the relative defect $|d(p,q) - ||\phi(p) - \phi(q)|||/d(p,q)$ must converge to 0 as $p \to q$. That is to say, as long as we only worry about pairs of points that are *close* in M, the stress function above places the correct restriction on X - the further they are apart, the less sense the corresponding contribution to σ makes. This motivates us to pick a positive weight function $w(p,q)$ of $d(p,q)$ that decays monotonically and sufficiently quickly to suppress those lengths that cannot be approximated well by an Euclidean embedding.

For example, imagine two points connected by a shortest geodesic (a great circle arc) of length $l \leq \pi$ on the unit sphere and let that sphere be embedded isometrically in $\mathbb{R}^3$. In $\mathbb{R}^3$, the shortest geodesic connecting the points is a chord of length $c(l) = 2\sin(l/2)$. The defect for small geodesic distances l is thus quite small, being $O(l^3)$. It reaches its maximum when the points are antipodal, with a relative error of $(\pi - 2)/\pi$. The weight function should suppress the contribution of the larger distances to the stress σ, in order to still recognize when an embedding of the distance graph is *locally* isometric.

Let $\phi : M \to \mathbb{R}^n$ be isometric and let w_k be a sequence of weight functions depending on the cardinality k of $V \subset M$. If $w_k(l) = o(1)$ for fixed l and the marginal defect, which is bounded by

$$\frac{k \sup_{p,q \in M} w_k(p,q) \left(d(p,q) - \|\phi(p) - \phi(q)\|\right)^2}{\inf_{|V|=k} \sum_{p,q \in V} w_k(p,q)},$$

is summable in k, we can at least be sure that the stress function σ converges to 0 for embeddings $X = \phi|_V$.

The optimal choice of w then depends (at least somewhat) on the geometry of M itself; the curvature $\inf_{\phi:M\to\mathbb{R}^n}\sup\{|d(p,q)-\|\phi(p)-\phi(q)\|| \mid p,q\in M, d(p,q)\le\epsilon\}$, as function of ϵ, together with the Hausdorff distance between V and M, determines the optimal behaviour of w.

Implementation

For $\dim M = 2$, we expect the length of the smallest edges to scale roughly as $k^{-1/2}$. For $w_k(l) = \exp\left(-\sqrt{k}l\right)$ the infimum in the denominator of the marginal defect, above, is roughly bounded from below by its value for an equidistributed V, which is of order $k^2\int_0^\pi \sin(l)w_k(l)dl \sim k$ as $k\to\infty$. The supremum in its numerator is $O(k^{-3/2})$, so this sequence w_k will do in the narrow sense that it will asymptotically detect when a sequence $\{\phi_k : V_k \to \mathbb{R}^n\}$ corresponds asymptotically to an isometric embedding of M, assuming the V_k are roughly equidistributed.

Given the choice of weights, minima of the resulting stress function can be found efficiently using the weighted SMACOF algorithm for stress majorization. A simple Python implementation of the weighted SMACOF algorithm is part of [SG19], but for more intensive use we recommend the more efficient FORTRAN version with Python bindings [Ste19a].

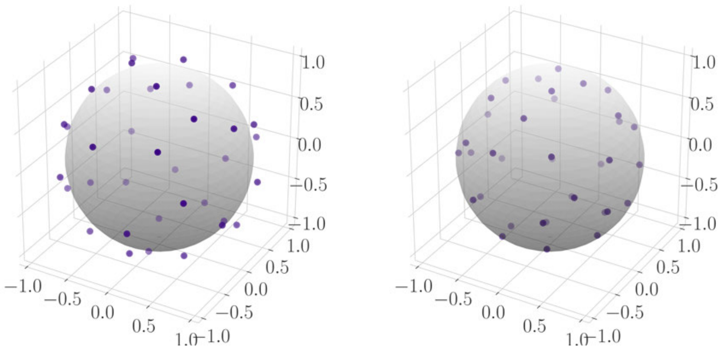
The D_c operator on the sphere

The results of Section 3.3 apply to any Dirac-type commutative spectral triple. In particular, they apply to perturbations $(C^\infty(M), L^2(M,\mathcal{S}), D_{\mathcal{S}}+B)$ of a Dirac spectral triple, as long as the perturbation B does not change the principal symbol of D. We will apply the PointForge and embedding algorithms both to the sphere and to a perturbation thereof that will arise in Chapter 4.

All spin manifolds of dimension ≤ 4 satisfy (the two-sided version of) the higher Heisenberg equation introduced in [CCM14]. Chapter 4 explores the constraint that existence of solutions to the one-sided higher Heisenberg equation,

$$\frac{1}{n!}\langle Y \underbrace{[Y,D]\ldots[Y,D]}_{\text{repeated } n \text{ times}}\rangle = \gamma, \tag{3.6}$$

places on the truncation of a spectral triple. There we found that (3.6), with Y and γ obtained from the Dirac spectral triple of S^2, is solved by a one-parameter class of



(a) D_{S^2} (b) $D_{S^2} + cB$

Figure 3.6: Locally almost-isometric embeddings corresponding to D_{S^2} and $D_{S^2} + cB$, with shaded S^2 for reference

operators $\{D_c \mid c \in \mathbb{R}\} \subset \mathscr{D}$, where

$$D_c = D_{S^2} + cB.$$

Here B is a bounded, self-adjoint operator $B = \text{sign}(D)\cos(\pi D_{S^2})$. This class of solutions does not strictly describe spectral triples, since the pseudo differential operator D_c does not satisfy the first order condition. As discussed there, however, failure of this condition is not detectable by standard methods at the level of truncations of spectral triples.

Result

The PointForge algorithm returns a metric graph, given an operator system spectral triple (A, H, D) and a designated element $\phi \in A^n$. We apply the locally isometric embedding above not only to the example from Section 3.5, but also (tentatively) to the triple $(C^\infty(S^2)_\Lambda, H_\Lambda, D_{c,\Lambda})$ of [GS20], in order to investigate the metric properties of the latter. Here $\Lambda = 5$, corresponding $\dim H_\Lambda = 84$, which leads to 35 states.

This leads to the results in Figure 3.6. There we can see that the embedded points for D_{S^2}, the left hand plot, lie outside the shaded S^2 that is included for reference, while on the other hand the points for $D_{S^2} + cB$, in the right hand plot, lie inside the shaded S^2. The transparency of the dots increases with distance to the viewer. Both embeddings show some deviation from the sphere: for D_{S^2}, the radii of the embedded points lie in $[1.06, 1.12]$, with an average of 1.09; for $D_{S^2} + cB$, in $[0.94, 0.98]$ averaging 0.96.

3.7 Final remarks

The PointForge algorithm we introduced in section 3.4 was designed to reconstruct metric spaces from truncations of commutative (Dirac) spectral triples. However, the ingredients of the algorithm need not originate as truncations of a commutative spectral triple at all; the steps apply verbatim to arbitrary operator system spectral triples, provided a special 'embedding' element ϕ is given. Obtaining such ϕ could either be related to the higher Heisenberg equation of [CCM14], or, computational resources allowing, be disposed of entirely as discussed in section 3.3. This would provide one with the means to construct finite metric spaces associated to an arbitrary noncommutative spectral triple.

It would be interesting to elaborate on this and relate it to quantization and e.g. fuzzy spaces, to get a geometric sense of the relation between a commutative spectral triple and its noncommutative deformations. For the case of the fuzzy sphere, which is not a truncation of a Dirac-type spectral triple but, unlike those, is a family of genuine spectral triples, it would be interesting to compare the *localized* states discussed here to the *coherent* states of [DLV13]. A proper generalization to the context of fuzzy spaces and other finite spectral triples could be particularly useful in connection to more physically inspired explorations thereof, such as [BG16]. The ensemble of finite, random spectral triples defined there has shown signs of a phase transition [BG16; Gla17] and can be characterized through spectral dimension measures [BDG19], which can be taken as an indication of possibly emergent geometric properties. The PointForge algorithm might then be an interesting tool to further explore some exemplary spectral triples from this class to gather further insights. It would also be instructive to test how the PointForge algorithm works for spectral triples of different topologies, e.g. the non-commutative torus [PS06] or a fuzzy torus [BG19].

Another possible application in this direction would be to exploit the explicit scale dependence of the present formalism in order to obtain a better understanding of the gravitational properties of noncommutative approaches (such as [CC96]) to quantum field theory.

Chapter 4

Understanding truncated noncommutative geometries through computer simulations[1]

When aiming to apply mathematical results of non-commutative geometry to physical problems the question arises how they translate to a context in which only a part of the spectrum is known. In this chapter we aim to detect when a finite-dimensional triple is the truncation of the Dirac spectral triple of a spin manifold. To that end, we numerically investigate the restriction that the higher Heisenberg equation [CCM14] places on a truncated Dirac operator. We find a bounded perturbation of the Dirac operator on the Riemann sphere that induces the same Chern class.

4.1 Introduction

The spectral viewpoint is central to non-commutative geometry, which makes it a natural framework to investigate the relation between energy and geometry. To understand low-energy (that is, physical) observations, we need to be able to distinguish commutative spectral triples from classically meaningless configurations, using only low-energy data.

Connes' spectral reconstruction theorem [Con13] tells us when a spectral triple (A, H, D) is the Dirac triple of a spin^c manifold. However, checking the conditions under which

[1]This work was written jointly with Lisa Glaser and published as [GS20].

the theorem holds requires knowledge of all spectral information: they can not be applied when we only consider a finite part of the frequency (energy) spectrum. That is, the usual spectral expressions do not reveal much about the nature of the universe to an observer with access to only finite spectral information.

This is highly relevant when applying non-commutative geometry to physical problems, since in realistic systems only approximate knowledge is available. It is also highly relevant when using spectral triples to discretize geometries through finite algebras and Hilbert spaces, and in most attempts to use numerical methods to explore spectral triples.

In order to engage the issue, we explore whether it is possible to use the higher Heisenberg equation [CCM14] to detect, at a finite frequency level, whether a given truncated spectral triple corresponds to a spin^{c}manifold. The analysis starts with a computer simulation of the higher Heisenberg constraint (introduced below) on the sphere, which leads to a new analytic solution of the corresponding equation. Lastly, the methods from Chapter 3 are applied to generate and visualise finite metric graphs that represent (what is argued to be) the metric space corresponding to the finite-scale geometries involved.

The remainder of this introduction is structured as follows. Section 4.1 briefly introduces the relevant concepts from noncommutative geometry, such as spectral triples, the spectral action principle, and the relation of the latter to observations based on finite spectra. Section 4.1 expands on the notion of information contained in finite spectra and introduces the problem of detecting 'commutativity' at finite scale, whereafter Section 4.1 introduces the higher Heisenberg relation as a possible approach to that problem and gives a brief overview of the structure of the chapter itself.

Background: noncommutative geometry and the cutoff scale

By Gelfand duality, a (compact Hausdorff) space X may be entirely understood in terms of the algebra $C(X)$ of continuous functions. Moreover, each commutative unital C^*-algebra is of this form $C(X)$ for some X.

Noncommutative geometry starts by the observation that we can extend this duality to spin^{c}manifolds: the spin^{c}manifold M (and, therefore, its metric) can be described uniquely in terms of the *spectral triple* $(C^\infty(M), H, D)$, where D is the associated Dirac operator and H is a Hilbert space of spinors.

This description of spin geometry in terms of operators on Hilbert spaces then allows one to extend many spin-geometric notions to the study of more general geometric objects, the noncommutative spectral triples[2] (A, H, D). In particular, the resulting flexibility allows one to use the same language to describe both ordinary spin geometry and the field theories common in particle physics [CCM07]. A very simple choice of algebra, together with the *spectral action principle* [CC97] (see below) leads to the standard model, minimally coupled to general relativity.

One interesting feature of this latter formulation is that while the classical (metric) geometry is described through an infinite-dimensional algebra and Hilbert space, the particles of the standard model are encoded in a finite-dimensional non-commutative algebra. Fundamentally finite dimensional spectral triples allow for a description of spaces that are discretized, but still retain their original symmetry group. Examples of these, often called fuzzy, spaces are the fuzzy sphere [GP95], fuzzy projective spaces [Bal+02] or the fuzzy torus [DO03]. General finite spectral triples have been classified [Kra98; Ćać11; CC08] and parametrized [Bar15]. In the present chapter, however, we will be concerned with *truncated*, not fundamentally finite, spectral triples.

By the assumption of diffeomorphism invariance, all observables in pure gravity – including the action – must be expressible in terms of global geometric invariants. The spectral action principle [CC97] in noncommutative geometry asserts that, moreover, the action should be formulated in terms of the spectrum of the Dirac operator D alone. The identification of such global invariants with zeta residues allows them to be written in terms of asymptotic traces of D, and this induces the prescription

$$S(D) = \mathrm{tr}(\chi(D/\Lambda))$$

for the bare action, where χ should be a suitable smooth cutoff function. The scale parameter Λ controls the relative contributions of Dirac eigenvalues. At finite cutoff scale Λ we are then automatically invited to think of the corresponding system as described by a finite-rank, *truncated* Dirac operator.

Recent work has started numerically exploring the path integral,

$$\mathcal{Z} = \int \mathrm{d}D e^{-\mathcal{S}(D)}, \tag{4.1}$$

with $\mathcal{S}$ a trace of powers of D, over finite-rank Dirac operators, as a possible nonperturbative description for quantum gravitational phenomena [BG16; Gla17; BDG19].

[2] Here, A is a possibly noncommutative C*-algebra, corresponding to the 'topological' aspect of the noncommutatige geometry, and D a possibly unbounded selfadjoint operator, corresponding to the 'metric' aspect thereof, both represented on a Hilbert space H. See [GVF01] for an introduction.

Geometry at finite scale

Spectral descriptions of continuum geometry involve infinite-dimensional algebras and Hilbert spaces. If these are to be applied to physics involving measurement at finite energies, to be captured in computer simulations or to be described approximately, we must understand *how (much) information can be contained in partial spectra*. This involves extending the tools that have been developed to understand infinite-dimensional non-commutative geometries to truncated spectral triples, as has been done for the residue functionals in [Ste19b].

A particular difficulty, which is central to the present chapter, relates to the recognition of (possibly almost-commutative) manifolds at the truncated level. In carrying out the path integral (4.1), for instance, one should in principle restrict to Dirac operators that actually correspond to (possibly almost-commutative) spin^c structures for the given (fixed) manifold M, just like the path integral in Euclidean quantum gravity restricts the integration to fields that describe Riemannian metrics as opposed to being fully arbitrary. However, it is a priori unclear what this restriction means for the integration variable D.

Although Connes' reconstruction theorem [Con13] allows us to detect when a spectral triple (A, H, D) corresponds to the Dirac triple on a spin manifold, it is not clear how to implement those conditions as a constraint on an integral over operators D. Moreover, it is not clear when a *finite-rank* Dirac operator D corresponds to a cutoff of such a spin geometry. This complicates the proposed identification of path integrals over finite-rank Dirac operators with finite-scale path integrals over spin geometries. The one-sided higher Heisenberg equation recalled below (and more generally, its two-sided version) offers a possible approach to constraining the domain of integration in (4.1).

The higher Heisenberg equation

In [CCM14] Chamseddine, Connes and Mukhanov introduce a non-commutative analogue to the Heisenberg relation of quantum mechanics. This 'higher Heisenberg equation' neatly captures the relation between the scalar fields (smooth functions) and the Dirac operator that is central to noncommutative geometry in a single *algebraic* equation. The one-sided version of this equation, applicable to (disjoint sums of) even-dimensional n-spheres, works as follows. For M a n-dimensional manifold, there trivially exists a covering $\phi : M \to S^n$; let its components be denoted by Y^i, $1 \leq i \leq n$. Then, the section $Y = Y^i \Gamma_i$ of the trivial Clifford bundle of rank $2^{n/2}$ satisfies $Y^2 = 1$

and $Y^* = Y$, and moreover

$$\frac{1}{n!}\langle Y \underbrace{[Y,D]\dots[Y,D]}_{\text{repeated } n \text{ times}}\rangle = \gamma, \tag{4.2}$$

where γ is the grading on the spinor bundle and $\langle\cdot\rangle$ denotes the $C^\infty(M)$-valued fiberwise trace on the Clifford algebra bundle. If a *general* Riemannian manifold M admits such Y, moreover, they must necessarily be of the form considered above, ensuring that M is a disjoint sum of even-dimensional spheres. The Y_i then generate $C^\infty(S^n)$ and the spectral triple $(C^\infty(M), H, D)$ is unitarily equivalent to the direct sum of any splitting of $(C^\infty(S^n), H, D)$ into irreducible components.

For more general (spin) M, the real structure on the spinor bundle induces a two-sided version of the equation above, corresponding to a map $\phi \times \phi'$ that induces a (not necessarily isometric) embedding $M \to S^n \times S^n$. We are presently concerned only with the one-sided equation as a first example.

In this chapter we propose to use the higher Heisenberg relation to constrain general selfadjoint matrices D, in order to induce them to correspond to truncated Dirac operators of reasonable Riemannian geometries on the underlying manifold. Computer simulations then allow us to explore numerically the effects of this constraint.

A real spectral triple consists of $(\mathscr{A}, \mathscr{H}, D)$ together with a real structure J and a chirality γ that satisfy a number of conditions. An introduction can be found e.g. in [GVF01]. One axiom that has special significance, is the first order condition

$$\left[[D,a], Jb^*J^{-1}\right] = 0 \qquad \forall\, a, b \in \mathscr{A},$$

which ensures that D acts as first-order differential operator in the commutative case, and is the second algebraic constraint (besides the one corresponding to the higher Heisenberg equation) appearing in Connes' reconstruction theorem.

To recover the metric on a spin^{c}manifold from the corresponding spectral triple, one can define a metric on the space of states $\omega_1, \omega_2 \in S(\mathscr{A})$,

$$d(\omega_1, \omega_2) = \sup_{a \in \mathscr{A}}\{\omega_1(a) - \omega_2(a) \,|\, ||[D,a]|| \le 1\}. \tag{4.3}$$

In the commutative case, the pure states correspond to atomic measures, that is, points, on the underlying manifold. In Chapter 3 we used this definition of distance, together with a notion of locality, to associate *finite* metric spaces to truncated non-commutative geometries.

In section 4.2 we explain the truncation and our simulations methods and present results for the circle and the two-sphere. This section in particular discusses the reasoning behind our choice of truncation, how it is implemented and some possible problems in this choice. In section 4.3 we show that one of the Dirac operators found in the previous section is a better solution to the Heisenberg relation, while not strictly belonging to a spectral triple in the infinite size limit. In our conclusion, section 4.4, we summarize the results and collect some questions that are opened by our work.

4.2 The Heisenberg relation in simulations

In noncommutative geometry one describes a spin manifold in terms of the associated spectral triple (A,H,D). From a mathematical perspective, it is desirable to be able to describe such a spectral triple as a limit of finite-dimensional data of increasing precision, just like one can describe a Riemannian manifold as a Gromov-Hausdorff limit of finite metric spaces. From a physical perspective, the same desire results from the view that one should be able to gain at least *some* information about the geometry by probing it at finite energies.

One natural approach to such a 'cutoff' of the geometric data (A,H,D) is to pick a scale Λ, then define

$$P_\Lambda \stackrel{\text{def}}{=} \chi_{[-\Lambda,\Lambda]}(D)$$

to be the spectral projection onto the eigenspaces of D of eigenvalue $|\lambda| \le \Lambda$, and then take the finite-dimensional data

$$(P_\Lambda A P_\Lambda, P_\Lambda H, P_\Lambda D) \tag{4.4}$$

as our starting point. As we will notice again later, these truncated triples do not neccesarily satisfy the conditions for a spectral triple using the truncated real structure J_Λ. We accept this as one of the limitations of the current program, constructing finite spectral triples with a finite real structure would lead back towards the fuzzy spaces defined by Barrett [Bar15]. In this setting, Chapter 3 reconstructed (asymptotically) spin manifolds M from the data $(P_\Lambda C^\infty(M) P_\Lambda, P_\Lambda H, P_\Lambda D_M)$ associated to the commutative spectral triple $(C^\infty(M), H, D_M)$. Some properties of the induced metric on the state spaces of $P_\Lambda A P_\Lambda$ and A were previously investigated in [DLM14b], and for the sphere specifically in [DLV13].

The truncated higher Heisenberg equation

All spin[3] manifolds of dimension ≤ 4 satisfy (the two-sided of) the higher Heisenberg equation (4.2), whereas clearly not all spectral triples do. This suggests to use the equation to recognize many cases in which a spectral triple does *not* correspond to a spin manifold, without needing to check the rather elusive conditions of the spectral reconstruction theorem. We will extend this tool to the finite-dimensional data $(P_\Lambda AP_\Lambda, P_\Lambda H, P_\Lambda D)$ introduced above, and explore what type of truncated triple solves the truncated higher Heisenberg relation.

Given a solution Y, D of equation (4.2) and the spectral projection $P_\Lambda = \chi_{[-\Lambda,\Lambda]}(D)$, the defect

$$\delta(Y_\Lambda, D_\Lambda, \gamma_\Lambda) \stackrel{\text{def}}{=} \langle Y_\Lambda [D_\Lambda, Y_\Lambda]^n \rangle - n! k \gamma_\Lambda \tag{4.5}$$

strongly converges (superpolynomially) to zero as $\Lambda \to \infty$. Simple examples like the circle (see below) show, however, that we cannot expect the defect to converge to zero in any Schatten p-norm including $p = \infty$. One wonders then which restrictions on the truncated triple are enforced by minimizing $\delta(Y_\Lambda, D_\Lambda, \gamma_\Lambda)$.

The direct approach to this question starts by searching for an operator D' on $P_\Lambda H$ that comes at least close to solving (4.2) in the sense of minimizing the constraint

$$\left\| \delta(Y_\Lambda, D', \gamma_\Lambda) \right\|_2^2 = \left\| \langle Y_\Lambda [D'_\Lambda, Y_\Lambda]^n \rangle - n! k \gamma_\Lambda \right\|_2^2, \tag{4.6}$$

for fixed Λ. The Hilbert-Schmidt norm is a natural choice here; all Schatten norms are equivalent in finite dimensions and this is the least computationally expensive among them. This, then, is the constraint whose solutions we investigate numerically below:

- Fix a cutoff Λ,
- Take $P_\Lambda, Y_\Lambda, \gamma_\Lambda$ from the corresponding commutative spectral triple (that is, here, from the circle and the (spin) sphere),
- Look for the arguments D'_Λ (matrices of dimension $\operatorname{rank} P_\Lambda$) that minimize (4.6).

The second step means that, for the sphere, the possible matrix size of the truncations will be restricted to the sums of multiplicities of eigenspaces. To have some more freedom in the choice of matrix size for D'_Λ one could, instead of P_Λ, use some other projection in its commutant. It seems, however, that in the cases of the circle and the sphere doing so would introduce a further defect in $\delta(Y_\Lambda, D_{S^n,\Lambda}, \gamma_\Lambda)$.

[3] Unlike equation (4.2), its more general two-sided version involves the spin structure. In the example $M = S^2$ considered here, the spin structure does not play a role.

Computation

In order to numerically investigate the behaviour of (4.6) in practice, we use an annealing type algorithm. Simulated annealing algorithms find optima of a given function by running a random walk in its domain, with transition probability depending on the value of the optimized function and a global 'temperature' parameter T that is decreased in time. The algorithm we use is called thermal annealing, and controls the temperature by postulating that the information theoretic and thermodynamic entropy densities must agree [VLH03]. This is a convenient choice for our problem since it has few free parameters, and we are only interested in the final result. The free parameters in question are a constant c which governs the speed at which the temperature is lowered and the final temperature T_f. Any choice of c that does not lead to freezing out of the system before equilibrium is reached is valid, while the final temperature governs how strongly the system is allowed to fluctuate around the final state. We set $T_f = 0.001$ and adjust c to the simulations in question, testing several c to ensure the results are equivalent.

The annealing algorithm runs until some $T < T_f$ is reached[4], and then simulate the system at this low temperature for a while. The quantities of interest to us are then the configuration with the lowest value of the constraint, as well as an average over the states at the final temperature.

The circle as a simple example

A first example of an algebraic relation, analogous to (4.2), whose solution describes a spin manifold is as follows [Con00]. Assume that $U \in B(H)$ is unitary and D is a self-adjoint unbounded operator on H such that $0 \in \sigma(D)$ and $D^{-1} \in L^{(1,\infty)}(H)$. Assume, moreover, that the pair U, D is represented irreducibly. Then, if U and D satisfy

$$U^*[D,U] = 1 , \tag{4.7}$$

the triple (A,H,D), where A is a dense subalgebra of the C*algebra generated by U, is unitarily isomorphic to the spectral triple $(C^\infty(S^1), L^2(S^1), D_{S^1})$ that describes the circle. Under such an isomorphism U is mapped to the generator $\theta \mapsto e^{i\theta}$ of $C(S^1)$ (up to the obvious phase ambiguity in equation (4.7)).

Given the spectral projection P_Λ as in section 4.2, the operator $U_\Lambda = P_\Lambda U P_\Lambda$ is no longer unitary and is even nilpotent, so (4.7), with U replaced by U_Λ, cannot be solved in D.

[4]The nature of the algorithm means that we do not have perfect control of the finite temperature, however the exact finite temperature is not important in our case.

The corresponding version of (4.6) is

$$\|\partial(U_\Lambda, D_\Lambda)\|_2^2 = ||U_\Lambda^*[D_\Lambda, U_\Lambda] - 1||_2^2 . \tag{4.8}$$

In order to counter the spurious symmetry $D \mapsto D + cI$ of (4.7), we demand that D_Λ additionally satisfies $DJ = JD$, where J is the real structure corresponding to the point-wise complex conjugation map on $L^2(S^1)$. This anticommutation with J ensures that the spectrum of D_Λ is symmetric around 0, and our computer simulations parametrize the space of such D_Λ as follows: Pick a eigenbasis of $P_\Lambda D_{S^2}$ and find unitary V in P_Λ such that V^*JV is just complex conjugation of the coefficients in this eigenbasis. Then, we parametrize D_Λ as iV^*H_0V, where $H_0 \in M_{\operatorname{rank} P_\Lambda}(\mathbb{R})$ is an arbitrary real antisymmetric matrix.

Using the constraint (4.8) as a weight for thermal annealing we collect two types of observations. On the one hand, we measure the Dirac operator that leads to the smallest value of the constraint. This is ideally going to be very close to the Dirac operator for the circle. To compensate for small numerical fluctuations, we also measure 500 times after the low final temperature is reached and average these measurements.

In Figure 4.1 we see that the eigenvalues of the simulated Dirac operators turn out very close to those of the circle Dirac. They can not be distinguished in the upper plot, while the lower plot shows the difference from the analytic spectrum for the average and the best eigenvalues. The small difference is an effect of the cutoff, which is also reinforced by the difference being larger for larger eigenvalues.

Examining thematrix entries $\partial(U_\Lambda, D_\Lambda)_{ij}$ of the constraint we find that the violations of this equation are minimal almost everywhere. The only sizable deviation is on the entry for $i, j = 0$ where the deviation is ~ -1. Since the defect $U_\Lambda^*[D_{S^1,\Lambda}, U_\Lambda] - P_\Lambda$ equals the projection onto that kernel, it is not surprising to find the maximum there.

Hence our simulations find the truncated circle Dirac operator, which we know to be the correct solution. This is a good test for the formalism, and encourages us to move on from the simple circle to the more complicated sphere.

S^2 simulations

The version of equation (4.2) corresponding to the sphere S^2 is

$$\partial(Y_\Lambda, D_\Lambda, \gamma_\Lambda) = \langle Y_\Lambda [D_\Lambda, Y_\Lambda][D_\Lambda, Y_\Lambda]\rangle - \gamma_\Lambda. \tag{4.9}$$

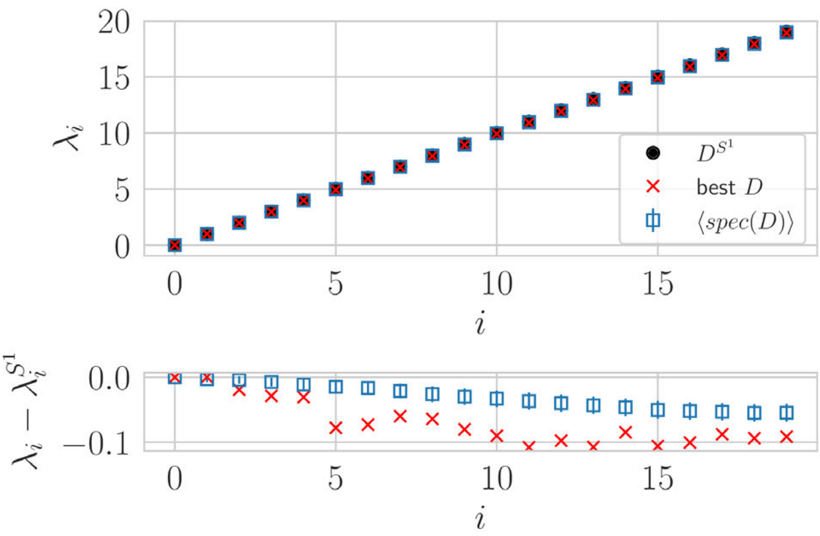


Figure 4.1: Comparing the eigenvalues of the Dirac operator with the smallest value of the constraint to that of the average over operators (with error indicating the statistical fluctutations) and the exact circle. The results are all so close together that we can not distinguish them in the upper plot, the lower plot shows only the difference between the simulation results and the exact numbers.

Here, $Y = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$, with x, y, z the standard coordinates on $\mathbb{R}^3$, viewed as functions on S^2 through its standard embedding. That is, $Y + 1$ is twice the Bott projector. The angular brackets denote the $B(P_\Lambda H)$-valued trace on $M_2(B(P_\Lambda))$ and γ_Λ is the truncation of the usual grading on $L^2(S^2, S)$. See the Appendix 4.A for the representation used in the numerical simulations.

For the sphere the Dirac operator has a few symmetries that the truncated operator should satisfy for the truncated operator to still interact correctly with the truncated chirality and real structure. This leads us to consider different parametrizations for the operator.

Parametrizing the Dirac operator

In order to cancel the symmetry $D \mapsto D + cI$ of (4.9) and to enforce symmetry of the spectrum of D_Λ, we have tested two different additional constraints. The first, stronger constraint is that D_Λ correspond to the (truncation of the) same K-cycle as D_{S^2} – that is, that $D_\Lambda|D_\Lambda|^{-1} = P_\Lambda D_{S^2}|D_{S^2}|^{-1}$, so that the bounded transform of a hypothetical even spectral triple $(C^\infty(S^2), H, D, \gamma)$ with $D_\Lambda = P_\Lambda D$ equals that of $(C^\infty(S^2), H, D_{S^2}, \gamma)$. The second, strictly weaker constraint is that D_Λ anticommute with γ_Λ, which is necessary for D_Λ to possibly correspond to part of an even spectral triple $(C^\infty(S^2), H, D, \gamma)$. These constraints lead to the parametrizations

$$D_\Lambda = \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \qquad \text{or} \qquad D_\Lambda = \begin{pmatrix} R & S \\ -S & -R \end{pmatrix},$$

respectively, where P is positive (ensuring that $D_\Lambda = D_\Lambda^*$ and $D_\Lambda|D_\Lambda|^{-1} = P_\Lambda D_{S^2}|D_{S^2}|^{-1}$) and R, iS selfadjoint (ensuring that $D_\Lambda = D_\Lambda^*$ and $D_\Lambda\gamma_\Lambda = -\gamma_\Lambda D_\Lambda$).

The former parametrization is faster than the latter since both eigendecompositions of P and the search for optimal D_Λ occur in a vector space of half the dimension. The geometries parametrized through P are strictly a subclass of those parametrized through R, S, hence we know that solutions arising in the first ensemble also exist in the second. Our simulations however show that to find the same optimal solutions in the R, S parametrization requires longer runtimes and much lower temperatures. This is because the larger configuration space takes longer to explore and lowers the relative fraction of the most optimal solutions. We have tested that the R, S simulations do not allow for additional, more optimal solutions than the P parametrization, hence the results shown will all use the P parametrization.

Results

To visualize the results of our simulations we will again look both at averages over roughly 150 measurements near the minimum as well as at the actual numerical minimum of equation (4.9) that was encountered. If we look at the operators as heatmaps, with each pixel in heatmap representing one entry of the matrix, as in see Figure 4.2, we see that the average Dirac operator in the $-P \oplus P$ parametrization commutes (up to numerical error) with D_{S^2}.

This simple structure of the simulated Dirac operators implies they are well described, quantitatively, by their spectrum. In Figure 4.3, we compare the measured eigenvalues

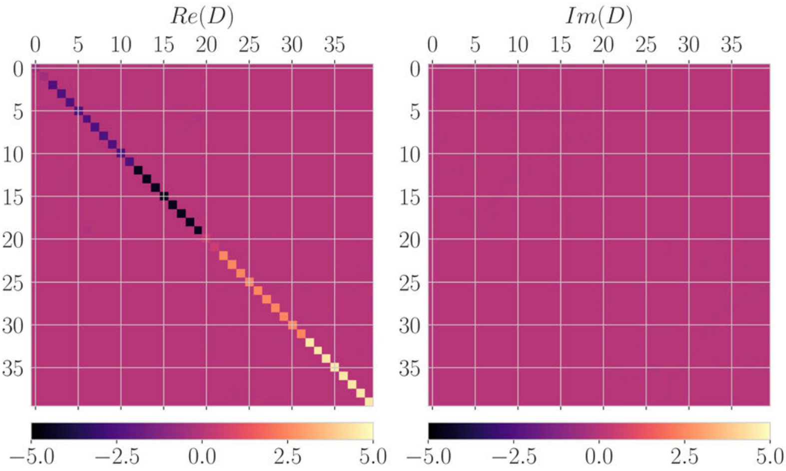


Figure 4.2: The average Dirac operator is almost entirely real, and completely diagonal.

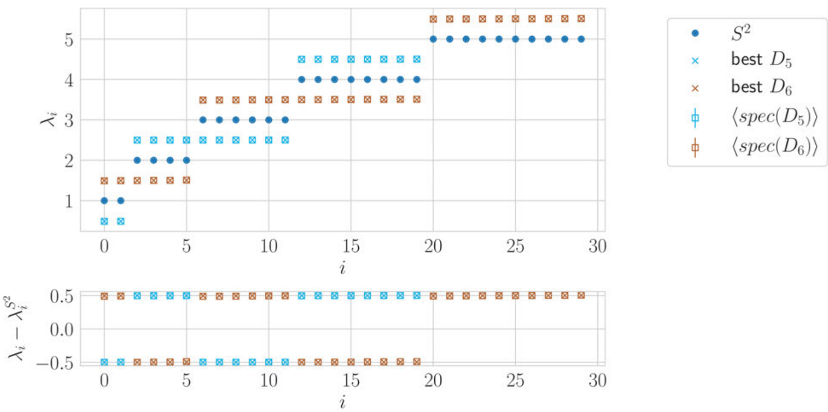


Figure 4.3: Comparing the average eigenvalues, and the best case eigenvalues of the simulations with those of the sphere. We can see that the results differ considerably between odd and even Λ, but that neither agrees with the sphere.

with those of the sphere. The Figure shows results for spectral cutoffs of $\Lambda = 5, 6$, which showcases a clear difference between odd and even cutoffs.

The simulated Dirac operators are (up to numerical error) diagonal in an eigenbasis of D_{S^2}, but the simulated eigenvalues are shifted up or down by roughly $\frac{1}{2}$. The direction of the shift appears dependent on the parity of the eigenvalue and of the cutoff Λ. That is to say, it seems we are be dealing with a bounded perturbation of D_{S^2} with particularly simple structure.

In particular, the localized zeta function asymptotics (which measure at least volume and dimension) must agree for this perturbation and the sphere. When we have identified the numerical solutions analytically, in Section 4.3 below, we will show in Figure 4.7 how this fact is reflected by the finite parts of the spectrum obtained.

Results for the Heisenberg equation

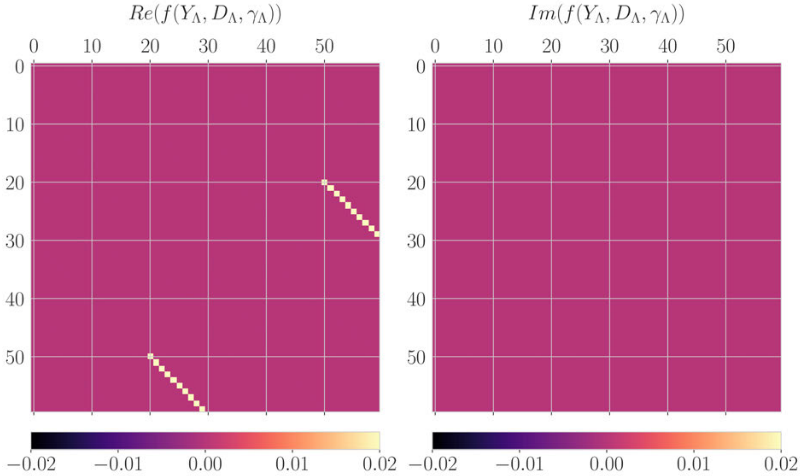
The operators in Figure 4.2 arise from minimization of the Heisenberg constraint given by the squared Hilbert-Schmidt norm $\|\partial(Y_\Lambda, \gamma_\Lambda, D)\|_2^2$, so it is interesting to see whether patterns arise in the corresponding matrix entries of $\partial(Y_\Lambda, \gamma_\Lambda, D)$; we show these in Figure 4.4. Clearly, the simulations come close to fully letting $\partial(Y_\Lambda, \gamma_\Lambda, D)$ vanish.

For the operator $D_{S^2,\Lambda}$, however, the defect $\partial(Y_\Lambda, D_{S^2,\Lambda}, \gamma_\Lambda)$ does not vanish and equals

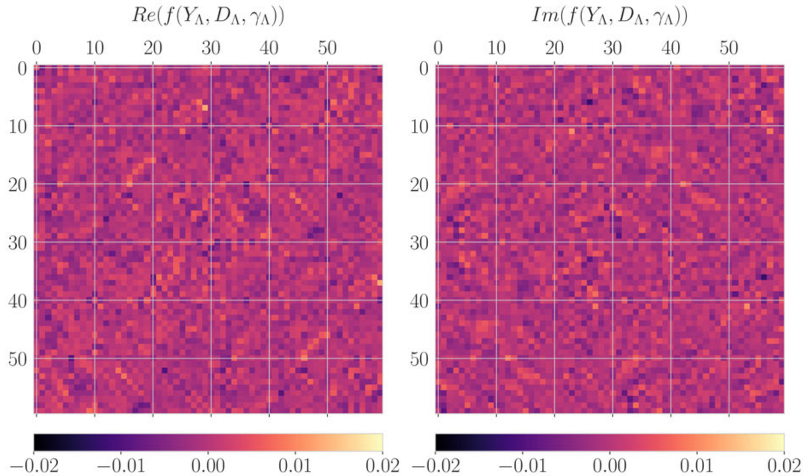
$$\partial(Y_\Lambda, D_{S^2,\Lambda}, \gamma_\Lambda) = -\frac{(1+\lambda)(1+4\lambda)}{2(1+2\lambda)^2}(E_\lambda + E_{-\lambda})\gamma \tag{4.10}$$

where E_λ projects onto the eigenspace corresponding to $\lambda = \max\{\lambda' \in \sigma(D) \mid |\lambda'| \leq \Lambda\}$; this is of norm $\sim \frac{1}{2}$ and of divergent $(O(\Lambda^{1/p}))$ p-Schatten norm for $p < \infty$ as $\Lambda \to \infty$.

For each Λ considered, we found a D_Λ with $\|\partial(Y_\Lambda, \gamma_\Lambda, D_\Lambda)\|_2 \approx 0$ and in particular the constraint satisfied $\|\partial(Y_\Lambda, \gamma_\Lambda, D)\|_2 \ll \|\partial(Y_\Lambda, \gamma_\Lambda, D_{S^2,\Lambda})\|_2$. Additionally these optimal D_Λ seem to be quite simple and symmetric, and shows a remarkable consistency across different sizes, as shown in Figure 4.5. Since the matrix size (the rank of P_Λ) grows as $O(\Lambda^2)$ it is hard to obtain reliable results for larger Λ, however the results we obtained suggest that there might be a similar type of solution for all sizes, i.e. a compatible chain of finite size Dirac operators that might arise as $P_\Lambda D' P_\Lambda$ for some D' that solves (4.2) exactly. It is thus useful to supplement the numerical results with some analytic explorations.



(a) The exact sphere



(b) Average operator

Figure 4.4: Heatmap plot of the Heisenberg relation for the operator parametrized through P and the sphere for $\Lambda = 6$. The uppermost plot shows the finite size defects in the sphere, while the lower plots show the defect generated by an averaged Dirac operator.

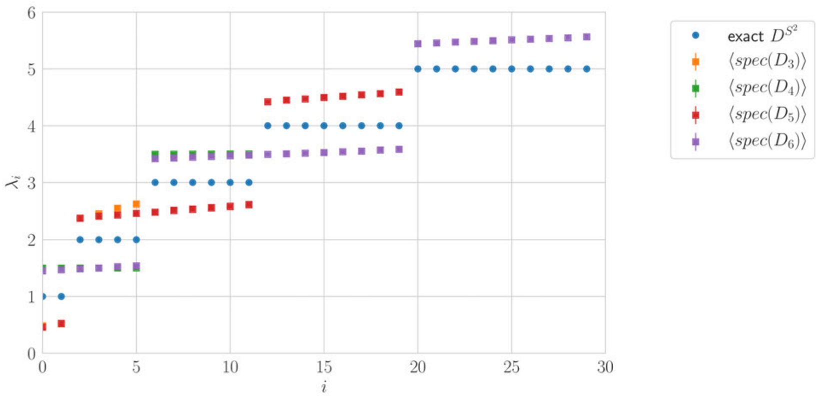


Figure 4.5: Average eigenvalues for the 4 smallest truncations of the sphere.

4.3 An alternative analytic solution to the Heisenberg relation

The simulations above suggest that, for finite Λ, there might be a class of operators $D \in B(P_\Lambda H)$ that lead to lower values of the constraint $\|\delta(Y_\Lambda, \gamma_\Lambda, D)\|_2$ than the truncations of D_{S^2} do. Since the D that show up commute with D_{S^2} and seem to be compatible across alternating choices of Λ (see Figure 4.5), we are led to look analytically for a corresponding general solution of $\delta(Y, \gamma, D) = 0$ inside the commutant of D_{S^2}.

Let us denote by $\mathscr{D}$ the space of selfadjoint operators with discrete spectrum that commute with D_{S^2} and anticommute with γ, that is, those of the form $f(D)$ for some antisymmetric $f \in C(\mathbb{R}, \mathbb{R})$. Is there an analytic solution $D \in \mathscr{D}$ to equation (4.9)?

The Appendix 4.A exhibits the coefficients of the representation of Y, γ, D on H in the basis chosen for the simulations. Since the generators Y_i are laddering, i.e. band, matrices in this basis, the resulting version of equation (4.9) is easy to solve analytically.

It leads to the following recursion for the sequence μ_l of positive eigenvalues of D, labeled by the spinor momenta $l = \frac{1}{2}, \frac{3}{2}, \ldots$,

$$\mu_l^2 - 2a_l\mu_l\mu_{l-1} + 2b_l\mu_{l+1}\mu_l = a_l\mu_{l-1}^2 + b_l\mu_{l+1}^2 + 16l^2(1+l)^2$$

where $a_l = (1+l)^2(2l-1)$, $b_l = l^2(3+2l)$ The corresponding recursion equation, with

$\mu_{-1/2} = 0$, has the unique one-parameter solution

$$\mu_l = \left(l + \frac{1}{2}\right) + c\sin(\pi l).$$

That is to say, the unique one-parameter solution $\{D_c \mid c \in \mathbb{R}\} \subset \mathscr{D}$ to equation (4.9) is

$$D_c = D_{S^2} + cB,$$

where the bounded and selfadjoint operator B equals $\cos(\pi D_{S^2})(D_{S^2}|D_{S^2}|^{-1})$.

Looking at this we see that it agrees with the Dirac operator we found in our simulations using the parametrization with P. In particular for $c = \pm 1/2$ this agrees with the simulations with an even/ odd maximal eigenvalue, as shown in Figures 4.3 and 4.5.

Spectral triple axioms

For nonzero c, the full operator $D_{S^2} + cB$ does not satisfy the first-order axiom (condition 2 in the reconstruction theorem of [Con13]) because $[[B, Y_i], Y_j]$ is not zero for all i, j; B, although pseudodifferential of order zero, is not an endomorphism of the spinor bundle. The defect $[[B, Y_i], Y_j]$, however, is compact (it is in fact in $L^{(1,\infty)}(H)$). As we will see in the next subsection the boundary effects caused by the truncation to finite matrix sizes mask this difference and lead to violations of the first order axiom for D_{S^2} alone that are of the same order of magnitude as the violation for B.

Boundary defects

As mentioned in the introduction, replacing a solution Y, γ, D of the one-sided higher Heisenberg equation (4.2) by $Y_\Lambda, \gamma_\Lambda, D_\Lambda$ leads to a nontrivial defect.

For operators in $\mathscr{D}$ this introduces an additional term in $\delta(Y_\Lambda, \gamma_\Lambda, D)$ that is not present in $\delta(Y, \gamma, D)$. This term is a multiple of the γ operator projected onto the highest eigenspace of $|D|$, where the coefficient equals

$$c_l \mu_l^2 + \frac{(1-2l)}{16 l^2} \mu_{l-1}(\mu_{l-1} + 2\mu_l) - 1,$$

with $c_l = \frac{1+9l^2+6l^3}{16 l^2 (l+1)^2}$. In terms of the parameter c, above, this means that additionally to solving $\delta(Y, \gamma, D_c) = 0$ we can solve the finite-cutoff equation $\delta(Y_\Lambda, \gamma_\Lambda, D_c) = 0$ (uniquely) by $c = s(\Lambda)/2$, where the sign $s(\Lambda)$ equals the parity $\cos(\pi \lambda_{\max})$ of the highest eigenvalue $\lambda_{\max}$ of $|D_{S^2}|$ below Λ (so that the corresponding eigenvalue of cB is $+\frac{1}{2}$): see Figure 4.6.

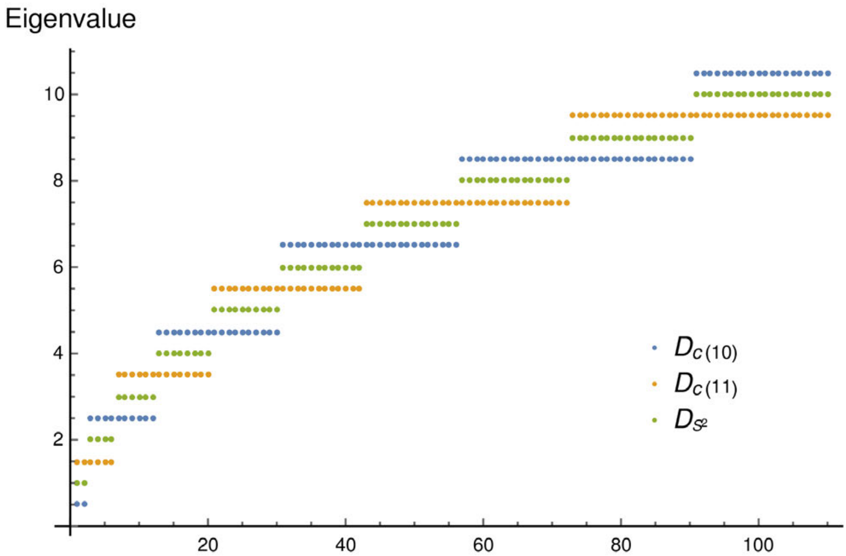


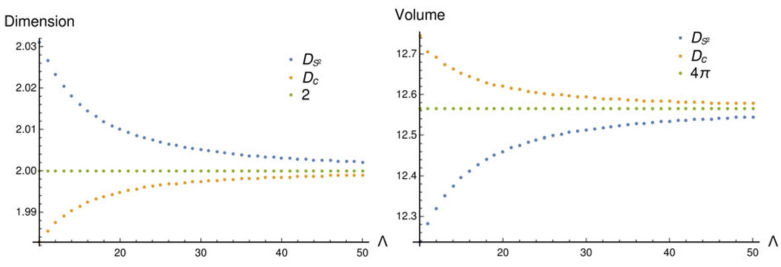
Figure 4.6: Spectra of D_{S^2} and $D_{S^2} + cB$ for even/odd $\lambda_{\max}$.

The finite-rank operators $D_{c,\Lambda}$, for any $c \in \mathbb{R}$, never satisfy the first-order condition that $[[D_\Lambda, Y_\Lambda], Y_\Lambda]$ should vanish. For $D = D_{S^2}$, for which the defect vanishes in the strong limit $\Lambda \to \infty$, there is a boundary defect of asymptotically constant norm and of unbounded trace norm as $\Lambda \to \infty$ – that is, $\|[[D_{S^2,\Lambda}, Y_\Lambda], Y_\Lambda\| \sim 1$ and $\|[[D_{S^2,\Lambda}, Y_\Lambda], Y_\Lambda\|_1 = O(\Lambda)$.

As mentioned above, the defect $[[B_\Lambda, Y_\Lambda], Y_\Lambda]$ does not vanish in the strong limit $\Lambda \to \infty$. However, precisely when $c = s(\Lambda)/2$ as above, the highest-order terms of $[[cB_\Lambda, Y_\Lambda], Y_\Lambda]$ and $[[D_{S^2,\Lambda}, Y_\Lambda], Y_\Lambda]$ cancel each other. As a result, the defect $[[D_{c,\Lambda}, Y_\Lambda], Y_\Lambda]$ is of norm $O(\Lambda^{-1})$ and trace norm $O(1)$. In this sense it is hard to computationally detect the fact that $D_{S^2,\Lambda}$ comes from a spectral triple while (for nonzero c) $D_{c,\Lambda}$ does not.

Visualisation: a locally isometric graph embedding

The operator $D_{S^2} + cB$ seems, at least on $P_\Lambda H$ for finite Λ, to come closer to satisfying the higher Heisenberg equation (4.2) than the original solution D_{S^2} does, and neither its spectral asymptotics nor the first-order equation allow us to discern at the finite



(a) Dimension estimate

(b) Volume estimate

Figure 4.7: Finite-rank estimates of the spectral asymptotics of D_{S^2} and $D_{S^2} + cB$, where c is $\pm\frac{1}{2}$ for Λ even/ odd.

level that it does not form a commutative spectral triple with $C^\infty(S^2)$ and $L^2(S^2, S)$. This suggests to pretend it *does* arise from a spin geometry and to compare at least the resulting metric on S^2 to the standard one.

First of all, since the difference B is bounded, the Weyl asymptotics agree in the sense that the first zeta residues must be equal in both value and argument. This is already detectable at the truncated level, e.g. using the finite-rank zeta approximations from [Ste19b]: see Figure 4.7. One interesting feature of these figures is that the dimension and volume estimators converge faster for the $D_{S^2} + cB$ operator than for the truncated sphere.

The asymptotics corresponding to total scalar curvature, however, are completely different for $D_{S^2} + cB$ (the corresponding residue is not $\frac{2\cdot 4\pi}{6\cdot 4\pi}$ but rather $\frac{-4\pi}{6\cdot 4\pi}$) because it is the $O(t^{-n/2+1})$ term in the asymptotics of $\operatorname{tr} e^{-tD^2}$ and is therefore highly sensitive to bounded shifts when the dimension equals 2.

Chapter 3 developed a method to associate a finite metric space to 'operator system spectral triples' $(P_\Lambda C^\infty(M) P_\Lambda, P_\Lambda H, P_\Lambda D)$. The method, briefly, is as follows.

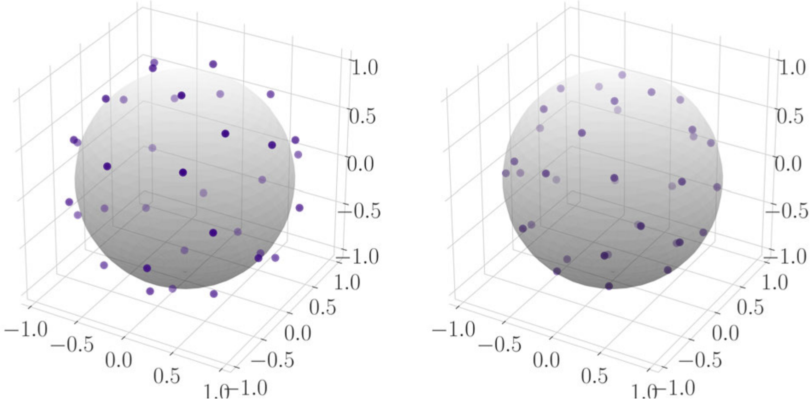
- The embedding Y is used to define the *dispersion* $\eth(v) \overset{\text{def}}{=} \sum_i \langle v, Y_i^2 v\rangle - \langle v, Y_i v\rangle^2$ of a vector $v \in H$, which measures the degree to which the corresponding vector state is localized. In the commutative case, this corresponds to the statistical variance of the position variable Y under the measure induced by v.

- One iteratively constructs a reasonably dense (finite) set of localized states by minimizing the dispersion, combined with an electrostatic repulsion to avoid repetition. Up to the distortion induced by imperfect localization, this results in the commutative case in generating a set of roughly equidistributed points on the underlying manifold.
- The Connes distance formula (4.3) is used to calculate the distance between the generated states, in order to obtain a metric graph. In the commutative case, those distances correspond to the Kantorovich-Wasserstein distance between the measures induced by the localized states, which reduces to the geodesic distance in the limit of perfect localization.
- The SMACOF algorithm is utilized to embed the obtained metric graph in $\mathbb{R}^n$ in an asymptotically locally isometric way. This means that, asymptotically as $\Lambda \to \infty$, the embedding is pressured to be Riemannian.

For D_{S^2} and $D_{S^2} + cB$, this procedure yields the images displayed in Figure 4.8. There we can see that the embedded points for D_{S^2}, the left hand plot, lie outside the shaded S^2 that is included for reference, while on the other hand the points for $D_{S^2} + cB$, in the right hand plot, lie inside the shaded S^2. The transparency of the dots increases with distance to the viewer. Both embeddings show some deviation from the sphere: for D_{S^2}, the radii of the embedded points lie in $[1.06, 1.12]$, with an average of 1.09; for $D_{S^2} + cB$, in $[0.94, 0.98]$ averaging 0.96.

4.4 Conclusions

In this chapter we explored the behaviour of the truncated one-sided higher Heisenberg relation in dimensions 1 and 2. In the one-dimensional case the simulations yielded the expected result, showing that the truncation of the Dirac operator on the circle is closest to solving the corresponding truncated relation. The two-dimensional version of the truncated Heisenberg relation, however, lead to a new minimum that differs from (but commutes with) the truncated Dirac operator on the sphere. We found analytically that this numerical minimum corresponds to the truncation at $c = \pm\frac{1}{2}$ of a new one-parameter family $D_c = D_{S^2} + cB$ of exact solutions to the non-truncated higher Heisenberg equation. While these bounded perturbations D_c of D_{S^2} satisfy most conditions of the reconstruction theorem, they fail to satisfy the first-order condition. Unlike many other geometric properties, however, this defect turns out to not be detectable at the truncated level.



(a) D_{S^2}

(b) $D_{S^2} + cB$

Figure 4.8: Locally almost-isometric embeddings corresponding to D_{S^2} and $D_{S^2} + cB$, with shaded S^2 for reference

An interesting comparison here is the case of the four-dimensional version of the higher Heisenberg relation. That relation is solved not only by the four-sphere, but also by an additional, genuinely non-commutative, spectral triple, the Connes-Landi sphere [CL01]. This similarity invites the question whether the Heisenberg relation might invite more freedom the larger the dimension becomes.

There are many interesting extensions of this work waiting to be explored. In particular the Heisenberg relation needs to be understood in more detail. It is unclear how its one-sided version behaves in higher dimensions and, just as importantly, when more freedom is allowed for the parameter Y. Our results, seen in context with the Connes-Landi sphere, suggest that more conditions are required to ensure that we deal with truncations of genuine Dirac spectral triples. In addition, it would be interesting to explore the two-sided equation of the Heisenberg relation. In that context, allowing the embedding maps Y to vary as well as the Dirac operator enlarges the resulting ensemble to contain *all* spin manifolds of the dimensions considered. With additional conditions, this would be a solid basis for a spectral version of random geometry, which could be compared to and begin a dialogue with results in quantum gravity, such as those of dynamical triangulations [Lol98] and spinfoams [Per13].

4.A Representation of Y, γ, D_{S^2}

Let S be the standard spinor bundle over S^2, with Dirac operator D_{S^2}, and let x, y, z be the standard coordinate functions on $S^2 \subset \mathbb{R}^3$. Then, the Dirac-type spectral triple given by $(C^\infty(S^2), L^2(S^2; S), D_{S^2})$ can be represented as follows. Let $\{|l,m\rangle_\pm \mid l \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, -l \leq m \leq l\}$ be an orthonormal basis of the Hilbert space H. Then, we represent the generators $a = x - iy$ and $b = z$ of the algebra $C^\infty(S^2)$, the grading γ of S and the Dirac operator D_{S^2} as follows:

$$
\begin{aligned}
a\,|l,m\rangle_\pm &= -\frac{\sqrt{(l+m+1)(l-m)}}{2l(l+1)}\,|l,m+1\rangle_\mp \\
&\quad + \frac{\sqrt{(l+m+1)(l+m+2)}}{2(l+1)}\,|l+1,m+1\rangle_\pm \\
&\quad - \frac{\sqrt{(l-m)(l-m-1)}}{2l}\,|l-1,m\rangle_\pm, \\
b\,|l,m\rangle_\pm &= \frac{m}{2l(l+1)}\,|l,m\rangle_\mp \\
&\quad + \frac{\sqrt{(l-m+1)(l+m+1)}}{2(l+1)}\,|l+1,m\rangle_\pm \\
&\quad + \frac{\sqrt{(l-m)(l+m)}}{2l}\,|l-1,m\rangle_\pm, \\
\gamma\,|l,m\rangle_\pm &= |l,m\rangle_\mp, \\
D_{S^2}\,|l,m\rangle_\pm &= \pm\,(l+\frac{1}{2})\,|l,m\rangle_\pm .
\end{aligned}
$$

This representation was chosen to align well with that of [Dab+05]. We then write the matrix Y as $\begin{pmatrix} b & a \\ a^* & -b \end{pmatrix}$.

Part II

Schatten classes for Hilbert modules over commutative C^*-algebras

Chapter 5

Introduction

Words can never trace out all the fibers
that knit us to the old.

George Eliot, *Letter to Charles Bray*,
Christmas Day, 1858

The trace is a fundamental and highly versatile invariant of operators on Hilbert spaces. In many applications, however, one is rather concerned with continuous *families* of such operators. From the perspective of Gelfand duality, the natural framework for such continuous families is that of Hilbert C*-modules over an abelian base. The present study provides a systematic construction of trace and Schatten classes in this setting.

The finite-rank trace

The $*$-algebra $M(A)$ of finite matrices over a C*-algebra A comes naturally equipped with a positive linear map

$$\mathrm{tr}\colon M(A) \to A,\ (a_{ij})_{ij} \to \sum_i a_{ii},$$

which clearly commutes with the entrywise lift of linear maps between C*-algebras, and is cyclic if and only if the algebra A is commutative.

If E is a finitely generated projective Hilbert A-module, any compact adjointable endomorphism of E can be represented as an element of $M(A)$ by a choice of isomorphism between E and a complemented submodule of A^n. Whenever A is commutative, cyclicity implies that the trace of the resulting matrix is invariant under the choice of isomorphism and so constitutes the A-valued (Hattori-Stallings) trace on $\mathrm{End}_A(E)$.

Chapter 6, below, introduces a robust framework that generalizes the construction to *countably* generated Hilbert modules.

Continuous families of Schatten-class operators

Given a finite-rank Hermitian vector bundle V over a locally compact Hausdorff space X, Gelfand duality and the Serre-Swan theorm imply that the finite-rank trace is just the fiberwise trace $\mathrm{tr}\colon \Gamma(\mathrm{End}\,V) \to C_0(X)$ of continuous sections of the endomorphism bundle.

If $V \to X$ is instead a continuous field of separable Hilbert spaces, then there is still a trace map on the fiberwise trace classes. The challenge is then to unify these fiberwise trace classes in such a way as to yield a $C_0(X)$-valued fiberwise trace that retains the fundamental properties of the trace class on a fixed Hilbert space H.

As a fundamental example, consider the trivial bundle $H \times X \to X$ with fiber a fixed separable Hilbert space H. The C*-algebra of its adjointable endomorphisms consists of all $*$-stronglycontinuous, bounded families of bounded operators on H. One wonders whether this algebra, denoted $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$, contains two-sided ideals of “continuous Schatten-class operators” such that some or most of the usual theory of Schatten classes on Hilbert spaces is preserved.

In order to ensure continuity of the trace, the least one should demand of such an ideal is that the pointwise Schatten norms lie in $C_0(X)$. On the other hand, the strongest reasonable condition at hand is that the families are themselves Schatten-norm continuous, that is, that they lie in the Banach space $C_0(X, \mathscr{L}^p(H))$. Through careful control over the relation between the Schatten classes on the standard Hilbert space $l^2(\mathbb{C})$ and the complex matrix algebras $M_n(\mathbb{C})$, we are able in Theorem 6.2.11 to show that these conditions do in fact coincide on $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$ and yield a two-sided ideal that is contained in the compact operators and is closed under its Banach norm.

Kasparov’s stabilization theorem and unitary invariance of the Schatten norms allow us to easily generalize the trivial bundle example to all continuous fields of separable Hilbert spaces in Theorem 6.3.1. A further upshot of this approach is that much of the pointwise Schatten-class theory, including the Hölder-von Neumann inequality, carries over easily to the general case, cf. Theorem 6.3.4.

Frames and the fiberwise trace

The theory of Schatten classes on Hilbert spaces is often mediated through the language of orthonormal bases and diagonalization. The approach of Section 6.2 shows that one may very well work with (standard normalized) *frames* [FL02; RT03] instead. This allows for straightforward generalization of the familiar formulas to the theory of Hilbert modules, and indeed, the result is what one would hope for: the fiberwise trace turns out to be the norm-convergent sum over the diagonal in a given frame, cf. Theorem 6.1.5. In the context of frames, it was earlier remarked in [DH94],[FL02, Proposition 4.8] that the obvious notion of Hilbert-Schmidt inner product is invariant both under the choice of frame and under the adjoint. That observation is supplied with the necessary context as a special case of our Schatten-class operators in Section 6.4.

Applications

By the same principles as for the finite-rank trace, [Alm73] introduced, in the context of K-theory, the *Fredholm determinant* of endomorphisms of finitely generated modules over unital commutative algebras. As the Fredholm determinant is interesting in its own right, Section 7.1 uses the result on the Schatten classes to extend its definition and basic properties to the setting of countably generated modules over unital commutative C^*-algebras. A straightforward generalization of [Alm74] remains however elusive, due to the conceptual problems in generalizing the relevant category.

Spectral geometry is the study of Riemannian manifolds M via the spectra of differential operators, such as spin^c-Dirac operators D, on M. An important example of spectral invariant is the localized heat trace $t \mapsto \operatorname{tr} f e^{-tD^2}, f \in C(M)$. It determines the volume and total scalar curvature of M, is strongly related to the Atiyah-Singer Index Theorem [Gil95], and is able to describe classical field theories on M through the spectral action principle [CM08, Chapter 11]. The first step in generalizing the above to unbounded Kasparov cycles, that is, certain $(C_0(M), C_0(N))$-Hilbert bimodules carrying a selfadjoint, regular, $C_0(N)$-invariant unbounded operator S, is to make sense of the expressions $\operatorname{tr} f e^{-tS^2}$ and $\operatorname{tr} f|S|^{-z}$ as elements of $C_0(N)$. Sections 7.2,7.3 embark on the necessary theory. Open questions for further research remain, particularly in the direction of zeta residues and compatibility with the interior product in unbounded KK-theory.

The noncommutative case

As briefly discussed in Chapter 8, the case of *noncommutative* C*-algebras A is markedly different. We adapt a basic example from [FL02] to show that one is induced to contemplate not a trace ideal in the adjointable operators, but a trace ideal inside some smaller $*$-algebra dependent on a choice of Hilbert *bi*module structure. In the noncommutative case, it is more natural not to work with an A-valued "trace" but instead to consider generalizing the induced map $\mathrm{tr}_*\colon H_*(M_N(A), M_N(M)) \to H_*(A, M)$ in Hochschild (or cyclic) homology, as initiated in [Nis91]. Although this cleanly addresses the limitation of noncyclicity, it introduces additional complications to the issue of convergence. This in turn further reinforces the idea that one should investigate trace classes on bimodules instead. This direction of research, however, remains wide-open.

5.1 Preliminaries

We start by recalling the notion of frames on Hilbert C^*-modules over C^*-algebras. For basic definitions on Hilbert C^*-modules, adjointable maps, tensor products, *et cetera* we refer to e.g. [Weg93]. We also recall the definition of unbounded Kasparov cycles [BJ83].

Keep in mind that we will, in later sections, specialize to the case of *abelian* C^*-algebras, that is, those of the form $C_0(X)$ for X a locally compact Hausdorff space. Hilbert C^*-modules over such C^*-algebras are given by the sections of continuous fields of Hilbert spaces; cf. [Tak79; DD63].

Frames on Hilbert C*-modules

We start this section by recalling two well-known results on Hilbert C*-modules. For completeness we include their (short) proofs.

Proposition 5.1.1. Let A be a C*-algebra and let E_A be a Hilbert A-module. Then $E_A A$ is dense in E, and the map $u\colon v \otimes_A a \mapsto va$, $E_A \otimes_A A \to E_A$, is unitary.

Proof. Let $\{e_\lambda\}$ be an approximate unit of A. For $v \in E_A$ one has $\langle v - ve_\lambda, v - ve_\lambda\rangle = \langle v, v\rangle - e_\lambda \langle v, v\rangle - \langle v, v\rangle e_\lambda + e_\lambda \langle v, v\rangle e_\lambda$, which converges to 0; thus, v is the norm limit of the sequence $ve_\lambda \in E_A A$.

Clearly u is isometric, so that its range is closed. As the range is dense, it must be surjective and, therefore, unitary. $\square$

Proposition 5.1.2. Let A and B be C*-algebras and let E_A be a Hilbert A-module. If $\phi : A \to B$ is a $*$-homomorphism, then B is a left A-module with the action $a \cdot b \stackrel{\text{def}}{=} \phi(a)b$. Then, there is an adjointable map

$$\phi_* : E_A \to E_A \otimes_A B,$$

such that $\langle \phi_* v, \phi_* w \rangle_B = \phi(\langle v, w \rangle_A)$.

Moreover, if $T \in \mathscr{L}(E_A)$ then $\phi_* T := T \otimes 1$ is an adjointable endomorphism of $E_A \otimes_A B$, *i.e.* there is an induced map $\phi_* : \mathscr{L}(E_A) \to \mathscr{L}(E_A \otimes_A B)$.

Proof. Recall that the map $\mathrm{id}_* : v \cdot a \mapsto v \otimes_A a$, $E_A \to E_A \otimes_A A$, is an isomorphism. We set $\phi_* \stackrel{\text{def}}{=} (\mathrm{id} \otimes_A \phi) \circ \mathrm{id}_*$ and find that for $v \in E_A, a \in A$ one has $\langle \phi_* v, \phi_* v \rangle_B = \langle \phi(a), \langle v, v \rangle \cdot \phi(a) \rangle = \phi(a)^* \phi(\langle v, v \rangle) \phi(a) = \phi(\langle v \cdot a, v \cdot a \rangle)$. □

A convenient basic fact about separable Hilbert spaces H is that they possess countable orthonormal bases $\{e_i\}$. For one thing, this allows one to explicitly relate the compact operators $B_0(H)$ to the direct limit $M(\mathbb{C})$ of matrix algebras over the base field $\mathbb{C}$ and in particular to treat the trace on $L^1(H)$ using the series expression $\operatorname{tr} T = \sum_i \langle e_i, T e_i \rangle$.

The situation for Hilbert C*-modules is slightly less straightforward: we will introduce the analogous but strictly weaker concept of a *frame*. In spite of the increased generality, we will see that frames provide sufficient flexibility to mimic standard treatments of trace-class operators on Hilbert spaces in the setting of Hilbert $C_0(X)$-modules.

Definition 5.1.3. Let E_A be a countably generated Hilbert C*-module over a C*-algebra A. A *frame* e of E_A is a sequence e_i of elements of E_A, such that

$$\langle v, w \rangle = \sum_{i=1}^{\infty} \langle v, e_i \rangle \langle e_i, w \rangle,$$

in norm, for all $v, w \in E_A$.

Such objects e were called 'standard normalized frames' in [FL02]. Note that the subsequent treatment in [RT03], which is very similar to the definition used here, is different for non-unital C*-algebras: we require the e_i to be in E_A, not in the 'multiplier module' $\mathscr{L}(A_A, E_A)$. This choice will later imply, for instance, that we do *not* consider the identity on the $C_0(X)$-module $C_0(X)$, for noncompact spaces X, to be in the trace class.

Example 5.1.4. Let H be a separable Hilbert space. Let $P \in B(H)$ be a projection and $K = PH \subset H$. Then, if $\{e_i\}$ is an orthonormal basis of H, we have

$$\langle v,w\rangle = \sum_{i=1}^{\infty} \langle Pv,e_i\rangle \langle e_i,Pw\rangle = \sum_{i=1}^{\infty} \langle v,Pe_i\rangle \langle Pe_i,w\rangle ,$$

for all $v,w \in K$. That is, $e = \{Pe_i\}$ is a frame of K. Note that e is not an orthonormal basis, because the e_i might be neither orthogonal nor of norm 1.

Now, in the context of trace-class operators on a separable Hilbert space H, frames 'are as good as orthonormal bases', in the sense of Corollary 5.1.6 below.

Lemma 5.1.5. Let e,f be frames of a separable Hilbert space H and let T be a bounded endomorphism of H. Then, the series $\sum_{i=1}^{\infty} \left\langle T^*f_i,T^*f_i\right\rangle$ converges if and only if $\sum_{i=1}^{\infty} \langle Te_i,Te_i\rangle$ converges, and the limits agree.

Proof. Assume that $\sum_{i=1}^{\infty} \langle Te_i,Te_i\rangle < \infty$. Then for finite subsets $F \subset \mathbb{N}$,

$$\begin{aligned}\sum_{i\in F} \left\langle T^*f_i,T^*f_i\right\rangle &= \sum_{i\in F}\sum_{j=1}^{\infty} \left\langle T^*f,e\right\rangle \left\langle e,T^*f\right\rangle \overset{!}{=} \sum_{i\in F}\sum_{j=1}^{\infty} \left\langle e,T^*f\right\rangle \left\langle T^*f,e\right\rangle \\ &\leq \sum_{j=1}^{\infty} \langle Te,Te\rangle .\end{aligned}$$

Being bounded and monotone, the series $\sum_{i=1}^{\infty} \left\langle T^*f_i,T^*f_i\right\rangle$ must converge. If we now switch T and T^*, e and f and repeat the calculation, we see that the limits must in fact agree. □

Corollary 5.1.6. Let e be a frame of a separable Hilbert space H. Then, for bounded endomorphisms T of H, we have $T \in \mathscr{L}^2(H)$ whenever $\sum_{i=1}^{\infty} \langle Te_i,Te_i\rangle < \infty$. Moreover, for $T \in \mathscr{L}^1(H)$, one has $\operatorname{tr} T = \sum_{i=1}^{\infty} \langle e_i,Te_i\rangle$.

Proof. For the first part, let f be an orthonormal basis and note that $\sum_{i=1}^{\infty} \langle Te_i,Te_i\rangle$ converges whenever $\sum_{i=1}^{\infty} \left\langle T^*f_i,T^*f_i\right\rangle$ does by Lemma 5.1.5. That, then, is equivalent to $T^* \in \mathscr{L}^2(H)$, which in turn is equivalent to $T \in \mathscr{L}^2(H)$.

For the second part, note that $\mathscr{L}^1(H) = \mathscr{L}^2(H)\mathscr{L}^2(H)$. It is thus sufficient to consider an element $T = |S|^2 \in \mathscr{L}^1(H)$ with $S \in \mathscr{L}^2(H)$. Then, $\operatorname{tr}|S|^2 = \operatorname{tr}|S^*|^2 = \sum_{i=1}^{\infty} \left\langle S^*f_i,S^*f_i\right\rangle$, which equals $\sum_{i=1}^{\infty} \langle Se_i,Se_i\rangle$ by Lemma 5.1.5. □

We will see later that the Example 5.1.4 is a very good prototype for the general situation for Hilbert C*-modules as well.

Example 5.1.7. Let A be a unital C*-algebra and let $l^2(A) = l^2 \otimes_{\mathbb{C}} A_A$ be its standard module. Let $\{e_i\}$ be the standard orthonormal basis of l^2 and define $\{e_i \overset{\text{def}}{=} e_i \otimes 1_A\}$ in $l^2(A)$. Then clearly

$$\langle v, w\rangle = \sum_{i=1}^{\infty} \langle v, e_i\rangle \langle e_i, w\rangle ; \qquad (v, w \in l^2(A)).$$

If $P \in \mathscr{L}(E_A)$ is a projection (i.e. $P^2 = P^* = P$) and $F_A = P(E_A)$, then

$$\langle Pv, Pw\rangle = \sum_{i=1}^{\infty} \langle Pv, e_i\rangle \langle e_i, Pw\rangle = \sum_{i=1}^{\infty} \langle v, Pe_i\rangle \langle Pe_i, w\rangle ,$$

so $\{Pe_i\}$ is a frame of F_A.

Each frame of E_A gives rise to a unitary $\theta_e \colon E_A \to (\theta_e\theta_e^*)l^2(A)$, as follows.

Proposition 5.1.8. Let e be a frame of E_A. The *frame transform* $\theta_e \colon E_A \to l^2(A)$, given by

$$\theta_e(v) := (\langle e_i, v\rangle)_i$$

is adjointable, and its adjoint satisfies $\theta_e^*(e_k \otimes a) = e_k \cdot a$. Moreover, e is a frame if and only if $\theta_e^*\theta_e = \mathrm{id}_E$.

For the proof we refer to [RT03, Theorem 3.5].

Remark 5.1.9. Note that, unless A is unital, the converse does not hold: not every isometry sending E_A to a complemented submodule of $l^2(A)$ is induced by a frame. The frame elements e_i would be given by $\theta_e^*(\delta_{ij}1_A)_j$, but the latter is not an element of $l^2(A)$ unless A is unital. This is where our treatment differs from that of [RT03], which works with frames (in the present sense) of the multiplier module $\mathscr{L}(E_A, A)$ instead.

Frames are compatible with $*$-homomorphisms. In particular, this means that characters of a commutative C*-algebra map frames of Hilbert C^*-modules to frames of Hilbert spaces.

Proposition 5.1.10. Let A, B be C*-algebras, E_A a Hilbert A-module and $\phi : A \to B$ a $*$-homomorphism. If e is a frame of E_A and if ϕ is surjective, then $\phi_*(e)$ is a frame of $E_A \otimes_A B$.

Proof. Consider $f \stackrel{\text{def}}{=} \phi_*(e) = \{\phi_*(e_i)\}_i \in E_A \otimes_A B$. Note that $\theta_f(\phi_* v) = \phi_* \theta_e(v)$ inside $l^2(B)$, so that $\left\langle \theta_f(\phi_* v), \theta_f(\phi_* w) \right\rangle = \phi(\langle \theta_e(v), \theta_e(w) \rangle) = \phi(\langle v, w \rangle) = \left\langle \phi_* v, \phi_* w \right\rangle$. Thus, with Proposition 5.1.8 it follows that f is a frame. □

Existence of frames

Kasparov's stabilization Theorem [Kas80] shows that example 5.1.7 describes the general unital (and, as we will see, the non-unital) case very well.

Theorem 5.1.11. Let A be a unital C*-algebra and let E_A be a countably generated Hilbert A-module. Then there exists a projection $P^2 = P = P^*$ in $\mathscr{L}\left(l^2(A)\right)$ such that $E_A \simeq P\left(l^2(A)\right)$. In particular, E_A possesses a frame.

For the proof we refer to [Kas80] (see also [Lan95, Theorem 6.2]).

The non-unital case requires more effort, but the end result is the same. We refer to [Kaa17, Section 2] for a proof.

Proposition 5.1.12 ([Kaa17, Proposition 2.6]). Let A be a C*-algebra. Then, all countably generated Hilbert A-modules possess a frame.

The standard module over abelian C*-algebras

For the rest of this subsection, let X be a locally compact Hausdorff space. The C*-algebra $C_0(X)$ is abelian – and, by Gelfand duality, all abelian C*-algebras are of this type.

We will investigate the Hilbert $C_0(X)$-module $C_0(X, H)$, for H a separable Hilbert space, which will later provide a useful tool in investigating the Schatten classes of operators on more general Hilbert $C_0(X)$-modules. We start with some basic definitions and results, whose proof we leave to the reader.

Definition 5.1.13. Let f be a map from a locally compact topological space X to a normed space. We say that f *vanishes at infinity* whenever for all $\epsilon > 0$, the set $\{x \in X : \left\|f(x)\right\| \geq \epsilon\}$ is compact.

Definition 5.1.14. Let Y be a Banach space, equipped with its norm topology. The space $C_0(X,Y)$ consists of the continuous functions from X to Y that vanish at infinity.

Proposition 5.1.15. Let Y be a Banach space and X be a locally compact topological space. Then, $C_0(X,Y)$ is a Banach space when equipped with the norm $\|f\| \stackrel{\text{def}}{=} \sup_{x\in X} \|f(x)\|$. Moreover, for $f \in C_0(X,Y)$, the map $x \mapsto \|f(x)\|$ lies in $C_0(X)$.

Proposition 5.1.16. Let H be a separable Hilbert space. Then, the Banach space $C_0(X,H)$ has the structure of a Hilbert C*-module when equipped with the $C_0(X)$-valued inner product $\langle v,w\rangle (x) \stackrel{\text{def}}{=} \langle v(x),w(x)\rangle_H$.

Proposition 5.1.17. The Hilbert C^*-module $C_0(X,H)$ is unitarily equivalent to the tensor product $H \otimes_{\mathbb{C}} C_0(X)$ of Hilbert C^*-modules.

Proof. Let $\{e_i\}_i$ be an orthonormal basis of H, and for $v \in C_0(X,H)$ write $v_i\colon x \mapsto \langle v(x),e_i\rangle$; this defines a sequence of functions in $C_0(X)$. Because $\langle v,v\rangle = \sum_{i=1}^{\infty} v_i^* v_i$ converges pointwise, is positive and lies in $C_0(X)$, it converges in the norm of $C_0(X)$ by Dini's theorem.

Consider then the $C_0(X)$-linear map $\theta\colon C_0(X,H) \to H \otimes_{\mathbb{C}} C_0(X), v \mapsto \sum_i e_i \otimes v_i$. This series converges in $H \otimes_{\mathbb{C}} C_0(X)$ because

$$\left\| \sum_{i\in F} e_i \otimes v_i \right\| = \left\| \sum_{i,j\in F} \left\langle v_i, \left\langle e_i, e_j \right\rangle v_j \right\rangle \right\| = \left\| \sum_{i\in F} \langle v_i, v_i\rangle \right\|.$$

Moreover, taking limits on both sides we see that θ is isometric.

Now, the map $m\colon H \otimes_{\mathbb{C}} C_0(X) \to C_0(X,H), h \otimes a \mapsto ha$, is isometric because

$$\left\langle \sum_i h_i \otimes a_i, \sum_i h_i \otimes a_i \right\rangle = \sum_{ij} \left\langle h_i, h_j \right\rangle a_i^* a_j = \left\langle \sum_i m(h_i \otimes a_i), \sum_i m(h_i \otimes a_i) \right\rangle.$$

As m inverts θ by orthonormality of the $\{e_i\}_i$, we conclude that θ is a surjective $C_0(X)$-linear isometry, so that it is adjointable (with adjoint m) and moreover unitary. Therefore, $C_0(X,H)$ is unitarily equivalent to $H \otimes_{\mathbb{C}} C_0(X)$. □

Proposition 5.1.18. The C*-algebra $\mathscr{L}(C_0(X,H))$ of adjointable endomorphisms of the Hilbert $C_0(X)$-module $C_0(X,H)$ is isomorphic to the C*-algebra of bounded, $*$-strongly continuous maps from X to $B(H)$, denoted $C_{\mathrm{b}}^{\mathrm{str}}(X,B(H))$.

Proof. Consider a point x in X as a character $\chi : C_0(X) \to \mathbb{C}$ given by evaluation in x. By Proposition 5.1.2 there is an adjointable map $\chi_* : C_0(X,H) \to C_0(X,H) \otimes_{C_0(X,H)} \mathbb{C}$ whose image $\chi_*(C_0(X,H))$ can be canonically identified with H. Accordingly, in terms of the map $\chi_* : \mathscr{L}(C_0(X,H)) \to \mathscr{L}(C_0(X,H))$ given by $T \mapsto T \otimes 1$, we have canonically $\chi_* \mathscr{L}(C_0(X,H)) \simeq B(H)$.

Now take $T \in \mathscr{L}(C_0(X,H))$ and a convergent sequence $x_i \to x$ in X. Let $h \in H$ and let $v_0 \in C_0(X,H)$ be constant in a neighborhood of x with $v_0(x) = h$. Since T is an adjointable endomorphism, both Tv_0 and $T^* v_0$ lie in $C_0(X,H)$, so that $Tv_0(x_i)$ converges to $Tv_0(x)$ in norm. That is to say, $\left\| (\chi_{i_*} T)h - (\chi_* T)h \right\| \to 0$ (and similarly for T^*). As x and h were arbitrary, we conclude that $T \in C^{\mathrm{str}}(X, B(H))$. Moreover, if $\|T(x)\| > C$ for some $x \in X$ and $C > 0$, there is v_0 as above with $\|v_0\| \leq 1$ and $\|Tv_0\| > C - \epsilon$ for all $\epsilon > 0$, so that we conclude that $\|T\| \geq C$. Thus, if T preserves $C_0(X,H)$ it must lie in $C^{\mathrm{str}}_{\mathrm{b}}(X, B(H))$.

Conversely, for all $T \in C^{\mathrm{str}}_{\mathrm{b}}(X, B(H))$ we have that $x \mapsto T(x)v(x)$ and $x \mapsto T^*(x)v(x)$ lie in $C_0(X,H)$ for all $v \in C_0(X,H)$. As $\langle Tv, w\rangle(x) = \langle T(x)v(x), w(x)\rangle = \langle v(x), T^*(x)w(x)\rangle$ we conclude that the pointwise adjoint provides an adjoint of T as an operator of $C_0(X,H)$. That is, all such T are adjointable operators on $C_0(X,H)$. □

General Hilbert C*-modules over a commutative base

We now apply the above results to the case of general Hilbert $C_0(X)$-modules. For a deeper topological understanding of such modules, see [Tak79; DD63].

Proposition 5.1.19. Let E be a Hilbert $C_0(X)$-module. Then there exists a $*$-strongly continuous projection $P \in C^{\mathrm{str}}_{\mathrm{b}}(X, B(H))$ such that E is isomorphic to the subset $\Gamma_0(X,P) \subset C_0(X,H)$ of those elements h of $C_0(X,H)$ satisfying $h(x) \in \operatorname{ran} P_x$. Moreover, under this identification $\mathscr{L}(E)$ is isomorphic to the set of those elements $T \in C^{\mathrm{str}}_{\mathrm{b}}(X, B(H))$ that satisfy $PT = TP = T$.

Proof. If X is compact, this is a direct consequence of Kasparov's Stabilization Theorem (*cf.* Theorem 5.1.11) and Propositions 5.1.17 and 5.1.18. If not, consider E as an $C_0(X)^+$-module and note that the endomorphism P from Theorem 5.1.11 lies in $C^{\mathrm{str}}_{\mathrm{b}}(X^+, B(H))$ in terms of the one-point compactification X^+ of X. But since $(e,e) \in C_0(X)$ for $e \in E$, the map P must project into a subspace of $C_0(X,H)$.

For the last statement, let $T \in C^{\mathrm{str}}_{\mathrm{b}}(X, B(H))$ such that $PT = TP = T$. Then $PTP = T$ so that T preserves $\Gamma_0(X,P)$. Conversely, let $S \in \mathscr{L}(\Gamma_0(X,P))$. Then, the map $T = SP$ is a composition of adjointable operators and is therefore an element of $\mathscr{L}(C_0(X,H)) \simeq C_b^{\mathrm{str}}(X,B(H))$ using Proposition 5.1.18. We clearly have $PT = TP = T$ so that the claim follows. $\square$

Note that the fibers P_xH of E may vary quite wildly with $x \in X$, as the following example shows.

Example 5.1.20. Let $U \subset X$ be open and let P be the orthogonal projection onto a closed subspace $V \subset H$. Then, $E = C_0(U,V) \subset C_0(X,V)$ is (in particular) a Hilbert $C_0(X)$-module because the action of $C_0(X)$ on $C_0(X,V)$ by pointwise multiplication preserves the subspace $C_0(U,V)$. The fibers of E are V for $x \in U$ and $\{0\}$, for $x \notin U$.

The following example illustrates how the projections associated to such bundles behave.

Example 5.1.21. Let $U \subset X$ be open and let $V = \operatorname{span} v_0 \subset H$, with $\|v_0\|_H = 1$. We will investigate the projection associated to the Hilbert $C_0(X)$-module $E = C_0(U,V)$ by Proposition 5.1.19.

Let $\{\eta_i\}_i$ be a compactly supported partition of unity on U, so that $\sum_i \eta_i^2(x) = 1_U(x)$ for $x \in X$. Then, $v = \sum_i v_0\eta_i^2 \langle v, v_0\rangle$ for all $v \in E$, so that $\{e_i \overset{\text{def}}{=} v_0\eta_i\}_i$ is a frame of $C_0(U,V)$. In fact, any frame f is of this form: we have $\eta_i = \langle v_0, f_i\rangle$.

Now, let $\{w_i\}_i$ be an orthonormal basis of H. Then, θ_e, the frame transform of e, maps $w \in C_0(U,V)$ to $\sum_i w_i \langle e_i, w\rangle$.

Although the image $\Gamma_0(X,\theta_e\theta_e^*)$ of θ_e (consisting of those elements w of $C_0(X,H)$ for which $\langle w_i, w\rangle$ has support contained in that of e_i) is isomorphic to $C_0(U,V)$ through the map θ_e^*, it looks decidedly different from the isomorphic subspace $C_0(U,V)$ of $C_0(X,H)$ we started with. The associated projection $P = \theta_e\theta_e^*$ maps elements w to $\sum_i w_i\eta_i \sum_j \eta_j \langle w, w_j\rangle$.

In the previous Example, note that $\|P(x)\| = 1$ for $x \in U$ and $\|P(x)\| = 0$ for $x \notin U$. Thus, $P \in C_b^{\mathrm{str}}(X,B(H))$ lies in $C_b(X,B(H))$ if and only if U is clopen, that is, if and only if the bundle $\{(x,h) \mid h = \mathbb{1}_U(x)h\}$ is locally trivial. This illustrates a general criterion for local triviality:

Remark 5.1.22. If $P \in C_b(X,B(H)) \subset C_b^{\mathrm{str}}(X,B(H))$ is a projection, then each $x \in X$ has a neighbourhood on which $\|P(y) - P(x)\| < 1$, so that there exists a continuous map $y \mapsto u_y$ with $P(y) = u_yP(x)u_y^*$ by [Weg93, Proposition 5.2.6]. We conclude that the bundle $\{(x,h) \mid h \in P(x)H\}$ is locally trivial.

Conversely, if the bundle is locally trivial, at least when the fibers are constant, we can choose P to be norm continuous. More precisely, if we let $p : F \to X$ be a locally trivial bundle of Hilbert spaces with separable, infinite-dimensional fiber H. By [RW98, Corollary 4.79] F is isomorphic to $X \times H$, and $\Gamma_0(F)$ is isomorphic to $C_0(X,H)$. We may thus choose $P = \mathrm{id}_H \in C_b(X,B(H))$ in Proposition 5.1.19.

If instead H is finite-dimensional, then by the Serre-Swan theorem [GVF01, Theorem 2.10] there exists a projection $p \in M_n(A)$ with $pA^n \simeq \Gamma_0(F)$. Thus, in Proposition 5.1.19 we may choose $H = \mathbb{C}^n$ and $P = p$.

Chapter 6

Schatten classes for Hilbert $C_0(X)$-modules

We introduce a new definition of Schatten classes of endomorphisms on Hilbert modules over separable abelian C*-algebras $A = C_0(X)$, such that the resulting trace class is equipped with an A-valued trace. A detailed investigation of the case of the standard module $H \otimes_{\mathbb{C}} A \simeq C_0(X, H)$ leads us to establish several desirable properties that are familiar from the Hilbert space case.

6.1 Introduction and definition

When A is abelian, *i.e.* $A \simeq C_0(X)$ for some locally compact Hausdorff space X, each adjointable operator T on a Hilbert A-module E_A can be localized by the pure states χ of A to yield a family $\chi_* T$ of operators on the Hilbert spaces $\chi_* E_A$. We will unify the pointwise Schatten classes $\mathscr{L}^p\left(\chi_* E_A\right)$ into a two-sided ideal $\mathscr{L}^p\left(E_A\right) \subset \mathscr{L}\left(E_A\right)$ and define an A-valued trace on $\mathscr{L}^1\left(E_A\right)$.

Assumption 6.1.1. We will require all of our Hilbert A-modules to be countably generated in order to ensure access to frames using Proposition 5.1.12.

We will denote the character space of a commutative C^*-algebra A equipped with the weak* topology by $\widehat{A}$, so that $A \simeq C_0(\widehat{A})$ with $\widehat{A}$ a locally compact Hausdorff space.

The *least* we should demand of 'Schatten-class operators' T on E_A is that their pointwise

Schatten norm, *i.e.* the trace of $|\chi_* T|^p$, varies continuously with $\chi \in \widehat{A}$. In fact, this is the way to ensure that the 'trace-class operators' have traces with values in A and that the other Schatten classes respect this property in their pairing. The *most* we could reasonably demand, in contrast, is that the operators $\chi_* T$ are continuous in Schatten norm with respect to some trivialization of E_A (see Definition 6.2.1, below). It will turn out that these requirements, properly understood, are equivalent and yield a well-behaved Schatten class.

Definition 6.1.2. The *p-th Schatten class* $\mathscr{L}^p(E_A)$ for $1 \le p < \infty$ is the space of all endomorphisms $T \in \mathscr{L}(E_A)$ for which the function $\operatorname{tr}|T|^p : \widehat{A} \to \mathbb{R} \cup \{\infty\}, \chi \mapsto \operatorname{tr}|\chi_* T|^p$ lies in A.

The following proposition, familiar from the Hilbert space case, is immediate from the definition:

Proposition 6.1.3. Let $1 \le p < \infty$ and let $T \in \mathscr{L}(E_A)$. Then

$$T \in \mathscr{L}^p(E_A) \iff |T| \in \mathscr{L}^p(E_A) \iff |T|^p \in \mathscr{L}^1(E_A).$$

Remark 6.1.4. Recall that Dini's theorem, translated to the abelian C*-algebraic context, states the following: if a_i is a sequence of positive elements in A, then $\sum_i a_i$ converges in norm if and only if the function $x \mapsto \sum_i x(a_i)$ is an element of $C_0(X) \simeq A$. This theorem plays a crucial role throughout, because it allows us to relate the fiberwise Schatten norms on bundles of Hilbert spaces to various expressions for the element $\operatorname{tr}|T|^p \in A$, for $T \in \mathscr{L}^p(E_A)$.

We will use the existence of frames (*cf.* Proposition 5.1.12), to relate $\mathscr{L}^p(E_A)$ to the Schatten classes $\mathscr{L}^p\left(l^2(A)\right)$ on the standard module $l^2(A)$ and to relate the trace $\operatorname{tr}|T|^p$ to a series expression in terms of (arbitrary) frames.

Theorem 6.1.5. Let $T \in \mathscr{L}(E_A)$. Then $T \in \mathscr{L}^p(E_A)$ if and only if $\theta_e T \theta_e^* \in \mathscr{L}^p\left(l^2(A)\right)$ for any frame e of E_A with frame transform θ_e. Equivalently, $T \in \mathscr{L}^p(E_A)$ if and only if the series $\sum_{i=1}^\infty \langle e_i, |T|^p e_i \rangle$ converges in norm; the limit equals $\operatorname{tr}|T|^p = \operatorname{tr} \theta_e |T|^p \theta_e^*$.

Proof. We start with the second part. As $\chi_*(e)$ is a frame of the Hilbert space $\chi_* E_A$, one has $\operatorname{tr}|\chi_* T|^p = \sum_{i=1}^\infty \langle \chi_*(e_i), |\chi_* T|^p \chi_*(e_i) \rangle = \sum_{i=1}^\infty \chi(\langle e_i, |T|^p e_i \rangle)$. Hence if $T \in \mathscr{L}^p(E_A)$, the positive series $\sum_{i=1}^\infty \langle e_i, |T|^p e_i \rangle$ converges in norm to an element $\operatorname{tr}|T|^p$ of A by Dini's theorem.

Conversely, if the series $\sum_{i=1}^{\infty} \langle e_i, |T|^p e_i \rangle$ converges in norm, then the limit provides an element $\operatorname{tr}|T|^p \in A$ such that $\chi(\operatorname{tr}|T|^p) = \operatorname{tr}|\chi_* T|^p$ for all characters χ of A, so that $T \in \mathcal{L}^p(E_A)$.

For the first part, because $\theta_e^* \theta_e = \mathrm{id}_{E_A}$, we have $|\theta_e T \theta_e^*|^p = \theta_e |T|^p \theta_e^*$. Furthermore, note that $\{f_i \stackrel{\text{def}}{=} \theta_e(e_i)\}_i$ is a frame of $\theta_e \theta_e^* l^2(A) \simeq E_A$ so that the elements $\{h_i \stackrel{\text{def}}{=} \chi_* f_i\}_i$ form a frame of $\chi_* \theta_e \theta_e^* l^2(A)$ by Proposition 5.1.10. Now, by definition of the h_i we have $\chi_*(\theta_e |T|^p \theta_e^*)(h_i) = \chi_*(\theta_e |T|^p e_i)$ and so $\langle h_i, \chi_*(\theta_e |T|^p \theta_e^*)(h_i) \rangle = \chi(\langle \theta_e e_i, \theta_e |T|^p e_i \rangle)$ $= \chi(\langle e_i, |T|^p e_i \rangle)$. That is,

$$\operatorname{tr} \chi_* \theta_e |T|^p \theta_e^* = \sum_{i=1}^{\infty} \langle h_i, \chi_*(\theta_e |T|^p \theta_e^*)(h_i) \rangle = \sum_{i=1}^{\infty} \chi\left(\langle e_i, |T|^p e_i \rangle\right).$$

Thus, if $\sum_{i=1}^{\infty} \langle e_i, |T|^p e_i \rangle$ converges to an element of A, we have $\theta_e |T|^p \theta_e^* \in \mathcal{L}^p\left(l^2(A)\right)$. Conversely, if $\theta_e |T|^p \theta_e^* \in \mathcal{L}^p\left(l^2(A)\right)$ then the function $\chi \mapsto \sum_{i=1}^{\infty} \chi\left(\langle e_i, |T|^p e_i \rangle\right)$ for $\chi \in \widehat{A}$ lies in $C_0(X) \simeq A$. The series must then converge in norm by Dini's theorem. □

Corollary 6.1.6. Let $S \in \mathcal{L}(E_A)$ and $T \in \mathcal{L}^p(E_A)$. If $|S|^p \leq |T|^p$, then $S \in \mathcal{L}^p(E_A)$ and in particular $\operatorname{tr}|S|^p \leq \operatorname{tr}|T|^p$.

Proof. Let e be a frame of E_A. Then $\sum_{i \in F} \langle e_i, |S|^p e_i \rangle \leq \sum_{i \in F} \langle e_i, |T|^p e_i \rangle$ for all finite $F \subset \mathbb{N}$; in particular, the left-hand side is Cauchy whenever the right-hand side is. By Theorem 6.1.5, this will suffice. □

Remark 6.1.7. The above Corollary is weaker than the Hilbert space version, cf. Lemma 6.2.7(3) below. Instead, Corollary 6.2.16 below gives a stronger result but an additional assumption on S is required. Note that this is the only point in the treatment of this chapter where such a difference between the Hilbert module and Hilbert space Schatten classes appears.

The most straightforward road to analyzing the structure and properties of $\mathcal{L}^p(E_A)$ now lies open: we will investigate $\mathcal{L}^p\left(l^2(A)\right)$ and use the pullback by the frame transforms to transfer its properties to $\mathcal{L}^p(E_A)$. It will turn out that $\mathcal{L}^p\left(l^2(A)\right)$ is indeed very well-behaved, so that this allows us to recover many of the familiar properties of the Schatten classes of operators on Hilbert spaces.

6.2 The Schatten class on the standard module

Let H be a separable Hilbert space and let $A = C_0(X)$ be an abelian C^*-algebra. Recall from Section 5.1 that the Hilbert A-module $H \otimes_{\mathbb{C}} A$ is isomorphic to $C_0(X, H)$ through the canonical isomorphism $\chi_*(H \otimes_{\mathbb{C}} A) \simeq H$ (with a character $\chi \in \widehat{A}$ always corresponding via the Gelfand transform to a point $x \in X$). Recall, moreover, that its endomorphism space is given by $C_{\mathrm{b}}^{\mathrm{str}}(X, H)$ with the fiberwise action given simply by $T(h)(x) \overset{\mathrm{def}}{=} T(x)(h(x))$ for $x \in X$.

Because all fibers are identified canonically with H, we may canonically compare the localizations of an operator $T \in \mathscr{L}(H \otimes_{\mathbb{C}} A)$ between *different* fibers. This technically useful difference between $H \otimes_{\mathbb{C}} A$ and other Hilbert A-modules will allow us to use topologies on $B(H)$ to define particular subsets of $\mathscr{L}(H \otimes_{\mathbb{C}} A)$. Most importantly,

Definition 6.2.1. The space of *continuous Schatten-class operators* on $C_0(X, H) \simeq H \otimes_{\mathbb{C}} A$ is the subspace $C_0(X, \mathscr{L}^p(H))$ of $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$.

Note that where the requirement that $\operatorname{tr} |T|^p \in A$ is the least restrictive among reasonable criteria for a 'Schatten-class operator' T, as discussed above Definition 6.1.2, the condition of Definition 6.2.1 is arguably the *most* restrictive.

However, we will prove that the demands are, in fact, equivalent, so that the p-th Schatten class $\mathscr{L}^p(H \otimes_{\mathbb{C}} A)$ on the standard module can be identified with the Banach space $C_0(X, \mathscr{L}^p(H))$ of continuous Schatten-class operators. This will later —specifically in Theorem 6.3.1—allow us to combine the properties that follow straightforwardly from either of the two definitions.

Remark 6.2.2. Clearly, one has $C_0(X, \mathscr{L}^p(H)) \subset \mathscr{L}^p(C_0(X, H))$ because, for continuous maps T from X to $\mathscr{L}^p(H)$ and $x, y \in X$, we have $\left\|T(x) - T(y)\right\|_p \geq \left| \|T(x)\|_p - \|T(y)\|_p \right|$ so that $x \mapsto \|T(x)\|_p \in A$.

The ostensibly more restrictive definition of the continuous Schatten class has some advantages to that of $\mathscr{L}^p(C_0(X, H))$. For instance, it is immediate from the definition that $C_0(X, \mathscr{L}^p(H))$ is closed under addition, as we have *not* yet shown for $\mathscr{L}^p(C_0(X, H))$. Moreover, we can easily obtain a continuous version of the Hölder–von Neumann inequality:

Proposition 6.2.3. Let $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and let $S \in C_0(X, \mathscr{L}^p(H))$ and $T \in C_0(X, \mathscr{L}^q(H))$. Then $ST \in C_0(X, \mathscr{L}^r(H))$ and $\|ST\|_r \leq \|S\|_p \|T\|_q$.

Proof. For any $x, y \in X$ we have

$$\left\|ST(x) - ST(y)\right\|_r \leq \left\|S(x) - S(y)\right\|_p \|T(x)\|_q + \left\|S(y)\right\|_p \left\|T(x) - T(y)\right\|_q,$$

by the Hölder–von Neumann inequality [Sim05, Theorem 2.8]. We conclude that ST is continuous as a map from X to $\mathscr{L}^r(H)$. Moreover, since $\|ST(x)\|_r \leq \|S(x)\|_p \|T(x)\|_q$ for all x, the statement on the norms follows as well as the claim that $ST \in C_0(X, \mathscr{L}^r(H))$. □

We now identify a subset of $C_0(X, \mathscr{L}^p(H)) \subseteq \mathscr{L}^p(H \otimes_{\mathbb{C}} A)$ (*cf.* Remark 6.2.2 for the latter inclusion) whose completion in the Banach norm of $C_0(X, \mathscr{L}^p(H))$ is all of $\mathscr{L}^p(H \otimes_{\mathbb{C}} A)$. This will show that $\mathscr{L}^p(H \otimes_{\mathbb{C}} A)$ coincides with the Banach space $C_0(X, \mathscr{L}^p(H))$. Moreover, the fact that this common subset consists of finite-rank operators, in the Hilbert module sense, allows us to show that $\mathscr{L}^p(E_A) \subset \mathscr{K}(E_A)$ in Theorem 6.3.1.

Proposition 6.2.4. The finite-rank operators on $C_0(X, H)$ (in the Hilbert C*-module sense) lie in $C_0(X, \mathscr{L}^p(H))$.

Proof. Let $T = |v\rangle\langle w|$ with $v, w \in C_0(X, H)$. Then, for $x, y \in X$, we have

$$\left\|(T_x - T_y)\right\|_p \leq \left\| |v_x - v_y\rangle\langle w_y| \right\|_p + \left\| |v_y\rangle\langle w_x - w_y| \right\|_p.$$

Pointwise in H, however, we have $\left\| |\xi\rangle\langle\eta| \right\|_p = \|\xi\| \, \|\eta\|$, so that norm continuity of v and w finish the proof. □

Lemma 6.2.5. Let $V \subset H$ be finite-dimensional and consider $C_0(X, B(V))$ as a subspace of $C_b^{\mathrm{str}}(X, B(H))$ by the map $T \mapsto T \oplus 0 \in B(V) \oplus B(V^\perp) \subset B(H)$. Then, all elements of $C_0(X, B(V))$ are finite rank operators on the Hilbert modules $C_0(X, H)$. In particular, we have $C_0(X, B(V)) \subset C_0(X, \mathscr{L}^p(H))$ for all $1 \leq p < \infty$.

Proof. Let $T \in C_0(X, B(V))$ and decompose so that $T = S|T|^{\frac{1}{2}}$. Then let $\{e_i\}_{i=1}^n$ be an orthonormal basis of V. Because $\sum_i |e_i\rangle\langle e_i| = \mathrm{id}_V$, we see that $T = S\sum_i |e_i\rangle\langle e_i| \, |T|^{\frac{1}{2}} = \sum_i |Se_i\rangle\langle |T|^{\frac{1}{2}} e_i|$, where we denote for $R \in C_0(X, B(V))$ by Re_i the element $x \mapsto R(x)e_i$ of $C_0(X, V)$. We conclude that T is of finite rank. □

Remark 6.2.6. By a theorem of Fell [Fel61, Theorem 4.1], compact operators of bounded rank on $C_0(X, H)$ automatically have continuous trace. See remark 6.3.2, below, for the link to the study of continuous-trace C*-algebras.

Some properties of the Schatten classes on Hilbert spaces

We assemble here some more or less well-known properties of the ordinary Schatten classes on $B(H)$. The purpose is to show that one can use the series that defines the trace of $|T|^p$ to control the rate at which certain finite-rank approximations of T will converge to T in Schatten norm.

Lemma 6.2.7. Let $T \in \mathscr{L}^p(H)$.

1. For $1 \leq p < 2$ one has $\|T\|_p^p = \inf_{\{e_i\}_i} \sum_{i=1}^{\infty} \|Te_i\|^p$, where the infimum is taken over orthonormal bases $\{e_i\}_i$ of H.
2. For $p \geq 2$, one has $\|T\|_p^p = \sup_{\{e_i\}_i} \sum_{i=1}^{\infty} \|Te_i\|^p$, where the supremum is over orthonormal bases $\{e_i\}_i$ of H.
3. Let $p \geq 2$. For any bounded endomorphism $S \in \mathscr{L}(H)$, if $|S|^2 \leq T$ for some $T \in \mathscr{L}^{p/2}(H)$, then $S \in \mathscr{L}^p(H)$ and $\|S\|_p^p \leq \|T\|_{p/2}^{p/2}$.

Proof. Let $\{e_i\}_i$ be an orthonormal eigenbasis of (the compact, normal operator) T^*T, ordered by decreasing of the corresponding eigenvalues $\{\lambda_i\}$.

First note that $\operatorname{tr}|T|^p = \sum_{i=1}^{\infty} \langle Te_i, Te_i\rangle^{p/2}$. Any other orthonormal basis $\{f_i = Ue_i\}$ of H is related to $\{e_i\}$ by some unitary operator $U \in B(H)$.

For $1 \leq p < 2$, the function $x \mapsto x^{p/2}$ is concave on $\mathbb{R}_+$. Thus, since

$$\left\langle Tf_i, Tf_i \right\rangle = \sum_j \left\langle f_i, T^*Te_j \right\rangle \left\langle e_j, f_i \right\rangle = \sum_j \lambda_j \left| \left\langle e_j, f_i \right\rangle \right|^2,$$

we find that $\left\|Tf_i\right\|^p \geq \sum_j \lambda_j^{p/2} \left|\left\langle e_j, f_i\right\rangle\right|^2 = \left\langle f_i, |T|^p f_i\right\rangle$. We conclude that

$$\sum_{i=1}^{\infty} \left\|Tf_i\right\|^p \geq \sum_{i=1}^{\infty} \left\langle f_i, |T|^p f_i \right\rangle = \operatorname{tr} U^* |T|^p U = \operatorname{tr} |T|^p.$$

For $p \geq 2$ the function $x \mapsto x^{p/2}$ is convex on $\mathbb{R}_+$ and we find, *mutatis mutandis* in the argument as above that now

$$\sum_{i=1}^{\infty} \left\|Tf_i\right\|^p \leq \sum_{i=1}^{\infty} \left\langle f_i, |T|^p f_i \right\rangle = \operatorname{tr} U^* |T|^p U = \operatorname{tr} |T|^p.$$

For the final claim, if $|S|^2 \leq R^*R$, one has $\|Se_i\|^p \leq \|Re_i\|^p$ so that $\|S\|_p^p \leq \|R\|_p^p = \|R^*R\|_{p/2}^{p/2}$. ☐

We will need the following Corollary in the proof of Lemma 6.2.9.

Corollary 6.2.8. Let $T \in B(H)$ and let e be a finite-rank projection. For any $p \geq 2$ we have $Te \in \mathscr{L}^p(H)$ and, in fact,

$$\|Te\|_p^p \leq \operatorname{tr} e|T|^p e.$$

Proof. As in Lemma 6.2.7, let eT^*Te have eigenbasis $\{g_i\}$ with eigenvalues $\{\lambda_i\}$. Then, for any $v \in H$, we have $\langle Tev, Tev\rangle = \langle e|T|^2 ev, v\rangle = \sum_j \lambda_j |\langle v, g_j\rangle|^2$. In particular,

$$\langle Tev, Tev\rangle^{p/2} = \|v\|^p \left(\sum_j \lambda_j \frac{|\langle v, g_j\rangle|^2}{\langle v, v\rangle}\right)^{p/2}.$$

By convexity of $x \mapsto x^{p/2}$ on $\mathbb{R}_+$ for $p \geq 2$ we have

$$\langle Tev, Tev\rangle^{p/2} \leq \|v\|^{p-2} \sum_j \lambda_j^{p/2} |\langle v, g_j\rangle|^2 = \|v\|^{p-2} \langle v, e|T|^p ev\rangle.$$

But from the Lemma it then follows that

$$\|Te\|_p^p = \sup_{\{f_i\}} \sum_i \|Tef_i\|^p \leq \sup_{\{f_i\}} \sum_i \|f_i\|^{p-2} \langle f_i, e|T|^p ef_i\rangle \leq \operatorname{tr} e|T|^p e. \qquad ☐$$

With respect to a choice of orthonormal basis on H, we can view $\mathscr{L}^p(H)$ as a completion of the direct limit $\operatorname{Mat}\mathbb{C}$ of finite matrix algebras $\operatorname{Mat}_n(\mathbb{C})$ in the Schatten norm. This can be done 'uniformly', where the convergence of the limit is controlled by the trace, as we will show in Proposition 6.2.10 below.

Given $T \in \mathscr{L}(H)$ and a sequence of increasing finite-dimensional subspaces $PH \subset H$, the operator PTP converges to T $*$-strongly as $P \to \mathrm{id}_H$. The following Lemma allows us to control the p-norm of the difference when increasing the rank of P by one, such that $PTP \to T$ in p-norm precisely when $T \in \mathscr{L}^p(H)$.

Lemma 6.2.9. Let $p \in [1, \infty)$ and let $T \in \mathscr{L}^p(H)$. Let e be a finite-rank projection in H, let P be a finite-rank projection with $Pe = eP = 0$ and let $Q = P + e$. Then,

$$\|QTQ - PTP\|_p \leq \|T\|_p^{1/2} \left(\left(\operatorname{tr} e|T^*|^p e\right)^{1/2p} + \left(\operatorname{tr} e|T|^p e\right)^{1/2p} \right)$$

Proof. One has $QTQ - PTP = PTe + eTe + eTP = eTQ + PTe$, so that $\|QTQ - PTP\|_p \leq \|eTQ\|_p + \|PTe\|_p$. Now compose T as $T = S|T|^{\frac{1}{2}}$ with $|S|^2 = |T|$. Then, by the Hölder-von Neumann inequality, $\|eTQ\|_p \leq \|eS\|_{2p}\||T|^{1/2}Q\|_{2p}$ and similarly $\|PTe\|_p \leq \|PS\|_{2p}\||T|^{1/2}e\|_{2p}$. Moreover, $\|PS\|_{2p} \leq \|T\|_p^{1/2}$ and $\||T|^{1/2}Q\|_{2p} \leq \|T\|_p^{1/2}$, and as $\|eS\|_{2p} = \|S^*e\|_{2p}$, we may now apply Corollary 6.2.8 directly to $\|S^*e\|_{2p}$ and $\||T|^{1/2}e\|_{2p}$ in order to finish the proof. $\square$

Identification of $\mathscr{L}^p\left(H \otimes_{\mathbb{C}} A\right)$ with $C_0(\widehat{A}, \mathscr{L}^p(H))$

It is a well-known fact that the finite-rank operators are dense in $\mathscr{L}^p(H)$ for any p, as can easily be seen from the spectral theorem for compact self-adjoint operators. This argument, however, does not extend uniformly to self-adjoint $T \in C_0(X, \mathscr{L}^p(H))$ unless T is continuously diagonalizable. The explicit, albeit apparently somewhat clumsy, result of Lemma 6.2.9 presents a solution to this problem. Namely, orthogonal projections $P \in B(H)$ can be lifted to constant projections $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$. This allows the Lemma to be applied uniformly to all of $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$, as we do in Proposition 6.2.10 below. This will then provide the main ingredient of main result of this section, to wit, the identification of $\mathscr{L}^p(H \otimes_{\mathbb{C}} A)$ with $C_0(X, \mathscr{L}^p(H))$ (Theorem 6.2.11).

Proposition 6.2.10. Let $T \in C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$, let $p \in [1, \infty)$ and assume that the function $\operatorname{tr}|T|^p : x \in X \mapsto \operatorname{tr}|T(x)|^p$ is defined everywhere and lies in $A = C_0(X)$.

Let $\{e_i\}_i$ be an orthonormal basis of H and let, for any $n \geq 0$, $P_n \stackrel{\text{def}}{=} \sum_{i=1}^n |e_i\rangle\langle e_i|$ be the corresponding spectral projections. Then the operators $T_n \stackrel{\text{def}}{=} P_n T P_n$ (as in Lemma 6.2.9) are elements in $C_0(X, \mathscr{L}^p(H))$ and, for $m \geq n$,

$$\|T_m - T_n\|_p^{2p} \leq 2^{2p-1} \sup_{x \in X} \operatorname{tr}|T(x)|^p \sum_{i=n+1}^{m} \left\langle e_i, \left(|T(x)|^p + |T^*(x)|^p\right) e_i \right\rangle$$

Proof. Note that T_n is a finite-rank operator and in particular $T_n \in C_0(X, \mathscr{L}^p(H))$, by Lemma 6.2.5. The result follows from Lemma 6.2.9 applied to the projections $P = P_n$, $Q = P_m$ and $e = P_m - P_n$ and by the simple equality $(x+y)^{2p} \leq 2^{2p-1}(x^{2p} + y^{2p})$. $\square$

Theorem 6.2.11. Let $T \in C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$. Then $T \in C_0(X, \mathscr{L}^p(H))$ if and only if $T \in \mathscr{L}^p(C_0(X, H))$.

Proof. The implication $\Rightarrow$ was already established in Remark 6.2.2. For the converse, assume that $T \in C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$ and that $x \mapsto \operatorname{tr}|T(x)|^p$ lies in $C_0(X)$. Then, pick an orthonormal basis $\{e_i\}_i$ of H and write again $T_n = P_nTP_n$ as in Proposition 6.2.10. Since $x \mapsto \operatorname{tr}|T(x)|^p = \operatorname{tr}|T^*(x)|^p$ is in $C_0(X)$, the series $\sum_{i=1}^{\infty} \langle e_i, |T(x)|^p e_i\rangle$ as well as the series $\sum_{i=1}^{\infty} \langle e_i, |T^*(x)|^p e_i\rangle$ must converge uniformly on compact subsets of X by Dini's theorem. That, in turn, implies that

$$\sup_{x\in X} \|T_{n+k}(x) - T_n(x)\|_p < \left[2^{2p-1} \sup_{x\in X} \operatorname{tr}|T(x)|^p \sum_{i=n+1}^{n+k} \langle e_i, (|T(x)|^p + |T^*(x)|^p)e_i\rangle\right]^{1/2p}$$

goes to zero for large n. Since $C_0(X, \mathscr{L}^p(H))$ is a Banach space, the sequence T_n thus converges (to T) in Schatten p-norm, so that $T \in C_0(X, \mathscr{L}^p(H))$. □

Remark 6.2.12. In terms of tensor products of Banach spaces, Theorem 6.2.11 translates to the statement that $\mathscr{L}^p(H \otimes_{\mathbb{C}} A) \simeq \mathscr{L}^p(H) \otimes_\epsilon A$, the injective tensor product.

Corollary 6.2.13. The continuous Schatten class $C_0(X, \mathscr{L}^p(H))$ forms a two-sided ideal in $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$.

Proof. Let $T \in C_0(X, \mathscr{L}^p(H))$ and let $T' \in C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$. Then, for any basis $\{e_i\}_i$ of H the operators $T_n \stackrel{\text{def}}{=} P_nTP_n$ $(p \geq 2)$ or $T_n \stackrel{\text{def}}{=} P_nSP_n|T|^{\frac{1}{2}}P_n$ $(p < 2)$, with P_n and S as in Proposition 6.2.10, converge to T in the continuous Schatten p-norm. Now, note that $\|T'(T_m - T_n)\|_p \leq \|T'\| \, \|T_m - T_n\|_p$ and $\|(T_m - T_n)T'\|_p \leq \|T'\| \, \|T_m - T_n\|_p$; thus, $T'T_n$ and T_nT' converge to $T'T$ and TT', respectively, in the norm of $C_0(X, \mathscr{L}^p(H))$. □

Remark 6.2.14. This result does not follow directly from the fact that $\mathscr{L}^p(H)$ is an ideal of $B(H)$ equipped with a Banach norm such that the inclusion into $B(H)$ is continuous: each such ideal induces a two-sided ideal $C_b(X, I) \subset C_b(X, B(H))$, but that is not necessarily an ideal of $C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$. An easy counterexample is given by $I = B(H)$ itself.

Corollary 6.2.15. The continuous Schatten class $C_0(X, \mathscr{L}^p(H))$ is contained in the compact operators on the Hilbert C^*-module $C_0(X, H)$.

Proof. The operators T_n of Proposition 6.2.10 are of finite rank in the Hilbert module sense, because they are contained in $C_0(X, B(V))$ for some finite-dimensional $V \subset H$ so that we can apply Lemma 6.2.5. As $T_n \to T$ for $T \in C_0(X, \mathscr{L}^p(H))$ in the Schatten p-norm, $T_n \to T$ in operator norm as well. We conclude that T is compact in the Hilbert module sense. □

The following slight strengthening of Corollary 6.1.6 to $C_0(X, \mathscr{L}^p(H))$ now translates to $\mathscr{L}^p(C_0(X,H))$.

Corollary 6.2.16. If $0 \le S \le T \in C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$ and $T \in \mathscr{L}^p(C_0(X,H))$ and additionally we have $S \in C_b^{\mathrm{norm}}(X, B(H))$, then $S \in \mathscr{L}^p(C_0(X,H))$ and $\operatorname{tr}|S|^p \le \operatorname{tr}|T|^p$.

Proof. The operators S, T are pointwise compact, norm continuous (this is where we use the additional assumption on S) and positive, so that their individual eigenvalues λ_k (ordered decreasingly) are continuous ([Kat95, p. IV.3.5], see also Lemma 7.2.2 below).

Then, by the min-max theorem, the k'th singular value of $\chi_* S$ is bounded by the k'th singular value of $\chi_* T$, so that the same holds for their p'th powers. As the Schatten norm of $\chi_* T$ is the sum of those p'th powers, which converges to a continuous function, the convergence must be uniform by Dini's theorem. Thus, the series $\sum_k \lambda_k(T)^p$ of elements of A is Cauchy, so that the series $\sum_k \lambda_k(S)^p$ must be Cauchy as well. We conclude that $x \mapsto \left\|\chi_* S\right\|_p$ lies in A. □

Remark 6.2.17. The additional assumption in Corollary 6.2.16 is necessary because the positive compact operators on a Hilbert C^*-module, in contrast to those on a Hilbert space, may not necessarily form an order ideal: there is an additional continuity requirement on the (pointwise compact) localizations. In contrast, as in Corollary 6.1.6, the positive trace-class operators on a Hilbert C^*-module *do* form an order ideal.

6.3 Properties of the Schatten classes on Hilbert modules

We now return to the general setup of countably generated Hilbert C^*-modules over commutative C^*-algebras. For the case of the standard module $l^2(A)$ with $A = C_0(X)$ Theorem 6.2.11 shows that $\mathscr{L}^p\left(l^2(A)\right)$ is a Banach space and a two-sided ideal of $\mathscr{L}\left(l^2(A)\right)$ that is moreover contained in $\mathscr{K}\left(l^2(A)\right)$. These are very desirable properties for general, countably generated Hilbert A-modules. Fortuitously, the existence of frames (Proposition 5.1.12) and the pull-back criterion of Theorem 6.1.5 allows us to easily establish equivalent properties of $\mathscr{L}^p(E_A)$ for all countably generated Hilbert A-modules E_A.

Theorem 6.3.1. The space $\mathscr{L}^p(E_A)$ is a two-sided ideal of $\mathscr{L}(E_A)$ that is contained in $\mathscr{K}(E_A)$.

Proof. Choose a frame e of E_A and let ϕ_e be the $*$-homomorphism induced by the frame transform: $\phi_e \colon \mathscr{L}(E_A) \to \mathscr{L}\left(l^2(A)\right), T \mapsto \theta_e T \theta_e^*$. Recall that $T \in \mathscr{L}^p(E_A)$ if and only if $\phi_e(T) \in \mathscr{L}^p\left(l^2(A)\right)$ by Theorem 6.1.5. By Theorem 6.2.11, then, $\mathscr{L}^p(E_A)$ is closed under finite linear combinations. Moreover, for $S \in \mathscr{L}(E_A), T \in \mathscr{L}^p(E_A)$, we have $\phi_e(ST) \in \mathscr{L}^p\left(l^2(A)\right)$ and $\phi_e(TS) \in \mathscr{L}^p\left(l^2(A)\right)$ by Corollary 6.2.13. We conclude that $\mathscr{L}^p(E_A)$ is a two-sided ideal.

Moreover, as in the previous paragraph, $T \in \mathscr{L}^p(E_A)$ iff $\theta_e T \theta_e^* \in \mathscr{L}^p\left(l^2(A)\right) \subset \mathscr{K}\left(l^2(A)\right)$ by Corollary 6.2.15. But if $\theta_e T \theta_e^*$ is compact, then the operator $T = \theta_e^* \theta_e T \theta_e^* \theta_e$ is compact as well (essentially because $\theta_e^*(|e_i\rangle\langle e_j|)\theta_e = |e_i\rangle\langle e_j|$ for any $i, j \in \mathbb{N}$). ☐

Remark 6.3.2. Dixmier's definition of *continuous-trace* C*-algebras [Dix77, Chapter 4.5] applies to $\mathscr{K}(E_A)$. By Theorem 6.3.1, the corresponding trace and Hilbert-Schmidt classes agree with our $\mathscr{L}^1(E_A)$ and $\mathscr{L}^2(E_A)$ respectively. The projections satisfying Fell's criterion are then the *compact* finite-rank ones, that is, those that correspond to finitely generated projective modules.

Remark 6.3.3. In the light of Theorem 6.3.1, we have obtained an *a fortiori* method of determine whether a positive operator T on a Hilbert C*-module over an abelian C^*-algebra is compact: it suffices that $\operatorname{tr} T^p$ lies in A for some $p \geq 1$. Compare e.g. the proof of [KS17, Proposition 7] and [ibid., Remark 6] to see that showing such compactness directly can be a nontrivial undertaking.

Using Theorems 6.1.5 and 6.2.11, we can pull back the Banach norm on $C_0(X, \mathscr{L}^p(H))$ to $\mathscr{L}^p(E_A)$, and this turns out rather well:

Theorem 6.3.4. The function $\|\cdot\|_p : T \mapsto \left\|\operatorname{tr}|T|^p\right\|_A^{1/p}$ is a norm that turns $\mathscr{L}^p(E_A)$ into a normed vector space. Moreover, for all $T \in \mathscr{L}^p(E_A)$,

1. $\|T\|_p = \sup_{\chi \in \widehat{A}} \|\chi_* T\|_p$
2. $\|T^*\|_p = \|T\|_p$
3. $\|ST\|_p \leq \|S\| \, \|T\|_p$ for all $S \in \mathscr{L}(E_A)$
4. $\|T\| \leq \|T\|_p$
5. For $p, q, r \geq 1$, if $S \in \mathscr{L}^q(E_A)$ and $T \in \mathscr{L}^p(E_A)$ with $\frac{1}{p} + \frac{1}{q} = 1/r$ then $ST \in \mathscr{L}^r(E_A)$ and $\|ST\|_r \leq \|S\|_q \|T\|_p$.

Moreover, $\mathscr{L}^p(E_A)$ is a Banach space.

Proof. Let e be a frame of E_A and consider the projection $P_e \overset{\text{def}}{=} \theta_e\theta_e^*$ in $\mathscr{L}\left(l^2(A)\right)$. Recall that the invertible $*$-homomorphism $\phi_e \colon T \mapsto \theta_e T\theta_e^*$ maps $\mathscr{L}^p\,(E_A)$ into the subspace $P_e\mathscr{L}^p\left(l^2(A)\right)P_e$ by Theorem 6.1.5. Moreover, elementary calculation shows that $\phi_e\,|_{\mathscr{L}^p(E_A)}$ is an isomorphism of normed spaces.

For the properties of the norm, recall that $\chi(\operatorname{tr}|T|^p) = \operatorname{tr}|\chi_* T|^p = \left\|\chi_* T\right\|_p^p$. Thus, the norm satisfies $\left\|\operatorname{tr}|T|^p\right\|_A^{1/p} = \sup_{\chi\in\widehat{A}}\left\|\chi_* T\right\|_p^{1/p}$. Therefore, by the analogous properties of $\mathscr{L}^p\,(H)$, we see that $\|T^*\|_p = \|T\|_p$, $\|T\|_p \geq \|T\| = \sup_{\chi\in\widehat{A}}\left\|\chi_* T\right\|$ and $\|ST\|_p \leq \sup_{\chi\in\widehat{A}}\left\|\chi_*(S)\right\|\left\|\chi_* T\right\|_p$ which is bounded by $\|S\|\,\|T\|_p$.

For property 5, recall that $\theta_e ST\theta_e^* = \theta_e S\theta_e^*\theta_e T\theta_e^* \in \mathscr{L}^r\left(l^2(A)\right)$ by Proposition 6.2.3, and so $ST \in \mathscr{L}^r\,(E_A)$ by Theorem 6.1.5. The norm inequality then follows from the Hölder–von Neumann inequality for operators on Hilbert spaces.

Finally, to establish completeness of $\mathscr{L}^p\,(E_A)$ it is enough to prove that its pullback to $l^2(A)$, which equals $P_e\mathscr{L}^p\left(l^2(A)\right)P_e$, is a closed subspace of $\mathscr{L}^p\left(l^2(A)\right)$. But if $P_e T_n P_e \to T$ in $\mathscr{L}^p\left(l^2(A)\right)$ then

$$\|P_e T_n P_e - P_e T P_e\|_p = \|P_e(P_e T_n P_e - T)P_e\|_p \leq \|P_e\|^2\|(P_e T_n P_e - T)\|_p \to 0$$

as $n\to\infty$, in virtue of the just-proved inequality 3. Hence, $P_e T_n P_e$ converges to $P_e T P_e \in P_e\mathscr{L}^p\left(l^2(A)\right)P_e$ as desired. ☐

6.4 The Hilbert module of Hilbert–Schmidt operators

The Hilbert–Schmidt class $\mathscr{L}^2\,(E_A)$ is a somewhat special case among the Schatten classes, because the map $T \mapsto \operatorname{tr} T^*T$ is a positive definite quadratic form. That is, it induces an inner product as we will now explore.

Definition 6.4.1. The pairing $\langle\cdot,\cdot\rangle_2 : \mathscr{L}^2\,(E_A)\times\mathscr{L}^2\,(E_A) \to A$ is given by

$$\langle S,T\rangle_2 \overset{\text{def}}{=} \frac{1}{4}\sum_{k\in\mathbb{Z}/4\mathbb{Z}} i^k \operatorname{tr}|T+i^k S|^2$$

When viewed fiberwise, this is just the ordinary Hilbert–Schmidt inner product:

Proposition 6.4.2. For $S,T \in \mathscr{L}^2\,(E_A)$ and a character χ of A the pairing $\langle S,T\rangle_2$ satisfies $\chi\left(\langle S,T\rangle_2\right) = \operatorname{tr}((\chi_* S)^*\chi_* T)$. Moreover, the series $\sum_{i=1}^{\infty}\langle Se_i, Te_i\rangle$ converges in norm to $\langle S,T\rangle_2$ for any frame e of E_A.

Proof. Since χ_* is a homomorphism, the first part follows from the fact that $\operatorname{tr}\chi_*(|T+i^kS|^2) = \chi(\operatorname{tr}|T+i^kS|^2)$ by the polarization identity for the fiberwise Hilbert–Schmidt inner product. Since

$$\langle Se_i, Te_i\rangle = \langle e_i, S^*Te_i\rangle = \frac{1}{4}\sum_{k\in\mathbb{Z}/4\mathbb{Z}} i^k \left\langle e_i, |T+i^kS|^2 e_i\right\rangle,$$

the second part follows from Theorem 6.1.5. □

Corollary 6.4.3. The pairing $\langle\cdot,\cdot\rangle_2$ on $\mathscr{L}^2(E_A)$ is non-degenerate and sesquilinear.

Next, because A is commutative E_A is automatically an A-bimodule. In fact, there is a $*$-homomorphism $\rho\colon A \to \mathscr{L}(E_A)$ given by $\rho(a)(v) \stackrel{\text{def}}{=} v\cdot a$. This makes $\mathscr{L}(E_A)$ an A-bimodule with $a\cdot T\cdot b = \rho(a)\circ T\circ\rho(b)$ for all $a,b\in A$ and $T\in\mathscr{L}(E_A)$.

Proposition 6.4.4. The A-bimodule structure of $\mathscr{L}(E_A)$ restricts to $\mathscr{L}^p(E_A)$ for all $1\le p<\infty$ and satisfies $\|T\cdot a\|_p \le \|a\|\,\|T\|_p$ as well as $\|a\cdot T\|_p \le \|a\|\,\|T\|_p$.

Proof. Since any $*$-homomorphism between C^*-algebras is norm decreasing, this follows from Theorem 6.3.1 since $\|T\rho(a)\|_p \le \|\rho(a)\|\,\|T\|_p \le \|a\|\,\|T\|_p$. □

All this leads to the following result for the case that $p=2$:

Proposition 6.4.5. With the above right A-action and the inner product $\langle\cdot,\cdot\rangle_2$, $\mathscr{L}^2(E_A)$ becomes a Hilbert A-module.

Proof. Note that $\chi_*(T\circ\rho(a)) = \chi(a)\chi_*T$ for all $a\in A$ and characters χ of A. Thus, with Proposition 6.4.2, the inner product is A-sesquilinear. All that is left to show, therefore, is that $\mathscr{L}^2(E_A)$ is complete. That, however, was proven already in Theorem 6.3.1. □

Proposition 6.4.6. Let H be a separable Hilbert space. Then, $\mathscr{L}^2(H\otimes_{\mathbb{C}}A)$ is isomorphic, as a Hilbert A-module, to $\mathscr{L}^2(H)\otimes_{\mathbb{C}}A$.

Proof. Under the isomorphism $H\otimes_{\mathbb{C}}A\simeq C_0(X,H)$, $\mathscr{L}^2(H\otimes_{\mathbb{C}}A)$ is mapped isometrically onto $C_0(X,\mathscr{L}^2(H))$ by Theorem 6.2.11. Under this identification, the Hilbert C^*-module structure of $\mathscr{L}^2(H\otimes_{\mathbb{C}}A)$ coincides with the canonical Hilbert C^*-module structure on $C_0(X,\mathscr{L}^2(H))$ induced by the inner product on the Hilbert space $\mathscr{L}^2(H)$. Thus, we have $\mathscr{L}^2(H\otimes_{\mathbb{C}}A)\simeq C_0(X,\mathscr{L}^2(H))$ as Hilbert C^*-modules. Now, invoke once again the isomorphism $\mathscr{L}^2(H)\otimes_{\mathbb{C}}A\simeq C_0(X,\mathscr{L}^2(H))$, this time for the Hilbert space $\mathscr{L}^2(H)$, to complete the proof. □

6.5 The trace class and the trace

We will refer to the ideal $\mathscr{L}^1(E_A) \subset \mathscr{L}(E_A)$ consisting of those operators T for which $\operatorname{tr}|T|$ is given by an element of A, as the *trace class*. In the case of Schatten classes of Hilbert spaces, *i.e.* $A = \mathbb{C}$ and $E_A = H$, it is customary to identify the trace class as the ideal generated by squares of elements of the Hilbert–Schmidt class, in order to relate the Hilbert–Schmidt inner product to a linear function, the *trace*, on $\mathscr{L}^1(H)$. The situation here is completely analogous:

Proposition 6.5.1. Let $\mathscr{L}^2(E_A)\mathscr{L}^2(E_A) = \{RS \mid R \in \mathscr{L}^2(E_A), S \in \mathscr{L}^2(E_A)\}$ as a subset of $\mathscr{L}(E_A)$. Then, $\mathscr{L}^2(E_A)\mathscr{L}^2(E_A) = \mathscr{L}^1(E_A)$.

Proof. The inclusion $\mathscr{L}^2(E_A)\mathscr{L}^2(E_A) \subset \mathscr{L}^1(E_A)$ is a direct consequence of the Hölder–von Neumann inequality in Theorem 6.3.4(5)Conversely, let $T \in \mathscr{L}^1(E_A)$. Then, $T = S|T|^{\frac{1}{2}}$ in the usual weak polar decomposition, with $|S| = |T|^{\frac{1}{2}}$. By Proposition 6.1.3, S and $|T|^{\frac{1}{2}}$ lie in $\mathscr{L}^2(E_A)$. □

This furnishes us with a way to turn the bilinear map $\langle \cdot, \cdot \rangle_2$ on $\mathscr{L}^2(E_A)$ into a linear map tr on $\mathscr{L}^1(E_A)$ called the *trace*:

Definition 6.5.2. The *trace* on $\mathscr{L}^1(E_A)$ is the map $\operatorname{tr}\colon T \mapsto \langle S^*, |T|^{\frac{1}{2}} \rangle_2$, where $T = S|T|^{\frac{1}{2}}$ is the weak polar decomposition.

Proposition 6.5.3. The trace is well-defined. Moreover, let e be a frame. Then, the series $\sum_i \langle e_i, Te_i \rangle$ converges in norm to $\operatorname{tr} T$.

Proof. Assume $T = RS$ with $R, S \in \mathscr{L}^2(E_A)$ and let e be a frame. Then $\langle R^*, S \rangle_2$ by definition equals $\sum_i \langle R^* e_i, Se_i \rangle$, which converges in norm by Proposition 6.4.2. As $\langle R^* e_i, Se_i \rangle = \langle e_i, Te_i \rangle$, we have two expressions for $\operatorname{tr} T$: one independent of the decomposition $T = RS$ and one independent of the choice e of frame. The proposition follows. □

Corollary 6.5.4. Let χ be a character of A and let $T \in \mathscr{L}^1(E_A)$. Then, $\operatorname{tr} \chi_* T = \chi(\operatorname{tr} T)$.

Proof. Note that $\chi(\langle e_i, Te_i \rangle) = \langle \chi_* e_i, \chi_* T \chi_* e_i \rangle$. Since $\chi_* e_i$ is a frame of $\chi_* E_A$ the result then follows from Corollary 5.1.6. □

Corollary 6.5.5. For $T \in \mathscr{L}^1(E_A)$, $|\operatorname{tr} T| \leq \operatorname{tr}|T|$, as elements of A. In particular, if A is unital $|\operatorname{tr} T| \leq \operatorname{tr}|T| \leq \|T\|_1 1_A$.

Proof. For all $\chi \in \widehat{A}$ we have $\chi(|\operatorname{tr} T|) = |\operatorname{tr} \chi_* T| \leq \operatorname{tr}|\chi_* T| = \chi(\operatorname{tr}|T|)$ by the inequality $|\operatorname{tr} S| \leq \operatorname{tr}|S|$ on $\mathscr{L}^1(H)$ for Hilbert spaces H. As the characters separate A, the first statement follows immediately. The last statement follows from the inequality $a \leq \|a\| 1_A$ for positive elements of any unital C*-algebra. □

Now, we can finally show that the trace is cyclic. The standard approach is as follows:

Proposition 6.5.6. If $S, T \in \mathscr{L}(E_A)$ are such that $ST \in \mathscr{L}^1(E_A)$ and $TS \in \mathscr{L}^1(E_A)$, then $\operatorname{tr} ST = \operatorname{tr} TS$.

Proof. Consider the value of a character $\chi \in \widehat{A}$ on the difference $\operatorname{tr} ST - \operatorname{tr} TS \in A$ and use Corollary 6.5.4 above:

$$\begin{aligned}
\chi(\operatorname{tr} ST - \operatorname{tr} TS) &= \chi(\operatorname{tr}(ST)) - \chi(\operatorname{tr}(TS)) \\
&= \operatorname{tr}(\chi_*(ST)) - \operatorname{tr}(\chi_*(TS)) \\
&= \operatorname{tr}(\chi_*(S)\chi_*(T)) - \operatorname{tr}(\chi_*(T)\chi_*(S))
\end{aligned}$$

We may now use the tracial property of the trace on $\mathscr{L}^1(\chi_* E_A)$ (*cf.* [Sim05, Corollary 3.8]) and the fact that χ separates A to conclude the proof. □

Chapter 7

Applications of Schatten classes

We investigate several applications of the just-developed theory of Schatten classes, to wit: the Fredholm determinant, the operator zeta functions, and a new definition of *summability* of unbounded Kasparov (A,B)-cycles when B is commutative.

7.1 The Fredholm determinant

As a first application of the above theory of Schatten classes for Hilbert modules over unital abelian C^*-algebras, we consider the Fredholm determinant. Let A be a commutative C*-algebra and let E_A be a countably generated A-module.

Definition 7.1.1. Let $G(E_A) \subset \mathscr{L}(E_A)$ be the set of bounded, invertible endomorphisms of E_A of the form $\mathrm{id}_{E_A} + T$, where $T \in \mathscr{L}^1(E_A)$.

Note that $G(E_A)$ is a group under the multiplication of $\mathscr{L}(E_A)$ because we have

$$(\mathrm{id} + T)^{-1} = \mathrm{id} - T(\mathrm{id} + T)^{-1},$$

and $\mathscr{L}^1(E_A)$ is an ideal. We will define the Fredholm determinant, first on the standard module, and then by pullback by a frame transform on general countably generated Hilbert A-modules.

The following definition of the Fredholm determinant on a Hilbert space H is well-known. See e.g. [Sim05, Chapter 3] for a brief discussion in the context of Lidskii's theorem.

Definition 7.1.2. Let $T \in \mathscr{L}^1(H_\mathbb{C})$. Then, the *Fredholm determinant* of $\mathrm{id} + T$ is

$$\det(\mathrm{id} + T) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \operatorname{tr} \bigwedge^k T.$$

Recall that the series converges by the estimate $\| \bigwedge^k T\|_1 \le \|T\|_1^k / k!$.

Remark 7.1.3. We have $|\det(\mathrm{id} + T_1) - \det(\mathrm{id} + T_2)| \le \|T_1 - T_2\|_1 \exp(\|T_1\|_1 + \|T_2\|_1 + 1)$, cf. [Sim05, Theorem 3.4]. Thus, $T \mapsto \det(\mathrm{id} + T)$ is a continuous function on $\mathscr{L}^1(H)$.

The Fredholm determinant of $\mathrm{id} + T$ is invariant under conjugation of T by partial isometries that commute with T:

Lemma 7.1.4. If $u \colon H \to K$ is a partial isometry of Hilbert spaces, and $T \in \mathscr{L}^1(H)$ is such that $u^*uT = Tu^*u = T$, then $\mathrm{id} + uTu^* \in G(K_\mathbb{C})$ and in fact $\det(\mathrm{id}_K + uTu^*) = \det(\mathrm{id}_H + T)$.

Proof. Note that we have

$$(\mathrm{id} + uTu^*)(\mathrm{id} - uT(\mathrm{id} + T)^{-1}u^*) = (\mathrm{id} + uTu^*) - u(\mathrm{id} + T)T(\mathrm{id} + T)^{-1}u^* = \mathrm{id},$$

so that $\mathrm{id} + uTu^* \in G(K_\mathbb{C})$.

Next, note that $\operatorname{tr} \bigwedge^k uTu^* = \operatorname{tr} \bigwedge^k u^*uT = \operatorname{tr} \bigwedge^k T$ so that indeed

$$\det(\mathrm{id}_H + T) = \sum_{k=0}^{\infty} \operatorname{tr} \bigwedge^k T = \sum_{k=0}^{\infty} \operatorname{tr} \bigwedge^k uTu^* = \det(\mathrm{id}_K + uTu^*).$$

□

Proposition 7.1.5. If K is a separable Hilbert space equipped with a frame e and $T \in \mathscr{L}^1(K)$, then $\det(1 + \theta_e T \theta_e^*) = \det(1 + T)$ in terms of the corresponding frame transform $\theta_e \colon K \to l^2$.

Proof. Since $\theta_e^* \theta_e = \mathrm{id}_K$ commutes with T we can apply Lemma 7.1.4. □

Definition 7.1.6. Let E_A be a countably generated Hilbert C^*-module over a unital and abelian C*-algebra A. For $T \in \mathscr{L}^1(E_A)$, the Fredholm determinant $\det(\mathrm{id} + T)$ of $\mathrm{id} + T$ is the function on $\widehat{A}$ given by $\chi \mapsto \det(\chi_*(\mathrm{id} + T))$.

Proposition 7.1.7. Let A be unital and abelian as above. For $T \in \mathscr{L}^1(E_A)$, the Fredholm determinant lies in $A \equiv C(\widehat{A})$ and as such we have $\chi(\det(\mathrm{id}+T)) = \det(\chi_*(\mathrm{id}+T))$.

Proof. Let e be a frame of E_A. Note that $\det(\chi_*(\mathrm{id}+T)) = \det(\chi_*(\mathrm{id}+\theta_e T\theta_e^*))$ for all $\chi \in \widehat{A}$ by Proposition 7.1.5. That is, $\det(\mathrm{id}+T) = \det(\mathrm{id}+\theta_e T\theta_e^*)$. Now, $\theta_e T\theta_e^* \in \mathscr{L}^1(l^2(A))$ by Theorem 6.1.5. With Remark 7.1.3 we see that $\chi \mapsto \det(\mathrm{id}+\chi_* S)$ is continuous whenever $S \in \mathscr{L}^1(l^2(A))$. Thus, since A is unital (and thus $\widehat{A}$ compact) we find that $\det(\mathrm{id}+T) \in C(\widehat{A}) = A$. □

Proposition 7.1.8. Let $T \in C(X, \mathscr{L}^1(H))$ with $X = \widehat{A}$. Then we have

$$\bigwedge^k(T) \in C\left(X, \mathscr{L}^1\left(\bigwedge^k H\right)\right).$$

In particular, one has $\det(\mathrm{id}+zT) = \sum_{k\geq 0} z^k \operatorname{tr} \bigwedge^k(T)$, and $z \mapsto \det(\mathrm{id}+zT)$ is entire (as an A-valued function on $\mathbb{C}$).

Proof. Let $A, B \in \mathscr{L}^1(H)$ and note that $\bigwedge^{k+1}(A) - \bigwedge^{k+1}(B) = (\bigwedge^k(A) - \bigwedge^k(B)) \wedge A + \bigwedge^k(B) \wedge (A-B)$, which can be iterated to yield

$$\left\|\bigwedge^k(A) - \bigwedge^k(B)\right\|_1 \leq \|A-B\|_1 \sum_{m=0}^{k-1} \|A\|_1^m \|B\|_1^{k-1-m}.$$

As a consequence, we see that $\bigwedge^k(T) \in C\left(X, \mathscr{L}^1\left(\bigwedge^k H\right)\right)$ whenever $T \in C(X, \mathscr{L}^1(H))$. Moreover, we have the pointwise series expression

$$\det(\chi_*(\mathrm{id}+zT)) = \det(\mathrm{id}+z\,\chi_*T) = \sum_{k\geq 0} z^k \operatorname{tr} \bigwedge^k(\chi_*T)$$

as in [Sim05, Lemma 3.3]. Since $\operatorname{tr} \bigwedge^k(\chi_*T) \leq \left\|\chi_*T\right\|_1^k / k!$ the series $\sum_{k\geq 0} z^k \operatorname{tr} \bigwedge^k(T)$ in A converges absolutely for all $z \in \mathbb{C}$. This implies in particular that $z \mapsto \det(\mathrm{id}+zT)$ is entire. □

Remark 7.1.9. If $f \in A$ is invertible, then $\det(\mathrm{id}-f^{-1}T) = 0$ in A if and only if $\chi(f) = f(x)$ is a (nonzero) eigenvalue of χ_*T for all $\chi \in \widehat{A}$ (corresponding to the point $x \in X$).

Proposition 7.1.10. The Fredholm determinant is multiplicative in the sense that

$$\det(\mathrm{id}+T)(\mathrm{id}+S) = \det(\mathrm{id}+T)\det(\mathrm{id}+S).$$

Proof. This follows simply from the analogous property of the Fredholm determinant on Hilbert spaces, since

$$\begin{aligned}\chi(\det((\mathrm{id}+T)(\mathrm{id}+S))) &= \det(\chi_*((\mathrm{id}+T)(\mathrm{id}+S)))\\ &= \det(\chi_*(\mathrm{id}+T))\det(\chi_*(\mathrm{id}+S)))\\ &= \chi(\det(\mathrm{id}+T)\chi(\det(\mathrm{id}+S)),\end{aligned}$$

for all $\mathrm{id}+T$, $\mathrm{id}+S$ in $G(E_A)$ and all $\chi \in \widehat{A}$. □

Proposition 7.1.11. For $0 \leq |z| < \|T\|_1^{-1}$, the Fredholm determinant satisfies

$$\det(\mathrm{id}+zT) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \operatorname{tr} T^n\right),$$

and the series converges absolutely.

Proof. Absolute convergence follows from the fact that $\|\operatorname{tr} T^n\| \leq \|T^n\|_1 \leq \|T\|_1^n$, so that the sum of the norms of the summands is bounded by $\log(1+z\|T\|_1)$.

As the characters separate A, it will suffice to prove the formula for the case $E_A = H_{\mathbb{C}}$, where it is well-known. For $|z|\,\|T\|_1 < 1$, by absolute convergence of the trace and Lidskii's theorem,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \operatorname{tr} T^n &= \sum_{n=1}^{\infty} (-1)^{n+1} z^n \sum_k \frac{\lambda_k(T)^n}{n}\\ &= \sum_k \log(1+z\lambda_k(T)),\end{aligned}$$

so that the exponential of the right-hand side equals $\det(\mathrm{id}+zT)$ by Definition 7.1.2. □

Proposition 7.1.12. Let $T \in \mathscr{L}^1(E_A)$. Then $\mathrm{id}+T$ is invertible (that is, $\mathrm{id}+T \in G(E_A)$) if and only if $\det(\mathrm{id}+T) \in A$ is invertible.

Proof. By Proposition 7.1.10, $\det(\mathrm{id}+T)$ is invertible whenever $\mathrm{id}+T$ is.

For the converse statement, in view of Proposition 7.1.5 we may assume without loss of generality that $E_A = l^2(A) \simeq C(X,H)$ - where X is compact by the assumption that A be unital. So, if $\det(\mathrm{id}+T)$ is invertible then it follows that $\mathrm{id}+\chi_* T$ is invertible for all

χ by [Sim05, Theorem 3.5b)]. Moreover, as for $S_1, S_2 \in G(H)$ we have $\|S_1^{-1} - S_2^{-1}\|_1 \le \|S_1^{-1}\| \, \|S_2^{-1}\| \, \|S_1 - S_2\|_1$, the $B(H)$-valued map $\chi \mapsto (\mathrm{id} + \chi_* T)^{-1}$ lies in $C_b(X, B(H)))$ whenever it is bounded.

Now, for all eigenvalues $\lambda \in \sigma(\chi_* T)$ we have $\det(\mathrm{id} - \lambda^{-1}\chi_* T) = 0$ so that $\det(\mathrm{id} - \lambda^{-1} T)$ is not invertible in A. Thus, if there exist a sequence $\lambda((\chi_i)_* T) \in \sigma((\chi_i)_* T)$ converging (in $\mathbb{C}$) to -1, the element $\det(\mathrm{id} + T) = \lim_i \det(\mathrm{id} - \lambda(x_i)^{-1} T)$ is contained in the closed set consisting of the non-invertible elements of A, which contradicts the assumpion.

We conclude that there exists $\mu > 0$ with $\inf_n |1 + \lambda_n(\chi_* T)| > \mu$ uniformly for $\chi \in \widehat{A}$. In particular, $\|(\mathrm{id} + \chi_* T)^{-1}\| = \sup_n |1 + \lambda_n(\chi_* T)|^{-1} < 1/\mu$ for all x. We conclude that $\chi \mapsto (\mathrm{id} + \chi_* T)^{-1}$ is bounded and therefore continuous. □

Corollary 7.1.13. The Fredholm determinant is a homomorphism from the group $G(E_A)$ to the group of invertible elements of A.

In particular, the map det extends the (matrix) determinant homomorphism $K_1^{\mathrm{alg}}(A) = GL_\infty(A)/[GL_\infty(A), GL_\infty(A)] \to A$ to all of $G(l^2(A))$.

Remark 7.1.14. In [Alm73; Alm74] it was shown that the Fredholm determinant is the unique additive invariant of endomorphisms $T : E \to E$ on finitely generated projective A-modules. It is an interesting open question to see under which additional conditions this result extends to the countably generated Hilbert module context. Clearly, the Fredholm determinant discussed above gives rise to a additive map from a countably generated Hilbert A-module E equipped with a trace-class operator T to analytic A-valued functions.

7.2 The zeta function

As a second application we consider zeta functions associated to positive Schatten class operators on a Hilbert module. Again, A is a commutative C^*-algebra and we identify $\widehat{A} = X$ so that $A \cong C_0(X)$.

Definition 7.2.1. Let $0 \le T \le 1 \in \mathscr{L}^p(E_A)$, for $p \ge 1$. For $z \in \mathbb{C}$ with $\Re z > p$, define T^z using the continuous functional calculus in $\mathscr{L}(E_A)$. Then, the associated *zeta function* is the function on the complex half-plane $\Re z > p$ given by

$$\zeta(z, T) \overset{\mathrm{def}}{=} \operatorname{tr} T^z.$$

We will show that the zeta function is in fact *holomorphic* (in the sense of Banach space-valued holomorphic functions, see e.g. [Rud91, Definition 3.30]) on the defining half-plane.

First, let us recall that the individual (fiberwise) eigenvalues of positive compact operators are themselves continuous.

Lemma 7.2.2. Let $0 \leq T \in \mathcal{K}(E_A)$. For $k \geq 0$ and $\chi \in \widehat{A}$ be the character corresponding to the point $x \in X$, let $\{\lambda_k(x)\}_k$ be the eigenvalues of $\chi_* T$, in decreasing order with multiplicity. Then, the map $x \mapsto \lambda_k(x)$ is an element of A.

Proof. A proof can be found in [MT05, Theorem 6.4.2] (see also *op.cit.* Section 6.6). □

Remark 7.2.3. Note that it is important to know that both T is pointwise compact and that T is norm continuous. Dropping the latter assumption is fatal (see e.g. Example 5.1.21).

By self-adjointness and norm continuity, the full spectrum of T is continuous in norm topology, cf. [Kat95, Remark IV.3.3]. However, the spectral projections (or even the eigenvectors) can in general *not* be continuously extended over any open neighbourhood, even in the case where the module is finitely generated: see e.g. [Kad84]. *A fortiori*, continuous diagonalizability ('diagonability') is entirely out of the question in general.

The classical treatment of zeta functions of operators on Hilbert spaces is in terms of the Dirichlet series $\mathrm{tr}\, T^z = \sum_k \lambda_k^z$. The Jensen–Cahen theorem (see e.g. [HR64]) shows that these series converge uniformly on angular regions contained in the defining half-plane, so that the limit is in fact holomorphic. The theorem translates very well to the Hilbert module setting.

Lemma 7.2.4. Suppose that $0 \leq T \leq 1 \in \mathcal{L}^p(E_A)$. For $0 < \alpha < \pi/2$, denote the angular region $\{z \in \mathbb{C} \mid |\operatorname{Arg}(z - p)| \leq \alpha\}$ by C_α. Then, for all $\epsilon > 0$ there exists $m_0 \geq 0$ such that, for all $n \geq m \geq m_0$ and all $z \in C_\alpha$,

$$\left\| \sum_{k=m}^{n} \lambda_k^z \right\|_A < \epsilon.$$

Proof. The proof is based on [HR64, Theorem 2]. Consider the series $\sum_{k=m}^{n} \lambda_k^p \lambda_k^{z-p}$. Write $A(p,q) \stackrel{\text{def}}{=} \sum_{k=p}^{q} \lambda_k^p$ and $\Delta_{z,k} \stackrel{\text{def}}{=} \lambda_{k+1}^z - \lambda_k^z$. Then, by Abel's lemma on partial summation [Abe26], we have

$$\sum_{k=m}^{n} \lambda_k^z = \sum_{k=m}^{n-1} A(m,k)\Delta_{z,k} + A(m,n)\lambda_n^z$$

Now, because the series $\sum_{k=0}^{\infty} \lambda_k(x)^p$ converges pointwise to $\operatorname{tr} T^p$, Dini's theorem shows that $\sum_{k=0}^{\infty} \lambda_k^p$ converges in norm. In particular, for all $\epsilon > 0$ there exists m_0 such that $\|A(m,q)\| < \epsilon \cos\alpha$ for all $q \geq m \geq m_0$.

By [HR64, Lemma 2], we have $\Delta_{z,k} \leq |z|/p\Delta_{p,k}$. Thus, as $|z|/p \leq \sec\alpha$ throughout C_α, we have

$$\left\| \sum_{k=m}^{n} \lambda_k^z \right\| < \epsilon \left(\sum_{k=m}^{n-1} \Delta_{p,k} + \lambda_n^p \right) = \epsilon \lambda_n^p < \epsilon \left\| T^p \right\|.$$

□

As in the classical case, the Jensen–Cahen theorem paves the way for a *holomorphic* zeta function.

Theorem 7.2.5. Let $0 \leq T \leq 1 \in \mathscr{L}^p(E_A)$. Then the map $z \mapsto \zeta(z,T)$ is holomorphic on the half-plane $\mathbb{C}_{\Re z > p} = \{z \mid \Re z > p\}$. Moreover, for all compact subsets $K \subset \mathbb{C}_{\Re z > p}$, all $x \in X$ and all $\epsilon > 0$, there is a neighbourhood U of x on which

$$\sup_{z \in K} |\zeta(z,T)(y) - \zeta(z,T)(x)| < \epsilon$$

for all $y \in U$.

Proof. For the first statement we consider the A-valued function $\zeta_n(\cdot,T) : z \mapsto \sum_{k=0}^{n} \lambda_k^z$ on $\mathbb{C}_{\Re z > p}$. Recall that all bounded functionals $A \to \mathbb{C}$ decompose in four positive linear functionals. By the Riesz representation theorem, all such positive linear functionals are given by positive, finite, regular Borel measures μ on $\widehat{A}$ under the identification $\phi_\mu(f) \stackrel{\text{def}}{=} \int f \, d\mu$. In particular, we have $\phi_\mu(\lambda_k^z) = \int \lambda_k(x)^z d\mu(x)$. Now, if C is a contour around z_0 in $\mathbb{C}_{\Re z > p}$, the contour integral $\oint_C \lambda_k(x)^z$ vanishes for all $x \in X$. Note that $|\lambda_k^z| = \lambda_k^{\Re z}$ and so, by Fubini's theorem, $\oint_C \int \lambda_k(x)^z d\mu(x) = \int \left(\oint_C \lambda_k(x)^z \right) d\mu(x) = 0$. By Morera's theorem, we conclude that $\phi_\mu(\zeta_n)$ is holomorphic on $\mathbb{C}_{\Re z > p}$. Moreover, as $\zeta_n(\cdot,T) \to \zeta(\cdot,T)$ uniformly on compact subsets of $\mathbb{C}_{\Re z > p}$ by Lemma 7.2.4, the function $\phi_\mu(\zeta(\cdot,T))$ is holomorphic on $\mathbb{C}_{\Re z>p}$ as well. By [Rud91, Theorem 3.31], we may conclude that $\zeta(\cdot,T)$ is holomorphic, as an A-valued function, on the half-plane $\Re z > p$.

For the second statement, note that by Lemma 7.2.4, for all $\epsilon > 0$ there is, for all compact subsets $K \subset \mathbb{C}_{\Re z>p}$, some m_0 with $\|\zeta_m(z,T) - \zeta(z,T)\|_A < \epsilon$ for all $z \in K$ and all $m \geq m_0$.

Assume without loss of generality that $\lambda_k(x) > 0$ and pick a neighbourhood U of x on which $\lambda_k(y) > 0$, for all $k = 1,\dots,m$. For any $\epsilon_0 > 0$ let $V \subset U$ be such that $|\ln\lambda_k(x) - \ln\lambda_k(y)| < \epsilon_0$ for all $x,y \in V$. Then, $\lambda_k(y)^z = e^{z\ln\lambda_k(y)}$ for all $z \in K$, so that $|\lambda_k(x)^z - \lambda_k(y)^z| \leq |1-e^{zs}||\lambda_k^p(x)|$ for some $s \in \mathbb{C}$ with $|s| < \epsilon_0$. Moreover, $|1-e^{zs}| \leq |1-e^{\epsilon_1}| \leq \epsilon_1 e^{\epsilon_1}$, where $\epsilon_1 = \epsilon_0 \sup_{z\in K}|z|$.

If we now pick ϵ_0 such that $\epsilon_1 e^{\epsilon_1}|\lambda_k^p(x)| < \epsilon/m$, we conclude that for all $y \in V$ and all $z \in K$ we have $|\lambda_k(x)^z - \lambda_k(y)^z| < \epsilon/m$. Consequently, we find that $\|\zeta_m(\cdot,T)(x) - \zeta_m(\cdot,T)(y)\| < \epsilon$ as desired. □

Remark 7.2.6. It would be desirable to extend the previous Lemma and Theorem to the functions $\operatorname{tr} aT^z$, for $a \in \mathscr{L}(E_A)$. However, since the spectral projections of T are in general not even weakly continuous, this would be a nontrivial extension. We expect that a possibility for such an extension would be to investigate the functions

$$x \mapsto \operatorname{tr} p_k(x)a(x)p_k(x)/\operatorname{rank}(p_k(x)),$$

where $p_k(x)$ is the spectral projection on the eigenspace belonging to the eigenvalue $\lambda_k(x)$ of $T(x)$, and then use these expressions as coefficients in the continuous Dirichlet series.

The next step in the classical case would be to show that certain operators have zeta functions that can be continued meromorphically to all of $\mathbb{C}$, with a discrete set of poles. The residues at these poles then yields interesting information about the operator T. For instance, if T is the bounded transform $(1+D^2)^{-1/2}$ of a pseudodifferential operator these residues give geometric information about the pertinent background manifold (for more details, *cf.* [BGV04]). For this reason, it would be very desirable to have a reasonable criterion under which our zeta function of operators on Hilbert C^*-modules can be continued meromorphically to all of $\mathbb{C}$, but further research in that direction is beyond the scope of the present work.

7.3 Summability of unbounded Kasparov cycles

To motivate investigating *summability* of unbounded KK-cycles, we will recall some foundational results that motivate the notion of summability of (the special case of) spectral triples.

Summability and spectral triples

We will aim to generalize the following property [GVF01, Definition 10.8] of certain spectral triples in noncommutative geometry.

Definition 7.3.1. A spectral triple (A,H,D) is said to be *finitely summable* if there exists $p \in \mathbb{R}_{\geq 0}$ such that $a(1+D^2)^{-p/2}$ is in $\mathscr{L}^1(H)$ for all $a \in A$.

By the Jensen-Cahen theorem, for such spectral triples the zeta function $\zeta(z,a) = \operatorname{tr} a(1+D^2)^{-z/2}$ is holomorphic on the half-plane $\Re z > p$, for any $a \in A$.

Example 7.3.2. Let M be a compact Riemannian manifold of dimension m and D be a Dirac-type operator on a smooth hermitian bundle $\mathcal{S}$ over M. Then, by the asymptotic expansion [BGV04, Theorem 2.30] of the heat kernel $k_t(x,y)$ of D^2, there exists smooth families Υ_i of morphisms of $\mathcal{S}$ such that, for y sufficiently close to x,

$$k_t(x,y) \sim (4\pi t)^{-m/2} e^{-d(x,y)^2/4t} \sum_{i=0}^{\infty} t^i \Upsilon_i(x,y),$$

in the sense that the latter is an asymptotic expansion of the former as $t \to 0$. In particular, for endomorphisms h of $\mathcal{S}$ this provides us with the expansion

$$\operatorname{tr} h e^{-tD^2} \sim (4\pi t)^{-m/2} \sum_{i=0}^{\infty} t^i \int_M \operatorname{tr}_{\mathcal{S}_x}(h\Upsilon_i(x,x))\, d\operatorname{vol}(x).$$

Under a Mellin transform, this proves that (A,H,D) is in fact m^+-summable, and that moreover $\zeta(a,\cdot)$ extends holomorphically to $\mathbb{C} \setminus \{m-2j \mid j \geq 0\}$. Moreover,

$$\operatorname{res}_{z=j-m} \zeta(z,a)\Gamma(z) = \int_M a(x) \operatorname{tr}_{\mathcal{S}_x}(h\Upsilon_j(x,x))\, d\operatorname{vol}(x),$$

where in particular $\operatorname{tr}_{\mathcal{S}_x} \Upsilon(x,x) = \operatorname{rk} \mathcal{S}$ so that

$$\operatorname{res}_{z=-m} \zeta(z,a)\Gamma(z) = \int_M a(x)\, d\operatorname{vol}(x)$$

and $(4\pi)^{m/2}\,\mathrm{tr}_{\mathcal{S}_x}\,\Psi_1(x,x) = s(x)/6 - \frac{1}{4}\int_M s(y)\,d\,\mathrm{vol}(y)$, where s denotes the scalar curvature of M.

The previous example shows at the very least that the residues of the zeta function (or, in analytically less tractable situations, the Dixmier traces) associated to D are of geometric interest (in the sense of *metric* geometry), and that moreover the infinimal degree p of summability itself already contains some geometric information. A further application of the relevance of these residues to metric geometry in the truly noncommutative setting can be seen in (the literature based on) [CC97].

An important motivation for the development of noncommutative geometry comes from the fact that it is a natural framework in which to consider vast generalizations of the Atiyah-Singer index theorem. The topic of zeta residues (or Dixmier traces) and summability interlocks with that story from the very beginning, so it may be informative to dwell upon it for a while.

The Chern character and zeta residues in differential topology

Let (A,H,F,γ) be an even Fredholm module that is $(n+1)$-summable, in the sense that $[F,a] \in \mathscr{L}^{n+1}(H)$ for all a in (a dense subalgebra[1] $\mathscr{A}$ of) A. To simplify the exposition, we assumed that the module is even. Then, the Chern character

$$\mathrm{Ch}_*(F)\colon\ (a_0,\dots,a_n) \mapsto \frac{\Gamma(\frac{n}{2}+1)}{2n!}\,\mathrm{tr}\left(\gamma F[F,a_0]\cdots[F,a_n]\right)$$

of the module defines a class in the periodic cyclic cohomology of $\mathscr{A}$, as in [Con94, Definition 4.1.β.3]. Under Connes' pairing [Con85, Corollary II.2.17] between cyclic cohomology and K-theory, this Chern character implements Atiyah's index map on the K-theory of $\mathscr{A}$ [Con94, Proposition 4.1.γ.4].

This, however, is far from sufficient to encapsulate the statement of the Atiyah-Singer index theorem. On a smooth vector bundle $E \to M$, if F is an elliptic psuedodifferential operator of order 0, it is not at all clear a priori how to interpret the trace above in terms of *local* quantities associated to F.

Now, the definition of a spectral triple (A,H,D) is precisely tailored to ensure that its *bounded transform* $(A,H,F_D \overset{\mathrm{def}}{=} D(1+D^2)^{-1/2})$ is a Fredholm module, and moreover we have $[F_D,a] \in \mathscr{L}^p(H)$ for all $a \in A$ whenever (A,H,D) is p-summable.

[1] Which is assumed to be closed under the holomorphic functional calculus, so that the K-theory of $\mathscr{A}$ agrees with that of A [Con94, Proposition 4.1.β.7]

Atiyah and Bott showed by a simple calculation that, for a graded spectral triple $D = D_- \oplus D_+$,

$$\operatorname{index} D_+ = \operatorname{res}_{z=0} \Gamma(z) \operatorname{tr} \gamma (1 + D^2)^{-z}.$$

In fact, the invariant on the left-hand side can be vastly refined. In this language, Connes and Moscovici showed [CM95] that there exist universal constants c_{pq} such that the cycle

$$\Phi_D(a_0, \dots, a_p) \stackrel{\text{def}}{=} \sum_{k_j \geq 0} c_{pk} \operatorname{res}_{z=0} \operatorname{tr} \left(\gamma a_0 [D, a_0]^{(k_1)} \cdots [D, a_p]^{(k_p)} |D|^{z-p-2\sum_j k_j} \right)$$

is cohomologous in (periodic) cyclic cohomology to $\mathrm{Ch}_*(F_D)$. Here $x^{(k)}$ denotes the kth iterated commutator $[D^2, \dots, [D^2, x]]$ with D^2.

All this goes to show that there is an important and natural role to play for zeta functions in the context of spectral triples, even if they are only viewed *topologically*, that is, as unbounded representatives of the associated Fredholm modules. As (unbounded) KK-cycles are the natural bivariant generalization of (spectral tripes) Fredholm modules, it is to be expected that traces and zeta residues will be instrumental in further developing KK-theory in the framework of noncommutative geometry.

Summability of right-commutative unbounded Kasparov cycles

We now apply the theory of Schatten classes on Hilbert modules to arrive at a notion of summability for unbounded Kasparov cycles over a commutative C^*-algebra. This notion is supposed to generalize summability for spectral triples (as unbounded Kasparov cycles over $\mathbb{C}$) as just defined. We refer to [BJ83; Mes14; KL12; MR16] for all relevant notions of unbounded Kasparov cycles, external and internal Kasparov product, and to [KS18; SV19] for the specific application to Riemannian submersions and immersions to be discussed below.

In order to set the notation, for A, B two C^*-algebras we let $({}_A E_B, S)$ be an *unbounded Kasparov $A - B$ cycle*, consisting of

- E_B is a (graded) Hilbert B-module
- A is represented on E_B by adjointable (even) operators.
- S is a regular, self-adjoint (and odd) operator, densely defined on $\operatorname{dom}(S) \subset E_A$.
- $a(1 + S^2)^{-1/2}$ is in $\mathscr{K}(E_B)$ for all $a \in A$.

- There exists a dense subalgebra $\mathscr{A} \subset A$ that preserves $\mathrm{dom}(S)$ and is such that for all $a \in \mathscr{A}$, the commutator $[S,a]$ extends to an adjointable operator on E_B.

Definition 7.3.3. Let B be a unital, commutative C*-algebra and let $({}_AE_B, S)$ be an unbounded Kasparov $A-B$ cycle. We say that $({}_AE_B, S)$ is *p-summable* if

$$(1+S^2)^{-\frac{1}{2}} \in \mathscr{L}^q(E_B).$$

For simplicity, we restrict to the case where A is unital. See [Car+14, Section 2] for an understanding of the nonunital case.

Example 7.3.4. If (A,H,D) is a p-summable spectral triple (*cf.* [GVF01, Definition 10.8]), then it is an unbounded p-summable Kasparov $(A,\mathbb{C})$ cycle.

By the very definition of the Schatten classes, the localization of an unbounded p-summable Kasparov (A,B)-cycle along a character of B yields a p-summable spectral triple. In the other direction, it is not true that if all such localized spectral triples are p-summable, then the original unbounded KK-cycle is p-summable: one needs the fiberwise $\mathscr{L}^p$-norms of the resolvents to be continuous over the base.

There is, however, a reasonably simple criterion that is sufficient for a *pointwise p-summable* spectral triple to be p-summable. All our examples will be of this type.

Proposition 7.3.5. Let B be a commutative C*-algebra and let $({}_AE_B, D)$ be an unbounded Kasparov (A,B)-cycle that is boundedly pointwise p-summable, in the sense that

$$\sup_{x\in\widehat{B}} \|x_*(1+D^2)^{-1/2}\|_p < \infty.$$

Assume moreover that $x \mapsto \|x_* a\|$ lies in B (as opposed to $M(B)$) for all $a \in A$.

Then, if there exists for each $x \in \widehat{B}$ a neighbourhood U and for each $y \in U$ a (not necessarily surjective) isometry $\tau_y : E_y \to E_x$ such that, with $A_y \stackrel{\text{def}}{=} D_x - \tau_y D_y \tau_y^*$,

- $\mathrm{dom}\, A_y$ is dense in E_x
- $A_y(D_x+i)^{-1}$ extends to a bounded operator on E_x,
- $\left\| A_y(D_x+i)^{-1} \right\| \to 0$ as $y \to x$,

then $({}_AE_B, D)$ is p-summable.

Proof. Write $T = \tau_x D_x \tau_x^*$ and $S = \tau_y D_y \tau_y^*$. As $\operatorname{dom} T \cap \operatorname{dom} A_y = \operatorname{dom} A_y$ is dense in H, the fact that A_y is T-bounded implies that the domain of the closure of A contains $\operatorname{dom} T$. Under the last assumption, let $0 < \epsilon < 1$. There exists a neighbourhood $V \subset U$ of $x \in \widehat{B}$ such that $\left\| A_y(T+i)^{-1} \right\| < \epsilon$, so that $1 + A_y(T+i)^{-1}$ has an inverse that satisfies $\left\| (1 + A_y(T+i)^{-1})^{-1} \right\| \le (1-\epsilon)^{-1}$. Thus, the operator $S + i = T + A_y + i$ is (selfadjoint and) invertible, and its inverse satisfies $(S+i)^{-1} = (T+i)^{-1}(1 + A_y(T+i)^{-1})^{-1}$, so that in particular $\left\| (S+i)^{-1} \right\|_q \le (1-\epsilon)^{-1} \left\| (T+i)^{-1} \right\|_q$ and $\left\| (T+i)^{-1} \right\|_q \le (1 + \epsilon) \left\| (S+i)^{-1} \right\|_q$. As $\left\| (\tau_y D_y \tau_y^* + i)^{-1} \right\|_q = \left\| (1 + \tau_y D_y^2 \tau_y^*)^{-1/2} \right\|_q$ for all y, the continuity of the trace follows from the pointwise existence. That the trace of $a(1+D^2)^{-q/2}$ then vanishes at infinity for $a \in A$ follows from the fact that $\operatorname{tr} x_* a(1+D^2)^{-q/2} \le \|x_* a\| \left\| x_*(1+D^2)^{-1/2} \right\|_q$. $\square$

Some remarks are in order here.

- The maps τ_b are to be thought of as a slightly weaker version of parallel transport: we need not assume that the bundle E_B be locally trivial.
- We have assumed here that $x_*(1+D^2)^{-q/2}$, instead of $x_* a(1+D^2)^{-q/2}$, is in the trace class; this corresponds to the 'compact fiber' assumption in the case of Riemannian submersions, below.

Example 7.3.6 (Differentiable frames). Let e be a frame such that $e_i \in \operatorname{dom}(D)$ for all i, so that $\operatorname{dom}(\theta_e D \theta_e^*)$ contains all finite sequences in B. Then, for all $y \in \widehat{B}$, $y(e)$ is a frame of E_y so that $\theta_{x(e)} \theta_{y(e)}^* : E_y \to E_x$ is an isometry such that $A_y = D_x - \tau_y D_y \tau_y^*$ is densely defined and symmetric. Now, the condition in the proposition amounts to $A_y(1 + D_x^2)^{-1/2} \in C_b^{\mathrm{norm}}(U, B(H))$. That is, in this setup the question is whether the relative perturbations of D_x are *norm* continuous.

Compatibility with the unbounded Kasparov product

We expect the summability of (a reasonably large class of) unbounded KK-cycles to correspond to some *fiber dimension* and, therefore, be *additive* under the internal and external unbounded Kasparov product.

Apart from the examples presented below, a very rough general analysis goes as follows. Let $({}_A E_B, D)$ be an unbounded KK-cycle. Consider the localizations $\rho_y : A \to \mathscr{L}\left(E_y\right)$

of the representation of A by elements $y \in \widehat{B}$. These will in general not be faithful, even if the representation $\rho\colon A \to \mathscr{L}(E)$ is. Instead, each ρ_y will define a closed ideal $I_y \overset{\text{def}}{=} \ker \rho_y$ of A, so that there is a closed subset F_y of $\widehat{A}$ with $I_y = \{f \in A \colon f(x) = 0 \text{ for all } x \in F_y\}$ and so ρ_y is in fact a faithful representation of $C_0(F_y) \simeq A/I_y$.

If we assume that the $C_0(F_y)$-spectral triples (y_*E, y_*D) are given by first-order elliptic operators on some vector bundle over F_y, then by the analysis of e.g. [See67], they are $\dim F_y$-summable. That is to say, under this assumption of local ellipticity and under for instance one of the assumptions of Proposition 7.3.5, the summability of (E, D) is equal to $\sup_{y \in \widehat{B}} \dim F_y$.

As for additivity of the fiber dimension under the unbounded Kasparov product: under the interior product of modules ${}_AE_B, {}_BE'_C$, the subspaces F_y dual to $\ker \rho_y$ and F'_z dual to $\ker \rho'_z$ satisfy $\ker(\rho \otimes_B \rho')_z = \cap_{y \in \complement F'_z} \ker \rho_y$ so that the closed subset of $\widehat{A}$ corresponding to this ideal is $\bigcup_{y \in F'_z} F_y$, which has Hausdorff dimension bounded from above by $\dim F'_z + \sup_y \dim F_y$.

Remark 7.3.7. The rather strong assumption of ellipticity of the localizations made here fails very much in general, but it may fail in a way that does not impact the rest of the story. For instance, if we consider the general construction of (bounded representatives) of the shriek map in [CS84, Definition 2.1], then we see that we might end up with a *vertically* elliptic operator over the neighbourhood $\{x \mid d(f(x), y) < \epsilon\}$ of the 'true' fiber $f^{-1}(y)$. Due to this vertical ellipticity, however, the principal symbols constructed there still correspond (to the bounded transforms of) $\dim f^{-1}(y)$-summable operators.

There is a large class of nontrivial examples associated to pseudodifferential operators, which deserves a somewhat more detailed treatment.

Example: Riemannian submersions

In [KS18] the factorization of the Dirac operator D_Y on Y in terms of a vertical operator S and the Dirac operator D_X on X was studied for a Riemannian submersion $Y \to X$ of compact spin^c manifolds (more general proper Riemannian submersions were considered in [KS17; Dun18; Dun20]).

We let $L^2(\mathcal{S}_X)$ and $L^2(\mathcal{S}_Y)$ denote the Hilbert space completions of the spinor modules $\Gamma^\infty(\mathcal{S}_X)$ and $\Gamma^\infty(\mathcal{S}_Y)$, respectively. Based on a certain $C^\infty(Y)$-module of smooth sections of the vertical spinor bundle $\mathcal{S}_V$ one then defines a Hilbert C^*-module $E_{C(X)}$

between $C(Y)$ and $C(X)$, together with a self-adjoint and regular unbounded operator D_V on E, such that

$$L^2(\mathcal{S}_Y) \cong E\widehat{\otimes}_{C(X)} L^2(\mathcal{S}_X)$$

and in such a way that the operator D_Y corresponds to the tensor sum $D_V \otimes \gamma_E + 1 \otimes_\nabla D_X$ for some metric connection ∇ on $E_{C(X)}$ and the grading operator γ_E on E (up to an explicit error term related to the curvature).

Let us analyze here the summability aspects of the operator $D_V : \mathrm{dom}(D_V) \to E$. The main property that we will use below is that D_V is the closure of a so-called *vertically elliptic operator* $\mathscr{D} : \Gamma^\infty(\mathcal{S}_V) \to \Gamma^\infty(\mathcal{S}_V)$. This means that for all $f \in C^\infty(Y)$, $[\mathscr{D}, f]$ is an endomorphism of $\Gamma^\infty(\mathcal{S}_V)$ that is invertible at all points where $df|_{\ker d\pi}$ is nonzero (see [KS18; KS17] for more details). In fact, this allows one to prove [KS17, Theorem 3] that the pair (E, D_V) is an unbounded Kasparov $C(Y) - C(X)$ cycle.

As far as summability is concerned, note that the restrictions $\chi^* \mathscr{D} = \mathscr{D}_x$ of $\mathscr{D}$ to the fibers of $x \in X$ (for the character $\chi : C(X) \to \mathbb{C}$) are elliptic Dirac type operators. Since the dimension of the fibers is constant and equal to $\dim F$ for the typical fiber F, one has $\left\| (1 + D_x^2)^{-1/2} \right\|_p < \infty$ for all $p > \dim F$ [Con96] (*cf.* [GVF01, Theorem 11.1]). The question, then, is whether this pointwise trace is continuous.

Proposition 7.3.8. The $\mathscr{L}^p$-norm of $(1 + \mathscr{D}_x^2)^{-1/2}$ defines (as x varies over X) a continuous function on X for any $p > \dim F$. Consequently, the unbounded Kasparov $C(Y) - C(X)$ cycle (E, D_V) is p-summable for all $p > \dim F$.

Proof (based on [KS17, Section 2.3]). For simplicity of exposition, we will assume that the bundle $\mathcal{S}_Y \to Y$ is locally trivial over X; that is, we assume that around each point $x_0 \in X$ there exists a neighbourhood $U \subset X$ such that 1) $\pi^{-1}(U)$ is diffeomorphic to $U \times F$, and 2) the bundle $\mathcal{S}_Y \to Y$ can be smoothly and unitarily trivialized over U. If we pull sections of $\mathcal{S}_Y$ over U back through this trivialization, we obtain a family of Dirac-type differential operators $\{\mathscr{D}_x\}$ on the trivial bundle $\mathbb{C}^k$ over the compact Riemannian manifold F.

In particular, these operators can be written as

$$\mathscr{D}_x = \sum_{j=1}^{\dim F} A_j(x,z) \frac{\partial}{\partial z_j} + B(x,z)$$

with A_j, B symmetric matrix-valued smooth functions on $U \times F$ and A_j invertible. Thus, there are smooth families $Z_{x,x'}(z) = A_j(x', z) A_j^{-1}(x,z)$, $W_{x,x'}(z) = B(x', z) - Z_{x,x'} B(x,z)$

of matrices such that $\mathscr{D}_{x'} = Z_{x,x'}\mathscr{D}_x + W_{x,x'}$, with in particular $\lim_{x'\to x}(\mathrm{id} - Z_{x,x'}) = 0 = \lim_{x'\to x} W_{x,x'}$.

Now denote the closure of $\mathscr{D}_x$ by D_x. Note that D_x is selfadjoint by compactness of F. Moreover, because $\mathscr{D}_x$ is an elliptic differential operator of order 1, the resolvent $(D_x + i)^{-1}$ lies in $\mathscr{L}^p\left(L^2(F, \mathbb{C}^k)\right)$ for all $p > \dim F$.

Then, $D_x(D_x + i)^{-1}$ being bounded, we see that $\lim_{x'\to x} \|(D_{x'} - D_x)(D_x + i)^{-1}\| = 0$. Now, apply Proposition 7.3.5: that is, by the resolvent identity, we conclude that

$$\lim_{x'\to x} \|(D_x + i)^{-1} - (D_{x'} + i)^{-1}\|_p = 0$$

for all $p > \dim F$, and so $(1 + D^2)^{-1/2} \in \mathscr{L}^p(E)$ if and only if $p > \dim F$. □

Note that when we combine this result with the factorization result [KS18] of D_Y in terms of D_V and D_X we obtain for Riemannian submersions of spin^c manifolds the desired additivity of summability for the unbounded interior Kasparov product.

Example: Embedding spheres in Euclidean space

We consider a special class of immersions, given by the embedding of spheres S^n in Euclidean space $\mathbb{R}^{n+1}$. This is based on [CS84; SV19; Ver19]. As in [CS84] the embedding $S^n \to \mathbb{R}^{n+1}$ gives rise to an immersion class in KK-theory. For spheres, the unbounded representative is given by the module $C_0(S^n \times (-\epsilon, \epsilon))$ based on a normal neighborhood of $S^n \subset \mathbb{R}^{n+1}$, equipped with the regular self-adjoint operator S given by the multiplication operator with a suitable function $f : (-\epsilon, \epsilon) \to \mathbb{R}$. For convenience, we will take S to be multiplication by the function

$$f(s) = \frac{\pi}{2\epsilon}\tan\left(\frac{\pi s}{2\epsilon}\right); \qquad (s \in (-\epsilon, \epsilon)).$$

Since $(i + f)^{-1}$ is clearly a C_0-function on $S^n \times (-\epsilon, \epsilon)$, we find as in [SV19, Lemma 2.3] that $(i + S)^{-1}$ is a compact operator on the Hilbert module $C_0(S^n \times (-\epsilon, \epsilon))$ and so forms an unbounded Kasparov $C(S^n) - C_0(\mathbb{R}^{n+1})$ cycle.

Proposition 7.3.9. We have $(1 + S^2)^{-1/2} \in \mathscr{L}^p(C_0(S^n \times (-\epsilon, \epsilon)))$ for any $p > 0$. Hence the unbounded Kasparov $C(S^n) - C_0(\mathbb{R}^{n+1})$ is p-summable for all $p > 0$.

Proof. For any locally compact Hausdorff space X, the pointwise localizations of the Hilbert $C_0(X)$-module $C_0(X)$ are one-dimensional, so that the pointwise $\mathscr{L}^p$-norm of any $g \in \mathscr{L}(C_0(X)) = C_b(X)$ is given by pointwise evaluation of $|g|$. Hence, $\mathrm{tr}(1+S^2)^{-p/2} = (1+f^2)^{-p/2}$, which lies in C_0 for all $p > 0$. ☐

Again, this is a confirmation of additivity for summability under the unbounded interior Kasparov product. Indeed, in [SV19] it was shown that D_{S^n} can be related to the immersion class defined by S as above and $D_{\mathbb{R}^{n+1}}$ in the following way. Namely, the unbounded interior product of S and $D_{\mathbb{R}^{n+1}}$ is equal to the unbounded interior product of D_{S^n} with a so-called index cycle T. The latter represents the identity at the bounded level but is in fact a p-summable Kasparov cycle for all $p > 1$.

Proposition 7.3.10. The selfadjoint closure T of the operator

$$T_0 = \begin{pmatrix} 0 & -i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix}$$

on $C_c^\infty((-\epsilon,\epsilon),\mathbb{C}^2)$ is p-summable for all $p > 1$.

Proof. As in [SV19, Lemma 2.11], for $|\lambda| > \frac{\pi}{2\epsilon}$ we have $\Delta_\epsilon + 1 < T^2 + \lambda^2 + 1$, where Δ_ϵ is the closure of the Dirichlet Laplacian on $C_c^\infty((-\epsilon,\epsilon))^{\oplus 2}$. On the other hand, if $\Delta_{\epsilon/2}$ is the closure of the Dirichlet Laplacian on $C_c^\infty((-\epsilon/2,\epsilon/2))^{\oplus 2}$ and $c = f^2(\epsilon/2) + |f'(\epsilon/2)|$, then $\|T^2 - \Delta_{\epsilon/2}|_{L^2((-\epsilon/2,\epsilon/2))^{\oplus 2}}\| = c < \infty$ so that, by the min-max principle, the singular values $\sigma_n(T^2)$ are bounded from above by $\sigma_n(\Delta_{\epsilon/2}) + c$. Thus, one has $\|(\Delta_{\epsilon/2} + \lambda^2 + c + 1)^{-1}\|_p \leq \|(T^2 + \lambda^2 + 1)^{-1}\|_p \leq \|(\Delta + 1)^{-1}\|_p$ and so $(T \pm \lambda i)^{-1} \in \mathscr{L}^p$ if and only if $p > 1$. ☐

As such, the summability of $D_{\mathbb{R}^{n+1}}$ plus that of the immersion cycle (*i.e.* 0^+) indeed coincides with the summability of D_{S^n} plus that of the index cycle.

Example: Actions of $\mathbb{Z}$

Consider the standard $C(X)$-module $E = L^2(S^1) \otimes_{\mathbb{C}} C(X)$ equipped with the 'Dirac' operator

$$D = D_{S^1} \otimes 1.$$

Then $\mathrm{tr}\,\chi_* |D|^{-p} = \mathrm{tr}\,|D_{S^1}|^{-p} < \infty$ for all characters χ and $p > 1$ so that we have $|D|^{-1} \in \mathscr{L}^p(E)$ for $p > 1$.

As an example of an unbounded Kasparov cycle, consider a homeomorphism on X and consider the action $n \cdot f \overset{\mathrm{def}}{=} f \circ \phi^n$ of $\mathbb{Z}$ on $C(X)$. Let $C(X) \rtimes_\phi \mathbb{Z}$ be the corresponding full crossed product C*-algebra. Consider the unitary $U \in \mathscr{L}(E)$ given by $U = S \otimes 1$, where S is multiplication by $\theta \mapsto e^{i\pi\theta}$ on $L^2(S^1)$. Then, the map ρ defined on finite sums in $C(X) \rtimes_\phi \mathbb{Z}$ by

$$\rho\colon \sum_k f_k u_k \mapsto \sum_k f_k U^k,$$

where the left action of $C(X)$ is just given by pointwise multiplication, extends by universality to a representation of $C(X) \rtimes_\phi \mathbb{Z}$. Moreover, $[D, \sum_k f_k U^k] = \sum_k k f_k U^k$ because $[D_{S^1}, S^k] = kS$, so that there is a dense subalgebra of $C(X) \rtimes_\phi \mathbb{Z}$ with $[D,a]$ bounded. We conclude that $(E_{C(X)}, D)$ is an unbounded p-summable Kasparov $C(X) \rtimes_\phi \mathbb{Z} - C(X)$ cycle for all $p > 1$.

In particular, one has

$$\zeta_D(f u_k, z)(x) = \operatorname{tr} \chi_*(f U^k D^{-z}) = f(x) \operatorname{tr}_{L^2(S^1)} S^k |D_{S^1}|^{-z},$$

so that $\zeta_D(\sum_k f_k u_k, z)$ extends meromorphically to $\mathbb{C} \setminus \{1\}$ and in fact

$$\operatorname{res}_{z=1} \zeta_D(\sum_k f_k u_k, z) \propto f_0.$$

Summability and the exterior Kasparov product

One of the key results in [BJ83] was an explicit and linear formula for the external Kasparov product. More precisely, they showed that two unbounded Kasparov cycles (restricting to the even-odd case for simplicity) (E_B, γ, S) and (F_C, T) can be combined into an *external product* unbounded KK-cycle over the minimal tensor product $B \otimes \mathbb{C}$:

$$((E \otimes F)_{B\otimes C}, S \otimes 1 + \gamma \otimes T).$$

For spectral triples this can be understood as the direct product of the corresponding (noncommutative) spaces. In any case, it is desirable to have an additive property of summability for this external product in the case of a commutative base.

Lemma 7.3.11. If a, b are positive, (resolvent) commuting, regular operators on a Hilbert C*-module, then for $p, q > 0$ one has

$$(1 + a + b)^{-p-q} \le (1+a)^{-p/2}(1+b)^{-q}(1+a)^{-p/2}$$

Proof. By positivity of a, b, we have $(b+1)^{-1} \ge (a+b+1)^{-1} \le (a+1)^{-1}$, and by commutativity of the C*-algebra generated by the resolvents of a, b, we have $(a+b+1)^{-p-q} \le (a+1)^{-p/2}(a+b+1)^{-q}(a+1)^{-p/2} \le (a+1)^{-p/2}(b+1)^{-q}(a+1)^{-p/2}$. □

Corollary 7.3.12. The summability of unbounded Kasparov modules (over commutative C^*-algebras) is additive under the *exterior* product.

Proof. The corresponding selfadjoint operators $(S \otimes 1)$ and $(\gamma \otimes T)$ on $E \otimes F$ anticommute, and the actions of B, C commute. Thus, we have $|S \otimes 1 + \gamma \otimes T|^2 = |S \otimes 1|^2 + |\gamma \otimes T|^2$, whose summands commute. Moreover, the exterior product $\{e_i \otimes f_j\}_{ij}$ of frames is a frame. We conclude, with the Lemma, that $|S \otimes 1 + 1 \otimes T + i|^{-p-q} \leq |S \otimes 1 + i|^{-p} |1 \otimes T + i|^{-q}$, and the $B \otimes C$-valued trace of the latter is just the tensor product of the traces of $|S + i|^{-p}$ and $|T + i|^{-q}$. □

Summability and the interior Kasparov product

Of course, the real challenge is to establish the compatibility of summability with the *internal* unbounded Kasparov product. Clearly, Lemma 7.3.11 is then not sufficient. We leave its (challenging) extension to further research, but provide a very first step if only to exhibit the easier part.

The following Lemma would appear to be relevant. The operators s and t in its statement should correspond to the pth and qth (absolute) powers of the respective resolvents.

Lemma 7.3.13. Let B, C be separable, commutative C*-algebras and let E_B, F_C be countably generated Hilbert C*-modules, and let e be any frame of E_B. Then, for all $s \in \mathscr{L}^2(E_B)$ and all $0 \leq t \in \mathscr{L}^1(F_C)$, one has

$$(s^* \otimes 1)(1 \otimes_e t)(s \otimes 1) \in \mathscr{L}^1(E_B \otimes_B F_C),$$

where $\otimes_e$ denotes the product along the Grassman connection associated to e,

$$(1 \otimes_e t)(x \otimes_B y) \stackrel{\text{def}}{=} \sum_i e_i \otimes_B \langle e_i, x \rangle \, t y.$$

Proof. Choose a frame f_j of F_C, so that $\eta_{ij} \stackrel{\text{def}}{=} e_i \otimes_B f_j$ is a frame of $E_B \otimes_B F_C$. Because

$$\left\langle x \otimes y, (1 \otimes_e t) x \otimes y \right\rangle = \sum_i \left\langle y, \langle x, e_i \rangle \, t \, \langle e_i, x \rangle \, y \right\rangle \geq 0,$$

the product operator is positive so that the statement of the Lemma is equivalent to the convergence of its trace along a frame of $E_B \otimes_B F_C$. Now,

$$\left\langle \eta_{ij}, (s^* \otimes 1)(1 \otimes_\nabla t)(s \otimes 1) \eta_{ij} \right\rangle = \sum_k \left\langle f_j, \langle e_i, s e_k \rangle \, t \, \langle s e_k, e_i \rangle f_j \right\rangle.$$

Write $s_{ik} \overset{\text{def}}{=} \langle e_i, se_k\rangle$. For finite subsets $F \subset \mathbb{N}^2$, one has $\sum_{ik\in F} s_{ik}^* t s_{ik} \in \mathscr{L}^1(F_C)$, and then $\left\|\operatorname{tr}\sum_{ik\in F} s_{ik}^* t s_{ik}\right\| = \left\|\operatorname{tr} t \sum_{ik\in F} s_{ik}^* s_{ik}\right\| \leq \|t\|_1 \left\|\sum_{ik\in F} s_{ik}^* s_{ik}\right\|_B$. As the series $\sum_{ik} s_{ik}^* s_{ik}$ and $\sum_k s_{ik}^* s_{ik}$ are Cauchy (with limits $\operatorname{tr} s^* s \in B$ and $\langle se_i, se_i\rangle \in B$, respectively), we conclude that $\sum_{ik} s_{ik}^* t s_{ik}$ converges in $\mathscr{L}^1(F_C)$, so that the statement follows. □

The rather extreme assumptions of the following Lemma, which is itself a simple corollary of Lemma 7.3.11, allow us to compare the $p+q$-th power of the resolvent of the unbounded Kasparov product to the expression in the statement of Lemma 7.3.13.

Lemma 7.3.14. Let ∇ be any T-connection on E_B. Then, if either

- (E_B, S) is graded, $\{S\otimes 1, \gamma\otimes_\nabla T\} = 0$ and $S \times_\nabla T \overset{\text{def}}{=} S\otimes 1 + \gamma\otimes_\nabla T$, or
- $[S\otimes 1, 1\otimes_\nabla T] = 0$ and $S\times_\nabla T \overset{\text{def}}{=} \begin{pmatrix} 0 & S\otimes 1 + i\otimes_\nabla T \\ S\otimes 1 - i\otimes_\nabla T & 0\end{pmatrix}$,

then $|S\times_\nabla T + i|^{-p-q} \leq |S\otimes 1 + i|^{-p/2} |1\otimes T + i|^{-q} |S\otimes 1 + i|^{-p/2}$.

As the resolvent of $S\otimes 1$ is just that of S tensored with 1, all that we need now in order to apply Lemma 7.3.13 is an estimate on the q-th absolute power of the resolvent of $1\otimes_\nabla T$ or $\gamma\otimes_\nabla T$.

In order to treat this resolvent when ∇ is a Grassman connection, we will work in the larger module $H\otimes_{\mathbb{C}} F$. To this end, let u be the unitary isomorphism between the Hilbert C-modules $H\otimes_{\mathbb{C}} F$ and $H\otimes_{\mathbb{C}} B\otimes_B F$. Then, for any frame e of E_B, let $Q_e \overset{\text{def}}{=} u^*(\theta_e^*\theta_e\otimes 1)u$, so that we have $Q_e(H\otimes_{\mathbb{C}} F) \simeq E\otimes_B F$. Let $\widetilde{T} \overset{\text{def}}{=} 1\otimes T\colon H\otimes_{\mathbb{C}} F \to H\otimes_{\mathbb{C}} F$.

Now, the conditions of the following Lemma are satisfied rather generally, cf. the analysis of [KL13, Lemma 5.3] and the differentiable frames constructed in [Kaa17].

Lemma 7.3.15. If there exists a frame e of E_B such that $[\widetilde{T}, Q_e]$ extends to a bounded, adjointable operator on $H\otimes_{\mathbb{C}} F$, then there exists bounded, adjointable $A = Q\widetilde{T}(1-Q) + (1-Q)\widetilde{T}Q \in \mathscr{L}(H\otimes_{\mathbb{C}} F)$ such that the operator $1\otimes_{\nabla_0} T = Q\widetilde{T}Q$ satisfies

$$1\otimes_{\nabla_0} T = Q(\widetilde{T} - A) = (\widetilde{T} - A)Q.$$

In particular, $|1\otimes_{\nabla_0} T + \mu|^{-2} \leq \|A\|^2\, Q|\widetilde{T} + \mu|^{-2} Q = \|A\|^2\, 1\otimes_{\nabla_0} |T+\mu|^{-2}$.

As in [KL13, Proposition 6.6]. Note that A is bounded and adjointable by the identity $A = [Q,\widetilde{T}](1-Q) + (1-Q)[\widetilde{T},Q]$. Now, we have $Q\widetilde{T}Q = Q(Q\widetilde{T}Q + (1-Q)\widetilde{T}(1-Q)) = Q(\widetilde{T}-A)$ and $Q\widetilde{T}Q = (Q\widetilde{T}Q + (1-Q)\widetilde{T}(1-Q))Q = (\widetilde{T}-A)Q$. The expression for the resolvent now follows from the resolvent identity. □

We conclude that, under the *joint* assumptions of Lemma 7.3.15 and Lemma 7.3.13, the unbounded Kasparov product is $p+q$-summable whenever $q \leq 2$.

Remark 7.3.16. There is a $\mathscr{L}(F)$-valued, unbounded, 'partial trace'

$$\tau_e \colon \operatorname{dom}\tau_e \subset \mathscr{L}(E \otimes_B F) \to \mathscr{L}(F),$$

given by $\tau_e(x)(f) = \sum_i \langle e_i | x | e_i \otimes f\rangle$, where $\langle e_i|$ is seen as an F-valued map on $E \otimes_B F$. Clearly we have $\operatorname{dom}\tau_e \supset \mathscr{L}^1(E \otimes_B F)$. Note that $\operatorname{tr}\tau_e(x) = \operatorname{tr} x$ so that for positive x we have $x \in \mathscr{L}^1(E \otimes_B F) \subset \operatorname{dom}\tau_e$ if and only if $\tau_e(x) \in \mathscr{L}^1(F)$.

As seen in Lemma 7.3.13, all operators of the form $s(1 \otimes_e t)s$ with $s \in \mathscr{L}^2(E)$ are contained in $\operatorname{dom}\tau_e$, and moreover $\tau_e(s(1 \otimes_e t)s) \in \mathscr{L}^1(F)$ whenever $t \in \mathscr{L}^1(F)$.

It follows trivially that when $0 \leq x^q \leq 1 \otimes_e x_0^q$, we have $\tau_e(sx^q s) \in \mathscr{L}^1(F)$ whenever $x_0 \in \mathscr{L}^q(F)$. However, we have no such guarantee whenever $0 \leq x \leq 1 \otimes_e x_0$ with $x_0 \in \mathscr{L}^q(F)$ instead: unfortunately, the range of the partial trace τ_e is the *non*commutative C*-algebra $\mathscr{L}(F)$, so that the techniques for traces with values in a commutative C*-algebra developed here (such as Corollary 6.2.16 in particular) do not apply. See Chapter 8.

Chapter 8

Outlook: the noncommutative case

The approach to the Schatten classes outlined in Chapter 6 relies heavily on the commutativity of the algebra A. Not just the common proof technique, which boils down to combining the properties of $\mathscr{L}^p(H_{\mathbb{C}})$ with the uniformity argument of Theorem 6.2.11, fails in the noncommutative case: key results no longer hold.

For instance, we can build on [FL02, Example 1.1] to show that the set $\{T \in \mathscr{L}\left(l^2(A)\right) \mid \operatorname{tr}|T|^p \in A\}$ is no longer a two-sided ideal in $\mathscr{L}\left(l^2(A)\right)$ when $A = B(H)$ in the following example. Moreover, the following example shows that the finite matrices $M(B(H))$ cannot be contained in a two-sided ideal in $\mathscr{L}\left(l^2(B(H))\right)$ on which the trace is even weakly convergent.

Example 8.0.1. Let A be a unital C*-algebra that contains a copy of the Cuntz algebra $\mathcal{O}_n$: a family $\{u_i\}_{i=1}^n$ of isometries such that $u_i^* u_i = 1$ and $\sum_j u_j u_j^* = 1$. Let $T_n \in \operatorname{Mat}_n(A)$ be given by

$$T_n = \begin{bmatrix} u_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ u_n & 0 & \dots & 0 \end{bmatrix}$$

so that in particular $\|T_n\|_{\operatorname{Mat}_n(A)} = 1$ and

$$T_n^* T_n = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 0 & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} \qquad\qquad T_n T_n^* = \begin{bmatrix} 1 & 0 & \dots \\ 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}.$$

Consider the "trace" tr on $\mathrm{Mat}_n(A)$ given by $T \mapsto \sum_i T_{ii}$. Then, the element $S = T_1^* T_1 \in \mathrm{Mat}_1(A)$ satisfies $\mathrm{tr}\, T_n S T_n^* = n$.

In the language of frames, note that $e = (u_1, \dots, u_n)$ is a frame of the Hilbert A-module A because $\sum_i a^* u_i^* u_i b = a^* b$ for $a, b \in A$. The frame transform θ_e now sends the identity 1_A to $T_n T_n^*$, so that $\mathrm{tr}\, \theta_e 1_A \theta_e^* = n$.

In particular, if A contains a copy of $\mathcal{O}_\infty$ (as does $B(H)$ in particular), we must conclude that no trace ideal in $\mathscr{L}\left(l^2(A)\right)$ can contain $\mathrm{Mat}(A)$.

In the general case, then, there is no hope for an extension of Theorem 6.3.1 to provide for a two-sided trace ideal in $\mathscr{L}\left(l^2(A)\right)$ that at least contains $\mathrm{Mat}\, A$, as it should in order to really extend the "trace" on $\mathrm{Mat}_n A$.

8.1 Ideals inside a smaller endomorphism algebra

As a counterpoint to Example 8.0.1, we can assign a *subalgebra* of $\mathscr{L}\left(l^2(A)\right)$ with an associated trace ideal in the general case, as the following example shows.

Example 8.1.1 (The Schatten classes of[Nis91]). For $a \in A$ and $t \in B(H)$, we have $t \otimes a \in \mathscr{L}\left(l^2(A)\right)$ by the inclusion $\mathscr{L}\left(l^2\right) \otimes \mathscr{L}(A) \hookrightarrow \mathscr{L}\left(l^2 \otimes_{\mathbb{C}} A\right)$. Moreover, as the projective tensor norm on $B(H) \otimes A$ dominates the operator norm of $\mathscr{L}\left(l^2(A)\right)$, there is an inclusion $B(H) \otimes_\pi A \hookrightarrow \mathscr{L}\left(l^2(A)\right)$ of Banach algebras. This implies the inclusion $\mathscr{L}^p(H) \otimes_\pi A \subset B(H) \otimes_\pi A \subset \mathscr{L}\left(l^2(A)\right)$.

For $T \in \mathscr{L}^p(H) \otimes_\pi A$ there exist (cf. [Rya02, Chapter 2.1]) bounded sequences $\{a_i\}_i, \{t_i\}_i$ in the unit ball of A and $\mathscr{L}^p(H)$, respectively, and $\{\lambda_i\}_i \in l^1$ with $\sum_i \lambda_i t_i \otimes a_i = T$, and similar for $S \in B(H) \otimes_\pi A$. Thus, $\sum_{ij} \lambda_i \mu_j s_i t_j \otimes a_i b_j = ST$ converges in $\mathscr{L}^p(H) \otimes_\pi A$ because $\left\|s_i t_j\right\|_p \le \|s_i\| \left\|t_j\right\|_p$. Mutatis mutandis, we also see that $TS \in \mathscr{L}^p(H) \otimes_\pi A$ and therefore conclude that $\mathscr{L}^p(H) \otimes_\pi A$ is a two-sided ideal in $B(H) \otimes_\pi A$.

By projectivity, the map $\mathrm{tr}_{\mathscr{L}^p(H)} \otimes 1_A$ extends continuously to $\mathscr{L}^p(H) \otimes_\pi A$. With the algebraic identification $\mathrm{Mat}\, A = \mathrm{Mat}\, \mathbb{C} \otimes A$, this projective trace extends that on $\mathrm{Mat}\, A$.

The crucial difference between Example 8.0.1 and Example 8.1.1, is that the Schatten classes of the latter are only two-sided ideals inside $B(H) \otimes_\pi A$, not in all of $\mathscr{L}\left(l^2(A)\right)$.

Example 8.1.1, however, does not solve the problem fully. First of all, we now treat the commutative and noncommutative case very differently[1], without a clear bridge between the two cases. Second, there are likely to be many examples where the algebra $B(H) \otimes_\pi A$ is insufficiently large to accomodate a given representation, even when the A-valued "trace" may still be perfectly well-defined on a subset thereof that does form a two-sided ideal – as is of course the case when A is commutative, or even almost so.

In the light of this example, we should perhaps not require the domain of an A-valued "trace" to be an ideal inside all of $\mathscr{L}\left(l^2(A)\right)$, but rather inside some subalgebra. That would be particularly natural in the language of KK-cycles: if one has a KK-cycle $({}_AE_B, F)$, there is already a designated subalgebra $A \subset \mathscr{L}(E_B)$ and one might like to say the cycle is p-summable if $\{a \in A \colon [F,a] \in \mathscr{L}^p({}_AE_B)\}$ is dense inside A, or something to that effect. We would then only need $\mathscr{L}^p({}_AE_B)$, whatever its definition will be, to be a two-sided ideal in (or, at least, be closed under multiplication by) $A \subset \mathscr{L}(E_B)$, not in all of $\mathscr{L}(E_B)$.

Example 8.1.2. Note that the functional $T \mapsto \sqrt{\|\operatorname{tr} T^*T\|}$ on the matrix algebras is subadditive, bounded from below by the operator norm. Only from the left, it is even unitarily invariant and submultiplicative in the usual sense. This, given a normed subalgebra $C \subset \mathscr{L}\left(l^2(A)\right)$, might lead one to define a two-sided Hilbert-Schmidt ideal "relative to C" as the completion of $M(A)$ in the norm $\|T\|_{2;C}^2 = \sup\{\|\operatorname{tr} S^*T^*TS\| \mid S \in C, \|S\|_C \le 1\}$.

Remark 8.1.3. Building on the previous example, a rather different extension on the lines of the Hattori-Stallings trace would be to first compose with the quotient map $q \colon A \to A/[A,A]$, i.e. to define the trace with values in the latter as $\sum_i q(\langle e_i, Te_i\rangle)$ for any given frame. The resulting space of Hilbert-Schmidt operators (i.e. those for which the trace of the square converges in the norm on the abelianization) is then a two-sided ideal in the adjointable endomorphisms of a countably generated Hilbert A-module. The square of this Hilbert-Schmidt ideal then gives a two-sided trace class ideal. Unlike for the Hattori-Stallings trace it is not clear, however, that this construction possesses even a non-canonical lift to A itself.

[1] Note that $B(H) \otimes_\pi C_0(X) \subset C_0(X, B(H)) \subsetneq C_{\mathrm{b}}^{\mathrm{str}}(X, B(H))$ unless H is finite-dimensional or X is finite (assuming of course it is Hausdorff). Moreover, even though $\mathscr{L}^p(H) \otimes_\pi C_0(X) \subset C_0(X, \mathscr{L}^p(H))$ always, the spaces need not agree: the diagonal elements of $\mathscr{L}^1\left(l^2\right) \otimes_\pi C_0(X)$ consists of the *absolutely* summable sequences in $C_0(X)$, whereas the diagonal elements of $C_0(X, \mathscr{L}^1\left(l^2\right))$ consists of the *unconditionally* summable sequences, cf. [Rya02].

8.2 The bivariant Chern character

Because the trace on $\mathrm{Mat}(A)$ is not cyclic for noncommutative A, it is more natural to regard it as a map (the Dennis trace) between Hochschild or cyclic complexes instead. That brings us to a highly desired – but elusive – application: the *bivariant Chern character*. This introduces two additional complications, however: that of working with more general *locally convex algebras* $\mathscr{A} \subset A$ and that of the trace on $\mathrm{Mat}(\mathscr{A})^{\hat{\otimes} n}$ with values in the *tensor product* $\mathscr{A}^{\hat{\otimes} n}$.

If $({}_A H, \gamma, F)$ is a finitely summable (even) Fredholm module (with respect to the dense subalgebra $\mathscr{A} \subset A$), Connes' Chern character

$$\mathrm{Ch}_*(F)\colon (a_0, \dots, a_{2n}) \mapsto \frac{\Gamma(n+1)}{2n!} \operatorname{tr}\left(\gamma F[F, a_0] \cdots [F, a_{2n}]\right)$$

(see also pp. 130) and the Chern character of an idempotent $e \in M_k(\mathscr{A})$,

$$\mathrm{Ch}^*(e) = (n!)^{-1} \sum_{0 \le i_j \le k} e_{i_0 i_1} \otimes \cdots \otimes e_{i_{2m} i_0},$$

produce classes in $\mathrm{HC}^{2n}(\mathscr{A})$ and $\mathrm{HC}_{2n}(\mathscr{A})$ respectively that are independent of the choices of representative $({}_A H, F)$ and e of their respective classes in K-theory and K-homology. Moreover, their pairing $\langle \mathrm{Ch}^*(e), \mathrm{Ch}_*(F) \rangle$ equals the index of F on the K-theory class $[e]$, cf. [Con94, Chapter 4.1].

If $({}_B E_A, F)$ is a (bounded) Kasparov $(\mathscr{B}, \mathscr{A})$-module and $e \in M_k(\mathscr{B})$ a projection representing the classes $[E, F] \in \mathrm{KK}^i(\mathscr{B}, \mathscr{A})$ and $[e] \in K_0(\mathscr{B})$ respectively, then their Kasparov product $[e] \otimes_B [E, F]$ is an element of $K^i(\mathscr{A})$. The bivariant index problem, as posed by Connes in [Con83] and confronted by Nistor in [Nis93], is to determine a map $\mathrm{Ch}\colon \mathrm{KK}^i(\mathscr{B}, \mathscr{A}) \to \mathrm{HC}^{2n+i}(\mathscr{B}, \mathscr{A})$ such that

$$\left\langle \mathrm{Ch}^*\left([e] \otimes_B [E, F]\right), \phi \right\rangle = \left\langle \mathrm{Ch}^*[e], \mathrm{Ch}^*([E, F])(\phi) \right\rangle$$

for all $\phi \in \mathrm{HC}^{2n+i}(\mathscr{A})$. For a discussion of the precise definition of $\mathrm{HC}^{2n+i}(\mathscr{B}, \mathscr{A})$ in this context and the way in which to regard its elements as maps from $\mathrm{HC}^\bullet(\mathscr{A})$ to $\mathrm{HC}^\bullet(\mathscr{B})$, see [Nis93].

Now, the connection between the topic of the Connes-Chern character and the present discussion of the trace is that the construction of the Chern character proposed in [Nis91; Nis93] factors through a generalization of the (Dennis trace) map tr^n. That is, consider the map

$$\begin{aligned}\mathrm{tr}^n : M(\mathscr{A})^{\otimes(n+1)} &\to \mathscr{A}^{\otimes(n+1)},\\ \mathrm{tr}^n(T^0 \otimes \cdots \otimes T^n) &= \sum_{i_0,\cdots,i_n} T^0_{i_0,i_1} \otimes \cdots \otimes T^n_{i_n,i_0}.\end{aligned}$$

Given a choice $\hat{\otimes}$ of tensor product appropriate to the computation of $\mathrm{HC}(\mathscr{A})$, a proper generalization of tr^n would then be constructed from a pair $\left(\mathscr{L}^p_{\mathscr{A},\hat{\otimes}}, \mathscr{L}^\infty_{\mathscr{A},\hat{\otimes}}\right)$ of subalgebras of $\mathscr{L}(H \otimes_{\mathbb{C}} A)$ with at least the following properties:

- The domain $\operatorname{dom}\mathrm{tr}^\bullet$, which is spanned by all tensor products $\widehat{\bigotimes}_{i=1}^n \mathscr{L}^{p_i}_{\mathscr{A},\hat{\otimes}}$ such that the reciprocals of the p_i sum to $\sum_{i=1}^k p_i^{-1} = 1$, is closed under contractions, and
- the series defining tr^n converges to a continuous map $\operatorname{dom}\mathrm{tr}^\bullet \to \bigoplus_n \mathscr{A}^{\hat{\otimes}(n+1)}$.

Nistor's original construction takes $\mathscr{L}^p_{\mathscr{A},\hat{\otimes}} \overset{\text{def}}{=} \mathscr{L}^p(H) \otimes_\pi \mathscr{A}$ and $\mathscr{L}^\infty_{\mathscr{A},\hat{\otimes}} \overset{\text{def}}{=} B(H) \otimes_\pi \mathscr{A}$, as in Example 8.1.1. It is important to note that the resulting definition of a '1-summable bounded Kasparov module' $(H \otimes_{\mathbb{C}} A, F)$ requires both that the representation of $\mathscr{A} \subset A$ land in $B(H) \otimes_\pi \mathscr{A}$ and that the commutators $[F,a]$ land in $\mathscr{L}^1(H) \otimes_\pi \mathscr{A}$, again as in Example 8.1.1. This is more restrictive than one might desire, as we will see below.

The Chern character of the Toeplitz extension

An important case for the Chern character is that of the boundary map in the Pimsner-Voiculescu sequence. That is to say, consider the Toeplitz extension [Pim97]

$$0 \to \mathscr{K}\left(l^2(A)\right) \to \mathscr{T}(A,\alpha) \to A \rtimes_\alpha \mathbb{Z} \to 0.$$

Its class in $\mathrm{KK}^1(A \rtimes_\alpha \mathbb{Z}, A)$ is represented by the unbounded cycle (E,D) presented in Example 8.2.1 below, cf. [GMR18, Theorem 1] and [AM19, Section 4].

Example 8.2.1. Let A be a unital C^*-algebra equipped with an automorphism $\alpha\colon A \to A$. Then, consider the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{Z}$ generated by A and a unitary u satisfying $\alpha^n(a) = u^n a u^{*n}$.

Equip the Hilbert A-module $E = L^2(S^1) \otimes_{\mathbb{C}} A$ with the Dirac operator $D = D_{S^1} \otimes 1$ and consider the unitary $U = S \otimes 1 \in \mathscr{L}(E)$, where S is multiplication by $\theta \mapsto e^{i\pi\theta}$ on $L^2(S^1)$. Then, we may represent $A \rtimes_\alpha \mathbb{Z}$ on E by the map $\rho\colon \sum_k a_k u^k \mapsto \sum_k a_k U^k$.

Note that there is a dense subalgebra of $A \rtimes_\alpha \mathbb{Z}$ with $[D,b]$ bounded because $[D_{S^1}, S] = 1$, so that (E,D) is an unbounded Kasparov $(A \rtimes_\alpha \mathbb{Z}, A)$-module.

Taking the interior Kasparov product with this cycle, in turn, implements the boundary map ∂ in the Pimsner-Voiculescu six-term exact sequence [PV80]

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "1-\alpha_*^{-1}"] & \mathrm{K}_0(A) \arrow[r, "\iota_*"] & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "\partial"] \\
\mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\partial"] & \mathrm{K}_1(A) \arrow[l, "\iota_*"] & \mathrm{K}_1(A) \arrow[l, "1-\alpha_*^{-1}"]
\end{tikzcd} .$$

In [Nes88], Nest constructed a – notoriously complicated – map

$$\#\colon \mathrm{HC}^\bullet(\mathscr{A}) \to \mathrm{HC}^{\bullet+1}(\mathscr{A} \rtimes_\alpha \mathbb{Z}),$$

compatible with the boundary map ∂ discussed above, that yields the compatible diagram

$$\begin{tikzcd}
\mathrm{HC}^{\mathrm{ev}}(\mathscr{A}) \arrow[d, "\#"] & \mathrm{HC}^{\mathrm{ev}}(\mathscr{A}) \arrow[l, "1-\alpha_*^{-1}"'] & \mathrm{HC}^{\mathrm{ev}}(\mathscr{A} \rtimes_\alpha \mathbb{Z}) \arrow[l, "\iota_*"'] \\
\mathrm{HC}^{\mathrm{odd}}(\mathscr{A} \rtimes_\alpha \mathbb{Z}) \arrow[r, "\iota_*"] & \mathrm{HC}^{\mathrm{odd}}(\mathscr{A}) \arrow[r, "1-\alpha_*^{-1}"] & \mathrm{HC}^{\mathrm{odd}}(\mathscr{A}) \arrow[u, "\#"']
\end{tikzcd} .$$

It would be a beautiful result to realize both the boundary map ∂ in KK-theory and the corresponding map $\#$ in cyclic cohomology as resulting from the same, simple unbounded Kasparov cycle (E,D). The former is already known to be its class in KK-theory, and the latter should arise as its Chern character.

Nistor's theory of bivariant Chern characters, however, does not apply here in the desired generality. The problem is that the spaces $B(H) \otimes_\pi \mathscr{A}$ and $\mathscr{L}^p(H) \otimes_\pi \mathscr{A}$ may be too small, as the following example shows.

Example 8.2.2. Let M be a compact smooth manifold that is equipped with a diffeomorphism $\phi\colon M \to M$ and consider the unbounded Kasparov $(C(M) \rtimes_\phi \mathbb{Z}, C(M))$-cycle (E,D) as in Example 8.2.1 (note: this is precisely the 1^+-summable cycle of Section 7.3).

If ϕ is topologically 2-transitive and $f \in C^\infty(M)$ takes at least two different values z_1, z_2, then the element $fu_0 \in C_{\mathrm{b}}^{\mathrm{str}}(M, \mathscr{L}(L^2(S^1)))$ is not norm continuous. That is to say, in that case the image of fu_0 in $\mathscr{L}(E)$ does *not* lie in (the obvious representation of) $\mathscr{L}(L^2(S^1)) \otimes_\pi C(M)$.

Even though we have constructed well-behaved Schatten classes in the case where A is a commutative C*-algebra, that does not suffice to treat this example. We are likely to wish to consider the cyclic cohomology of some locally convex subalgebra $\mathscr{A} \subset A$, say, of smooth functions, where Dini's theorem no longer applies. Moreover, even if we really desire to work with A itself, the relation (if any) between convergence of the tensor trace tr^n and that of tr depends on the choice of topological tensor product. Given the fickle nature of these complications, it seems advisable for further treatment of the subject to embark with very specific examples in mind.

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Samenvatting

Dit proefschrift verkent twee facetten van de *spectrale meetkunde*: dat van *eindige benadering* en dat van *continue verandering*. In deze samenvatting, bedoeld voor de geïnteresseerde van buiten het vakgebied, probeer ik een idee te geven van wat de drie schuingedrukte termen uit die vorige zin inhouden in de context van dit proefschrift.

Spectrale meetkunde

Welk geluid een muziekinstrument maakt, wordt bepaald door de frequenties waarmee het kan trillen (als reactie op de handelingen van de muzikant). De hoogte van die frequenties is vervolgens afhankelijk van de vorm van het instrument. Zo kan je aan bijvoorbeeld de lengte van een pianosnaar of de straal van een trommel al zien hoe hoog die zal klinken. Wiskundig gezien zouden we zeggen: de *meetkunde* (de vorm, in dit geval van het instrument) zegt iets over het *spectrum* (de frequenties) van bepaalde *operatoren* (zoals de wiskundige bewerking die – in bovenstaand voorbeeld – de voortstuwing van geluidsgolven door het instrument beschrijft).

De *spectrale meetkunde* gaat over de omgekeerde kwestie: als we alleen dat spectrum (de frequenties dus) weten – of het nou van een instrument, een tafel, of een waterstofatoom is – wat kunnen we daarmee zeggen over de vorm?

Eindige benadering

Omdat we het in de praktijk moeten doen met een beperkt waarnemingsvermogen en een beperkte rekencapaciteit, introduceert dit proefschrift een methode om vormen te beschrijven aan de hand van *een klein deel van* de mogelijke resonantiefrequenties[2].

[2] Dat wil hier zeggen: van stationaire geluidsgolven of, algemener, van het spectrum van een operator van het Laplace-type.

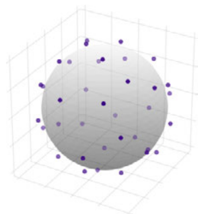
Het blijkt zo te zijn dat steeds hogere frequenties steeds meer detail toevoegen, maar juist steeds minder van doen hebben met de grote lijnen (zie figuur 1). Daarom kunnen we de lagere frequenties gebruiken om een soort schets te maken.

Figuur 1: Een klein deel van de frequenties overheerst het geluid: ook zonder de hoge frequenties (onder) blijven de grote lijnen behouden.

De hoofdstukken 2 en 3 gaan dus over de vraag: *hoe weet je welke vorm een object heeft, als je alleen maar deels weet hoe het klinkt?* Eerst wordt in hoofdstuk 2 een techniek ontwikkeld om – door het gedrag van hittestroom[3] op heel korte termijn te bekijken en in verband te brengen met het gedrag van stationaire hittegolven van steeds hogere frequentie – de meetkundige rol van de laagfrequente resonanties te isoleren.

Vervolgens legt hoofdstuk 3 een verband tussen *lokalisatie* – in de zin van op-één-plek-zijn – en de ingrediënten uit hoofdstuk 2. Door combinaties van laagfrequente golven te maken die zoveel mogelijk zijn gelokaliseerd, krijgen we een (schetsmatig) beeld van de kleinste stukjes waar de vorm uit bestaat (zie figuur 2). Die stukjes worden vervolgens aan elkaar gekoppeld door een formule van Connes, die de afstand tussen punten relateert aan de frequenties van golven die ertussen bewegen.

Figuur 2: Optimaal gelokaliseerde, laagfrequente golven op de bol.



Figuur 3: Een reconstructie door PointForge.

Op basis van deze wiskundige constructie schreven we vervolgens het computerprogramma

[3]Hittestroom is gerelateerd aan het gedrag van geluidsgolven: het gaat allebei om het verspreiden van trillingen door het materiaal.

PointForge, dat een reeks resonantiefrequenties kan vertalen in een afbeelding van het resonerende object (zie figuur 3).

De vraag die centraal staat in hoofdstuk 4 is: voor welke patronen van resonantiefrequenties bestaat er daadwerkelijk een vorm die zo klinkt? In dit hoofdstuk zetten we de eerste stappen in die richting, door een natuurkundig concept te gebruiken om zulke patronen te detecteren en dat concept aan te passen aan de situatie waar alleen kennis van de lage frequenties beschikbaar is. In een computerverificatie van deze methode – een experiment, zogezegd – vonden we een nieuwe meetkundige structuur die verrassend genoeg op lage frequenties beter de eigenschappen van een bol vertoont dan de bol dat zelf doet. Dat wil zeggen, als we niet al van tevoren hadden geweten welke van de getalletjes in onze computer daadwerkelijk een bol voorstelden, hadden we de verkeerde gekozen.

Als de vraag uit dit hoofdstuk in verder onderzoek wordt beantwoord, kunnen we de wereld van alle mogelijke vormen verkennen in termen van de wereld van alle 'correcte' patronen van resonantiefrequenties: we kunnen dan 'op gehoor' vormen construeren.

De beoogde toepassing van dit hele onderzoek naar de gedeeltelijke resonantiespectra ligt in de quantumzwaartekracht, omdat de daarvoor cruciale interactie tussen meetkunde en energie veel tastbaarder is in termen van resonantiefrequenties. Verder is er een sterk verband met de computerwetenschap, omdat in *machine learning* en *computer graphics* een zoektocht gaande is naar nieuwe methodes om grip te krijgen op de meetkunde van zowel abstracte vormen (zoals datasets) als concrete vormen (voor beeldmanipulatie en animatie).

Continue verandering

Het *spoor* is een fundamenteel en veelzijdig wiskundig gereedschap in de functionaalanalyse, en in het bijzonder in de spectrale meetkunde; bijvoorbeeld alle meetkundige informatie uit hoofdstuk 2 wordt uitgedrukt in termen van sporen. Kort en inadequaat gezegd is het spoor een manier om het spectrum van een operator in één getal samen te vatten, zoals je bijvoorbeeld een gemiddelde zou gebruiken om een hoop verschillende getallen te representeren. We komen zometeen op deze sporen terug.

In de spectrale meetkunde wordt het concept van *verandering* (van vorm) beschreven door een verder verfijnd concept van de vormen zelf – we zouden zeggen dat de vorm *langs* iets, zoals bijvoorbeeld langs een tijdslijn, verandert, en beschouwen vervolgens dat hele veranderende proces in één keer, zoals in figuur 4.

Figuur 4: Twee cirkels veranderen in één cirkel: een verandering van vorm is zelf ook een vorm.

Figuur 5: Een regenboog verliest zijn hoogfrequente kleuren.

Wiskundig gezien wordt zo'n veranderende vorm beschreven met zogenaamde Hilbert-modules over abelse C^*-algebras. In deze context, echter, bestond zoiets als een spoor nog niet, en dat belemmerde ons om onze spoorformules uit de spectrale meetkunde toe te passen in deze bredere sfeer en zo een beter begrip te krijgen van de relatie tussen verandering en eindpunt. Het laatste deel van dit proefschrift introduceert dan ook een systematische theorie van sporen in de context van zulke Hilbert-modules.

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...et haec olim meminisse iuvabit.

Virgil, Aeneid 1.203

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