Fermionic particle creation in asymptotically static Generalized Friedmann–Lemaître–Robertson–Walker spacetimes

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January 2021

Master thesis Mathematical Physics
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1 Introduction

> Anyone who has lost track of time when using a computer knows the propensity to dream, the urge to make dreams come true and the tendency to miss lunch.
>
> –Sir Tim Berners-Lee

The marriage of Quantum Field theory en General Relativity has always been an interesting one. Two of the most well-known quantum relativistic effects are Hawking radiation ([28]) and the Fulling–Davies–Unruh effect [21, 13, 48]. A closely related effect is that of particle creation in asymptotic static isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, as first studied by Parker [37, 38] even before Hawking, Fulling, Davies and Unruh published their results, and later extensively by Bernard and Duncan [9, 15].

Recently these results have seen a revival of attention from quantum information theorists, studying the effect of relativity on entanglement of quantum states (e.g. in [36, 34, 20, 35, 39]).

Inspired by this revival of interest, we have studied the entanglement, or in the language of the original papers, the particle creation of fermionic fields in asymptotically static FLRW spacetimes, generalizing the results to spatially compact asymptotically static Generalized Friedmann–Lemaître–Robertson–Walker (GFLRW) spacetimes.

We will proceed as follows. We will conclude this section by introducing the relevant notation and conventions, which is always challenging when working on the boundary of two disciplines and having to deal with two sets of not always matching conventions.

In Section 2 we define GFLRW spacetimes and the other relevant geometric notions. We will define Clifford algebras, the spinor bundle, the spin-connection on the spinor bundle, and the Dirac operator acting on sections of the spinor bundle. We also define the charge conjugation operator J on the spinor bundle. We have assumed knowledge of smooth manifolds and vector bundles, but we will introduce and prove all other notions in depth. For a good introduction into smooth manifolds and vector bundles, we refer to [33, 32]. Finally we will explain how the Dirac operator on a product spacetime $\mathbb{R} \times \Sigma$ is uniquely determined by the Dirac operator on the Riemannian spin manifold Σ, following the results of [2].

In Section 3 we find solutions to the Dirac operator on a static product spacetime $\mathbb{R} \times \Sigma$, where we assume Σ to be compact. We pay close attention to the role of the the charge conjugation operator J. We also describe the solutions of the Dirac equation on Minkowski space. Therefore we will also give a short introduction into Fourier theory in this section. We have assumed knowledge of Functional Analysis, although it is not necessary to get the general picture. For a good introduction see e.g. [41, 42, 12]. At the end of Section 3 we define the relevant mathematical notions to make the concept of solutions at infinity more precise.

In Section 4 we shorty describe the quantization of the Dirac field on a GFLRW, while we refer to Appendix A for a more detailed introduction into fermionic Fock spaces and canonical quantization of fermionic fields. We will find relations between the creation and annihilation operators around $t = -\infty$, called the *in*-region and the operators around $t = +\infty$, called the *out*-region. Using our more general setting, we are able to find rather elegant expressions for these transformations using the charge conjugation operator J. We show that these transformations give rise to a unitary map from the Fock space at $-\infty$ to the Fock space at ∞.

In Section 5 we will give a detailed exposition how the Bogoliubov transformation are computed in Minkowski space, thus connecting to the existing literature mentioned above, and how to explicitly perform the necessary calculations. We hope this will be useful for anyone interested in the subject. We will identify a small typographical error in [15] which since then has been plaguing the quantum informational literature. We will finish with a short conclusion in Section 6.

1.1 Conventions and notation

We write $\eta^{(r,s)}$ for the standard (indefinite) inner product on $\mathbb{R}^{r+s}$ of signature (r,s) , but with reversed order of the coordinates, i.e. for $x, y \in \mathbb{R}^n$

$$\eta^{(r,s)}(x,y) = -\sum_{i=1}^{s} x_i y_i + \sum_{i=s+1}^{r+s} x_i y_i.$$

This notation is somehow unconventional, but will turn out very convenient for our spinoral purposes. If we write $\mathbb{R}^{r,s}$ we mean the inner product space $(\mathbb{R}^{r+s}, \eta^{(r,s)})$. The first s coordinates are called the time-coordinates, and the last r coordinates are called the space-coordinates.

If $s = 0$ we will often write $\langle x, y\rangle = \eta^{(n,0)}(x,y)$. Using this convention, the Minkowski metric is given by

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We assume all manifolds to be equipped with a smooth structure and all trivializations, sections etc. are assumed to be smooth. If $E \to M$ is a vector bundle, we denote the space of (smooth) sections by $\Gamma(E)$. Given a local trivialization (U, ϕ) of an n-dimensional manifold M, we will use the convention that

$$\phi = (x^1, \ldots, x^n),$$

when there is no room for confusion about which chart is used. Often no explicit chart will be specified though. Sometimes we will denote it by $(U, (x^\mu))$. Given such a chart the coordinate frame of the tangent bundle TM is denoted by

$$(\partial_\mu) = (\partial_1, \ldots, \partial_n),$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Given a metric g on TM its components in this frame are denoted by

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu).$$

We call a frame (e_a) pseudo-orthonormal if

$$g(e_a, e_b) = \eta^{(r,s)}_{ab},$$

and we call it orthonormal if $s = 0$. In general we will use Greek indices when referring to a coordinate basis, and Latin indices when referring to pseudo-orthonormal frames or bases.

Given any basis or frame we denote the dual basis or frame by the same symbol with the index raised. That is, (e_a) is a frame of a vector bundle E, then (e^a) is the corresponding coframe of the covector bundle E^*, such that

$$e^b(e_a) = \delta^b_a.$$

For the coordinate frame of the cotangent bundle an exception is made. Here (dx^μ) is the dual frame corresponding to (∂_μ). Using these frames, the components of any (n, m)-tensor $A \in (\otimes^n TM) \otimes (\otimes^m TM)$ are given by

$$A^{a_1 \cdots a_n}{}_{b_1 \ldots b_m} = A(e^{a_1}, \ldots, e^{a_n}, e_{b_1}, \ldots, e_{b_m}).$$

When working with indices we will use the Einstein summation convention, i.e. a summation is implied over repeated indices, e.g.

$$g^{\mu\nu} X_\mu = \sum_{\mu=0}^{3} g^{\mu\nu} X_\mu,$$

but only in a strict way: summation over repeated lower or upper indices is not implied. Summation will take place over all possible values for the index, and otherwise an explicit summation symbol will be used. For a symmetric $(1, 1)$ tensor A we will often write A^a_b for its components instead of $A^a{}_b$, as it doesn't matter. Indeed

$$A^a{}_b X_a = \eta^{ac} A_{cb} X_a = \eta^{ac} A_{bc} X_a = A_b{}^a X_a.$$

Using the metric we can change vectors in covectors and vice versa. Using index notation, this boils down to raising and lowering of the indices. For example, when $X \in E$, with corresponding covector $\hat{X} \in E^*$, we have

$$X^a = X(e^a), \quad X_b = \hat{X}(e_b), \quad X^a = \eta^{ab} X_b.$$

When raising and lowering indices of tensors on the tangent bundle, we use the metric corresponding to the indices. That is if (M, g) is a pseudo-Riemannian manifold and X a (1,1)-tensor, then

$$X_{a\mu} = \eta_{ab} X^b{}_\mu = \eta_{ab} g_{\mu\nu} X^{b\mu}.$$

The Pauli matrices are denoted by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are hermitian matrices which square to the identity-matrix. They satisfy the anti-commutation relations given by

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}, \quad i, j = 1, 2, 3.$$

We will use $(\cdot)^*$ to denote complex conjugation, and $(\cdot)^\dagger$ do denote the adjoint of a linear operator. For a Hermitian inner product we follow the physics convention by defining them linear in their second slot, i.e.

$$\langle \alpha u, \beta v \rangle = \alpha^* \beta \langle u, v \rangle.$$

2 Geometry, spinor bundles and the Dirac operator

Aim for the sky, but move slowly,
enjoying every step along the way.
It is all those little steps that
make the journey complete

— Chanda Kochhar

2.1 Generalized FLRW spacetimes

We will start by introducing Friedmann–Lemaître–Robertson–Walker and globally hyperbolic spacetimes and other geometric notions needed for formulating quantum field theory in curved spacetimes. These will from the fabric on which our other constructions take place.

Definition 2.1. Let M be a pseudo-Riemannian manifold with metric g. A tangent vector $X \in TM$ is called

- *timelike* if $g(X, X) < 0$,
- *lightlike* if $g(X, X) = 0$,
- *spacelike* if $g(X, X) > 0$.

A differentiable curve in a manifold M is called timelike, lightlike, or spacelike, if its tangent vectors are timelike, lightlike, or spacelike at all points in the curve respectively.

A differentiable curve in M is called *inextendible* if no differentiable reparametrization of the curve can be continuously extended beyond any of the end points.

Definition 2.2. We say that a Riemannian (M, g) manifold is *complete* if for any $v \in TM$ the geodesic $\gamma : J \subseteq \mathbb{R} \to M$ with $\gamma'(0) = v$ is defined for all $t \in \mathbb{R}$, i.e. $J = \mathbb{R}$.

Definition 2.3. A metric g on a $n + 1$ dimensional manifold is called *stationary* if there is a timelike Killing vector field.

We say that the metric is *static* if it is stationary and there is a family of spacelike hypersurfaces orhogonal to the Killing vector field everywhere. In a globally hyperbolic case these are Cauchy surfaces.

Using coordinates, these two conditions boil down to that their are coordinates x^μ, with $t = x^0$ timelike such that

1. g is independent of t, and
2. $g_{0j} = 0 \quad \text{for } j = 1, \ldots, n.$

Using these coordinates the spacelike hypersurfaces orthogonal to the Killing vector are the $t =$ const. surfaces.

Definition 2.4. A *Cauchy hypersurface* Σ in a spacetime (M, g) is a subset of M which is met exactly once by every inextendible timelike curve.

Definition 2.5. A spacetime (M, g) is called *globally hyperbolic* if it contains a Cauchy hypersurface.

The following theorem states that any globally hyperbolic spacetime are isometric to a smooth product spacetime. This is a stronger version of Geroch's topological splitting of globally hyperbolic spacetimes, as stated in [23].

Theorem 2.6 (A. Bernal and M. Sánchez [8])**.** *Let (M,g) be a globally hyperbolic spacetime. Then there exists a smooth manifold Σ, a smooth one-parameter family of Riemannian metrics $(g_t)_{t\in\mathbb{R}}$ on Σ and a smooth positive function N on $\mathbb{R}\times\Sigma$ such that (M,g) is isometric to $(\mathbb{R}\times\Sigma, -N^2dt^2\oplus g_t)$. Each $\{t\}\times\Sigma$ corresponds to a smooth spacelike Cauchy hypersurface in (M,g).*

In the rest of this thesis we will restrict ourselves to the case that $N = 1$ and $g_t = a^2(t)g_\Sigma$, i.e. to the case of a globally hyperbolic GFLRW spacetime.

Definition 2.7. We say that a manifold is a *Generalized Friedmann–Lemaître–Robertson–Walker (GFLRW) spacetime* if it is a product manifold $M = I \times \Sigma$, $I \subseteq \mathbb{R}$ an interval, endowed with the Lorentzian metric

$$ds^2 = -dt^2 \oplus a^2(t)g_\Sigma,$$

where g_Σ is a Riemannian metric on Σ, and $a : I \to (0,\infty)$ is a smooth, positive function.

Remark 2.8. This spacetime is called a Generalized FLRW spacetime as it is a generalization of a FLRW spacetime, for which the space Σ is assumed to be complete and of constant curvature. The notion of a Generalized Friedmann–Lemaître–Robertson–Walker spacetime is introduced by M. Sánchez in [1]. ♢

Not all GFLRW spacetimes are globally hyperbolic, but if the metric g_Σ is complete, they are.

Theorem 2.9. *Let $(M,g) = (I \times \Sigma, -dt^2 \oplus a^2(t)g_\Sigma)$ be a GFLRW spacetime. (M,g) is globally hyperbolic if and only if (Σ, g_Σ) is complete.*

Proof. See e.g. [5, Thm. 3.66]). □

Definition 2.10. We say that a generalized Friedmann–Lemaître–Robertson–Walker spacetime $(I \times \Sigma, -dt^2 \oplus a^2(t)g_\Sigma)$ is *spatially closed* if the fibre Σ is compact.

Remark 2.11. Note that a stationary GFLRW spacetime is automatically static. It is stationary if $a(t)$ is constant and then ∂_t is a timelike Killing vector field. ♢

2.2 Spinor bundles

Before we can define the Dirac operator, we have to define the space its solutions live in: the spinor bundle, a specific vector bundle over a manifold M. In this section we follow [2, 43, 4].

We first recall some facts about the orthogonal, and special orthogonal groups. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $n = r + s$. We define the *orthogonal group* of $(\mathbb{F}^n, \eta^{(r,s)})$ by

$$\mathrm{O}(r, s, \mathbb{F}) = \{A \in \mathrm{GL}_n(n, \mathbb{F}) \mid \eta^{(r,s)}(Av, Aw) = \eta^{(r,s)}(v, w)\ \forall v, w \in \mathbb{F}^n\}$$

and the *special orthogonal group* by

$$\mathrm{SO}(r, s, \mathbb{F}) = \{A \in \mathrm{O}(r, s) \mid \det(A) = 1\}.$$

For $\mathbb{C}$ we only have $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$, as for a complex vector space there is only one inner product up to isomorphism. One can check that if $r = 0$ or $s = 0$, then $\mathrm{SO}(r, s, \mathbb{C})$ is connected and otherwise it has two connected components. We often write $\mathrm{SO}(r, s)$ for $\mathrm{SO}(r, s, \mathbb{R})$.

Definition 2.12. Let V be a vector space. Let $\{e_n\}$ and $\{f_n\}$ be two bases of V, and $A : V \to V$ a linear map such that $Ae_n = f_n \forall n$. We say that $\{e_n\}$ and $\{f_n\}$ have the same orientation if $\det(A) > 0$. Having the same orientation defines a equivalence relation on the set of all ordered bases of V, providing two equivalence classes. An orientation is an assignment of $+1$ two one equivalence class, and -1 to the other equivalence class. An orientation preserving map $A : V \to V$ is a map respecting the two equivalence classes, i.e. mapping a basis to a basis with the same orientation.

Definition 2.13. We consider $\mathbb{R}^{r,s} \cong \mathbb{R}^s \oplus \mathbb{R}^r$. We say that an orientation of $\mathbb{R}^{r,s}$ is a *space and time orientation* of $\mathbb{R}^{r,s}$, if its restrictions to bases of $\mathbb{R}^s$ and $\mathbb{R}^r$ define orientations on $\mathbb{R}^s$ and $\mathbb{R}^r$ respectively.

By definition $\mathrm{SO}(r, s)$ preserves orientations on $\mathbb{R}^{r+s}$. The connected component of the identity of $\mathrm{SO}(r, s, \mathbb{F})$ is denoted by $\mathrm{SO}_0(r, s, \mathbb{F})$. $\mathrm{SO}_0(r, s)$ preserves space and time orientations of $\mathbb{R}^{r,s}$.

Definition 2.14. Let (V, h) be a vector space V equipped with an inner product h. The Clifford algebra $\mathrm{Cl}(V, h)$ is the algebra generated by the vectors $v \in V$, with unit 1, subject to the relations

$$v \cdot w + w \cdot v = 2h(v, w). \tag{1}$$

The $\mathbb{Z}_2$ grading

$$\mathsf{p}(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k$$

on $\mathrm{Cl}(V, h)$ gives rise to a decomposition into an even and odd part

$$\mathrm{Cl}(V, h) = \mathrm{Cl}^0(V, h) \oplus \mathrm{Cl}^1(V, h).$$

We call $\mathsf{p} : \mathrm{Cl}(V, h) \to \mathrm{Cl}(V, h)$ the *parity automorphism.* We set

$$\mathrm{Cl}_{r,s} = \mathrm{Cl}(\mathbb{R}^{r+s}, \eta^{(r,s)}),$$

and define the special cases

$$\begin{aligned} \mathrm{Cl}_n^+ &= \mathrm{Cl}_{n,0}, \\ \mathrm{Cl}_n^- &= \mathrm{Cl}_{0,n}, \\ \mathbb{C}\mathrm{l}_n &= \mathrm{Cl}_n^+ \otimes_{\mathbb{R}} \mathbb{C}. \end{aligned}$$

If $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$, then the even part $(\mathrm{Cl}_{r,s})^0$ consists of products of an even number of e_i's and the odd part $(\mathrm{Cl}_{r,s})^1$ of products of an odd number of e_i's.

Remark 2.15. Note that for a complex vector space, there is only one inner-product up to isomorphism, i.e. for all r, s we have $\mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}_{r+s}$. When using $\mathbb{C}\mathrm{l}_n$, we will always use the standard positive definite inner product $\eta^{(n,0)}$, unless stated otherwise. ♢

Remark 2.16. In the other half of the literature a minus sign in the definition of a Clifford algebra is added in Eq. (1), i.e $2h(v, w)$ replaced by $-2h(v, w)$. The reader should be aware of this when comparing formulas between different articles. ♢

Proposition 2.17. *We have algebra isomorphisms*

$$\mathrm{Cl}_{r,s} \xrightarrow{\cong} \mathrm{Cl}^0_{r,s+1}, \quad \mathrm{Cl}_{r,s} \xrightarrow{\cong} \mathrm{Cl}^0_{s+1,r},$$

induced by the maps

$$\mathbb{R}^{r+s} \to \mathrm{Cl}^0_{r,s+1}\ (\mathrm{Cl}^0_{s+1,r}), \quad v \mapsto e_0 \cdot v.$$

Also

$$\mathrm{Cl}^0_{r,s} \cong \mathrm{Cl}^0_{s,r}\,.$$

Proof. We consider the inclusion $\mathbb{R}^{r,s} \subseteq \mathbb{R}^{r+s+1}$ such that if $e_1, \ldots, e_{r+s}$ is the standard basis of $\mathbb{R}^{r+s}$, then $e_0, \ldots, e_{r+s}$ is the standard basis of $\mathbb{R}^{r+s+1}$. We first construct the map $\Psi : \mathbb{R}^{r+s} \to \mathrm{Cl}^0_{r,s+1}$, given by $\Psi(e_i) = e_0 e_i$. This map extends to a homomorphim

$$\Psi : \mathrm{Cl}_{r,s} \to \mathrm{Cl}^0_{r,s+1}\,.$$

Indeed for $i, j \in \{1, \ldots, r + s\}$ we have

$$\begin{aligned}\Psi(e_ie_j + e_je_i) &= \Psi(e_i)\Psi(e_j) + \Psi(e_j)\Psi(e_i) = e_0e_ie_0e_j + e_0e_je_0e_i\\ &= -e_0^2(e_ie_j + e_je_i) = 2\eta^{(r,s)}_{ij} = \Psi(2\eta^{(r,s)}_{ij}),\end{aligned}$$

as $e_0^2 = -1$. Since Ψ sends basis vectors in $\mathrm{Cl}_{r,s}$ to basis vectors in $\mathrm{Cl}^0_{r,s+1}$ and the dimension of $\mathrm{Cl}_{r,s}$ and $\mathrm{Cl}^0_{r,s+1}$ coincide, as one can check, Ψ is an isomorphism.

To construct the second isomorphism, we define $\Psi : \mathbb{R}^{r+s} \to \mathrm{Cl}^0_{r,s+1}$, again as $\Psi(e_i) = e_0e_i$, which again extends to a homomorphism

$$\Psi : \mathrm{Cl}_{r,s} \to \mathrm{Cl}^0_{s+1,r}\,.$$

Indeed for $i, j \in \{1, \ldots, r + s\}$ we have

$$\begin{aligned}\Psi(e_ie_j + e_je_i) &= \Psi(e_i)\Psi(e_j) + \Psi(e_j)\Psi(e_i) = e_0e_ie_0e_j + e_0e_je_0e_i\\ &= -e_0^2(e_ie_j + e_je_i) = -2\eta^{(s,r)}_{ij} = 2\eta^{(r,s)}_{ij} = \Psi(2\eta^{(r,s)}_{ij}),\end{aligned}$$

as now $e_0^2 = 1$. Again by dimensional analysis this is a isomorphism. Since both $\mathrm{Cl}^0_{r,s+1}$ and $\mathrm{Cl}^0_{s+1,r}$ are isomorphic to $\mathrm{Cl}_{r,s}$ we conclude that for all $r, s \in \mathbb{N}$

$$\mathrm{Cl}^0_{r,s+1} \cong \mathrm{Cl}^0_{s+1,r},$$

explicitly given by

$$e_i \mapsto e_{r+s-i}.$$

This conclude the proof as $\mathrm{Cl}^-_0 = \mathrm{Cl}^+_0 = \mathbb{R}$ by definition. □

Remark 2.18. Note that if $r+s$ is odd, Ψ maps the volume element of $\mathrm{Cl}_{r,s}$ to the volume element $\mathrm{Cl}_{r,s+1}$. ♢

Definition 2.19. The *complex spin group* is given by

$$\mathrm{Spin}^c(n) = \{v_1 \cdots v_k \in \mathbb{C}\mathrm{l}_n^0 \mid v_j \in \mathbb{C}^n, |\langle v_j, v_j \rangle| = 1\}$$

The *spin group* is given by

$$\mathrm{Spin}(r,s) = \{v_1 \cdots v_k \in \mathrm{Cl}_{r,s}^0 \mid v_j \in \mathbb{R}^n, \eta^{(r,s)}(v_j, v_j) = \pm 1 \ \forall j \in 1, \ldots, n\}$$

The spin group $\mathrm{Spin}(r,s)$ is connected if $rs = 0$ and otherwise the connected component of the identity of the spin group is given by

$$\mathrm{Spin}_0(r,s) = \{v_1 \cdots v_k \in \mathrm{Spin}(r,s) \mid v_j \in \mathbb{R}^n, \prod_{j=1}^{k} \eta^{(r,s)}(v_j, v_j) = 1\}.$$

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. For a $v \in \mathbb{F}^n$, with $\eta^{(r,s)}(v,v) \neq 0$ we see from Eq. (1) that $v^{-1} = \frac{v}{\eta^{(r,s)}(v,v)}$ and that for $w \in \mathbb{F}^n$ arbitrary

$$\mathrm{Ad}(v)(w) := v^{-1} w v = w - 2\frac{\eta^{(r,s)}(v,w)}{\eta^{(r,s)}(v,v)} v.$$

Hence $\mathrm{Ad}(v)$ is the reflection across the hyperplane $v^{\perp}$. Since it is know that any element in $SO(r,s)$ is given by an even number of reflections, it follows that Ad is a homomorphism,

$$\mathrm{Ad} : \mathrm{Spin}^c(n) \to \mathrm{SO}(n, \mathbb{C}), \qquad \mathrm{Ad} : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s,\mathbb{R}),$$

by calculating the kernel of Ad one can check we have the following short exact sequences of groups

$$1 \to U(1) \to \mathrm{Spin}^c(n) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(n, \mathbb{C}) \to 1,$$

$$1 \to \{\pm 1\} \to \mathrm{Spin}(r,s) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(r,s,\mathbb{R}) \to 1,$$

One can also check that

$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n,0) \times U(1))/\sim,$$

where $(g,w) \sim (h,z)$ if and only if $g = -h$ and $w = -z$. Moreover we have that $\mathrm{Spin}_0(r,s)$ is a double covering of $\mathrm{SO}_0(r,s)$.

Proposition 2.20. *For every pair* (r,s) $\mathrm{Spin}_0(r,s)$ *is a double covering of the identity component* $\mathrm{SO}_0(r,s)$*, that is there is a short exact sequence*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_0(r,s) \xrightarrow{\mathrm{Ad}} \mathrm{SO}_0(r,s) \to 1.$$

Proof. See [31, Thm 2.10]. □

We will now study the representations of Clifford algebras and spin groups.

Definition 2.21. Let (V, η) be an inner product space over a commutative field $\boldsymbol{F}$ and let $\mathbb{F} \supseteq \boldsymbol{F}$ be a field containing $\boldsymbol{F}$. Let W be a finite-dimensional vector space over $\mathbb{F}$. A $\boldsymbol{F}$-*representation* of the Clifford algebra $\mathrm{Cl}(V, \eta)$ is a $\boldsymbol{F}$-algebra homomorphism

$$\rho : \mathrm{Cl}(V, \eta) \to \mathrm{End}_{\mathbb{F}}(W).$$

The representation space W is called a $\mathrm{Cl}(V, \eta)$-*module* over $\mathbb{F}$.

Proposition 2.22. $\mathbb{C}\mathrm{l}_{2k}$ *has a unique faithful irreducible representation*

$$\rho_{2k} : \mathbb{C}\mathrm{l}_{2k} \to \mathrm{End}(\varDelta_{2k}), \quad \varDelta_{2k} = \mathbb{C}^{2^k}$$

and $\mathbb{C}\mathrm{l}_{2k+1}$ *has two irreducible representation* ρ^+ *and* ρ^-*, such that*

$$\rho_{2k+1} = \rho^+ \oplus \rho^- : \mathbb{C}\mathrm{l}_{2k+1} \to \mathrm{End}(\varDelta_{2k+1}) \oplus \mathrm{End}(\varDelta_{2k+1}), \quad \varDelta_{2k+1} = \mathbb{C}^{2^k}$$

is faithful.

Proof. We follow [4, Satz 1.3], and give an explicit representation. We define the matrices

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We first assume $n = 2k$, then for $1 \leqslant j \leqslant n$ we define

$$\begin{aligned} \rho_n(e_{2j-1}) &= iW \otimes W \otimes \cdots \otimes W \otimes U \otimes \boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I}, \\ \rho_n(e_{2j}) &= iW \otimes W \otimes \cdots \otimes W \otimes V \otimes \boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I}. \end{aligned}$$

One can check that the matrices $\rho(e_j)$ satisfy the Clifford relations and that they generate $M(2^k, \mathbb{C})$. Hence ρ is an algebra-isomorphism.

In the odd case, if $n = 2k + 1$, we define

$$\begin{aligned} \rho_n(e_j) &= (\rho_{2k}(e_j), \rho_{2k}(e_j)) \quad 1 \leqslant j \leqslant m, \\ \rho_n(e_n) &= (W \otimes \cdots \otimes W, -W \otimes \cdots \otimes W). \end{aligned}$$

One can again check that the matrices $\rho(e_j)$ satisfy the clifford relations and that they generate $M(2^k, \mathbb{C})$. The result now follows. □

We call $\varDelta_n$ a n-spinor module. From these representations we can construct representations of $\mathbb{C}\mathrm{l}_{r,s} = \mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$, where by exception we use the the inner-product $\eta^{(r,s)}$ to define $\mathbb{C}\mathrm{l}_{r,s}$.

Proposition 2.23. *If* $r + s$ *is even* $\mathbb{C}\mathrm{l}_{r,s}$ *has a unique faithful irreducible* $\mathbb{C}$*-representation*

$$\rho_{r,s} : \mathbb{C}\mathrm{l}_{r,s} \to \mathrm{End}(\varDelta_{r+s}).$$

If $r + s$ *is odd* $\mathbb{C}\mathrm{l}_{r,s}$ *has two irreducible* $\mathbb{C}$*-representations* $\rho^-_{r,s}$,$\rho^+_{r,s}$ *such that*

$$\rho_{r,s} = \rho^+_{r,s} \oplus \rho^-_{r,s} : \mathbb{C}\mathrm{l}_{r,s} \to \mathrm{End}(\varDelta_{r+s}) \oplus \mathrm{End}(\varDelta_{r+s})$$

is faithful.

Proof. Given the action $\rho_n : \mathbb{C}\mathrm{l}_n \to \Delta_n$, we define the actions $\rho_{r,s} : \mathbb{C}\mathrm{l}_{r,s} \to \Delta_{r+s}$, as follows. Let $\{e_1, \dots, e_{r+s}\}$ be the standard pseudo-orthogonal basis of $\mathbb{R}^{r,s}$. Then we define

$$\rho_{r,s}(e_j) = i\rho_n(e_j)\ \forall\ 1 \leqslant j \leqslant s \qquad \rho_{r,s}(e_j) = \rho_n(e_j)\ \forall\ s+1 \leqslant j \leqslant r+s.$$

One can easily check that $\rho_{r,s}(e_j)$ satisfy the Clifford relations. Uniqueness follows from the fact that $\rho_{r,s}$ are algebra-isomorphisms, in the same way ρ_n are, as $\mathbb{C}\mathrm{l}_{r,s} \cong \mathbb{C}\mathrm{l}_n$. □

One can restrict $\rho_{r,s}$ to $\mathrm{Cl}_{r,s}$ to obtain irreducible complex representations of $\mathrm{Cl}_{r,s}$

Remark 2.24. If there can be no confusion we will we often drop the index n or indices r, s on the representation ρ. ♢

Remark 2.25. One can also look for real representations of $\mathrm{Cl}_{r,s}$. In fact $\mathrm{Cl}_{r,s}$ has two inequivalent irreducible real representations if

$$s + 1 - r = 0 \mod 4$$

and one otherwise, see [31, Thm. 5.7]. These are not relevant to us, as we are only looking for complex representations. ♢

For the real Clifford algebra's $\mathrm{Cl}_n^\pm$ there is also another equivalent way to obtain irreducible representations from the representations of its complexification $\mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n^+ \otimes_\mathbb{R} \mathbb{C}$. Instead of modifying the representation, we find another embedding of $\mathrm{Cl}_n^\pm$ in $\mathbb{C}\mathrm{l}_n$. This will be done by using anti-linear maps $J_n^\pm$ on the n-spinor modules.

Proposition 2.26. *For any $k \geqslant 1$ there exist anti-linear operators*

$$J_{2k}^\pm : \Delta_{2k} \to \Delta_{2k}, \quad J_{2k+1}^\pm : \Delta_{2k+1} \to \Delta_{2k+1}$$

such that

$$\mathrm{Cl}_{2k}^\pm \cong \{a \in \mathbb{C}\mathrm{l}_{2k} \mid [J_{2k}^\pm, \rho(a)] = 0\},$$
$$(\mathrm{Cl}_{2k+1}^\pm)^0 \cong \{a \in \mathbb{C}\mathrm{l}_{2k+1}^0 \mid [J_{2k+1}^\pm, \rho(a)] = 0\}.$$

The operator J_n^- satisfies for $x \in \mathbb{C}\mathrm{l}_n^1$

$$(J_n^-)^2 = \varepsilon \qquad J_n^- x = \varepsilon' x J_n^-, \qquad J_n^- \Gamma_n = \varepsilon'' \Gamma_n J_n^-,$$

where $\varepsilon, \varepsilon', \varepsilon''$ are given in Table 1 for n modulo 8.

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Table 1: The values of $\varepsilon, \varepsilon', \varepsilon''$ depending on the dimension n modulo 8.

Proof. We refer to e.g. [50, Prop. 4.7]. We note that

$$J_3^- = J_2^- : \mathbb{C}^2 \to \mathbb{C}^2$$
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -v_2^* \\ v_1^* \end{pmatrix},$$

and $J_4^- = J_2^- \oplus J_2^-$. □

Definition 2.27. Let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{C}^n$. The *chirality element* in $\mathbb{C}\mathrm{l}_n$ is defined by

$$\Gamma_n = (-i)^{\lceil (n-1)/2 \rceil} e_1 \cdots e_n = (-i)^m e_1 \cdots e_n \in \mathbb{C}\mathrm{l}_n,$$

where $n = 2m$ if n is even, and $n = 2m+1$ if n is odd.

Proposition 2.28. *The chirality element squares to the identity, i.e.* $\Gamma_n^2 = 1$. *Moreover, for* $v \in \mathbb{R}^n$, *we have*

$$v\Gamma_n = -\Gamma_n v \ (n \text{ even}), \qquad v\Gamma_n = \Gamma_n v \ (n \text{ odd}). \tag{2}$$

In general for $a \in \mathbb{C}\mathrm{l}_n$,

$$a\Gamma_n = \Gamma_n \mathsf{p}(a) \ (n \text{ even}), \qquad a\Gamma_n = \Gamma_n a \ (n \text{ odd}). \tag{3}$$

Proof. Let $n = 2m$ if n is even and $n = 2m+1$ if n is odd. We first note that

$$e_1 \cdots e_n = (-1)^{\sum_{k=1}^{n-1} k} e_n \cdots e_1 = (-1)^{n(n-1)/2} e_n \cdots e_1 = (-1)^m e_n \cdots e_1,$$

since $n(n-1)/2 = m \mod 2$. Now it follows immediately that

$$\Gamma_n^2 = (-i)^{2m} e_1 \cdots e_n \cdot e_1 \cdot e_n = (-1)^m (-1)^m e_1^2 \cdots e_n^2 = 1.$$

This proves the first claim. For every $i \in 1, \ldots, n$ we have $e_i \Gamma_n = (-1)^{n-1} \Gamma_n e_i$, since e_i anti-commutes with all e_j, except when $j = i$ and then they commute. The result now follows as $n-1$ is odd as n is even, and vice versa. □

Remark 2.29. When $e_1, \ldots, e_n$ is a pseudo-orthogonal basis of $\mathbb{C}^n$, such that

$$\langle e_j, e_k \rangle = \eta_{jk}^{(r,s)},$$

the chirality element is given by

$$\Gamma_n = (-i)^m i^s e_1 \cdot e_n,$$

where again $n = 2m$ if n is even, and $n = 2m+1$ if n is odd, since now

$$\{ie_1, \ldots, ie_s, e_{s+1}, \ldots, e_{r+s}\}$$

is an orthonormal basis. ♢

We can define projections

$$P^{\pm} = \frac{1}{2}(1 \pm \Gamma_n),$$

satisfying

$$P^{+} + P^{-} = \mathrm{id}, \quad P^{+}P^{-} = P^{-}P^{+} = 0,$$

as one can easily check.

If n is odd, this induces a decomposition

$$\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}_n^{+} \oplus \mathbb{C}\mathrm{l}_n^{-},$$

where

$$\mathbb{C}\mathrm{l}_n^{\pm} = P^{\pm} \cdot \mathbb{C}\mathrm{l}_n .$$

As $\mathsf{p}(\Gamma_n) = -\Gamma_n$, we have

$$\mathsf{p}(\mathbb{C}\mathrm{l}_n^{\pm}) = \mathbb{C}\mathrm{l}_n^{\mp} .$$

This means that $\mathbb{C}\mathrm{l}_n^{0}$ has to be diagonally embedded in this decomposition, i.e.

$$\mathbb{C}\mathrm{l}_n^{0} = \{a + \mathsf{p}(a) \mid a \in \mathbb{C}\mathrm{l}_n^{+}\}. \tag{4}$$

If n is even on the other hand we can decompose the spinor module into eigenspaces of $\rho(\Gamma_n)$ corresponding to the eigenvalues ± 1:

$$\Delta_n = \Delta_n^{+} \oplus \Delta_n^{-}, \quad \Delta_n^{\pm} = \{\alpha \in \Delta_n : \rho(\Gamma_n)\alpha = \pm\alpha\},$$

and define the representations $\rho^{\pm}$ with respect to this decomposition such that

$$\rho = (\rho^{+}, \rho^{-}) : \mathbb{C}\mathrm{l}_n \to \Delta_n^{+} \oplus \Delta_n^{-}.$$

The projection onto $\Delta_n^{\pm}$ are given by $\rho_n(P^{\pm})$.

Proposition 2.30. *The followings holds for the spin representations of* $\mathbb{C}\mathrm{l}_n$.

1. *If n is odd, ρ decomposes into two non-isomorphic inequivalent sub-representations*

$$\rho^{\pm} : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Delta_{n+1}^{\pm}).$$

The two representations ρ^{+}, ρ^{-} are distinguished by the action of the chirality element,

$$\rho^{+}(\Gamma_n) = \mathrm{id}, \quad \rho^{-}(\Gamma_n) = -\,\mathrm{id} .$$

When restricted to $\mathrm{Cl}_{r,s}^{0}$ the two representations become equivalent.

2. *If n is even, ρ restricted to $\mathbb{C}\mathrm{l}_n^{0}$, decomposes into two non-isomorphic inequivalent irreducible sub-representations*

$$\rho^{\pm} : \mathbb{C}\mathrm{l}_n^{0} \to \mathrm{End}(\Delta_n^{\pm}).$$

Proof. Let n be odd. By Proposition 2.17 we have an isomorphism

$$\Psi : \mathbb{C}\mathrm{l}_n \to \mathbb{C}\mathrm{l}_{n+1}^{0} .$$

By Remark 2.18 we have $\Psi(\Gamma_n) = \Gamma_{n+1}$. Moreover Ψ induces an isomorphism of representations

$$\tilde{\Psi} : \mathrm{End}(\Delta_n) \to \mathrm{End}^0(\Delta_{n+1}) = \mathrm{End}^0(\Delta_{n+1}^+ \oplus \Delta_{n+1}^-)$$

of representations of $\mathbb{C}\mathrm{l}_n$ and representations of $\mathbb{C}\mathrm{l}_n^0$. Note that

$$\mathrm{End}^0(\Delta_{n+1}^+ \oplus \Delta_{n+1}^-) = \begin{pmatrix} \mathrm{End}^0(\Delta_{n+1}^+) & \mathrm{Hom}^0(\Delta_{n+1}^+, \Delta_{n+1}^-) \\ \mathrm{Hom}^0(\Delta_{n+1}^-, \Delta_{n+1}^+) & \mathrm{End}^0(\Delta_{n+1}^-). \end{pmatrix}$$

Let $a \in \mathbb{C}\mathrm{l}_{n+1}^0$, then we have

$$\begin{aligned} \tilde{\Psi}\left(\rho_n(\Gamma_n)\rho_n(a)\right) &= \rho_{n+1}(\Gamma_{n+1})\rho_{n+1}(\Psi(a)) = -\rho_{n+1}(\Psi(a))\rho_{n+1}(\Gamma_{n+1}) \\ &= -\tilde{\Psi}\left(\rho_n(a)\rho_n(\Gamma_n)\right), \end{aligned}$$

using Proposition 2.28 and the fact that $\Psi(a)$ is odd. On the other hand

$$\rho_n(\Gamma_n)\rho_n(a) = \rho_n(\Gamma_n a) = \rho_n(a\Gamma_n) = \rho_n(a)\rho_n(\Gamma_n),$$

again by Proposition 2.28, hence $\tilde{\Psi}(\rho_n(a)) = 0$ for a even. This shows

$$\mathrm{Hom}^0(\Delta_{n+1}^-, \Delta_{n+1}^+) = \mathrm{Hom}^0(\Delta_{n+1}^+, \Delta_{n+1}^1) = \varnothing.$$

Moreover for $b \in \mathbb{C}\mathrm{l}_{n+1}$ and $\alpha_\pm \in \Delta_{n+1}^\pm$, we have

$$\rho_{n+1}(b)\alpha_\pm = \pm\rho_{n+1}(b)\rho_{n+1}(\Gamma_{n+1})\alpha_\pm = \pm\rho_{n+1}(\Gamma_{n+1})\rho_{n+1}(\mathsf{p}(b))\alpha_\pm = \mathsf{p}(b)\alpha_\pm.$$

So we find that if $b\Delta_{n+1}^\pm \subseteq \Delta_{n+1}^\pm$, then $b \in \mathbb{C}\mathrm{l}_n^0$. Therefore,

$$\mathrm{End}^0(\Delta_{n+1}^\pm) = \mathrm{End}(\Delta_{n+1}^\pm).$$

We thus have a decomposition of ρ_n:

$$\rho_n = \rho_n^+ \oplus \rho_n^- : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Delta_n) \cong \mathrm{End}(\Delta_{n+1}^+) \oplus \mathrm{End}(\Delta_{n+1}^-),$$

where

$$\rho_n^\pm(a) = \rho_{n+1}^\pm(\Psi(a)).$$

As Γ_n is in the center of $\mathbb{C}\mathrm{l}_n$, the two sub-representations

$$\rho_n^\pm : \mathbb{C}\mathrm{l}_n \to \mathrm{End}(\Delta_{n+1}^\pm)$$

are invariant under $\mathbb{C}\mathrm{l}_n$ and hence irreducible. They are also inequivalent as

$$\rho_n^\pm(\Gamma_n) = \rho_{n+1}^\pm(\Gamma_{n+1}) = \pm\,\mathrm{id}_{\Delta_{n+1}^\pm}.$$

But if we restrict them to $\mathbb{C}\mathrm{l}_n^0$ the representations become equivalent, because of the diagonal embedding given in Eq. (4).

Now let n be even. We have already done most of the work. Indeed we already found that

$$\mathrm{End}^0(\Delta_n) \cong \mathrm{End}(\Delta_n^+) \oplus \mathrm{End}(\Delta_n^-).$$

By Proposition 2.28 we see that $\Delta_n^\pm$ are invariant under $\mathbb{C}\mathrm{l}_n^0$, so they are irreducible representations of $\mathbb{C}\mathrm{l}_n^0$. They are obviously inequivalent by definition, as the Γ_n acts as $\pm\,\mathrm{id}$ on $\Delta_n^\pm$. □

When we restrict these representations to the spin group, we get faithful spin representations.

Definition 2.31. We define the *spin representations*

$$\rho^+ \oplus \rho^- : \mathrm{Spin}_0(r,s) \to \mathrm{Aut}(\Delta^+_{r+s}) \oplus \mathrm{Aut}(\Delta^-_{r+s}) \subseteq \mathrm{Aut}(\Delta_{r+s}) \qquad \text{for } r+s \text{ even,} \tag{5}$$

$$\rho := \rho^+ : \mathrm{Spin}_0(r,s) \to \mathrm{Aut}(\Delta_{r+s}) \qquad \text{for } r+s \text{ odd,} \tag{6}$$

as the restrictions of the Clifford representations given in Proposition 2.23.

Proposition 2.32. *The spin representations ρ^+, ρ^- for $r+s$ even, and ρ for $r+s$ even are irreducible.*

These representations extend to irreducible representations of $\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$ *via*

$$\rho^{\pm}([(g,z)])\alpha = z \cdot \rho^{\pm}(g)\alpha, \quad \rho([(g,z)])\alpha = z \cdot \rho(g)\alpha.$$

Proof. This follows almost immediately if one notices that the algebra generated by $\mathrm{Spin}_0(r,s)$ is isomorphic to $\mathbb{C}\mathrm{l}_{r+s}$. □

Proposition 2.33. *Let $r = 2m+1$,*

$$\rho^+_r \oplus \rho^-_r : \mathrm{Cl}_{r,0} \to \mathrm{End}(\Delta_r) \oplus \mathrm{End}(\Delta_r),$$

and

$$\rho^{\pm}_{r,1} : \mathrm{Cl}^0_{r,1} \to \mathrm{End}(\Delta^{\pm}_{r+1})$$

be the representations as defined above. Under the isomorphism

$$\mathrm{Cl}_{r,0} \cong \mathrm{Cl}^0_{r,1}$$

given by Proposition 2.17, we have

$$\rho^+_{r,1} \cong \rho^+_r, \quad \rho^-_{r,1} \cong \rho^-_r .$$

Proof. This follows immediately by considering the action of the chirality element Γ_n on both representations. □

Proposition 2.34. *For every representation*

$$\rho_{r,s} : \mathrm{Cl}_{r,s} \to \mathrm{End}(\Delta_{r+s}),$$

there is a $\mathrm{Spin}_0(r,s)$ invariant hermitian (possible indefinite) inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle \rho_{r,s}(v)\alpha, \beta \rangle = (-1)^s \langle \alpha, \rho_{r,s}(v)(\beta) \rangle. \tag{7}$$

Proof. Let h be the standard Hermitian inner product on $\Delta_{r+s} = \mathbb{C}^{2^k}$, given by

$$h(\alpha, \beta) = \alpha^{*T}\beta,$$

which is $\mathrm{Spin}(n)$ invariant. Let $\{e_1, \dots, e_n\}$ be a pseudo-orthogonal bases of $\mathbb{R}^{r,s}$. We have

$$h(\rho_{r,s}(e_i)\alpha, \beta) = h(\alpha, \rho_{r,s}(e_i)\beta) \quad \forall\, i \in \{s+1, \dots, s+r\} \tag{8}$$

$$h(\rho_{r,s}(e_j)\alpha, \beta) = -h(\alpha, \rho_{r,s}(e_j)\beta) \quad \forall\, j \in \{1, \dots, s\}. \tag{9}$$

Let $s = 2k$ if s is even and $s = 2k + 1$ if s is odd and set

$$\hat{\Gamma}_s = (-i)^{k+s} e_1 \cdots e_s. \tag{10}$$

For $i \in \{s+1, \ldots, s+r\},\ j \in \{1, \ldots, s\}$ we have

$$\hat{\Gamma}_s^2 = 1, \qquad e_i \hat{\Gamma}_s = (-1)^s \hat{\Gamma}_r e_i, \qquad e_j \hat{\Gamma}_s = (-1)^{s-1} \hat{\Gamma}_r e_j.$$

We also have

$$h(\rho_{r,s}(\hat{\Gamma}_s)\alpha, \beta) = h(\alpha, \rho_{r,s}(\hat{\Gamma}_s)\beta),$$

since

$$e_{r+1} \cdots e_{r+s} = (-1)^k e_{r+s} \cdots e_{r+1}$$

and Eq. (9). Now the inner product $\langle \cdot, \cdot \rangle : \Delta_{r+s} \to \mathbb{C}$, defined by

$$\langle \alpha, \beta \rangle = h(\rho_{r,s}(\hat{\Gamma}_s)\alpha, \beta)$$

is an hermitian inner product. Indeed,

$$\langle \alpha, \beta \rangle^* = h(\rho_{r,s}(\hat{\Gamma}_s)\alpha, \beta)^* = h(\beta, \rho_{r,s}(\hat{\Gamma}_s)\alpha) = h(\rho_{r,s}(\hat{\Gamma}_s)\beta, \alpha) = \langle \beta, \alpha \rangle.$$

For $i \in \{s+1, \ldots, s+r\}$, we have

$$\begin{aligned}\langle \rho_{r,s}(e_i)\alpha, \beta \rangle &= h(\rho_{r,s}(\hat{\Gamma}_s e_i)\alpha, \beta) = (-1)^s h(\rho_{r,s}(e_i \hat{\Gamma}_s)\alpha, \beta) \\ &= (-1)^s h(\rho_{r,s}(\hat{\Gamma}_s)\alpha, \rho_{r,s}(e_i)\beta) = (-1)^s \langle \alpha, \rho_{r,s}(e_i)\beta \rangle.\end{aligned}$$

And similarly, for $j \in \{1, \ldots, s\}$

$$\begin{aligned}\langle \rho_{r,s}(e_j)\alpha, \beta \rangle &= h(\rho_{r,s}(\hat{\Gamma}_s e_j)\alpha, \beta) = (-1)^{s-1} h(\rho_{r,s}(e_j \hat{\Gamma}_s)\alpha, \beta) \\ &= (-1)^s h(\rho_{r,s}(\hat{\Gamma}_s)\alpha, \rho_{r,s}(e_j)\beta) = (-1)^s \langle \alpha, \rho_{r,s}(e_j)\beta \rangle.\end{aligned}$$

This proves Eq. (7). To prove $\mathrm{Spin}_0(r, s)$-invariance, let $g = v_1 \cdots v_k \in \mathrm{Spin}_0(r, s)$. Using $v_k \cdots v_1 \cdot v_1 \cdots v_k = 1$ and the fact that k is even, we obtain

$$\langle \rho_{r,s}(g)\alpha, \rho_{r,s}(g)\beta \rangle = (-1)^{ks} \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle.$$

□

2.2.1 Principal bundles

We want to define the spinor bundle as an associated vector bundle of a $\mathrm{Spin}_0(r, s)$-principal bundle. We will first give an introduction into principal bundles, following [43].

We recall the following about Lie groups and Lie algebras. We will assume that G is a matrix-Lie group, although everything holds for general Lie groups, unless it is explicitly stated for matrix Lie groups. The Lie algebra of G is given by $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$ where e denotes the unit element of G. The exponential map $\exp : \mathfrak{g} \to G$ is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

for matrix Lie Groups. If $\phi : G \to H$ is a homomorphism of Lie Groups, we have

$$\phi \circ \exp_G = d_e\phi \circ \exp_H .$$

For any $g \in G$ we have the conjugation map

$$C_g : G \to G, \quad C_g(h) = g^{-1}hg.$$

The derivative of this map is denoted by

$$\mathrm{Ad}(g) = d_eC_g : \mathfrak{g} \to \mathfrak{g},$$

and the map

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}(g)$$

is called the adjoint represention of G. It is a Lie group homomorphism. For matrix Lie algebra's, we have

$$Ad(g)A = gAg^{-1}.$$

Differentiating Ad at the identity element, we get

$$\mathrm{ad} = d_e\,\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

For matrix Lie algebra's one can check that ad is given by

$$\mathrm{ad}(A)B = [A, B] = AB - BA.$$

Since Ad is a Lie group homomorphism we have

$$\mathrm{Ad} \circ \exp = \exp \circ\, \mathrm{ad},$$

as is easily checked for matrix Lie groups.

Example 2.35. The Lie algebra of $\mathrm{SO}(r, s)$, is given by

$$\mathfrak{so}_{r,s} = \{A \in M_n \mid \eta^{(r,s)}(Ax, y) = -\eta^{(r,s)}(x, Ay)\},$$

and the Lie algebra of $\mathrm{Spin}(r, s)$ is given by

$$\mathfrak{spin}_{r,s} = \mathrm{span}\{e_ie_j \mid 1 \leqslant i < j \leqslant r + s\} \subset \mathrm{Cl}^0_{r,s} .$$

In fact as the spin group is a double covering of the special orthogonal group, their Lie algebra's coincide. We will give an explicit isomorphism is Proposition 2.77. $\lhd$

Definition 2.36 (Principal bundle)**.** Let M be a manifold and let G be a Lie group. A *principal G-bundle* is given by a surjective submersion $\pi : P \to M$ with a free right action of G on P along the fibers of π such that $P/G \cong M$, which is locally trivial.

That is for every $x \in M$ there exists an open neighbourhood U of x and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$, such that

1. For every $g \in G$, $p \in \pi^{-1}(U)$, we have

$$\phi_U(pg) = \phi_U(p)g = (x, hg),$$

where $\phi_U(p) = (x, h)$.

2. $\mathrm{pr}_U \circ \phi_U(p) = \pi(p)$ for all $p \in \pi^{-1}(U)$, i.e. the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathrm{pr}_U} & \\ U & & \end{array}$$

commutes.

For a point $x \in M$ we call $P_x := \pi^{-1}(x)$ the *fiber* over x. The diffeomorphism ϕ_U is called a *local trivialization.* By definition one can choose a countable open covering $\{U_\alpha\}$ of M such that there are local trivializations $\phi_\alpha := \phi_{U_\alpha}$. The collection $\{U_\alpha, \phi_\alpha\}$ is called a *bundle atlas.*

Remark 2.37. One can easily check that each fiber carries a free and transitive action of G. ♢

Definition 2.38. Given two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β) we define the *transition function*

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

by the equation

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, g) = (x, g\varphi_{\alpha\beta}(x)).$$

Note that since the action of G is free and transitive on each fibre $\varphi_{\alpha\beta}$ is well-defined.

Definition 2.39. A (local) section of a principal bundle $\pi : P \to M$ is a smooth map

$$s : U \to P$$

for an open subset $U \subseteq M$, such that $\pi \circ s = \mathrm{id}\,|_U$. A global section is a section $s : M \to P$.

Proposition 2.40. *Local trivializations of P are in one-to-one correspondence with local sections.*

Proof. Let $\phi_U : \pi^{-1}(U) \to U \times G$ be a local trivialization. Then

$$s : U \to P \quad s(x) = \phi_U^{-1}(x, 1)$$

is a local section. Here we denoted the unit element of G by 1.

Conversely let $s : U \to P$ be a local section. Since the action of G on the fibres is free and transitive, for every $p \in P_x$ there is a unique g_p such that $p = s(x)g_p$. This defines a smooth map $\kappa : P \to G$ given by $\kappa(p) = g_p$. Now

$$\pi \times \kappa : \pi^{-1}(U) \to U \times G$$

is a local trivialization, since for $p \in P_x$

$$(\pi \times \kappa)(pg) = (\pi \times \kappa)(s(x)g_p g) = (x, g_p g).$$

□

Definition 2.41. Let $\pi_1 : P_1 \to M_1$ be a G_1-principal bundle and $\pi_2 : P_2 \to M_2$ be a G_2-principal bundle.

1. A morphism of principal bundles from P_1 to P_2 is a pair of mapping (θ, λ) where $\theta : P_1 \to P_2$ is smooth and $\lambda : G_1 \to G_2$ is a homomorphism of Lie groups, such that for all $p \in P_1, g \in G_1$.
$$\theta(pg) = \theta(p)\lambda(g). \tag{11}$$
2. (θ, λ) is called an isomorphism if θ is a diffeomorphism and λ an isomorphism of Lie groups.

Remark 2.42. By the condition Eq. (11) and the fact that a fiber carries a transitive action of G, θ maps fibres to fibres. Thus it induces a mapping $\bar{\theta} : M_1 \to M_2$, such that the following diagram commutes

$$\begin{array}{ccc} P_1 & \xrightarrow{\theta} & P_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\bar{\theta}} & M_2 \end{array}$$

◊

Definition 2.43. Let $\pi_1 : P_1 \to M$ be a G_1-principal bundle and $\pi_2 : P_2 \to M$ be a G_2-principal bundle over the same base manifold M and let (θ, λ) be a morphism between P_1 and P_2.

1. If $\bar{\theta} = \mathrm{id}_M$ then (θ, λ) is said to be a *vertical bundle morphism.*
2. If moreover $G_1 = G_2$ and $\lambda = \mathrm{id}_G$, then θ is called a *G-morphism.*

Given a principal G-bundle $\pi : P \to M$ and a representation $\rho : G \to \mathrm{GL}(V)$ we can form an associated vector bundle $P \times_G V$.

Proposition 2.44 (Associated vector bundle)**.** *Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G on a vector space V. The space*
$$P \times_G V := (P \times V)/\sim,$$
where
$$(p_1, v_1) \sim (p_2, v_2) \text{ if and only if } \exists g \in G \text{ s.t. } (p_1, v_1) = (p_2 g^{-1}, \rho(g)v_2),$$
has a canonical structure of a vector bundle.

Proof. Let $E = P \times_G V$. Elements in E are denoted by $[p, v] \in E$ for representatives $(p, v) \in P \times V$. We have $[pg, v] = [p, gv]$, where we have written gv instead of $\rho(g)v$. We define the projection
$$\tilde{\pi} : E \to M, \quad \tilde{\pi}([p, v]) = \pi(p).$$
This is well-defined as $\pi(pg) = \pi(p)$, hence we have fibres $E_x := \tilde{\pi}^{-1}(x) = \{[p, v] \mid p \in P_x, v \in V\}$ isomorphic to V. Let $x \in M$ arbitrary, then we have a open neighbourhood U of x and a smooth local trivialization $\phi_U = \phi_U^1 \times \phi_U^2 : \pi^{-1}(U) \to U \times G$, using which we can define a local trivialization of E over U. Indeed, we define $\tilde{\phi}_U : \pi^{-1}(U) \to U \times V$ by
$$\tilde{\phi}_U([p, v]) = (\phi_U^1(p), \phi_U^2(p)v) = (\pi(p), \phi_U^2(p)v).$$
This is well-defined, as
$$\tilde{\phi}_U([pg^{-1}, gv]) = (\pi(pg^{-1}), \phi_U^2(pg^{-1})gv) = (\pi(p), \phi_U^2(p)g^{-1}gv) = (\pi(p), \phi_U^2(p)v).$$
So we have defined a smooth local trivialization of $E \to M$ over U, hence we conclude that $E = P \times_G V$ is a smooth vector bundle. □

Sections of associated vector bundle are conveniently described by equivarient maps.

Definition 2.45. Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G on a vector space V. A map

$$\Psi : P \to V$$

is called equivariant if

$$\Psi(pg) = \rho(g^{-1})\Psi(p),$$

for all $p \in P, g \in G$. The set of all smooth equivariant maps $\Psi : P \to V$ is denoted by $\mathrm{Hom}_G(P, V)$.

Proposition 2.46. *Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G on a vector space V. Smooth sections of the associated vector bundle $P \times_G V$ are in one to one correspondence with smooth equivariant maps $\Psi : P \to V$.*

Proof. Let $\Psi \in \mathrm{Hom}_G(P, V)$. We will define a section

$$\sigma : M \to P \times_G V.$$

For any $x \in M$, choose $p \in P_x$ arbitrary (i.e. choose any local section around x), then we define

$$\sigma(x) = [p, \Psi(p)].$$

This is well-defined, since as we had taken another $q = pg \in P_x$, we have

$$[pg, \Psi(pg)] = [pg, \rho(g^{-1})\Psi(p)] = [pgg^{-1}, \Psi(p)] = [p, \Psi(p)].$$

Conversely, let $\sigma : M \to P \times_G V$ be a section. We will define a map $\Psi : P \to V$. Let $x \in M$, and $s(x) = [p, v]$ for any $q \in P_x$, there is a $g \in G$ such that $q = pg$. We define

$$\Psi(q) = \rho(g^{-1})v.$$

This is indeed equivariant, since for every $h \in G$ we have

$$\Psi(qh) = \Psi(pgh) = \rho((gh)^{-1})v = \rho(h^{-1})\rho(g^{-1})v = \rho(h^{-1})\Psi(q).$$

One easily checks that

$$\sigma \circ \pi = (\mathrm{id}_P, \Psi) : P \to P \times_G V.$$

□

Remark 2.47. 1. Sometimes the equivalence relation on $P \times V$ defining $P \times_G V$ is given by

$$(p_1, v_1) \sim (p_2, v_2) \text{ if and only if } \exists g \in G \text{ s.t. } (p_1, v_1) = (p_2 g, \rho(g)v_2).$$

Then a equivariant mapping has to be defined as map $\Psi : P \to V$ such that

$$\Psi(pg) = \rho(g)\Psi(p),$$

for the equivalence above to work

2. If $s : U \to P$ is a local section, we can pull back any equivariant mapping $\Psi : P \to V$, defining a map $\psi = s^*\Psi : M \to V$. This maps induces a section $\tau : U \to P \times_G V$ such that $\tau = \sigma\,|_U$, where σ is the global section induced by Ψ, because the definition of σ is independent of the section $s : M \to P$.

$\diamondsuit$

Given a vector bundle and a free and transitive right action of a Lie group G on a subset of its frames, one can also form a principal bundle from it, see e.g. [45, Par. 1.9] or [43, Par. 1.1].

Proposition 2.48. *Let G be any Lie subgroup $GL(k, \mathbb{R})$. Let $E \to M$ be a vector bundle of rank k over M, and let $F_G(E_x)$ be space of all bases in the fibre E_x, such that we have a free and transitive right action of G on $F_G(E_x)$. Then*

$$F_G(E) := \coprod_{x \in M} F(E_x)$$

carries the structure of a G-principal bundle over M.

Proof. Let $F_G(E_x)$ be the set of all bases of the vector space E_x such that we have a free and transitive right action of G on $F_G(E_x)$. Then define

$$F_G(E) := \coprod_{x \in M} F(E_x) = \{(x, s) \mid x \in M, s \in F_G(E_x)\}.$$

We have a canonical projection $\pi : F_G(E) \to M$, which assigns to every G-basis at x the point x. We often write $s \in F_G(E)$ instead of $(x, s) \in F_G(E)$ to simplify notation, and set $\pi(s) = x$. By definition for any G-basis at x, $s_x = (s_x^1, \ldots, s_x^k)$ and $A \in G$ the ordered set

$$(As_x)^j := \sum_{i=1}^{k} s_x^i A^i{}_j$$

is again a G-basis at x. We thus get a right action $R : F_G(E) \times G \to F_G(E)$, given by

$$R_A s = R(s, A) = sA,$$

which is obviously free. A G-frame over U is defined as an ordered set of sections

$$s = (s_1, \ldots, s_k) : U \to E,$$

such that for all $x \in M$, $s_x \in F_G(E_x)$. For any $x \in M$ there is a open neighbourhoud $U \subseteq M$ and a local G-frame $s : U \to E$. Given such a local G-frame s, for any $\sigma \in \pi^{-1}(U) \subseteq F_G(E)$ there is an $A_s(\sigma) \in G$ such that $\sigma = A_s(\sigma)s_{\pi(\sigma)}$. This defines a bijection

$$\phi_U : \pi^{-1}(U) \to U \times G, \quad \phi_U(\sigma) = (\pi(\sigma), A_s(\sigma)).$$

We equip $F_G(E)$ with a smooth structure by requiring that all such ϕ_U are diffeomorphisms. Then $F_G(E)$ with local trivializations ϕ_U has the structure of a principal bundle. This follows by definition of ϕ_U as,

$$\phi_U(\sigma g) = (\pi(\sigma), A_s(\sigma)g) = \phi_U(\sigma)g,$$

and

$$\mathrm{pr}_U \circ \phi_U(\sigma) = \pi(\sigma).$$

$\square$

Taking the tangent space as the vector bundle and $G = O(r,s,\mathbb{R})$ or $\mathrm{SO}_0(r,s,\mathbb{R})$, we get the following result.

Corollary 2.49. *Let (M,g) be a n-dimensional pseudo-Riemannian manifold of signature (r,s), where $r+s=n$. Let $F_{\mathrm{O}}(T_xM)$ be the set of pseudo-orthonormal bases of the tangent space T_xM. Then*

$$F_{\mathrm{O}}(TM) := \coprod_{x\in M} F_{\mathrm{O}}(T_xM)$$

carries the structure of a $O(r,s,\mathbb{R})$-principal bundle. If moreover M is orientable, we can choose an orientation. Let $F_{\mathrm{SO}_0}(T_xM)$ be set of positively time and space oriented pseudo-orthonormal bases, then

$$F_{\mathrm{SO}_0}(TM) := \coprod_{x\in M} F_{\mathrm{SO}_0}(T_xM)$$

carries the structure of a $\mathrm{SO}_0(r,s,\mathbb{R})$-principal bundle.

Definition 2.50. The principal bundle $F_{\mathrm{GL}(k,\mathbb{R})}(E)$ is called the *frame bundle* of E. The principal bundle $F_{\mathrm{O}}(TM)$ is the *orthonormal frame bundle* of TM, and $F_{\mathrm{SO}_0}(TM)$ is the *(space and time) oriented orthonormal frame bundle* of TM.

Definition 2.51. Let M be an oriented pseudo-Riemannian manifold of signature (r,s). If there exists a Spin^c-principal bundle $S^c \to M$ together with a smooth map: $\theta : S^c \to F_{\mathrm{SO}_0}(TM)$ such that (θ, Ad) is a vertical bundle morphism, we say that M is a *spin^c manifold.* We refer to (S^c, θ) as a spin$^{\mathrm{c}}$ structure.

Definition 2.52 (Spin Manifold)**.** Let M be an oriented pseudo-Riemannian manifold of signature (r,s). If there exists a $\mathrm{Spin}_0(r,s)$-principal bundle $S \to M$ together with a smooth map: $\theta : S \to F_{\mathrm{SO}_0}(TM)$ such that (θ, Ad) is a verticle bundle morphism, we say that M is a *spin manifold.* We refer to (S,θ) as a spin structure.

Remark 2.53.

1. One can check that $TM \cong F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathbb{R}^{r,s}$, and that

$$TM \cong S \times_{\mathrm{Spin}_0(r,s)} \mathbb{R}^{r,s},$$

is an equivalent condition for S to be a spin structure.

2. Any spin manifold is also a spin$^{\mathrm{c}}$ manifold. Indeed if (S,θ) is a spin-structure, then

$$S^c = S \times P_0,$$

defines a spin$^{\mathrm{c}}$ structure, where $P_0 = M \times U(1)$ is the trivial principal $U(1)$-bundle over M.

♢

Definition 2.54 (Spinor bundle)**.** Give a pseudo-Riemannian spin$^{\mathrm{c}}$ manifold with a fixed spin$^{\mathrm{c}}$ structure (S,θ). The associated vector bundle

$$\mathcal{S} := S^c \times_{\mathrm{Spin}^c} \varDelta_n,$$

where Spin^c acts on $\varDelta_n$ via the spin representation, is called the *spinor bundle.*

Remark 2.55. If we have a spin manifold with spin structure (S, θ), then the spinor bundle is isomorphic to

$$S \times_{\mathrm{Spin}_0(r,s)} \Delta_n,$$

using the isomorphism

$$\begin{aligned} S \times_{\mathrm{Spin}_0(r,s)} \Delta_n &\to (S \times P_0) \times_{\mathrm{Spin}^c} \Delta_n \\ [a, w] &\mapsto [(a, (\pi_S(a), 1)), w], \end{aligned}$$

where $\mathrm{Spin}^c(n) = \mathrm{Spin}_0(r, s) \times_{\mathbb{Z}^2} U(1)$. This is well-defined since

$$[(a, (\pi_S(a), -1)), w] = [(-a, (\pi_S(a), 1)), w] = [(a, (\pi_S(a), 1)), -w].$$

When we have a spin manifold, we will use this definition for the spinor bundle.

$\diamondsuit$

Definition 2.56. Let (M, g) be a pseudo-Riemannian manifold of signature (r, s). We define the Clifford bundles over M as associated vector bundles of the oriented orthonormal frame bundle:

$$\begin{aligned} \mathrm{Cl}(TM) &= F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathrm{Cl}_{r,s} \\ \mathrm{Cl}^-(TM) &= F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathrm{Cl}_{s,r} \\ \mathbb{C}\mathrm{l}(TM) &= F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathbb{C}\mathrm{l}_{r,s} . \end{aligned}$$

Here $A \in \mathrm{SO}_0(r, s)$ acts on $\mathrm{Cl}_{r,s}$ in the following way:

$$A \cdot (v_1 \cdots v_k) = Av_1 \cdots Av_k.$$

This action is well-defined for $\mathrm{Cl}(TM)$ and its complexification $\mathbb{C}\mathrm{l}(TM)$, as for $A \in \mathrm{SO}_0(r, s)$

$$\eta^{(r,s)}(Av, Aw) = \eta^{(r,s)}(v, w), \qquad v, w \in \mathbb{R}^{r+s}$$

For $\mathrm{Cl}^-(TM)$ it also is well-defined as

$$\eta^{(s,r)}(Av, Aw) = -\eta^{(r,s)}(Av, Aw) = -\eta^{(r,s)}(v, w) = \eta^{(s,r)}(v, w).$$

Definition 2.57. A vector bundle $E \to M$ of dimension k with metric g, with a mapping

$$\gamma : \Gamma(TM) \to \mathrm{End}(\Gamma(E)),$$

fulfilling $\gamma(X)^2 = g(X, X)$ is called a *Clifford module bundle*.

Remark 2.58. One can show that a oriented Riemannian manifold M, g is is a spin^c manifold if and only if there is a Clifford module bundle E such that

$$\mathrm{End}(E) = \mathbb{C}\mathrm{l}(TM) \text{ (n even)} \quad \text{or} \quad \mathrm{End}(E) = \mathbb{C}\mathrm{l}(TM)^0 \text{ (n odd)}.$$

The spinor bundle is the - up to isomorphism - unique Clifford bundle E such that this holds. See [40, Thm. 2.11]. This justifies using this as an alternative definition of the spinor bundle, which is widely used settings like Noncommutative Geometry. $\diamondsuit$

Definition 2.59. *Clifford multiplication*

$$c : \Gamma(TM) \times \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}),$$

is fibrewise defined as

$$c([\theta(b), v], [b, \alpha]) = [b, \rho(v)\alpha],$$

where $[\theta(b), v] \in F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathbb{R}^{r,s} \cong TM$. The *Clifford mapping*

$$\gamma : \Gamma(TM) \to \mathrm{End}(\Gamma(\mathcal{S}))$$

is given by

$$\gamma(X)\phi = c(X, \phi)$$

for $X \in \Gamma(TM), \phi \in \Gamma(\mathcal{S})$.

Remark 2.60. 1. This is well-defined, as for $g \in \mathrm{Spin}_0(r, s)$ we have

$$\begin{aligned} c([\theta(bg), v], [bg, \alpha]) &= c([\theta(b)\,\mathrm{Ad}(g), v], [bg, \alpha]) = c([\theta(b), \mathrm{Ad}(g)v], [b, g\alpha]) \\ &= [b, gvg^{-1}g\alpha] = [b, gv\alpha] = [bg, v\sigma]. \end{aligned}$$

2. Since $TM \subseteq \mathrm{Cl}(TM)$ generates $\mathrm{Cl}(TM)$ fibrewise, γ induces a unique homomorphism

$$\hat{\gamma} : \Gamma(\mathrm{Cl}(TM)) \to \mathrm{End}(\Gamma(\mathcal{S})),$$

which justifies why γ is called the Clifford representation.

♢

Definition 2.61. Let V be an inner product space. We say that an operator $J : V \to V$ is anti-unitary if for all $u, v \in V$

$$\langle Ju, Jv\rangle = \langle v, u\rangle.$$

Proposition 2.62. *Let M be a Riemannian n-dimensional spin manifold, with spin structure (S, θ) and*

$$\mathcal{S} = S \times_{\mathrm{Spin}(n,0)} \Delta_n.$$

There is a globally defined anti-unitary operator $J_M : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$, such that for $\psi \in \Gamma(\mathcal{S})$

$$(J_M\psi)(x) = J_n^-(\psi(x)).$$

The following two conditions hold

1. *J_M commutes with the action of real-valued continuous functions on $\Gamma(\mathcal{S})$;*
2. *J_M commutes with $\Gamma(\mathrm{Cl}^-(TM))$ if n is even, and with $\Gamma(\mathrm{Cl}^-(TM))^0$ if n is odd.*

Proof. Let $\psi \in \Gamma(\mathcal{S})$. Let $x \in M$ and $\psi(x) = [b, \alpha]$. We define J_M fibrewise by

$$(J_M\psi)(x) = [b, J_n^-\alpha].$$

This is well-defined, as for $g = v_1 \cdots v_k \in \mathrm{Spin}(n, 0)$, considered as element of $\mathbb{C}\mathrm{l}_n$, we have

$$v_1 \cdots v_k = (-1)^{k/2}(iv_1) \cdots (iv_k),$$

and because

$$(iv_j)(iv_k) + (iv_k)(iv_j) = -2\langle v_k, v_j\rangle,$$

we see $g \in (\mathrm{Cl}_n^-)^0$ and hence $J_n^-\rho(a) = \rho(a)J_n^-$. The two requirements on J_M follow immediately from the properties of J_n^-

□

This operator J_M is called the *charge conjugation* operator.

Remark 2.63. It can be shown that a Riemannian spin^c manifold is a spin manifold if and only if such a charge conjugation operator exits. See [40, Sec. 2.12] and Theorem 9.6 and the subsequent discussion in [26]. ♢

Proposition 2.64. *Let (M, g) be a spin manifold of signature (r, s). There is a* $\mathrm{Spin}_0(r, s)$*-invariant (possibly indefinite) Hermitian metric h on $\mathcal{S}$ such that*

$$h(\gamma(X)\phi_1, \phi_2) = (-1)^s h(\phi_1, \gamma(X)\phi_2), \tag{12}$$

for $\sigma_1, \sigma_2 \in \Gamma(\mathcal{S}), X \in \Gamma(TM)$.

Proof. We define $h : \Gamma(\mathcal{S}) \times \Gamma(\mathcal{S}) \to C^\infty(M)$ fibrewise by

$$h_x : \mathcal{S}_x \times \mathcal{S}_x \to \mathbb{C},$$
$$h_x([b, \alpha], [b, \beta]) = \langle \alpha, \beta \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product given by Proposition 2.34 and $\pi(b) = x$. This is well-defined since for $g \in \mathrm{Spin}_0(r, s)$ we have

$$h_x([bg^{-1}, \rho_{r,s}(g)\alpha], [bg^{-1}, \rho_{r,s}(g)\beta]) = \langle \rho_{r,s}(g)\alpha, \rho_{r,s}(g)\beta \rangle = \langle \alpha, \beta \rangle.$$

The fact that it is $\mathrm{Spin}_0(r, s)$-invariant and that Eq. (12) holds, follow immediately from the fact that these hold for $\langle \cdot, \cdot \rangle$. □

We often denote the metric h by $\langle \cdot, \cdot \rangle$, when there is little room for confusion.

Remark 2.65. For a Riemannian manifold the metric given by Proposition 2.64 is positive definite, and we set $\langle \cdot, \cdot \rangle_{pos} = \langle \cdot, \cdot \rangle$. For a pseudo-Riemannian manifold this is not the the case. Then we define the positive definite metric $\langle \cdot, \cdot \rangle_{pos}$ fibrewise

$$\langle [b, \alpha], [b, \beta] \rangle_{pos} = \alpha^* \beta.$$

This metric is in general not $\mathrm{Spin}_0(r, s)$-invariant. ♢

Proposition 2.66. *In even dimensions the spinor bundle splits into the positive and negative half-spinor bundles*

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-,$$

where $\mathcal{S}^\pm = S \times_{\rho^\pm} \Delta^\pm_{r+s}$. Clifford multiplication by a tangent vector maps $\mathcal{S}^+$ to $\mathcal{S}^-$ and vice versa.

Proof. This splitting follows immediately as

$$S \times_\rho \Delta_{r+s} = S \times_{(\rho^+ \oplus \rho^-)} (\Delta^+_{r+s} \oplus \Delta^-_{r+s}) = (S \times_{\rho^+} \Delta^+_{r+s}) \oplus (S \times_{\rho^-} \Delta^-_{r+s}).$$

For any $X \in TM$, we have $\gamma(X) : S^\pm \to S^\mp$ as $X\Gamma_n = -\Gamma_n X$ by Eq. (2). □

2.2.2 The spinor bundle on a product spacetime

In this section we closely follow [2]. We assume (Σ, g_Σ) to be a 3-dimensional Riemannian spin Manifold. We want to study the spinor bundle on the Lorentzian product spacetime $(M := \mathbb{R} \times \Sigma, -dt^2 \oplus g_\Sigma)$, with spin structure $(\mathcal{S}_M, \theta)$. Let $e_0 = \partial_t$. The bundle of oriented orthonormal frames of $\Sigma_t := \{t\} \times \Sigma \cong \Sigma$ can be embedded into the bundle of space and time oriented orthonormal frames on M restricted to Σ_t, by the map

$$i : F_{\mathrm{SO}}(\Sigma) \to F_{\mathrm{SO}_0}(M), \quad i : (e_1, e_2, e_3) \mapsto (e_0, e_1, e_2, e_3).$$

Now $\mathcal{S}_\Sigma := \theta^{-1}(i(F_{\mathrm{SO}}(\Sigma))$ defines a spin structure on Σ. We will assume that this spin structure has been taken on Σ. Since $n + 1$ is even, we have

$$\mathcal{S}_M = \mathcal{S}_M^+ \oplus \mathcal{S}_M^-,$$

where by Proposition 2.33 we have $\mathcal{S}_M^+ |_\Sigma = \mathcal{S}_\Sigma$, and that Clifford multiplication is given by

$$\gamma_\Sigma(X)\alpha = \gamma(\nu)\gamma(X)\alpha,$$

where $X \in T\Sigma$ and $\gamma(\cdot)$ is Clifford multiplication with respect to M. On the other hand $\mathcal{S}_M^- |_\Sigma = \mathcal{S}_\Sigma$, and Clifford multiplication is given by

$$\gamma_\Sigma(X)\alpha = -\gamma(\nu)\gamma(X)\alpha,$$

where $X \in T\Sigma$. The minus sign follows from the way we defined the Clifford module Δ_{r+s} in odd dimensions, see Eq. (6).

Remark 2.67. Given the Lorentzian-spin manifold $(M := \mathbb{R} \times \Sigma, -dt^2 \oplus g)$ we will often use the following explicit construction.

Given a spin representation $\rho_3 : \mathbb{C}\mathrm{l}_3 \to \mathrm{Aut}(\Delta_3)$, we choose the spin representation

$$\rho : \mathbb{C}\mathrm{l}_4 \to \mathrm{Aut}(\Delta_4)$$

on $\Delta_4 = \Delta_3 \oplus \Delta_3$ to be explicitly given by

$$\rho(e_0) = -i\sigma_1 \otimes \boldsymbol{I}, \quad \rho(v) = \sigma_2 \otimes \rho_3(v),$$

where $v \in \mathbb{R}^3$. Lifting this to a Clifford representation using Definition 2.59 we get

$$\begin{gathered} \gamma : \Gamma(TM) \to \mathrm{Aut}(\Gamma(\mathcal{S})), \\ \gamma(e_0) = -i\sigma_1 \otimes \boldsymbol{I}, \quad \gamma(X) = \sigma_2 \otimes \gamma_\Sigma(X), \end{gathered} \tag{13}$$

where $X \in \Gamma(TM)$.

In the following we will often use the following explicit local realisation of the Clifford representation γ, given an pseudo-orthogonal basis (e_0, e_1, e_e, e_3).

$$\begin{aligned} \gamma_0 &= \gamma(\nu) = -i\sigma_1 \otimes \boldsymbol{I}, \\ \gamma_a &= \gamma(e_a) = \sigma_2 \otimes \sigma_a \quad a = 1, 2, 3. \end{aligned} \tag{14}$$

We refer to them as the constant gamma-matrices in the Weyl representation. Raising indices happens with the Minkowski-metric η, i.e $\gamma^a = \eta^{ab}\gamma_b$. This boils down to

$$\gamma^0 = i\sigma_1 \otimes \boldsymbol{I}, \quad \gamma^a = \gamma_a.$$

Having explicitly chosen the spin representation, we also define the the curved gamma-matrices,

$$\tilde{\gamma}_\mu = \gamma(\partial_\mu). \tag{15}$$

Notice that $\gamma_0 = \tilde{\gamma}_0$. Raising indices happens with the metric g, i.e $\gamma^\mu = g^{\mu\nu}\gamma_\nu$. Defining coefficients e_a^μ such that

$$e_a = e_a^\mu \partial_\mu, \quad \text{i.e. } e_a^\mu = g^{\mu\nu} g(e_a, \partial_\mu).$$

we find that the constant and curved gamma matrices are related by

$$\gamma_a = e_a^\mu \tilde{\gamma}_\mu.$$

♢

2.3 Connections

Now we have defined the Spinor bundle and Clifford multiplication, we are only one step away from defining the Dirac operator. In this section we will define the notion of a connection an principal bundles and vector bundles, and we will lift the Levi-Civita connection to the spinor bundle.

Let $\pi : P \to M$ be a G-principal bundle, and denote the right-action of G of P by R, i.e. $R_g p = pg$. Now every element A of the Lie algebra of G defines a vector field $A_* \in \Gamma(TP)$, given by

$$(A_*)_p = \frac{d}{dt}\bigg|_{t=0} R_{\exp(tA)}(p), \quad \forall p \in P.$$

Definition 2.68. A *connection form* on a principal G-bundle P is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying

1. $\omega(A_*) = A$ for all $A \in \mathfrak{g}$
2. $R_g^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for all $g \in G$

Note that these forms are defined over P instead of M. It is often easier to work with *local connection* forms, which are pull-backs of the connections forms to M.

Definition 2.69. Given a connection form ω and a local section $s : U \to P$, the *local connection form* $\mathcal{A} \in \Omega^1(M, \mathfrak{g})$ (w.r.t. this local section) is given by

$$\mathcal{A} = s^* \omega.$$

A complete collection of local connection forms contains exactly the same information as a connection form.

Proposition 2.70. *Let $\pi : P \to M$ be a G-principal bundle and $\{U_\alpha, \phi_\alpha\}$ a bundle atlas for P. Denote the local section corresponding to ϕ_α by s_α. If ω is a connection form then the local connection forms $\mathcal{A}_\alpha := s_\alpha^* \omega$ satisfy*

$$\mathcal{A}_\alpha = \varphi_{\alpha\beta} \mathcal{A}_\beta \varphi_{\alpha\beta}^{-1} + \varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta} \tag{16}$$

Conversely every collection $\{\mathcal{A}_\alpha\}$ of $\mathfrak{g}$-valued one-forms subordinate to a bundle atlas $\{U_\alpha, \phi_\alpha\}$, satisfying Eq. (16) defines a connection form ω on P.

Proof. We will assume G to be a matrix Lie Group, see [43, Prop. 1.3.12] for the general case. Let $(U_\alpha, s_\alpha), (U_\beta, s_\beta)$ be two local sections such that $U_\alpha \cap U_\beta \neq \varnothing, 0$ and let $\varphi_{\alpha\beta}$ be the corresponding transition function, such that $s_\alpha = s_\beta \varphi_{\alpha\beta}$. Let $x \in U, X \in T_x M$, and $\gamma : (-\epsilon, \epsilon) \to M$,

$$\gamma(0) = x, \quad \left.\frac{d}{dt}\right|_{t=0} \gamma(t) = X.$$

We now compute

$$\begin{aligned} d_x s_\alpha(X) &= \left.\frac{d}{dt}\right|_{t=0} s_\alpha(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} s_\beta(\gamma(t))\varphi_{\alpha\beta}(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} s_\beta(\gamma(t))\varphi_{\alpha\beta}(x) + \left.\frac{d}{dt}\right|_{t=0} s_\beta(x)\varphi_{\alpha\beta}(\gamma(t)) \\ &= d_x s_\beta(X)\varphi_{\alpha\beta}(x) + s_\beta(x)\varphi_{\alpha\beta}^{-1}(x) d\varphi_{\alpha\beta}(X). \end{aligned}$$

Using this we get

$$\begin{aligned} \mathcal{A}_{\alpha,x}(X) &= (s_\alpha^*\omega)_x(X) = \omega_{s_\alpha(x)}(d_x s_\alpha(X)) \\ &= \omega_{s_\alpha(x)}\left(d_x s_\beta(X)\varphi_{\alpha\beta}(x)\right) + \omega_{s_\alpha(x)}\left(s_\beta(x)\varphi_{\alpha\beta}^{-1}(x)d\varphi_{\alpha\beta}(X)\right) \\ &= R_{\varphi_{\alpha\beta}(x)}\omega_{s_\beta(x)}\left(d_x s_\beta(X)\right) + \omega\left(\left(\varphi_{\alpha\beta}^{-1}(x)d\varphi_{\alpha\beta}(X)\right)_*\right)_{s_\alpha(x)} \\ &= \varphi_{\alpha\beta}(x)\omega_{s_\beta(x)}\left(d_x s_\beta(X)\right)\varphi_{\alpha\beta}^{-1}(x) + \varphi_{\alpha\beta}^{-1}(x)d\varphi_{\alpha\beta}(X) \\ &= \varphi_{\alpha\beta})(x)\mathcal{A}_{\beta,x}(X)\varphi_{\alpha\beta}^{-1}(x) + \varphi_{\alpha\beta}^{-1}(x)d\varphi_{\alpha\beta}(X), \end{aligned}$$

where we have used the two properties of Definition 2.68 and the fact that $\varphi_{\alpha\beta}(x)d\varphi_{\alpha\beta}(X) \in \mathfrak{g}$.
For the converse statement note that such a collection of local connection forms $\{U_\alpha, s_\alpha, \mathcal{A}_\alpha\}$ can be patched together to form a unique and well-defined connection form as Eq. (16) ensures $\{\mathcal{A}_\alpha\}$ agree on overlaps. □

In the following proposition we will use the following multiple times.

Lemma 2.71. *Let $\pi_1 : M_1 \to P_1$ be a G_1-principal bundle, and $\pi_2 : M_2 \to P_2$ a G_2-principal bundle and let (θ, λ) be a bundle morphism from P_1 to P_2. Let ω be a connection form on P_2. If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, we have*

$$\phi \circ \theta^*\omega = \theta^*(\phi \circ \omega).$$

Proof. This follows immediately by definition, as for any $X \in TP_1$

$$(\phi \circ \theta^*\omega)(X) = \phi(\omega(d\theta X)) = (\phi \circ \omega)(d\theta X) = (\theta^*(\phi \circ \omega))(X).$$

□

Proposition 2.72. *Let $\pi_1 : M \to P_1$ be a G_1-principal bundle, and $\pi_2 : M \to P_2$ a G_2-principal bundle and let (θ, λ) be a vertical bundle morphism from P_1 to P_2. Let ω_2 be a connection form on P_2. If $d\lambda : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a isomorphism of Lie algebra's, the following holds:*

1. ω_2 *induces a unique connection form* ω_1 *on* P_1 *such that for any* $p \in P_1, X \in \Gamma(TP_1)$

$$d\theta(\omega_1(X)_*)_p = (\omega_2(d\theta X)_*)_{\theta(p)}.$$

We have

$$\omega_1 = (d\lambda)^{-1} \circ \theta^* \omega_2.$$

2. If $s_1 : U \to P_1$ *is a section of* P_1 *and* $s_2 = \theta \circ s_1 : U \to P_2$*, is the corresponding section of* P_2 *and we write*

$$\mathcal{A}_1 = s_1^* \omega_1, \quad \mathcal{A}_2 = s_2^* \omega_2,$$

for the local connection forms, then these are related by

$$\mathcal{A}_1 = (d\lambda)^{-1} \circ \mathcal{A}_2.$$

Proof. 1. First we check that ω_1 is indeed a connection form. For any $A \in \mathfrak{g}_1$ we have

$$\begin{aligned} d\theta(A_*)_p &= d\theta \left(\frac{d}{dt}\bigg|_{t=0} R_{\exp(tA(X))}(p) \right) = \frac{d}{dt}\bigg|_{t=0} \theta\left(R_{\exp(tA(X))}(p)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} R_{\exp(t\, d\lambda\, A(X))}(\theta(p)) = ((d\lambda\ A)_*)_{\theta(p)}. \end{aligned} \tag{17}$$

Therefore we have

$$d\lambda(A) = \omega_2((d\lambda(A))_*) = \omega_2(d\theta A_*) = \theta^* \omega_2(A_*) = d\lambda^{-1} \omega_1(A_*)$$

and since $d\lambda^{-1}$ is a isomorphism, we find that $\omega(A_*) = A$. To check that ω_1 also satisfies the second condition for being a connection form, we observe that for $X \in TP_1$

$$\begin{aligned} d\lambda \circ R_g^* \omega_1 &= R_g^*(d\lambda \circ \omega_1) = R_g^* \theta^* \omega_2 = (\theta \circ R_g)^* \omega_2 = (R_{\lambda(g)} \circ \theta)^* \omega_2 = \theta^* R_{\lambda(g)}^* \omega_2 \\ &= \theta^*(\mathrm{Ad}(\lambda(g)^{-1}) \circ \omega_2) = \mathrm{Ad}(\lambda(g^{-1})) \circ \theta^* \omega_2 = \mathrm{Ad}(\lambda(g^{-1})) \circ \theta^* \omega_2 \\ &= dC_{\lambda(g^{-1})} \circ d\lambda \circ \omega_1 = d(C_{\lambda(g^{-1})} \circ \lambda) \circ \omega_1 = d(\lambda \circ C_{g^{-1}}) \omega_1 \\ &= d\lambda \circ \mathrm{Ad}(g^{-1}) \circ \omega_1, \end{aligned}$$

and again because $d\lambda$ is a isomorphism it follows that $R_g^* \omega_1 = \mathrm{Ad}(g^{-1}) \circ \omega_1$. To check that ω_1 satisfies the condition from the proposition we rewrite the definition of ω_1 into

$$d\lambda(\omega_1(X)) = \theta^* \omega_2(X) = \omega_2(d\theta X),$$

hence $(d\lambda(\omega_1(X)))_* = (\omega_2(d\theta X))_*$. Combining this with Eq. (17) for $A = \omega_1(X)$ gives the desired result. Uniqueness now follows from the fact that $d\lambda$ is an isomorphism.

2. Indeed,

$$\begin{aligned} (d\lambda)^{-1} \circ \mathcal{A}_2 &= (d\lambda)^{-1} \circ s_2^* \omega_2 = (d\lambda)^{-1} \circ (\theta \circ s_1)^* \omega_2 \\ &= (d\lambda)^{-1} \circ s_1^* \theta^* \omega_2 = s_1^*((d\lambda)^{-1} \circ \theta^* \omega_2) = s_1^* \omega_1 = \mathcal{A}_1. \end{aligned}$$

□

Definition 2.73. A *connection* on a vector bundle $E \to M$ is given by a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E),$$

that satisfies the Leibniz rule

$$\nabla(f\sigma) = f\nabla(\sigma) + \mathrm{d}f \otimes \sigma,$$

for all $f \in C^\infty(M), \sigma \in \Gamma(E)$.
We often write

$$\nabla_X Y = (\nabla Y)(X) \quad \text{and} \quad \nabla_\mu = \nabla_{\partial_\mu}.$$

If $h : \Gamma(E) \times \Gamma(E) \to C(M)$ is metric on E, then ∇ is said to be *compatible* or *Hermitian* in the case of a complex vector bundle, if

$$h(\nabla\sigma, \tau) + h(\sigma, \nabla\tau) = \mathrm{d}h(\sigma, \tau),$$

for all $\sigma, \tau \in \Gamma(E)$.

Proposition 2.74. *Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G on a vector space V*

1. *Any connection form ω defines a connection on the associated vector bundle $P \times_G V$. Let $\Psi : P \to V$ be an equivariant map defining a section of $P \times_G V$, this connection is given by given by*
$$\nabla\Psi = d\Psi + d\rho \circ \omega \circ \Psi.$$
2. *Let $s : U \to P$ be a local section of P and write $\psi = s^*\Psi$. Then this connection is given by*
$$\nabla\psi = d\psi + A \circ \psi$$
where $A = d\rho \circ s^*\omega$.
3. *Given a section $X = [s, \psi] : U \to P \times_G V$, this connection is given by*
$$\nabla X = [s, \nabla\psi] = [s, d\psi + A \circ \psi].$$
4. *Let (U, x) be a local chart of M and (U, ϕ) a local trivialization of P with corresponding section $s : U \to P$. Let $\{\hat{e}_a\}$ be a basis of V. By writing $\psi = \psi^a \hat{e}_a$ and $A = A_\mu dx^\mu$, we can write*
$$\nabla_\mu \psi^a = \partial_\mu \psi^a + A^a_{\mu b}\psi^b$$
where $A^a_{\mu b} = \hat{e}^a(A_\mu \hat{e}_b)$.

Proof. Since all maps used in the definition of ∇ are linear, ∇ is linear. To see it satifies the Leibniz rule, we compute for $f \in C^\infty(M)$

$$\begin{aligned}\nabla(f\Psi) &= d(f\Psi) + d\rho \circ \omega \circ f\Psi \\ &= df \otimes \Psi + f d\Psi + f d\rho \circ \omega \circ \Psi \\ &= f\nabla(\Psi) + df \otimes \Psi.\end{aligned}$$

The other points follow from trivial computations. □

Remark 2.75. 1. Note that although a section $X : M \to P \times_G V$ is globally defined, their need not to be globally defined representatives $s : M \to P, \psi : M \to V$ such that $X = [s, \psi]$.

2. By abuse of notation we will often write $\nabla X = dX + AX$ instead of $\nabla X = [s, d\psi + A \circ \psi]$ for $X = [s, \psi] \in \Gamma(P \times_G V)$.

$\diamondsuit$

For a pseudo-Riemannian Manifold (M, g) there is a unique connection ∇ on TM compatible with g which is torsion free, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

This connection is called the *Levi-Civita connection.* Choosing a local coördinate basis $\{x_\mu\}$, we define the Christoffel symbols by

$$\nabla \partial_\nu = \Gamma^\lambda_{\mu\nu}\, \mathrm{d}x^\mu \otimes \partial_\lambda.$$

They are explicitly given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).$$

We can also choose a local pseudo-orthonormal frame (e_a) of TM, using which we define the spin-Christoffel symbols by $\tilde{\Gamma}^b_{\mu a} = \tilde{\Gamma}_\mu{}^b{}_a$.

$$\nabla e_a = \tilde{\Gamma}^b_{\mu a} dx^\mu \otimes e_b.$$

The spin-Christoffel symbols are related to the ordinary Christoffel symbols by

$$\tilde{\Gamma}^b_{\mu a} = e_\nu{}^b e^\lambda{}_a \Gamma^\nu_{\mu\lambda} - e^\lambda{}_a \partial_\mu e_\lambda{}^b,$$

where $e^\lambda{}_a$ is defined by $\partial_\mu = e_\mu{}^a e_a$.

Proposition 2.76. *Let (M, g) be a pseudo-Riemmanina Manifold. The Levi-Civita connection defines a unique local connection form of $F_{\mathrm{SO}_0}(TM)$, given by*

$$\tilde{\Gamma} = \tilde{\Gamma}^b_{\mu a} \hat{e}_b \otimes \hat{e}^a \otimes dx^\mu, \tag{18}$$

where $\tilde{\Gamma}^a_{\mu b}$ are the spin-Christoffel symbols, such that the corresponding connection on $F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathbb{R}^{r,s} \cong TM$ is again the Levi-Civita connection.

Proof. We use the notation of Proposition 2.74, with $P = F_{\mathrm{SO}_0}(TM)$ and $V = \mathbb{R}^{r,s}$. Since $TM \cong F_{\mathrm{SO}_0}(TM) \times_{\mathrm{SO}_0(r,s)} \mathbb{R}^{r,s}$, where $\mathrm{SO}_0(r, s)$ acts on $\mathbb{R}^{r,s}$ by just matrix-multiplication it follows that $d\rho = \boldsymbol{I}$. Then Eq. (18) defines a local connection form using the identifications $\mathrm{Lie}(\mathrm{SO}_0(r, s)) = \mathfrak{so}_{r,s} \subseteq M_{r+s}(\mathbb{R}) \cong \mathbb{R}^{r+s} \otimes (\mathbb{R}^{r+s})^*$. Indeed $\tilde{\Gamma}_\mu \in \mathfrak{so}_{r,s}$, since

$$\eta_{ac} \tilde{\Gamma}^c_{\mu b} = g(e_a, \nabla_\mu e_b) = -g(\nabla_\mu e_a, e_b) = -\eta_{bc} \tilde{\Gamma}^c_{\mu a},$$

by metric compatibility. $\square$

Proposition 2.77. *Let M be a spin manifold with spin structure $(\mathcal{S}, \theta)$ such that (θ, Ad) is the vertical bundle morphism between $\mathcal{S}$ and $F_{\mathrm{SO}_0}(TM)$. Then $(d\,\mathrm{Ad})^{-1}$ is given by*

$$\begin{aligned}(d\,\mathrm{Ad})^{-1} : \mathfrak{so}_{r,s} &\to \mathfrak{spin}_{r,s} \subseteq \mathrm{Cl}^0_{r,s} \\ A &\mapsto \frac{1}{4} A^{ab} e_a e_b.\end{aligned}$$

Proof. We denote the map $A \mapsto \frac{1}{4}A^{ab}e_a e_b$ by φ. It is easy to see that φ is an isomorphism of vector spaces. Let $\{e_a\}$ be a pseudo-orthogonal basis of $\mathbb{R}^{r,s}$. A acts on $v \in \mathbb{R}^{r,s}$ by matrix multplication, i.e. if $v = v^a e_a$ then

$$A(v) = A^a{}_b v^b e_a = \eta_{bc} A^{ab} v^c e_a.$$

Under the isomomorphism ϕ this action is given by

$$A(v) = \mathrm{ad}(\phi(A))(v) = [\phi(A), v].$$

Indeed

$$\begin{aligned}[\phi(A), v] &= \frac{1}{4}A^{ab}v^c e_a e_b e_c - \frac{1}{4}A^{ab}v^c e_c e_a e_b = \frac{1}{4}A^{ab}(e_b e_c e_a + e_c e_b e_a) \\ &= \frac{1}{4}A^{ab}v^c(2\{e_b, e_c\}e_a) = \eta_{bc}A^{ab}v^c e_a.\end{aligned}$$

Hence we have found

$$A = \mathrm{ad}(\phi(A)) = (d\,\mathrm{Ad})(\phi(A)),$$

or equivalently $(d\,\mathrm{Ad})^{-1} = \phi$. □

If $\tilde{\Gamma}$ is a $\mathfrak{so}_{r,s}$-valued one-form, we can also apply ϕ fibre-wise, hence for a vector field $Y \in \Gamma(S)$ we have

$$[\phi(\tilde{\Gamma}), Y] = \tilde{\Gamma}(Y).$$

We will use this proving the following proposition.

Proposition 2.78. *The Levi-Civita connection ∇ uniquely lifts to a Hermitian connection $\nabla^{\mathcal{S}}$ on the spinor bundle. It satisfies the following Leibniz rule*

$$\nabla^S(\gamma(Y)\varphi) = \gamma(\nabla Y)\varphi + \gamma(Y)\nabla^S(\varphi),$$

for all $Y \in \Gamma(TM), \varphi \in \Gamma(\mathcal{S})$. It commutes with the charge conjugation operator J_M. It is locally given by

$$\nabla^S_X = d_X - \frac{1}{4}g(\nabla_X e_a, e_b)\gamma^a\gamma^b,$$

or using the spin-Christoffel symbols

$$\nabla^S_\mu = \partial_\mu - \frac{1}{4}\tilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b.$$

Proof. Let

$$\mathcal{S} := S \times_{\mathrm{Spin}_0(r,s)} \Delta_n$$

be the spinor bundle. By combining the last two proposition with Proposition 2.72, we obtain a local connection form on S, given by

$$\phi(\tilde{\Gamma}) = -\frac{1}{4}\tilde{\Gamma}^{b}_{\mu a} e^a e_b \otimes dx^\mu.$$

Apply Proposition 2.74 and using the canonical identification $T\varDelta_n \cong \varDelta_n$, so that $d\gamma = \gamma$, we get that the unique lift of the Levi-Civita connection to the spinor bundle is locally given

$$\nabla^S_\mu = \partial_\mu - \frac{1}{4}\tilde{\eta}^{ac}\,\Gamma^{b}_{\mu a}\,\gamma(e_c e_b) = \partial_\mu - \frac{1}{4}\tilde{\Gamma}^{b}_{\mu a}\gamma^a\gamma_b.$$

It is Hermitian as the spin-Chirstoffel symbols are real and skew-symmetric. It commutes with J_M by Proposition 2.62. To see that it satisfies the given Leibniz rule, we compute:

$$\begin{aligned}\nabla^S(\gamma(Y)\varphi) &= d(\gamma(Y)\varphi) + \phi(\tilde{\Gamma})\gamma(Y)\varphi = \gamma(dY)\varphi + \gamma(Y)d\varphi + \phi(\tilde{\Gamma})\varphi \\ &= \Big(\gamma(dY) + [\phi(\tilde{\Gamma}), \gamma(Y)]\Big)\,\varphi + \gamma(Y)(d\varphi + \phi(\tilde{\Gamma})\varphi) \\ &= \gamma(dY + \tilde{\Gamma}Y)\varphi + \gamma(Y)\nabla^S\varphi = \gamma(\nabla Y)\varphi + \gamma(Y)\nabla^S\varphi.\end{aligned}$$

□

Definition 2.79. We call the connection $\nabla^{\mathcal{S}}$ of Proposition 2.78 the *spin-connection.*

2.4 Dirac operators

We now are finally ready to define Dirac operator on the spinor bundle.

Definition 2.80. Let $\langle\cdot,\cdot\rangle$ be the (possibly indefinite) inner product on the spinor bundle $\mathcal{S}$ over M, given by Proposition 2.64. Then we define an inner product on $\Gamma(\mathcal{S})$ by

$$(\phi, \psi) = \int_M \langle\phi, \psi\rangle\, \mathrm{d}V_g, \tag{19}$$

where $\mathrm{d}V_g$ is the Riemannian volume form of (M, g). We also define

$$(\phi, \psi)_{pos} = \int_M \langle\phi, \psi\rangle_{pos}\, \mathrm{d}V_g.$$

In the Riemmanian case we have $(\cdot,\cdot) = (\cdot,\cdot)_{pos}$. The completion of $\Gamma(\mathcal{S})$ w.r.t. this positive definite inner product is denoted by $L^2(M, \mathcal{S})$ or simply $L^2(\mathcal{S})$.

Remark 2.81.

1. In the pseudo-Riemmannian case with signature (r, s), $s > 0$, $\Gamma(\mathcal{S})$ is a so-called Krein-space with inner-products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{pos}$ related by

$$(\cdot,\cdot) = \Big(\cdot, \gamma(\hat{\varGamma}_s)\cdot\Big)_{pos},$$

where $\hat{\varGamma}_s$ is defined in Eq. (10). For more details, see [4, sect. 3.3.1].

2. It is possible to restrict the spinor bundle to certain sections such that the invariant inner product becomes positive definite, as we will do for Minkowski-space in Proposition 3.41.

♢

Definition 2.82 (Dirac operator)**.** Let (M, g) be pseudo-Riemannian manifold of signature (r, s) and let $\mathcal{S}$ be its spinor bundle. The canonical Dirac operator $D : \Gamma^\infty(\mathcal{S}) \to \Gamma^\infty(\mathcal{S})$ is given by the composition

$$\Gamma^\infty(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes \mathcal{S}) \xrightarrow{T^*M \overset{g}{\cong} TM} \Gamma^\infty(TM \otimes \mathcal{S}) \xrightarrow{-i^{s+1}c(\cdot)} \Gamma^\infty(\mathcal{S})$$

It is locally given by

$$D = -i^{s+1}\tilde{\gamma}^\mu \nabla^S_\mu.$$

Remark 2.83. We can also write the Dirac operator locally without the use of a coordinate basis. Indeed let (e_a) be a pseudo-orthogonal local frame, then we have

$$D = -i^{s+1}\gamma^a \nabla^S_{e_a}$$

or without using indices-notation and Einstein-summation convention

$$D = -i^{s+1} \sum_{j=1}^{n} \varepsilon_j \gamma(e_j) \nabla^S_{e_j},$$

where $\varepsilon_j = \eta(e_j, e_j)$ replaces the raising of the index on the gamma matrix. The factor i^{1+s} is added to make the Dirac operator self-adjoint. When using the opposing definition for the Clifford algebra (c.f. Remark 2.16), one factor i is already incorporated in the gamma-matrices. The factor -1 in front is somewhat arbitrary and can be incorporated in the gamma-matrices, but it is added to make sure the plane wave solutions e^{ikx} have eigenvalue k, when using 1 as gamma matrix in the one-dimensional case. I.e. if we have $M = \mathbb{R}$ and we choose

$$\gamma(\partial_x) = 1,$$

then

$$De^{ikx} = -i\partial_x e^{ikx} = ke^{ikx}.$$

♢

Definition 2.84. Let M be a spin manifold of dimension n and let D be the Dirac operator on M. A *real structure* is an anti-unitary operator $J : L^2(\mathcal{S}) \to L^2(\mathcal{S})$, such that

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \qquad J\Gamma_n = \varepsilon'' \Gamma_n J \text{ if n is even.}$$

Here $\varepsilon, \varepsilon', \varepsilon'' \in \{1, -1\}$ are given as a function of n modulo 8, as defined in Table 2.

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Table 2: The values of $\varepsilon, \varepsilon', \varepsilon''$ depending on the dimension n modulo 8.

Proposition 2.85. *Every spin manifold admits a canonical real structure, which is given by the charge conjugation operator J_M.*

Proof. This follows immediately by the definition of the charge conjugatation operator and Proposition 2.26 □

2.5 The Dirac operator on generalized Lorentzian cylinder

The Dirac operator on a generalized Lorentzian cylinder is uniquely determined by the Dirac operators on Σ. We follow [2, 49]. We first need to define the Weingarten map.

Definition 2.86. Let $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus g_t)$ be a product spacetime of dimension $n + 1$. The *Weingarten map* W with respect to $\nu = \partial_t$ is defined by

$$W : T\Sigma \to T\Sigma, \quad X \mapsto \nabla^M_X \nu.$$

The *mean curvature* $H(t)$ of (Σ, g_t) is given by

$$H = \frac{1}{n} \operatorname{tr}^\Sigma(W) = \frac{1}{n} \sum_{j=1}^{n} g(e_j, W(e_j)),$$

for a frame (e_j) of $T\Sigma$.

Lemma 2.87. *If $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus g_t)$ we have $g(\nu, \nabla_X \nu) = 0$ for all $X \in TM$.*

Proof. This follows by metric compatibility of the connection ∇. Indeed since $g(\nu, \nu) = -1$

$$0 = d_X g(\nu, \nu) = g(\nabla_X \nu, \nu) + g(\nu, \nabla_X \nu),$$

it follows that $g(\nabla_X \nu, \nu) = -g(\nabla_X \nu, \nu) = 0$. □

Lemma 2.88. *Let $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus g_t)$ be a product spacetime, where (Σ, g_t) is a Riemannian spin manifold, and let $n = \dim \Sigma$. We have*

$$\nabla^{S_M}_X = \nabla^{S_\Sigma}_X - \frac{1}{2}\gamma^0 \gamma(W(X)).$$

Proof.

$$\begin{aligned}
\nabla^{S_M}_X &= d_X - \frac{1}{4} g(\nabla_X e_a, e_b)\gamma^a\gamma^b \\
&= d_X - \frac{1}{4} \sum_{1 \leqslant a,b \leqslant n} g(\nabla_X e_a, e_b)\gamma^a\gamma^b - \frac{1}{4} g(\nabla_X e_0, e_b)\gamma^0\gamma^b - \frac{1}{4} g(\nabla_X e_a, e_0)\gamma^a\gamma^0 \\
&= \nabla^{S_\Sigma}_X - \frac{1}{2} g(\nabla_X e_0, e_b)\gamma^0\gamma^b = \nabla^{S_\Sigma}_X - \frac{1}{2} g(W(X), e_b)\gamma^0\gamma^b \\
&= \nabla^{S_\Sigma}_X - \frac{1}{2}\gamma^0\gamma(W(X)).
\end{aligned}$$

Here we have used that

$$g(\nabla_X e_a, e_0)\gamma^a\gamma^0 = -g(\nabla_X e_0, e_a)\gamma^a\gamma^0 = g(\nabla_X e_0, e_a)\gamma^0\gamma^a$$

because of metric compatibilty, and that

$$W(X) = \sum_{a=1}^{3} g(W(X), e_a)e^a = g(\nabla_X e_0, e_a)e^a,$$

since $\nabla_X e_0 = \nabla_X \partial_t = 0$ by Lemma 2.87. □

Theorem 2.89 ([2]). *Let $(M,g) = (\mathbb{R}\times\Sigma, -dt^2\oplus g_t)$ be a product spacetime, where (Σ, g_t) is a Riemannian spin Manifold, and let $n = \dim\Sigma$. Let $(D_t)_t$ be a smooth family of Dirac operators on (Σ, g_t). Using the explicit embedding Eq. (13), the canonical Dirac operator on $L^2(M, \mathcal{S}_M)$ is equal to*

$$\gamma^0(\partial_t + \frac{n}{2}H) + i\sigma_2\otimes D_t$$

Proof. Using Eq. (13), we have

$$\begin{aligned}
\tilde{\gamma}^\mu\nabla_\mu^{S_M} &= \tilde{\gamma}^0\nabla_t^{S_M} + \sum_{\mu=1}^{4}\tilde{\gamma}^m u\nabla_\mu^{S_M}\\
&= \gamma^0\nabla_t^{S_M} + \sum_{\mu=1}^{4}\tilde{\gamma}^\mu\nabla_\mu^{S_\Sigma} - \frac{1}{2}\sum_{\mu=1}^{3}\tilde{\gamma}^\mu\gamma^0\gamma(W(\partial_\mu))\\
&= \gamma^0\nabla_t^{S_M} + \sum_{\mu=1}^{4}\tilde{\gamma}^\mu\nabla_\mu^{S_\Sigma} + \frac{1}{2}\gamma^0\sum_{\mu=1}^{3}\tilde{\gamma}^\mu\tilde{\gamma}^\nu W_{\mu\nu}\\
&= \gamma^0\nabla_t^{S_M} + \sigma_2\otimes\sum_{\mu=1}^{4}\tilde{\gamma}_\Sigma^\mu\nabla_\mu^{S_\Sigma} + \frac{1}{2}\gamma^0\sum_{\mu=1}^{4}g^{\mu\nu}W_{\mu\nu}\\
&= \gamma^0\nabla_t^{S_M} + \sigma_2\otimes\sum_{\mu=1}^{4}\tilde{\gamma}_\Sigma^\mu\nabla_\mu^{S_\Sigma} + \frac{1}{2}\gamma^0\operatorname{tr}^\Sigma W
\end{aligned}$$

Therefore by writing $\operatorname{tr}^\Sigma W = nH$, and $\nabla_t^{S_M} = \partial_t$ we get

$$D = -i^2\tilde{\gamma}^\mu\nabla_\mu^S = \gamma^0(\partial_t + \frac{nH}{2}) + i\sigma_2\otimes D_t.$$ □

Proposition 2.90. *Let $(M,g) = (\mathbb{R}\times\Sigma, -dt\oplus a^2(t)h)$ be a GFLRW spacetime, where Σ is a Riemannian spin manifold. Using the representation given by Eq. (14) the Dirac operator is given by*

$$D = i\sigma_1\otimes\left(\partial_t + \frac{3}{2}\frac{\dot{a}(t)}{a(t)}\right) + i\sigma_2\otimes\frac{1}{a(t)}D_\Sigma. \tag{20}$$

Proof. To find the Dirac equation on M we need to calculate the main curvature H. Choosing coordinates $(x^0 = t, x^i)$, we calculate

$$\nabla_j\nu^\rho = \Gamma^\rho_{j0} = \frac{1}{2}g^{\rho\sigma}(\partial_j g_{\sigma 0} + \partial_0 g_{\sigma j} - \partial_\sigma g_{j0}) = \frac{1}{2}g^{\rho\sigma}(\partial_0 g_{\sigma j}) = \frac{1}{2}g^{\rho\sigma}(\partial_0 g_{\sigma j}).$$

And we see this is only nonzero if $\rho = 1,2,3$, and

$$\nabla_j\nu^k = \frac{1}{2}\frac{1}{a^2(t)}h^{ki}(\partial_t a^2(t))h_{ij} = \frac{2\dot{a}(t)a(t)}{2a^2(t)}h^{ki}h_{ij} = \frac{\dot{a}(t)}{a(t)}\delta_j^k.$$

The result follows. □

Example 2.91. For a Friedmann–Lemaître–Robertson–Walker spacetime with $\varSigma = \mathbb{R}^3$ we have

$$D_{\mathbb{R}^3} = -i\sigma_1\partial_x - i\sigma_2\partial_y - i\sigma_1\partial_z,$$

hence the Dirac operator on a FLRW-spacetime is equal to

$$\gamma^0\left(\partial_t + \frac{3}{2}\frac{\dot{a}}{a}\right) + \frac{1}{a(t)}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} = i\sigma_1 \otimes \left(\partial_t + \frac{3}{2}\frac{\dot{a}(t)}{a(t)}\right) + \frac{1}{a(t)}\sigma_2 \otimes \boldsymbol{\sigma}\cdot\nabla. \qquad \triangleleft$$

Proposition 2.92. *We assume we are in the situation of Proposition 2.90. The chirality element is given by*

$$\varGamma = \begin{pmatrix} \boldsymbol{I}_2 & 0 \\ 0 & -\boldsymbol{I}_2 \end{pmatrix},$$

defining a splitting of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. Regarding to this splitting the Dirac operator takes the form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

where $D_\pm : \Gamma(\mathcal{S}^\pm) \to \Gamma(\mathcal{S}^\mp)$ is given by

$$D_\pm = i\partial_t + i\frac{3}{2}\frac{\dot{a}(t)}{a(t)} \mp \frac{1}{a(t)}D_\varSigma.$$

Proof. According to Definition 2.27 the chirality element is given by

$$\varGamma = (-i)^2 \cdot i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes \boldsymbol{I}_2\,.$$

Using

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we see that Eq. (20) entails

$$D = \begin{pmatrix} 0 & i\partial_t + i\frac{3}{2}\frac{\dot{a}(t)}{a(t)} + \frac{1}{a(t)}D_\varSigma \\ i\partial_t + i\frac{3}{2}\frac{\dot{a}(t)}{a(t)} - \frac{1}{a(t)}D_\varSigma & \end{pmatrix}.$$

□

2.6 Analytical aspects

In this section we follow [19].

Proposition 2.93. *Let (M, g) be a spin manifold of signature (r, s). The Dirac operator with domain $\Gamma_c(\mathcal{S})$ is a symmetric operator w.r.t to the (possible indefinite) inner product Eq. (19). That is for $\phi, \psi \in \Gamma(\mathcal{S})$ with $\operatorname{supp}(\phi) \cap \operatorname{supp}(\psi)$ compact we have*

$$(D\phi, \psi) = (\phi, D\psi)\,.$$

Moreover, for $f \in C^\infty(M)$ we have

$$[D, f] = -i^{s+1}\gamma(\,\mathrm{d}f).$$

Proof. By using the local expression for the Dirac operator Proposition 2.78 and Proposition 2.64 we get

$$\begin{aligned}(i^s D\phi, \psi) &= \left(-i^{s+1}\gamma(\,\mathrm{d}x^\mu)\nabla^{\mathcal{S}}_\mu\phi, \psi\right) = -(-1)^{2s+1}\int_M \langle\nabla^{\mathcal{S}}_\mu\phi, i^{s+2}\gamma(\,\mathrm{d}x^\mu)\psi\rangle\,\mathrm{d}V_g \\ &= \int_M \langle\phi, -i^{s+1}\gamma(\,\mathrm{d}x^\mu)\nabla^{\mathcal{S}}_\mu\psi\rangle\,\mathrm{d}V_g - i^{s+1}\int_M \langle\phi, \gamma(\nabla_\mu(\,\mathrm{d}x^\mu))\psi\rangle\,\mathrm{d}V_g \\ &\quad + i^{s+1}\int_M \partial_\mu\langle\phi, \gamma(\,\mathrm{d}x^\mu)\psi\rangle\,\mathrm{d}V_g\end{aligned} \tag{21}$$

We observe that the first term is exactly $(\phi, i^s D\psi)$.

The volume form $\mathrm{d}V_g$ is given by

$$\mathrm{d}V_g = \sqrt{|\det g|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

By differentiating the identity $\ln(\det g) = \operatorname{tr}(\ln g)$ with respect to x^ρ we get

$$\frac{1}{\det g}\partial_\rho \det(g) = \operatorname{tr}(g^{-1}\partial_\rho g) = g^{\mu\nu}\partial_\rho g_{\mu\nu}.$$

Hence,

$$\partial_\rho\sqrt{|\det g|} = \frac{1}{2}\sqrt{|\det g|}g^{\mu\nu}\partial_\rho g_{\mu\nu}$$

By taking the trace over the second and third indices of the Christoffel symbols we get

$$\Gamma^\mu_{\rho\mu} = \frac{1}{2}g^{\mu\nu}\partial_\rho g_{\mu\nu}.$$

A simple computation shows $\nabla_\mu(\,\mathrm{d}x^\mu) = -\Gamma^\mu_{\rho\mu}\,\mathrm{d}x^\rho$. Combining our results we get

$$\nabla_\mu(\,\mathrm{d}x^\mu)\sqrt{|\det g|} = -\sqrt{|\det g|}\;\Gamma^\mu_{\rho\mu} = -\partial_\rho\sqrt{|\det g|}\,\mathrm{d}x^\mu,$$

From this and partial integration, it follows that

$$i^{s+1}\int_M \langle\phi, \gamma(\nabla_\mu(\,\mathrm{d}x^\mu))\psi\rangle\,\mathrm{d}V_g - i^{s+1}\int_M \partial_\mu\langle\phi, \gamma(\,\mathrm{d}x^\mu)\psi\rangle\,\mathrm{d}V_g = 0.$$

This proves the first statement.

For the second statement let $\phi \in \Gamma(\mathcal{S})$, then we have by the Leibniz rule for the spinor connection

$$[D, f](\phi) = -i^{s+1}\gamma(\,\mathrm{d}x^\mu)[\nabla^{\mathcal{S}}_\mu, f]\phi = -i^{s+1}\gamma(\,\mathrm{d}x^\mu)(\partial_\mu f)\phi = -i^{s+1}\gamma(\,\mathrm{d}f)\phi.$$

□

From now on we will assume that Σ is compact and hence complete - c.f. Theorem 2.9 and Definition 2.10 - , since then there is a complete set of eigenspinors of the Dirac operator.

Theorem 2.94. *Let D be a Dirac operator on a compact Riemannian spin manifold Σ. Then there exists a complete orthonormal basis $(e_n)_{n=1}^{\infty}$ of the Hilbert space $L^2(\mathcal{S}_\Sigma)$ consisting of eigenspinors of the Dirac operator D,*

$$De_n = \lambda_n e_n.$$

Moreover,

1. *The set $\sigma(D)$ is a closed subset of $\mathbb{R}$ consisting of an unbounded discrete sequence of eigenvalues, i.e $\lim_{n\to\infty} |\lambda_n| = \infty$.*
2. *The eigenspinors e_n are smooth sections.*
3. *The eigenspaces of D form a complete orthonormal decomposition of $L^2(\mathcal{S}_\Sigma)$.*
4. *Each eigenspace V_λ of D is finite-dimensional.*
5. *The set $\sigma(D)$ is unbounded on both sides of $\mathbb{R}$ and, if moreover $n \neq 3 \ (mod\ 4)$, then it is symmetric about the origin.*

Proof. See [24, Lem. 1.6.3] for the proof of the main statement and items 1 - 3, which is applicable as the closure of a Dirac operator on a complete Riemannian manifold is elliptic (see e.g. [25, Prop. 1.3.5]). For the other statements, see [25, Thm. 1.3.7] □

Recall that an unbounded operator T is essential self-adjoint if its closure $\bar{T}$ is self-adjoint.

Proposition 2.95. *Let (Σ, g) be a compact Riemannian spin Manifold. The Dirac operator D is essentially self-adjoint in $L^2(\mathcal{S}_\Sigma)$. Its closure $\bar{D}$ has domain the first Sobolev space $H^1(\mathcal{S}_\Sigma) \subseteq L^2(\mathcal{S}_\Sigma)$.*

Proof. See e.g. [19, Sec. 4.1, 4.2]. □

We refer to [24] for the definition of Sobolev spaces. For our purposes, the following characterization of the Sobolev spaces $H^k(\mathcal{S}_\Sigma)$ is sufficient.

Proposition 2.96. *Let $(e_n)_{n=1}^{\infty}$ be a complete orthonormal basis of $L^2(\mathcal{S}_\Sigma)$ consisting of eigenspinors of the Dirac operator. For an arbitrary $\psi \in L^2(\mathcal{S}_\Sigma)$ we write*

$$\psi = \sum_{n=1}^{\infty} a_n e_n.$$

Then $\psi \in H^k(\mathcal{S}_\Sigma)$ if and only if

$$\sum_{n=1}^{\infty} |a_n|^2 (\lambda_n)^{2k} < \infty.$$

Proof. See [19, Sec. 4.2]. □

Definition 2.97. We denote the eigenspace corresponding to the eigenvalue λ by $\mathcal{H}_\lambda^\Sigma$, and the basis of $\mathcal{H}_\lambda^\Sigma$ of smooth eigensections by $\{e_n^\lambda\}_{n\in\mathbb{N}_\lambda}$, where $\mathbb{N}_\lambda = \{1, \dots, \dim \mathcal{H}_\lambda^\Sigma\}$. The previous theorems provides us with the following isomorphism

$$L^2(\mathcal{S}_\Sigma) \xrightarrow{\cong} l^2(\mathbb{N}),$$
$$\sum_{\lambda\in\sigma(D), n\in\mathbb{N}_\lambda} a_n^\lambda e_n^\lambda \mapsto (a_{j(\lambda,n)}),$$

here $j : \bigoplus_{\lambda\in\sigma(D)}\{\lambda \times \mathbb{N}_\lambda\} \xrightarrow{\cong} \mathbb{N}$ inducted by the orthonormal decomposition of $L^2(\mathcal{S}_\Sigma)$ into finite dimensional eigenspaces $\mathcal{H}_\lambda^\Sigma$ with orthonormal bases $\{e_n^\lambda\}$. We also define

$$\lambda : \mathbb{N} \to \mathbb{R}$$

such that $\lambda(j(\lambda', n)) = \lambda'$. The Dirac operator acts on square summable sequences $(a_n)_{n=1}^\infty$ by

$$D_\Sigma(a_n)_{n=1}^\infty = (\lambda(n)a_n)_{n=1}^\infty.$$

3 Solutions of the Dirac equation

If you give a hacker a new toy,
the first thing he ll do is take it
apart to figure out how it works.

— Jamie Zawinski

Definition 3.1. Let D be the Dirac operator of a spin manifold M. The Dirac equation with mass m is given by

$$(D + m)\psi = 0.$$

Remark 3.2. The Dirac equation was first formulated by Dirac in [14] as follows

$$i\partial_t \psi = \left(-i \sum_{j=1}^{3} \alpha^j \partial_j + m\beta \right) \psi, \quad \psi \in L^2(\mathbb{R}^{3,1}; \mathbb{C}^4).$$

Here β and α^j are 4 by 4 matrices given by

$$\beta = \begin{pmatrix} \boldsymbol{I}_2 & 0 \\ 0 & -\boldsymbol{I}_2 \end{pmatrix}, \qquad \alpha^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

By multiplying the equation by β, it becomes

$$\hat{\gamma}^j \partial_j \psi + m\psi = 0,$$

where

$$\hat{\gamma}^0 = -i\beta, \quad \hat{\gamma}^j = -i\beta \cdot \alpha^j = \sigma_2 \otimes \sigma_j,$$

are gamma-matrices in the so-called Dirac representation. Writing $D = -i\hat{\gamma}^j \partial_j$ for the Dirac operator, we get the formula from the definition for $M = \mathbb{R}^{3,1}$. ♢

In this chapter we will find solutions of the Dirac equation on a generalized n-dimensional cylinder $\mathbb{R} \times \Sigma$ with constant metric, where Σ is a compact odd-dimensional Riemannian spin manifold. But first we have to recall some functional analysis, with regard to Fourier Theory and Stone's theorem.

3.1 Fourier Theory

In this section we follow [41, Ch. IX], [12, Par. X.6] and [42, Ch. 7].

Definition 3.3. The *normalized Lebesgue measure* on $\mathbb{R}^n$ is the measure m_n given by

$$dm_n(\boldsymbol{x}) = \frac{1}{(2\pi)^{(n/2)}} d\boldsymbol{x}.$$

We will norm the Lebesgue spaces $L^n(\mathbb{R}^n)$ using this measure, that is for $f \in L^p(\mathbb{R}^n)$ we have

$$\|f\|_p = \frac{1}{(2\pi)^{(n/2)}} \left(\int_{\mathbb{R}^n} |f|^p d\boldsymbol{x} \right)^{1/p}.$$

Definition 3.4. A function $f \in C^\infty(\mathbb{R}^n)$ is called *rapidly decreasing* if

$$\sup_{\alpha,\beta} \sup_{x\in\mathbb{R}^n} \left| x^\beta D_\alpha f^{(n)} \right| < \infty,$$

where α is a multi-index and

$$D_\alpha = (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n}.$$

The set of all rapidly decreasing functions over $\mathbb{R}^n$ is called the *Schwartz space* and is denoted by $\mathcal{S}_n = \mathcal{S}(\mathbb{R}^n)$.

Proposition 3.5. *If $1 \leqslant p < \infty$, then $\mathcal{S}_n$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. See [12, Prop. 6.5]. □

Definition 3.6. Let h be an pseudo-Riemannian metric on $\mathbb{R}^n$. The Fourier transform (with respect to the metric h) is the map

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C(\mathbb{R}^n),$$
$$f \mapsto \hat{f},$$

where

$$\hat{f}(\boldsymbol{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-ih(\boldsymbol{k},\boldsymbol{x})} d\boldsymbol{x} \tag{22}$$

Proposition 3.7. *For $f \in L^1(\mathbb{R}^n)$, we have $\hat{f} \in C_0(\mathbb{R}^n)$ and $\left\|\hat{f}\right\|_\infty \leqslant \|f\|_1$.*

Proof. See [42, Thm. 7.5]. □

Lemma 3.8. *If $f \in \mathcal{S}_n$ then $\hat{f} \in \mathcal{S}_n$.*

Proof. See [41, Lemma on the second page of Ch. IX]. □

If we restrict the Fourier transform to $\mathcal{S}_n$ we thus get a map $\mathcal{F} : \mathcal{S}_n \to \mathcal{S}_n$. This map is invertible.

Theorem 3.9. *The Fourier transform restricted to the Schwartz space,*

$$\mathcal{F} : \mathcal{S}_n \to \mathcal{S}_n$$

is a linear bicontinous bijection with inverse

$$(\mathcal{F}^{-1} f)(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\boldsymbol{k}) e^{ih(\boldsymbol{k},\boldsymbol{x})} d\boldsymbol{k}.$$

Proof. See [41, Thm. IX.1]. □

Theorem 3.10 (Plancherel's Theorem). *1. If $f \in \mathcal{S}_n$, then $\|f\|_2 = \left\|\hat{f}\right\|_2$.*

2. The Fourier transform extends to a unitary operator on $L^2(\mathbb{R}^n)$.

Proof. See [41, Thm. IX.6]. □

Definition 3.11. The unitary extension of the Fourier transform from Theorem 3.10 is called the *Fourier-Plancherel transform.* It will sometimes be denoted by $\mathcal{F}$.

Remark 3.12.
1. The formula for the Fourier-Plancherel transform is only given by Eq. (22) if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, as it does not make sense for $f \notin L^1(\mathbb{R}^n)$.
2. In the following we will often restrict our function to the Schwartz space for notational clarity, although all calculations can be generalized to $L^2(\mathbb{R}^n)$ by using the Fourier-Plancherel transform instead of the Fourier transform.

$\diamondsuit$

Remark 3.13. When considering Minkowski space $\mathbb{R}^{3,1}$, we will often perform the Fourier transform in the time and special components separately. The Fourier transform in the time component, called the *temporal Fourier transform,* is then given by

$$f(t) \mapsto \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ik^0 x_0} dx_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{i\omega t} dt,$$

where $t = x^0 = -x_0$ and $\omega = k^0 = -k_0$. We often denote the temporal Fourier transform by $\mathcal{F}$ if there is no confusion with the full Fourier transform.

$\diamondsuit$

Theorem 3.14. *Let $D = i\partial_t : \mathrm{dom}(D) \to L^2(\mathbb{R}^n)$ on $\mathrm{dom}(D) \subseteq L^2(\mathbb{R}^n)$ (where we parameterize $\mathbb{R}$ by t) and let $M : \mathrm{dom}(M) \to L^2(\mathbb{R}^n)$ be the operator defined by $Mf = \omega f$ (where we parameterize $\mathbb{R}$ by ω). Let $\mathcal{F}$ be the temporal Fourier transform. Then $\mathrm{dom}(M) = \mathcal{F}\,\mathrm{dom}(D)$ and*

$$\mathcal{F}D = M\mathcal{F}.$$

Proof. See [12, Thm 6.18]. $\square$

Remark 3.15. Sometimes, there is a more canonical measure then the normalized Lebesgue measure resulting from an (invariant) inner product on $L^2(\mathbb{R}^n)$, given by

$$(f, g)_{inv} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f^*(\boldsymbol{k}) g(\boldsymbol{k}) \mu(\boldsymbol{k}) d\boldsymbol{k}, \tag{23}$$

where $\mu(\boldsymbol{k})$ is a smooth function. We then define the inverse Fourier transform as

$$\mathcal{F}_\mu^{-1} f(\boldsymbol{k}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-ih(\boldsymbol{k},\boldsymbol{x})} \sqrt{\mu(\boldsymbol{k})} dm_n(\boldsymbol{k}),$$

such that the standard inner product on $L^2(\mathbb{R}^n, dm_n)$ of two Fourier transformed functions leads to an expression of the from Eq. (23), i.e.

$$\left(\mathcal{F}_\mu^{-1} f(\boldsymbol{x}), \mathcal{F}_\mu^{-1} g(\boldsymbol{x})\right)_{std} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f^*(\boldsymbol{k}) g(\boldsymbol{k}) \mu(\boldsymbol{k}) d\boldsymbol{k} = (f, g)_{inv}.$$

$\diamondsuit$

3.2 The Schrödinger equation and Stone's theorem

In this section we follow [41, 27, 47].

Definition 3.16. The Schrödinger equation on a Hilbert space $\mathcal{H}$ for a self-adjoint operator H_0 is given by

$$i\partial_t\psi(t) = H_0\psi(t), \qquad \psi(t) \in \mathcal{H}\ \forall t \in \mathbb{R}, \qquad \psi(0) = \psi_0 \in \mathcal{H}. \tag{24}$$

Definition 3.17. An operator valued function

$$U : \mathbb{R} \to \mathcal{B}(\mathcal{H})$$

is called a *strongly continuous one-parameter unitary group* on $\mathcal{H}$ if

1. For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator on $\mathcal{H}$.
2. For all $s, t \in \mathbb{R}$, we have $U(t + s) = U(t)U(s)$.
3. If $\phi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\phi \to U(t_0)\phi$.

From a physics point of view, it is a reasonable assumption to consider only solutions generated by strongly continuous one-parameter unitary groups.

Definition 3.18. A solution to the Schrödinger equation, Eq. (24), is called a *strong solution* if it is given by

$$\psi(t) = U(t)\psi_0,$$

with $U(t)$ a strongly continuous one-parameter unitary group with $U(0) = \mathrm{id}$.

Recall that using functional calculus, for a self-adjoint (possibly unbounded) operator T, we can define the unitary operator e^{itT}, for any $t \in \mathbb{R}$.

Theorem 3.19 (Stone's theorem)**.** *Let H be a self-adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(H)$ and define $U(t) = e^{-iHt}$. Then $U(t)$ is a strongly continuous one-parameter unitary group, and*

1. *For $\phi \in \mathcal{D}(H)$,*

$$i\frac{d}{dt}\bigg|_{t=s} U(t)\phi := \lim_{h\to 0} \frac{U(s+h)\phi - U(s)\phi}{h} = HU(s)\phi.$$

2. *If $\frac{d}{dt}|_{t=0}U(t)\phi$ exists, $\phi \in \mathcal{D}(H)$.*

Conversely, let $U(t)$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then there is a densely defined and self-adjoint operator H on $\mathcal{H}$ such that $U(t) = e^{-iHt}$.

Proof. See e.g. [41, Thm. VIII.7, VIII.8] or [27, Thm. 10.15] □

The following is an immediate corollary, and provides us with existence and uniqueness of strong solutions of the Schrödinger equation.

Corollary 3.20. *The Schrödinger equation* (24) *$i\partial_t\psi(t) = H_0\psi(t)$ with initial value $\psi(0) = \psi_0 \in \mathcal{D}(H_0)$ has a unique strong solution given by*

$$\psi(t) = e^{-iH_0t}\psi_0.$$

Remark 3.21. As e^{-iH_0t} is a bounded operator, $e^{-iH_0t}\psi_0$ is defined for all $\psi_0 \in \mathcal{H}$. This allows us to interpret $e^{-iH_0t}\psi_0$ as a solution to the Schrödinger equation, even if $\psi_0 \notin \mathcal{D}(H_0)$. ♢

3.3 Initial value problems

In this section we follow [47, 18, 41]. Let us assume that $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus g_\Sigma)$, with (Σ, g_Σ) a compact odd-dimensional Riemannian spin manifold. We now want to find solutions of the Dirac equation on M, given by

$$((i\sigma_1 \otimes \boldsymbol{I})\partial_t + i\sigma_2 \otimes D_\Sigma + m)\,\psi = 0.$$

There is of course not a unique solution, but if we require it to satisfy the initial value condition $\psi(0, \cdot) = f \in H^1(\mathcal{S}_\Sigma)$ for a fixed $f \in H^1(\mathcal{S}_\Sigma)$ the problem becomes well-posed.

There are multiple ways to find its solutions, one is based on Stone's theorem and will be given first. The second, more heuristic approach is based on Fourier transformations.

By multiplying the Dirac equation by σ_1, the Dirac equation transforms into (c.f. Remark 3.2)

$$i\partial_t\psi = (\sigma_3 \otimes D_\Sigma - m(\sigma_1 \otimes \mathrm{id}))\,\psi.$$

If we define $H_0 = \sigma_3 \otimes D_\Sigma + m(\sigma_1 \otimes \mathrm{id})$, we recognize this to be of the form of the Schrödinger equation. H_0 is essential self-adjoint on $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ as D_Σ is essential self-adjoint and the Pauli matrices are Hermitian (see [41, Thm. VIII.33]). Its self-adjoint closure $\bar{H}_0$ is defined on $H^1(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$. We will show that there is a orthonormal basis of $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ consisting of eigenvalues of H_0. We call H_0 the *Dirac Hamiltonian*. But before we proof this, we recall that the real structure

$$J : \Gamma(\mathcal{S}_\Sigma) \to \Gamma(\mathcal{S}_\Sigma),$$

is fibrewise given by

$$\begin{aligned} J_3^- &: \mathbb{C}^2 \to \mathbb{C}^2, \\ &\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -v_2^* \\ v_1^* \end{pmatrix}. \end{aligned}$$

It satisfies

$$\begin{aligned} JD_\Sigma &= D_\Sigma J, \\ J^\dagger &= J^{-1} = -J, \quad J^2 = -1 \\ Ji\sigma_i &= i\sigma_i J, \quad J\sigma_i = -\sigma_i J, \quad i = 1, 2, 3. \end{aligned} \tag{25}$$

Proposition 3.22. *Let $(e_n)_{n=1}^\infty$ be a orthonormal basisof $L^2(\mathcal{S}_\Sigma)$ consisting of eigenspinors with eigenvalues $\lambda(n)$. The Dirac Hamiltonian*

$$H_0 = \sigma_3 \otimes D_\Sigma + m(\sigma_1 \otimes \mathrm{id}),$$

on $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$, has an orthonormal basis of eigenvectors, given by

$$\left(\sqrt{\frac{m}{\omega_{\lambda(n)}}} w^{\lambda(n)} \otimes e_n, \sqrt{\frac{m}{\omega_{\lambda(n)}}} \bar{w}^{\lambda(n)} \otimes \bar{e}_n\right)_{n=1}^\infty, \tag{26}$$

with corresponding eigenvalues $\omega_{\lambda(n)}, -\omega_{\lambda(n)}$, where $\omega_\lambda = \sqrt{\lambda^2+m^2}$, $\bar{e}_n = Je_n$, and

$$w^\lambda = \frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix} = \frac{1}{2\sqrt{m(\omega_\lambda+m)}}\begin{pmatrix}\omega_\lambda+\lambda+m\\ -\omega_\lambda+\lambda-m\end{pmatrix},$$
$$\bar{w}^\lambda = Jw^\lambda = \frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\begin{pmatrix}m\\ \omega_\lambda+\lambda\end{pmatrix} = \frac{1}{2\sqrt{m(\omega_\lambda+m)}}\begin{pmatrix}\omega_\lambda-\lambda+m\\ \omega_\lambda+\lambda+m\end{pmatrix}.$$

Proof. Note that for all $\lambda \in \mathbb{R}$, $\left(\sqrt{\frac{m}{\omega_\lambda}}w^\lambda, \sqrt{\frac{m}{\omega_\lambda}}\bar{w}^\lambda\right)$ is a basis of $\mathbb{C}^2$, hence Eq. (26) provides an orthonormal basis of $L^2(\mathcal{S}_\Sigma)\otimes\mathbb{C}^2$. To check that $w^{\lambda(n)}\otimes e_n$ is an eigenvector, we check

$$\begin{aligned} H_0\left(\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}\otimes e_n\right) &= \begin{pmatrix}\lambda & -m\\ -m & -\lambda\end{pmatrix}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}\otimes e_n \\ &= \begin{pmatrix}\omega_\lambda+\lambda^2+m^2\\ -m\omega_\lambda-m\lambda+m\lambda\end{pmatrix}\otimes e_n = \omega_\lambda\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}\otimes e_n, \end{aligned}$$

where we have written $\lambda = \lambda(n)$. As $J\otimes J$ anti-commutes with H_0 it now immediately follows that $\bar{w}^{\lambda(n)}\otimes\bar{e}_n$ is an eigenvector with eigenvalue $-\omega_\lambda$. The two expressions for w^λ are equivalent, as

$$\begin{aligned} \frac{1}{2\sqrt{m(\omega_\lambda+m)}}\begin{pmatrix}\omega_\lambda+\lambda+m\\ -\omega_\lambda+\lambda-m\end{pmatrix} &= \frac{1}{2\sqrt{m(\omega_\lambda+m)}}\left(\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}+\begin{pmatrix}m\\ -\omega_\lambda+\lambda\end{pmatrix}\right) \\ &= \frac{1}{2\sqrt{m(\omega_\lambda+m)}}\left(\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}+\frac{m}{\omega_\lambda+\lambda}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}\right) \\ &= \frac{m+\omega_\lambda+\lambda}{2(\omega_\lambda+\lambda)\sqrt{m(\omega_\lambda+m)}}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix} \\ &= \frac{m+\omega_\lambda+\lambda}{\sqrt{2(\omega_\lambda+\lambda)(\omega_\lambda+m)}}\frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix} \\ &= \frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix}, \end{aligned}$$

where in the last step we have used

$$\begin{aligned} 2(\omega_\lambda+\lambda)(\omega_\lambda+m) &= 2\omega_\lambda^2+2\lambda m+2\lambda\omega_\lambda+2m\omega_\lambda \\ &= \omega_\lambda^2+\lambda^2+m^2+2\lambda m+2\lambda\omega_\lambda+2m\omega_\lambda = (m+\omega_\lambda+\lambda)^2. \end{aligned}$$

□

Note that by Proposition 2.96 an arbitrary $\psi_0 \in L^2(\mathcal{S}_\Sigma)\otimes\mathbb{C}^2$ is in $H^1(\mathcal{S}_\Sigma)\otimes\mathbb{C}^2$ if and only if

$$\psi_0 = \sum_{n=1}^{\infty}\sqrt{\frac{m}{\omega_{\lambda(n)}}}\left(a_n w^{\lambda(n)}\otimes e_n + b_n\bar{w}^{\lambda(n)}\otimes\bar{e}_n\right),$$

with

$$\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)(\lambda_n)^2<\infty.$$

Theorem 3.23. *Let $(M,g)=(\mathbb{R}\times\Sigma,-dt^2\oplus g_\Sigma)$, with (Σ,g_Σ) a compact Riemannian spin manifold. The Dirac equation*

$$((i\sigma_1\otimes\boldsymbol{I})\partial_t+i\sigma_2\otimes D_\Sigma+m)\,\psi=0,$$

with initial value $\psi(0)=\psi_0\in H^1(\mathcal{S}_\Sigma)\otimes\mathbb{C}^2$ for

$$\psi_0=\sum_{n=1}^{\infty}\sqrt{\frac{m}{\omega_{\lambda(n)}}}\left(a_n w^{\lambda(n)}\otimes e_n+b_n\bar{w}^{\lambda(n)}\otimes\bar{e}_n\right),$$

has a unique strong solution given by

$$\psi(t)=e^{-iHt}\psi_0=\sum_{n=1}^{\infty}\sqrt{\frac{m}{\omega_{\lambda(n)}}}\left(a_n w^{\lambda(n)}\otimes e_n e^{-i\omega_{\lambda(n)}t}+b_n\bar{w}^{\lambda(n)}\otimes\bar{e}_n e^{i\omega_{\lambda(n)}t}\right).$$

Proof. This follows immediately from Corollary 3.20, Proposition 3.22 and functional calculus. □

We will now obtain this result again by using the Fourier transform. By applying the Fourier transformation and Theorem 3.14 to the Dirac equation we get the temporal Fourier transformed Dirac equation

$$(\omega(\sigma_1\otimes\boldsymbol{I})+i\sigma_2\otimes D_\Sigma+m)\,\hat{\psi}(\omega)=0.$$

In the following we will often use the tempororal Fourier transformed version of sections of the spinor bundle. We will also use the notation introduced by Definition 2.97. That is in local coordinates we will use $\omega\in\mathbb{R}, j\in\mathbb{N}$ to describe a local section, instead of $t\in\mathbb{R},\boldsymbol{x}\in\Sigma$. Solutions of the temporal Fourier transformed Dirac equation will be defined on so called *mass shells.*

Definition 3.24. The *positive mass shell* of mass m is given by

$$X_m^+=\{(\omega,j)\in\mathbb{R}\times\mathbb{N}\mid\omega^2=\lambda(j)^2+m^2,\omega>0\}.$$

The *negative mass shell* of mass m is given by

$$X_m^-=\{(\omega,j)\in\mathbb{R}\times\mathbb{N}\mid\omega^2=\lambda(j)^2+m^2,\omega<0\}.$$

Here $\lambda:\mathbb{N}\to\mathbb{R}$ is as defined in Definition 2.97.

Proposition 3.25. *Let Σ be a compact 3-dimensional Riemannian spin-manifold with Dirac operator D_Σ. We assume that the Dirac operator has a symmetric spectrum $\{\lambda\}\subseteq\mathbb{R}$ with corresponding normalized eigenfunctions $e_n\in L^2(\mathcal{S}_\Sigma)$,that is*

$$D_\Sigma e_n^\lambda=\lambda(n)e_n$$

Let J be the real structure on Σ and write

$$\bar{e}_n^\lambda=Je_n^\lambda.$$

Then the solution of the temporal Fourier transformed Dirac equation

$$(\omega\sigma_1 \otimes \boldsymbol{I} + i\sigma_2 \otimes D_\Sigma + m)\hat{\psi}(\omega) = 0$$

on X_m^+ is given by

$$\hat{\psi}(\omega) = \left(w^{\lambda(n)} a_n\right)_{n=1}^{\infty},$$

while the solution on X_m^- is given by

$$\hat{\psi}(-\omega) = \left(w^{\lambda(n)} b_n\right)_{n=1}^{\infty}.$$

In both equations $\omega > 0$ and

$$w^\lambda = \frac{1}{\sqrt{2m(\omega_\lambda + \lambda)}}\begin{pmatrix}\omega_\lambda + \lambda \\ -m\end{pmatrix} = \frac{1}{2\sqrt{m(\omega_\lambda + m)}}\begin{pmatrix}\omega_\lambda + \lambda + m \\ -\omega_\lambda + \lambda - m\end{pmatrix},$$

$$\bar{w}^\lambda = Jw^\lambda = \frac{1}{\sqrt{2m(\omega_\lambda + \lambda)}}\begin{pmatrix}m \\ \omega_\lambda + \lambda\end{pmatrix} = \frac{1}{2\sqrt{m(\omega_\lambda + m)}}\begin{pmatrix}\omega_\lambda - \lambda + m \\ \omega_\lambda + \lambda + m\end{pmatrix}.$$

Proof. Every $\hat{\psi}(\omega) \in l^2(\mathbb{N}) \otimes \mathbb{C}^2$ can be written as

$$\hat{\psi}(\omega) = (a_n w_n)_{n=1}^{\infty},$$

with $a_n(\omega) \in \mathbb{C}$ and $w_n^\lambda(\omega) \in \mathbb{C}^2$. By inserting this into the Dirac equation we see we need to have

$$\omega\sigma_1 w_n + i\sigma_2\lambda(n)w_n + mw_n = \begin{pmatrix} m & \omega + \lambda(n) \\ \omega - \lambda(n) & m\end{pmatrix} w_n = 0.$$

For solutions to exist, the determinant of this matrix has to be zero, which is the case if and only if

$$\omega = \pm\omega_{\lambda(n)} = \pm\sqrt{m^2 + \lambda(n)^2}.$$

Hence we are only able to define solutions on the mass shells $X_m^\pm$. Then, the (up to normalisation) unique solution is given by

$$w^\lambda = \frac{1}{\sqrt{2m(\omega_\lambda + \lambda)}}\begin{pmatrix}\omega_\lambda + \lambda \\ -m\end{pmatrix} = \frac{1}{2\sqrt{m(\omega_\lambda + m)}}\begin{pmatrix}\omega_\lambda + \lambda + m \\ -\omega_\lambda + \lambda - m\end{pmatrix},$$

with $\lambda = \lambda(n)$.

This justifies us to choose $w_n = w^\lambda$ for all $n \in \mathbb{N}$ such that $\lambda(n) = \lambda$ as we can pull every scalar into a_n. Hence the solution on X_m^+ is given by

$$\hat{\psi}(\omega) = \left(w^{\lambda(n)} a_n\right)_{n=1}^{\infty}.$$

Here and in the future we will assume $\omega > 0$, and extract the minus sign from a negative ω into the solution, that is, if $\omega < 0$, we will perform the substitution of variables $\omega \mapsto -\omega$.

The solutions corresponding to a positive ω will be called the positive frequency solutions, and the solutions corresponding to a negative ω (before the substitution) will be called the negative frequency solutions. For the solutions w^λ this entails

$$\begin{aligned}\frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\begin{pmatrix}\omega_\lambda+\lambda\\ -m\end{pmatrix} &\mapsto \frac{1}{\sqrt{2m(-\omega_\lambda+\lambda)}}\begin{pmatrix}-\omega_\lambda+\lambda\\ -m\end{pmatrix}\\ &= -\frac{\sqrt{\lambda+\omega_\lambda}}{\sqrt{\lambda-\omega_\lambda}}\frac{1}{\sqrt{2m(\omega_\lambda+\lambda)}}\frac{\lambda-\omega_\lambda}{m}\begin{pmatrix}m\\ \omega_\lambda+\lambda\end{pmatrix}\\ &= \frac{\sqrt{(\lambda+\omega_\lambda)(\lambda-\omega_\lambda)}}{m}Jw^\lambda = iJw^\lambda.\end{aligned}$$

We set $\bar{w}^\lambda = Jw^\lambda$, and absorb the factor i in the coefficients. We thus see the operator J is an convenient way to switch from positive frequency solutions to negative frequency solutions, and we can expand an arbitrary negative frequency solutions as

$$\hat{\psi}(-\omega) = \left(w^{\lambda(n)}b_n\right)_{n=1}^{\infty}.$$

This is indeed a solution as

$$(J\otimes J)\left((i\sigma_1\otimes 1)M_\omega + i\sigma_2\otimes D_\Sigma + m\right) = \left((i\sigma_1\otimes 1)M_\omega + i\sigma_2\otimes D_\Sigma + m\right)(J\otimes J),$$

The coefficients a_n^λ, b_n^λ have to be determined by an initial value condition. □

Integrating any function $f(\omega, j)$ over the mass shells is heuristically equal to integrating $f(\omega,j)\delta(\omega^2-m^2-\lambda(j)^2)$ over the whole space. Here δ is a distribution called the Dirac delta function, as defined in Eq. (87). Since this is a delta function with a non linear argument, we have to take care to define this rigorously.

Proposition 3.26. *For any square integrable function $f(\omega, j)$ on the mass shells we have*

$$\sum\!\!\!\!\!\!\int_{X_m^\pm} |f(\omega,j)|^2 d\omega = \sum_{j\in\mathbb{N}} \frac{\left|f(\omega_{\lambda(j)}, j)\right|^2}{2\omega_{\lambda(j)}}$$

Proof. We will proof this only for the positive mass shell, as a completely similar proof holds for the negative mass shell. Let

$$X^+ = \bigcup_{m\in(0,\infty)} X_m^+.$$

We define the map $\phi : (0,\infty)\times\mathbb{N} \to X^+$ by

$$\phi(y,j) = (\sqrt{y+\lambda(j)^2}, j).$$

We have

$$\frac{\partial\phi_1}{\partial y} = \frac{1}{2\sqrt{y+\lambda(j)^2}}$$

where ϕ_1 is the first component of ϕ. Hence for any compactly supported function g we have

$$g(\omega^2-\lambda(j)^2)d\omega = \frac{g(y)dy}{2\sqrt{y+\lambda(j)^2}}.$$

If we now take the limit where $g(y)$ converges to a function with a pole at $y = \lambda(j)^2$ we obtain for any integrable function $f(\omega, j)$ on the positive mass shell

$$\begin{aligned}\sum\!\!\!\!\!\!\int_{X_m^+} |f(\omega,j)|^2 d\omega &= \int_{\mathbb{R}} \sum_{j\in\mathbb{N}} |f(\omega,j)|^2 \delta(\omega^2 - m^2 - \lambda(j)^2) d\omega \\ &= \int_{\mathbb{R}} \sum_{j\in\mathbb{N}} \left| f(\sqrt{y+\lambda(j)^2}, j)\right|^2 \frac{\delta(y-m^2)dy}{2\sqrt{y+\lambda(j)^2}} \\ &= \sum_{j\in\mathbb{N}} \left| f(\omega_{\lambda(j)}, j)\right|^2 \frac{1}{2\sqrt{m^2+\lambda(j)^2}} = \sum_{j\in\mathbb{N}} \left| f(\omega_{\lambda(j)}, j)\right|^2 \frac{1}{2\omega_{\lambda(j)}}. \qquad \square\end{aligned}$$

For fixed $(\omega, n) \in X_m^+$, for solutions of Proposition 3.32 we can write

$$\hat{\psi}(\omega, n) = a_n^\lambda w^\lambda \otimes e_n^\lambda \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$$

using the isomorphism Definition 2.97, and similarly for the negatieve mass shell.

Proposition 3.27. *The spinors $w^\lambda \otimes e_n^\lambda, \bar{w}^\lambda \otimes \bar{e}_n^\lambda$ are normalized with respect to the invariant inner product induced from the invariant inner product on $\Gamma(\mathcal{S})$. That is we have*

$$\left(w^\lambda \otimes e_n^\lambda, w^\lambda \otimes e_n^\lambda\right) = -\left(\bar{w}^\lambda \otimes \bar{e}_n^\lambda, \bar{w}^\lambda \otimes \bar{e}_n^\lambda\right) = 1.$$

In particular w^λ and $\bar{w}^\lambda$ satisfy

$$-w^{\lambda\dagger}\sigma_1 w^\lambda = \bar{w}^{\lambda\dagger}\sigma_1 \bar{w}^\lambda = 1.$$

Proof. The second statement follows from a straightforward calculation. Indeed we have

$$\begin{aligned} w^{\lambda\dagger}\sigma_1 w^\lambda &= \frac{1}{2m(\omega+\lambda)} \begin{pmatrix}\omega+\lambda & -m\end{pmatrix} \sigma_1 \begin{pmatrix}\omega+\lambda \\ -m\end{pmatrix} \\ &= \frac{-2m(\omega+\lambda)}{2m(\omega+\lambda)} = -1\end{aligned}$$

Hence by equations Eq. (25)

$$\bar{w}^{\lambda\dagger}\sigma_1\bar{w}^\lambda = w^{\lambda\dagger}J^\dagger\sigma_1 J w^\lambda = w^{\lambda\dagger}J^\dagger\sigma_1 J w^\lambda = w^{\lambda\dagger}\sigma_1 J^2 w^\lambda = -w^{\lambda\dagger}\sigma_1 w^\lambda = 1.$$

Now the first statement follows by definition of the invariant inner product Eq. (19), as

$$\left(w^\lambda \otimes e_n^\lambda, w^\lambda \otimes e_n^\lambda\right) = -w^{\lambda\dagger}\sigma_1 w^\lambda \left\langle e_n^\lambda \,\middle|\, e_n^\lambda\right\rangle = 1,$$

and

$$\left(\bar{w}^\lambda \otimes \bar{e}_n^\lambda, \bar{w}^\lambda \otimes \bar{e}_n^\lambda\right) = -\bar{w}^{\lambda\dagger}\sigma_1 \bar{w}^\lambda \left\langle \bar{e}_n^\lambda \,\middle|\, \bar{e}_n^\lambda\right\rangle = -1. \qquad \square$$

Definition 3.28. The space of positive (negative) frequency solutions is given by

$$V^\pm = \{\psi(\omega, j) \in \Gamma(X_m^\pm) \mid (\omega(\sigma_1 \otimes \boldsymbol{I}) + i\sigma_2 \otimes D_\Sigma + m)\,\hat{\psi}(\omega) = 0\},$$

It will be convenient to write $\psi = \sum_{n,\lambda} a_n^\lambda w^\lambda \otimes e_n^\lambda$ for $\psi \in V^+$, and $\psi = \sum_{n,\lambda} b_n^\lambda \bar{w}^\lambda \otimes \bar{e}_n^\lambda$ for $\psi \in V^+$, using the isomorphism given by Definition 2.97, although the domain of these sections are not conveniently described in this representation.

The $\mathrm{Spin}_0(n-1,1)$ invariant inner product $(\cdot,\cdot)$ on $\Gamma(\mathcal{S})$ as defined in Eq. (19), is not a definite inner product, hence it is not suitable to define a Hilbert space. But restricting ourselves to the sections in $V^\pm$, it is!

Proposition 3.29. *The* $\mathrm{Spin}_0(n-1,1)$ *invariant inner product* $(\cdot,\cdot)$ *on* $\Gamma(\mathcal{S})$ *restricted to* $V^\pm$ *is positive (negative) definite. Using Fourier coordinates it is given by*

$$(v,w) = \int_{X_m^+} \frac{m}{\omega} \langle v, w\rangle_{pos}\ d\omega\ dV_{g_\Sigma}.$$

Moreover for $v = \sum_{\lambda,n} a_n^\lambda w^\lambda e_n^\lambda, w = \sum_{\lambda,n} c_n^\lambda w^\lambda e_n^\lambda$*, we have*

$$(v,w) = \sum_{\lambda,n} \frac{1}{2\omega_\lambda} a_n^{\lambda *} c_n^\lambda.$$

Proof. Let $(\cdot,\cdot)_{pos}$ be the standard positive definite inner product on $\Gamma(\mathcal{S})$. We have that the $\mathrm{Spin}_0(n-1,1)$ invariant inner product is related to this by

$$(\cdot,\cdot) = \left(\cdot, i\gamma^0 \cdot\right)_{pos}.$$

Note that for $v \in V_\lambda^\pm$ $(v, i\sigma_2 \otimes D_\Sigma v)_{pos} = 0$, as

$$\begin{pmatrix} \omega_\lambda + \lambda \\ -m \end{pmatrix} i\sigma_2 \begin{pmatrix} \omega_\lambda + \lambda \\ -m \end{pmatrix} = 0.$$

Hence

$$m\|v\|_{pos}^2 = (v, mv)_{pos} = \left(v, (\pm i\gamma^0\omega_\lambda - i\sigma_2 \otimes D_\Sigma)v\right)_{pos} = \pm\omega_\lambda \left(v, i\gamma^0 v\right)_{pos} = \pm\omega_\lambda (v,v).$$

Therefore $(v,v) = \frac{m\|v\|_{pos}^2}{\pm\omega_\lambda}$ is a positive definite inner product on V_λ^+, and a negative definite inner product on V_λ^-. Hence it is a positive definite inner product on V^+ and a negative definite inner product on V^-. For simplicity we will restrict ourselves in the rest of the proof to V^+, but a similar calculation can also be performed on V^-. Hence we assume

$$v = \sum_{\lambda,n} a_n^\lambda w^\lambda e_n^\lambda, \quad w = \sum_{\lambda,n} c_n^\lambda w^\lambda e_n^\lambda.$$

Then we have

$$\begin{aligned}
(v,w) &= \int_{X_m^+} \frac{m}{\omega} \langle v, w\rangle_{pos}\ d\omega\ dV_{g_\Sigma} \\
&= \int_{X_m^+} \frac{m}{\omega} \left(\sum_{\lambda,n} a_n^\lambda w^\lambda e_n^\lambda\right)^\dagger \left(\sum_{\lambda',n'} c_{n'}^{\lambda'} w^{\lambda'} e_{n'}^{\lambda'}\right) d\omega\ dV_{g_\Sigma} \\
&= \int_{\mathbb{R}} \sum_{\lambda,n} \frac{m}{\omega} a_n^{\lambda *} c_n^\lambda \frac{\omega_\lambda}{m} \delta(\omega^2 - m^2 - \lambda^2) d\omega \\
&= \int_{\mathbb{R}} \sum_{\lambda,n} a_n^{\lambda *} c_n^\lambda \delta(\omega^2 - m^2 - \lambda^2) d\omega = \sum_{\lambda,n} \frac{a_n^{\lambda *} c_n^\lambda}{2\omega_\lambda}.
\end{aligned}$$

Here we have used
$$w^{\lambda\dagger}w^{\lambda} = \frac{\omega_\lambda}{m}.$$
□

Remark 3.30. It is common practice to incorporate the factor $\frac{1}{2\omega_\lambda}$ into the coefficients a_n^λ, c_n^λ. That is we redefine
$$v = \sum_{\lambda,n} \sqrt{2\omega_\lambda} a_n^\lambda w^\lambda e_n^\lambda, \quad w = \sum_{\lambda,n} \sqrt{2\omega_\lambda} c_n^\lambda w^\lambda e_n^\lambda,$$
such that
$$(v,w) = \sum_{\lambda,n} a_n^{\lambda *} c_n^\lambda.$$
♢

Definition 3.31. We define the Hilbert space $\mathcal{H}^+$ of particles as the completion of V^+ with respect to the positive definite invariant inner product on V^+.
We define the Hilbert space $\mathcal{H}^-$ of antiparticles as the completion of V^- with respect to the negative definite invariant inner product on V^-.
We define the Hilbert space of solutions $\mathcal{H}$, as the direct sum of both Hilbert spaces
$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.$$

Note that we have a unitary maps $U^\pm : \mathcal{H}^\pm \to L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$, induced by
$$U^+(\sqrt{2\omega_\lambda} w^\lambda \otimes e_n^\lambda) = \sqrt{\frac{m}{\omega_\lambda}} w^\lambda \otimes e_n^\lambda, \quad U^-(\sqrt{2\omega_\lambda} \bar{w}^\lambda \otimes \bar{e}_n^\lambda) = \sqrt{\frac{m}{\omega_\lambda}} \bar{w}^\lambda \otimes \bar{e}_n^\lambda \tag{27}$$

Now its time to return from Fourier transformed coordinates and finish this section.

Proposition 3.32. *Every strong solution of the initial value problem of the Dirac equation*
$$\psi(t) = \sum_{\lambda,n} \sqrt{\frac{m}{\omega_\lambda}} \left(a_n^\lambda w^\lambda \otimes e_n^\lambda e^{-i\omega_\lambda t} + b_n^\lambda \bar{w}^\lambda \otimes \bar{e}_n^\lambda e^{i\omega_\lambda t} \right)$$
as stated in Theorem 3.23, is the inverse temporal Fourier transform of $v = v^+ + v^-$, with $v^\pm \in V^\pm$ given by
$$v^+ = \sum_{\lambda,n} \sqrt{2\omega_\lambda} a_n^\lambda w^\lambda e_n^\lambda, \quad v^- = \sum_{\lambda,n} \sqrt{2\omega_\lambda} b_n^\lambda \bar{w}^\lambda \bar{e}_n^\lambda.$$

Proof. This almost immediately follows from the previous propositions, and the Fourier inversion theorem Theorem 3.9. Note that we have added a factor $\sqrt{\frac{m}{\omega_\lambda}} \frac{1}{\sqrt{2\omega_\lambda}}$ to the inverse Fourier transform as explained in Remark 3.15. □

Remark 3.33. Analog to Remark 3.45 we will call
$$\begin{aligned} \psi_{\lambda,n}(x) &= e^{-i\omega t} w^\lambda \otimes e_n^\lambda, \\ \bar{\psi}_{\lambda,n}(x) &= (J \otimes J)(\psi_{\lambda,n}) = e^{i\omega t} \bar{w}^\lambda \otimes \bar{e}_n^\lambda. \end{aligned} \tag{28}$$
plane wave solutions. ♢

3.3.1 Minkowski-space equivalent

This construction of solutions can be performed in an equivalent manner on Minkowski space, but one has to take all analytical aspects into account as $\mathbb{R}^3$ is not compact. We refer to [47, Sec. 1.4] for a rigorous approach, while we focus here on heuristically constructing solutions using the full Fourier transform.

We will first study solutions of the Dirac operator on $\mathbb{R}^3$, which is given by

$$D_{\mathbb{R}^3} = -i\sigma_1\partial_x - i\sigma_2\partial_y - i\sigma_3\partial_z.$$

By applying Theorem 3.14 the Fourier transformed Dirac operator is given by

$$\mathcal{F}D_{\mathbb{R}^3}\mathcal{F}^{-1} = M_{k_1}\sigma_1 + M_{k_2}\sigma_2 + M_{k_3}\sigma_3.$$

For a fixed $\boldsymbol{k}$, this is just a 2 by 2 matrix.

Lemma 3.34. *The matrix*

$$\boldsymbol{k}\cdot\boldsymbol{\sigma} = k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3 = \begin{pmatrix} k_3 & k_1 - ik_2 \\ k_1 + ik_2 & -k_3 \end{pmatrix}$$

has eigenvalues $\pm|\boldsymbol{k}|$ *with corresponding eigenvectors*

$$w_2(\boldsymbol{k}) = \frac{1}{\sqrt{2|\boldsymbol{k}|(|\boldsymbol{k}|+k_3)}}\begin{pmatrix} |\boldsymbol{k}| + k_3 \\ k_1 + ik_2 \end{pmatrix},$$

$$\bar{w}_2(\boldsymbol{k}) = Jw_2(\boldsymbol{k}) = \frac{1}{\sqrt{2|\boldsymbol{k}|(|\boldsymbol{k}|+k_3)}}\begin{pmatrix} -k_1 + ik_2 \\ |\boldsymbol{k}| + k_3 \end{pmatrix} = \frac{-k_1+ik_2}{|-k_1+ik_2|}w_2(-\boldsymbol{k})$$

respectively. Here $\kappa(-\boldsymbol{k}) = \frac{-k_1+ik_2}{|-k_1+ik_2|}$ *is just a phase factor. When* $\boldsymbol{k} = 0$*, we set*

$$w_2(\boldsymbol{k}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{w}_2(\boldsymbol{k}) = Jw_2(\boldsymbol{k}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Proof. Indeed,

$$\begin{pmatrix} k_3 & k_1 - ik_2 \\ k_1 + ik_2 & -k_3 \end{pmatrix}\begin{pmatrix} |\boldsymbol{k}| + k_3 \\ k_1 + ik_2 \end{pmatrix} = \begin{pmatrix} |\boldsymbol{k}|k_3 + k_3^2 + k_1^2 + k_2^2 \\ |\boldsymbol{k}|(k_1+ik_2) + k_3(k_1+ik_2) - k_3(k_1+ik_2) \end{pmatrix}$$
$$= |\boldsymbol{k}|\begin{pmatrix} |\boldsymbol{k}| + k_3 \\ k_1 + ik_2 \end{pmatrix}.$$

□

The Dirac equation of Minkowski space is given by

$$(\gamma^\mu\partial_\mu + m)\psi = 0.$$

We use the concrete representation for the gamma-matrices given by Eq. (14), i.e.

$$\gamma^0 = i\sigma_1\otimes\boldsymbol{I}, \quad \gamma^a = \sigma_2\otimes\sigma_a \quad a = 1,2,3.$$

In this representation the Dirac equation becomes

$$((i\sigma_1 \otimes \boldsymbol{I})\partial_t + i\sigma_2 \otimes D_{\mathbb{R}^3} + m)\, \psi(t, \boldsymbol{x}).$$

Transforming both sides with the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^4) \to \mathcal{S}(\mathbb{R}^4)$ and Theorem 3.14, we find

$$(\omega(\sigma_1 \otimes \boldsymbol{I}) + i\sigma_2 \otimes \boldsymbol{k} \cdot \boldsymbol{\sigma} + m)\, \hat{\psi}(\omega, \boldsymbol{k}) = 0.$$

As we shall see, solutions of the Fourier transformed Dirac equation are supported on the so-called mass shells $X_m^{\pm}$.

Definition 3.35. The positive mass shell is given by

$$X_m^+ = \{k \in \mathbb{R}^4 \mid k^\mu k_\mu = m^2, k^0 > 0\}.$$

The negative mass shell is given by

$$X_m^- = \{k \in \mathbb{R}^4 \mid k^\mu k_\mu = m^2, k^0 < 0\}.$$

Using similar calculations as in Proposition 3.26, one can show that

$$\frac{d^3\boldsymbol{k}}{2\omega(|\boldsymbol{k}|)}$$

is the (up to a constant) unique invariant measure on both mass shells, where

$$\omega(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + m^2}.$$

It is invariant with respect to the action of $\mathrm{SO}_0(3,1)$. For a more details we refer to [41, Appendix: Lorentz invariant measures].

Proposition 3.36. *The solutions of the Fourier transformed Dirac equation on Minkowski space*

$$(i\gamma^\mu k_\mu + m)\psi(\omega, \boldsymbol{k}) = (\omega(\sigma_1 \otimes \boldsymbol{I}) + i\sigma_2 \otimes \boldsymbol{k} \cdot \boldsymbol{\sigma} + m)\, \hat{\psi}(\omega, \boldsymbol{k}) = 0$$

are given by

$$\hat{\psi}(\omega, \boldsymbol{k}) = \sum_{s=\pm 1} \delta(\omega^2 - |\boldsymbol{k}|^2 - m^2) a^s_{\boldsymbol{k}} u(\boldsymbol{k}, s)$$
$$\hat{\psi}(-\omega, \boldsymbol{k}) = \sum_{s=\pm 1} \delta(\omega^2 - |\boldsymbol{k}|^2 - m^2) b^s_{\boldsymbol{k}} v(\boldsymbol{k}, s)$$

where $\omega > 0$

$$u(\boldsymbol{k}, +1) = w_1(|\boldsymbol{k}|) \otimes w_2(\boldsymbol{k}), \qquad v(\boldsymbol{k}, +1) = \bar{w}_1(|\boldsymbol{k}|) \otimes w_2(\boldsymbol{k}),$$
$$u(\boldsymbol{k}, -1) = w_1(-|\boldsymbol{k}|) \otimes \bar{w}_2(\boldsymbol{k}), \qquad v(\boldsymbol{k}, -1) = \bar{w}_1(-|\boldsymbol{k}|) \otimes \bar{w}_2(\boldsymbol{k}),$$

and

$$
\begin{aligned}
w_1(\pm|\boldsymbol{k}|) &= \frac{1}{\sqrt{2m(\omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}|)}} \begin{pmatrix} \omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}| \\ -m \end{pmatrix} \\
&= \frac{1}{2\sqrt{m(\omega(|\boldsymbol{k}|) + m)}} \begin{pmatrix} \omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}| + m \\ -\omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}| - m \end{pmatrix}, \\
\bar{w}_1(\pm|\boldsymbol{k}|) = Jw_1(\pm|\boldsymbol{k}|) &= \frac{1}{\sqrt{2m(\omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}|)}} \begin{pmatrix} m \\ \omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}| \end{pmatrix} \\
&= \frac{1}{2\sqrt{m(\omega(|\boldsymbol{k}|) + m)}} \begin{pmatrix} \omega(|\boldsymbol{k}|) \mp |\boldsymbol{k}| + m \\ \omega(|\boldsymbol{k}|) \pm |\boldsymbol{k}| + m \end{pmatrix},
\end{aligned}
\tag{29}
$$

with $\omega(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + m^2}$.

Proof. The proof is completely similar to the proof of Proposition 3.32, with $w_1(\pm|\boldsymbol{k}|) = w^{\lambda=\pm|\boldsymbol{k}|}$.

□

We see that the solutions corresponding to $u(\boldsymbol{k}, s)$ live on the positive mass shell, while the solutions corresponding to $v(\boldsymbol{k}, s)$ live on the negative mass shell.

Definition 3.37. The space of positive (negative) frequency solutions is given by

$$V^{\pm} = \{\psi(\omega, \boldsymbol{k}) \in \Gamma(X_m^{\pm}) \mid (i\gamma^{\mu}k_{\mu} + m)\psi = 0.\}$$

Proposition 3.36 tells us that any $f \in V^+$ can be written as a linear combination of $u(\boldsymbol{k}, \pm)$ with scalar coefficients that depend on $\boldsymbol{k}$. Similarly any $f \in V^-$ can be written as a linear combination of $u(\boldsymbol{k}, \pm)$ with scalar coefficients that depend on $\boldsymbol{k}$.

Remark 3.38. The solutions $u(\boldsymbol{k}, \pm), v(\boldsymbol{k}, \pm)$ found in previous proposition are not the standard solutions which are normally found (see e.g. [44, Ch. 38]) applying Lorentz boosts to the zero momentum solutions given by

$$u_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes e_s, \quad v_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e'_s$$

where $\{e_+, e_-\}$ and $\{e'_+, e'_-\}$ are two, often equal, orthonormal bases of $\mathbb{C}^2$. A pure Lorentz boost with a speed of $\frac{\boldsymbol{k}}{m}$ is given by

$$\exp\Big(i\eta\hat{\boldsymbol{k}} \cdot \boldsymbol{B}\Big),$$

where where $\eta = \sinh^{-1}(\frac{\boldsymbol{k}}{m})$ is called the rapidity, $B^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{2}\gamma^j\gamma^0$ is the boost matrix, and $\hat{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$ is the unit vector in the direction of $\boldsymbol{k}$. Using the Chiral representation for the gamma-matrices we find

$$2i\hat{\boldsymbol{k}} \cdot \boldsymbol{B} = -\hat{k}^j(i\sigma_1 \otimes I_2)(\sigma_2 \otimes \sigma_j) = \hat{k}^j(\sigma_3 \otimes \sigma_j.)$$

Using the trigonometric identities

$$\cosh\left(\frac{1}{2}\sinh^{-1}\left(\frac{|\boldsymbol{k}|}{m}\right)\right) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{|\boldsymbol{k}|^2}{m^2}+1}+1} = \frac{1}{\sqrt{2m}}\sqrt{\omega+m},$$

$$\sinh\left(\frac{1}{2}\sinh^{-1}\left(\frac{|\boldsymbol{k}|}{m}\right)\right) = \frac{|\boldsymbol{k}|}{m\sqrt{2}}\frac{1}{\sqrt{\sqrt{\frac{|\boldsymbol{k}|^2}{m^2}+1}+1}} = \frac{|\boldsymbol{k}|}{\sqrt{2m}}\frac{1}{\sqrt{\omega+m}},$$

we then find

$$\begin{aligned}\exp\Big(i\eta\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\Big) &= \cosh\Big(\frac{\eta}{2}\Big) + \sinh\Big(\frac{\eta}{2}\Big)(2i\hat{\boldsymbol{k}}\cdot\boldsymbol{B}) \\ &= \frac{1}{\sqrt{2m}}\sqrt{\omega+m} + \frac{|\boldsymbol{k}|}{\sqrt{2m}}\frac{1}{\sqrt{\omega+m}}\hat{k}^j(\sigma_3\otimes\sigma_j) \\ &= \frac{1}{\sqrt{2m(\omega+m)}}\left((\omega+m)\,\boldsymbol{I}_4 + k^j(\sigma_3\otimes\sigma_j)\right).\end{aligned}$$

The standard boosted solutions are then given by

$$\tilde{u}(\boldsymbol{k},s) = \exp\Big(i\eta\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\Big)u_s, \quad \tilde{v}(\boldsymbol{k},s) = \exp\Big(i\eta\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\Big)v_s. \tag{30}$$

These are even easier found by noticing $(-ik_\mu\gamma^\mu + m)(ik_\nu\gamma^\nu + m) = -\omega^2 + |\boldsymbol{k}|^2 + m = 0$, causing

$$\tilde{u}(\boldsymbol{k},s) = \frac{1}{\sqrt{2m(\omega+m)}}(-ik_\mu\gamma^\mu + m)u_s, \quad \tilde{v}(\boldsymbol{k},s) = \frac{1}{\sqrt{2m(\omega+m)}}(ik_\mu\gamma^\mu - m)v_s,$$

to be the same solutions. The solutions found in Proposition 3.36 are also boosted solutions, but of zero-momentum vectors based on the eigenvectors $w_2(\boldsymbol{k})$, that is

$$u(\boldsymbol{k},s) = \exp\Big(i\eta\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\Big)\begin{pmatrix}1\\-1\end{pmatrix}\otimes w_2^s(\boldsymbol{k}), \quad v(\boldsymbol{k},s) = \exp\Big(i\eta\hat{\boldsymbol{k}}\cdot\boldsymbol{B}\Big)\begin{pmatrix}1\\1\end{pmatrix}\otimes w_2^s(\boldsymbol{k}).$$

We can also obtain them using the same trick as before by

$$\begin{aligned}u(\boldsymbol{k},s) &= \frac{1}{2\sqrt{m(\omega+m)}}(-ik_\mu\gamma^\mu + m)\begin{pmatrix}1\\-1\end{pmatrix}\otimes w_2^s(\boldsymbol{k}), \\ v(\boldsymbol{k},s) &= \frac{1}{2\sqrt{m(\omega+m)}}(ik_\mu\gamma^\mu + m)\begin{pmatrix}1\\1\end{pmatrix}\otimes w_2^s(\boldsymbol{k}).\end{aligned}$$

Here we have written $w_2^+ = w_2, w_2^- = \bar{w}_2$ for notational convenience. ♢

Definition 3.39. We define the *Dirac adjoint* of a solution $u(k)$ in momentum space by

$$\bar{u}(k) = iu^\dagger\gamma^0.$$

Remark 3.40. The solutions w_1 and $\bar{w}_1$ are normalized such that

$$-w_1(\pm|\boldsymbol{k}|)^\dagger\sigma_1 w_1(\pm|\boldsymbol{k}|) = \bar{w}_1(\pm|\boldsymbol{k}|)^\dagger\sigma_1\bar{w}_1(\pm|\boldsymbol{k}|) = 1,$$

for later convenience. Using the Dirac adjoint the orthonormality relations can be written as

$$\bar{u}(\boldsymbol{k}, s)u(\boldsymbol{k}, s') = -\bar{v}(\boldsymbol{k}, s)v(\boldsymbol{k}, s') = \delta_{ss'}.$$

The Dirac adjoint is just a convenient way to write the $\mathrm{Spin}_0(3,1)$ invariant inner product, as

$$\langle u(k), v(k)\rangle = \langle u(k), i\gamma^0 v(k)\rangle_{pos} = \bar{u}(k)v(k). \qquad \diamondsuit$$

Proposition 3.41. *The* $\mathrm{Spin}_0(3,1)$ *invariant inner product* $(\cdot,\cdot)$ *on* $\Gamma(\mathcal{S})$ *restricted to functions supported on the mass shells* $X_m^\pm$ *is definite. Using Fourier coordinates it is given by*

$$\begin{aligned}(f,g) &= \int_{X_m^+}\langle f(\omega(|\boldsymbol{k}|),\boldsymbol{k}), g(\omega(|\boldsymbol{k}|),\boldsymbol{k})\rangle\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}\omega(|\boldsymbol{k}|)}\\ &= \int_{\mathbb{R}^3}\frac{mf(\omega(|\boldsymbol{k}|),\boldsymbol{k})^* g(\omega(|\boldsymbol{k}|),\boldsymbol{k})}{2\omega(|\boldsymbol{k}|)^2}\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}.\end{aligned} \tag{31}$$

Proof. The proof is almost equal to the proof of Proposition 3.29. ☐

Remark 3.42. We will incorporate a factor $\frac{1}{\sqrt{2\omega(|\boldsymbol{k}|)}}$ into sections $f(\omega(|\boldsymbol{k}|),\boldsymbol{k})$ such that

$$(f,g) = \int_{\mathbb{R}^3}\frac{mf(\omega(|\boldsymbol{k}|),\boldsymbol{k})^* g(\omega(|\boldsymbol{k}|),\boldsymbol{k})}{\omega(|\boldsymbol{k}|)}\frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}. \qquad \diamondsuit$$

Definition 3.43. We define the Hilbert space $\mathcal{H}^+$ of particles as the completion of V^+ with respect to the positive definite invariant inner product on V^+.
We define the Hilbert space $\mathcal{H}^-$ of antiparticles as the completion of V^- with respect to the negative definite invariant inner product on V^-.
We define the Hilbert space of solutions $\mathcal{H}$, as the direct sum of both Hilbert spaces

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.$$

Following the standard convention we have chosen to incorporate the minus signs from solutions on the negative mass shell into the formula of the solutions when performing the inverse Fourier transform, while keeping $k \in X_m^+$ for all solutions. This makes integration easier, as we only have to integrate over the positive mass shell.

Theorem 3.44. *The solution of the intial value problem*

$$\begin{aligned}((i\sigma_1\otimes \boldsymbol{I})\partial_t + i\sigma_2\otimes D_{\mathbb{R}^3} + m)\,\psi = 0, &\quad \psi\in\mathcal{S}(\mathbb{R}^4)\otimes\mathbb{C}^4\\ \psi(0,\cdot) = \psi_0 &\in \mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^4\end{aligned}$$

is given by

$$\psi = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\sum_{s=\pm1}\sqrt{\frac{m}{\omega(|\boldsymbol{k}|)}}\left(a_{\boldsymbol{k}}^s u(\boldsymbol{k},s)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{-i\omega(|\boldsymbol{k}|)t} + b_{\boldsymbol{k}}^s v(\boldsymbol{k},s)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\omega(|\boldsymbol{k}|)t}\right)d\boldsymbol{k},$$

where $a_{\boldsymbol{k}}^s, b_{\boldsymbol{k}}^s$ *are uniquely determined by solving*

$$\psi_0 = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\sum_{s=\pm1}\sqrt{\frac{m}{\omega(|\boldsymbol{k}|)}}\left(a_{\boldsymbol{k}}^s u(\boldsymbol{k},s)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + b_{\boldsymbol{k}}^s v(\boldsymbol{k},s)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)d\boldsymbol{k}.$$

Remark 3.45. A physicist would say that the plane wave solutions of the Dirac equation are given by

$$\begin{aligned}\psi(x) &= e^{ik^0 x_0} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} u(\boldsymbol{k}, s) = e^{ik^\mu x_\mu} u(\boldsymbol{k}, s),\\ \bar{\psi}(x) &= e^{-ik^0 x_0} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v(\boldsymbol{k}, s) = e^{-ik^\mu x_\mu} v(\boldsymbol{k}, s),\end{aligned} \tag{32}$$

corresponding to positive and negative frequency solutions respectively. In both equations $k^0 = \omega(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + m^2} > 0$.

♢

3.4 Asymptotic solutions of differential equations

In this section we introduce the mathematics needed to speak rigorously of *solutions at infinity* of a differential equation, which is needed in next chapter. We begin with a definition when two functions are asymptotically equivalent. In the following statements $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, see [10, Par. 5.9] how results for real-valued differential equations can be extended to complex valued differential equations. Reference materials are [7, Sec. 3.4], [10, 16, 6, 46].

Definition 3.46. Let $f, g : \mathbb{R} \to \mathbb{F}$ be two continuous functions. We say that $f(t)$ is asymptotically equivalent, or asymptotic to $g(t)$ as x goes to $t_0 \in [-\infty, +\infty]$ if

$$\lim_{t \to t_0} f(t)/g(t) = 1,$$

and we write

$$f(t) \sim g(t) \quad (t \to t_0).$$

Remark 3.47. Note that $f(t)$ and $g(t)$ being asymptotically equivalent is different from having their difference go to zero in the limit. For example $e^t \sim e^t + t$ $(t \to \infty)$, while

$$\lim_{t \to \infty} \left| e^t + t - e^t \right| = \infty.$$

On the other hand, while $\lim_{t \to 0} \left| t - t^2 \right| = 0$, we don't have $t \sim t^2$ $(t \to 0)$ as they approach zero at different rates. But, when one of the two functions is constant, the notions are the same. ♢

Proposition 3.48. *Let $f : \mathbb{R} \to \mathbb{F}$ be a continuous function and $g = C$ the constant function with value $0 \neq C \in \mathbb{F}$. Then*

$$\lim_{t \to t_0} f(t) = C$$

if and only if

$$f(t) \sim g(t) \quad (t \to t_0).$$

Proof. This is trivial, as

$$\lim_{t \to t_0} f(t) = C \iff \lim_{t \to t_0} f(t)/C = 1.$$

□

Using the notion of asymptotic equivalence we can state the following theorem, as given in [16, Theorem 1.9.2]. This is a special case of the Levinson theorem [11, Theorem 8.1].

Theorem 3.49. *Consider the n^{th}-order linear homogeneous differential equation*

$$y^{(n)}(t) + (c_1 + r_1(t))y^{(n-1)}(t) + \cdots + (c_n + r_n(t))y(t) = 0 \tag{33}$$

where c_j are constants such that the polynomial

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

has n distinct zeros λ_k $(1 \leqslant k \leqslant n)$, and $r_j(t)$ are functions that satisfy

$$\int_a^\infty |r_j(t)| \, \mathrm{d}x < \infty,$$

for some $a \in \mathbb{R}$. Then (33) has n solutions $y_k(t)$, which satisfy

$$y_k^{(i-1)}(t) \sim \lambda_k^{i-1} e^{\lambda_k x} \quad (x \to \infty) \qquad \forall\ 1 \leqslant i \leqslant n.$$

The following corollary is a special case of this theorem, for $n = 2$, see [16, Example 1.9.1].

Corollary 3.50. *Let*

$$\phi''(t) + (\omega^2 + r(t))\phi(t) = 0,$$

be a second order homogeneous differential equation with

$$\int_a^\infty |r(t)| \, \mathrm{d}t < \infty,$$

for some $a \in \mathbb{R}$. There are solutions $\phi_1^{out}, \phi_2^{out}$ such that

$$\begin{aligned} \phi_1^{out} &\sim e^{i\omega t} \quad (t \to \infty), & \partial_t \phi_1^{out} &\sim i\omega e^{i\omega t} \quad (t \to \infty), \\ \phi_2^{out} &\sim e^{-i\omega t} \quad (t \to \infty), & \partial_t \phi_2^{out} &\sim -i\omega e^{-i\omega t} \quad (t \to \infty). \end{aligned}$$

If we also assume $\int_{-\infty}^b |r(t)| \, \mathrm{d}t < \infty$, for some $b \in \mathbb{R}$, we also have solutions ϕ_1^{in}, ϕ_2^{in} being asymptotic to $e^{i\omega t}, e^{-i\omega t}$ resp. in the limit $t \to -\infty$.

We will also recall some standard facts about homogeneous linear differential equations.

Definition 3.51. Let f, g be two differentiable functions. Their *Wronskian* is given by

$$W[f, g] = fg' - gf'.$$

Proposition 3.52. *Let $q : \mathbb{R} \to \mathbb{F}$ be an arbitrary real valued continuous function. Let $y_1(t), y_2(t)$ be two solutions of the homogeneous second-order linear differential equation*

$$y'' + q(t)y = 0.$$

Then the Wronskian

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

is constant.

Proof. See e.g. [29, Lemma VI-1-4]. □

Proposition 3.53. *Let*

$$y'' + p(t)y' + q(t)y = 0$$

be a second order linear homogenous differential equation, where p and q are continuous function. If $y_1(t)$ and $y_2(t)$ are two solutions of this differential equation, such that the Wronksian $W[y_1, y_2]$ is not identically zero, then every solution of the differential equation can be written as

$$y = C_1 y_1 + C_2 y_2,$$

for constants $C_1, C_2 \in \mathbb{C}$.

Proof. See, e.g. [46, Thm. 65.5]. □

Definition 3.54. Solutions $y_1(t), y_2(t)$ of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

satisfying

$$W[y_1, y_2] \not\equiv 0,$$

are called a *fundamental set of solutions.*

4 Evolution of quantized Dirac fields in asymptotically static GFLRW spacetimes

If it's not tested, it's broken.

— Bruce Eckel

In this section we study fermionic particle creation due to the expanding of spacetime, as first studied by Parker [37, 38] and further studied by [15] and more recently in [20, 39]. We generalize the results from FLRW spacetime to Generalized Friedmann–Lemaître–Robertson–Walker (GFLRW) spacetimes.

So let us assume that $(M, g) = (\mathbb{R} \times \Sigma, -dt^2 \oplus a^2(t)h)$ is a spatially closed, globally hyperbolic GFLRW spacetime, with Σ a Riemannian spin manifold. We recall that the Dirac equation on M is equal to

$$\left(i\sigma_1 \otimes \left(\partial_t + \frac{3}{2}\frac{\dot{a}(t)}{a(t)}\right) + i\sigma_2 \otimes \frac{1}{a(t)}D_\Sigma + m\right)\psi = 0.$$

We can simplify this equation, by introducing a coordinate transformation on time, given by

$$\partial_t = \frac{1}{a(t)}\partial_\eta, \quad \text{i.e.} \quad \eta = \int \frac{\mathrm{d}t}{a(t)},$$

which we will call *conformal* time. We will also write $C(\eta) = a^2(t)$. The Dirac equation now transforms into

$$(i\sigma_1 \otimes (C^{-1/2}(\eta)\partial_\eta + \frac{3}{4}C^{-1/2}(\eta)\frac{\dot{C}(\eta)}{C(\eta)}) + i\sigma_2 \otimes C^{-1/2}(\eta)D_\Sigma + m)\psi = 0, \tag{34}$$

and by multiplying with $C^{1/2}(\eta)$ and writing $\mu(\eta) = mC^{1/2}(\eta)$ we get

$$\left(i\sigma_1 \otimes \left(\partial_\eta + \frac{3}{4}\frac{\dot{C}(\eta)}{C(\eta)}\right) + i\sigma_2 \otimes D_\Sigma + \mu(\eta)\right)\psi = 0, \tag{35}$$

Proposition 4.1. *Let (e_n^λ) be a complete orthonormal set of eigenspinors of the Dirac operator D_Σ on Σ, with corresponding eigenvalues λ, $n = 1, \dots, \dim V_\lambda$. The functions $\psi_{\lambda,n}$ given by*

$$\psi_{\lambda,n} = \frac{C^{-3/4}N}{\sqrt{2}}(i\sigma_1\partial_\eta + \lambda i\sigma_2 - \mu)\phi_\lambda^\pm \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \otimes e_n^\lambda,$$

where $\phi_\lambda^\pm \in C(\mathbb{R})^2$ satisfies the second order homogeneous differential equation

$$\left(\partial_\eta^2 + \mu^2(\eta) \pm im\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + \lambda^2\right)\phi_\lambda^\pm(\eta) = 0,$$

are solutions of the Dirac equation, Eq. (35).

Proof. We choose the ansatz

$$\psi_{\lambda,n} = C^{-3/4}N\tilde{\phi}_\lambda^n e_n^\lambda, \quad \tilde{\phi}_\lambda^n \in C(\mathbb{R}) \otimes \mathbb{C}^2,$$

where we have omitted the tensor product for notational brevity. Inserting this into the Dirac equation, we get

$$\begin{aligned}0 &= \left(i\sigma_1\left(\partial_\eta + \frac{3}{4}\frac{\dot{C}(\eta)}{C(\eta)}\right) + i\sigma_2 D_\Sigma + \mu\right) e_n^\lambda C^{-3/4}\tilde{\phi}_\lambda^n(\eta) = \\ &= C^{-3/4}e_n^\lambda\big(- \frac{3}{4}i\sigma_1 C^{-1}\dot{C} + i\sigma_1\partial_\eta + \frac{3}{4}i\sigma_1 C^{-1}\dot{C} + i\sigma_2\lambda + \mu\big)\tilde{\phi}_\lambda^n \\ &= C^{-3/4}e_n^\lambda\big(i\sigma_1\partial_\eta + i\sigma_2\lambda + \mu\big)\tilde{\phi}_\lambda^n.\end{aligned}$$

We see that $\tilde{\phi}_\lambda^n$ do only depend on λ, i.e. we can choose $\tilde{\phi}_\lambda^n = \tilde{\phi}_\lambda$ for all n. We conclude that $\psi_{\lambda,n}$ is a solution if and only if

$$(i\sigma_1\partial_\eta + \lambda i\sigma_2 + \mu)\tilde{\phi}_\lambda = 0. \tag{36}$$

We choose the ansatz

$$\tilde{\phi}_\lambda = (i\sigma_1\partial_\eta + \lambda i\sigma_2 - \mu)\bar{\phi}_\lambda, \quad \bar{\phi}_\lambda \in C(\mathbb{R}) \otimes \mathbb{C}^2.$$

Inserting this into Eq. (36), we get

$$\begin{aligned}0 &= (i\sigma_1\partial_\eta + \lambda i\sigma_2 + \mu)(i\sigma_1\partial_\eta + \lambda i\sigma_2 - \mu)\bar{\phi}_\lambda = \\ &= \big(- \partial_\eta^2 - \sigma_1\sigma_2\lambda\partial_\eta - i\sigma_1\mu\partial_\eta - i\sigma_1(\partial_\eta\mu) \\ &\qquad - \sigma_2\sigma_1\lambda\partial_\eta - \lambda^2 - i\sigma_2\lambda\mu + i\sigma_1\mu\partial_\eta + i\sigma_2\lambda\mu - \mu^2\big)\bar{\phi}_\lambda,\end{aligned}$$

and hence we find that $\bar{\phi}_\lambda(\eta)$ has to satisfy

$$(\partial_\eta^2 + m^2C(\eta) + i\sigma_1 m\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + \lambda^2)\bar{\phi}_\lambda(\eta) = 0.$$

By choosing $\bar{\phi}_\lambda^{(\pm)} = \frac{1}{\sqrt{2}}\phi_\lambda^{(\pm)}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$ in the ± 1 eigenspace of σ_1 we get

$$\left(\partial_\eta^2 + m^2C(\eta) \pm im\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + \lambda^2\right)\phi_\lambda^{(\pm)}(\eta) = 0. \tag{37}$$

□

Remark 4.2. When $\dot{C}(\eta) = 0$, Eq. (37) reduces to the harmonic oscillator and we have solutions

$$\phi_\lambda(\eta) = e^{\pm i\omega\eta},$$

where

$$\omega_\lambda = \sqrt{\mu^2 + \lambda^2}, \qquad \mu = mC^{1/2}.$$

♢

Proposition 4.3. *Suppose that*

$$\int_a^\infty \left| m^2(C(\eta) - C_{out}) \pm im\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)}\right| \mathrm{d}\eta < \infty$$

and

$$\int_{-\infty}^{b} \left| m^2(C(\eta) - C_{in}) \pm im\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} \right| \mathrm{d}\eta < \infty,$$

for some $a, b \in \mathbb{R}$, where $C_{out} = \lim_{\eta\to\infty} C(\eta)$ and $C_{in} = \lim_{\eta\to-\infty} C(\eta)$. Then we have asymptotic positive frequency solutions $\phi_\lambda^{in(\pm)}, \phi_\lambda^{out(\pm)}$ of Eq. (37), satisfying

$$\phi_\lambda^{in(\pm)} \sim e^{-i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty) \quad \textit{and} \quad \phi_\lambda^{out(\pm)} \sim e^{-i\omega_{out}(\lambda)\eta} \quad (\eta \to \infty),$$

where

$$\omega_{in}(\lambda) = \sqrt{\mu_{in}^2 + \lambda^2}, \qquad \omega_{out}(\lambda) = \sqrt{\mu_{out}^2 + \lambda^2},$$
$$\mu_{in} = \lim_{\eta\to-\infty} m\sqrt{C(\eta)}, \qquad \mu_{out} = \lim_{\eta\to\infty} m\sqrt{C(\eta)}.$$

The corresponding asymptotic negative frequency solutions of $\phi_\lambda^{in(\pm)}, \phi_\lambda^{out(\pm)}$ are given by $\phi_\lambda^{in(\mp)*}, \phi_\lambda^{out(\mp)*}$ respectively.

Proof. This follows immediately by applying Corollary 3.50. The sign-flip of $\pm$ in the negative frequency solutions originates from the explicit factor i in Eq. (37). □

Assumption 4.4. We will assume that $C(\eta)$ satisfies the conditions stated in the proposition above.

Remark 4.5. Note that for solutions $\phi_\lambda^{(\pm)}$ of Eq. (37), their complex conjugated variants $\phi_\lambda^{(\pm)*}$ satisfy the complex conjugated version of Eq. (37), that is

$$\left(\partial_\eta^2 + m^2C(\eta) \mp im\frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + \lambda^2\right)\phi_\lambda^{(\pm)*}(\eta) = 0.$$

This means that $\phi_\lambda^{(-)}$ and $\phi_\lambda^{(+)*}$ are solutions to the same equation, and the equivalent statement with the signs flipped also holds . Therefore

$$\frac{1}{\sqrt{2}}\phi_\lambda^{(-)}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\phi_\lambda^{(-)*}\begin{pmatrix}1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\phi_\lambda^{(+)}\begin{pmatrix}1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\phi_\lambda^{(+)*}\begin{pmatrix}1\\-1\end{pmatrix}$$

are solutions to Eq. (65). ♢

Proposition 4.6. *Defining*

$$N_{in}^\lambda = \frac{-1}{2\sqrt{\mu_{in}(\omega_{in}(\lambda) + \mu_{in})}}$$

we have for the asymptotic limits

$$N_{in}^\lambda(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)}\begin{pmatrix}1\\-1\end{pmatrix} \sim w_{in}^\lambda e^{-i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty)$$
$$N_{in}^\lambda(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)*}\begin{pmatrix}1\\1\end{pmatrix} \sim \bar{w}_{in}^\lambda e^{i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty)$$

Equivalent statements hold for $\phi^{out(-)}$ and $\phi^{out(-)*}$

Proof. Using $\phi_\lambda^{in(-)} \sim e^{-i\omega_{in}(\lambda)\eta}$ and $\partial_\eta \phi_\lambda^{in(-)} \sim -i\omega_{in}(\lambda)e^{-i\omega_{in}(\lambda)\eta}$ we get

$$\begin{aligned}
N_{in}^\lambda (i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} &\sim N_{in}^\lambda e^{-i\omega_{in}(\lambda)\eta}(\omega_{in}(\lambda)\sigma_1 + i\lambda\sigma_2 - \mu) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= N_{in}^\lambda e^{-i\omega_{in}(\lambda)\eta} \begin{pmatrix} -\mu & \omega_{in}(\lambda) + \lambda \\ \omega_{in}(\lambda) - \lambda & -\mu \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{2\sqrt{\mu_{in}(\omega_{in}(\lambda) + \mu_{in})}} e^{-i\omega_{in}(\lambda)\eta} \begin{pmatrix} \omega_{in}(\lambda) + \lambda + \mu \\ -\omega_{in}(\lambda) + \lambda - \mu \end{pmatrix} \\
&= w_{in}^\lambda e^{-i\omega_{in}(\lambda)\eta}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
N_{in}^\lambda (i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\sim N_{in}^\lambda e^{i\omega_{in}(\lambda)\eta}(-\omega_{in}(\lambda)\sigma_1 + i\lambda\sigma_2 - \mu) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= N_{in}^\lambda e^{i\omega_{in}(\lambda)\eta} \begin{pmatrix} -\mu & -\omega_{in}(\lambda) + \lambda \\ -\omega_{in}(\lambda) - \lambda & -\mu \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{2\sqrt{\mu_{in}(\omega_{in}(\lambda) + \mu_{in})}} e^{i\omega_{in}(\lambda)\eta} \begin{pmatrix} \omega_{in}(\lambda) - \lambda + \mu \\ \omega_{in}(\lambda) + \lambda + \mu \end{pmatrix} \\
&= w_{in}^\lambda e^{i\omega_{in}(\lambda)\eta}.
\end{aligned}$$

□

We also proof the following two lemmas for future use.

Lemma 4.7. *The following identities hold.*

1.

$$(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)e^{-i\omega_{out}(\lambda)\eta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \omega_{out}(\lambda) + \lambda - \mu_{out} \\ \omega_{out}(\lambda) - \lambda - \mu_{out} \end{pmatrix} e^{-i\omega_{out}(\lambda)\eta}.$$

2.

$$(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)e^{i\omega_{out}(\lambda)\eta} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \omega_{out}(\lambda) - \lambda - \mu_{out} \\ -\omega_{out}(\lambda) - \lambda + \mu_{out} \end{pmatrix} e^{i\omega_{out}(\lambda)\eta}.$$

Proof. This follows from completely straightforward computations. Indeed for the first identity we have

$$\begin{aligned}
(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)e^{-i\omega_{out}(\lambda)\eta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= (\omega_{out}(\lambda)\sigma_1 + i\lambda\sigma_2 - \mu) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_{out}(\lambda)\eta} \\
&= \begin{pmatrix} -\mu & \omega_{out}(\lambda) + \lambda \\ \omega_{out}(\lambda) - \lambda & -\mu \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_{out}(\lambda)\eta} \\
&= \begin{pmatrix} \omega_{out}(\lambda) + \lambda - \mu_{out} \\ \omega_{out}(\lambda) - \lambda - \mu_{out} \end{pmatrix} e^{-i\omega_{out}(\lambda)\eta}.
\end{aligned}$$

Similarly, we have for the second identity

$$\begin{aligned}(i\sigma_1\partial_\eta + i\lambda\sigma_2 - \mu)e^{i\omega_{out}(\lambda)\eta}\begin{pmatrix}1\\-1\end{pmatrix} &= (-\omega_{out}(\lambda)\sigma_1 + i\lambda\sigma_2 - \mu)\begin{pmatrix}1\\-1\end{pmatrix}e^{i\omega_{out}(\lambda)\eta}\\ &= \begin{pmatrix}-\mu & -\omega_{out}(\lambda)+\lambda\\ -\omega_{out}(\lambda)-\lambda & -\mu\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}e^{i\omega_{out}(\lambda)\eta}\\ &= \begin{pmatrix}\omega_{out}(\lambda)-\lambda-\mu_{out}\\ -\omega_{out}(\lambda)-\lambda+\mu_{out}\end{pmatrix}e^{i\omega_{out}(\lambda)\eta}.\end{aligned}$$

□

Lemma 4.8. *The following identities hold.*

1.
$$-N_{out}^{\lambda}w^{\lambda\dagger}\sigma_1\begin{pmatrix}\omega_{out}(\lambda)+\lambda-\mu_{out}\\ \omega_{out}(\lambda)-\lambda-\mu_{out}\end{pmatrix} = \frac{\lambda}{\omega_{out}(\lambda)+\mu_{out}}.$$

2.
$$N_{out}^{\lambda}\bar{w}^{\lambda\dagger}\sigma_1\begin{pmatrix}\omega_{out}(\lambda)-\lambda-\mu_{out}\\ -\omega_{out}(\lambda)-\lambda+\mu_{out}\end{pmatrix} = \frac{-\lambda}{\omega_{out}(\lambda)+\mu_{out}}.$$

Proof. To check the first identity, we compute

$$\begin{aligned}&-N_{out}^{\lambda}w^{\lambda\dagger}\sigma_1\begin{pmatrix}\omega_{out}(\lambda)+\lambda-\mu_{out}\\ \omega_{out}(\lambda)-\lambda-\mu_{out}\end{pmatrix}\\ &= -\left(N_{out}^{\lambda}\right)^2\begin{pmatrix}\omega_{out}(\lambda)+\lambda+\mu_{out} & -\omega_{out}(\lambda)+\lambda-\mu_{out}\end{pmatrix}\begin{pmatrix}\omega_{out}(\lambda)-\lambda-\mu_{out}\\ \omega_{out}(\lambda)+\lambda-\mu_{out}\end{pmatrix}\\ &= -\left(N_{out}^{\lambda}\right)^2\left(\omega_{out}(\lambda)^2-\lambda^2-\mu_{out}^2-2\mu_{out}\lambda-\omega_{out}(\lambda)^2+\lambda^2+\mu_{out}^2-2\mu_{out}\lambda\right)\\ &= -\frac{-4\mu_{out}\lambda}{4\mu_{out}(\omega_{out}(\lambda)+\mu_{out})} = \frac{\lambda}{\omega_{out}(\lambda)+\mu_{out}}.\end{aligned}$$

Similarly, we check the second identity by computing

$$\begin{aligned}&N_{out}^{\lambda}\bar{w}^{\lambda\dagger}\sigma_1\begin{pmatrix}\omega_{out}(\lambda)-\lambda-\mu_{out}\\ -\omega_{out}(\lambda)-\lambda+\mu_{out}\end{pmatrix}\\ &= -\left(N_{out}^{\lambda}\right)^2\begin{pmatrix}\omega_{out}(\lambda)-\lambda+\mu_{out} & \omega_{out}(\lambda)+\lambda+\mu_{out}\end{pmatrix}\begin{pmatrix}-\omega_{out}(\lambda)-\lambda+\mu_{out}\\ \omega_{out}(\lambda)-\lambda-\mu_{out}\end{pmatrix}\\ &= -\left(N_{out}^{\lambda}\right)^2\left(-\omega_{out}(\lambda)^2+\lambda^2+\mu_{out}^2-2\mu_{out}\lambda+\omega_{out}(\lambda)^2-\lambda^2-\mu_{out}^2-2\mu_{out}\lambda\right)\\ &= \frac{-4\mu_{out}\lambda}{4\mu_{out}(\omega_{out}(\lambda)+\mu_{out})} = \frac{-\lambda}{\omega_{out}(\lambda)+\mu_{out}}.\end{aligned}$$

□

Proposition 4.9. *Let $\phi_{\lambda}^{in(\pm)}, \phi_{\lambda}^{out(\pm)}$ be the solutions given by Proposition 4.3. Then*

$$W[\phi_{\lambda}^{in(-)}, \phi_{\lambda}^{in(+)*}] = 2i\omega_{in}(\lambda), \qquad W[\phi_{\lambda}^{out(-)}, \phi_{\lambda}^{out(+)*}] = 2i\omega_{out}(\lambda)$$

In particular, the Wronskians are non-zero.

Proof. We will only show the first equation, since the second follows by a completely similar computation. By Proposition 3.52 we know the Wronskian is constant. So to compute its value, it is sufficient we to compute its limit at $-\infty$. Since by Corollary 3.50 we have

$$\phi_\lambda^{in(-)} \sim e^{-i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty), \qquad \partial_\eta \phi_\lambda^{in(+)*} \sim -i\omega_{in}(\lambda) e^{i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty),$$

it follows that

$$\phi_\lambda^{in(-)} \partial_\eta \phi_\lambda^{in(+)*} \sim i\omega_{in}(\lambda), \quad \phi_\lambda^{in(+)*} \partial_\eta \phi_\lambda^{in(-)} \sim -i\omega_{in}(\lambda),$$

and using Proposition 3.48 we find

$$W[\phi_\lambda^{in(-)}, \phi_\lambda^{in(+)*}] = \lim_{\eta \to -\infty} W[\phi_\lambda^{in(-)}, \phi_\lambda^{in(+)*}](\eta) = 2i\omega_{in}(\lambda).$$

□

Proposition 4.10. *There are coefficients* $\alpha_\lambda^{(\pm)}, \beta_\lambda^{(\pm)}$ *satisfying*

$$\phi_\lambda^{in(\pm)}(\eta) = \alpha_\lambda^{(\pm)} \phi_\lambda^{out(\pm)}(\eta) + \beta_\lambda^{(\pm)} \phi_\lambda^{out(\mp)*}(\eta). \tag{38}$$

Proof. By Proposition 4.9 we know that $\{\phi_\lambda^{in(\pm)}, \phi_\lambda^{in(\mp)*}\}$ and $\{\phi_\lambda^{out(\pm)}, \phi_\lambda^{out(\mp)*}\}$ are two fundamental set of solutions of the same differential Eq. (37). Hence by Proposition 3.53 we can find constants $\alpha_\lambda^{(\pm)}, \beta_\lambda^{(\pm)}$ such that Eq. (38) holds. □

Definition 4.11. The coefficients $\alpha_\lambda^{(\pm)}, \beta_\lambda^{(\pm)}$ from Proposition 4.10 are called *Bogoliubov coefficients.*

Following appendix A of [15], we will prove some relations, relating the Bogoliubov coefficients for future use.

Proposition 4.12. *We have the following relations between the Bogoliubov coefficients.*

$$\frac{\alpha_\lambda^{(+)}}{\alpha_\lambda^{(-)}} = \frac{\omega_{in}(\lambda) - \mu_{in}}{\omega_{out}(\lambda) - \mu_{out}} = \frac{\omega_{out}(\lambda) + \mu_{out}}{\omega_{in}(\lambda) + \mu_{in}}, \tag{39}$$

$$\frac{\beta_\lambda^{(+)}}{\beta_\lambda^{(-)}} = -\frac{\omega_{in}(\lambda) - \mu_{in}}{\omega_{out}(\lambda) + \mu_{out}} = -\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{in}(\lambda) + \mu_{in}}, \tag{40}$$

$$\alpha_\lambda^{(-)} \alpha_\lambda^{(+)*} - \beta_\lambda^{(-)} \beta_\lambda^{(+)*} = \frac{\omega_{in}(\lambda)}{\omega_{out}(\lambda)}, \tag{41}$$

$$\left|\alpha_\lambda^{(-)}\right|^2 - \frac{-\omega_{out}(\lambda) + \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}} \left|\beta_\lambda^{(-)}\right|^2 = \frac{\mu_{out}\omega_{in}(\lambda)}{\mu_{in}\omega_{out}(\lambda)} \left(\frac{N_{out}}{N_{in}}\right)^2. \tag{42}$$

Proof. By defining $\mathcal{D}_\pm = i\partial_\eta \pm m\sqrt{C(\eta)}$, Eq. (37) becomes

$$\mathcal{D}_\pm \mathcal{D}_\mp \phi_\lambda^{(\pm)} = \lambda^2 \phi_\lambda^{(\pm)}.$$

Applying $\mathcal{D}_\mp$ again, we get

$$\mathcal{D}_\mp \mathcal{D}_\pm (\mathcal{D}_\mp \phi_\lambda^{(\pm)}) = \lambda^2 (\mathcal{D}_\mp \phi_\lambda^{(\pm)}).$$

As every solution of $\mathcal{D}_\mp\mathcal{D}_\pm\phi = \lambda^2\phi$ is a linear combination of $\phi^{(\mp)}$ and $\phi^{(\pm)*}$ it follows that there are constants $A_\mp, B_\mp$ such that

$$\mathcal{D}_\mp\phi^{(\pm)} = A_\mp\phi^{(\mp)} + B_\mp\phi^{(\pm)*}.$$

But as $\phi^\pm$ and $\partial_\eta\phi^\pm$ are negative frequency solutions in the asymptotic limits we need to have $B_\mp = 0$, hence $\mathcal{D}_\mp\phi_\lambda^{(\pm)} = A_\mp\phi_\lambda^{(\mp)}$, for some constants $A_\mp$. And since by Theorem 3.49 we have

$$\mathcal{D}_\mp\phi_\lambda^{in(\pm)} \sim (\omega_{in}(\lambda) \mp \mu_{in})e^{-i\omega_{in}(\lambda)\eta} \quad (\eta \to -\infty),$$

we need $\mathcal{D}_\mp\phi^{in(\pm)} = (\omega_{in}(\lambda) \mp \mu_{in})\phi^{in(\mp)}$. We now apply the Bogoliubov transformation on both sides of this equality, and get

$$\mathcal{D}_\mp(\alpha_\lambda^{(\pm)}\phi_\lambda^{out(\pm)} + \beta_\lambda^{(\pm)}\phi_\lambda^{out(\mp)*}) = (\omega_{in}(\lambda) \mp \mu_{in})(\alpha_\lambda^{(\mp)}\phi_\lambda^{out(\mp)} + \beta_\lambda^{(\mp)}\phi_\lambda^{out(\pm)*}) \tag{43}$$

And since again by Theorem 3.49 we have

$$\partial_\eta\phi_\lambda^{out(\pm)} \sim -i\omega_{out}(\lambda)e^{-i\omega_{out}(\lambda)\eta} \quad (\eta \to \infty).$$

Therefore we see that Eq. (43) is asymptotically equivalent to

$$\begin{aligned}\alpha_\lambda^{(\pm)}(\omega_{out}(\lambda) \mp \mu_{out})e^{-i\omega_{out}(\lambda)\eta} &+ \beta_\lambda^{(\pm)}(-\omega_{out}(\lambda) \mp \mu_{out})e^{i\omega_{out}(\lambda)\eta} \\ &= (\omega_{in}(\lambda) \mp \mu_{in})(\alpha_\lambda^{(\mp)}e^{-i\omega_{out}(\lambda)\eta} + \beta_\lambda^{(\mp)}e^{i\omega_{out}(\lambda)\eta}),\end{aligned}$$

as $\eta \to \infty$. Equating coefficients, we immediately obtain the relations Eq. (39) and Eq. (40).

To obtain Eq. (41) we calculate the Wronskians of Proposition 3.52 in two different ways, using the Bogoliubov transfomations

$$\begin{aligned}2i\omega_{in}(\lambda) &= \lim_{\eta\to-\infty} W[\phi_\lambda^{in(-)}, \phi_\lambda^{in(+)*}] = \lim_{\eta\to\infty} W[\phi_\lambda^{in(-)}, \phi_\lambda^{in(+)*}] \\ &= \lim_{\eta\to\infty} W[\alpha_\lambda^{(-)}\phi_\lambda^{out(-)} + \beta_\lambda^{(-)}\phi_\lambda^{out(+)*}, \alpha_\lambda^{(+)*}\phi_\lambda^{out(+)*} + \beta_\lambda^{(+)*}\phi_\lambda^{out(-)}] \\ &= 2i\omega_{out}(\lambda)\left(\alpha_\lambda^{(-)}\alpha_\lambda^{(+)*} - \beta_\lambda^{(-)}\beta_\lambda^{(+)*}\right),\end{aligned}$$

from which Eq. (41) immediately follows.

Now we will show Eq. (42). By observing

$$(N^\lambda)^2 = \frac{1}{(\omega_\lambda - \mu)(\omega_\lambda + \mu)}\frac{\omega_\lambda - \mu}{4\mu} = \frac{1}{\omega_\lambda^2 - \mu^2}\frac{\omega_\lambda - \mu}{4\mu} = \frac{1}{\lambda^2}\frac{\omega_\lambda - \mu}{4\mu}$$

and using the relations Eq. (39), Eq. (40), Eq. (41), that we have just shown, we obtain:

$$\begin{aligned}\frac{\mu_{out}\omega_{in}(\lambda)}{\mu_{in}\omega_{out}(\lambda)}\left(\frac{N_{out}}{N_{in}}\right)^2 &= \frac{\omega_{in}(\lambda)}{\omega_{out}(\lambda)}\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{in}(\lambda) - \mu_{in}} \\ &= \frac{\alpha_\lambda^{(-)}}{\alpha_\lambda^{(+)}}(\alpha_\lambda^{(-)*}\alpha_\lambda^{(+)} - \beta_\lambda^{(-)*}\beta_\lambda^{(+)}) \\ &= \left|\alpha_\lambda^{(-)}\right|^2 - \frac{\beta_\lambda^{(-)*}\beta_\lambda^{(+)}\alpha_\lambda^{(-)}}{\alpha_\lambda^{(+)}} \\ &= \left|\alpha_\lambda^{(-)}\right|^2 - \left|\beta_\lambda^{(-)}\right|^2\left(\frac{\omega_{in}(\lambda) + \mu_{in}}{\omega_{out}(\lambda) + \mu_{out}}\right)\left(-\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{in}(\lambda) + \mu_{in}}\right) \\ &= \left|\alpha_\lambda^{(-)}\right|^2 + \left|\beta_\lambda^{(-)}\right|^2\left(\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}}\right),\end{aligned}$$

which proves Eq. (42). □

Proposition 4.13. *Let Σ is a compact Riemannian spin Manifold. Let $\mathcal{H}_\Sigma = L^2(\Sigma, \mathcal{S}_\Sigma)$. We assume that the Dirac operator $D_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$ has a symmetric spectrum $\{\lambda\} \subseteq \mathbb{R}$ with normalized eigenfunctions e_n^λ, $n = 1, \ldots, \dim V_\lambda$. Let J be the real structure on Σ and write*

$$\bar{e}_n^\lambda = J e_n^\lambda.$$

Then the Dirac Eq. (35) has two sets of solutions given by $\{\psi_{\lambda,n}^{in}, \bar{\psi}_{\lambda,n}^{in}\}$ and $\{\psi_{\lambda,n}^{out}, \bar{\psi}_{\lambda,n}^{out}\}$ that go over to the plane wave solutions given by Eq. (28) in the aysmptotic regions. Here

$$\psi_{\lambda,n}^{in} = C^{-3/4} N_{in}^\lambda (i\sigma_1 \partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes e_n^\lambda,$$

$$\bar{\psi}_{\lambda,n}^{in} = C^{-3/4} N_{in}^\lambda (i\sigma_1 \partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \bar{e}_n^\lambda$$

and mutatis mutandis for $\psi_{\lambda,n}^{out}, \bar{\psi}_{\lambda,n}^{out}$.

Proof. This follows immediately from Proposition 4.1, Remark 4.5 and Proposition 4.6, and the fact that $\bar{e}_n^\lambda$ has the same eigenvalue λ as e_n^λ does. □

Let us denote the unique strong solution of the static Dirac equation with initial value $f \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ and mass $m = \mu$, as given by Theorem 3.23, by $\psi_0^\mu(f)$.

Now let $f_{in} \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ given by

$$f_{in} = C_{in}^{-3/4} \sum_{\lambda,n} K_{in}^\lambda \left(a_{\lambda,n}^{in} w_{in}^\lambda \otimes e_n^\lambda + b_{\lambda,n}^{in} \bar{w}_{in}^\lambda \otimes \bar{e}_n^\lambda \right).$$

Here and in the future we will write $K_{in/out}^\lambda = \sqrt{\frac{\mu_{in/out}}{\omega_{in/out}(\lambda)}}$ for the normalization factor. We also adjust the inner product $\langle \cdot | \cdot \rangle_{in} = C_{in}^{3/2} \langle \cdot | \cdot \rangle$, to compensate for the explicit factor $C_{in}^{-3/4}$. Using the previous proposition, we can construct a solution to the Dirac equation Eq. (35),

$$\begin{aligned}\psi_{in}(f_{in}) = \; & C^{-3/4} \sum_\lambda K_{in}^\lambda N_{in}^\lambda \sum_{n=1}^{\dim V_\lambda} \left(a_{\lambda,n}^{in} (i\sigma_1 \partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)} e_n^\lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right. \\ & \left. + b_{\lambda,n}^{in} (i\sigma_1 \partial_\eta + i\lambda\sigma_2 - \mu)\phi_\lambda^{in(-)*} \bar{e}_n^\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),\end{aligned} \tag{44}$$

that is asymptotically equivalent with $\psi_0^{\mu_{in}}(f_{in})$ for $\eta \to -\infty$ In exactly the same way we have for a $f_{out} \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ a solution $\psi_{out}(f_{out})$ such that

$$\psi_{out}(f_{out}) \sim \psi_0^{\mu_{out}}(f_{out}) \quad (\eta \to \infty).$$

Using the Bogoliubov transformation given by Proposition 4.10, we will now construct a map

$$\mathcal{U} : L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2 \to L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2,$$

that assigns to a $f_{out} \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ the vector $\mathcal{U} f_{out}$, such that

$$\psi_{in}(\mathcal{U}f) \sim \psi_0^{\mu_{out}}(f).$$

This map represents the evolution of a field from the asymptotic *out* region to the asymptotic *in* region. Before we state and proof the theorem defining the map $\mathcal{U}$, we define the *polarization tensor.*

Definition 4.14. The *polarization tensor* X^λ is the square matrix of order $\dim V_\lambda$ with entries

$$(X^\lambda)_{nm} = X^\lambda_{nm} = \frac{-\lambda \left\langle e^\lambda_n \,\middle|\, J e^\lambda_m \right\rangle}{\omega_{out}(\lambda) + \mu_{out}}$$

We also set

$$\mathcal{C}^\lambda = \frac{K^\lambda_{in}}{K^\lambda_{out}} \frac{N^\lambda_{in}}{N^\lambda_{out}}.$$

Theorem 4.15. *Let*

$$\mathcal{U} : L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2 \to L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2,$$

be the map that assigns to a $f \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ *the element* $\mathcal{U}f$*, such that*

$$\psi_{in}(\mathcal{U}f) \sim \psi_0^{\mu_{out}}(f) \quad (\eta \to \infty).$$

Then the map $\mathcal{U}$ *is given by* $\mathcal{U} = \bigoplus_\lambda U^\lambda$ *, for* $U^\lambda = U^\lambda_{even} + U^\lambda_{odd}$ *with*

$$U^\lambda_{even} = \mathcal{C}^\lambda \begin{pmatrix} \alpha^{(-)*}_\lambda \boldsymbol{I}_N & 0 \\ 0 & \alpha^{(-)}_\lambda \boldsymbol{I}_N \end{pmatrix}, \quad U^\lambda_{odd} = \mathcal{C}^\lambda \begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^\dagger & 0 \end{pmatrix}$$

with respect to the ordered orthonormal bases $((C_{in}^{-3/4} K^\lambda_{in} w^\lambda_{in} \otimes e^\lambda_n), (C_{in}^{-3/4} K^\lambda_{in} \bar{w}^\lambda_{in} \otimes \bar{e}^\lambda_n))$ *and* $((C_{out}^{-3/2} K^\lambda_{out} w^\lambda_{out} \otimes e^\lambda_n), (C_{out}^{-3/2} K^\lambda_{out} \bar{w}^\lambda_{out} \otimes \bar{e}^\lambda_n))$ *of* $\mathcal{H}^\Sigma_\lambda \otimes \mathbb{C}^2$.
Here $N = \dim \mathcal{H}^\Sigma_\lambda$ *and* $\mathcal{B} = \beta^{(-)}_\lambda X^{\lambda *}$.

Proof. Let $f \in L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ arbitrary. As we know that $\psi_{out}(\mathcal{U}f) \sim \psi_0^{\mu_{out}}(\mathcal{U}f) \quad (\eta \to \infty)$, this reduces finding $\mathcal{U}$ to finding coefficients $a^{out}_{\lambda,n}, b^{out}_{\lambda,n}$, such that $\psi_{in}(f) = \psi_{out}$, with

$$\psi_{out} = C^{-3/4} \sum_\lambda K^\lambda_{out} N^\lambda_{out} \sum_{n=1}^{\dim V_\lambda} \left(a^{out}_{\lambda,n} D_\lambda \phi^{out(-)}_\lambda e^\lambda_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b^{out}_{\lambda,n} D_\lambda \phi^{out(+)*}_\lambda \bar{e}^\lambda_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$

To shorten the equations we will write $D_\lambda = (i\sigma_1 \partial_\eta + i\lambda\sigma_2 - \mu)$. By applying the Bogoliubov transformations given by Proposition 4.10 to $\psi_{in}(f)$ and equating $\psi_{in}(f) = \psi_{out}$ we get

$$\begin{aligned} &\sum_\lambda K^\lambda_{in} N^\lambda_{in} \sum_{n=1}^{\dim V_\lambda} \Bigg(a^{in}_{\lambda,n} D_\lambda \left(\alpha^{(-)}_\lambda \phi^{out(-)}_\lambda + \beta^{(-)}_\lambda \phi^{out(+)*}_\lambda \right) e^\lambda_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &\qquad + b^{in}_{\lambda,n} D_\lambda \left(\alpha^{(-)*}_\lambda \phi^{out(-)*}_\lambda + \beta^{(-)*}_\lambda \phi^{out(+)}_\lambda \right) \bar{e}^\lambda_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Bigg) \\ &= \sum_\lambda K^\lambda_{out} N^\lambda_{out} \sum_{n=1}^{\dim V_\lambda} \left(a^{out}_{\lambda,n} D_\lambda \phi^{out(-)}_\lambda e^\lambda_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b^{out}_{\lambda,n} D_\lambda \phi^{out(+)*}_\lambda \bar{e}^\lambda_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right). \end{aligned} \tag{45}$$

Now we transfer to the asymptotic *out*-region. Using the identities given in Proposition 4.6 and Lemma 4.7, we compare the coefficients of $\phi_\lambda^{out(\pm)} \sim e^{-i\omega_{out}(\lambda)\eta}$:

$$\sum_\lambda K_{in}^\lambda \frac{N_{in}^\lambda}{N_{out}^\lambda} \sum_{n=1}^{\dim V_\lambda} \left(a_{\lambda,n}^{in} \alpha_\lambda^{(-)} w_{out}^\lambda e_n^\lambda + b_{\lambda,n}^{in} \beta_\lambda^{(-)*} J e_n^\lambda N_{out}^\lambda \begin{pmatrix} \omega_{out}(\lambda) + \lambda - \mu_{out} \\ \omega_{out}(\lambda) - \lambda - \mu_{out} \end{pmatrix} \right)$$
$$= \sum_\lambda K_{out}^\lambda \sum_{n=1}^{\dim V_\lambda} a_{\lambda,n}^{out} w_{out}^\lambda e_n^\lambda.$$

In the same way, we have by comparing coefficients of $\phi_\lambda^{out(\pm)*} \sim e^{i\omega_{out}(\lambda)\eta}$:

$$\sum_\lambda K_{in}^\lambda \frac{N_{in}^\lambda}{N_{out}^\lambda} \sum_{n=1}^{\dim V_\lambda} \left(a_{\lambda,n}^{in} \beta_\lambda^{(-)} e_n^\lambda N_{out}^\lambda \begin{pmatrix} \omega_{out}(\lambda) - \lambda - \mu_{out} \\ -\omega_{out}(\lambda) - \lambda + \mu_{out} \end{pmatrix} + b_{\lambda,n}^{in} \alpha_\lambda^{(-)*} \bar{w}_{out}^\lambda \bar{e}_n^\lambda \right)$$
$$= \sum_\lambda K_{out}^\lambda \sum_{n=1}^{\dim V_\lambda} b_{\lambda,n}^{out} \bar{w}_{out}^\lambda \bar{e}_n^\lambda.$$

Hence by using the orthonormality relations for w^λ (Proposition 3.27) and e_n^λ, and the results of Lemma 4.8 we get

$$a_{\lambda,n}^{out} = \frac{K_{in}^\lambda}{K_{out}^\lambda} \frac{N_{in}^\lambda}{N_{out}^\lambda} \left(\alpha_\lambda^{(-)} a_{\lambda,n}^{in} + \beta_\lambda^{(-)*} \frac{-\lambda}{\omega_{out}(\lambda) + \mu_{out}} \sum_{m=1}^{\dim V_\lambda} \left\langle e_n^\lambda \,\middle|\, J e_m^\lambda \right\rangle b_{\lambda,m}^{in} \right),$$
$$b_{\lambda,n}^{out} = \frac{K_{in}^\lambda}{K_{out}^\lambda} \frac{N_{in}^\lambda}{N_{out}^\lambda} \left(\alpha_\lambda^{(-)*} b_{\lambda,n}^{in} - \beta_\lambda^{(-)} \frac{-\lambda}{\omega_{out}(\lambda) + \mu_{out}} \sum_{m=1}^{\dim V_\lambda} \left\langle J e_n^\lambda \,\middle|\, e_m^\lambda \right\rangle a_{\lambda,m}^{in} \right).$$

Using the just defined polarisation tensor, we get

$$a_{\lambda,n}^{out} = \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)} a_{\lambda,n}^{in} + \sum_{m=1}^{\dim V_\lambda} \beta_\lambda^{(-)*} X_{nm}^\lambda b_{\lambda,m}^{in} \right),$$
$$b_{\lambda,n}^{out} = \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} b_{\lambda,n}^{in} - \sum_{m=1}^{\dim V_\lambda} \beta_\lambda^{(-)} X_{mn}^{\lambda*} a_{\lambda,m}^{in} \right).$$

As this holds for arbitrary f, and we have for the coefficients

$$a_{\lambda,n}^{in} = \left\langle C_{in}^{-3/4} K_{in}^\lambda w_{in}^\lambda \otimes e_n^\lambda \,\middle|\, \mathcal{U} f \right\rangle_{in}, \quad b_{\lambda,n}^{in} = \left\langle C_{in}^{-3/4} K_{in}^\lambda \bar{w}_{in}^\lambda \otimes \bar{e}_n^\lambda \,\middle|\, \mathcal{U} f \right\rangle_{in},$$

$$a_{\lambda,n}^{out} = \left\langle C_{out}^{-3/4} K_{out}^\lambda w_{out}^\lambda \otimes e_n^\lambda \,\middle|\, f \right\rangle_{out}, \quad b_{\lambda,n}^{out} = \left\langle C_{out}^{-3/4} K_{out}^\lambda \bar{w}_{out}^\lambda \otimes \bar{e}_n^\lambda \,\middle|\, f \right\rangle_{out},$$

the result now follows. □

Let us now recall some facts about quantization. We will refer the reader to Appendix A for an introduction into canonical quantization of fermionic fields and more details about the construction.

Using asymptotic equivalences of solutions in the *in* and *out* asymptotic regions to solutions of the static Dirac equation, we can define the quantized Dirac field in the *in* and *out*

Fock space as follows. The following construction will be carried out for the *in* Fock space, while equivalent results for the *out* Fock space are obtained mutatis mutandis. Let us define the *in* Fock space as $\mathcal{F}(L^2(\mathcal{S}_\Sigma)\otimes\mathbb{C}^2)$. For any solution ψ_{in} of the Dirac operator such that

$$\psi_{in} \sim \psi_0^{\mu_{in}} \quad (\eta \to -\infty)$$

we define the quantized Dirac field at infinity as an operator on the *in* Fock space as

$$\Psi(\psi_0^{\mu_{in}}) = C_{in}^{-3/4}\sum_{\lambda,n}\sqrt{\frac{m}{\omega_\lambda}}\Big(\big(\mathcal{F}\psi_0^{\mu_{in}}, w_{in}^\lambda e_n^\lambda\big)\, e^{-i\omega_\lambda t} a_{\lambda,n}^{in} + \big(\mathcal{F}\psi_0^{\mu_{in}}, \bar{w}_{in}^\lambda \bar{e}_n^\lambda\big)\, e^{i\omega_\lambda t} b_{\lambda,n}^{in\dagger}\Big),$$

where $\mathcal{F}$ is the temporal Fourier transform, and

$$a_{\lambda,n}^{in} = a\left(C_{in}^{-3/4}K_{in}^\lambda w_{in}^\lambda \otimes e_n^\lambda\right), \quad b_{\lambda,n}^{in\dagger} = b^\dagger\left(C_{in}^{-3/4}K_{in}^\lambda \bar{w}_{in}^\lambda \otimes \bar{e}_n^\lambda\right)$$

are the annihilation and creation operators for particles and anti-particles respectively. Using the language that distributions are functions (see Definition A.6), we can write

$$\Psi(x) = C_{in}^{-3/4}\sum_{\lambda,n}\sqrt{\frac{m}{\omega_\lambda}}\Big(e^{-i\omega_\lambda t} w^\lambda e_n^\lambda a_{\lambda,n}^{in} + e^{i\omega_\lambda t}\bar{w}^\lambda \bar{e}_n^\lambda b_{\lambda,n}^{in\dagger}\Big). \tag{46}$$

We will use the map $\mathcal{U}$ as defined above to define transformations between the Fock spaces in the *in* and *out* regions.

4.1 Abstract Bogoliubov transformations

To generalize and formalize the idea of a Bogoliubov transformation, we proof the following results. We refer the reader to Appendix A for an introduction into the canonical quantization of fermionic fields, as we will use basic facts and terminology without explanation. We follow [41, Thm. XI.108], [30, Prop. 10.12] and [47, Sec. 10.3].

Proposition 4.16. *Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a Hilbert space, and*

$$U_{even} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad U_{odd} = \begin{pmatrix} 0 & U_{+-} \\ U_{-+} & 0 \end{pmatrix}$$

be operators on $\mathcal{H}$. Let for $v \in \mathcal{H}_+, w \in \mathcal{H}_-$, $a(v), a^\dagger(v)$ and $b(w), b^\dagger(w)$ be the annihilation and creation operators on the Fermionic Fock Space $\mathcal{F}(\mathcal{H})$ for particles and antiparticles resp., satisfying the CAR. We define the ***Bogoliubov transformation***

$$a'(v) = a(U_+v) + b^\dagger(U_{-+}v),$$
$$b'(w) = a^\dagger(U_{+-}w) + b(U_-w).$$

Then $a'(v), b'(w)$ satisfy the CAR if and only if

$$U_{even}^\dagger U_{even} + U_{odd}^\dagger U_{odd} = \boldsymbol{I}, \tag{47}$$

$$U_{even}^\dagger U_{odd} + U_{odd}^\dagger U_{even} = 0. \tag{48}$$

Moreover the Bogoliubov transformation is invertible with inverse

$$a(v) = a'(U_+^\dagger v) + b'^\dagger(U_{-+}^\dagger v), \tag{49}$$
$$b(w) = a'^\dagger(U_{+-}^\dagger w) + b'(U_-^\dagger w) \tag{50}$$

if and only if

$$U_{even}U_{even}^\dagger + U_{odd}U_{odd}^\dagger = \boldsymbol{I},$$
$$U_{even}U_{odd}^\dagger + U_{odd}U_{even}^\dagger = 0.$$

Proof. The following proof is an adaptation of [30, Prop. 10.12] to a Fermionic Fock space with both particles and anti-particles. We will show only the first statement, as the second follows from completely analog computations.

$$\begin{aligned}\{a'(v), b'(w)\} &= \{a(U_+v) + b^\dagger(U_{-+}v), a^\dagger(U_{+-}w) + b(U_-w)\} \\ &= \langle U_+v \,|\, U_{+-}w\rangle + \langle U_{-+}v \,|\, U_-w\rangle = \left\langle (U_{+-}^\dagger U_+ + U_-^\dagger U_{-+})v \,\middle|\, w \right\rangle\end{aligned}$$

and

$$\{a'^\dagger(v), b'^\dagger(w)\} = \left\langle (U_{-+}^\dagger U_- + U_+^\dagger U_{+-})v \,\middle|\, w \right\rangle$$

are both zero if and only if (47) holds. Moreover for arbitrary $v_1, v_2 \in \mathcal{H}_+$ and $w_1, w_2 \in \mathcal{H}_-$

$$\{a'(v_1), a'^\dagger(v_2)\} = \left\langle (U_+^\dagger U_+ + U_{-+}^\dagger U_{-+})v_1 \,\middle|\, v_2 \right\rangle$$

and

$$\{b'(w_1), b'^\dagger(w_2)\} = \left\langle (U_-^\dagger U_- + U_{+-}^\dagger U_{+-})w_1 \,\middle|\, w_2 \right\rangle$$

are equal to $\langle v_1 \,|\, v_2\rangle$ and $\langle w_1 \,|\, w_2\rangle$ respectively if and only if (48) holds. □

Any unitary transformation $U : \mathcal{H} \to \mathcal{H}$ can be decomposed into even an odd parts $U = U_{even} + U_{odd}$, where

$$U_{\text{even}} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad U_{\text{odd}} = \begin{pmatrix} 0 & U_{+-} \\ U_{-+} & 0 \end{pmatrix}.$$

Here

$$U_+ = P_+UP_+, \quad U_- = P_-UP_-, \quad U_{+-} = P_+UP_-, \quad U_{-+} = P_-UP_+.$$

Hence U induces a transformation of the annihilation operators:

$$a'(u) = a(U_+u) + b^\dagger(U_{-+}u), \quad \text{for } u \in \mathcal{H}_+,$$
$$b'(v) = a^\dagger(U_{+-}v) + b(U_-v), \quad \text{for } v \in \mathcal{H}_-,$$

and hence a transformation of the field operator by

$$\Psi(v) \mapsto \Psi'(v) = a'(v) + b'^\dagger(v). \tag{51}$$

Using the elementary facts that $U^\dagger = U^{-1}$ and $P_\pm^\dagger = P_\pm^2 = P_\pm$ one easily checks that U_+, U_-, U_{+-}, U_{-+} satisfy (47) - (50) and hence the transformed operators $a'(u), b'(v)$ satisfy the CAR.

Definition 4.17. The transformation given by Eq. (51) induced by U is called *unitarily implementable* if there exists a unitary operator $\mathbb{U} : \mathcal{F} \to \mathcal{F}$ such that

$$\Psi'(v) = \mathbb{U}\Psi(v)\mathbb{U}^\dagger,$$

for any $v \in \mathcal{H}$.

Definition 4.18. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis (e_j). An operator

$$A : \mathcal{H} \to \mathcal{H}$$

is called *Hilbert-Schmidt* if

$$\sum_{n=1}^{\infty} \|Ae_j\|^2 < \infty.$$

We have the following result, stating when Bogoliubov transformations are unitarily implementable.

Theorem 4.19 (Shale-Stinespring)**.** *The transformation $U : \mathcal{H} \to \mathcal{H}$ with $U = U_{even} + U_{odd}$, where*

$$U_{even} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad U_{odd} = \begin{pmatrix} 0 & U_{+-} \\ U_{-+} & 0 \end{pmatrix},$$

is unitarily implementable if and only if U_{+-} and U_{-+} are Hilbert-Schmidt operators.

Proof. See e.g. [47, Thm. 10.7]. □

4.2 Transformation of the operators in a GFLRW spacetime

Weaponized by our knowledge about abstract Bogoliubov transformation, we can compute how our operators transform under the transformations induced by the metric on the GFLRW spacetime.

Theorem 4.20. *The transformations of creation and annihilation operators as induced by Bogoliubov transformations on the Hilbert space $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ as given by Theorem 4.15, are given by*

$$\begin{aligned} a^{out}_{\lambda,n} &= \mathcal{C}^\lambda \left(\alpha^{(-)}_\lambda a^{in}_{\lambda,n} + \sum_{m=1}^{\dim V_\lambda} \beta^{(-)*}_\lambda X^\lambda_{nm} b^{in\dagger}_{\lambda,m} \right), \\ b^{out}_{\lambda,n} &= \mathcal{C}^\lambda \left(\alpha^{(-)}_\lambda b^{in}_{\lambda,n} - \sum_{m=1}^{\dim V_\lambda} \beta^{(-)*}_\lambda X^\lambda_{mn} a^{in}_{\lambda,m} \right). \end{aligned} \tag{52}$$

Proof. This follows immediately from Proposition 4.16 and Theorem 4.15. Note that $a^\dagger$ and b depend linearly on their arguments, while a and $b^\dagger$ depend anti-linearly on their arguments. □

To now show that the conditions (47) - (50) hold in this case we prove the following lemma, which will aid in the calculations.

Lemma 4.21. *The following relations hold for the polarization tensor*

$$(X^\lambda)^\dagger X^\lambda = X^\lambda (X^\lambda)^\dagger = \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \boldsymbol{I} = \frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}} \boldsymbol{I} \,. \tag{53}$$

Or equivalently

$$\sum_m X^\lambda_{nm} X^{\lambda *}_{km} = \sum_m X^{\lambda *}_{km} X^\lambda_{nm} = \frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}} \delta_{nk}$$

Proof. This follows from straightforward computations. Indeed,

$$\begin{aligned}
\sum_m X^\lambda_{nm} X^{\lambda *}_{km} &= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \sum_m \langle e^\lambda_n \,|\, J e^\lambda_m \rangle \langle e^\lambda_m \,|\, J^\dagger e^\lambda_k \rangle \\
&= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \langle J^\dagger e^\lambda_n | \left(\sum_m |e^\lambda_m\rangle \langle e^\lambda_m| \right) |J^\dagger e^\lambda_k \rangle \\
&= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \langle e^\lambda_n \,|\, e^\lambda_k \rangle = \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \delta_{nk},
\end{aligned}$$

where we have used the identity

$$\sum_m |e^\lambda_m\rangle \langle e^\lambda_m| = \mathrm{id}\,.$$

Moreover we have,

$$\begin{aligned}
\sum_m X^{\lambda *}_{km} X^\lambda_{nm} &= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \sum_m \langle J e^\lambda_m \,|\, e^\lambda_n \rangle \langle e^\lambda_k \,|\, J e^\lambda_m \rangle \\
&= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \mathrm{Tr}\big(|e^\lambda_n\rangle \langle e^\lambda_k| \big) \\
&= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \sum_m \langle e^\lambda_m \,|\, e^\lambda_n \rangle \langle e^\lambda_k \,|\, e^\lambda_m \rangle \\
&= \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \sum_m \delta_{mn} \delta_{km} = \frac{\lambda^2}{(\omega_{out}(\lambda) + \mu_{out})^2} \delta_{nk},
\end{aligned}$$

where we have used the fact that the trace is base independent. The result now follows from

$$\lambda^2 = (\omega_{out}(\lambda) - \mu_{out})(\omega_{out}(\lambda) + \mu_{out}).$$

□

Using Proposition 4.16 we now show that the commutation relations of $a^{out}_{\lambda,n}$ and $b^{out}_{\lambda,n}$ are retained under the transformations in Eq. (52) and compute the inverse transformations.

Proposition 4.22. *The CAR of $a^{out}_{\lambda,n}$ and $b^{out}_{\lambda,n}$ are retained under the transformations given by Eq. (52), if and only if $\alpha^{(-)}_\lambda = \alpha^{(-)*}_\lambda$.*

Moreover, if that is the case, the inverse transformations of Eq. (52) are given by

$$\begin{aligned}
a^{in}_{\lambda,n} &= \mathcal{C}^\lambda \left(\alpha^{(-)*}_\lambda a^{out}_{\lambda,n} - \sum_{m=1}^{\dim V_\lambda} \beta^{(-)*}_\lambda X^\lambda_{nm} b^{out\dagger}_{\lambda,m} \right), \\
b^{in}_{\lambda,n} &= \mathcal{C}^\lambda \left(\alpha^{(-)*}_\lambda b^{out}_{\lambda,n} + \sum_{m=1}^{\dim V_\lambda} \beta^{(-)*}_\lambda X^\lambda_{mn} a^{out\dagger}_{\lambda,m} \right),
\end{aligned}$$

Proof. Note that we can do the calculation for fixed λ, as $\mathcal{U}$ does not mix elements corresponding to different λ. Hence we set λ fixed in the following calculations. Let

$$U_{even} = \mathcal{C}^\lambda \begin{pmatrix} \alpha_\lambda^{(-)*} \boldsymbol{I}_N & 0 \\ 0 & \alpha_\lambda^{(-)} \boldsymbol{I}_N \end{pmatrix}, \quad U_{odd} = \mathcal{C}^\lambda \begin{pmatrix} 0 & \mathcal{B} \\ -B^\dagger & 0 \end{pmatrix}$$

with $N = \dim V_\lambda$ and $\mathcal{B} = \beta_\lambda^{(-)} X^{\lambda *}$, as before. We recall Eq. (42), which is using our current notation equal to

$$\left(\mathcal{C}^\lambda\right)^{-2} = \left|\alpha_\lambda^{(-)}\right|^2 + \left|\beta_\lambda^{(-)}\right|^2 \left(\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}}\right). \tag{54}$$

To see that Eq. (47) holds, we check

$$\begin{aligned} U_{\text{even}}^\dagger U_{\text{even}} + U_{\text{odd}}^\dagger U_{\text{odd}} &= \left(\mathcal{C}^\lambda\right)^2 \left|\alpha_\lambda^{(-)}\right|^2 \boldsymbol{I}_{2N} + \left(\mathcal{C}^\lambda\right)^2 \left|\beta_\lambda^{(-)}\right|^2 \begin{pmatrix} (X^{\lambda\dagger} X^\lambda)^T & 0 \\ (X^\lambda X^{\lambda\dagger})^T & 0 \end{pmatrix} \\ &= \left(\mathcal{C}^\lambda\right)^2 \left(\left|\alpha_\lambda^{(-)}\right|^2 + \left|\beta_\lambda^{(-)}\right|^2 \left(\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}}\right)\right) \boldsymbol{I}_{2N} = \boldsymbol{I}_{2N}, \end{aligned}$$

where we have used Lemma 4.21. To check when Eq. (48) holds, we compute

$$\begin{aligned} U_{\text{even}}^\dagger U_{\text{odd}} + U_{\text{odd}}^\dagger U_{\text{even}} = \left(\mathcal{C}^\lambda\right)^2 \Bigg(&\begin{pmatrix} \alpha_\lambda^{(-)*} \boldsymbol{I}_N & 0 \\ 0 & \alpha_\lambda^{(-)} \boldsymbol{I}_N \end{pmatrix} \begin{pmatrix} 0 & \mathcal{B} \\ -B^\dagger & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \alpha_\lambda^{(-)} \boldsymbol{I}_N & 0 \\ 0 & \alpha_\lambda^{(-)*} \boldsymbol{I}_N \end{pmatrix} \begin{pmatrix} 0 & -\mathcal{B} \\ B^\dagger & 0 \end{pmatrix} \Bigg), \end{aligned}$$

which vanishes if and only if $\alpha_\lambda^{(-)} = \alpha_\lambda^{(-)*}$. Note that if $\alpha_\lambda^{(-)}$ is real, U_{even} is a multiple of the identity and hence central, so the relations Eq. (49) and Eq. (50) immediately follow as

$$(AB^\dagger)^\dagger = (B^\dagger A)^\dagger = A^\dagger B,$$

for A central.

As a result of Proposition 4.16 the inverse transformations are now given by

$$\begin{aligned} a^{in}(v) &= a^{out}(U_+^\dagger v) + b^{out\dagger}(U_{-+}^\dagger v), \\ b^{in}(w) &= a^{out\dagger}(U_{+-}^\dagger w) + b^{out}(U_-^\dagger w), \end{aligned}$$

with

$$U_{\text{even}} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad U_{\text{odd}} = \begin{pmatrix} 0 & U_{+-} \\ U_{-+} & 0 \end{pmatrix}.$$ □

By choosing a specific basis for the eigenspaces V_λ we can simplify the polarization tensor. Since the eigenspaces V_λ are finite-dimensional and $D_\Sigma J = J D_\Sigma$, we can apply the following proposition to the eigenspaces.

Proposition 4.23. *Let J be an anti-unitary operator on a finite-dimensional Hilbert space $\mathcal{H}$ with $J^2 = -1$, then there is an orthonormal basis $\{f_n, f_{-n}\}$ of $\mathcal{H}$ such that*

$$J f_n = f_{-n}, \quad J f_{-n} = -f_n.$$

In particular, $\mathcal{H}$ is even dimensional.

Proof. We follow [50, Lem. 3.8]. Let $f_1 \in \mathcal{H}$ be an arbitrary normalized element. Set $f_{-1} = Jf_1$. It is orthogonal to f_1 since

$$\langle f_{-1} \,|\, f_1 \rangle = \langle Jf_1 \,|\, f_1 \rangle = \langle Jf_1 \,|\, J^2 f_1 \rangle = -\langle Jf_1 \,|\, f_1 \rangle = -\langle f_{-1} \,|\, f_1 \rangle .$$

Next take another normalized $f_2 \perp f_1, f_{-1}$ and set $f_{-2} = Jf_2$. As before f_{-2} is orthogonal to f_2 and also to f_1 and f_{-1} :

$$\begin{aligned}
\langle f_1 \,|\, f_{-2} \rangle &= \langle f_1 \,|\, Jf_2 \rangle = \langle J^2 f_2 \,|\, Jf_1 \rangle = -\langle f_2 \,|\, f_{-1} \rangle = 0 \\
\langle f_{-1} \,|\, f_{-2} \rangle &= \langle Jf_1 \,|\, Jf_2 \rangle = \langle f_2 \,|\, f_1 \rangle = 0.
\end{aligned}$$

Continuing in this way gives a basis $\{f_n, f_{-n}\}$ for $\mathcal{H}$ with $Jf_n = f_{-n}$. □

Proposition 4.24. *We assume that the Dirac operator D_Σ has a symmetric spectrum $\{\lambda\} \subseteq \mathbb{R}$. Then there is a basis of normalized eigenfunctions f_n^λ,*

$$n = -\frac{1}{2}\dim V_\lambda, \ldots, -2, -1, 1, 2, \ldots, \frac{1}{2}\dim V_\lambda,$$

such that

$$f_{-n}^\lambda = Jf_n^\lambda.$$

Moreover we have

$$\begin{aligned}
a_{\lambda,n}^{in} &= \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} a_{\lambda,n}^{out} - \beta_\lambda^{(-)*} \frac{\operatorname{sgn}(n)\lambda}{\omega_{out}(\lambda) + \mu_{out}} b_{\lambda,-n}^{out\dagger} \right), \\
b_{\lambda,n}^{in} &= \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} b_{\lambda,n}^{out} - \beta_\lambda^{(-)*} \frac{\operatorname{sgn}(n)\lambda}{\omega_{out}(\lambda) + \mu_{out}} a_{\lambda,-n}^{out\dagger} \right),
\end{aligned} \tag{55}$$

Proof. Let $\{e_n^\lambda\}$ be an orthonormal basis of eigenvectors of the Dirac operator D_Σ. Applying Proposition 4.23 to every eigenspace V_λ, provides us with the basis $\{f_n^\lambda\}$. Note that this is still a basis of eigenvectors. Using this basis we have for the entries of the polarisation tensor

$$X_{nm}^\lambda = \frac{-\lambda \, \langle f_n^\lambda \,|\, Jf_m^\lambda \rangle}{\omega_{out}(\lambda) + \mu_{out}} = \frac{-\operatorname{sgn}(m)\lambda \, \langle f_n^\lambda \,|\, f_{-m}^\lambda \rangle}{\omega_{out}(\lambda) + \mu_{out}} = \frac{\operatorname{sgn}(n)\lambda \, \delta_{n,-m}}{\omega_{out}(\lambda) + \mu_{out}}$$

Changing the sums in Eq. (52)

$$\sum_{n=1}^{\dim V_\lambda} \rightsquigarrow \sum_{n=-\frac{1}{2}\dim V_\lambda}^{\frac{1}{2}\dim V_\lambda},$$

we get the transformations:

$$\begin{aligned}
a_{\lambda,n}^{in} &= \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} a_{\lambda,n}^{out} - \beta_\lambda^{(-)*} X_{n,-n}^\lambda b_{\lambda,-n}^{out\dagger} \right) = \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} a_{\lambda,n}^{out} - \beta_\lambda^{(-)*} \frac{\operatorname{sgn}(n)\lambda}{\omega_{out}(\lambda) + \mu_{out}} b_{\lambda,-n}^{out\dagger} \right), \\
b_{\lambda,n}^{in} &= \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} b_{\lambda,n}^{out} + \beta_\lambda^{(-)*} X_{-n,n}^\lambda a_{\lambda,-n}^{out\dagger} \right) = \mathcal{C}^\lambda \left(\alpha_\lambda^{(-)*} b_{\lambda,n}^{out} - \beta_\lambda^{(-)*} \frac{\operatorname{sgn}(n)\lambda}{\omega_{out}(\lambda) + \mu_{out}} a_{\lambda,-n}^{out\dagger} \right).
\end{aligned}$$

□

4.3 Unitary implementation of the Bogoliubov transformation

In this section we will be looking for a unitary operator that implements the Bogoliubov transformations found in previous section. That is we want to find a unitary map

$$\mathbb{U} : \mathcal{F} \to \mathcal{F}$$

such that

$$\Psi'(v) = \mathbb{U}\Psi(v)\mathbb{U}^\dagger,$$

for any $v \in \mathcal{H}$. The Shale-Stinespring theorem (Theorem 4.19) puts conditions on the coefficients $\beta_\lambda^{(-)}$ for the transformation to be unitarily implementable. For the rest of this section, we will assume these conditions are satisfied. For the following proposition, we have adapted [39, Appendix A] to our situation.

Proposition 4.25. *The Bogoliubov transformation given by Theorem 4.20 are unitarily implementable if $\alpha_\lambda^{(-)}$ is real for all $\lambda \in \sigma(D_\Sigma)$. Then the unitary operator $\mathbb{U} : \mathcal{F} \to \mathcal{F}$ which implements the transformation is given by*

$$\mathbb{U} = \bigoplus_\lambda \mathbb{U}_\lambda.$$

Here $\mathbb{U}_\lambda : \mathcal{F} \to \mathcal{F}$, is given by

$$\mathbb{U}_\lambda = \exp\left(L_{\mathbb{U}_\lambda}\right),$$

where

$$L_{\mathbb{U}_\lambda} = \frac{1}{2}\sum_{i_1,i_2}\left(\theta_{i_1 i_2} d_{i_1}^\dagger d_{i_2}^\dagger + \theta_{i_1 i_2}^* d_{i_1} d_{i_2}\right),$$

and

$$d_i = \begin{cases} a_{\lambda,i}^{out} & 1 \leqslant i \leqslant n \\ b_{\lambda,i-n}^{out} & n+1 \leqslant i \leqslant 2n, \end{cases}$$

and

$$\theta = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}.$$

Here A is the square matrix of order n given by

$$A = -\frac{\arccos\left(\mathcal{C}^\lambda \alpha_\lambda^{(-)*}\right)}{\sqrt{1 - (\mathcal{C}^\lambda \alpha_\lambda^{(-)*})^2}} \mathcal{C}^\lambda \beta_\lambda^{(-)*} X^\lambda \tag{56}$$

Proof. For notational simplicity we will write

$$\mathcal{A} = \mathcal{C}^\lambda \alpha_\lambda^{(-)*}, \quad \mathcal{B} = \mathcal{C}^\lambda \beta_\lambda^{(-)*} X^\lambda, \tag{57}$$

and

$$d_i = \begin{cases} a_{\lambda,i}^{out} & 1 \leqslant i \leqslant n \\ b_{\lambda,i-n}^{out} & n+1 \leqslant i \leqslant 2n, \end{cases}.$$

Using this notation the inverse Bogoliubov transformations given by Proposition 4.22, can be written as

$$\boldsymbol{d}^{in} = \mathcal{A}\,\boldsymbol{I}_{2n}\,\boldsymbol{d}^{out} - \begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix} \boldsymbol{d}^{out\dagger}. \tag{58}$$

We claim there is a unitary map

$$\mathbb{U}_\lambda = \exp\left(L_{\mathbb{U}_\lambda}\right),$$

with

$$L_{\mathbb{U}_\lambda} = \frac{1}{2}\sum_{i_1,i_2}\left(\theta_{i_1 i_2} d_{i_1}^\dagger d_{i_2}^\dagger + \theta_{i_1 i_2}^* d_{i_1} d_{i_2}\right),$$

and θ an anti-symmetric matrix given by

$$\theta = \frac{\arccos(\mathcal{A})}{\sin(\arccos(\mathcal{A}))}\begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix} = \frac{\arccos(\mathcal{A})}{\sqrt{1-\mathcal{A}^2}}\begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix}.$$

such that

$$d_i^{in} = \mathbb{U}_\lambda d_i^{out} \mathbb{U}_\lambda^\dagger.$$

To proof this, let us first compute

$$|\theta| = \sqrt{\theta^\dagger\theta} = \arccos(\mathcal{A})\,\boldsymbol{I}_{2n}\,. \tag{59}$$

We recall Eq. (42), which is using our current notation equal to

$$1 = |\mathcal{A}|^2 + \left|\mathcal{C}^\lambda \beta_\lambda^{(-)}\right|^2 \left(\frac{\omega_{out}(\lambda) - \mu_{out}}{\omega_{out}(\lambda) + \mu_{out}}\right),$$

and using Lemma 4.21 this becomes

$$\mathcal{B}^\dagger\mathcal{B} = \mathcal{B}\mathcal{B}^\dagger = (1 - |\mathcal{A}|^2)\,\boldsymbol{I}_n\,.$$

Now we compute $\theta^\dagger\theta$, and see that Eq. (59) holds if $\alpha_\lambda^{(-)}$ is real. Indeed

$$\begin{aligned}\theta^\dagger\theta &= \frac{\arccos^2(\mathcal{A})}{1-\mathcal{A}^2}\begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix} = \frac{\arccos^2(\mathcal{A})}{1-\mathcal{A}^2}\begin{pmatrix} 0 & -\mathcal{B}^* \\ \mathcal{B}^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix} \\ &= \frac{\arccos^2(\mathcal{A})}{1-\mathcal{A}^2}\begin{pmatrix} (\mathcal{B}\mathcal{B}^\dagger)^T & 0 \\ 0 & \mathcal{B}^\dagger\mathcal{B} \end{pmatrix} = \arccos^2(\mathcal{A})\frac{1-|\mathcal{A}|^2}{1-\mathcal{A}^2}\,\boldsymbol{I}_{2n} = \arccos^2(\mathcal{A})\,\boldsymbol{I}_{2n}\,.\end{aligned}$$

if $\mathcal{A}$ is real. Therefore

$$\cos(|\theta|) = \mathcal{A}\,\boldsymbol{I}_{2n}, \tag{60}$$

$$\frac{\sin(|\theta|)}{|\theta|}\theta = \begin{pmatrix} 0 & \mathcal{B} \\ -\mathcal{B}^T & 0 \end{pmatrix}. \tag{61}$$

We can now compute

$$\mathbb{U}_\lambda d_j \mathbb{U}_\lambda^\dagger = \mathrm{Ad}\circ\exp(L_{\mathbb{U}_\lambda})(d_j) = \exp\circ\,\mathrm{ad}(L_{\mathbb{U}_\lambda})(d_j) = \sum_{k=0}^{\infty}\frac{1}{k!}\,\mathrm{ad}(L_{\mathbb{U}_\lambda})^k(d_j),$$

where $\mathrm{ad}(L_{\mathbb{U}_\lambda})(d_i) = [L_{\mathbb{U}_\lambda}, d_i] = L_{\mathbb{U}_\lambda} d_i - d_i L_{\mathbb{U}_\lambda}$. One can check that

$$\mathrm{ad}(L_{\mathbb{U}_\lambda})^{2k}(d_j) = (-1)^k \sum_i (|\theta|^{2k})_{ji} d_i,$$

and

$$\mathrm{ad}(L_{\mathbb{U}_\lambda})^{2k+1}(d_j) = -(-1)^k \sum_i (|\theta|^{2k+1}|\theta|^{-1}\theta)_{ji} d_i^\dagger.$$

We refer to [39] for the explicit calculations. Hence we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}(L_{\mathbb{U}_\lambda})^k(d_j) &= \sum_i \left(\sum_{k=0}^{\infty} \frac{(-1)^k |\theta|^{2k}}{(2k)!} \right)_{ji} d_i - \sum_i \left(\sum_{k=0}^{\infty} \frac{(-1)^k |\theta|^{2k+1}|\theta|^{-1}\theta}{(2k+1)!} \right)_{ji} d_i^\dagger \\ &= \sum_i \left((\cos(|\theta|))_{ji}\, d_i - \left(\sin(|\theta|)|\theta|^{-1}\theta\right)_{ji} d_i^\dagger \right).\end{aligned}$$

Using Eq. (60) and Eq. (61) we see that this is agrees with Eq. (58). The result now follows. □

When we choose another basis of $L^2(\mathcal{S}_\Sigma)$ such that we have the simplified Bogoliubov transformations, given by Eq. (55), the operators $\mathbb{U}_\lambda$ also become less involved. We will assume this in the following propositions.

One interesting transformation is the transformation of the vacuum. That is we want to find an expression for the the *in-vacuum* $|0_{in}\rangle$ in terms of $a^\dagger_{out}, b^\dagger_{out}$ and the *out-vacuum* $|0_{out}\rangle$. This can be done using the unitary operator found in the previous proposition, see [39, Appendix B]. We will take a more direct approach, using the method as described in [9] for a scalar boson field.

Proposition 4.26. *Using the Bogoliubov transformation as given in Eq. (55), the in-vacuum is formally given by*

$$|0_{in}\rangle = \mathbb{U}\,|0_{out}\rangle = \langle 0_{out} \mid 0_{in}\rangle \prod_\lambda \exp\left\{ \sum_m \frac{\beta_\lambda^{(-)*}}{\alpha_\lambda^{(-)*}} X^\lambda_{m,-m} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda,-m} \right\} |0_{out}\rangle .$$

Proof. By conservation of charge, that is the difference between the number of particles and anti-particles must be constant, the vacuum must be of the form:

$$\begin{aligned}|0_{in}\rangle =& A_0\,|0_{out}\rangle + \sum_{n=1}^{\infty} \sum_{\lambda_1,\ldots,\lambda_{2n}} \sum_{m_1,\ldots,m_{2n}} A_n(\lambda_1, m_1, \ldots, \lambda_{2n}, m_{2n}) \\ & \cdot a^{out\dagger}_{\lambda_1,m_1} b^{out\dagger}_{\lambda_2,m_2} \cdots a^{out\dagger}_{\lambda_{2n-1},m_{2n-1}} b^{out\dagger}_{\lambda_{2n},m_{2n}}\,|0_{out}\rangle\end{aligned} \tag{62}$$

Here and in the following the sum of m_i is assumed to be over the all

$$m_i \in \{-\frac{1}{2}\dim V_{\lambda_i}, \ldots, \frac{1}{2}\dim V_{\lambda_i}\}.$$

Let us denote symmetric group of n elements by S_n. Note that by symmetry we have for all $(\sigma, \bar\sigma) \in S_n \times S_n$

$$\begin{aligned}A_n(\lambda_1, m_1, \ldots, \lambda_{2n}, m_{2n}) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\bar\sigma) A_n(&\lambda_{\sigma(1)}, m_{\sigma(1)}, \lambda_{2\bar\sigma(1)}, m_{2\bar\sigma(1)}, \ldots, \\ &\lambda_{2\sigma(n)-1}, m_{2\sigma(n)-1} \lambda_{2\bar\sigma(n)}, m_{2\bar\sigma(n)}).\end{aligned}$$

Because $\langle 0_{out} \,|\, 0_{out} \rangle = 1$, we have

$$A_0 = \langle 0_{out} \,|\, 0_{in} \rangle .$$

By using *Eq.* (55), $a^{in}_{\lambda,m} \left| 0_{in} \right\rangle = 0$ and comparing terms in Eq. (62), we have

$$\begin{aligned} 0 = &-A_0 \beta^{(-)*}_{\lambda} X^{\lambda}_{m,-m} b^{out\dagger}_{\lambda,-m} \left| 0_{out} \right\rangle \\ &+ \sum_{\lambda'} \sum_{m'} A_1(\lambda, m, \lambda', m') \alpha^{(-)*}_{\lambda} a^{out}_{\lambda,m} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda',m'} \left| 0_{out} \right\rangle \end{aligned}$$

and hence

$$\begin{aligned} A_1(\lambda, m, \lambda, -m) &= \frac{\beta^{(-)*}_{\lambda}}{\alpha^{(-)*}_{\lambda}} X^{\lambda}_{m,-m} A_0, \\ A_1(\lambda, m, \kappa, n) &= 0 \quad \text{if } \lambda \neq \kappa \text{ or } m \neq -n. \end{aligned}$$

From higher terms in $a^{in}_{\lambda,m} \left| 0_{in} \right\rangle = 0$ we get the recursion relation

$$\begin{aligned} &A_n(\lambda_1, m_1, \lambda_1, -m_1, \ldots, \lambda_n, m_n, \lambda_n, -m_n) \\ &= \frac{1}{2n} \sum_{i=1}^{n-1} \frac{\beta^{(-)*}_{\lambda_i}}{\alpha^{(-)*}_{\lambda_i}} X^{\lambda_i}_{m_i,-m_i} A_{n-1}(\lambda_1, m_1, \lambda_1, -m_1, \ldots, \\ &\qquad \lambda_{i-1}, -m_{i-1}, \lambda_{i+1}, m_{i+1}, \ldots, \lambda_n, m_n, \lambda_n, -m_n), \end{aligned}$$

and all $A_n(\cdots) = 0$ for other combination of λ_i and m_i, that are not related to a A_n of the form above by a permutation $(\sigma, \bar{\sigma}) \in S_n \times S_n$.

From this it follows that the in-vacuum is formally given by

$$\begin{aligned} \left| 0_{in} \right\rangle &= \langle 0_{out} \,|\, 0_{in} \rangle \exp\left\{ \sum_{\lambda} \sum_{m} \frac{\beta^{(-)*}_{\lambda}}{\alpha^{(-)*}_{\lambda}} X^{\lambda}_{m,-m} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda,-m} \right\} \left| 0_{out} \right\rangle \\ &= \langle 0_{out} \,|\, 0_{in} \rangle \prod_{\lambda} \exp\left\{ \sum_{m} \frac{\beta^{(-)*}_{\lambda}}{\alpha^{(-)*}_{\lambda}} X^{\lambda}_{m,-m} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda,-m} \right\} \left| 0_{out} \right\rangle . \end{aligned}$$ □

This provides us with an another way to calculate the effect of the unitary map $\mathbb{U}$ on a arbitrary pure state $\left| \psi_{out} \right\rangle \in \mathcal{F}_0$. Note that such a state can be written as

$$\left| \psi_{out} \right\rangle = \prod_{\lambda \in \varLambda, \lambda' \in \varLambda'} \prod_{m \in M_\lambda, n \in N_\lambda} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda',n} \left| 0_{out} \right\rangle ,$$

for finite sets

$$\varLambda, \varLambda' \subseteq \sigma(D_\Sigma), M_\lambda, N_\lambda \subseteq \mathbb{N}_\lambda .$$

Corollary 4.27. *Let*

$$\left| \psi_{out} \right\rangle = \prod_{\lambda \in \varLambda, \lambda' \in \varLambda'} \prod_{m \in M_\lambda, n \in N_\lambda} a^{out\dagger}_{\lambda,m} b^{out\dagger}_{\lambda',n} \left| 0_{out} \right\rangle ,$$

for finite sets

$$\Lambda, \Lambda' \subseteq \sigma(D_\Sigma), M_\lambda, N_\lambda \subseteq \mathbb{N}_\lambda.$$

Using the Bogoliubov transformations as given in Eq. (55), we have

$$\mathbb{U}\,|\psi_{out}\rangle = \langle 0_{out}\,|\,0_{in}\rangle \prod_{\lambda\in\Lambda,\lambda'\in\Lambda'} \prod_{m\in M_\lambda, n\in N_\lambda} a^{in\dagger}_{\lambda,m} b^{in\dagger}_{\lambda',n} \prod_{\lambda''} \exp\left\{ \sum_m \frac{\beta^{(-)*}_{\lambda''}}{\alpha^{(-)*}_{\lambda''}} X^{\lambda''}_{k,-k} a^{out\dagger}_{\lambda'',k} b^{out\dagger}_{\lambda'',-k} \right\} |0_{out}\rangle\,.$$

Proof. This follows immediately from Proposition 4.26 and

$$\mathbb{U} a^{out\dagger}_{\lambda,k} = a^{in\dagger}_{\lambda,k}\mathbb{U}$$

by construction of $\mathbb{U}$ in Proposition 4.25. □

5 Evolution of quantized Dirac fields in a spatially flat FLRW spacetime

> A bug is never just a mistake. It represents something bigger. An error of thinking. That makes you who you are.
>
> — Elliot Alderson (Rami Malek)

In this chapter we transfer our results of the previous chapter to the Minkowski case, relating our findings to the results presented in [15, 39, 20]. We will omit some details if the calculations are almost identical to those in the previous chapter, and elaborate when they are not. We will pay extra attention to calculations when considered useful to the literature, and correct a small typographical error in [15] which since then has been plaguing the literature. We have changed our notation and presentation of results to follow the literature more closely.

We consider a Dirac field ψ with mass m on a 4-dimensional spatially flat Friedmann–Lemaître–Robertson–Walker spacetime, with metric $g_{\mu\nu}$ given by the line element

$$ds^2 = -dt^2 + a^2(t)dx_i dx^i = C(\eta)(-d\eta^2 + dx_i dx^i), \tag{63}$$

where x_i are the spacial coordinates and η is the so-called conformal time, related to the time t by $\eta = \int^t \frac{dt'}{a(t')}$. The dynamics of the field given by the Dirac equation

$$(\bar{\gamma}^\mu \partial_\mu + m)\psi = 0, \tag{64}$$

where $\bar{\gamma}^\mu$ are the curved-space γ-matrices, satisfying

$$\bar{\gamma}^\mu \bar{\gamma}^\nu + \bar{\gamma}^\nu \bar{\gamma}^\mu = 2g^{\mu\nu}.$$

One can show that for this metric this boils down to

$$\left(\gamma^0 \partial_t + \frac{3}{2}\frac{\dot{a}(t)}{a(t)}\gamma^0 + \frac{1}{a(t)}\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m\right)\psi = 0,$$

or equivalently

$$\left(\gamma^0 \partial_\eta + \frac{3}{4}\frac{\dot{C}(\eta)}{C(\eta)}\gamma^0 + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + mC^{1/2}(\eta)\right)\psi = 0,$$

where γ^μ are the constant γ-matrices. Since $C(\eta)$ is independent of the spatial coordinates, we can sepeparate the solutions, by introducing the ansatz

$$\psi_{\boldsymbol{k}}(\eta, \boldsymbol{x}) = C^{-3/4}(\gamma^\mu \partial_\mu - mC^{1/2})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{\boldsymbol{k}}(\eta) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}C^{-3/4}(\gamma^0 \partial_\eta + i\boldsymbol{k} \cdot \boldsymbol{\gamma} - mC^{1/2})\phi_{\boldsymbol{k}}(\eta).$$

We see $\phi_{\boldsymbol{k}}$ has to satisfy

$$\left(\partial_\eta^2 + m^2 C(\eta) + \gamma^0 \frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + |\boldsymbol{k}|^2\right)\phi_{\boldsymbol{k}}^{(\pm)}(\eta) = 0. \tag{65}$$

Choosing $\phi_{\boldsymbol{k}} = \phi_{\boldsymbol{k}}^{(+)} v$ or $\phi_{\boldsymbol{k}} = \phi_{\boldsymbol{k}}^{(-)} u$, with $u_s, v_s \in \mathbb{C}^4$ satisfying

$$\gamma^0 u_s = -i u_s, \quad \gamma^0 v_s = i v_s \tag{66}$$

and inserting this into the Dirac equation we find that $\phi_{\boldsymbol{k}}^{(\pm)}$ have to satisfy

$$\left(\partial_\eta^2 + m^2 C(\eta) \pm im \frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + |\boldsymbol{k}|^2 \right) \phi_{\boldsymbol{k}}^{(\pm)}(\eta) = 0. \tag{67}$$

Remark 5.1. Note that for solutions $\phi_{\boldsymbol{k}}^{(\pm)}$ of Eq. (67), their complex conjugated variants $\phi_{\boldsymbol{k}}^{(\pm)*}$ satisfy the complex conjugated version of Eq. (67), that is

$$\left(\partial_\eta^2 + m^2 C(\eta) \mp im \frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} + |\boldsymbol{k}|^2 \right) \phi_{\boldsymbol{k}}^{(\pm)*}(\eta) = 0.$$

This means that $\phi_{\boldsymbol{k}}^{(-)}$ and $\phi_{\boldsymbol{k}}^{(+)*}$ are solutions to the same equation, and the equivalent statement with the signs flipped also holds. Therefore

$$\phi_{\boldsymbol{k}}^{(-)} u_s, \quad \phi_{\boldsymbol{k}}^{(-)*} v_s, \quad \phi_{\boldsymbol{k}}^{(+)} v_s, \quad \phi_{\boldsymbol{k}}^{(+)*} u_s$$

are solutions to Eq. (65). In the relevant literature these solutions are plagued by typographical errors, as the non-solution $\phi_{\boldsymbol{k}}^{(+)*} v_s$ often appears in formulas. This is probably due to a small typo in [15, Eq. 3.14]. ♢

If we assume

$$\int_a^\infty \left| m^2 (C(\eta) - C_{out}) \pm im \frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} \right| \mathrm{d}\eta < \infty$$

and

$$\int_{-\infty}^b \left| m^2 (C(\eta) - C_{in}) \pm im \frac{\dot{C}(\eta)}{2C^{1/2}(\eta)} \right| \mathrm{d}\eta < \infty,$$

for some $a, b \in \mathbb{R}$, where $C_{out} = \lim_{\eta\to\infty} C(\eta)$ and $C_{in} = \lim_{\eta\to-\infty} C(\eta)$, we are in the same situation as in Corollary 3.50. Hence we have positive frequency solutions $\phi_{\boldsymbol{k}}^{in(\pm)}, \phi_{\boldsymbol{k}}^{out(\pm)}$, satisfying

$$\phi_{\boldsymbol{k}}^{in(\pm)} \sim e^{-i\omega_{in}(|\boldsymbol{k}|)\eta} \quad (\eta \to -\infty) \quad \text{and} \quad \phi_{\boldsymbol{k}}^{out(\pm)} \sim e^{-i\omega_{out}(|\boldsymbol{k}|)\eta} \quad (\eta \to \infty),$$

where

$$\omega_{in}(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + \mu_{in}^2}, \qquad \omega_{out}(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + \mu_{out}^2},$$
$$\mu_{in} = \lim_{\eta\to-\infty} m\sqrt{C(\eta)}, \qquad \mu_{out} = \lim_{\eta\to\infty} m\sqrt{C(\eta)}.$$

The corresponding negative-frequency solutions are then given by $\phi_{\boldsymbol{k}}^{in(\mp)*}, \phi_{\boldsymbol{k}}^{out(\mp)*}$, where the sign of $\pm$ has flipped as explained in Remark 5.1. Since the in and out solutions are both a complete set of solutions of the same differential equation, we can define Bogoliubov coefficients $\alpha_{\boldsymbol{k}}^{(\pm)} = \alpha^{(\pm)}(|\boldsymbol{k}|), \beta_{\boldsymbol{k}}^{(\pm)} = \beta^{(\pm)}(|\boldsymbol{k}|)$ satisfying:

$$\phi_{\boldsymbol{k}}^{in(\pm)}(\eta) = \alpha_{\boldsymbol{k}}^{(\pm)} \phi_{\boldsymbol{k}}^{out(\pm)}(\eta) + \beta_{\boldsymbol{k}}^{(\pm)} \phi_{\boldsymbol{k}}^{out(\mp)*}(\eta) \tag{68}$$

The curved space spinor solutions to the Dirac equation are then given by

$$\begin{aligned} N_{\boldsymbol{k}}^{in} U_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) &= N_{\boldsymbol{k}}^{in} C(\eta)^{-3/4} (\gamma^\mu \partial_\mu - \mu(\eta)) \phi_{\boldsymbol{k}}^{in(-)}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} u_s, \\ V_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) &= N_{\boldsymbol{k}}^{in} C(\eta)^{-3/4} (\gamma^\mu \partial_\mu - \mu(\eta)) \phi_{\boldsymbol{k}}^{in(-)*}(\eta) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v_s, \end{aligned}$$

with similar equations for U_{out} and V_{out}. Here

$$N_{\boldsymbol{k}}^{in} = -\frac{1}{|\boldsymbol{k}|} \left(\frac{\omega_{in}(|\boldsymbol{k}|) - \mu_{in}}{2\mu_{in}} \right)^{1/2} = \frac{-1}{\sqrt{2\mu_{in}(\omega_{in}(|\boldsymbol{k}|) + \mu_{in})}}$$

for normalization. To relate these solutions to the solutions in the flat case, we recall that the flat-space spinors with polarization s, momentum $\boldsymbol{k}$ and energy $\omega(|\boldsymbol{k}|) = \sqrt{|\boldsymbol{k}|^2 + \mu^2} = k^0$, are given by

$$\begin{aligned} u(\boldsymbol{k}, s) &= N_{\boldsymbol{k}} (i\not{k} - \mu) u_s, \\ v(\boldsymbol{k}, s) &= N_{\boldsymbol{k}} (-i\not{k} - \mu) v_s, \end{aligned}$$

c.f. Remark 3.38. The dirac adjoint of a flat-space spinor is defined as

$$\bar{u}(\boldsymbol{k}, s) = iu(\boldsymbol{k}, s)^\dagger \gamma^0, \quad \bar{v}(\boldsymbol{k}, s) = iv(\boldsymbol{k}, s)^\dagger \gamma^0.$$

The spinors satisfy the orthogonality relations

$$\bar{u}(\boldsymbol{k}, s) u(\boldsymbol{k}, s') = -\bar{v}(\boldsymbol{k}, s) v(\boldsymbol{k}, s') = \delta_{ss'}. \tag{69}$$

Now it's easy to check that the curved space solutions go over to the corresponding flat-space spinors in the asymptotic limits

$$\begin{aligned} U_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) &\sim C^{-3/4}(-\infty) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-i\omega_{in}(|\boldsymbol{k}|)\eta} u_{in}(\boldsymbol{k}, s) \quad (\eta \to -\infty), \\ V_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) &\sim C^{-3/4}(-\infty) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{in}(|\boldsymbol{k}|)\eta} v_{in}(\boldsymbol{k}, s) \quad (\eta \to -\infty), \end{aligned}$$

with similar expressions for U_{out} and V_{out} in the limit $\eta \to \infty$.

5.1 Bogoliubov transformations

Using the theory of fermionic quantization as explained in Appendix A, the field can be expanded in two ways:

$$\begin{aligned} \psi &= \sqrt{\frac{\mu_{out}}{\omega_{out}(|\boldsymbol{k}|)}} \sum\!\!\!\!\!\!\int \left(a_{out}(\boldsymbol{k}, s) U_{out}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) + b_{out}^\dagger(\boldsymbol{k}, s) V_{out}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) \right), \\ \psi &= \sqrt{\frac{\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)}} \sum\!\!\!\!\!\!\int \left(a_{in}(\boldsymbol{k}, s) U_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta) + b_{in}^\dagger(\boldsymbol{k}, s) V_{in}(\boldsymbol{k}, s, \boldsymbol{x}, \eta). \right) \end{aligned} \tag{70}$$

To simplify notation we have written $\sum\!\!\!\!\!\!\int$ for

$$\frac{1}{2\pi}^{(3/2)} \int d^3\boldsymbol{k} \sum_s .$$

We will now first prove the following lemma, which is a straightforward but lengthy spin calculation, as it is needed in the next step.

Lemma 5.2. *We have*

$$\bar{v}(\boldsymbol{k}, s')(-i\not{k} - m)u_s = -2imNv(0, s')^\dagger \boldsymbol{k} \cdot \boldsymbol{\gamma} u_s. \tag{71}$$

Proof. Indeed,

$$\begin{aligned}
\bar{v}(\boldsymbol{k}, s')(-i\not{k} - m)u_s &= Niv_{s'}^\dagger(-i\eta_{\mu\nu}k^\mu\gamma^\nu - m)^\dagger\gamma^0(-i\eta_{\rho\sigma}k^\rho\gamma^\sigma - m)u_s \\
&= Niv_{s'}^\dagger(i\delta_{\mu\nu}k^\mu\gamma^\nu - m)\gamma^0(-i\eta_{\rho\sigma}k^\rho\gamma^\sigma - m)u_s \\
&= Niv_{s'}^\dagger(i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(i\eta_{\rho\sigma}k^\rho\gamma^\sigma\gamma^0 - i\eta_{\rho\sigma}\{\gamma^\sigma, \gamma^0\}k^\rho - m\gamma^0)u_s \\
&= Niv_{s'}^\dagger(i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(i\eta_{\rho\sigma}k^\rho\gamma^\sigma\gamma^0 - 2i\eta_{\rho\sigma}\eta^{\sigma 0}k^\rho - m\gamma^0)u_s \\
&= Niv_{s'}^\dagger(i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(i\eta_{\rho\sigma}k^\rho\gamma^\sigma\gamma^0 + 2ik^0(\gamma^0)^2 - m\gamma^0)u_s \\
&= Niv_{s'}^\dagger\left((i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(i\eta_{\rho\sigma}k^\rho\gamma^\sigma - m) + (i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(2ik^0\gamma^0)\right)\gamma^0 u_s \\
&= Niv_{s'}^\dagger\left((i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(i\eta_{\rho\sigma}k^\rho\gamma^\sigma - m) + (i\delta_{\mu\nu}k^\mu\gamma^\nu - m)(2ik^0\gamma^0)\right)\gamma^0 u_s \\
&= Niv_{s'}^\dagger\left(-m^2 - 2\gamma^0k^0\boldsymbol{k}\cdot\boldsymbol{\gamma} + m^2 - 2im\boldsymbol{k}\cdot\boldsymbol{\gamma} - 2k^0\delta_{\mu\nu}k^\mu\gamma^\nu\gamma^0 - 2imk^0\gamma^0)\right)\gamma^0 u_s \\
&= Niv_{s'}^\dagger\left(-2im\boldsymbol{k}\cdot\boldsymbol{\gamma} - 2\gamma^0k^0\boldsymbol{k}\cdot\boldsymbol{\gamma} + 2\gamma^0k^0\boldsymbol{k}\cdot\boldsymbol{\gamma} + 2E^2 - 2imk^0\gamma^0)\right)\gamma^0 u_s \\
&= Niv_{s'}^\dagger\left(-2im\boldsymbol{k}\cdot\boldsymbol{\gamma} + 2(k^0)^2 - 2imk^0\gamma^0)\right)\gamma^0 u_s \\
&= Niv_{s'}^\dagger\left(-2im\boldsymbol{k}\cdot\boldsymbol{\gamma})\right)(-i)u_s \\
&= -2imNv_{s'}^\dagger\boldsymbol{k}\cdot\boldsymbol{\gamma}u_s,
\end{aligned}$$

where in the penultimate step we have used the orthogonality of $v_{s'}^\dagger$ and u_s, for all s, s'. We also have used the identity

$$\delta_{\mu\nu}\eta_{\rho\sigma}k^\mu k^\rho\gamma^\nu\gamma^\sigma = (k^0)^2 - |\boldsymbol{k}|^2 + 2\gamma^0k^0\boldsymbol{k}\cdot\boldsymbol{\gamma} = m^2 + 2\gamma^0k^0\boldsymbol{k}\cdot\boldsymbol{\gamma}.$$

□

Proposition 5.3. *The annihilation and creation operators as used in Eq. (70) are related by*

$$b_{out}(\boldsymbol{k}, s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\left(\alpha_{\boldsymbol{k}}^{(-)}b_{in}(\boldsymbol{k}, s) + \beta_{\boldsymbol{k}}^{(-)*}\sum_{s'}X_{ss'}(-\boldsymbol{k})a_{in}^\dagger(-\boldsymbol{k}, s')\right).$$

and

$$a_{out}(\boldsymbol{k}, s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\left(\alpha_{\boldsymbol{k}}^{(-)}a_{in}(\boldsymbol{k}, s) + \beta_{\boldsymbol{k}}^{(-)*}\sum_{s'}X_{s's}(-\boldsymbol{k})b_{in}^\dagger(-\boldsymbol{k}, s')\right)$$

Proof. Using the Bogoliubov transformations Eq. (68) we will write a_{out} and $b^{\dagger}_{out}$ in terms of a_{in} and $b^{\dagger}_{in}$. Indeed by inserting the Bogoliubov transformations in Eq. (70) we get

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\!\int \Big(a_{out}(\boldsymbol{k},s) U_{out}(\boldsymbol{k},s,\boldsymbol{x},\eta) + b^{\dagger}_{out}(\boldsymbol{k},s) V_{out}(\boldsymbol{k},s,\boldsymbol{x},\eta) \Big) = \\
&\sum\!\!\!\!\!\!\!\int \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \Big(a_{in}(\boldsymbol{k},s) C(\eta)^{-3/4} N^{in}_{\boldsymbol{k}} \cdot \\
&\qquad (\gamma^{\mu}\partial_{\mu} - \mu(\eta)) \left(\alpha^{(-)}_{\boldsymbol{k}} \phi^{out(-)}_{\boldsymbol{k}}(\eta) + \beta^{(-)}_{\boldsymbol{k}} \phi^{out(+)*}_{\boldsymbol{k}} \right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} u_s \\
&\quad + b^{\dagger}_{in}(\boldsymbol{k},s) C(\eta)^{-3/4} N^{in}_{\boldsymbol{k}} (\gamma^{\mu}\partial_{\mu} - \mu(\eta)) \left(\alpha^{(-)*}_{\boldsymbol{k}} \phi^{out(-)*}_{\boldsymbol{k}}(\eta) + \beta^{(-)*}_{\boldsymbol{k}} \phi^{out(+)}_{\boldsymbol{k}}(\eta) \right) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v_s \Big).
\end{aligned}
$$

We will only calculate the transformation for $b^{\dagger}_{out}$ explicitly, as the calculations for a_{out} are quite similar. Hence collecting only the terms containing $\phi^{out(\pm)*}_{\boldsymbol{k}}$ we get:

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\!\int b^{\dagger}_{out}(\boldsymbol{k},s) V_{out}(\boldsymbol{k},s,\boldsymbol{x},\eta)) = \\
&\sum\!\!\!\!\!\!\!\int \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \Big(a_{in}(\boldsymbol{k},s) C(\eta)^{-3/4} N^{in}_{\boldsymbol{k}} (\gamma^{\mu}\partial_{\mu} - \mu(\eta)) \beta^{(-)}_{\boldsymbol{k}} \phi^{out(+)*}_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} u_s \\
&\qquad + b^{\dagger}_{in}(\boldsymbol{k},s) C(\eta)^{-3/4} N^{in}_{\boldsymbol{k}} (\gamma^{\mu}\partial_{\mu} - \mu(\eta)) \alpha^{(-)*}_{\boldsymbol{k}} \phi^{out(-)*}_{\boldsymbol{k}}(\eta) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v_s \Big)
\end{aligned}
$$

We will now go to the asymptotic limit $\eta \to \infty$, where $\partial_{\eta}\phi^{out(\pm)*}_{\boldsymbol{k}} \sim i\omega_{out}(|\boldsymbol{k}|) e^{i\omega_{out}(|\boldsymbol{k}|)\eta}$ and we get

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\!\int b^{\dagger}_{out}(\boldsymbol{k},s) v_{out}(\boldsymbol{k},s) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} = \\
&\sum\!\!\!\!\!\!\!\int \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \Big(a_{in}(\boldsymbol{k},s) N^{in}_{\boldsymbol{k}} (i\omega_{out}(|\boldsymbol{k}|)\gamma^0 + i\boldsymbol{k}\cdot\boldsymbol{\gamma} - \mu_{out}) \beta^{(-)}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} u_s \\
&\qquad + b^{\dagger}_{in}(\boldsymbol{k},s) N^{in}_{\boldsymbol{k}} (i\omega_{out}(|\boldsymbol{k}|)\gamma^0 - i\boldsymbol{k}\cdot\boldsymbol{\gamma} - \mu_{out}) \alpha^{(-)*}_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} v_s \Big) = \\
&\sum\!\!\!\!\!\!\!\int \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \Big(a_{in}(\boldsymbol{k},s) N^{in}_{\boldsymbol{k}} (i\omega_{out}(|\boldsymbol{k}|)\gamma^0 + i\boldsymbol{k}\cdot\boldsymbol{\gamma} - \mu_{out}) \beta^{(-)}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} u_s \\
&\qquad + b^{\dagger}_{in}(\boldsymbol{k},s) \frac{N^{in}_{\boldsymbol{k}}}{N^{out}_{\boldsymbol{k}}} \alpha^{(-)*}_{\boldsymbol{k}} v_{out}(\boldsymbol{k},s) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} \Big).
\end{aligned}
$$

We will consider both terms on the right hand side separately. By orthonormality of $v_{out}(\boldsymbol{k},s)$ and $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$ it follows that

$$
b^{\dagger}_{out}(\boldsymbol{k},s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \frac{N^{in}_{\boldsymbol{k}}}{N^{out}_{\boldsymbol{k}}} \alpha^{(-)*}_{\boldsymbol{k}} b^{\dagger}_{in}(\boldsymbol{k},s) + B(\boldsymbol{k},s)
$$

Here $B(\boldsymbol{k},s)$ has to be determined by solving:

$$
\begin{aligned}
&\sum\!\!\!\!\!\!\!\int B(\boldsymbol{k},s) v_{out}(\boldsymbol{k},s) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} = \\
&\sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \sum\!\!\!\!\!\!\!\int a_{in}(\boldsymbol{k},s) N^{in}_{\boldsymbol{k}} (i\omega_{out}(|\boldsymbol{k}|)\gamma^0 + i\boldsymbol{k}\cdot\boldsymbol{\gamma} - \mu_{out}) \beta^{(-)}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\omega_{out}(|\boldsymbol{k}|)\eta} u_s.
\end{aligned}
$$

As we integrate over $\mathbb{R}^3$, we can take $\boldsymbol{k} \mapsto -\boldsymbol{k}$ on the right-hand side, and get

$$\begin{aligned}&\sum\!\!\!\!\!\!\int B(\boldsymbol{k},s)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\omega_{out}(|\boldsymbol{k}|)\eta}v_{out}(\boldsymbol{k},s)\\&\qquad= \sum\!\!\!\!\!\!\int \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}a_{in}(-\boldsymbol{k},s)N_{\boldsymbol{k}}^{in}(-i\not{k}-\mu_{out})\beta_{\boldsymbol{k}}^{(-)}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\omega_{out}(|\boldsymbol{k}|)\eta}u_s.\end{aligned}$$

By using the orthogonality relations for $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$, multiplying with $\bar{v}_{out}(\boldsymbol{k},s')$ on both sides, and using the spin orthonormality relations Eq. (69) we obtain

$$\begin{aligned}B(\boldsymbol{k},s') &= -\sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\beta_{\boldsymbol{k}}^{(-)}\sum_s a_{in}(-\boldsymbol{k},s)N_{\boldsymbol{k}}^{in}\bar{v}_{out}(\boldsymbol{k},s')(-i\not{k}-\mu_{out})u_s\\&= -\sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\beta_{\boldsymbol{k}}^{(-)}\sum_s a_{in}(-\boldsymbol{k},s)N_{\boldsymbol{k}}^{in}(-2i\mu_{out}N_{\boldsymbol{k}}^{out}v_{s'}^{\dagger}\boldsymbol{k}\cdot\boldsymbol{\gamma}u_s)\\&= \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}2N_{\boldsymbol{k}}^{in}N_{\boldsymbol{k}}^{out}\mu_{out}(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})\beta_{\boldsymbol{k}}^{(-)}\sum_s a_{in}(-\boldsymbol{k},s)X_{s's}^{*}(-\boldsymbol{k})\\&= \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\beta_{\boldsymbol{k}}^{(-)}\sum_s a_{in}(-\boldsymbol{k},s)X_{s's}^{*}(-\boldsymbol{k}).\end{aligned}$$

Here we have used the result of Lemma 5.2,

$$\bar{v}_{out}(\boldsymbol{k},s')(-i\not{k}-\mu_{out})u_s = -2i\mu_{out}N_{\boldsymbol{k}}^{out}v_{s'}^{\dagger}\boldsymbol{k}\cdot\boldsymbol{\gamma}u_s, \tag{72}$$

on the second line, and

$$2{N_{\boldsymbol{k}}^{out}}^2\mu_{out}(\omega_{out}(|\boldsymbol{k}|)+\mu_{out}) = 1$$

in the final step.

We have also defined the polarisation tensor by

$$X_{ss'}(\boldsymbol{k}) = -2\mu_{out}N_{\boldsymbol{k}}^{out}\bar{u}_{out}(-\boldsymbol{k},s')v_s = \frac{-iu_{s'}^{\dagger}\boldsymbol{k}\cdot\boldsymbol{\gamma}v_s}{\omega_{out}(|\boldsymbol{k}|)+\mu_{out}}, \tag{73}$$

such that

$$X_{ss'}^{*}(-\boldsymbol{k}) = \frac{-iv_s^{\dagger}\boldsymbol{k}\cdot\boldsymbol{\gamma}u_{s'}}{\omega_{out}(|\boldsymbol{k}|)+\mu_{out}}, \tag{74}$$

Combining our results we find

$$b_{out}^{\dagger}(\boldsymbol{k},s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\left(\alpha_{\boldsymbol{k}}^{(-)*}b_{in}^{\dagger}(\boldsymbol{k},s)+\beta_{\boldsymbol{k}}^{(-)}\sum_{s'}X_{ss'}^{*}(-\boldsymbol{k})a_{in}(-\boldsymbol{k},s')\right), \tag{75}$$

or equivalently

$$b_{out}(\boldsymbol{k},s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\left(\alpha_{\boldsymbol{k}}^{(-)}b_{in}(\boldsymbol{k},s)+\beta_{\boldsymbol{k}}^{(-)*}\sum_{s'}X_{ss'}(-\boldsymbol{k})a_{in}^{\dagger}(-\boldsymbol{k},s')\right). \tag{76}$$

In a similar way one can find that

$$a_{out}(\boldsymbol{k},s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}}\frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}}\left(\alpha_{\boldsymbol{k}}^{(-)}a_{in}(\boldsymbol{k},s)+\beta_{\boldsymbol{k}}^{(-)*}\sum_{s'}X_{s's}(-\boldsymbol{k})b_{in}^{\dagger}(-\boldsymbol{k},s').\right) \tag{77}$$

□

Corollary 5.4. *The inverse transformations of Proposition 5.3 are given by*

$$a_{in}(\boldsymbol{k}, s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}} \left(\alpha_{\boldsymbol{k}}^{(-)*} a_{out}(\boldsymbol{k}, s) + \beta_{\boldsymbol{k}}^{(-)*} \sum_{s'} X_{ss'}(-\boldsymbol{k}) b_{out}^{\dagger}(-\boldsymbol{k}, s') \right)$$

and

$$b_{in}(\boldsymbol{k}, s) = \sqrt{\frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}}} \frac{N_{\boldsymbol{k}}^{in}}{N_{\boldsymbol{k}}^{out}} \left(\alpha_{\boldsymbol{k}}^{(-)*} b_{out}(\boldsymbol{k}, s) + \beta_{\boldsymbol{k}}^{(-)*} \sum_{s'} X_{s's}(-\boldsymbol{k}) a_{out}^{\dagger}(-\boldsymbol{k}, s') \right).$$

Proof. This is completely similar to Proposition 4.22. □

Remark 5.5. There is a small difference between the results of Proposition 5.3 and Corollary 5.4 in comparison with Theorem 4.20 and Proposition 4.22, regarding the minus sign in front of the second term of the transformations. This is due to a small difference in the definition of the polarisation tensor. In this chapter it contains a factor i, while in Definition 4.14 it does not. ♢

5.1.1 Checking the anti-commutation relations

We will show that the anti-commutation relations of the creation and annihilation operators are maintained under the Bogoliubov transformations.

Proposition 5.6. *We have the following relations between the Bogoliubov coefficients.*

$$\frac{\alpha_{\boldsymbol{k}}^{(+)}}{\alpha_{\boldsymbol{k}}^{(-)}} = \frac{\omega_{in}(|\boldsymbol{k}|) - \mu_{in}}{\omega_{out}(|\boldsymbol{k}|) - \mu_{out}} = \frac{\omega_{out}(|\boldsymbol{k}|) + \mu_{out}}{\omega_{in}(|\boldsymbol{k}|) + \mu_{in}}, \tag{78}$$

$$\frac{\beta_{\boldsymbol{k}}^{(+)}}{\beta_{\boldsymbol{k}}^{(-)}} = -\frac{\omega_{in}(|\boldsymbol{k}|) - \mu_{in}}{\omega_{out}(|\boldsymbol{k}|) + \mu_{out}} = -\frac{\omega_{out}(|\boldsymbol{k}|) - \mu_{out}}{\omega_{in}(|\boldsymbol{k}|) + \mu_{in}}, \tag{79}$$

$$\alpha_{\boldsymbol{k}}^{(-)} \alpha_{\boldsymbol{k}}^{(+)*} - \beta_{\boldsymbol{k}}^{(-)} \beta_{\boldsymbol{k}}^{(+)*} = \frac{\omega_{in}(|\boldsymbol{k}|)}{\omega_{out}(|\boldsymbol{k}|)}, \tag{80}$$

$$\left|\alpha_{\boldsymbol{k}}^{(-)}\right|^2 + \frac{\omega_{out}(|\boldsymbol{k}|) - \mu_{out}}{\omega_{out}(|\boldsymbol{k}|) + \mu_{out}} \left|\beta_{\boldsymbol{k}}^{(-)}\right|^2 = \frac{\mu_{out}\omega_{in}(|\boldsymbol{k}|)}{\mu_{in}\omega_{out}(|\boldsymbol{k}|)} \left(\frac{N_{out}}{N_{in}}\right)^2. \tag{81}$$

Proof. See Proposition 4.12. □

Proposition 5.7.

$$\sum_{s'} X_{ss'}^*(-\boldsymbol{k}) X_{ss'}(-\boldsymbol{k}) = \sum_{s'} X_{ss'}(-\boldsymbol{k}) X_{ss'}^*(-\boldsymbol{k}) = \frac{\omega_{out}(|\boldsymbol{k}|) - \mu_{out}}{\omega_{out}(|\boldsymbol{k}|) + \mu_{out}}.$$

Proof. Using the Weyl representations for the gamma matrices, i.e. $\gamma^0 = i\sigma_1 \otimes \boldsymbol{I}$ we see that the constant zero-momentum spinors, as defined by Eq. (66), are of the form

$$u_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes e_{\pm}, \qquad v_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e_{\pm},$$

where $\{e_+, e_-\}$ is an orthonormal basis of $\mathbb{C}^2$. Using

$$\sum_{s'} u_{s'} u_{s'}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \boldsymbol{I}$$

and

$$(\boldsymbol{k}\cdot\boldsymbol{\sigma})^2 = |\boldsymbol{k}|^2 = \omega_{out}(|\boldsymbol{k}|)^2 - \mu_{out}^2 = (\omega_{out}(|\boldsymbol{k}|) - \mu_{out})(\omega_{out}(|\boldsymbol{k}|) + \mu_{out}),$$

we find

$$\begin{aligned}
\sum_{s'} X_{ss'}^*(-\boldsymbol{k}) X_{ss'}(-\boldsymbol{k}) &= \frac{v_s^\dagger(\boldsymbol{k}\cdot\boldsymbol{\gamma})\left(\sum_{s'} u_{s'}u_{s'}^\dagger\right)(\boldsymbol{k}\cdot\boldsymbol{\gamma})v_s}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&= \frac{\frac{1}{4}\begin{pmatrix} 1 & 1 \end{pmatrix}\sigma_2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\sigma_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e_s^\dagger(\boldsymbol{k}\cdot\boldsymbol{\sigma})\,\boldsymbol{I}(\boldsymbol{k}\cdot\boldsymbol{\sigma})e_s}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&= \frac{\frac{1}{4}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes |\boldsymbol{k}|^2}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} = \frac{|\boldsymbol{k}|^2}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&= \frac{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})(\omega_{out}(|\boldsymbol{k}|)-\mu_{out})}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} = \frac{\omega_{out}(|\boldsymbol{k}|)-\mu_{out}}{\omega_{out}(|\boldsymbol{k}|)+\mu_{out}}.
\end{aligned}$$

For the other equation we note that

$$v_s v_s^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \frac{1}{2}(\boldsymbol{I} + s\sigma_3),$$

and

$$\sum_{s'} e_{s'}^\dagger(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{I}+s\sigma_3)(\boldsymbol{k}\cdot\boldsymbol{\sigma})e_{s'} = \sum_{s'} e_{s'}^\dagger(\boldsymbol{k}\cdot\boldsymbol{\sigma})\,\boldsymbol{I}(\boldsymbol{k}\cdot\boldsymbol{\sigma})e_{s'} = 2|\boldsymbol{k}|^2.$$

Therefore,

$$\begin{aligned}
&\sum_{s'} X_{ss'}(-\boldsymbol{k}) X_{ss'}^*(-\boldsymbol{k}) \\
&\quad= \sum_{s'} \frac{u_{s'}(\boldsymbol{k}\cdot\boldsymbol{\gamma})v_s v_s^\dagger(\boldsymbol{k}\cdot\boldsymbol{\gamma})u_{s'}^\dagger}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&\quad= \frac{\frac{1}{4}\begin{pmatrix} 1 & -1 \end{pmatrix}\sigma_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\sigma_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{2}\sum_{s'} e_{s'}^\dagger(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{I}+s\sigma_3)(\boldsymbol{k}\cdot\boldsymbol{\sigma})e_{s'}}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&\quad= \frac{\frac{1}{4}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{2}{2}|\boldsymbol{k}|^2}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} = \frac{|\boldsymbol{k}|^2}{(\omega_{out}(|\boldsymbol{k}|)+\mu_{out})^2} \\
&\quad= \frac{\omega_{out}(|\boldsymbol{k}|)-\mu_{out}}{\omega_{out}(|\boldsymbol{k}|)+\mu_{out}}.
\end{aligned}$$

□

Using previous proposition we now show that the commutation relations of a_{out} and b_{out} are retained under the transformations Eq. (75) and Eq. (75).

Proposition 5.8. *The CAR of a_{out} and b_{out} are retained under the transformations given by Eq. (75) and Eq. (77), if $\alpha_{\boldsymbol{k}}^{(-)}$ is real.*

Proof. This follows from similar calculations as done in Proposition 4.22. For example, using the CAR for b_{in} and a_{in} we have

$$\begin{aligned}\{b_{out}, b_{out}^\dagger\} &= \frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}} \frac{N_{\boldsymbol{k}}^{in\,2}}{N_{\boldsymbol{k}}^{out\,2}} \Bigg(\left|\alpha_{\boldsymbol{k}}^{(-)}\right|^2 \{b_{in}, b_{in}^\dagger\} \\ &\qquad + \left|\beta_{\boldsymbol{k}}^{(-)}\right|^2 \Big(\sum_{s'} X_{ss'}^* X_{ss'} a_{in} a_{in}^\dagger + \sum_{s'} X_{ss'} X_{ss'}^* a_{in}^\dagger a_{in}\Big) \Bigg) \\ &= \frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}} \frac{N_{\boldsymbol{k}}^{in\,2}}{N_{\boldsymbol{k}}^{out\,2}} \left(\left|\alpha_{\boldsymbol{k}}^{(-)}\right|^2 \{b_{in}, b_{in}^\dagger\} + \left|\beta_{\boldsymbol{k}}^{(-)}\right|^2 \frac{\omega_{out}(|\boldsymbol{k}|) - \mu_{out}}{\omega_{out}(|\boldsymbol{k}|) + \mu_{out}} \{a_{in}, a_{in}^\dagger\} \right) \\ &= \frac{\omega_{out}(|\boldsymbol{k}|)\mu_{in}}{\omega_{in}(|\boldsymbol{k}|)\mu_{out}} \frac{N_{\boldsymbol{k}}^{in\,2}}{N_{\boldsymbol{k}}^{out\,2}} \frac{\mu_{out}\omega_{in}(|\boldsymbol{k}|)}{\mu_{in}\omega_{out}(|\boldsymbol{k}|)} \left(\frac{N_{\boldsymbol{k}}^{out}}{N_{\boldsymbol{k}}^{in}} \right)^2 = 1,\end{aligned}$$

where in the second step we used Eq. (81). □

In the following section, we will assume $\alpha_{\boldsymbol{k}}^{(-)}$ to be real.

5.2 Unitarily implementation of the Bogoliubov transformations

The Bogoliubov transformations provide us with a unitary implementation

$$\mathbb{U} : \mathcal{F} \to \mathcal{F}$$

such that

$$\Psi'(v) = \mathbb{U}\Psi(v)\mathbb{U}^\dagger,$$

for any $v \in \mathcal{H}$, exactly as in Proposition 4.25. We will give a similar statement to Proposition 4.26 using a more physical notation.

Proposition 5.9. *Under the Bogoliubov transformations given by Corollary 5.4 the in-vacuum transforms to*

$$|0_{in}\rangle = \prod_{\boldsymbol{k}} \frac{|0_{out}\rangle - \gamma_{\boldsymbol{k}\downarrow\uparrow}^* |\uparrow_{\boldsymbol{k}}; \downarrow_{-\boldsymbol{k}}\rangle_{out} - \gamma_{\boldsymbol{k}\downarrow\uparrow}^* |\downarrow_{\boldsymbol{k}}; \uparrow_{-\boldsymbol{k}}\rangle_{out} + \gamma_{\boldsymbol{k}\downarrow\uparrow}^* \gamma_{\boldsymbol{k}\uparrow\downarrow}^* |\uparrow\downarrow_{\boldsymbol{k}}; \uparrow\downarrow_{-\boldsymbol{k}}\rangle_{out}}{\sqrt{1 + \left|\gamma_{\boldsymbol{k}\downarrow\uparrow}^*\right|^2 + \left|\gamma_{\boldsymbol{k}\uparrow\downarrow}^*\right|^2 + \left|\gamma_{\boldsymbol{k}\downarrow\uparrow}^* \gamma_{\boldsymbol{k}\uparrow\downarrow}^*\right|^2}}.$$

Proof. The inverse transformation of *Eq.* (77) is, up to some constant which will drop out, given by

$$a_{in}(\boldsymbol{k}, s) \propto \alpha_{\boldsymbol{k}}^{(-)*} a_{out}(\boldsymbol{k}, s) + \sum_{s'} X_{s's}(\boldsymbol{k}) \beta_{\boldsymbol{k}}^{(-)*} b_{out}^\dagger(-\boldsymbol{k}, s'),$$

as follows from Corollary 5.4. To have conservation of spin we must drop the terms containing $X_{ss'}$, where $s' \neq -s.$, and hence the inverse transformations boil down to

$$a_{in}(\boldsymbol{k},s) \propto \alpha_{\boldsymbol{k}}^{(-)*} a_{out}(\boldsymbol{k},s) - X_{-s,s}(-\boldsymbol{k})\beta_{\boldsymbol{k}}^{(-)*} b_{out}^{\dagger}(-\boldsymbol{k},-s), \tag{82}$$

$$b_{in}(\boldsymbol{k},s) \propto \alpha_{\boldsymbol{k}}^{(-)*} b_{out}(\boldsymbol{k},s) - X_{s,-s}(-\boldsymbol{k})\beta_{\boldsymbol{k}}^{(-)*} a_{out}^{\dagger}(-\boldsymbol{k},-s). \tag{83}$$

Using the method as described in [9] for a scalar boson field, we can compute the *in-vacuum* $|0_{in}\rangle$ in terms of $a_{out}^{\dagger}, b_{out}^{\dagger}$ and the *out-vacuum* $|0_{out}\rangle$. By conservation of spin, charge and momentum the vacuum must be of the form:

$$\begin{aligned} |0_{in}\rangle =& A_0 \, |0_{out}\rangle \\ &+ \sum_{n=1}^{\infty} \int d^3\boldsymbol{k}_1 \cdots d^3\boldsymbol{k}_{2n} \sum_{s_1,\dots,s_{2n}} \delta(\sum_{i=1}^{2n} s_i)\delta(\sum_{i=1}^{2n} k_i) A_n(\boldsymbol{k}_1, s_1, \dots, \boldsymbol{k}_{2n}, s_{2n}) \\ &\quad \cdot a_{out}^{\dagger}(\boldsymbol{k}_1, s_1) b_{out}^{\dagger}(\boldsymbol{k}_2, s_2) \cdots a_{out}^{\dagger}(\boldsymbol{k}_{2n-1}, s_{2n-1}) b_{out}^{\dagger}(\boldsymbol{k}_{2n}, s_{2n}) \, |0_{out}\rangle \end{aligned} \tag{84}$$

Because $\langle 0_{out} \,|\, 0_{out}\rangle = 1$, we have

$$A_0 = \langle 0_{out} \,|\, 0_{in}\rangle .$$

By using *Eq.* (82), $a_{in}(\boldsymbol{k},s)\,|0_{in}\rangle = 0$ and comparing terms in Eq. (84), we have

$$\begin{aligned} 0 =& -A_0 \beta_{\boldsymbol{k}}^{(-)*} X_{-s,s}(-\boldsymbol{k}) b_{out}^{\dagger}(-\boldsymbol{k},-s)\, |0_{out}\rangle \\ &+ A_1(\boldsymbol{k}, s, -\boldsymbol{k}, -s)\alpha_{\boldsymbol{k}}^{(-)*} a_{out}(\boldsymbol{k},s) a_{out}^{\dagger}(\boldsymbol{k},s) b_{out}^{\dagger}(-\boldsymbol{k},-s)\, |0_{out}\rangle \end{aligned}$$

and hence

$$A_1(\boldsymbol{k}, s, -\boldsymbol{k}, -s) = \frac{\beta_{\boldsymbol{k}}^{(-)*}}{\alpha_{\boldsymbol{k}}^{(-)*}} X_{-s,s}(-\boldsymbol{k}) A_0.$$

From higher terms we get the recursion relations

$$\begin{aligned} A_n(\boldsymbol{k}_1, s_1, \dots, \boldsymbol{k}_{2n-1}, s_{2n-1}, -\sum_{i=1}^{2n-1} \boldsymbol{k}_i, -\sum_{i=1}^{2n-1} s_i) &= \frac{1}{2n-1} \sum_{j=1}^{2n-1} \frac{\beta_{\boldsymbol{k}_j}^{(-)*}}{\alpha_{\boldsymbol{k}_j}^{(-)*}} X_{-s_j,s_j} \\ \cdot\, \delta\left(\sum_{i=1, i\neq j}^{2n-1} \boldsymbol{k}_i\right) \delta\left(\sum_{i=1, i\neq j}^{2n-1} s_i\right) & A_{n-1}(\boldsymbol{k}_1, s_1, \dots, \boldsymbol{k}_{j-1}, s_{j-1}, \boldsymbol{k}_{j+1}, s_{j+1}, \dots, \boldsymbol{k}_{2n-1}, s_{2n-1}) \end{aligned}$$

from which it follows that the in-vacuum is formally given by

$$|0_{in}\rangle = \langle 0_{out} \,|\, 0_{in}\rangle \exp\left\{ \int d^3\boldsymbol{k} \sum_s \frac{\beta_{\boldsymbol{k}}^{(-)*}}{\alpha_{\boldsymbol{k}}^{(-)*}} X_{-s,s}(-\boldsymbol{k}) a_{out}^{\dagger}(\boldsymbol{k},s) b_{out}^{\dagger}(-\boldsymbol{k},-s) \right\} |0_{out}\rangle .$$

For the following, we generalize the calculations in [20] from two to four dimensions. From the previous result, we see that different $\boldsymbol{k}$ does not mix, i.e. if there is a particle with momentum $\boldsymbol{k}$, there is always an anti-particle with momentum $-\boldsymbol{k}$, and hence the vacuum must be of the form

$$|0_{in}\rangle = \prod_{\boldsymbol{k}} \left(B_0(\boldsymbol{k})\, |0_{out}\rangle + B_1(\boldsymbol{k})\, |{\uparrow_{\boldsymbol{k}}}; {\downarrow_{-\boldsymbol{k}}}\rangle_{out} + B_2(\boldsymbol{k})\, |{\downarrow_{\boldsymbol{k}}}; {\uparrow_{-\boldsymbol{k}}}\rangle_{out} + B_3(\boldsymbol{k})\, |{\uparrow\downarrow_{\boldsymbol{k}}}; {\uparrow\downarrow_{-\boldsymbol{k}}}\rangle_{out} \right),$$

where

$$
\begin{aligned}
\left|\uparrow_{\boldsymbol{k}};\downarrow_{-\boldsymbol{k}}\right\rangle_{out} &:= a^{\dagger}_{out}(\boldsymbol{k},1)b^{\dagger}_{out}(-\boldsymbol{k},-1)\left|0_{out}\right\rangle, \\
\left|\downarrow_{\boldsymbol{k}};\uparrow_{-\boldsymbol{k}}\right\rangle_{out} &:= a^{\dagger}_{out}(\boldsymbol{k},-1)b^{\dagger}_{out}(-\boldsymbol{k},1)\left|0_{out}\right\rangle, \\
\left|\uparrow\downarrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out} &:= a^{\dagger}_{out}(\boldsymbol{k},1)a^{\dagger}_{out}(\boldsymbol{k},-1)b^{\dagger}_{out}(-\boldsymbol{k},-1)b^{\dagger}_{out}(-\boldsymbol{k},1)\left|0_{out}\right\rangle,
\end{aligned}
$$

Since different $\boldsymbol{k}$ do not mix, we only have to consider only one frequency. By applying Eq. (82) to one frequency part, we find that we need to have

$$
\begin{aligned}
\beta^{(-)*}_{\boldsymbol{k}}X_{-1,1}(-\boldsymbol{k})B_0(\boldsymbol{k})\left|\downarrow_{-\boldsymbol{k}}\right\rangle_{out} &= -\alpha^{(-)*}_{\boldsymbol{k}}B_1(\boldsymbol{k})\left|\downarrow_{-\boldsymbol{k}}\right\rangle_{out}, \\
\beta^{(-)*}_{\boldsymbol{k}}X_{1,-1}(-\boldsymbol{k})B_0(\boldsymbol{k})\left|\uparrow_{-\boldsymbol{k}}\right\rangle_{out} &= -\alpha^{(-)*}_{\boldsymbol{k}}B_2(\boldsymbol{k})\left|\uparrow_{-\boldsymbol{k}}\right\rangle_{out}, \\
\alpha^{(-)*}_{\boldsymbol{k}}B_3(\boldsymbol{k})\left|\downarrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out} &= -\beta^{(-)*}_{\boldsymbol{k}}X_{-1,1}(-\boldsymbol{k})B_2(\boldsymbol{k})\left|\downarrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out}, \\
\alpha^{(-)*}_{\boldsymbol{k}}B_3(\boldsymbol{k})\left|\uparrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out} &= -\beta^{(-)*}_{\boldsymbol{k}}X_{1,-1}(-\boldsymbol{k})B_1(\boldsymbol{k})\left|\uparrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|0_{in}\right\rangle = \prod_{\boldsymbol{k}} B_0(\boldsymbol{k})\Bigg(&\left|0_{out}\right\rangle - \frac{\beta^{(-)*}_{\boldsymbol{k}}}{\alpha^{(-)*}_{\boldsymbol{k}}}X_{\downarrow\uparrow}(-\boldsymbol{k})\left|\uparrow_{\boldsymbol{k}};\downarrow_{-\boldsymbol{k}}\right\rangle_{out} - \frac{\beta^{(-)*}_{\boldsymbol{k}}}{\alpha^{(-)*}_{\boldsymbol{k}}}X_{\uparrow\downarrow}(-\boldsymbol{k})\left|\downarrow_{\boldsymbol{k}};\uparrow_{-\boldsymbol{k}}\right\rangle_{out} \\
&+ \left(\frac{\beta^{(-)*}_{\boldsymbol{k}}}{\alpha^{(-)*}_{\boldsymbol{k}}}\right)^2 X_{\downarrow\uparrow}(-\boldsymbol{k})X_{\uparrow\downarrow}(-\boldsymbol{k})\left|\uparrow\downarrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out}\Bigg),
\end{aligned}
$$

where $X_{\uparrow\downarrow}(-\boldsymbol{k}) = X_{1,-1}(-\boldsymbol{k})$ and $X_{\downarrow\uparrow}(-\boldsymbol{k}) = X_{-1,1}(-\boldsymbol{k})$. To simplify notation, lets introduce

$$
\gamma^*_{\boldsymbol{k}\uparrow\downarrow} = \frac{\beta^{(-)*}_{\boldsymbol{k}}}{\alpha^{(-)*}_{\boldsymbol{k}}}X_{\uparrow\downarrow}(-\boldsymbol{k}).
$$

Normalization of the vacuum $\left\langle 0_{in}\,|\,0_{in}\right\rangle = 1$ gives

$$
\left|0_{in}\right\rangle = \prod_{\boldsymbol{k}} \frac{\left|0_{out}\right\rangle - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\left|\uparrow_{\boldsymbol{k}};\downarrow_{-\boldsymbol{k}}\right\rangle_{out} - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\left|\downarrow_{\boldsymbol{k}};\uparrow_{-\boldsymbol{k}}\right\rangle_{out} + \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\gamma^*_{\boldsymbol{k}\uparrow\downarrow}\left|\uparrow\downarrow_{\boldsymbol{k}};\uparrow\downarrow_{-\boldsymbol{k}}\right\rangle_{out}}{\sqrt{1 + \left|\gamma^*_{\boldsymbol{k}\downarrow\uparrow}\right|^2 + \left|\gamma^*_{\boldsymbol{k}\uparrow\downarrow}\right|^2 + \left|\gamma^*_{\boldsymbol{k}\downarrow\uparrow}\gamma^*_{\boldsymbol{k}\uparrow\downarrow}\right|^2}},
$$

□

This result has also be found in [39, eq. 23] by uncoupling the operator found in Proposition 4.25. We can now define a unitary mapping of the Fock space, exactly as in Corollary 4.27. As this is not enlightening to repeat, we will conclude be given a simple example of such a transformation.

Example 5.10. We can compute the evolution of any vector, e.g. for

$$
\left|\uparrow_{\boldsymbol{l}}\right\rangle_{in} = a^{\dagger}_{in}(\boldsymbol{l},1)\left|0_{in}\right\rangle,
$$

by using Eq. (82) and the Pauli exclusion principle:

$$\begin{aligned}
|{\uparrow}_{\boldsymbol{l}}\rangle_{in} &= B_0(\boldsymbol{l})\big(\alpha_{\boldsymbol{l}}^{(-)}\,|{\uparrow}_{\boldsymbol{l}}\rangle_{out} + X^*_{\boldsymbol{l}\downarrow\uparrow}\beta_{\boldsymbol{l}}^{(-)}\gamma^*_{\boldsymbol{l}\downarrow\uparrow}\,|{\uparrow}_{\boldsymbol{l}}\rangle_{out} - \alpha_{\boldsymbol{l}}^{(-)}\gamma^*_{\boldsymbol{l}\downarrow\uparrow}\,|{\uparrow\downarrow}_{\boldsymbol{l}};{\uparrow}_{-\boldsymbol{l}}\rangle_{out} \\
&\quad - X^*_{\boldsymbol{l}\downarrow\uparrow}\beta_{\boldsymbol{l}}^{(-)}\gamma^*_{\boldsymbol{l}\downarrow\uparrow}\gamma^*_{\boldsymbol{l}\uparrow\downarrow}\,|{\uparrow\downarrow}_{\boldsymbol{l}};{\uparrow}_{-\boldsymbol{l}}\rangle_{out}\big) + \prod_{\boldsymbol{k}\neq\boldsymbol{l}} B_0(\boldsymbol{k})\alpha_{\boldsymbol{l}}^{(-)}\big(\,|{\uparrow}_{\boldsymbol{l}}\rangle_{out} - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\,|{\uparrow}_{\boldsymbol{l}};{\uparrow}_{\boldsymbol{k}};{\downarrow}_{-\boldsymbol{k}}\rangle_{out} \\
&\qquad - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\,|{\uparrow}_{\boldsymbol{l}};{\downarrow}_{\boldsymbol{k}};{\uparrow}_{-\boldsymbol{k}}\rangle_{out} + \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\gamma^*_{\boldsymbol{k}\uparrow\downarrow}\,|{\uparrow}_{\boldsymbol{l}};{\uparrow\downarrow}_{\boldsymbol{k}};{\uparrow\downarrow}_{-\boldsymbol{k}}\rangle_{out}\big) \\
&= B_0(\boldsymbol{l})\alpha_{\boldsymbol{l}}^{(-)}\big((1+|\gamma_{\boldsymbol{l}\downarrow\uparrow}|^2)\,|{\uparrow}_{\boldsymbol{l}}\rangle_{out} - (\gamma^*_{\boldsymbol{l}\downarrow\uparrow} + |\gamma_{\boldsymbol{l}\downarrow\uparrow}|^2\gamma^*_{\boldsymbol{l}\uparrow\downarrow})\,|{\uparrow\downarrow}_{\boldsymbol{l}};{\uparrow}_{-\boldsymbol{l}}\rangle_{out}\big) \\
&\quad + \prod_{\boldsymbol{k}\neq\boldsymbol{l}} B_0(\boldsymbol{k})\alpha_{\boldsymbol{l}}^{(-)}\big(\,|{\uparrow}_{\boldsymbol{l}}\rangle_{out} - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\,|{\uparrow}_{\boldsymbol{l}};{\uparrow}_{\boldsymbol{k}};{\downarrow}_{-\boldsymbol{k}}\rangle_{out} - \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\,|{\uparrow}_{\boldsymbol{l}};{\downarrow}_{\boldsymbol{k}};{\uparrow}_{-\boldsymbol{k}}\rangle_{out} \\
&\qquad + \gamma^*_{\boldsymbol{k}\downarrow\uparrow}\gamma^*_{\boldsymbol{k}\uparrow\downarrow}\,|{\uparrow}_{\boldsymbol{l}};{\uparrow\downarrow}_{\boldsymbol{k}};{\uparrow\downarrow}_{-\boldsymbol{k}}\rangle_{out}\big). \qquad \triangleleft
\end{aligned}$$

6 Conclusion

We have generalized the creation of fermionic particles as described in [15] for an isotropic FLRW spacetime, to an asymptotically static GFLRW spacetime

$$(\mathbb{R} \times \Sigma, g = -dt^2 \oplus a^2(t) g_\Sigma),$$

with Σ a complete and compact Riemannian spin manifold. Future investigations could try to generalize this even more, to encompass all asasymptotically ymptotic static hyperbolic spacetimes.

- A generalization to a non-compact Σ is probably not too troublesome, following the methods as explained in [22].
- A generalization enlarging the family of metrics to all smooth one-parameter family of Riemannian metrics $(g_t)_{t\in\mathbb{R}}$ on Σ would be more troublesome, as this would complicate the mean curvature H of (Σ, g_t), disabling the possibility to find uncoupled solutions to the Dirac equation. Increasing the family of allowed metrics, while decreasing the family of allowed spaces Σ could be a way forward here.
- A generalization beyond globally hyperbolic spacetimes by allowing a non-static Σ, e.g. a pair of pants, looks out of reach at the moment.

A Quantization

If you can't give me poetry, can't you give me poetical science?

— Ada Lovelace

A.1 Fermionic Fock space

In this section we follow [47, 18, 3]. For now we assume we have a Hilbert space $\mathcal{H}^+$ as the state space of a particle, and a Hilbert space $\mathcal{H}^- = J\mathcal{H}^+$ as the state space of an antiparticle. Let $Q^\pm : \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \to \mathcal{H}^\pm$ be the orthogonal projection onto these two Hilbert spaces. In the next section we will make these Hilbert spaces concrete. But first, we will define the Fock space corresponding to these Hilbert spaces. We will define the Fock Spaces $\mathcal{F}(\mathcal{H}^\pm)$ related to these Hilbert spaces. If there can be no confusion about the Hilbert spaces in use we will write $\mathcal{F}_\pm = \mathcal{F}(\mathcal{H}^\pm)$.
We will define these Fock spaces in multiple steps. We first define

$$\mathcal{F}_+^{(1)} := \mathcal{H}_+, \qquad \mathcal{F}_-^{(1)} := J\mathcal{H}_- = \mathcal{H}^+,$$

and the n-(anti-)particle Fermionic Fock space $\mathcal{F}_\pm^{(n)}$ as the the antisymmetrized tensor product of n copies of $\mathcal{F}_\pm^{(1)}$, i.e.

$$\mathcal{F}_+^{(n)} = \bigwedge^n \mathcal{F}_+^{(1)}, \qquad \mathcal{F}_-^{(n)} = \bigwedge^n \mathcal{F}_-^{(1)}.$$

Here we have used the antisymmetrized tensor product or *wedge product* $\wedge$ on the exterior algebra. The exterior algebra $\bigwedge(V)$ of a vector space V is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $v \otimes v$ for all $v \in V$. The wedge product is the product induced by the tensor product on $T(V)$.

We define the orthogonal projections on these spaces $P_\pm : \bigotimes^n \mathcal{F}_\pm^{(1)} \to \bigwedge^n \mathcal{F}_\pm^{(1)}$, given by

$$P_\pm(u_1 \otimes \cdots \otimes u_n) = u_1 \wedge \cdots \wedge u_n = \frac{1}{k!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\ u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)}.$$

To accommodate an arbitrary number of particles and anti-particles we define the Fermionic Fock space as the Hilbert space direct sum (denoted by $\hat{\bigoplus}$)

$$\mathcal{F} = \hat{\bigoplus}_{n,m=0}^{\infty} \mathcal{F}^{(n,m)} = \mathcal{F}_+ \otimes \mathcal{F}_- = \hat{\bigoplus}_n^{\infty} \mathcal{F}_+^{(n)} \otimes \hat{\bigoplus}_m^{\infty} \mathcal{F}_-^{(m)},$$

where $\mathcal{F}^{(n,m)} = \mathcal{F}^{(n)} \otimes \mathcal{F}^{(m)}$ and $\mathcal{F}_\pm^{(0,0)} = \mathcal{F}_\pm^{(0)} := \mathbb{C}$. The elements in the Fock space are sequences

$$\xi = (\xi^{(n,m)})_{n,m \in \mathbb{N}}, \qquad \xi^{(n,m)} \in \mathcal{F}^{(n,m)}.$$

We also define the *finite-particle subspace* $\mathcal{F}_0$ of states in which the total number number of particles is finite, i.e. instead of the Hilbert space direct sum, one takes the algebraic direct sum. Note that $\mathcal{F}_0$ is dense in $\mathcal{F}$. In the same way we define $\mathcal{F}_{+0}$ and $\mathcal{F}_{-0}$. To count how many particles (anti-particles) are in a state we define the *number operators* $N_\pm : \mathcal{F}_{\pm 0} \to \mathcal{F}_{\pm 0}$ by

$$N_\pm \mid_{\mathcal{F}_\pm^{(n)}} = n\, \boldsymbol{I}$$

on $\mathcal{F}_\pm^{(n)}$ and extend them linearly to $\mathcal{F}_{\pm 0}$. We also define the total number operator

$$N = N_+ + N_- : \mathcal{F}_0 \to \mathcal{F}_0$$

and the charge operator

$$Q = N_+ - N_- : \mathcal{F}_0 \to \mathcal{F}_0.$$

If we assume *conservation of charge*, this thus means the difference between the number of particles and anti-particles is constant. A state of the form

$$\Omega = (e^{i\lambda}, 0, 0, \ldots), \quad \lambda \in \mathbb{R},$$

is called a vacuum state and describes the case when there are no particles or anti-particles. To go from the vacuum state to a state with particles we will define creation operators. For $v \in \mathcal{F}_\pm^{(1)}$ we define $C(v), C^\dagger(v) : \bigoplus_{n=0}^\infty \bigotimes^n \mathcal{F}_\pm^{(1)} \to \bigoplus_{n=0}^\infty \bigotimes^n \mathcal{F}_\pm^{(1)}$ by

$$\begin{aligned} C(v)(u_1 \otimes \cdots \otimes u_n) &= \langle v \,|\, u_1 \rangle \, u_2 \otimes \cdots u_n, \\ C^\dagger(v)(u_1 \otimes \cdots \otimes u_n) &= v \otimes u_1 \otimes \cdots \otimes u_n. \end{aligned}$$

The annihilation and creation operator for particles

$$A(v), A^\dagger(v) : \mathcal{F}_{+0} \to \mathcal{F}_{+0}$$

are given by

$$A(v) = \sqrt{N_+} C(v), \qquad A^\dagger(v) = P_+ \sqrt{N_+} C^\dagger(v),$$

for $v \in \mathcal{H}^+$. Notices that $C(v)$ respects the Fermionic Fock space, hence we don't have to project onto it. Using the wedge product notation this entails

$$\begin{aligned} A(v)(u_1 \wedge \cdots \wedge u_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n (-1)^j \langle v \,|\, u_j \rangle \; u_1 \wedge \cdots u_{j-1} \wedge u_{j+1} \wedge \cdots \wedge u_k, \\ A^\dagger(v)(u_1 \wedge \cdots \wedge u_n) &= \sqrt{n+1} \; v \wedge u_1 \wedge \cdots \wedge u_k. \end{aligned}$$

Note that $v \to A(v)$ is anti-linear, while $v \to A^\dagger(v)$ is linear.

Similarly for any $v \in \mathcal{H}^-$ the annihilation and creation operator for anti-particles

$$B(v), B^\dagger(v) : \mathcal{F}_{-0} \to \mathcal{F}_{-0}$$

are given by

$$B(v) = C(Jv)\sqrt{N_-}, \qquad B^\dagger(v) = P_- \sqrt{N_-} C^\dagger(Jv).$$

Using the wedge product notation this entails

$$\begin{aligned} B(v)(u_1 \wedge \cdots \wedge u_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n (-1)^j \langle Jv \,|\, u_j \rangle \; u_1 \wedge \cdots u_{j-1} \wedge u_{j+1} \wedge \cdots \wedge u_k, \\ B^\dagger(v)(u_1 \wedge \cdots \wedge u_n) &= \sqrt{n+1} \; Jv \wedge u_1 \wedge \cdots \wedge u_k, \end{aligned}$$

Note hat $v \mapsto B(v)$ is linear, while $v \mapsto B^\dagger(v)$ is anti-linear.

Definition A.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F}_0(\mathcal{H})$ the corresponding finite Fermionic Fock subspace. Operators $A(u), A^\dagger(v) : \mathcal{F}_0(\mathcal{H}) \to \mathcal{F}_0(\mathcal{H})$ for $u, v \in \mathcal{H}$, are said to satisfy the *canonical anticommutation relations* (CAR) if

$$\{A(u), A^\dagger(v)\} = \langle u \,|\, v\rangle \boldsymbol{I},$$
$$\{A(u), A(v)\} = \{A^\dagger(u), A^\dagger(v)\} = 0.$$

One can check that $A(u), A^\dagger(u)$ satisfy the CAR for $\mathcal{H} = \mathcal{H}^+$, and also $B(v), B^\dagger(v)$ for $\mathcal{H} = \mathcal{H}^-$.

We can even extend $A(v), A^\dagger(v)$ to bounded operators on the whole Fermionic Fock space $\mathcal{F}_+$. Indeed for any $\xi \in \mathcal{F}_0$ the CAR implies that

$$\|A(v)\xi\|^2 + \left\|A^\dagger(v)\xi\right\|^2 = \left\langle \xi \,\middle|\, A^\dagger(v)A(v)\xi \right\rangle + \left\langle \xi \,\middle|\, A(v)A^\dagger(v)\xi \right\rangle = \|v\|^2 \|\xi\|^2,$$

hence

$$\|A(v)\| \leqslant \|v\|, \quad \left\|A(v)^\dagger\right\| \leqslant \|v\|,$$

and similarly we extend $B(v), B^\dagger(v)$ to $\mathcal{F}_-$. Moreover with a bit abuse of notation we extend $A(u), A^\dagger(u), B(v), B^\dagger(v)$ for $u \in \mathcal{H}^+, v \in \mathcal{H}^-$ to $\mathcal{F} = \mathcal{F}_+ \otimes \mathcal{F}_-$ by

$$A^\#(u) = A^\#(u) \otimes \boldsymbol{I}, \quad B^\#(v) = (-1)^{N_+}\, \boldsymbol{I} \otimes B^\#(v),$$

where $A^\#$ either means A or $A^\dagger$. The factor $(-1)^{N_+}$ is added to satisfy the canonical commutation relations

$$\{A(u), B(v)\} = \{A^\dagger(u), B^\dagger(v)\} = \{A^\dagger(u), B(v)\} = \{A(u), B^\dagger(v)\} = 0.$$

Notice also that for any $\xi \in \mathcal{F}$ and $v \in \mathcal{F}_+$ we have

$$A^2(v)\xi = \frac{1}{2}\{A(v), A(v)\}\xi = 0,$$

which shows that ξ contains at most one particle in the state v, the so called *Pauli exclusion principle.* For $B(v)$ exactly the same holds. If we assume that for a vector $\xi \in \mathcal{F}$ we have

$$A(u)\xi = 0, \quad B(v)\xi = 0 \quad \forall u \in \mathcal{H}^+, v \in \mathcal{H}^-,$$

then it follows $\xi = \lambda\Omega$ for $\lambda \in \mathbb{C}$, that is we see that the vacuum is unique up to a constant.

Definition A.2. For any $v \in \mathcal{H}$ we define the *field operator*

$$\Psi(v) : \mathcal{F} \to \mathcal{F}$$

by

$$\Psi(v) = A(Q^+ v) + B^\dagger(Q^- v). \tag{85}$$

Note that this is an anti-linear mapping.

Proposition A.3. *The operator $\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{F})$ is an isometry. In particular it is bounded.*

Proof. Using the CAR for A and $B^\dagger$ one can also check that $\Psi(v)$ satisfies the CAR for any $v \in \mathcal{H}$. And in the same way as we did for $A(v)$ we have

$$\|\Psi(v)\xi\|^2 + \left\|\Psi^\dagger(v)\xi\right\|^2 = \left\langle \xi \,\middle|\, \Psi^\dagger(v)\Psi(v)\xi \right\rangle + \left\langle \xi \,\middle|\, \Psi(v)\Psi^\dagger(v)\xi \right\rangle = \|v\|^2\|\xi\|^2,$$

hence $\|\Psi(v)\| = \|\Psi^*(v)\| = \|v\|$. □

Often it is useful to consider the Fock spaces in multiple equivalent ways based on isomorphic Hilbert spaces. The following lemma provides us with the canonical isomorphism of Fock spaces based on the underlying isomorphism of Hilbert spaces.

Lemma A.4. *A isomorphism of Hilbert spaces $U : \mathcal{H}_1 \xrightarrow{\cong} \mathcal{H}_2$ induces a isomorphism of Fock spaces*

$$\mathcal{F}(U) : \mathcal{F}(\mathcal{H}_1) \to \mathcal{F}(\mathcal{H}_2),$$

induced by

$$\mathcal{F}(U)(u_1 \otimes \cdots \otimes u_k) = Uu_1 \otimes \cdots \otimes Uu_k.$$

Moreover for any $v \in \mathcal{H}_1$

$$\mathcal{F}(U)A_1(v)F(U)^{-1} = A_2(Uv), \quad \mathcal{F}(U)A_1^\dagger(v)F(U)^{-1} = A_2^\dagger(Uv),$$

for the annihilation-creation operators $A_i, A_i^\dagger : \mathcal{F}(\mathcal{H}_i) \to \mathcal{F}(\mathcal{H}_i), i = 1, 2$. Moreover,

$$\mathcal{F}(U)\Psi_1(v)\mathcal{F}(U)^{-1} = \Psi_2(Uv),$$

for field operators $\Psi_i : \mathcal{F}(\mathcal{H}_i) \to \mathcal{F}(\mathcal{H}_i)$

Proof. This follows from elementary calculations. □

A.2 Choosing a basis

It is often useful to express things in terms of an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^+$ and an orthonormal basis $(\bar{e}_n)_{n\in\mathbb{N}}$ of $\mathcal{H}^-$. Now for $n_i \in \{0, 1\}$ define

$$|n_1, n_2, n_3, \ldots\rangle_+ = \sqrt{k!}e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad |n_1, n_2, n_3, \ldots\rangle_- = \sqrt{k!}J\bar{e}_{i_1} \wedge \cdots \wedge J\bar{e}_{i_k},$$

where $i_1 < \cdots < i_k$ are the indices i for which $n_i = 1$. Then

$$\{|n_1, n_2, n_3, \ldots\rangle_+ \mid n_i \in \{0, 1\}, \sum n_i < \infty\}, \quad \{|n_1, n_2, n_3, \ldots\rangle_- \mid n_i \in \{0, 1\}, \sum n_i < \infty\}$$

are orthonormal bases for $\mathcal{F}_+$ and $\mathcal{F}_-$ respectively. The fact that n_i can only be 0 or 1 is because of the Pauli exclusion principle. We define

$$a_k = a(e_k), \quad a_k^\dagger = a^\dagger(e_k), \quad b_k = b(\bar{e}_k), \quad b_k^\dagger = b^\dagger(\bar{e}_k),$$

and the CAR now entails

$$\{a_j, a_k^\dagger\} = \delta_{jk}\,\boldsymbol{I}, \quad \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0,$$

and exactly the same holds for $b_k, b_k^\dagger$. One can check that

$$a_k \, |n_1, \ldots, n_k, \ldots\rangle_+ = \begin{cases} (-1)^{\sum_{i=1}^{k-1} n_i} \, |n_1, \ldots, n_k - 1, \ldots\rangle_+ & n_k = 1 \\ 0 & n_k = 0, \end{cases}$$

$$a_k^\dagger \, |n_1, \ldots, n_k, \ldots\rangle_+ = \begin{cases} (-1)^{\sum_{i=1}^{k-1} n_i} \, |n_1, \ldots, n_k + 1, \ldots\rangle_+ & n_k = 0 \\ 0 & n_k = 1, \end{cases}$$

and similar for $b_k, b_k^\dagger$. Hence we have $a_k^\dagger a_k \, |n_1, n_2, \ldots\rangle_+ = n_k \, |n_1, n_2, \ldots\rangle_+$ and we have

$$N_+ = \sum_{k=1}^{\infty} a_k^\dagger a_k, \quad N_- = \sum_{k=1}^{\infty} b_k^\dagger b_k,$$

$$N = N_+ + N_- = \sum_{k=1}^{\infty} (a_k^\dagger a_k + b_k^\dagger b_k),$$

and

$$Q = N_+ - N_- = \sum_{k=1}^{\infty} (a_k^\dagger a_k - b_k^\dagger b_k)$$

as bounded operators with domain $\mathcal{F}_0$. Now since for every $v \in \mathcal{H}$

$$v = \sum_{n=1}^{\infty} (\langle e_n \,|\, v\rangle e_n + \langle \bar{e}_n \,|\, v\rangle \bar{e}_n),$$

we can write

$$\begin{aligned} \Psi(v) &= a\left(\sum_{n=1}^{\infty} \langle e_n \,|\, v\rangle e_n\right) + b^\dagger \left(\sum_{n=1}^{\infty} \langle \bar{e}_n \,|\, v\rangle \bar{e}_n\right) \\ &= \sum_{n=1}^{\infty} \left(\langle v \,|\, e_n\rangle a_n + \langle v \,|\, \bar{e}_n\rangle b_n^\dagger\right) \end{aligned} \tag{86}$$

A.3 The quantization of fermionic fields on concrete spaces

In this section we will concertize the field operator by taking a concrete Hilbert space for $\mathcal{H}^\pm$. We will start by defining the notion of a distribution, which is needed to rigorously define the quantum field. We follow [17, Ch. 9].

Definition A.5. A distribution is a linear functional acting on a class of *test-functions* $D(\mathbb{R}^n)$.

$$T : D(\mathbb{R}^n) \to \mathbb{R}.$$

Two common choices are smooth functions with compact support, and the Schwartz space, as defined in Definition 3.4. Distributions acting on rapidly decreasing functions are called *tempered distributions.*

An example of a distribution is the *Dirac delta function* defined by

$$\delta(\phi) = \phi(0).$$

A locally integrable function $f : \mathbb{R}^n \to \mathbb{R}^n$ gives rise to a distribution T_f, given by

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x) d^n x.$$

Because of this it is common in physics to adopt a *notational fiction* that any distribution is a function.

Definition A.6. For distribution $T : D(\mathbb{R}^n) \to \mathbb{R}$, we introduce the formal notation $T(x), x \in \mathbb{R}^n$, as if T were a function. This has to be interpreted in the following way:

$$\int_{\mathbb{R}^n} T(x)\phi(x) d^n x = T(\phi)$$

for a test function f. Any notation containing $T(x)$ without integration paired with a test function has to be interpreted in this way.

For example. for the we write the Dirac delta function as if it were a function $\delta(x)$, which makes only sense when integrated over paired with a test function in the following way

$$\int_{\mathbb{R}^n} \delta(x)\phi(x) d^n x = \phi(0). \tag{87}$$

It is possible to rigorously define the derivative of a distribution and, if we consider only tempered distributions, also the Fourier transform of a distribution.

Definition A.7. For a distribution

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$$

its derivative T' and Fourier transform $\mathcal{F}T$ are defined by

$$T'(\phi) = -T(\phi'), \quad \mathcal{F}(T)(\phi) = T(\mathcal{F}\phi).$$

Note that the Fourier transform gives an automorphism of the Schwartz-space, as stated in Theorem 3.9.

A.3.1 Minkowski space

Now back to quantization. We first consider the Dirac operator $D = \gamma^\mu \partial_\mu + m$ on Minkowski space, with domain contained in $L^2(\mathbb{R}^4, dx) \otimes \mathbb{C}^4 \cong L^2(\mathbb{R}^4, dk)^4$. We are only interested in the *on shell* solutions, nicely expressed in momentum-space by the Hilbert space

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

as defined in Definition 3.43. We first define the field operator following Eq. (85), as the mapping that that maps any vector in the Hilbert space $\mathcal{H}$ to a bounded operator on the Fock space $\mathcal{F}(\mathcal{H})$:

$$\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{F}(\mathcal{H})).$$

For various reasons and to coincide with the leading physical literature, we want to make two changes:

- Define the field operator for a $f \in \mathcal{S}(\mathbb{R}^4)^4$ instead of a vector w in the Hilbert space $\mathcal{H}$. We will use the Fourier transformation to achieve this.
- Use the Fock space based on the Hilbert space $L^2(\mathbb{R}^3)^4$ instead of $\mathcal{H}^\pm$. We will define isomorphisms $U^\pm : \mathcal{H}^\pm \to L^2(\mathbb{R}^3, \frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}})^4$ and Lemma A.4 to achieve this.

Remember that every $w^+ \in \mathcal{H}^+$ and every $w^- \in \mathcal{H}^-$ can be written as

$$w^+(k) = \sum_{s=\pm} \alpha_s(\boldsymbol{k}) u(\boldsymbol{k}, s), \quad w^-(k) = \sum_{s=\pm} \beta_s(\boldsymbol{k}) v(\boldsymbol{k}, s), \tag{88}$$

where $\alpha_s : \mathbb{R}^3 \to \mathbb{C}$ and $\beta_s : \mathbb{R}^3 \to \mathbb{C}$ are given by

$$\alpha_s(\boldsymbol{k}) = \langle u_s(\boldsymbol{k}), w(k)\rangle, \quad \beta_s(\boldsymbol{k}) = \langle v_s(\boldsymbol{k}), w(k)\rangle.$$

We define the operator

$$R : \mathcal{S}(\mathbb{R}^4)^4 \to \mathcal{H}$$

as the combination of the Fourier-transform, the inclusion into $L^2(\mathbb{R}^4) \otimes \mathbb{C}^4$ and the orthogonal projection onto $\mathcal{H}$. Let $Q^\pm : \mathcal{H} \to \mathcal{H}^\pm$ the projection onto the two subspaces of $\mathcal{H}$, as before. We define

$$R^\pm = Q^\pm \circ R : \mathcal{S}(\mathbb{R}^4)^4 \to \mathcal{H}^\pm.$$

Any $f \in \mathcal{S}(\mathbb{R}^4)^4 \cap \mathcal{F}^{-1}\mathcal{H}$ can be written as the inverse Fourier transform of a $w \in \mathcal{H} \cap \mathcal{S}(\mathbb{R})^4$, such that $Rf = w$, that is

$$f(x) = \int_{X_m^+ \cup X_m^-} e^{ik_\mu x^\mu} w(k) \frac{\sqrt{m} d^3\boldsymbol{k}}{\omega_{\boldsymbol{k}} (2\pi)^{3/2}} \tag{89}$$

$$= \int_{X_m^+} \left(e^{ik_\mu x^\mu} w(k) + e^{-ik_\mu x^\mu} w(-k) \right) \frac{\sqrt{m} d^3\boldsymbol{k}}{\omega_{\boldsymbol{k}} (2\pi)^{3/2}} \tag{90}$$

$$= \int_{X_m^+} \left(e^{ik_\mu x^\mu} w^+(k) + e^{-ik_\mu x^\mu} w^-(-k) \right) \frac{\sqrt{m} d^3\boldsymbol{k}}{\omega_{\boldsymbol{k}} (2\pi)^{3/2}} \tag{91}$$

$$= \int_{X_m^+} \sum_{s=\pm} \left(e^{ik_\mu x^\mu} \alpha_s(\boldsymbol{k}) u(\boldsymbol{k}, s) + e^{-ik_\mu x^\mu} \beta_s(\boldsymbol{k}) v(\boldsymbol{k}, s) \right) \frac{\sqrt{m} d^3\boldsymbol{k}}{\omega_{\boldsymbol{k}} (2\pi)^{3/2}}, \tag{92}$$

where

$$w^+ = w \mid_{X_m^+}, \quad w^- = w \mid_{X_m^-}.$$

Note that factor $\frac{\sqrt{m} d^3\boldsymbol{k}}{\omega_{\boldsymbol{k}} (2\pi)^{3/2}}$ is due to the inner product on $\mathcal{H}$, see Eq. (31).

We have the isomorphism of Hilbert spaces

$$U^\pm : \mathcal{H}_\pm \xrightarrow{\cong} L^2(\mathbb{R}^3)^4$$

given by

$$U^+ w(\boldsymbol{k}) = \frac{\sqrt{m}}{\omega_{\boldsymbol{k}}} w((\omega_{\boldsymbol{k}}, \boldsymbol{k}))$$

and

$$U^- w(\boldsymbol{k}) = \frac{\sqrt{m}}{\omega_{\boldsymbol{k}}} w((-\omega_{\boldsymbol{k}}, \boldsymbol{k})).$$

Note the appearance of a factor of $\frac{\sqrt{m}}{\omega_{\boldsymbol{k}}}$, due to the invariant inner product on $\mathcal{H}$, given by Eq. (31). Using Lemma A.4 these isomorphisms induce isomorphisms of Fock spaces

$$\mathcal{F}(U^{\pm}) : \mathcal{F}(\mathcal{H}^{\pm}) \to \mathcal{F}(L^2(\mathbb{R}^3)^4).$$

By construction we now have the identities:

$$U^+R^+f(\boldsymbol{k}) = \sum_{s=\pm} \frac{\sqrt{m}}{\omega_{\boldsymbol{k}}} \alpha_s(\boldsymbol{k}) u(\boldsymbol{k}, s), \quad U^-R^-f(\boldsymbol{k}) = \sum_{s=\pm} \frac{\sqrt{m}}{\omega_{\boldsymbol{k}}} \beta_s(\boldsymbol{k}) v(\boldsymbol{k}, s).$$

We are now ready to redefine the quantized field.

Definition A.8. For any $f \in \mathcal{S}(\mathbb{R}^4)^4$ the *quantized Dirac Field* is given by

$$\psi(f) = \mathcal{F}(U)\Psi(Rf)\mathcal{F}(U)^{-1}. \tag{93}$$

We will expand this definition to obtain an expression for the quantized Dirac field that is commonly found in quantum field theory books. We denote the annihilation operators for particles and antiparticles on $\mathcal{F}(L^2(\mathbb{R}^3)^4)$ by a and b respectivily, that is we have

$$a(Uv) = \mathcal{F}(U)A(v)\mathcal{F}(U^{-1}),$$

and similarly for b. We will interpret a, b as operator valued distributions on $\mathbb{R}^3$, by restricting their argument to $\mathcal{S}(\mathbb{R}^3)^4$. Adopting the notational fiction that a distribution is a function we write

$$b^\dagger(w) = \int_{\mathbb{R}^3} \sum_{s=\pm} b(\boldsymbol{k}, s)^\dagger \langle w(\boldsymbol{k}), v(\boldsymbol{k}, s)\rangle \frac{\sqrt{\omega_{\boldsymbol{k}}} d^3\boldsymbol{k}}{(2\pi)^{3/2}},$$

and similarly,

$$a(w) = \int_{\mathbb{R}^3} \sum_{s=\pm} a(\boldsymbol{k}, s)\langle w(\boldsymbol{k}), u(\boldsymbol{k}, s)\rangle \frac{\sqrt{\omega_{\boldsymbol{k}}} d^3\boldsymbol{k}}{(2\pi)^{3/2}}.$$

Here we have incorporated an arbitrary factor $\frac{1}{\sqrt{\omega_{\boldsymbol{k}}}}$ in $a(\boldsymbol{k}, s), b^\dagger(\boldsymbol{k}, s)$.

Unraveling the definitions we find that the quantized Dirac field for $f \in \mathcal{S}(\mathbb{R}^4)^4$ is given by

$$\begin{aligned}
\psi(f) &= \mathcal{F}(U)\left(A(R^+f) + B^\dagger(R^-f)\right)\mathcal{F}(U^{-1}) = a(UR^+f) + b^\dagger(UR^-f) \\
&= \int \sum_{s=\pm} \sqrt{\frac{m}{\omega_{\boldsymbol{k}}}} \left(\alpha_s^*(\boldsymbol{k})a(\boldsymbol{k},s) + \beta_s^*(\boldsymbol{k})b^\dagger(\boldsymbol{k},s)\right)\frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}} \\
&= \int \sum_{s=\pm} \sqrt{\frac{m}{\omega_{\boldsymbol{k}}}} \Big(\langle Rf(\boldsymbol{k}), u(\boldsymbol{k},s)\rangle a(\boldsymbol{k},s) \\
&\qquad\qquad + \langle Rf(\boldsymbol{k}), v(\boldsymbol{k},s)\rangle b^\dagger(\boldsymbol{k},s)\Big)\frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}} \\
&= \int \sum_{s=\pm} \sqrt{\frac{m}{\omega_{\boldsymbol{k}}}} \left(\left\langle \int_{\mathbb{R}^4} f(x)e^{-ik_\mu x^\mu}\frac{d^4x}{4\pi^2}, u(\boldsymbol{k},s)\right\rangle a(\boldsymbol{k},s)\right. \\
&\qquad\qquad \left. + \left\langle \int_{\mathbb{R}^4} f(x)e^{ik_\mu x^\mu}\frac{d^4x}{4\pi^2}, v(-\boldsymbol{k},s)\right\rangle b^\dagger(-\boldsymbol{k},s)\right)\frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}} \\
&= \int\int \sum_{s=\pm} \sqrt{\frac{m}{\omega_{\boldsymbol{k}}}} \Big(\langle f(x), u(\boldsymbol{k},s)\rangle e^{ik_\mu x^\mu} a(\boldsymbol{k},s) \\
&\qquad\qquad + \langle f(x), v(\boldsymbol{k},s)\rangle e^{-ik_\mu x^\mu} b^\dagger(\boldsymbol{k},s)\Big)\frac{d^4x}{4\pi^2}\frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}}.
\end{aligned} \tag{94}$$

In the tradition of writing a distribution as if it were a function, we write

$$\psi(x) = \int_{\mathbb{R}^3} \sum_{s=\pm} \sqrt{\frac{m}{\omega_{\boldsymbol{k}}}} \left(a(\boldsymbol{k},s)e^{ik_\mu x^\mu}u(\boldsymbol{k},s) + b^\dagger(\boldsymbol{k},s)e^{-ik_\mu x^\mu}v(\boldsymbol{k},s)\right)\frac{d^3\boldsymbol{k}}{(2\pi)^{2/3}},$$

which has to be interpreted in the sense of Eq. (94), i.e.

$$\int_{\mathbb{R}^4} \langle f(x), \psi(x)\rangle \frac{d^4x}{4\pi^2} := \psi(f).$$

A.3.2 A generalized static Lorentzian cylinder

In this example consider the Dirac operator

$$\left((i\sigma_1 \otimes \boldsymbol{I})\partial_t + i\sigma_2 \otimes D_\Sigma + m\right)\psi = 0$$

on a generalized static Lorentzian cylinder, $M = \mathbb{R} \times \Sigma$, with metric $g = -dt^2 \oplus g_\Sigma$, where (Σ, g_Σ) is a compact Riemmannian spin manifold. The solution space is given by the Hilbert space

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

as defined in Definition 3.31. Let $Q^\pm : \mathcal{H} \to \mathcal{H}^\pm$ the projection onto the two subspaces of $\mathcal{H}$, as before.

We first define the field operator following Eq. (85), as the mapping that that maps any vector in the Hilbert space $\mathcal{H}$ to a bounded operator on the Fock space $\mathcal{F}(\mathcal{H})$:

$$\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{F}(\mathcal{H})).$$

We now want to define the quantized Dirac field for a section $f \in L^2(\mathcal{S}_M)$ instead of a vector w in the Hilbert space $\mathcal{H}$. We will use the temporal Fourier transformation to achieve this. Moreover, we want to use the Fock space based on the Hilbert space $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ instead of $\mathcal{H}^\pm$. We will use the unitaries $U^\pm : \mathcal{H}^\pm \to L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$, as given in Eq. (27) to achieve this. Let us denote the inner product on $L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2$ by $\langle \cdot | \cdot \rangle$ and the inner product on $\mathcal{H}^\pm$ by $(\cdot, \cdot)$.

Remember that every $h \in \mathcal{H}$, can be written as

$$h = h^+ + h^-,$$

with $h^+ = Q^+ h,\ h^- = Q^- h$ given by

$$h^+ = \sum_{\lambda,n} \sqrt{2\omega_\lambda} a_n^\lambda w^\lambda \otimes e_n^\lambda, \quad h^- = \sum_{\lambda,n} \sqrt{2\omega_\lambda} b_n^\lambda \bar{w}^\lambda \otimes \bar{e}_n^\lambda. \tag{95}$$

where

$$a_n^\lambda = \left(\sqrt{2\omega_\lambda} w^\lambda \otimes e_n^\lambda, h^+\right), \quad b_n^\lambda = \left(\sqrt{2\omega_\lambda} \bar{w}^\lambda \otimes \bar{e}_n^\lambda, h^-\right).$$

We define the operator

$$R : L^2(\mathcal{S}_M) \to \mathcal{H}$$

as the combination of the temporal Fourier transform and the orthogonal projection onto $\mathcal{H}$. We define

$$R^\pm = Q^\pm \circ R : L^2(\mathcal{S}_M) \to \mathcal{H}^\pm.$$

Note that the following Remark 3.15 and Proposition 3.32 we have

$$\mathcal{F}^{-1}(\sqrt{2\omega_\lambda} w^\lambda \otimes e_n^\lambda) = \sqrt{\frac{m}{\omega}} w^\lambda \otimes e_n^\lambda e^{-i\omega_\lambda t}$$

and similarly

$$\mathcal{F}^{-1}(\sqrt{2\omega_\lambda} \bar{w}^\lambda \otimes \bar{e}_n^\lambda) = \sqrt{\frac{m}{\omega}} \bar{w}^\lambda \otimes \bar{e}_n^\lambda e^{i\omega_\lambda t},$$

where $\mathcal{F}$ is the temporal Fourier transform.

We are now ready to redefine the quantized Dirac field.

Definition A.9. For any $f \in \Gamma(\mathcal{S}_M)$ the *quantized Dirac Field* is given by

$$\psi(f) = \mathcal{F}(U)\Psi(Rf)\mathcal{F}(U)^{-1}, \tag{96}$$

with $U = U^+ \oplus U^-$.

We denote the annihilation operators for particles and antiparticles on $\mathcal{F}(L^2(\mathcal{S}_\Sigma) \otimes \mathbb{C}^2)$ by a and b respectively and we write

$$a_{\lambda,n} = a\left(\sqrt{\frac{m}{\omega_\lambda}} w^\lambda \otimes e_n^\lambda\right), \quad b_{\lambda,n} = b\left(\sqrt{\frac{m}{\omega_\lambda}} \bar{w}^\lambda \otimes \bar{e}_n^\lambda\right),$$

such that for $h = Rf$

$$\begin{aligned} a(U^+ h^+) &= \sum_{\lambda,n} a_{\lambda,n} \left\langle U^+ h^+ \middle| \sqrt{\frac{m}{\omega_\lambda}} w^\lambda \otimes e_n^\lambda \right\rangle = \sum_{\lambda,n} a_{\lambda,n} \left(\mathcal{F}f, \sqrt{2\omega_\lambda} w^\lambda \otimes e_n^\lambda\right) \\ &= \sum_{\lambda,n} a_{\lambda,n} \left(f, \mathcal{F}^{-1}(\sqrt{2\omega_\lambda} w^\lambda \otimes e_n^\lambda)\right) = \sum_{\lambda,n} a_{\lambda,n} \left(f, \sqrt{\frac{m}{\omega_\lambda}} w^\lambda \otimes e_n^\lambda e^{-i\omega_\lambda t}\right). \end{aligned}$$

and similarly

$$b^\dagger(U^- h^-) = \sum_{\lambda,n} b^\dagger_{\lambda,n} \left(f, \sqrt{\frac{m}{\omega_\lambda}} \bar{w}^\lambda \otimes \bar{e}^\lambda_n e^{i\omega_\lambda t} \right).$$

Unraveling the definitions we find that the quantized Dirac field is given by

$$\begin{aligned}
\psi(f) &= a(U^+ R^+ f) + b^\dagger(U^- R^- f) \\
&= \sum_{\lambda,n} \sqrt{\frac{m}{\omega_\lambda}} \Big(a_{\lambda,n} \left(f, w^\lambda \otimes e^\lambda_n e^{-i\omega_\lambda t} \right) a_{\lambda,n} \\
&\qquad\qquad + b^\dagger_{\lambda,n} \left(f, \bar{w}^\lambda \otimes \bar{e}^\lambda_n e^{i\omega_\lambda t} \right) \Big) \\
&= \left(f, \sum_{\lambda,n} \sqrt{\frac{m}{\omega_\lambda}} \Big(w^\lambda \otimes e^\lambda_n e^{-i\omega_\lambda t} a_{\lambda,n} + \bar{w}^\lambda \otimes \bar{e}^\lambda_n e^{i\omega_\lambda t} \bar{w}^\lambda \otimes \bar{e}^\lambda_n b^\dagger_{\lambda,n} \Big) \right).
\end{aligned} \tag{97}$$

Using the language that distributions are functions, we write

$$\psi(x) = \sum_{\lambda,n} \sqrt{\frac{m}{\omega_\lambda}} \Big(e^{-i\omega_\lambda t} w^\lambda \otimes e^\lambda_n a_{\lambda,n} + e^{i\omega_\lambda t} \bar{w}^\lambda \otimes \bar{e}^\lambda_n b^\dagger_{\lambda,n} \Big).$$

which has to be interpreted in the sense of Eq. (97), i.e.

$$\int_M \langle f(x), \psi(x) \rangle dV_g := \psi(f).$$

Acknowledgement

I would like to truly thank my supervisor Walter van Suijlekom for his idea to look into this subject, his support and our motivating and thought provoking conversations.

I must also thank him for his patience this summer, when I was teaching myself to be an Ethical Hacker and I had temporarily sidetracked my thesis. The epigraphs at the start of each section are a reminder to this period, as I acquired each as a small reward for hacking a box at Hack the Box.

I would also like to thank Klaas Landsman for acting as the second reader of my thesis

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