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Properties of the Propagation number

tensor products and dual spaces

Master's thesis Mathematics

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Introduction

In their article [8], Connes and Van Suijlekom define the *propagation number*, an invariant of operator systems that describes how far the operator system is removed from being a C^*-algebra. As this is a relatively new concept, not many properties are known. In a seminar on the article in June of 2020 hosted by UC Berkeley, the question arose how the propagation number behaves under standard constructions of new operator systems. The aim of this thesis is to begin uncovering the answer to this question.

The initial goal of this thesis was specifically to investigate the tensor product of operator systems, or in other words, see if the propagation number of the tensor product of two operator systems can be expressed in terms of the propagation number of the original operator systems. It quickly became clear that the difficulty in answering this question lied in the behaviour of the C^*-envelope.

After this the focus shifted to the operator system structure on the dual space of an operator system. In order to get an indication of the behaviour of the propagation number, the examples of truncated C^*-algebras put forward in [8] were generalized for a source of new, relatively simple operator systems. The results of these explorations are also recorded in this thesis.

Chapter 1 is a very brief introduction into operator systems and the C^*-envelope. It ends with the primary concern of this thesis, namely the propagation number of operator systems. Its level and scope may be somewhat limited for the more experienced reader, but it also introduces the viewpoint we will be taking in the rest of the thesis, and so might still be worth a look.

Chapter 2 covers the results concerning the C^*-envelope and the tensor product, and culminates in the proof of the fact that the C^*-envelope behaves nicely under the tensor product. It is fairly self-contained, and the reader already familiar with operator system theory might want to focus on this chapter, as it contains the main result of this thesis in Theorem 2.23. This theorem, as well as Theorem 2.20 on which it is based, are new results.

In Chapter 3 a step is made into the exploration of the relation between the propagation number and the dual operator system, as we discuss operator systems defined in [8] for which a duality was proved in [14]. We extend the definitions to a different context, calculate the C^*-envelopes and propagation numbers, and discuss the possibility of a duality there.

Finally, in the Chapter 4 we take a look at possible further investigations based on this thesis.

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Chapter 1

Preliminary definitions and results

1.1 C^*-algebras, operator systems and operator spaces

We start with recalling some basic definitions:

Definition 1.1. *A **Banach *-algebra** is a Banach space $\mathcal{A}$, equipped with an algebra structure (i.e. a product that is bilinear in both its arguments) and a *-structure (i.e. an anti-linear involution such that $(xy)^* = y^*x^*$), with the property that*

$$\|xy\| \leq \|x\|\|y\| \text{ and } \|x^*\| = \|x\|$$

for all $x, y \in \mathcal{A}$

Definition 1.2. *A C^***-algebra** is a Banach *-algebra for which $\|x^*x\| = \|x\|^2$ holds.*

It can be easily seen that the bounded operators on a Hilbert space H (which we denote as $B(H)$ from here on) is a C^*-algebra, as well as any closed subspace of $B(H)$ that is closed under the *-operation (which we call a **self-adjoint** subspace) and under the product. In fact, *any* C^*-algebra actually arises in this way (though not uniquely); more precisely we can define the following

Definition 1.3. *A linear map π between C^*-algebras that is also a homomorphism and respects the *-structure (i.e. $\pi(x^*) = \pi(x)^*$) is called a ***-homomorphism**. A **representation** of a C^*-algebra is a *-homomorphism into $B(H)$ for some Hilbert space H. A representation is called **faithful** if it is injective.*

For structures like vector spaces and algebras, we have a canonical correspondence between ideals and kernels of homomorphisms; a C^*-algebra, however, also has some analytic structure in the form of its norm, and it is therefore not immediately clear that this correspondence also holds. Although the proof is nontrivial, the result *does* actually hold.

Proposition 1.4. *Let $\pi : A \longrightarrow B$ be a *-homomorphism between C^*-algebra. Then $\pi(A) \subseteq B$ is a C^*-subalgebra and $A/\ker\pi \cong \pi(A)$.*

Theorem 1.5. *Every C^*-algebra has a faithful representation.*

Proofs of these statements, and more general theory of C^*-algebras can for example be found [29], among many other places. Combining Proposition 1.4 and Theorem 1.5, we can make our earlier statement more precise: that every C^*-algebra is *-isomorphic to a C^*-subalgebra of some $B(H)$.

In other words, (unital) C^*-algebras are closed subspaces of some $B(H)$ that are also closed under multiplication and the *-operation (and contain the unit). It turns out that the construction above also works when we drop some of these requirements; specifically, in this thesis we will examine *operator spaces* and *operator systems.*

Definition 1.6. *A (concrete)* ***operator space*** *is a closed subspace of $B(H)$ for some Hilbert space H. A (concrete)* ***operator system*** *is a self-adjoint closed unital subspace of $B(H)$ for some Hilbert space H.*

It easily seen that the role of $B(H)$ in the above definition can be replaced by 'some C^*-algebra': a subspace of a C^*-algebra is in particular a subspace of some $B(H)$ (since every C^*-algebra can be realized as a subalgebra of some $B(H)$), and $B(H)$ is a C^*-algebra. So operator spaces are equivalently subspaces of C^*-algebras. Similarly, operator systems are self-adjoint subspaces of C^*-algebras. In this thesis the two definitions will be used interchangeably.

Note the fact that operator spaces are usually not assumed to have units, while operator systems are usually assumed to be unital. Nonunital operator systems are for example treated in [31]. The unitality of operator systems makes it so that we have access to some useful properties of C^*-algebras, such as the following (based on the beginning of [22, Chapter 2]).

Proposition 1.7. *Let $E \subseteq B(H)$ be an operator system. Then for all $e \in E$ there are positive $e_i \in E_+$, $i = 1, 2, 3, 4$, such that*

$$e = (e_1 - e_2) + i(e_3 - e_4)$$

Proof. First, we can decompose e into self-adjoint parts in E through

$$e = \tfrac{1}{2}(e + e^*) + \tfrac{1}{2}(e - e^*).$$

So we have reduced the problem to decomposing self-adjoint elements in E into two positive elements. For $e \in E_{sa}$, write

$$e = \tfrac{1}{2}(\|e\|1 + e) - \tfrac{1}{2}(\|e\|1 - e).$$

The fact that $\|e\|1 \geq e$ and $e \geq -\|e\|1$ are true in C^*-algebras (this is usually proven through the Continuous Functional Calculus; see for example [9, Definition VIII.2.5], among many other places). □

Remark. Note that this is not quite the result we have for C^*-algebras, where we can decompose a self-adjoint element a as $a_+ - a_-$ where a_+ and a_- are positive and also $a_+a_- = 0$.

As we have defined them now, operator spaces and operator systems are a specific kind of subspace of C^*-algebras. We would like to introduce morphisms to make the collections of operator spaces and operator systems into categories. For this, we discuss the abstract characterisation of operator spaces and operator systems, from which it becomes clear what the 'proper' morphisms should be.

1.1.1 Abstract characterization of operator spaces and systems

For any vector space V, let $M_n(V)$ be the vector space of n by n matrices with entries in V. A general element of $M_n(V)$ is $(v_{ij})_{i,j=1}^n$, which we will denote as (v_{ij}) if there is no confusion over the indices. We also will denote $M_n := M_n(\mathbb{C})$. If V carries an algebra structure, its multiplication induces a multiplication on $M_n(V)$ through the standard matrix multiplication, and if V carries a conjugate-linear involution, this induces an involution through $(v_{ij})^* = (v_{ji}^*)$.

Lemma 1.8. *Let H be a Hilbert space, and let H^n be the direct sum of n copies of H. Then*

$$M_n(B(H)) \cong B(H^n)$$

*as *-algebras.*

Proof. Note that each element of H^n is of the form $(v_1, \dots, v_n)$. In particular, we can define a linear map a through $(a_{ij}) \in M_n(B(H))$ by setting $(av)_i = \sum_j a_{ij}v_j$. To see that this is bounded, note that

$$\|(av)_i\| \leq \sum_j \|a_{ij}v_j\| \leq \sum_j \|a_{ij}\|\|v_j\| \leq \left(\sum_j \|a_{ij}\|^2\right)^{1/2} \|v\|$$

by the triangle inequality, definition of the norm of a bounded operator and Cauchy-Schwarz. So we have

$$\|av\|^2 = \sum_i \|(av)_i\|^2 \leq \|v\|^2 \cdot \sum_{ij} \|a_{ij}\|^2$$

meaning that a is bounded. Conversely, let $A \in B(H^n)$. Let $I_j : H \longrightarrow H^n$ be the injection into the j'th coordinate, and let $P_j : H^n \longrightarrow H$ be the projection onto the j'the coordinate. Then we define $A_{ij} = P_i \circ A \circ I_j$, which is a composition of bounded operators, and therefore a bounded operator itself, and so we have $(A_{ij}) \in M_n(B(H)$. It is easily seen that the maps $(A_{ij}) \mapsto a$ and $a \mapsto (a_{ij})$ are linear and each others inverses, and that they respect multiplication and the *-operation. □

In particular Lemma 1.8 means that we can give a norm-structure to the vector space $M_n(B(H))$ by giving each element the same norm as its counterpart in $B(H^n)$. As we discussed in the previous section, any C^*-algebra A can be embedded in some $B(H)$, and so we have $M_n(A) \subseteq M_n(B(H))$ as a *-subalgebra. It therefore carries a norm, and is indeed a closed subspace with respect to this norm: if $(a_{ij}^{(n)}) \xrightarrow{n\to\infty} (b_{ij})$, then a close examination of the proof of Lemma 1.8 yields that the norm of each entry is bounded by the norm of the matrix, so that $a_{ij}^{(n)} \xrightarrow{n\to\infty} b_{ij}$ in $B(H)$ for all i, j. But A is a closed subspace, so $b_{ij} \in A$ for all i, j and indeed $M_n(A) \subseteq M_n(B(H))$ is a closed subspace. So $M_n(A)$ is a C^*-algebra with respect to the induced structure. Moreover, since C^*-algebra structures are unique (see for example [29, Chapter I, Corrolary 5.4]), we do not have to specify which norm $M_n(A)$ takes and can say that $M_n(A)$ is the C^*-algebra for the algebraic operations induced by A. In other words, every C^*-algebra carries cannonical C^*-algebra structures on its matrix spaces (independent of any concrete realization of the C^*-algebra).

Similarly, for $E \subseteq B(H)$ an operator space (so a closed subspace), we have

$$M_n(E) \subseteq M_n(B(H))$$

and so $M_n(E)$ is actually a normed space. However, these norms are *not* solely determined by the algebraic structure on E (i.e. the normed vector space structure), since one can construct isometrically isomorphic spaces $E_1 \subseteq B(H_1)$ and $E_2 \subseteq B(H_2)$ such that the induced normed spaces $M_n(E_1)$ and $M_n(E_2)$ are not all isometrically isomorphic. With this in mind we define the following:

Definition 1.9. *An (abstract)* ***operator space*** *is a Banach space E, together with norms $\|\cdot\|_n$ on $M_n(E)$ for all n, such that*

- *All $M_n(E)$ are complete;*
- $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$ *for $x \in M_n(E)$ and $y \in M_m(E)$;*
- $\|\alpha x \beta\|_n \leq \|\alpha\|_{M_n} \|x\|_n \|\beta\|_{M_n}$ *for $\alpha, \beta \in M_n$, and $x \in M_n(E)$.*

It is easily verified that the implicit structure we found above satisfies these criteria: we can therefore conclude that any *concrete* operator space is also an *abstract* operator space. The converse is much less obvious, but also turns out to be true; more specifically, we introduce a notion of a morphism for operator spaces, and conclude that every *abstract* operator space is isomorphic in this sense to a *concrete* operator space.

Definition 1.10. *Let E and F be abstract operator spaces, and let $\phi : E \longrightarrow F$ be a bounded linear map. Define*

$$\phi^{(n)} : M_n(E) \longrightarrow M_n(F) \text{ with}$$

$$\begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \phi(e_{11}) & \phi(e_{12}) & \cdots & \phi(e_{1n}) \\ \phi(e_{21}) & \phi(e_{22}) & \cdots & \phi(e_{2n}) \\ \vdots & \vdots & \ddots & \cdots \\ \phi(e_{n1}) & \phi(e_{n2}) & \cdots & \phi(e_{nn}) \end{pmatrix}.$$

Let $\|\phi^{(n)}\|_n$ *be the operator norm of* $\phi^{(n)} : M_n(E) \longrightarrow M_n(F)$*. We say* ϕ *is* ***completely bounded*** *if* $\sup_{n\to\infty} \|\phi^{(n)}\|_n < \infty$*. We say* ϕ *is* ***completely isometric*** *if* $\|\phi^{(n)}(x)\| = \|x\|$ *for all* $n \in \mathbb{N}$*,* $x \in M_n(E)$*.*

Theorem 1.11 (Ruan)**.** *Every abstract operator space is completely isometric to a concrete operator space.*

For a proof, see for example [12, section 2.3], [22, Theorem 13.4].

Finally, we turn to operator systems, for which the situation is very similar. Since $M_n(B(H))$ is a C^*-algebra, there are notions of self-adjoint elements and of positive elements (which agree with the notions on $B(H^n)$ since *-isomorphisms are in particular *-preserving and positive). This means that for a self-adjoint unital closed subspace $E \subseteq B(H)$ the *-operation descends to a *-operation on E and we can identify the cones

$$M_n(E)_+ = M_n(E) \cap M_n(B(H))_+$$

which lie in the self-adjoint part of $M_n(E)$. We therefore define the following.

Definition 1.12. *We call a vector space with conjugate linear involution* $*$ *a* $*$***-vector space****. Let* E *be a* $*$*-vector space. Define*

$$E_{sa} := \{x \in E \mid x^* = x\}.$$

Definition 1.13. *A *-vector space is called* ***matrix-ordered*** *if it is supplied with real cones* $M_n(E)_+$ *for each* $M_n(E)$ *such that*

- $M_n(E)_+ \cap (-M_n(E)_+) = \{0\}$ *for all* $n \in \mathbb{N}$*;*
- *For* $A \in M_{n\times m}$*, we have that* $x \in M_n(E)_+$ *implies* $A^*xA \in M_m(E)_+$*.*

Alternatively we will write $x \geq 0$ *if* $x \in M_n(E)_+$*, and* $x \geq y$ *if* $x - y \in M_n(E)_+$*.*

Definition 1.14. *An element* $e \in E_{sa}$ *is called an* ***order unit*** *if for every* $x \in E_{sa}$ *there exists a* $r \in [0, \infty)$ *such that* $-re \leq x \leq re$*. We call* e ***Archimedean*** *if* $x \geq -re$ *for all* $r \in (0, \infty)$ *implies* $x \geq 0$*. We call* $e \in E_{sa}$ *an* ***Archimedean matrix order unit*** *if* $(\delta_{ij}e)$ *is an Archimedean order unit in each* $M_n(E)$*. An (abstract)* ***operator system*** *is a matrix-ordered space with archimedean matrix order unit.*

Again, we define a morphism to introduce a notion of isomorphism, and conclude that all abstract operator systems are in fact concrete operator systems.

Definition 1.15. *Let* $\phi : E \longrightarrow F$ *be a linear map between abstract operator systems. It is called* ***completely positive*** *if* $\phi^{(n)}(M_n(E)_+) \subseteq M_n(F)_+$ *for all* n*. If* ϕ *is bijective, and both* ϕ *and* ϕ^{-1} *are completely positive, then* ϕ *is called a complete order isomorphism.*

Theorem 1.16 (Choi-Effros)**.** *Every abstract operator system is completely order isomorphic to a concrete operator system.*

For a proof, see for example [22, Theorem 13.1].

Clearly, every concrete operator system is in particular a concrete operator space. For the abstract structures, this is much less clear: we need to retrieve a norm from the ordering. This is where the unitality of the operator systems is very useful. The following proposition can for example be found in [22, Lemma 3.1] and [12, Proposition 1.3.2]; the proof is based on the former.

Proposition 1.17. *Let A be a unital C^*-algebra. For all $a \in A$ we have*

$$\|a\| \leq 1 \Leftrightarrow \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \geq 0$$

Proof. It suffices to check the statement for a C^*-algebra represented on some Hilbert space H. We see that for $v_1, v_2 \in H$ we have

$$\left\langle \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} v_1 + av_2 \\ a^*v_1 + v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle$$

$$= \|v_1\|^2 + \langle av_2, v_1 \rangle + \langle v_1, av_2 \rangle + \|v_2\|^2$$

By Cauchy-Schwarz we have

$$|\langle av_2, v_1 \rangle| \leq \|av_2\| \|v_1\| \leq \|a\| \|v_2\| \|v_1\|$$

and in general we have

$$-2|\langle av_2, v_1 \rangle| \leq 2\mathrm{Re}\langle av_2, v_1 \rangle = \langle av_2, v_1 \rangle + \langle v_1, av_2 \rangle.$$

So in particular we can conclude that

$$\left\langle \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \geq \|v_1\|^2 + \|v_2\|^2 - 2\|a\| \|v_2\| \|v_1\|.$$

Now if $\|a\| \leq 1$ then the lower bound is bounded below by $(\|v_1\| - \|v_2\|)^2$ and therefore positive. Conversely, because $\|a\| = \sup |\langle av_1, v_2 \rangle|$, if $\|a\| > 1$ then there exist unit vectors v_1, v_2 such that $|\langle av_1, v_2 \rangle| > 1$. By multiplying one of the vectors with a complex phase we can realize $2\mathrm{Re}\langle av_2, v_1 \rangle < -2$ so that

$$\left\langle \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle < 0$$

□

By this proposition we can see that the order structure of an abstract operator system defines a norm structure on the matrix spaces, so that indeed an abstract operator system is in particular an abstract operator space.

Using this abstract characterization, we can construct new operator spaces and operator systems; we simply have to supply a vector space with the corresponding structures on its matrix spaces. We will make use of this approach in the next section.

1.2 Space of Completely Bounded maps

Let E and F be two operator spaces, and let $CB(E,F)$ denote the space of completely bounded maps from E to F. It is easily seen that this is a vector space under pointwise operations. We are actually able to supply it with an operator space structure; this construction can for example be found in [12, p. 45 - 46], [24, Section 2.3] and [22, Proposition 14.7].

Note that $M_m(M_n(F)) \cong M_{mn}(F)$ as linear spaces, and so we can give $M_n(F)$ the induced operator space structure. Note also that in this way we have a norm on $M_n(M_m(F))$, and that $M_m(M_n(F))$ is linearly isomorphic to it through rearrangement of indices. This is in fact an isometry, because the rearrangement of indices is implemented by a unitary base change, which have unit norm, and so by definition of a matrix norm this leaves the norm invariant.

Lemma 1.18. *Let E and F be operator spaces. Then*

$$M_n(CB(E,F)) \cong CB(E, M_n(F))$$

as vector spaces, where $(\phi_{ij}) \in M_n(CB(E,F))$ is mapped to $\phi : e \mapsto (\phi_{ij}(e)) \in M_n(F)$, and conversely the map $\phi \in CB(E, M_n(F))$ is sent to (ϕ_{ij}) with $\phi_{ij}(e) = \phi(e)_{ij}$.

Proof. As noted in the proof of Lemma 1.8, we have for $F_{ij} \in M_n(F)$ (so that $(F_{ij})_{i,j=1}^m \in M_m(M_n(F))$ that

$$\|F_{ij}\| \leq \|(F_{ij})_{i,j=1}^n\| \leq \left(\sum_{ij} \|F_{ij}\|^2\right)^{1/2}$$

(where the left inequality holds for all i,j).

First, let $(\phi_{ij})_{i,j=1}^n \in M_n(CB(E,F))$. We need to show that the map $\phi : E \longrightarrow M_n(F)$ given by $\phi(e) = (\phi_{ij}(e))_{i,j=1}^n$ is completely bounded. For this, let $(e_{kl})_{k,l=1}^m \in M_m(E)$. Then

$$\phi^{(m)}((e_{kl})_{k,l=1}^m) = (\phi(e_{kl}))_{k,l=1}^m = ((\phi_{ij}(e_{kl})_{i,j=1}^n)_{k,l=1}^m \in M_m(M_n(F)).$$

By the remark preceding this lemma, we can calculate the norm of this element in $M_n(M_m(F))$ to conclude

$$\|\phi^{(m)}((e_{kl})_{k,l=1}^m)\|^2 = \|((\phi_{ij}(e_{kl})_{i,j=1}^n)_{k,l=1}^m\|^2 = \|((\phi_{ij}(e_{kl})_{k,l=1}^m)_{i,j=1}^n\|^2$$

$$\leq \sum_{i,j=1}^n \|(\phi_{ij}(e_{kl}))_{k,l=1}^m\|^2 = \sum_{i,j=1}^n \|\phi_{ij}^{(m)}((e_{kl})_{k,l=1}^m)\|^2$$

However, since the ϕ_{ij} are completely bounded, this means that

$$\|\phi^{(m)}((e_{kl})_{k,l=1}^m)\|^2 \leq \sum_{i,j=1}^n \|\phi_{ij}\|_{cb}^2 \|(e_{kl})_{k,l=1}^m\|^2 = \|(e_{kl})_{k,l=1}^m\|^2 \sum_{i,j=1}^n \|\phi_{ij}\|_{cb}^2$$

so that indeed ϕ is completely bounded.

Conversely, suppose that $\phi : E \to M_n(F)$ is completely bounded. We need to show that the map $\phi_{ij} : e \mapsto \phi(e)_{ij}$ is completely bounded. For this we note that

$$\|\phi_{ij}^{(m)}((e_{kl})_{k,l=1}^m)\| = \|(\phi_{ij}(e_{kl}))_{k,l=1}^m\| \leq \|((\phi_{ij}(e_{kl}))_{k,l=1}^m)_{i,j=1}^n\|$$

and using again the remark preceding this lemma we have that

$$\|((\phi_{ij}(e_{kl}))_{k,l=1}^m)_{i,j=1}^n\| = \|((\phi_{ij}(e_{kl}))_{i,j=1}^n)_{k,l=1}^m\|$$

$$= \|(\phi(e_{kl}))_{k,l=1}^m\| = \|\phi^{(m)}((e_{kl})_{k,l=1}^m)\| \leq \|\phi\|_{cb}\|((e_{kl})_{k,l=1}^m)\|.$$

So $\|\phi_{ij}^{(m)}\| \leq \|\phi\|_{cb}$ and therefore ϕ_{ij} is indeed completely bounded.

It is now easy to check that the maps $\phi \mapsto (\phi_{ij})_{i,j=1}^n$ and $(\phi_{ij})_{i,j=1}^n \mapsto \phi$ are linear and eachothers inverses.

□

Since $CB(E, M_n(F))$ is a normed space using the completely-bounded norm, we have induced norms on the matrix spaces of $CB(E, F)$. The properties of operator space norms are easily checked: they follow from the properties for the norms on $M_n(F)$.

The following shows that for bounded maps into M_n it is very easy to be completely bounded. The proof is adapted from [24, Prop. 1.12] and [12, Lem. 2.2.1, Prop. 2.2.2]. We use the fact that $M_m(M_n)$ acts on $(\mathbb{C}^n)^m$; elements of this latter space are of the form $x = (x_i)_{i=1}^m$ for $x_i \in \mathbb{C}^n$, with inner product $\langle x, y\rangle = \sum_i \langle x_i, y_i\rangle$.

Lemma 1.19. *Let $x_1, \ldots, x_m \in \mathbb{C}^n$, for $m \geq n$. Then there exist $y_1, \ldots, y_n \in \mathbb{C}^n$ and an isometric matrix $b \in M_{m\times n}$ such that*

$$x_i = \sum_{j=1}^n b_{ij} y_j.$$

Moreover, we have that $\sum_{i=1}^m \|x_i\|^2 = \sum_{i=1}^n \|y_i\|^2$.

Proof. Define vectors $\tilde{x}_1, \ldots \tilde{x}_n \in \mathbb{C}^m$ through $(\tilde{x}_i)_j = (x_j)_i$. Then the $\tilde{x}_1, \ldots \tilde{x}_n$ span a subspace of $\mathbb{C}^m$ of dimension at most n, and so there exists an isometry $b : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ such that the span of the $\tilde{x}_j$ lies in its image. So there are $\tilde{y}_1, \ldots \tilde{y}_n \in \mathbb{C}^n$ such that $b(\tilde{y}_j) = \tilde{x}_j$. So in particular

$$(x_i)_j = (\tilde{x}_j)_i = \sum_{k=1}^n b_{ik}(\tilde{y}_j)_k$$

and we therefore define $y_1, \ldots, y_n$ by $(y_k)_j = (\tilde{y}_j)_k$, to see that

$$(x_i)_j = \sum_{k=1}^n b_{ik}(y_k)_j.$$

In other words, we have $x_i = \sum_{k=1}^n b_{ij} y_j$. Moreover

$$\sum_{i=1}^m \|x_i\|^2 = \sum_{i=1}^m \sum_{j=1}^n |(x_i)_j|^2 = \sum_{j=1}^n \|\tilde{x}_j\|^2 = \sum_{j=1}^n \|b(\tilde{y}_j)\|^2$$

$$= \sum_{j=1}^n \|\tilde{y}_j\|^2 = \sum_{j=1}^n \sum_{i=1}^n |(y_i)_j)|^2 = \sum_{i=1}^n \|y_i\|^2$$

□

Proposition 1.20. *Let E be an operator space, and $\phi : E \to M_n$ be a bounded map. Then $\|\phi\|_{cb} = \|\phi^{(n)}\|$. In particular we have that ϕ is completely bounded.*

Proof. We show that for $m \geq n$ we have $\|\phi^{(m)}\| \leq \|\phi^{(n)}\|$. Since $\|\phi^{(n)}\| \leq \|\phi^{(m)}\|$ is automatic because

$$\|\phi^{(n)}(e)\| = \|\phi^{(m)}(e \oplus 0)\| \leq \|\phi^{(m)}\|\|e \oplus 0\| = \|\phi^{(m)}\|\|e\|$$

for $e \in M_n(E)$, this suffices.

For $e \in M_m(E)$, and $x, y \in (\mathbb{C}^n)^m$, then

$$\langle \phi^{(m)}(e)x, y\rangle = \sum_{i=1}^m \langle (\phi^{(m)}(e)x)_i, y_i\rangle = \sum_{i,j=1}^m \langle \phi(e_{ij})x_j, y_i\rangle.$$

Using Lemma 1.19 we now find $a, b \in M_{m\times n}$ and $v, w \in (\mathbb{C}^n)^n$ such that $x_j = \sum_{k=1}^n a_{jk} v_k$ and $y_i = \sum_{l=1}^n b_{il} w_l$. So

$$\langle \phi^{(m)}(e)x, y\rangle = \sum_{i,j=1}^m \sum_{k,l=1}^n \langle \phi(e_{ij}) a_{jk} v_k, b_{il} w_l\rangle = \sum_{k,l=1}^n \langle \phi((b^* e a)_{lk}) v_k, w_l\rangle$$

$$= \langle \phi^{(n)}(b^* e a) v, w\rangle.$$

Also by Lemma 1.19 we have that if x and y are unit vectors, then so are v and w. So in that case we have

$$|\langle \phi^{(m)}(e)x, y\rangle| \leq \|\phi^{(n)}(b^* e a)\| \leq \|\phi^{(n)}\|\|b^* e a\| \leq \|\phi^{(n)}\|\|e\|$$

and by taking the supremum on the lefthand side, we see that $\|\phi^{(m)}(e)\| \leq \|\phi^{(n)}\|\|e\|$, proving our claim and therefore the proposition. □

Corollary 1.21. *For any operator space E, with E^* the space of continuous linear maps, we have*

$$E^* = CB(E, \mathbb{C})$$

So E induces an operator space structure on $CB(E, \mathbb{C})$ (the operator space on $\mathbb{C}$ is the unique one, given simply by the standard operator norm of matrices), which means that E^* has a canonical operator space structure. We will make use of these constructions in chapter 2.

1.3 Dual operator system

In the previous section we were able to give an operator space structure to the dual space of an operator space through the canonical operator space structure on the space of completely bounded maps. In this section we apply the same tactic to operator systems using completely positive maps, although we will eventually restrict to the finite-dimensional case for simplicity. This discussion is based on [8, Section 2.3] and [22, Proposition 13.2], with some additional details filled in.

Let E and F be operator systems: we cannot replace the space of completely bounded maps by completely positive maps (the latter do not even form a vector space), but we can define a partial order on $CB(E,F)$ by letting the positive cone be given by the completely positive maps. For this we first need to make $CB(E,F)$ into a *-vector space, which we do by defining

$$\Phi^*(e) := \Phi(e^*)^*$$

which is indeed conjugate linear. Then a completely positive map Φ has (using the decomposition in Proposition 1.7)

$$\Phi^*(e_1 - e_2 + ie_3 - ie_4) = \Phi(e_1^*)^* - \Phi(e_2^*)^* + \Phi((ie_3)^*) - \Phi((ie_4)^*)^*$$

$$= \Phi(e_1) - \Phi(e_2) + (-i\Phi(e_3))^* - (-i\Phi(e_4))^* = \Phi(e_1) - \Phi(e_2) + i\Phi(e_3) - i\Phi(e_4)$$

since Φ is in particular positive and so both the e_i and the $\Phi(e_i)$ are self-adjoint. So indeed the completely positive maps lie in $CB(E,F)_{sa}$.

So we define

$$CB(E,F)_+ = CP(E,F)$$

from which we induce

$$M_n(CB(E,F))_+ :\cong CB(E, M_n(F))_+ = CP(E, M_n(F))$$

and by corollary 1.21 this gives a matrix ordering on E^*. We will denote E^* with this matrix ordering as E^d. We can see that positivity in $M_1(E^d)_+ = E^d_+$ agrees with our normal notion of positive functionals, since all positive functionals are completely positive as a consequence of the following proposition (note the similarities to Proposition 1.20).

Proposition 1.22. *Let E be an operator system, and $\phi : E \longrightarrow M_n$ be a linear map. Then ϕ is completely positive if and only if it is n-positive.*

Proof. Clearly, if ϕ is completely positive then it is n-positive. Suppose now that $\phi^{(n)}$ is positive, and let $e \in M_m(E)_+$, $x \in (\mathbb{C}^n)^m$ for $m \geq n$. Then by Lemma 1.19 there is an isometry $b \in M_{m\times n}$ and vectors $y_1, \ldots, y_n \in \mathbb{C}^n$ such that $x_i = \sum_{j=1}^n b_{ij} y_j$. This means that

$$\langle \phi^{(m)}(e)x, x\rangle = \sum_{i=1}^m \langle (\phi^{(m)}(e)x)_i, x_i\rangle = \sum_{i,j=1}^m \langle \phi(e_{ij})x_j, x_i\rangle$$

$$= \sum_{i,j=1}^{m} \sum_{k,l=1}^{n} \langle \phi(e_{ij}) b_{jk} y_k, b_{il} y_l \rangle = \sum_{k,l=1}^{n} \langle \phi((b^* e b)_{lk}) y_k, y_l \rangle = \langle \phi^{(n)}(b^* e b) y, y \rangle$$

Since $e \in M_m(E)_+$ we also have that $b^*eb \in M_n(E)_+$, and because $\phi^{(n)}$ is positive, this means that $\langle \phi^{(n)}(b^*eb)y, y \rangle \geq 0$, so that indeed $\phi^{(m)}$ is positive for all $m \geq n$. So ϕ is completely positive. □

Corollary 1.23. *Let E be an operator system. Then $f : E \longrightarrow \mathbb{C}$ is positive if and only if it is completely positive. Furthermore, $g : E \longrightarrow C(X)$ for X a compact Hausdorff space is positive if and only if it is completely positive.*

Proof. The fact that f is completely positive if it is positive follows directly from Proposition 1.22. For g, define $g_x : E \longrightarrow \mathbb{C}$ by $g_x(e) = g(e)(x)$. Note that $M_n(C(X)) \cong C(X, M_n)$, so we consider $g^{(n)} : E \longrightarrow C(X, M_n)$. Then we define $(g^{(n)})_x(e) = g^{(n)}(e)(x)$. Note that

$$(g^{(n)})_x(e) = (g(e_{ij})(x)) = (g_x(e_{ij})) = g_x^{(n)}(e)$$

so we simply write $g_x^{(n)}$. But g_x is positive, so by the above $g_x^{(n)}$ is positive, and therefore we have for positive e that $g^{(n)}(e)(x) \geq 0$ for all x. So $g^{(n)}(e)$ is positive for all positive e, and so g is completely positive. □

Concretely, the matrix ordering on E^d is given by

$$M_n(E^d)_+ = \{(\phi_{ij}) \in M_n(E^d) \mid e \mapsto (\phi_{ij}(e)) \text{ is completely positive}\}.$$

For practical reasons we can slightly reformulate this description, as explained in [22, Chapter 6]. For $\phi = (\phi_{ij}) \in M_n(E^d)$, we define a representation

$$R[\phi] : E \longrightarrow M_n, e \mapsto (\phi_{ij}(e))$$

and a scalar action

$$S[\phi] : M_n(E) \longrightarrow \mathbb{C}, (e_{ij}) \mapsto \sum_{i,j} \phi_{ij}(e_{ij}).$$

Recall from the previous section that $M_n(M_n)$ acts on $(\mathbb{C}^n)^n$.

Lemma 1.24. *Let $\epsilon^{i,j} \in (\mathbb{C}^n)^n$ be the vector given by $(\epsilon^{i,j})_k = \delta_{ik} e_j$, where e_j is a standard basis vector for $\mathbb{C}^n$. Also, let $\phi = (\phi_{ij}) \in M_n(E^*)$ and $y = (y_{ij}) \in M_n(E)$. Then we have*

$$\langle R[\phi]^{(n)}(y)\epsilon^{i,j}, \epsilon^{k,l} \rangle = \phi_{lj}(y_{ki}).$$

Proof. We calculate

$$\langle R[\phi]^{(n)}(y)\epsilon^{i,j}, \epsilon^{k,l} \rangle = \sum_{s=1}^{n} \langle (R[\phi]^{(n)}(y)\epsilon^{i,j})_s, (\epsilon^{k,l})_s \rangle$$

$$= \sum_{s=1}^{n} \delta_{ks} \langle (R[\phi]^{(n)}(y)\epsilon^{i,j})_s, e_l \rangle = \langle (R[\phi]^{(n)}(y)\epsilon^{i,j})_k, e_l \rangle$$

$$= \sum_{s=1}^{n} \langle R[\phi](y_{ks})(\epsilon^{i,j})_s, e_l \rangle = \sum_{s=1}^{n} \delta_{is} \langle R[\phi](y_{ks})e_j, e_l \rangle = \langle R[\phi](y_{ki})e_j, e_l \rangle$$

which is indeed by definition equal to $\phi_{lj}(y_{ki})$. □

Proposition 1.25. *Let E be an operator system, and $S[\phi]$ as mentioned above for all $\phi \in M_n(E^d)$. Then*

$$M_n(E^d)_+ = \{\phi \in M_n(E^d) \mid S[\phi] \text{ is a positive functional}\}$$

Proof. First, suppose ϕ is completely positive, and let $\eta = \sum_i \epsilon^{i,i} \in (\mathbb{C}^n)^n$. Now for $y \in M_n(E)$, by Lemma 1.24, we have

$$\langle R[\phi]^{(n)}(y)\eta, \eta \rangle = \sum_{i,j} \langle R[\phi]^{(n)}(y)\epsilon^{j,j}, \epsilon^{i,i} \rangle = \sum_{i,j} \phi_{ij}(y_{ij}) = S[\phi](y).$$

From this identity we can see that if ϕ is completely positive, then for all $y \in M_n(E)_+$ we have that $R[\phi]^{(n)}(y)$ is a positive operator, so in particular we have that $S[\phi](y) \geq 0$.

Conversely, suppose that $S[\phi]$ is positive. For arbitrary $\mu \in (\mathbb{C}^n)^n$, we expand $\mu = \sum_{i,j=1}^{n} M_{ij}\epsilon^{j,i}$, so that we have a matrix $M \in M_n$. Then for $y \in M_n(E)$ we have

$$\langle R[\phi]^{(n)}(y)\mu, \mu \rangle = \sum_{i,j,k,l} M_{ji}\overline{M_{lk}} \langle R[\phi]^{(n)}(y)\epsilon^{i,j}, \epsilon^{k,l} \rangle = \sum_{i,j,k,l} M_{ji}\overline{M_{lk}}\phi_{lj}(y_{ki}).$$

However, we note that

$$(M^*yM)_{ki} = \sum_j (M^*y)_{kj}M_{ji} = \sum_{lj} (M^*)_{kl}y_{lj}M_{ji} = \sum_{lj} \overline{M_{lk}}y_{lj}M_{ji}$$

so that in fact

$$\langle R[\phi]^{(n)}(y)\mu, \mu \rangle = \sum_{ki} \phi_{ki}(M^*yM)_{ki} = S[\phi](M^*yM).$$

So if $y \in M_n(E)_+$, then $M^*yM \in M_n(E)_+$ (by definition of an abstract operator space). Since by our assumption $S[\phi]$ is positive, we have $\langle R[\phi]^{(n)}(y)\mu, \mu \rangle$ is positive. So $R[\phi]^{(n)}$ is positive.

Note that because $R[\phi]$ maps into M_n, by Proposition 1.22 it is completely positive if it is n-positive. So $R[\phi]$ is completely positive, meaning that $\phi \in M_n(E^*)$. This proves the proposition. □

Corollary 1.26. *Let E be an operator system. Then*

$$S : M_n(E^d) \longrightarrow M_n(E)^d, \varphi \mapsto S[\varphi]$$

is an order isomorphism, i.e. it is positive with positive inverse.

Proof. We see that the inverse of S is given by $\varphi \mapsto (\varphi \circ \iota_{i,j})$ for $\varphi \in M_n(E)^d$, where $\iota_{i,j} : E \longrightarrow M_n(E)$ is the injection into the (i, j) spot. Indeed,

$$SS^{-1}[\varphi](e) = S[(\varphi \circ \iota_{i,j})](e) = \varphi\left(\sum \iota_{i,j}(e)\right) = \varphi(e)$$

$$S^{-1}S[\Phi] = (S[\Phi] \circ \iota_{i,j}) = (\Phi_{ij}) = \Phi$$

for $\varphi \in M_n(E)^d$, and $\Phi \in M_n(E^d)$. Note that by the above we have that $\varphi \in M_n(E^d)_+$ if and only if $S[\varphi] \in M_n(E)^d_+$, so indeed both S and S^{-1} are positive. □

We have now supplied the *-vector space E^* with a matrix ordering. In order to make it into an operator system we need to include an Archimedean matrix order unit. Such units are not too difficult to find, provided that E is finite-dimensional, as will be the case in this thesis when discussing the dual operator system.

Proposition 1.27. *Let E be a finite-dimensional operator system, and let χ be a faithful state, i.e. $\chi \in E^d_+$ of norm 1 with $x > 0$ implies $\varphi(x) > 0$. Then χ is an Archimedean matrix order unit for E^d.*

Proof. Let $\chi^{(n)} = (\delta_{ik}\chi) \in M_n(E^d)$. First, note that $S[\chi^{(n)}]$ is a faithful on $M_n(E)$; indeed, for any $e \in M_n(E)_+$ we have

$$e_{11} = (1, 0, \dots, 0)e(1, 0, \dots, 0)^* \in M_1(E)_+ = E_+$$

and similarly that $e_{ii} \in E_+$, so that

$$S[\chi^{(n)}](e) = \sum_{i=1}^{n} \chi(e_{ii}) > 0$$

because all the terms are strictly greater than zero.

Next, we show that $\chi^{(n)}$ is an order unit. We can consider $E \subseteq B(H)$ as a concrete operator system; since then $M_n(E)$ is a finite-dimensional normed space, its set of unit vectors $M_n(E)_1$ is compact. Also, $M_n(E)_+ = M_n(E) \cap B(H)_+$ is closed as the intersection of closed sets, so $M_n(E)_+ \cap M_n(E)_1$ is compact. This means that we have

$$0 < M := \inf\{S[\chi^{(n)}](e) \mid e \in M_n(E)_+ \cap M_n(E)_1\}$$

because if we would have a sequence $e_i \in M_n(E)_+ \cap M_n(E)_1$ such that $S[\chi^{(n)}](e_i)$ converges to zero, then because $M_n(E)_+ \cap M_n(E)_1$ is compact there is a convergent subsequence $e_{i_j} \xrightarrow{j\to\infty} e$, which means $S[\chi^{(n)}](e) = 0$ for some $e > 0$, which is a contradiction with the faithfullness of $S[\chi^{(n)}]$. Let $\varphi \in M_n(E^d)_{sa}$; then $S[\varphi]$ has an induced norm since $M_n(E)$ is a normed space, and taking $t := \|S[\varphi]\|/M$ we have for all $e \in M_n(E)_1 \cap M_n(E)_+$ that

$$S[t\chi^{(n)} \pm \varphi](e) = tS[\chi^{(n)}](e) \pm S[\varphi](e) \geq \|S[\varphi]\| - \|S[\varphi]\| \geq 0.$$

Here we have used that $S[\varphi](M_n(E)_{sa}) \subseteq \mathbb{R}$ because

$$S[\varphi](e) = \sum_{i=1}^{n} \varphi_{ii}(e_{ii}) + \sum_{i<j} \varphi_{ij}(e_{ij}) + \varphi_{ij}^*(e_{ij}^*)$$

$$= \sum_{i=1}^{n} \varphi_{ii}(e_{ii}) + \sum_{i<j} \varphi_{ij}(e_{ij}) + \overline{\varphi_{ij}(e_{ij})} \in \mathbb{R}$$

because $\overline{\varphi_{ii}(e_{ii})} = \varphi_{ii}(e_{ii})$. So for every $\varphi \in M_n(E)_{sa}$ there is a $t \geq 0$ such that $-t\chi^{(n)} \leq \varphi \leq t\chi^{(n)}$.

Lastly, we prove that $\chi^{(n)}$ is Archimedean. Suppose that

$$S[r\chi^{(n)} + \varphi](e) = rS[\chi^{(n)}](e) + S[\varphi](e) \geq 0$$

for all $r > 0$ and $e \in M_n(E)_+$. Then in particular

$$S[\varphi](e) \geq -rS[\chi^{(n)}](e) \geq -rM\|e\|$$

for all $r > 0$, so that indeed $S[\varphi](e) \geq 0$ for all $e \in M_n(E)_+$, which means that $\varphi \geq 0$. □

Lemma 1.28. *Every finite-dimensional operator system E admits a faithful state χ.*

Proof. Note that $E_+ \cap E_1$ is compact for a finite-dimensional concrete operator system $E \subseteq B(H)$, as discussed in the proof of Proposition 1.27 (here E_1 is the set of unit vectors in E). We have that for each $e \in E_+ \cap E_1$ there exists a positive $\varphi_e \in E^*$ such that $\varphi_e(e) > 0$, since this holds for $B(H)$. By continuity of φ_e there is then a neighbourhood U_e containing e such that $0 \notin \varphi_e(U_e)$. We therefore have an open cover

$$\{U_e \mid e \in E_+ \cap E_1\}$$

of $E_+ \cap E_1$ which by compactness has a finite subcover; say that these correspond to the elements $e_1, \ldots, e_n$. So then $\sum_{i=1}^n \varphi_{e_i}$ is a positive functional, and for each $e \in E_+$ there is one of the φ_{e_i} that is nonzero, so indeed (after normalizing) we have that $\sum_{i=1}^n \varphi_{e_i}$ is a faithful state. □

Corollary 1.29. *Let E be a finite-dimensional operator system. Then there exists an Archemedian order unit $\chi \in E^d$ that makes E^d into an abstract operator system.*

We now prove two properties of the dual operator system for later use (specifically in chapter 3). Note that a map $f : E \longrightarrow F$ induces a pullback $f^d : F^d \longrightarrow E^d$ through

$$f^d(\varphi)(x) = \varphi(f(x))$$

Lemma 1.30. *For any finite-dimensional operator system E we have $E \cong (E^d)^d$*

Proof. The isomorphism is given by the map $f : e \mapsto (\hat{e} : \varphi \mapsto \varphi(e))$. Since both E and $(E^d)^d$ are finite-dimensional, this is a linear bijection. What remains is to show that f and f^{-1} are completely positive.

So suppose $e \in M_n(E)_+$. Then for all $\varphi \in M_n(E^d)_+$ we have

$$S[f^{(n)}(e)](\varphi) = \sum f^{(n)}(e)_{ij}(\varphi_{ij}) = \sum \varphi_{ij}(e_{ij}) = S[\varphi](e) \geq 0$$

so $f^{(n)}$ is positive. Note that $f^{(n)}$ is also bijective with inverse $(f^{(n)})^{-1} = (f^{-1})^{(n)}$ since $(f \circ g)^{(n)} = f^{(n)} \circ g^{(n)}$. So for $\alpha \in M_n((E^d)^d)_+$ we have

$$S[\varphi]((f^{-1})^{(n)}(\alpha)) = S[\alpha](\varphi) \geq 0$$

for all $\varphi \in M_n(E^d)_+$. So since S is an order isomorphism by 1.26, we have that all positive functionals on $M_n(E)$ are positive on $(f^{-1})^{(n)}(\alpha)$. By the bipolar theorem, this means that $(f^{-1})^{(n)}(\alpha)$ is positive. □

Proposition 1.31. *A map $f : E \longrightarrow F$ between finite-dimensional operator systems is n-positive (resp. completely positive) if and only if $f^d : F^d \longrightarrow E^d$ is n-positive (resp. completely positive).*

Proof. Note that the statement for complete positivity follows for all the statements of n-positivity. Due to Lemma 1.30, it suffices to show that $f^d : F^d \longrightarrow E^d$ is n-positive if f is. Let $\varphi \in M_n(F^d)_+$; then for all $e \in M_n(E)_+$ we have that $f^{(n)}(e) \in M_n(F)_+$, and therefore

$$S[(f^d)^{(n)}(\varphi)](e) = \sum_{i,j}((f^d)^{(n)}(\varphi))_{ij}(e_{ij}) = \sum_{i,j} f^d(\varphi_{ij})(e_{ij}) = S[\varphi](f^{(n)}(e)) \geq 0$$

so that indeed $(f^d)^{(n)}(\varphi)$ is positive. So $(f^d)^{(n)}$ is positive. □

1.4 C^*-envelopes

We introduces operator systems as concrete (closed) subspaces of C^*-algebras (that are self-adjoint and unital). After that, we introduced morphisms to give a notion of isomorphism between these subspaces (more specifically, we identified an abstract structure on a vector space that characterizes operator systems with respect to this notion of isomorphism). However, it can happen that two operator systems in non-isomorphic C^*-algebras are completely order isomorphic, or in other words, that we can find the same operator system in two different C^*-algebras.

For example, let $D \subseteq \mathbb{C}$ be the unit disc, and let $C_{harm}(D) \subseteq C(D)$ be the subspace of harmonic functions on the disk, meaning all $f = f_1 + if_2 : D \longrightarrow \mathbb{C}$ such that

$$\Delta f_1 := \frac{\partial^2 f_1}{(\partial z_1)^2} + \frac{\partial^2 f_1}{(\partial z_2)^2} = 0, \quad \Delta f_2 = 0$$

where we expand $z = z_1 + iz_2$ in the domain. Through PDE theory (see for example [13, Section 2.2.4.c]) we know that each harmonic function is continuous on the boundary $\partial D = S^1$, and that for each continuous function $f \in C(S^1)$ there is a harmonic function on the disc such that its restriction to the circle is f. So as vector spaces, we have $C_{harm}(D) \cong C(S^1)$. What's more, by the maximum principle each harmonic function assumes its maximum and minimum on the boundary, so a harmonic function is positive if and only if its restriction to the circle is positive. So the correspondence $C_{harm}(D) \cong C(S^1)$ is actually an order isomorphism. What's more, since both spaces are embedded in commutative C^*-algebras, we can apply corollary 1.23 to conclude that $C_{harm}(D)$ and $C(S^1)$ are completely order isomorphic; but clearly the C^*-algebras in which they lie ($C(D)$ and $C(S^1)$) are not isomorphic (note also that this shows that an operator system that is completely order isomorphic to a C^*-algebra is not necessarily a C^*-algebra, and a completely positive map between two C^*-algebras is not necessarily a *-homomorphism).

It turns out that for a given operator system, it is possible to find realizations into C^*-algebras with useful properties. Specifically, we turn to the C^*-envelope. We will see that this is in some ways the 'minimal' C^*-algebra that contains some given operator system. We define this C^*-algebra by its universal property.

Definition 1.32. *Let E be an operator system. The C^***-envelope** of E, denoted as $C^*_{env}(E)$, is a C^*-algebra together with unital completely positive map $\pi : E \longrightarrow C^*_{env}(E)$ such that*

- *$C^*(\pi(E)) = C^*_{env}(E)$ and π is a complete order isomorphism onto its image $\pi(E)$;*
- *if $\lambda : E \longrightarrow B$ is a unital complete order isomorphism onto its image with $C^*(\lambda(E)) = B$, where B is a unital C^*-algebra, then there is a surjective *-homomorphism $\rho : B \longrightarrow C^*_{env}(E)$ such that $\pi = \rho \circ \lambda$.*

*Such a λ is called a C^***-extension**.*

This C^*-envelope exists, and is essentially unique, as is shown in [17]. For our purposes, we also need a more concrete description, and in order to arrive there, we take a slight detour.

Let X be a compact Hausdorff space, and let $E \subseteq C(X)$ be a closed subspace. A subset $S \subseteq X$ is called a **boundary** if for all $f \in E$, there is an $s_f \in S$ such that $f(s_f) = \|f\|$; formulated differently, a boundary for a subspace E of $C(X)$ is a set on which every function in E achieves its maximal value. If E separates points, there exists a unique minimal (closed) boundary, which is called the **Šilov boundary** (see for example [5, Sec. 4.1]).

We can translate this back to the algebra setting by realizing that closed sets $K \subseteq X$ correspond to vanishing ideals

$$I_K = \{f \in C(X) \mid f(K) = 0\}.$$

(see for example [9, Proposition VIII.4.9]). The vanishing ideal of the Šilov boundary of a closed subspace $E \subseteq C(X)$ can now be seen to be the largest ideal which we can 'throw away' without affecting the norm of elements in E. This brings us to the following:

Definition 1.33. *Let $E \subseteq A$ be a concrete operator system. An ideal I for which the projection $\pi : A \longrightarrow A/I$ is completely isometric on E is called a* ***boundary ideal****. The maximal boundary ideal is called the* ***Šilov boundary ideal*** *(or simply Šilov ideal).*

Proposition 1.34. *Let $E \subseteq A$ be a concrete operator system that generates A. Then there exists a Šilov boundary ideal I for E in A, and $C^*_{env}(E) \cong A/I$.*

This was also proven in [17]. So we have already made the C^*-envelope more concrete by showing that we can realize it in *any* concrete realization of the operator system (after possible restricting the ambient C^*-algebra to the subalgebra which the operator system generates). We can actually do better still: even the Šilov boundary ideal has a concrete description.

Definition 1.35. *Let $E \subseteq A$ be a concrete operator system where E generates A. A* ***boundary representation*** *is an irreducible representation $\sigma : A \longrightarrow B(H)$ such that for all completely positive maps $\rho : A \longrightarrow B(H)$ with $\sigma(e) = \rho(e)$ for all $e \in E$, we have that $\rho = \sigma$. We denote the set of all boundary representations of E as ∂E.*

Proposition 1.36. *The Šilov boundary K of an operator system S is equal to $\bigcap_{\sigma \in \partial S} \ker \sigma$, where ∂S is the collection of boundary representations of S.*

The history of these characterizations is quite interesting (see [3]). In [1, p. 171], Arveson shows this proposition for operator systems that are 'admissible subspaces', or as denoted in [2, p. 1066], operator systems that have 'sufficiently many boundary representations' (roughly this means that we can characterize the norms on $M_n(E)$ through boundary representations). Actually, Arveson tried to show that *all* operator systems have sufficiently many boundary representations, and that therefore the Šilov boundary ideal and C^*-envelope must exist. A decade later, Hamana *did* show the existence of both the Šilov ideal and the C^*-envelope; however, Hamana did so through a different method altogether, thus leaving the problem of whether the above description always holds open. In 2008, Arveson made a last attempt to prove the characterization, and he did so for separable operator systems, though using some fairly involved techniques. It was in 2013 ([10]) that Davidson and Kennedy solved the problem, using techniques going back to Arveson's original 1969 paper. Sadly, Arveson had passed away in 2011. However, Davidson and Kennedy were actually a student and grand-student of Arveson, so it is only fitting that they would finish Arveson's work.

1.4.1 Finite dimensions

Finite-dimensional C^*-algebras A are really well-behaved. Namely, we have for each such A that there are $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$A \cong \bigoplus_{i=1}^{m} M_{n_i}.$$

Moreover, each M_{n_i} is simple, meaning that it has no nontrivial ideals (see for example [21, Theorem 6.3.8], [29, Section I.11], among many others). We can use this to significantly simplify the description of C^*-envelopes for operator systems that we find in a finite-dimensional C^*-algebra.

Proposition 1.37. *Let A be a finite-dimensional C^*-algebra, and $E \subseteq A$ be a concrete operator system. Then $C^*_{env}(E) \cong C^*(E)$.*

Proof. Let $\pi : C^*(E) \mapsto C^*_{env}(E)$ be the surjective *-homomorphism induced by the universal property of $C^*_{env}(E)$. Specifically, $\ker \pi \subseteq C^*(E)$ is an ideal. Since $C^*(E)$ is finite-dimensional, we can write it as

$$C^*(E) = \bigoplus_{i=1}^{m} A_i$$

for A_i simple. But $A_i \cap \ker \pi$ is an ideal in A_i, and since A_i is simple, this means that either $A_i \subseteq \ker \pi$ or $A_i \cap \ker \pi = \{0\}$.

Suppose there is an $A_i \subseteq \ker \pi$ (without loss of generality, assume it to be A_m). Since $\pi|_E : E \longrightarrow C^*_{env}(E)$ is isometric, we must have that $\ker \pi \cap E = \{0\}$. But then $E \cap A_m = \{0\}$, so that

$$E \subseteq \bigoplus_{i=1}^{m-1} A_i =: A'.$$

But then $E \subseteq A' \subseteq C^*(E)$, with A' a C^*-algebra, so by the fact that E generates $C^*(E)$, we have that $A' = C^*(E)$, and $A_m = 0$.

So for all A_i we have $A_i \cap \ker \pi = \{0\}$ for all A_i, which means that $\ker \pi = \{0\}$. So π is a *-isomorphism. □

In other words, *if* an operator system has a finite-dimensional extension, then it's envelope is also finite-dimensional, and all finite-dimensional extensions are isomorphic. This corresponds nicely to the intuition that the C^*-envelope is somehow a 'minimal' extension.

When E can be seen as a subspace of a finite-dimensional space, it is definitely finite-dimensional. The converse, however, is in general not true: for example, if we take the self-adjoint subspace E spanned by the function $z \mapsto z$ in $C(S^1)$, (i.e. the subspace spanned by z, $\overline{z} = z^{-1}$ and the unit), then $C^*(z, \overline{z}) = C(S^1)$ since the Laurent series are dense in the continuous functions on S^1. Moreover, the ideals in $C(S^1)$ are vanishing ideals $I_C = \{f \in C(S^1) \mid f(C) = \{0\}\}$ for some closed subset $C \subseteq S^1$. But, for $z = e^{\phi i}$ we have

$$|1 - e^{\theta i} z| = 1 - e^{(\theta-\phi)i} - e^{(\phi-\theta)i} + 1 = 2 - 2\cos(\theta - \phi)$$

so that the Šilov boundary of the subspace is the entire space S^1, making the Šilov boundary ideal of E the trivial ideal $\{0\}$. So $C^*_{env}(E) \cong C^*(z, \overline{z}) = C(S^1)$, and so all C^* extensions of E are infinite-dimensional, even though E is finite-dimensional.

1.5 The Propagation number

As we have seen throughout this chapter, an operator system can be viewed as a subspace of a C^*-algebra that is not necessarily closed under multiplication (while still respecting the other structures of the C^*-algebra). In [8], Connes and Van Suijlekom define a measure of how far an operator system is from being a C^*-algebra.

The idea is that, given an operator system, we extend it by allowing products of two elements in the operator system. This is in general a larger space, and might or might not be closed under multiplication.

Definition 1.38. *Let $E \subseteq C^*_{env}(E)$ be the concrete operator system given by the C^*-envelope of E. We set*

$$E^{\circ n} = \overline{\text{span}}\{e_1 \cdot e_2 \cdot \ldots \cdot e_n \mid e_i \in E \text{ for all } 1 \leq i \leq n\}.$$

*The **propagation number** $\text{prop}(E)$ is the smallest number n such that $E^{\circ n} = C^*_{env}(E)$, if it exists. Otherwise, $\text{prop}(E) = \infty$.*

Proposition 1.39. *Let E and F be completely order isomorphic operator systems. Then $\text{prop}(E) = \text{prop}(F)$.*

Proof. We prove that $\text{prop}(E) \leq \text{prop}(F)$, since the reverse inequality follows in exactly the same way. Let $\varphi : E \longrightarrow F$ be the complete order isomorphism. Note that we have the commuting diagram

$$\begin{array}{ccc} E & \xhookrightarrow{i_E} & C^*_{env}(E) \\ \Big\downarrow \varphi & & \Big\uparrow \widehat{i_F \circ \varphi} \\ F & \xhookrightarrow{i_F} & C^*_{env}(F) \end{array}$$

where $\widehat{i_F \circ \varphi}$ is the map given by the universal property of $C^*_{env}(E)$. Since $\widehat{i_F \circ \varphi}$ respects multiplication and is continuous, we have that

$$\widehat{i_F \circ \varphi}(i_F(F)^{\circ n}) = \widehat{i_F \circ \varphi}(i_F(F))^{\circ n} = \widehat{i_F \circ \varphi} \circ i_F(F)^{\circ n}$$

and then by the commutativity of the diagram and invertibility of φ we have

$$\widehat{i_F \circ \varphi}(i_F(F)^{\circ n}) = i_E \circ \varphi(F)^{\circ n} = i_E(E)^{\circ n}$$

So if $n \geq \text{prop}(F)$ then $i_F(F)^{\circ n} = C^*_{env}(F)$ and so $i_E(E)^{\circ n} = C^*_{env}(E)$ by surjectivity. So $n \geq \text{prop}(F)$ implies $n \geq \text{prop}(E)$, meaning that $\text{prop}(F) \geq \text{prop}(E)$. □

By Proposition 1.39 we can see that the propagation number is an invariant for operator systems. In [8] Connes and Van Suijlekom actually show that the propagation number is also invariant under stable equivalence, which is stronger than complete order isomorphism. Interestingly, this would follow from

Theorem 2.23 (which we will prove later), were it not for the fact that the compact operators are non-unital, and in this thesis we only concerned ourselves with unital operator systems. A proof of Theorem 2.23 for non-unital operator systems would therefore directly show that the propagation number is invariant under stable equivalence.

Chapter 2

Tensor product and C^*-algebras

After having laid some foundations in the previous chapter, we turn our attention to answering the principal question of this thesis: how does the propagation number of operator systems behave under the tensor product? We will proceed as follows:

- First, we examine the definition of the tensor product for operator systems, and discuss some basic properties;
- Next, we describe the Šilov boundary ideal of the tensor product of operator systems in terms of the Šilov ideals of the factors (specifically, we identify a map in terms of the factors' Šilov ideals whose kernel we show is the Šilov ideal). From this we can immediately construct the C^*-envelope;
- Finally, we use the C^*-envelope to derive an expression for the propagation number of the tensor product of two operator systems.

2.1 Tensor Products in C^*-algebras, operator spaces and operator systems

2.1.1 Tensor product of C^*-algebras

For vector spaces A and B we can form the tensor product vector space $A \odot B$ (see Appendix A). If A and B are in addition *-algebras, we can enrich the vector space structure on $A \odot B$ to a *-algebra structure, which we will also denote by $A \odot B$; we do this by setting

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2)$$

$$(a \otimes b)^* = a^* \otimes b^*.$$

The natural question then arises: by giving $A \odot B$ a norm, can it also be completed to a C^*-algebra? The answer turns out to be yes. In fact, there are in general multiple norms with which we can endow $A \odot B$ so it completes to a C^*-algebra. In this thesis we will be examining the *minimal* tensor product norm. When $A \odot B$ is supplied with this norm and completed, we will denote it as $A \otimes B$. Note that in other literature $A \otimes B$ is often the *algebraic* tensor product, with $A \otimes_{\min} B$ being its completion with respect to the minimal tensor product norm.

In order to define the minimal tensor norm, we first examine a concrete construction. Given operators $A_i \in B(H_i)$ for $i = 1, 2$, where H_i are Hilbert spaces, we can define a map $A_1 \otimes A_2 : x \otimes y \mapsto A_1(x) \otimes A_2(y)$ on elementary tensors in $H_1 \otimes H_2$, and then extend linearly to the whole space. We can do this because by linearity it is well-defined on the dense subspace $H_1 \odot H_2 \subseteq H_1 \otimes H_2$, and it is easily verified that on that space it is continuous, so that we can extend continuously. We therefore have an element $A_1 \otimes A_2 \in B(H_1 \otimes H_2)$, meaning that we can consider $B(H_1) \odot B(H_2) \subseteq B(H_1 \otimes H_2)$. As a subspace of a normed space $B(H_1) \odot B(H_2)$ carries a norm, and the norm-closure of this space will be denoted by $B(H_1) \otimes B(H_2)$. It can be shown that $B(H_1) \otimes B(H_2) = B(H_1 \otimes H_2)$ as C^*-algebras.

The above norm, which is defined on subspaces of $B(H_1 \otimes H_2)$, is called the *spatial* tensor norm. As we have seen in the previous chapter, all C^*-algebras arise as a closed *-subalgebra of some $B(H)$. So given $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ C^*-algebras, we let $A_1 \otimes A_2$ be the norm-closed linear span of elementary tensors $a_1 \otimes a_2$, with $a_1 \in A_1$ and $a_2 \in A_2$. In principle, this construction depends on the chosen $B(H_i)$, but the following Proposition shows that actually this tensor product norm only depends on the internal C^*-algebra structure.

Proposition 2.1. *Let $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ be C^*-algebras, and let $\|\cdot\|_{\text{spat}}$ be the spatial tensor norm on $A_1 \otimes A_2$. Then*

$$\|x\|_{\text{spat}} = \sup\{\|\pi_1 \otimes \pi_2(x)\| \mid \pi_i \text{ representation of } A_i,\, i = 1, 2\}$$

A proof can for example be found in [25, section 4.1], [29, section IV.4] and [22, chapter 12]. In the context of abstract C^*-algebras, where an explicit reference to a faithful representation on a Hilbert space is not made, this is called the **minimal** tensor product. Often the above abstract characterisation of this norm is used as the definition, after which it is shown that the two definitions are equivalent.

2.1.2 Tensor product of operator spaces/systems

As discussed in the previous chapter, we can view operator spaces and operator systems either as concrete subspaces of C^*-algebras, or as (*-)vector spaces with additional structure on their matrix spaces. As such, the tensor product of two operator spaces/systems can also be defined in either of these settings. In this thesis we will restrict ourselves to introducing the tensor product in the concrete context. For an abstract characterization of the tensor product of operator systems, see [19].

Recall that operator spaces are simply closed subspaces of C^*-algebras. So in order to define a tensor product on operator spaces, we need to define a C^*-algebra and a subspace in it. Given $E \subseteq A$ and $F \subseteq B$ operator spaces, we set

$$E \otimes F := \overline{E \odot F} \subseteq A \otimes B.$$

Since this construction is based on the minimal tensor norm on $A \otimes B$, we will call this *the minimal tensor product of operator spaces.* The exact same applies to operator systems.

The following results shows that the resulting structure indeed depends only on the structures of the factors. The proofs require a little more operator space theory and therefore fall a bit outside the scope of this thesis, but they can be found in [22, Corrolary 12.4] and [24, Sections 1,2.1 and 2.2].

Proposition 2.2. *Let E_1 and E_2 be operator spaces, and let $\|\cdot\|_{\min}$ be the norm of $E_1 \otimes E_2$. Then*

$$\|x\|_{\min} = \sup\{\|L_1 \otimes L_2(x)\| \mid L_i : E_i \longrightarrow B(H_i), \|L_i\|_{cb} \leq 1, i = 1, 2\}$$

If E_1 and E_2 are operator systems, then

$$\|x\|_{\min} = \sup\{\|L_1 \otimes L_2(x)\| \mid L_i : E_i \longrightarrow B(H_i) \text{ completely positive}\}$$

Lemma 2.3. *In the above expression we restrict the suprememum to those L_i for which H_i is finite.*

As noted in Appendix A, we can naturally identify $E \odot M_n \cong M_n(E)$. Using the definitions above, we can see that this correspondence actually goes further for operator spaces.

Lemma 2.4. *For an operator space E we have*

$$E \otimes M_n \cong M_n(E)$$

as normed spaces.

Proof. For a concrete operator space $E \subseteq B(H)$, this follows directly from the isometric identifications

$$B(H \otimes \mathbb{C}^n) \cong B(H) \otimes B(\mathbb{C}^n) \cong B(H) \otimes M_n \cong M_n(B(H))$$

and the fact that $E \otimes M_n$ gets its norm from the left-hand side, and $M_n(E)$ from the right hand side. □

Recall furthermore that for linear spaces we have that the space of linear maps $X \longrightarrow Y$ can be identified with $X^* \odot Y$. This still holds in nice way for operator spaces. The proof of Proposition 2.5 is a compilation of results referenced in [24, Section 2.3].

Proposition 2.5. *Let E_1 and E_2 be operator spaces. Then the map*

$$E_1 \odot E_2 \hookrightarrow CB(E_1^d, E_2), \ \sum x_i \otimes y_i \mapsto (e \mapsto \sum e(x_i)y_i)$$

extends to a completely isometric map on $E_1 \otimes E_2$.

Proof. Write $z := \sum x_i \otimes y_i$, and denote $f_z : e \mapsto \sum e(x_i)y_i$. Note that for $e \in M_n(E_1^d)$ with $\|e\| \leq 1$ we have

$$f_z^{(n)}(e) = (f_z(e_{kl})) = (\sum_i e_{kl}(x_i)y_i)_{k,l=1}^n \in M_n(E_2) \cong M_n \otimes E_2.$$

So we consider the map $R[e] \otimes I_{E_2} : E_1 \otimes E_2 \longrightarrow M_n \otimes E_2$, and using the fact that $\|R[e]\|_{cb} = \|e\|$ by definition of the operator space structure on the dual space, we see that

$$\|f_z^{(n)}(e)\| = \|R[e] \otimes I_{E_2}(z)\| \leq \|z\|_{\min}$$

by Proposition 2.2. So $\|f_z\|_{cb} \leq \|z\|_{\min}$ (which in particular means that we indeed map into $CB(E_1^d, E_2)$).

Conversely, let $L_1 : E_1 \longrightarrow M_n$ and $L_2 : E_2 \longrightarrow M_m$ be linear maps with $\|L_i\|_{cb} \leq 1$. Then under the identification $M_n \otimes M_m \cong M_n(M_m)$ we have for $z = \sum x_i \otimes y_i$ that

$$\|L_1 \otimes I_{E_2}(z)\| = \|\sum L_1(x_i) \otimes y_i\| = \|(\sum L_1(x_i)_{kl}y_i)_{k,l=1}^n\|$$

$$= \|f_z^{(n)}((L_1(\cdot)_{kl})_{k,l=1}^n)\| \leq \|f_z\|_{cb}\|(L_1(\cdot)_{kl})_{k,l=1}^n\|$$

since $(L_1(\cdot)_{kl})_{k,l=1}^n \in M_n(E_1^d)$. But we have normed $M_n(E_1^d) = M_n(CB(E_1, \mathbb{C}))$ through $CB(E_1, M_n)$ so that indeed $\|(L_1(\cdot)_{kl})_{k,l=1}^n\| = \|L_1\| \leq 1$ and $\|L_1 \otimes I_{E_2}(z)\| \leq \|f_z\|_{cb}$. We now see that

$$\|L_1 \otimes L_2(z)\| \leq \|I_{E_1} \otimes L_2\|\|L_1 \otimes I_{E_2}(z)\| \leq \|L_1 \otimes I_{E_2}(z)\| \leq \|f_z\|_{cb}$$

so taking the supremum on the left side, we see that $\|z\|_{min} = \|f_z\|_{cb}$. We have therefore proven that f is isometric, and by replacing E_2 by $M_n \otimes E_2$ we see that

$$M_n \otimes E_1 \otimes E_2 \cong E_1 \otimes M_n \otimes E_2 \cong E_1 \otimes M_n(E_2)$$

$$\longrightarrow CB(E_1^d, M_n(E_2)) \cong M_n(CB(E_1^d, E_2))$$

which proves that f is indeed a *complete* isometry. □

2.2 Ideals and homomorphisms in the algebraic tensor product

The universal property of the tensor product of vector spaces can be extended to the following:

Proposition 2.6. *Let A,B and C be (not necessarily unital) *-algebras, and $f : A \longrightarrow C$ and $g : B \longrightarrow C$ be *-homomorphisms. Let $\widehat{h_{f,g}} : A \odot B \longrightarrow C$ be the linear map induced by the bilinear map*

$$h_{f,g} : A \times B \longrightarrow C, (a, b) \mapsto f(a) \cdot g(b).$$

1. *$\widehat{h_{f,g}}$ is a *-homomorphism if and only if $[f(A), g(B)] = 0$.*
2. *If A, B are unital, then every *-homomorphism $h' : A \odot B \longrightarrow C$ is given by $\widehat{h_{f,g}}$ for $f(a) = h(a \otimes 1)$ and $g(b) = h(1 \otimes b)$.*

Proof. 1. By the universal property of the tensor product of vector spaces, $\widehat{h_{f,g}}$ is a linear map. We see that

$$\widehat{h_{f,g}}((a \otimes b)^*) - (\widehat{h_{f,g}}(a \otimes b))^* = (f(a)g(b))^* - f(a^*)g(b^*).$$

Since $(a^*)^* = a$ and $(f(a)g(b))^* = g(b^*)f(a^*)$, we see that $[f(A), g(B)] = 0$ is a necessary condition. To see that it is sufficient, note that

$$\begin{aligned}\widehat{h_{f,g}}((a_1 \otimes b_1)(a_2 \otimes b_2)) &= f(a_1a_2)g(b_1b_2) = f(a_1)g(b_1)f(a_2)g(b_2)\\ &= \widehat{h_{f,g}}(a_1 \otimes b_1)\widehat{h_{f,g}}(a_2 \otimes b_2)\end{aligned}$$

so that $\widehat{h_{f,g}}$ is indeed a *-homomorphism.

2. It is easily verified that both $f : a \mapsto h(a \otimes 1)$ and $g : b \mapsto h(1 \otimes b)$ are *-homomorphisms, and $h_{f,g}(a, b) = h(a \otimes b)$, so that indeed $\widehat{h_{f,g}} = h$.

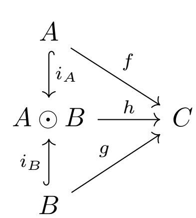
□

For general non-unital *-algebras, Proposition 2.6.2 does not necessarily hold: take for example $A = B = C = \mathbb{C}$ as a vector space, for clarity written as $\mathbb{C}x_1$ for a generator x_1, with multiplication $x_1 \cdot x_1 = 0$, and $(\lambda x_1)^* = \overline{\lambda} x_1$. Then take the *-homomorphism

$$h : A \odot B \longrightarrow C, \qquad \lambda x_1 \otimes \mu x_1 \mapsto \lambda\mu x_1.$$

Since no nonzero element in C can be expressed as the product of two elements, we can never have that $h(a \otimes b) = f(a)g(b)$, so h can never be of the form described in Proposition 2.6. However, note that the algebraic structure in this example cannot be made into a C^*-algebra, since that would mean that $\|x^*x\| = \|0\| = 0 = \|x\|^2$ for all x.

So from now on, let A and B be unital *-algebras. We now know that *-homomorphisms $h : A \odot B \longrightarrow C$ correspond exactly to pairs of maps $f : A \longrightarrow C$ and $g : B \longrightarrow C$ such that $[f(a), g(b)] = 0$ for all $a \in A$ and $b \in B$, according to the following commutative diagram:



Here i_A denotes the inclusion $i_A(a) = a \otimes 1$, and similarly $i_B(b) = 1 \otimes b$. From this, we can immediately conclude the following:

Proposition 2.7. *Let $h : A \odot B \longrightarrow C$ be a *-homomorphism, and let $f = h \circ i_A$ and $g = h \circ i_B$. Let $I = \ker f$ and $J = \ker g$. Then*

$$I \odot B + A \odot J \subseteq \ker h.$$

Correspondingly, every ideal in $A \odot B$ contains an ideal of the form $I \odot B + A \odot J$.

Proof. Let $\sum x_i \otimes y_i \in I \odot B + A \odot J$. Since $h(\sum x_i \otimes y_i) = \sum f(x_i)g(y_i)$, and in each term either $f(x_i) = 0$ or $g(y_i) = 0$, we have that $\sum x_i \otimes y_i \in \ker h$. The last claim follows from the fact that each ideal is the kernel of a homomorphism. □

If C is actually a tensor product itself, with the maps given by the inclusion into each factor, we can say more.

Proposition 2.8. *Let $h : A \odot B \longrightarrow A' \odot B'$ given by $f : A \longrightarrow A' \hookrightarrow A' \odot B'$ and similarly $g : B \longrightarrow B' \hookrightarrow A' \odot B'$. Let $I = \ker f$ and $J = \ker g$. Then*

$$\ker h = I \odot B + A \odot J$$

Proof. By Lemma A.1, we have that if $z \in \ker h$, then it must be a linear combination of elements $x_i \otimes y_i$ where $x_i \in \ker f$ or $y_i \in \ker g$, proving the statement. □

2.2.1 Completing the algebraic tensor product

Having considered the purely algebraic side in the previous section, i.e. *-algebras and algebraic tensor products, we now expand our view to C^*-algebras and the minimal tensor product. First, we note that maps on the algebraic tensor product can at least be extended to the completion:

Proposition 2.9. *Let $\pi_1 : A_1 \longrightarrow B_1$ and $\pi_2 : A_2 \longrightarrow B_2$ be *-homomorphisms between C^*-algebras. Then the homomorphism $\pi_1 \odot \pi_2 : A_1 \odot A_2 \longrightarrow B_1 \odot B_2$ extends to a *-homomorphism $A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2$.*

Proof. The proof is given in [29, Prop. 4.22]. □

A natural question is now whether Proposition 2.8 can be extended to this setting. However, it is not at all clear that

$$\ker \pi_1 \otimes \pi_2 = \overline{\ker \pi_1 \odot \pi_2}. \tag{2.1}$$

In fact, in [30, Theorem 5] Tomiyama shows that this is true for all π_1 *-homomorphism on A and π_2 *-homomorphism on B if and only if $A \otimes_{\min} B$ satisfy a property Tomiyama calls (F). In [20], several equivalent formulations of this property are given. It could still be possible that while property (F) might not hold, meaning that not *all* *-homomorphisms π_1 and π_2 satisfy (2.1), that it *does* hold for the specific projections to the Šilov ideals. However, I am not aware of evidence pointing in this direction. So there is no indication that we might be able to express the Šilov ideal of $E \otimes F$ in terms of the Šilov ideals of the factors.

Instead, we use the quotient maps given by the Šilov ideals; in the next section, we will construct the Šilov ideal as the kernel of the tensor product of these quotient maps, which in turn allows us to characterize the C^*-envelope of the tensor product.

2.3 Identifying the Šilov ideal

For any C^*-algebra A and ideal $I \subseteq A$, let $q_I : A \longrightarrow A/I$ be the canonical quotient map. In this section, we will identify the Šilov ideal of the tensor product of two operator systems making use of such quotient maps. Let $E \subseteq A$ and $F \subseteq B$ be concrete operator spaces, where $A = C^*(E)$ and $B = C^*(F)$. In the rest of this chapter, I, J and K are the Šilov ideals for E, F, and $E \otimes F$ respectively. The goal is to show that the ideal $\ker q_I \otimes q_J$ satisfies the properties of a Šilov ideal, so that we must have that $K = \ker q_I \otimes q_J$.

2.3.1 Isometric quotient

First, we show that $\ker q_I \otimes q_J$ is a boundary ideal. So, using the notation described above, we wish to show that the map

$$q_{\ker q_I \otimes q_J} : A \otimes B \longrightarrow \frac{A \otimes B}{\ker q_I \otimes q_J}$$

is isometric on $E \otimes F$. We derive this from the fact that $q_I \otimes q_J$ is isometric on $E \otimes F$ since I and J are Šilov ideals, and so in particular boundary ideals.

Lemma 2.10. *Let A, B be C^*-algebras, and $\pi : A \longrightarrow B$ a *-homomorphism. Let $E \subseteq A$ be a closed subspace. If $\pi|_E$ is completely isometric, then so is $q_{\ker \pi}|_E$*

Proof. Note that the $*$-isomorphism $\widehat{\pi} : A/\ker \pi \xrightarrow{\sim} \pi(A)$ induced by π satisfies $\pi = \widehat{\pi} \circ q_{\ker \pi}$, meaning that $q_{\ker \pi} = \widehat{\pi}^{-1} \circ \pi$. By assumption π_E is completely isometric, and $\widehat{\pi}^{-1}$ is a *-isomorphism, so in particular it is completely isometric. So $q_{\ker \pi}|_E$ is completely isometric as the composition of completely isometric maps. □

Lemma 2.11. *Let $\phi : E_1 \longrightarrow E_2$ be a completely isometric map. Then its pushforward $\phi_* : CB(F, E_1) \longrightarrow CB(F, E_2)$ given by $f \mapsto \phi \circ f$ is completely isometric.*

Proof. Note that with the induced map

$$(\phi_*)^{(n)} : M_n(CB(F, E_1)) \longrightarrow M_n(CB(F, E_2))$$

we can construct the commuting diagram

$$\begin{array}{ccc} M_n(CB(F,E_1)) & \xrightarrow{(\phi_*)^{(n)}} & M_n(CB(F,E_2)) \\ \big\downarrow\sim & & \sim\big\uparrow \\ CB(F,M_n(E_1)) & \xrightarrow{(\phi^{(n)})_*} & CB(F,M_n(E_2)) \end{array}$$

where we have used the (by definition isometric) bijections given in Lemma 1.18. We note that

$$\|(\phi^{(n)})_*(f)\|_{cb} = \|\phi^{(n)} \circ f\|_{cb} = \sup\{\|(\phi^{(n)} \circ f)^{(m)}\|_m \mid 0 \leq m\}$$

and

$$\|(\phi^{(n)} \circ f)^{(m)}\|_m = \sup\{\|\phi^{(nm)}(f^{(m)}(x))\|_m \mid x \in M_m(F), \|x\|_m \leq 1\}.$$

Because ϕ is completely isometric, $\phi^{(nm)}$ is isometric, and so $\|(\phi^{(n)} \circ f)^{(m)}\|_m = \|f^{(m)}\|_m$. So $(\phi^{(n)})_*$ is isometric, and $(\phi_*)^{(n)}$ is then the composition of isometric maps, making it isometric. So ϕ_* is completely isometric. □

The proof of Proposition 2.12 is based on [5, Sec. 1.5.1].

Proposition 2.12. *If $\phi : E_1 \longrightarrow E_2$ and $\psi : F_1 \longrightarrow F_2$ are complete isometries between operator spaces, then $\phi\otimes\psi : E_1\otimes F_1 \longrightarrow E_2\otimes F_2$ is a complete isometry.*

Proof. First, note that $\phi \otimes \psi = \phi \otimes I_{F_2} \circ I_{E_1} \otimes \psi$, where I_{E_1} and I_{F_2} is the identity on E_1 and F_2, respectively. It therefore suffices to show that $\phi \otimes I_{F_2}$ is a complete isometry.

So let $\phi : E_1 \longrightarrow E_2$ be a complete isometry, and F another operator space. Then because ϕ is completely isometric, it induces the completely isometric pushforward

$$\phi_* : CB(F^*, E_1) \longrightarrow CB(F^*, E_2), f \mapsto \phi \circ f.$$

Using the correspondence in Proposition 2.5, we have the commuting diagram

$$\begin{array}{ccc} CB(F^*,E_1) & \xrightarrow{\phi_*} & CB(F^*,E_2) \\ {\scriptstyle i_1}\big\uparrow & & {\scriptstyle i_2}\big\uparrow \\ E_1 \otimes F & \xrightarrow{\phi\otimes I_F} & E_2 \otimes F \end{array}$$

Since $\phi_* \circ i_1$ is completely isometric as the composition of completely isometric maps, so $i_2 \circ \phi \otimes I_F$ is also completely isometric. So $\phi \otimes I_F$ is completely isometric. □

Corollary 2.13. *Let I, J and K be the Šilov ideals for the operator spaces E, F and $E \otimes F$ in A, B and $A \otimes B$, respectively. Let $q_I : A \longrightarrow A/I$ and $q_J : B \longrightarrow B/J$ be the canonical projections. Then*

$$\ker q_I \otimes q_J \subseteq K.$$

Proof. Since I is a Šilov ideal, we have that q_I is completely isometric on E, and similarly q_J is completely isometric on F. By Proposition 2.12 we have therefore that $q_I \otimes q_J$ is completely isometric on $E \otimes F$. Then by Lemma 2.10 we have that $q_{\ker q_I \otimes q_J}$ is completely isometric on $E \otimes F$. Since the Šilov ideal is the maximal ideal for which the quotient is isometric on the operator space, we must have that $\ker q_I \otimes q_J \subseteq K$. □

2.3.2 Maximality

Since the Šilov boundary ideal is the *maximal* ideal for which the quotient is isometric on the operator space in question, we now need to show that the Šilov ideal is included in the ideal $\ker q_I \otimes q_J$. Recall from section 1.4 that we can characterise the Šilov ideal as an intersection of kernels of boundary representations. These boundary representations behave nicely with respect to the minimal tensor product. Let ∂E be the set of boundary representations for an operator system E.

Lemma 2.14. *For $E \subseteq A$, $F \subseteq B$ as above, we have that $\sigma_1 \in \partial E$ and $\sigma_2 \in \partial F$ implies $\sigma_1 \otimes \sigma_2 \in \partial(E \otimes F)$.*

Proof. This is Lemma 3 in [18]. □

Note that this means that

$$\bigcap_{\sigma \in \partial(E \otimes F)} \ker \sigma \subseteq \bigcap_{(\sigma_1, \sigma_2) \in \partial E \times \partial F} \ker \sigma_1 \otimes \sigma_2.$$

We will now need some technical results. The proof of Lemma 2.15 is rather long and technical, but can be found in [26].

Lemma 2.15 (Kirchberg's slice lemma)**.** *Let A, B be C^*-algebras, and let $D \subseteq A \otimes B$ be a hereditary C^*-subalgebra. Then there is a $z \in A \otimes B$ such that $zz^* \in D$ and $z^*z = a \otimes b$ for $a \in A$ and $b \in B$.*

Lemma 2.16. *Let A and B be C^*-algebras. Every nonzero ideal $I \subseteq A \otimes B$ contains a nonzero elementary tensor $a \otimes b$ for some $a \in A$, $b \in B$.*

Proof. Since every ideal is in particular a hereditary C^*-subalgebra, by Kirchberg's slice lemma we can find a $z \in I$ such that $zz^* \in I$ and $z^*z = a \otimes b$. Then

$$a^2 \otimes b^2 = (z^*z)^2 = z^*zz^*z \in z^*Iz \subseteq I$$

so there is indeed an elementary tensor in I. □

The proof of Lemma 2.17 is adapted from [20, Lemma 2.2]. That proof references [4, Lemma 2.12 (i)], which in turn references a statement in [28]. However, I was not able to precisely locate this statement or its proof in [28], so instead we prove the statement used in [20, Lemma 2.2] through Lemma 2.16.

Lemma 2.17. *Let A and B be C^*-algebras, and let $\mathcal{K}$ and $\mathcal{L}$ be families of ideals in A and B, respectively. Now define*

$$I = \bigcap_{K \in \mathcal{K}} K \text{ and } J = \bigcap_{L \in \mathcal{L}} L$$

Then

$$\ker q_I \otimes q_J = \bigcap \{\ker q_K \otimes q_L \mid (K, L) \in \mathcal{K} \times \mathcal{L}\}$$

Proof. For readability, we will write $M = \bigcap \{\ker q_K \otimes q_L \mid (K, L) \in \mathcal{K} \times \mathcal{L}\}$.

- First, we prove the inclusion $\ker q_I \otimes q_J \subseteq M$. For all $K \in \mathcal{K}$ and $L \in \mathcal{L}$, we have that $I \subsetneq K$ and $J \subsetneq L$. So we can apply the isomorphisms $\rho : (A/I)/(K/I) \xrightarrow{\sim} A/K$ and $\sigma : (B/J)/(L/J) \xrightarrow{\sim} B/L$ to see that

  $$q_K \otimes q_L = (\rho \otimes \sigma)(q_{K/I} \otimes q_{L/J})(q_I \otimes q_J)$$

  so $\ker q_I \otimes q_J \subseteq \ker q_K \otimes q_L$, proving the inclusion.

- Second, for the inclusion $\ker q_I \otimes q_J \supseteq M$, we define the seminorm

  $$N(x) = \sup\{\|q_{K/I} \otimes q_{L/J}(x)\| \mid (K, L) \in \mathcal{K} \times \mathcal{L}\}.$$

  on $A/I \otimes B/J$. It is easily seen that $N(xy) \leq N(x)N(y)$ and $N(x^*x) = N(x)^2$. Note that if $N(x) = 0$ if and only if $x \in \ker q_{K/I} \otimes q_{L/J}$ for all $(K, L) \in \mathcal{K} \times \mathcal{L}$. In particular we have that $\ker N$ is the intersection of ideals, and so it is an ideal itself.

  Note that if $[x] \otimes [y] \in A/I \otimes B/J$ is nonzero, then there are K and L such that $x \notin K$ and $y \notin L$. But

  $$\begin{aligned} 0 \neq q_K \otimes q_L(x \otimes y) &= (\rho \otimes \sigma)(q_{K/I} \otimes q_{L/J})(q_I \otimes q_J)(x \otimes y) \\ &= (\rho \otimes \sigma)(q_{K/I} \otimes q_{L/J})([x] \otimes [y]) \end{aligned}$$

  so

  $$N([x] \otimes [y]) \geq \|(q_{K/I} \otimes q_{L/J})([x] \otimes [y])\| = \|q_K \otimes q_L(x \otimes y)\| > 0.$$

  Specifically, $\ker N$ does not contain any simple tensors. But by Lemma 2.16 this means that $\ker N$ is trivial, and so N is actually a C^*-norm. But those are unique, so $N(x) = \|x\|$ on $A/I \otimes B/J$.

  Finally, as we remarked above, we have

  $$q_K \otimes q_L(x) = (\rho \otimes \sigma)(q_{K/I} \otimes q_{L/J})(q_I \otimes q_J)(x).$$

  So if $x \in M$, then the left-hand side is zero for all $(K, L) \in \mathcal{K} \times \mathcal{L}$, and $(q_I \otimes q_J)(x) \in \ker q_{K/I} \otimes q_{L/J}$ for all $(K, L) \in \mathcal{K} \times \mathcal{L}$ because $\rho \otimes \sigma$ is injective. We therefore have that

  $$\|q_I \otimes q_J(x)\| = N(q_I \otimes q_J(x)) = 0,$$

  so $x \in \ker q_I \otimes q_J$, so that indeed $M \subseteq \ker q_I \otimes q_J$.

In conclusion, we have that $M = \ker q_I \otimes q_J$, proving the lemma. $\square$

This finishes the heavy lifting: in the following Proposition we simply use that we know that the Šilov ideal is the intersection of specific kernels, so that we can apply Lemma 2.17.

Proposition 2.18. *Let $E \subseteq A$ and $F \subseteq B$ be concrete operator systems, and q_I and q_J the quotients by their Šilov ideals. Then we have that*

$$\ker q_I \otimes q_J = \bigcap_{(\sigma_1,\sigma_2)\in\partial E\times\partial F} \ker \sigma_1 \otimes \sigma_2.$$

Proof. Note that for the Šilov ideals I and J we have $I = \ker q_I$ and $J = \ker q_J$, but also that $I = \bigcap_{\sigma_1\in\partial E} \ker \sigma_1$ and $J = \bigcap_{\sigma_2\in\partial F} \ker \sigma_2$.

For any $\sigma_1 \in \partial E$ and $\sigma_2 \in \partial F$, we can apply Proposition 1.4 to induce *-isomorphisms $\widehat{\sigma_i}$ such that $\widehat{\sigma_i} \circ q_{\ker \sigma_i} = \sigma_i$, from which we conclude that

$$\ker[q_{\ker \sigma_1} \otimes q_{\ker \sigma_2}] = \ker[(\widehat{\sigma_1} \otimes \widehat{\sigma_2}) \circ (q_{\ker \sigma_1} \otimes q_{\ker \sigma_2})] = \ker \sigma_1 \otimes \sigma_2.$$

since the tensor product of two *-isomorphism is again a *-isomorphism. We can now apply Lemma 2.17 to conclude that

$$\ker q_I \otimes q_J = \bigcap_{(\sigma_1,\sigma_2)\in\partial E\times\partial F} \ker \sigma_1 \otimes \sigma_2$$

proving the statement. $\square$

We can now combine our results with the results of the previous section to conclude the following:

Proposition 2.19. *Let $E \subseteq A$ and $F \subseteq B$ be concrete operator systems, with corresponding Šilov ideals I and J. Also, let K be the Šilov ideal corresponding to $E \otimes F \subseteq A \otimes B$. Furthermore, let $q_I : A \longrightarrow A/I$ and $q_J : B \longrightarrow B/J$ be the canonical projections. Then*

$$K = \ker q_I \otimes q_J.$$

Proof. By Corollary 2.13, we have that $\ker q_I \otimes q_J \subseteq K$. Also, by Lemma 2.14 we have that

$$\{\sigma_1 \otimes \sigma_2 \mid (\sigma_1, \sigma_2) \in \partial E \times \partial F\} \subseteq \partial(E \otimes F),$$

so, using Proposition 2.18, we can see that

$$K = \bigcap_{\sigma\in\partial E\otimes F} \ker \sigma \subseteq \bigcap_{(\sigma_1,\sigma_2)\in\partial E\times\partial F} \ker \sigma_1 \otimes \sigma_2 = \ker q_I \otimes q_J$$

which proves the theorem. $\square$

Theorem 2.20. *Let E and F be operator systems. Then*

$$C^*_{env}(E \otimes F) \cong C^*_{env}(E) \otimes C^*_{env}(F)$$

Proof. Consider $E \subseteq C^*_{env}(E)$, $F \subseteq C^*_{env}(F)$ and therefore $E \otimes F \subseteq C^*_{env}(E) \otimes C^*_{env}(F)$. This in particular means that the Šilov ideals of E and F are trivial. By Proposition 2.19 we then have that the Šilov ideal of $E \otimes F$ is equal to $\ker q_0 \otimes q_0 = 0$. The result follows directly. □

2.4 Propagation number of the tensor product

Because of Theorem 2.20, in order to calculate the propagation number of $E \otimes F$, we can consider $E \otimes F \subseteq C^*_{env}(E) \otimes C^*_{env}(F)$ as a concrete operator system.

Lemma 2.21. *For $E \subseteq A$ and $F \subseteq B$ concrete operator systems. Then*

$$E^{(n)} \otimes F^{(n)} = (E \otimes F)^{(n)} \subseteq A \otimes B$$

Proof. First, for $(E \otimes F)^{(n)} \subseteq E^{(n)} \otimes F^{(n)}$, we note that

$$\left(\sum_i e_{1i} \otimes f_{1i}\right)\left(\sum_j e_{2j} \otimes f_{2j}\right) = \sum_{i,j} (e_{1i}e_{2j}) \otimes (f_{1i}f_{2j}).$$

So

$$\{x_1 x_2 \cdots x_n \mid x_i \in E \odot F\} \subseteq E^{(n)} \odot F^{(n)} \subseteq E^{(n)} \otimes F^{(n)}.$$

Also, if for $1 \leq i \leq n$ we have that $(x_{ij})_{j=1}^{\infty}$ is a sequence converging to x_i, then by continuity of multiplication we have that $x_{1j}x_{2j} \dots x_{nj} \longrightarrow x_1 x_2 \dots x_n$ as $j \to \infty$. We therefore have that

$$\{x_1 x_2 \cdots x_n \mid x_i \in E \otimes F\} \subseteq \overline{\{x_1 x_2 \cdots x_n \mid x_i \in E \odot F\}}.$$

Since $E^{(n)} \otimes F^{(n)}$ is closed, we can conclude that

$$\{x_1 x_2 \cdots x_n \mid x_i \in E \otimes F\} \subseteq E^{(n)} \otimes F^{(n)}.$$

Conversely, we prove that $E^{(n)} \otimes F^{(n)} \subseteq (E \otimes F)^{(n)}$. An element of the form $\sum_i (e_{i1}e_{i2} \dots e_{in}) \otimes (f_{i1}f_{i2} \dots f_{in})$ can be seen to lie in the linear span of elements of the form $(e_{i1} \otimes f_{i1})(e_{2i} \otimes f_{2i}) \dots (e_{in} \otimes f_{in}) \in (E \otimes F)^{(n)}$. So also $E^{(n)} \odot F^{(n)} \subseteq (E \otimes F)^{(n)}$. The inclusion now follows by noting that $(E \otimes F)^{(n)}$ is closed. □

Note also that for $m \geq \text{prop}(E)$, then $E^{(m)} = C^*_{env}(E)$. Using the following observation, we can answer the principal question of this thesis.

Lemma 2.22. *Let A and B be C^*-algebras, and $S \subsetneq A$ a proper closed subspace. Then $S \otimes B \subsetneq A \otimes B$ is a proper subspace. If $T \subsetneq B$ is also a proper closed subspace, then $S \otimes T \subsetneq A \otimes B$ is a proper subspace.*

Proof. Take $x \in A \setminus S$. Then by the standard separation result for Locally Convex Spaces (e.g. [9, IV.3.15]) we can find a functional $f_x \in A^*$ such that $f_x(S) = 0$ and $f_x(x) = 1$. Take any $b \in B$ nonzero together with $f_b \in B^*$ such that $f_b(b) = 1$. Then $f_x \otimes f_b : A \otimes B \longrightarrow \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ is again continuous; however, $f_x \otimes f_b(x \otimes b) = 1$ and $f_x \otimes f_b(S \odot B) = \{0\}$. So by continuity we have that $f_x \otimes f_b(S \otimes B) = \{0\}$ and therefore $x \otimes b \in (A \otimes B) \setminus (S \otimes B)$. The second statement follows from the fact that a subspace of a proper subspace is a proper subspace. □

Theorem 2.23. *Let E and F be operator systems. If* $\mathrm{prop}(E), \mathrm{prop}(F) < \infty$ *then*

$$\mathrm{prop}(E \otimes F) = \max\{\mathrm{prop}(E), \mathrm{prop}(F)\}.$$

If either $\mathrm{prop}(E) = \infty$ *or* $\mathrm{prop}(F) = \infty$*, then* $\mathrm{prop}(E \otimes F) = \infty$.

Proof. If $m \geq \max\{\mathrm{prop}(E), \mathrm{prop}(F)\}$, then we have that

$$(E \otimes F)^{(m)} = E^{(m)} \otimes F^{(m)} = C^*_{env}(E) \otimes C^*_{env}(F)$$

so $\mathrm{prop}(E \otimes F) \leq \max\{\mathrm{prop}(E), \mathrm{prop}(F)\}$. Conversely, if we have that $m := \mathrm{prop}(E \otimes F) < \max\{\mathrm{prop}(E), \mathrm{prop}(F)\}$, then either $E^{(m)} \subsetneq C^*_{env}(F)$ or $F^{(m)} \subsetneq C^*_{env}(F)$. So we can use Lemma 2.22 to conclude that $E^{(m)} \otimes F^{(m)} \subsetneq C^*_{env}(E) \otimes C^*_{env}(F)$, and conclude that $m = \mathrm{prop}(E \otimes F) < \mathrm{prop}(E \otimes F)$, which is a contradiction. So $\mathrm{prop}(E \otimes F) \geq \max\{\mathrm{prop}(E), \mathrm{prop}(F)\}$, and we have proven the theorem. □

Chapter 3

Examples of operator systems

In this chapter, we will examine some examples of operator systems that arise from the C^*-algebras $C(S^1)$ and $C(C_k)$, i.e. the continuous functions on the circle and the cyclic group of order k, respectively, with values in $\mathbb{C}$. They act on $L^2(S^1)$ and $L^2(C_k)$ as multiplication operators. We can use Fourier theory to 'truncate' the C^*-algebras to operator systems: on the one hand, Fourier theory gives a natural basis for $L^2(S^1)$ and $L^2(C_k)$, and we can project to subspaces induced by this basis; on the other hand, through Fourier theory we have the isomorphisms $C(S^1) \cong C^*(\mathbb{Z})$ and $C(C_k) \cong C^*(C_k)$, and we can examine elements with restricted support. In [8] Connes and Van Suijlekom define these operator systems for the S^1 case, and introduce an interesting duality between them, which was recently verified by Farenick in [14]. In this chapter, we will first discuss their definition and sharpen a result used for the calculation of the propagation number, after which we will give a reformulated version of Farenick's proof of the duality. Finally we extend the definitions to C_k, derive analogous results in order to calculate the propagation number, and examine the duality in this context.

3.1 Toeplitz matrices

In general, given a C^*-algebra $A \subseteq B(H)$ represented on a Hilbert space H, we see that any orthogonal projection P on H induces the operator system $PAP \subseteq B(PH)$. In this section we consider the case $A = C(S^1)$, $H = L^2(S^1)$, where A acts on H through multiplication, together with projections P_n defined as follows: let $(\epsilon_i)_{i\in\mathbb{Z}}$ be the orthogonal basis of $L^2(S^1)$ given through the functions $\epsilon_i : z \mapsto z^i$. Then we let P_n be the projection onto $\overline{\text{span}}\{\epsilon_i \mid 0 \leq i \leq n-1\}$. We define

$$C(S^1)^{(n)} := P_n C(S^1) P_n \subseteq B(P_n L^2(S^1)).$$

For any $f \in L^2(S^1)$, let $\hat{f} : \mathbb{Z} \longrightarrow \mathbb{C}$ denote its Fourier transform. Then for $f \in C(S^1) \subseteq L^2(S^1)$ and $g \in L^2(S^1)$ we have $\widehat{fg} = \widehat{f} * \widehat{g}$ (here the * denotes convolution). So in particular

$$fg = \sum_{k=-\infty}^{\infty} \widehat{f} * \widehat{g}(k)\epsilon_k.$$

Additionally, we have that

$$P_n g = \sum_{k=0}^{n-1} \widehat{g}(k)\epsilon_k.$$

We therefore see that we have an isomorphism $P_n L^2(S^1) \cong \mathbb{C}^n$ given by the basis $\epsilon_0, \ldots, \epsilon_{n-1}$. This then in turn induces an isomorphism $B(P_n L^2(S^1)) \cong M_n$, and we now characterize the image of $C(S^1)^{(n)}$ under this correspondence.

Note that we have

$$\widehat{P_n g}(k) = \begin{cases} \hat{g}(k) & \text{if } 0 \leq k \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$\widehat{f} * \widehat{P_n g}(k) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{P_n g}(k-j) = \sum_{j=0}^{n-1} \widehat{g}(j)\widehat{f}(k-j)$$

and

$$P_n f P_n(P_n g) = P_n f(P_n g) = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \hat{f}(k-j)\hat{g}(j)\epsilon_k$$

In other words, in the correspondence induced by the Fourier basis the element $P_n f P_n$ corresponds to the matrix

$$\begin{pmatrix} \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \hat{f}(-3) & \cdots & \hat{f}(-n+1) \\ \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \cdots & \hat{f}(-n+2) \\ \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \cdots & \hat{f}(-n+3) \\ \hat{f}(3) & \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \cdots & \hat{f}(-n+4) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{f}(n-1) & \hat{f}(n-2) & \hat{f}(n-3) & \hat{f}(n-4) & \cdots & \hat{f}(0) \end{pmatrix}.$$

Matrices of this form are called **Toeplitz matrices**, and we will therefore call $C(S^1)^{(n)}$ the operator system of n by n Toeplitz matrices. The identification of elements in $C(S^1)^{(n)}$ with their matrix representation will be made implicitly throughout this chapter.

We define

$$t_i := P_n \epsilon_i P_n \in C(S^1)^{(n)}$$

for $-n+1 \leq i \leq n-1$, which corresponds to the matrix with 1 on the i'th diagonal, and 0 everywhere else. From the characterization as matrices it is immediately clear that the t_i form a basis. Expressed in terms of standard basis elements e_{kl} in M_n, we have that

$$t_i = \sum_{k-l=i} e_{kl}.$$

3.1.1 Propagation number of the Toeplitz matrices

In [8], Connes and Van Suijlekom calculate the propagation number of $C(S^1)^{(n)}$ by giving a general description of elements of the form $t_i t_j$. In this section we fill in some details and show that there is a simple relationship between these types of elements and the standard matrix basis elements e_{kl}.

Note that

$$t_i t_j = \sum_{k-l=i} \sum_{r-s=j} e_{kl} e_{rs} = \sum_{k-l=i} \sum_{r-s=j} \delta_{lr} e_{ks} = \sum_{k-l=i} \sum_{l-s=j} e_{ks}$$

$$= \sum_{\substack{k-s=i+j \\ 1 \leq k-i \leq n}} e_{ks} = \sum_{\substack{k-s=i+j \\ 1 \leq s+j \leq n}} e_{ks}$$

where k, l, r and s are understood to run from 1 to n, subject to the condition under the sum. Specifically, we have

$$t_i t_{j-i} = \sum_{\substack{k-s=j \\ 1 \leq k-i \leq n}} e_{ks} = \sum_{\substack{k-s=j \\ 1 \leq s+j-i \leq n}} e_{ks}.$$

This is just t_j, but with some 1's removed, depending on i and j. Note that since $1 \leq k \leq n$ and $1 \leq s \leq n$, we discern six cases:

1. $0 \leq i \leq j$: here we have $t_i t_{j-i} = t_j$.

2. $i \leq 0 \leq j$: here we have that $t_i t_{j-i}$ has 1's along the j-diagonal, but the last $-i$ places actually have a 0.

3. $i \leq j \leq 0$: here we have that $t_i t_{j-i}$ has 1's along the j-diagonal, but the last $-i + j$ places actually have a 0.

4. $j \leq i \leq 0$: here we have that $t_i t_{j-i} = t_j$.

5. $j \leq 0 \leq i$: here we have that $t_i t_{j-i}$ has 1's along the j-diagonal, but the first i places actually have a 0.

6. $0 \leq j \leq i$: here we have that $t_i t_{j-i}$ has 1's along the j-diagonal, but the first $i - j$ places actually have a 0.

Applying the transformtation $i \mapsto j - i$ we get similar expressions for $t_{i-j} t_i$. We can summarize the multiplication of the Toeplitz basis elements as follows:

Lemma 3.1. *Let $\{t_i\}_{i=-n+1}^{n-1}$ be the standard basis for n by n Toeplitz matrices. Then for $i \leq 0 \leq j$ and $j \leq 0 \leq i$ we have that $t_i t_{j-i}$ is equal to t_j with i zeroes from above, and $t_{j-i} t_i$ is equal to t_j with i zeroes from below (negative amounts of zeroes are counted from the opposite side). For all other basis elements we have $t_a t_b = t_{a+b}$.*

This means that to calculate $t_a t_b$, we do the following: calculate $a + b$, and if either a or b is of opposite sign, we remove the corresponding zeroes from t_{a+b} (otherwise we simply keep t_{a+b}). We then quickly arrive at the following characterization.

Proposition 3.2. *Let $\{t_i\}$ denote the standard basis for the n by n Toeplitz matrices, and $\{e_{kl}\}$ the standard basis for the n by n matrices. Then*

$$e_{kl} = t_{k-n} t_{n-l} + t_{k-1} t_{1-l} - t_{k-l}$$

There is actually a more intuitive way to think about the multiplication of the standard Toeplitz matrices t_i, which we can more easily generalise later. Notice that to each operator $T \in B(L^2(S^1)) \cong B(\ell^2(\mathbb{Z}))$ we can associate the matrix elements $\langle T\epsilon_j, \epsilon_i \rangle$ for $i, j \in \mathbb{Z}$, giving us a map $m_T : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C}$. This can be viewed as an infinite matrix, in the sense that the rows extend infinitely far to the left and right, and the columns extend infinitely far upwards and downwards. Given two operators $T, S \in B(L^2(S^1))$, we have that the matrix corresponding to the operator TS is indeed given by 'matrix multiplication' of m_T and m_S, i.e.

$$m_{TS}(i, j) = \sum_{k \in \mathbb{Z}} m_T(i, k) m_S(k, j).$$

See for example [23, Ex. 3.2.16].

The projection P_n to the subspace spanned by $e_0, \ldots, e_{n-1} \in \ell^2(\mathbb{Z})$ is then given by the map

$$m_{P_n}(i, j) = \begin{cases} 1 & \text{if } 0 \leq i = j < n; \\ 0 & \text{else.} \end{cases}$$

We can display this in accordance with the traditional notion of a matrix as

$$\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & \multicolumn{1}{|c}{1} & 0 & 0 & \multicolumn{1}{c|}{0} & 0 & 0 & 0 & & \\
\cdots & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 1 & 0 & \multicolumn{1}{c|}{0} & 0 & 0 & 0 & \cdots & \\
 & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 0 & 1 & \multicolumn{1}{c|}{0} & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 0 & 0 & \multicolumn{1}{c|}{1} & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & & & & & \vdots & & & & & &
\end{array}$$

where the upper left corner of the square is the $(0,0)$ position, and the lower right corner is the $(n-1,n-1)$ position (in this example we have $n=4$). As can be seen from the discussion at the start of section 3.1, if we see $f \in C(S^1) \subseteq B(L^2(S^1))$ as a multiplication operator, then $m_f(i,j) = \hat{f}(i-j)$. For a cannoncial basis element ϵ_i of $L^2(S^1)$ (which is a continuous function; see the start of section 3.1) the associated map is given by

$$m_{\epsilon_i}(j,k) = \begin{cases} 1 & \text{if } j-k=i; \\ 0 & \text{else.} \end{cases}$$

As in the finite-dimensional case, for any $T \in L^2(S^1)$ we have that the infinite matrix associated to $\epsilon_i T$ is the matrix associated to T, but shifted down i places; in other words, $m_{\epsilon_i T}(j,k) = m_T(j-i,k)$. Similarly we have $m_{T\epsilon_i}(j,k) = m_T(j,k+i)$, which corresponds to shifting the matrix i places to the left.

With this we can describe the multiplication of standard Toeplitz basis elements. Recall that $t_i = P_n \epsilon_i P_n$, so $t_i t_j = P_n \epsilon_i P_n \epsilon_j P_n$. In general, conjugating a map by P_n has the effect of setting all matrix elements outside of the square between $(0,0)$ and $(n-1,n-1)$ to zero. So to calculate the matrix of $t_i t_j$ we start with the matrix for P_n, then shift it i places down and j places to the left, and restrict the matrix to the square with corners $(0,0)$ and $(n-1,n-1)$ (since we make the identification $P_n L^2(S^1) \cong \mathbb{C}^n$).

With this intuition we can try to make the standard matrices e_{kl}: we first shift the matrix of P_n so that the 1's lie on the $k-l$'th diagonal and the lowest 1 lies in the (k,l) spot, after which we cut out the square between $(0,0)$ and $(n-1,n-1)$. Then, do the same, except shift the highest 1 to the (k,l) spot, and add them to form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(here we have taken the example for $k=1$ and $l=2$). What remains is to subtract the full diagonal. Following this recipe indeed gives the formula in Proposition 3.2.

Theorem 3.3. *Let $C(S^1)^{(n)} \subseteq M_n$ be the operator system of n by n Toeplitz matrices. Then $C^*_{env}(C(S^1)^{(n)}) \cong M_n$ and* $\text{prop}(C(S^1)^{(n)}) = 2$.

Proof. By Proposition 3.2, we have that $C^*(C(S^1)^{(n)}) = M_n$. By Proposition 1.37 we then have that $C^*_{env}(C(S^1)^{(n)}) = M_n(\mathbb{C})$. Again by Proposition 3.2, we see that

$$(C(S^1)^{(n)})^{\circ 2} = M_n$$

so that indeed $\text{prop}(C(S^1)^{(n)}) = 2$. $\square$

3.1.2 Block Tensor product

As an application, let us look at **block Toeplitz matrices**. These are matrices of the form

$$\begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-n+2} & A_{-n+1} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-n+3} & A_{-n+2} \\ A_2 & A_1 & A_0 & \cdots & A_{-n+4} & A_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-2} & A_{n-3} & A_{n-4} & \cdots & A_0 & A_{-1} \\ A_{n-1} & A_{n-2} & A_{n-3} & \cdots & A_1 & A_0 \end{pmatrix}$$

where $A_i \in M_m$. In other words, these are matrices A such that

$$A = \sum_{i=-n+1}^{n-1} A_i \otimes t_i \in M_m \otimes C(S^1)^{(n)}.$$

By theorem 2.23 and theorem 3.3 we can now immediately conclude that

$$\text{prop}(M_m \otimes C(S^1)^{(n)}) = \max\{1, 2\} = 2.$$

There is an alternative way to verify this: note that we can view a n by n block Toeplitz matrix with blocks of size m as a nm by nm matrix. It is now easy to verify that the nm by nm Toeplitz matrices are actualy *block* Toeplitz matrices: indeed, for $A \in M_m \otimes C(S^1)^{(n)}$, each $(A_i)_{jk}$ appears on exactly 1 diagonal, so for any Toeplitz matrix $B \in C(S^1)^{(mn)}$ set $(A_i)_{jk}$ equal to the value of B on that diagonal. Then we have found matrices A_i such that the block-Toeplitz matrix A which they compose is equal to B. So indeed

$$C(S^1)^{(mn)} \subseteq C(S^1)^{(n)} \otimes M_m \subseteq M_{nm}.$$

Since clearly $E \subseteq F$ implies $E^{\circ n} \subseteq F^{\circ n}$, we have that

$$M_{nm} = (C(S^1)^{(mn)})^{\circ 2} \subseteq (C(S^1)^{(n)} \otimes M_m)^{\circ 2} \subseteq M_{nm}$$

so that indeed $\text{prop}(C(S^1)^{(n)} \otimes M_m) = 2$. The same argument also works for the more narrow definition of block Toeplitz matrices where the blocks are also assumed to be Toeplitz matrices, i.e. the space $C(S^1)^{(n)} \otimes C(S^1)^{(m)}$, which by theorem 2.23 also must have a propagation number of 2.

3.2 The Fejér-Riesz operator system

As another example of an operator system, consider the convolution C^*-algebra $C^*(\mathbb{Z})$, given as a normed space by $\ell^1(\mathbb{Z})$ and with product

$$(a * b)_i = \sum_{j=-\infty}^{\infty} a_k b_{i-k}.$$

Let $\operatorname{supp}(a) := \sup\{|i| \mid a_i \neq 0\}$. We define the operator system

$$C^*(\mathbb{Z})_{(n)} := \{a \in C^*(\mathbb{Z}) \mid \operatorname{supp}(a) \leq n-1\} \subseteq C^*(\mathbb{Z}).$$

As noted in the previous section, through Fourier theory, an element $f \in C(S^1)$ actually corresponds to an element $\hat{f} \in C^*(\mathbb{Z})$. Through this correspondence we see that $C^*(\mathbb{Z})_{(n)}$ is identified with functions of the form $\sum_{i=-n+1}^{n-1} a_i z^i$, i.e. trigonometric polynomials of degree at most $n-1$.

Proposition 3.4. *The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is equal to $C^*(\mathbb{Z})$, and its propagation number is infinite.*

Proof. Since $C^*(\mathbb{Z}) \cong C(S^1)$ is a commutative C^*-algebra, the traditional notion of the Šilov boundary applies: the Šilov boundary is the minimal set on which every function in $C^*(\mathbb{Z})_{(n)}$ takes its maximal value (in terms of absolute value). Note that

$$\cos(2\pi x) = \tfrac{1}{2}(e^{2\pi i x} + e^{-2\pi i x})$$

and

$$\cos(2\pi(x+\theta)) = \tfrac{1}{2}(e^{2\pi i\theta}e^{2\pi i x} + e^{-2\pi i\theta}e^{-2\pi i x}).$$

So for $\lambda = e^{2\pi i\theta}$, the function $z \mapsto \frac{1}{2}(\lambda z + \overline{\lambda}\overline{z})$ takes its maximal value on λ and $-\lambda$ (as the cosine function takes its maximal values on 0 and π), and so all points must lie in the Šilov boundary. So the Šilov boundary ideal is empty, meaning that $C^*_{env}(C^*(\mathbb{Z})_{(n)}) = C^*(\mathbb{Z})$. Clearly there are functions in $C(S^1)$ that are not a finite polynomial in z and z^{-1}, so all $(C^*(\mathbb{Z})_{(n)})^{\circ m}$ are strictly contained in $C(S^1)$. ☐

Since positivity in an operator space is defined through positivity in the C^*-algebra it lies in, we have that $a \in (C^*(\mathbb{Z})_{(n)})_+$ if and only if the function $z \mapsto \sum_{i=-n+1}^{n-1} a_i z^i$ is positive. However, in order to describe the operator system structure, we need to find positive elements in the matrix spaces. For this we use the following.

Lemma 3.5. *We have a *-isomorphism*

$$M_n(C(S^1)) \cong C(S^1, M_n)$$

where the latter is the space of continuous functions from S^1 to M_n.

Proof. The isomorphism is given by mapping $(f_{ij}) \in M_n(C(S^1))$ to $z \mapsto (f_{ij}(z))$. The latter is indeed a continuous map, since

$$\|(f_{ij}(x)) - (f_{ij}(y))\|^2 \leq \sum_{ij} \|f_{ij}(x) - f_{ij}(y)\|^2$$

so that by taking x and y close enough, their images lie arbitrarily close. The inverse is given by mapping $f \in C(S^1, M_n)$ to $(z \mapsto f(z)_{ij}) \in M_n(C(S^1))$; indeed,

$$\|f(x)_{ij} - f(y)_{ij}\| \leq \|(f(x)_{ij} - f(y)_{ij})\| = \|f(x) - f(y)\|$$

and the latter can be made arbitrarily small due to continuity of f. It is easily verified that these are indeed *-homomorphisms. □

By this identification we have

$$M_n(C^*(\mathbb{Z})_{(m)}) \subseteq M_n(C^*(\mathbb{Z})) \cong M_n(C(S^1)) \cong C(S^1, M_n)$$

and for elements of $M_n(C^*(\mathbb{Z})_{(m)})$ the correspondence looks like

$$M_n(C^*(\mathbb{Z})_{(m)}) \ni (a_{ij})_{i,j=1}^m \longmapsto \left(z \mapsto \sum_{k=-n+1}^{n-1} z^k ((a_{ij})_k)_{i,j=1}^m \right)$$

(note that $((a_{ij})_k)_{i,j=1}^m$ is a m by m matrix of complex numbers for each $k \in \{-n+1, \ldots, n-1\}$, so the right hand side is a polynomial with matrix coefficients). We can now introduce a characterization, originally due to Fejér and Riesz for $H = \mathbb{C}$ (hence the name for the operator system and the theorem).

Theorem 3.6 (Operator Fejér-Riesz)**.** *Let H be a Hilbert space, and let $F_k \in B(H)$ for $k \in \{-n+1, \ldots, n-1\}$ be such that*

$$F(z) = \sum_{k=1-n}^{n-1} z^k F_k \geq 0$$

for all $z \in S^1$. Then there are operators $G_k \in B(H)$ for $k \in \{0, \ldots, n-1\}$ such that

$$G(z) = \sum_{k=0}^{n-1} z^k G_k$$

satisfies $G(z)^* G(z) = F(z)$ for all z.

A proof can for example be found in [11], and is out of the scope of this thesis. For the interested reader, the proof for $H = \mathbb{C}$ in [16] is conceptually very clear and easy to follow.

3.3 Duality

Examining $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$, we note that both are subspaces of dimension $2n-1$. As such, one might wonder whether there is a relation between the two. They are certainly not isomorphic (we simply need to look at their propagation numbers or C^*-envelopes to conclude this), but it turns out that the two are **dual** to each other:

Theorem 3.7. *For all $n \in \mathbb{N}$, we have that $C(S^1)^{(n)} \cong (C^*(\mathbb{Z})_{(n)})^d$ as operator systems, i.e. they are completely order isomorphic.*

In this section we will prove the result. It was first stated in [8], but only proven that the two are order isomorphic (instead of *completely* order isomorphic). During work on generalizing the given proof to the completely order isomorphic case by the author, as luck would have it, [14] was posted, with precisely the solution to this problem. The results in this section are a largely along the lines of their proof, presented from a slightly different perspective with different notation.

First, for $T \in C(S^1)^{(n)}$, we will write $T_i \in \mathbb{C}$ for the element on the i'th diagonal. Through this we will also identify T with the element in $C^*(\mathbb{Z})_{(n)}$ given by T_i for $-n+1 \leq i \leq n-1$ (and 0 elsewhere). There is then a non-degenerate pairing

$$\phi : C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \longrightarrow \mathbb{C}\ ,\ (T, a) \mapsto \sum_{i=-n+1}^{n-1} T_{-i} a_i = (T * a)_0$$

By abuse of notation, we will denote the map $T \mapsto \phi_T$ also as ϕ. We will show that this is a complete order isomorphism.

At this point it is important to make some remarks about the notation. Elements of $C(S^1)^{(n)}$ are matrices, but we will often write T_i for the value on the i-th diagonal as we have done here. Through this, we can also identify T as an element in $C^*(\mathbb{Z})_{(n)}$. Moreover, we can identify the set

$$C^*(\mathbb{Z})^A_{(n)} := \{a \in C^*(\mathbb{Z})_{(n)} \mid a_{-n+1} = a_{-n+2} = \ldots = a_{-1} = 0\}$$

with $\mathbb{C}^n$ (here the A stands for 'analytic', since the set corresponds to those Laurent polynomials in $C(S^1)$ that are analytic on $\mathbb{C}$). We index the entries of the matrices in $M_n(\mathbb{C})$ and $\mathbb{C}^n$ from 0 to $n-1$, so that this correspondence is given by $a \leftrightarrow (a_i)_{i=0}^{n-1}$.

Lemma 3.8. *Let $a, b \in C^*(\mathbb{Z})^A_{(n)} \subseteq C^*(\mathbb{Z})$ and $T \in C(S^1)^{(n)}$. Furthermore, let ϕ be the pairing above. Then*

$$\phi_T(b^* * a) = \langle Ta, b\rangle.$$

Proof. As noted above, we have by the associativity and commutativity of the convolution that

$$\phi_T(b^* * a) = (T * b^* * a)_0 = (b^* * T * a)_0 = \sum_{i=0}^{n-1} \overline{b_i}(T * a)_i$$

Also, we note that

$$(Ta)_i = \sum_{j=0}^{n-1} T_{ij} a_j = \sum_{j=-n+1}^{n-1} a_j T_{i-j} = (a * T)_i = (T * a)_i$$

so that indeed

$$\phi_T(b^* * a) = \sum_{i=0}^{n-1} \overline{b_i}(T * a)_i = \sum_{i=0}^{n-1} \overline{b_i}(Ta)_i = \langle Ta, b\rangle.$$

□

We will now try to generalize this to the matrix spaces of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$, i.e. the spaces $C(S^1)^{(n)} \otimes M_p$ and $C^*(\mathbb{Z})_{(n)} \otimes M_p$ (recall from Lemma 2.4 that these spaces are isomorphic to $M_p(C(S^1)^{(n)})$ and $M_p(C^*(\mathbb{Z})_{(n)})$). All algebras have canonical bases, in which we can express the elements (see Appendix A). So in general we have

$$C(S^1)^{(n)} \otimes M_p \ni \mathcal{T} = \sum_{i=-n+1}^{n-1} t_i \otimes \mathcal{T}_i = \sum_{j,k=0}^{p-1} \mathcal{T}_{kl} \otimes e_{kl}$$

for $\mathcal{T}_i \in M_p$ and $\mathcal{T}_{kl} \in C(S^1)^{(n)}$ and

$$C^*(\mathbb{Z})_{(n)} \otimes M_p \ni \mathcal{A} = \sum_{i=-n+1}^{n-1} \delta_i \otimes \mathcal{A}_i = \sum_{j,k=0}^{p-1} \mathcal{A}_{jk} \otimes e_{jk}$$

for $\mathcal{A}_i \in M_p$ and $\mathcal{A}_{jk} \in C^*(\mathbb{Z})_{(n)}$, meaning that $\mathcal{A}$ is either a collection of matrices indexed by $\mathbb{Z}$, or a matrix of sequences in $C^*(\mathbb{Z})_{(n)}$. For example, since the adjoint is defined as the adjoint on each factor in the tensor product, we have

$$(\mathcal{A}^*)_i = (\mathcal{A}_{-i})^* \text{ and } (\mathcal{A}^*)_{jk} = (\mathcal{A}_{kj})^*$$

The map ϕ induces maps

$$\phi^{(p)} := \phi \otimes I_{M_p} : C(S^1)^{(n)} \otimes M_p \longrightarrow (C^*(\mathbb{Z})_{(n)})^d \otimes M_p$$

and again, for $\mathcal{T} \in C(S^1)^{(n)} \otimes M_p$ we write $\phi^{(p)}_{\mathcal{T}} \in (C^*(\mathbb{Z})_{(n)})^d \otimes M_p$. So in particular we have

$$\phi^{(p)}_{\mathcal{T}} = \sum_{j,k=0}^{p-1} \phi_{\mathcal{T}_{jk}} \otimes e_{jk},$$

or in other words, $(\phi^{(p)}_{\mathcal{T}})_{jk} = \phi_{\mathcal{T}_{jk}}$.

The standard inclusion $C^*(\mathbb{Z})_{(n)} \subseteq C^*(\mathbb{Z})$ can be generalized to the matrix algebras, so that we get an induced multiplication, which we shall write as $*$. Applying the above we then get for $\mathcal{A}, \mathcal{B} \in C^*(\mathbb{Z})_{(n)} \otimes M_p$

$$(\mathcal{A} * \mathcal{B})_i = \sum_{j\in\mathbb{Z}} \mathcal{A}_j \cdot \mathcal{B}_{i-j}$$

$$(\mathcal{A} * \mathcal{B})_{jk} = \sum_{l=0}^{p-1} \mathcal{A}_{jl} * \mathcal{B}_{lk}$$

Finally, we define the space

$$M_p(C^*(\mathbb{Z})_{(n)})^A := \{\mathcal{A} \in M_p(C^*(\mathbb{Z})_{(n)}) \mid \mathcal{A}_{jk} \in C^*(\mathbb{Z})^A_{(n)}\}$$

The isomorphism $C^*(\mathbb{Z})^A_{(n)} \cong \mathbb{C}^n$ means that each element $\mathcal{A}_{ij}$ associated to an element $\mathcal{A} \in M_p(C^*(\mathbb{Z})_{(n)})$ corresponds to an element in $\mathbb{C}^n$, which means we can construct vectors

$$\vec{\mathcal{A}}_i = \sum_{j=0}^{p-1} \mathcal{A}_{ij} \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^p$$

for $0 \leq i \leq n-1$.

Recall from Proposition 1.25 that we have

$$M_n(E^*)_+ = \{\psi \in M_n(E^*) \mid S[\psi] \text{ is a positive functional}\}.$$

We can now generalize Lemma 3.8.

Proposition 3.9. *Let* $\mathcal{A}, \mathcal{B} \in M_p(C^*(\mathbb{Z})_{(n)})^A$, *and* $\mathcal{T} \in M_p(C(S^1)^{(n)}$. *We then have*

$$S[\phi^{(p)}_{\mathcal{T}}](\mathcal{B}^* * \mathcal{A}) = \sum_{i=0}^{p-1} \langle \mathcal{T}(\vec{\mathcal{A}}_i), \vec{\mathcal{B}}_i \rangle.$$

Proof. Note that

$$S[\phi^{(p)}_{\mathcal{T}}](\mathcal{B}^* * \mathcal{A}) = \sum_{i,j=0}^{n-1} (\phi^{(p)}_{\mathcal{T}})_{ij}((\mathcal{B}^* * \mathcal{A})_{ij}) = \sum_{i,j=0}^{p-1} \phi_{\mathcal{T}_{ij}}((\mathcal{B}^* * \mathcal{A})_{ij})$$

$$= \sum_{i,j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{\mathcal{T}_{ij}}((\mathcal{B}^*)_{ik} * \mathcal{A}_{kj}) = \sum_{i,j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{\mathcal{T}_{ij}}((\mathcal{B}_{ki})^* * \mathcal{A}_{kj})$$

We now use the fact that we can view $\mathcal{A}_{kj}$ and $\mathcal{B}_{ki}$ as elements of $\mathbb{C}^n$ to invoke Lemma 3.8 to conclude that

$$S[\phi^{(p)}_{\mathcal{T}}](\mathcal{B}^* * \mathcal{A}) = \sum_{i,j=0}^{p-1} \sum_{k=0}^{p-1} \langle \mathcal{T}_{ij} \mathcal{A}_{kj}, \mathcal{B}_{kj} \rangle.$$

We also note that

$$\langle \mathcal{T}(\vec{\mathcal{A}}_k), \vec{\mathcal{B}}_k \rangle = \sum_{i,j=0}^{p-1} \langle (\mathcal{T}_{ij} \otimes e_{ij}) \vec{\mathcal{A}}_k), \vec{\mathcal{B}}_k) \rangle = \sum_{i,j=0}^{p-1} \langle \mathcal{T}_{ij} \mathcal{A}_{kj}, \mathcal{B}_{kj} \rangle$$

so that the result follows □

Summarizing what we have until now, we see that for any $\mathcal{T} \in M_p(C(S^1)^{(n)})$ we know the action of $S[\phi^{(p)}_{\mathcal{T}}]$ on elements in $M_p(C^*(\mathbb{Z})_{(n)})$ of a specific form, namely $\mathcal{B}^* * \mathcal{A}$ with $\mathcal{A}, \mathcal{B} \in M_p(C^*(\mathbb{Z})_{(n)})^A$. We want to show that $\phi^{(p)}$ is positive, and so we need to show that for positive $\mathcal{T} \in M_p(C(S^1)^{(n)})$ the functional $S[\phi^{(p)}_{\mathcal{T}}]$ is positive on positive elements of $M_p(C^*(\mathbb{Z})_{(n)})$. Of course every positive element $\mathcal{A}$ in $M_p(C^*(\mathbb{Z})_{(n)})$ can be written as $\mathcal{B}^* * \mathcal{B}$, since it is in particular a positive element of the C^*-algebra $M_p(C^*(\mathbb{Z}))$, but it would be very convenient if we could choose $\mathcal{B} \in M_p(C^*(\mathbb{Z})_{(n)})^A$ so that we can use 3.9. Luckily, this is precisely what the operator-valued Fejér-Riesz lemma (theorem 3.6) tells us:

Proposition 3.10. *Suppose $\mathcal{A} \in M_p(C^*(\mathbb{Z})_{(n)})_+$. Then there exists a $\mathcal{B} \in M_p(C^*(\mathbb{Z})_{(n)})^A$ such that*

$$\mathcal{A} = \mathcal{B}^* * \mathcal{B}.$$

Proof. We note that $\mathcal{A}$ corresponds to a positive element in $C(S^1, M_p)$ given by

$$z \mapsto \sum_{i=-n+1}^{n-1} \mathcal{A}_i z^i.$$

By the operator-valued Fejér-Riesz lemma, we now have that there exist $\mathcal{B}_i \in M_p$ for $0 \leq i \leq m-1$ such that

$$\sum_{i=-n+1}^{n-1} \mathcal{A}_i z^i = \left(\sum_{i=0}^{n-1} \mathcal{B}_i z^i\right)^* \left(\sum_{i=0}^{n-1} \mathcal{B}_i z^i\right) = \sum_{i=-n+1}^{n-1} (\mathcal{B}^* * \mathcal{B})_i z^i$$

by defining $\mathcal{B} = \sum_{i=0}^{n-1} \delta_i \otimes \mathcal{B}_i \in M_p(C^*(\mathbb{Z})_{(n)})^A$. □

Corollary 3.11. *The map $\phi : C(S^1)^{(n)} \longrightarrow (C^*(\mathbb{Z})_{(n)})^*$ given by*

$$T \mapsto \left(\phi_T : a \mapsto \sum_{i=-n+1}^{n-1} T_{-i} a_i\right)$$

is completely positive.

Proof. For any $p \in \mathbb{N}$, let $\mathcal{T} \in M_p(C(S^1)^{(n)})_+$. We need to show that $\phi_{\mathcal{T}}^{(p)}$ is positive, i.e. that for $\mathcal{A} \in M_p(C^*(\mathbb{Z})_{(n)})$ we have $S[\phi_{\mathcal{T}}^{(p)}](\mathcal{A}) \geq 0$. But by Proposition 3.10 we have that $\mathcal{A} = \mathcal{B}^* * \mathcal{B}$ for some $\mathcal{B} \in M_p(C^*(\mathbb{Z})_{(n)})$ with $B_{-i} = 0$ for $i > 0$. But then we have

$$S[\phi_{\mathcal{T}}^{(p)}](\mathcal{A}) = \sum_{i=0}^{p-1} \langle \mathcal{T}\vec{\mathcal{B}}_i, \vec{\mathcal{B}}_i \rangle \geq 0$$

because $\mathcal{T}$ is assumed to be positive. So indeed, $\phi^{(p)}$ is positive, and therefore ϕ is completely positive. □

Proof of Theorem 3.7. Clearly the map ϕ is bijective, with inverse

$$\left(f : a \mapsto \sum_{i=-n+1}^{n-1} f_i a_i\right) \mapsto \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-n+1} \\ f_1 & f_0 & \cdots & f_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-1} & f_{n-2} & \cdot & f_0 \end{pmatrix}$$

for $f \in C^*(\mathbb{Z})_{(n)}$. We need to show that ϕ is a complete order isomorphism. The fact that ϕ is completely positive was shown in corollary 3.11.

We claim that if ϕ^{-1} is positive, then it is actually completely positive. Indeed, if ϕ^{-1} is positive, then by Proposition 1.31 we have that $(\phi^{-1})^d : (C(S^1)^{(n)})^d \longrightarrow C^*(\mathbb{Z})_{(n)}$ is positive. But this can be viewed as a positive map into a commutative C^*-algebra, so that by corollary 1.23 it is actually completely positive. Now we apply Proposition 1.31 again to conclude that ϕ^{-1} is completely positive.

So it remains to be shown that ϕ^{-1} is positive. Note that $\phi^{-1}(\phi_T) = T$ by definition, and every element of $(C^*(\mathbb{Z})_{(n)})^d$ is of the form ϕ_T for some T. Note also that if $x \in \mathbb{C}^n$ then we can view it as an element of $C^*(\mathbb{Z})^A_{(n)}$ through

$$\begin{cases} x_i & \text{for } 0 \leq i \leq n-1 \\ 0 & \text{for } 1-n \leq i < 0 \end{cases}$$

(we will write this element as x by abuse of notation). Then $x^* * x$ is a positive element in $C^*(\mathbb{Z})_{(n)} \subseteq C^*(\mathbb{Z})$. So if ϕ_T is a positive element in $(C^*(\mathbb{Z})_{(n)})^d$, then by Lemma 3.8 we have

$$0 \leq \phi_T(x^* * x) = \langle Tx, x\rangle$$

and so $T = \phi^{-1}(\phi_T)$ is positive. So ϕ^{-1} is a positive map. □

3.4 $C(C_k)^{(n)}$

The above examples are based on the fact that we can develop Fourier theory on S^1. There is however also a finite Fourier theory on C_k, the cyclic group of order k, and we can apply the same constructions. For a more in depth introduction to finite Fourier theory, see for example [27, Ch. 7].

Consider first $C(C_k)$, the continuous functions from C_k to $\mathbb{C}$, and $L^2(C_k)$, the square-integrable functions from C_k to $\mathbb{C}$. Since C_k is a finite discrete space, any function $f : C_k \longrightarrow \mathbb{C}$ is in $C(C_k)$ and in $L^2(C_k)$. Given some principal k'th root of unity ζ, it turns out that any function $f : C_k \longrightarrow \mathbb{C}$ can be written as

$$f(n) = \sum_{m=0}^{k-1} \hat{f}(m)\zeta^{nm}$$

for some function $\hat{f} : C_k \longrightarrow \mathbb{C}$ (see [27, Thm. 7.1.2]). In other words, the functions $n \mapsto \zeta^{mn}$ form a basis for the space $L^2(C_k)$.

As in Section 3.1, we can cut off the functions in Fourier space: let $H = L^2(C_k)$, and let $\epsilon_i = (n \mapsto \zeta^{in})$ be the Fourier basis. Note that the index set can be considered to be C_k, with $\epsilon_i\epsilon_j = \epsilon_{i+j}$. Define P_n as the orthogonal projection to the space spanned by $\epsilon_0, \dots \epsilon_{n-1}$. We define the operator system

$$C(C_k)^{(n)} := P_n C(C_k) P_n \subseteq B(P_n L^2(C_k)) \cong B(\mathbb{C}^n)$$

Comparable to the infinite-dimensional case, we have

$$\widehat{P_n g}(m) = \begin{cases} \hat{g}(m) & \text{for } 0 \leq m \leq n-1 \\ 0 & \text{for } n \leq m \leq k-1 \end{cases}$$

so that

$$\widehat{fP_n g}(m) = \sum_{i=0}^{n-1} \hat{g}(i)\hat{f}(m-i).$$

Identifying $P_n L^2(C_k) \cong \mathbb{C}^n$ means that we can view $P_n f P_n$ as an element of the n by n matrices, specifically the matrix

$$\begin{pmatrix} \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \hat{f}(-3) & \cdots & \hat{f}(-n+1) \\ \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \cdots & \hat{f}(-n+2) \\ \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \cdots & \hat{f}(-n+3) \\ \hat{f}(3) & \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \cdots & \hat{f}(-n+4) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{f}(n-1) & \hat{f}(n-2) & \hat{f}(n-3) & \hat{f}(n-4) & \cdots & \hat{f}(0) \end{pmatrix}$$

However, note that $\hat{f} : C_k \longrightarrow \mathbb{C}$, so there is a k-periodicity in the diagonals. Specifically, $\dim C(C_k)^{(n)} = \min\{2n-1, k\}$. Also, if $k = n$, then $C(C_k)^{(n)}$ are actually the circulant matrices. Since there are only k basis vectors in $L^2(C_k)$ we restrict the definition to $k \geq n$, because if $k < n$ we cannot construct a space with n basisvectors to project to.

3.4.1 Propagation number of $C(C_k)^{(n)}$

Recall from section 3.1.1 that we can directly express the canonical matrix basis $\{e_{kl}\}$ as a linear combination of products of Toeplitz matrices, which allows us to directly calculate the propagation number. It turns out that we can follow a similar approach for $C(C_k)^{(n)}$, as we will show in this section. However, the construction is slightly more involved because not all t_i are in $C(C_k)^{(n)}$.

We define basis elements $c_i = P_n \epsilon_i P_n$, which can be expressed as

$$c_i = \begin{cases} t_i + t_{i+k} & \text{if } -n+1 \leq i < n-k \\ t_i & \text{if } n-k \leq i \leq k-n \end{cases}$$

We extend the definition of c_i by reducing i modulo k to one of the cases above.

An operator $T \in B(L^2(C_k))$ defines a map $m_T : C_k \times C_k \longrightarrow \mathbb{C}$ through $m_T(i,j) = \langle T\epsilon_j, \epsilon_i \rangle$. By composing this map with the projection $\mathbb{Z} \longrightarrow C_k$ this then also defines a map $m_T : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C}$, and so we can draw grids to represent such operators as we did for operators on $L^2(S^1)$. Note however that this map is not the infinite matrix for some operator $T \in B(L^2(S^1))$ because the rows and columns are not square-summable as was the case in section 3.1.1 (see [23, Ex. 3.2.16]). For example, the map $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{C}$ associated to $P_n \in B(L^2(C_k))$ can be displayed as

$$
\begin{array}{c}
\vdots \\
\cdots \;
\begin{array}{ccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{4-7}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \cline{4-7}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
\; \cdots \\
\vdots
\end{array}
$$

where the square has upper left corner $(0, 0)$ and lower right corner $(n-1, n-1)$, here with $n = 4$ and $k = 5$. These infinite 'matrices' have a k-periodicity in both directions, and any square with side k is a fundamental domain. Note that through the identification $P_n L^2(C_k) \cong \mathbb{C}^n$ any operator $P_n T P_n$ for $T \in B(L^2(C_k))$ is also associated to an n by n matrix, which can by found by taking the infinite matrix as above, and restricting it to the square with corners $(0, 0)$ and $(n - 1, n - 1)$.

As far as multiplication is concerned, we have

$$m_{TS}(i, j) = \langle TS\epsilon_j, \epsilon_i\rangle = \sum_{l \in C_k} \langle T\epsilon_l, \epsilon_i\rangle\langle S\epsilon_j, \epsilon_l\rangle = \sum_{l \in C_k} m_T(i, l)m_S(l, j).$$

So the multiplying an operator T by ϵ_i from the left (where ϵ_i is seen as a multiplication operator) has the effect of shifting the infinite matrix associated to T i places to the left. Similarly, multiplying by ϵ_i from the right has the effect of shifting the matrix down i places.

So in order to calculate $c_i c_j = P_n\epsilon_i P_n\epsilon_j P_n$, we note that the infinite matrix associated to $\epsilon_i P_n\epsilon_j$ is the same as the one associated to P_n, but shifted i places to the left and j places down. Conjugating with P_n has the effect of setting all elements in the fundamental domain $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i, j \leq k - 1\}$ but outside of $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i, j \leq n - 1\}$ to zero. When considering $c_i c_j$ as a finite matrix (i.e. making the correspondence $P_n L^2(C_k) \cong \mathbb{C}^n$) we then restrict to the square with corners $(0, 0)$ and $(n - 1, n - 1)$.

The main idea of the rest of this section is the following: the difference between $c_{i+p}c_{-p}$ and c_i is only nonzero on the i'th diagonal, and there it is a sequence of at most $k - n$ ones. With enough of these sequences we can recreate the partial diagonals $t_{i+p}t_{-p}$ we encountered in section 3.1.1, and use the formula for the Toeplitz matrices to recreate the standard matrix basis elements e_{ab}.

For notational convenience, we define the partial diagonals

$$AD_n(i; a_1, a_2, \ldots)$$

as the matrices which are zero everywhere except on the i'th diagonal (here we count the main diagonal as 0, diagonals below 0 as positive and diagonals above 0 as negative, so that the notation agrees with the start of this section), and on the i'th diagonal we have ones in the a_1'th place, the a_2'th place, etc. Here we count the places from above, and the first place is 1. For example,

$$AD_5(1;1,2,4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In order to simplify notation later on, we will allow a_i to be smaller than 1 or larger than the length of the diagonal; these positions will simply be ignored. We also define $BD_n(i;b_1,b_2,\ldots)$ as the same but counting from below, so

$$BD_5(1;1,2,4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Lemma 3.12. *In $C(C_k)^{(n)}$, we have that*

$$\begin{aligned} c_{i+p}c_{-p} &= c_i - BD_n(i;p-(k-n)+1,\ldots,p) \\ c_{-p}c_{i+p} &= c_i - AD_n(i;p-(k-n)+1,\ldots,p) \\ c_{-i-p}c_p &= c_i - AD_n(-i;p-(k-n)+1,\ldots,p) \\ c_p c_{-i-p} &= c_i - BD_n(-i;p-(k-n)+1,\ldots,p) \end{aligned}$$

for $0 \le i \le n-1$ *and* $0 \le p \le n-|i|$.

Proof. We prove the first equation, the rest are proven similarly. Note that

$$c_{i+p}c_p = P_n\epsilon_{i+p}P_n\epsilon_p P_n = P_n\epsilon_p\epsilon_i P_n\epsilon_p P_n.$$

Also note that the infinite matrix associated to P_n has sequences of $k-n$ zeroes on the $k\cdot\mathbb{Z}$ diagonals. The matrix associated to $\epsilon_i P_n$ is the same, but shifted to the left i places. This means that the ones are on the i'th diagonal, on which there now are $k-n$ zeroes below the $(0,0)$ to $(n-1,n-1)$ square.

We now move the matrix diagonally p places (left and up) to arrive at $\epsilon_p\epsilon_i P_n\epsilon_{-p}$: this means that the sequence of $k-n$ zeroes has now shifted p places into the square with corners $(0,0)$ and $(n-1,n-1)$. So on the i'th diagonal $c_i - BD_n(i;p-(k-n)+1,\ldots,p)$ and $c_{i+p}c_{-p}$ agree.

What remains is to check the possible nonzero $i-k$'th diagonal, which we have when $i \ge k-n$. The $i-k$'th diagonal is of length $n-|i-k| = n-k+i$. But when the matrix is shifted i places left, there are $n-(n-k+i) = k-i$ ones left to the upper left of $i-k$'th diagonal. But $k \ge n$, so $0 \le p \le n-i \le k-i$, so we never shift the zeroes into the $(0,0)$ to $(n-1,n-1)$-square. □

We can rephrase the results in 3.1.1 to see that

$$\begin{aligned} t_{i+p}t_{-p} &= AD_n(i;1,\ldots,n-i-p) \\ t_pt_{-i-p} &= BD_n(i;1,\ldots,n-i-p) \\ t_{-i-p}t_p &= AD_n(-i;1,\ldots,n-i-p) \\ t_{-p}t_{i+p} &= BD_n(-i;1,\ldots,n-i-p) \end{aligned}$$

for i and p positive (note that the i'th diagonal has $n-|i|$ elements, and we shift in p zeroes). We also note that we can cover a part of the diagonal in multiple steps, i.e.

$$BD_n(i;1,\ldots,p) = \sum_{j=0}^{m_p-1} BD_n(i;p+1-(k-n)(j+1),\ldots,p-(k-n)j)$$

for $m_p = \lceil p/(k-n)\rceil$ (and similarly for AD_n). Combining this with Lemma 3.12, we have the following:

Lemma 3.13. *Let $c_j \in C(C_k)^{(n)}$ and $t_j \in C(S^1)^{(n)}$. For $0 \le i \le n-1$ and $0 \le p \le n-|i|$, define $p' = n-i-p$. Then*

$$t_{i+p}t_{-p} = \sum_{j=0}^{m_{p'}-1} c_i - c_{-p'-j(k-n)}c_{i+p'+j(k-n)}$$

$$t_{-p}t_{i+p} = \sum_{j=0}^{m_{p'}-1} c_i - c_{i+p'+j(k-n)}c_{-p'-j(k-n)}$$

$$t_{-i-p}t_p = \sum_{j=0}^{m_{p'}-1} c_{-i} - c_{-i-p'-j(k-n)}c_{p'+j(k-n)}$$

$$t_pt_{-i-p} = \sum_{j=0}^{m_{p'}-1} c_{-i} - c_{p'+j(k-n)}c_{-i-p'-j(k-n)}$$

where $m_{p'} = \lceil (n-i-p)/(k-n)\rceil$.

In particular we have

$$t_i = t_{i+0}t_0 = \sum_{j=0}^{m_{n-i}} c_i - c_{i-n-j(k-n)}c_{n+j(k-n)}$$

$$t_{-i} = \sum_{j=1}^{m_{n-i}} c_{-i} - c_{-n-j(k-n)}c_{n-i+j(k-n)}$$

In essence we are now done, because we can now apply Proposition 3.2 to write every standard matrix-basis element as a linear combination of products of elements in $C(C_k)^{(n)}$. We summarize this in the following:

Proposition 3.14. *Let $1 \leq a \leq n$ and $1 \leq b \leq n$. Then*

$$e_{ab} = \sum_{j=0}^{M_1-1} \left(c_{a-b} - c_{-b-j(k-n)} c_{a+j(k-n)}\right)$$

$$+ \sum_{j=0}^{M_2-1} \left(c_{a-b} - c_{a-(n+1)-j(k-n)} c_{-b+(n+1)+j(k-n)}\right) - t_{a-b}$$

for c_i the standard basis for $C(C_k)^{(n)}$, and

$$M_1 = \left\lceil \frac{\min\{a,b\}}{k-n} \right\rceil, \quad M_2 = \left\lceil \frac{n+1-\max\{a,b\}}{k-n} \right\rceil$$

This is not a very pretty formula, certainly not as clean as Proposition 3.2, but the upshot is that we have proven the following:

Proposition 3.15. *For $n < k \leq 2n-1$, we have*

$$C^*_{env}(C(C_k)^{(n)}) \cong M_n \text{ and } \operatorname{prop} C(C_k)^{(n)} = 2.$$

3.5 $C^*(C_k)_{(n)}$ and duality

In the previous sections we generalized $C(S^1)^{(n)}$ to $C(C_k)^{(n)}$. We can also generalize $C^*(\mathbb{Z})_{(n)}$ to $C^*(C_k)_{(n)}$: we simply set

$$C^*(C_k)_{(n)} := \{a \in C^*(C_k) \mid a_i = 0 \text{ for } n \leq i \leq k-n\} \subseteq C^*(C_k).$$

Note that $C^*(C_k)$ can be realized as the **circulant matrices**, which are matrices of the form

$$\begin{pmatrix} a_0 & a_{k-1} & a_{k-2} & \cdots & a_2 & a_1 \\ a_1 & a_0 & a_{k-1} & \cdots & a_3 & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_4 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-2} & a_{k-3} & a_{k-4} & \cdots & a_0 & a_{k-1} \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & a_0 \end{pmatrix}$$

In this picture, the $C^*(C_k)_{(n)}$ are those circulant matrices which have a nonzero strip of width at most $2n-1$ diagonally, together with corresponding nonzero corners in the upper right and lower left, or alternatively, circulant matrices which have two strips of width $k-2n+1$ containing only zeroes parallel to the diagonal.

Proposition 3.16. *For $k > 2n-1$, we have that*

$$C^*_{env}(C^*(C_k)_{(n)}) \cong C^*(C_k) \text{ and } \operatorname{prop}(C^*(C_k)_{(n)}) = \left\lceil \frac{k-1}{2(n-1)} \right\rceil.$$

Proof. Since $C^*(C_k)_{(n)} \subseteq C^*(C_k)$ and since $C^*(C_k)$ is finite-dimensional, we have that

$$C^*_{env}(C^*(C_k)_{(n)}) \cong C^*(C^*(C_k)_{(n)}) = C^*(C_k)$$

Note that in $C^*(C_k)$ we have $\delta_i * \delta_j = \delta_{i+j}$, and in $(C^*(C_k)_{(n)})$ we have at the most extreme δ_{n-1} and δ_{1-n}. For those we have $(\delta_{n-1})^m = \delta_{m(n-1)}$ and $(\delta_{1-n})^m = \delta_{m(1-n)}$. So

$$(C^*(C_k)_{(n)})^{\circ m} \subseteq C^*(C_k)_{(m(n-1)+1)}.$$

By noting that all basis elements δ_i can easily be written as a product of m elements, we see that this is actually an equality.

We also have that $C^*(C_k)_{(n)}$ has elements with support of size at most $2n-1$, while in $C^*(C_k)$ each elements has k entries, so we have that

$$C^*(C_k)_{(n)} = C^*(C_k) \Longleftrightarrow k \leq 2n-1,$$

or equivalently, $\frac{k+1}{2} \leq n$. Consequently we have

$$(C^*(C_k)_{(n)})^{\circ m} = C^*(C_k) \Longleftrightarrow \frac{k+1}{2} \leq m(n-1)+1,$$

or equivalently, $(\frac{k+1}{2} - 1)/(n-1) \leq m$. So

$$\mathrm{prop}(C^*(C_k)_{(n)}) = \left\lceil \frac{k-1}{2(n-1)} \right\rceil$$

□

At this point it is natural to ask whether the duality in section 3.3 also holds in this finite setting, i.e. if $C(C_k)^{(n)} \cong (C^*(\mathbb{Z})_{(n)})^d$. However, there is no hope for this: first, note that both $C(C_k)^{(n)}$ and $C^*(C_k)_{(n)}$ reduce to trivial cases for certain k and n. Recall that $C(C_k)^{(n)}$ are the n by n Toeplitz matrices which a k-periodicity in the diagonals. But if $k \geq 2n-1$, this is no extra requirement at all. Alternatively, $C^*(C_k)_{(n)}$ are sequences indexed by C_k with a restriction on elements a_i for which $n \leq i \leq k-n$. But if $k-n < n$, or in other words $k \leq 2n-1$, then there is no such i, and so there are no restrictions on the values a_i. Summarized:

$$C^*(C_k)_{(n)} = C^*(C_k) \quad \text{for} \quad k \leq 2n-1$$

$$C(C_k)^{(n)} = C(S^1)^{(n)} \quad \text{for} \quad k \geq 2n-1$$

If we would have $C(C_k)^{(n)} \cong (C^*(C_k)_{(n)})^d$ for all k and n, then in particular $(C(S^1)^{(n)})^d \cong C^*(C_k)_{(n)}$ for all $k \geq 2n-1$, which is impossible, since those $C^*(C_k)_{(n)}$ have non-isomorphic operator system structures, which can be concluded from the propagation number.

Even the proof breaks down in a very early stage. Consider the Fejér-Riesz lemma, which was essential in the proof; it states that any polynomial in z and

z^* that is non-negative on the unit circle can be expressed as the modulo-squared of a polynomial in z with the same degree. However, $C^*(C_k) \cong C(C_k) \cong \mathbb{C}^k$ corresponds to maps

$$n \mapsto \sum_{i \in C_k} a_i \zeta^{in}$$

for some principle k'th root of unity. For all k'th roots of unity we have that $\zeta^{-n} = \zeta^{k-n}$, so there is no distinction between positive and negative powers in the polynomials. Even trying to get some zero coefficients for an element in $C^*(C_k)_+$ is problematic: consider the positive element $(1,1,1) \in C^*(C_3)$, which is positive because it corresponds to the map

$$n \mapsto 1 + \zeta^n + \zeta^{2n} = 3\delta_{n0}$$

in $C(C_3)$ which is positive. Using the correspondence between $C^*(C_k)$ and the k by k circulant matrices, we see that the square-modulus of an element in $C^*(C_3)$ is of the form

$$\begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix}^* \begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} & 0 \\ 0 & \overline{a} & \overline{b} \\ \overline{b} & 0 & \overline{a} \end{pmatrix} \begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix}$$
$$= \begin{pmatrix} |a|^2 + |b|^2 & \overline{b}a & \overline{a}b \\ \overline{a}b & |a|^2 + |b|^2 & \overline{b}a \\ \overline{b}a & \overline{a}b & |a|^2 + |b|^2 \end{pmatrix}.$$

But we can never have $(1,1,1) = (|a|^2 + |b|^2, \overline{a}b, \overline{b}a)$ because $|\overline{a}b| = |\overline{a}||b|$, so we would need to find $x, y \in \mathbb{R}$ such that $x^2 + y^2 = 1$ and $xy = 1$, which doesn't exist.

One might wonder about the reason that the duality does hold for S^1 and $\mathbb{Z}$, but does not hold for C_k. One interesting remark in this regard is the following: the space

$$H^2 := \{f \in L^2(S^1) \mid \hat{f}(n) = 0 \text{ for } n < 0\}$$

is called the Hardy space, and for $\phi \in L^\infty(S^1)$ we call the operator $P_{H^2}\phi|_{H^2}$ a *Toeplitz operator*, where ϕ acts as multiplication operator and P_{H^2} is the projection to H^2. Murphy notes in [21, Ch. 3, Addenda] that the theory of Toeplitz operators generalizes to the setting of *ordered groups* and their Pontryagin duals, where ordered groups are Abelian groups with a partial order such that $x \leq y$ implies $x + z \leq y + z$. He also notes that such an order exists if and only if the Pontryagin dual $\hat{G}$ is connected. This is the case in the setting for which we do have a duality, because $\widehat{\mathbb{Z}} = S^1$ is connected, and it is not the case in the setting for which we do not have a duality, because $\widehat{C_k} = C_k$ is not connected. Sadly there was no time left during the writing of this thesis to investigate the connection with the operator systems discussed in this chapter.

Chapter 4

Outlook

We finish this thesis with an outlook to possible avenues through which the results of this thesis might be extended, and obstacles that need to be overcome to do so.

I think that the most obvious way to proceed would be trying to generalize Theorem 2.20 and Theorem 2.23 to non-unital operator systems. As noted in section 1.5, a proof of this would directly show that if $E \otimes \mathcal{K} \cong F \otimes \mathcal{K}$, then

$$\text{prop}(E) = \text{prop}(E \otimes \mathcal{K}) = \text{prop}(F \otimes \mathcal{K}) = \text{prop}(F).$$

In other words, it would provide a very clear and direct proof of the fact that the propagation number is invariant under stable equivalence (although this has already been proven in [8]). Another reason to investigate this possibility is that the concepts used in the proof of Theorem 2.20 do not seem to rely on the unitality of the operator system in an essential way. In [8] Connes and van Suijlekom define a generalization of the C^*-envelope for the definition of a non-unital operator system given by Werner in [31], and prove that it always exists. If the theory of Šilov boundary ideals translates to this setting, then the proof would probably go through. Additionally, for proving Theorem 2.23 once Theorem 2.20 is known, the argument carries over verbatim to a nonunital setting.

Alternatively, an effort could be made to investigate the validity of Theorem 2.20 and Theorem 2.23 for other tensor norms than the minimal tensor norm. Here the way to proceed seems less obvious: Lemma 2.14 and Lemma 2.15 rely in a seemingly essential way on the fact that the minimal tensor norm for $E \subseteq B(H)$ and $F \subseteq B(K)$ is given by the induced norm from $B(H \otimes K)$. The proof of Corollary 2.13 is built upon the completely isometric inclusion $E \otimes F \hookrightarrow CB(E^*, F)$; clearly if this holds for any other tensor norm then it must equal the minimal one.

In this thesis we also discussed some examples of operator systems and the possible dualities between them, in order to get an idea of how the propagation number behaves under dualities. The results of this discussion are displayed in Table 4.1. The C^*-envelopes and propagation numbers are calculated in

	envelope	propagation number	dual
$C(S^1)^{(n)}$	M_n	2	$C^*(\mathbb{Z})_{(n)}$
$C(C_k)^{(n)}$	M_n	2	?
$C^*(\mathbb{Z})_{(n)}$	$C^*(\mathbb{Z})$	∞	$C(S^1)^{(n)}$
$C^*(C_k)_{(n)}$	$C^*(C_k)$	$\left\lceil \frac{k-1}{2(n-1)} \right\rceil$	?

Table 4.1: An overview of the properties derived in this chapter.

Theorem 3.3, Proposition 3.15, Proposition 3.4 and Proposition 3.16, and the duality between $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ is proven in section 3.3. The duals of $C(C_k)^{(n)}$ and $C^*(C_k)_{(n)}$ are currently unknown, but as explained in section 3.5, they are definitely not dual to each other, as $C(C_k)^{(n)} \neq C(S^1)^{(n)}$ only for $k \leq 2n-1$, while $C^*(C_k)_{(n)} \neq C^*(C_k)$ only for $k > 2n-1$.

In general, the dual construction seems to be quite subtle. For example, in [6, Thm 5.6] it is shown that for E and F operator spaces, we have that $(E \otimes F)^d$ is isomorphic to the completion of $E^d \odot F^d$ with respect to the *maximal* tensor norm (in this setting also often called the *projective* tensor norm). As remarked above, the relationship between the propagation number and the maximal tensor product is still unclear, and if there is no simple relationship, then there would also likely be no simple relationship between the propagation number and the dual operator system. Another reason why the dual operator system is less likely to be well-behaved with respect to the propagation number is the fact that we defined it using the abstract characterization of operator systems. Theorem 1.16 guarantees a concrete embedding for the dual operator system, but it in general doesn't guarantee that this embedding is somehow related to the original embedding. Effros and Ruan construct a concrete representation for the dual operator space in [12, p. 46], so perhaps a similar construction could be applied to operator systems. Perhaps the Šilov ideal can then be characterized concretely.

Finally, by further investigating the duals to $C(C_k)^{(n)}$ and $C^*(C_k)_{(n)}$, one might be able to find counterexamples to the hypothesis that the propagation number of the dual operator system can be expressed as a function of the original operator system. If it turns out that the dual to $C(C_k)^{(n)}$ does not have a propagation number of ∞, then clearly the propagation number of the dual is not only determined by the propagation number of the original. I think that this is the approach that is most likely to resolve this question.

Appendix A

Tensor product of vector spaces

In this appendix we recall the basics for the tensor product of vector spaces.

In a very abstract sense, the tensor product of two vector spaces V and W (or modules over rings in general, see [7, Ch. 15]) is a vector space $V \odot W$ with bilinear map $\otimes : V \times W \to V \odot W$ with the universal property that any bilinear map $f : V \times W \longrightarrow X$ to any vector space X factors as $f(v, w) = \hat{f}(v \otimes w)$ (here we denote $\otimes(v, w) = v \otimes w$).

However, this can be realised as a concrete construction: for this we take the free vector space over all pairs (v, w) and take the quotient space with respect to the space spanned by the vectors

$$(\lambda v_1 + \mu v_2, w) - \lambda(v_1, w) + \mu(v_2, w) \text{ for } v_1, v_2 \in V, w \in W, \lambda, \mu \in \mathbb{C} \quad \text{(A.1)}$$

$$(v, \lambda w_1 + \mu w_2) - \lambda(v, w_1) + \mu(v, w_2) \text{ for } v \in V, w_1, w_2 \in W, \lambda, \mu \in \mathbb{C}. \quad \text{(A.2)}$$

For the image of (v, w) under this quotient we write $v \otimes w$.

We can think of $V \odot W$ as the space of all finite linear combinations of elements of the form $v \otimes w$, keeping in mind the fact that $\otimes$ is bilinear. Note that there are many ways of representing an element in $x \in V \odot W$ as a finite linear combination $\sum_i v_i \otimes w_i$; the minimal number of such terms is called the **rank** of the tensor, and elements of rank 1 (i.e. elements that can be written as $v \otimes w$) are called **simple tensors**. Note that for each linear map $\phi : V \to \mathbb{C}$ and $\psi : W \to \mathbb{C}$ we can construct the bilinear map $(v, w) \mapsto \phi(v)\psi(w)$. So if $\sum_i v_i \otimes w_i = 0$, and the v_i and w_i are linearly independent, then take a linear map ϕ_i for each i that is nonzero only on v_i, and ψ_i for each i that is nonzero only on w_i; since $\sum_i v_i \otimes w_i = 0$ we must have that $\sum_i \phi_j(v_i)\psi_k(w_j) = \phi_j(v_j)\psi_k(w_k) = 0$, which is a contradiction. So if $\sum v_i \otimes w_i = 0$, then either the v_i or the w_i must be linearly dependent.

A nice fact about linear maps on the tensor product of vector spaces which follows from this is the following:

Lemma A.1. *For linear maps $f : X_1 \longrightarrow X_2$ and $g : Y_1 \longrightarrow Y_2$, if $f \odot g(z) = 0$ for $z \in X_1 \odot Y_1$ then there are $x_i \in X_1$ and $y_i \in Y_1$ such that $z = \sum_i x_i \otimes y_i$ and $f \odot g(x_i \otimes y_i) = 0$ for all i.*

Proof. We claim that if $\sum_{i=1}^n f(x_i) \otimes g(y_i)$ with $x_1, \ldots, x_n$ linearly independent over $\ker f$, then $g(y_i) = 0$ for all y_i. We prove this by induction on n.

For $n = 1$ this is true because $f(x_1) \otimes g(y_i) = 0$ with x_1 not in $\ker f$ means that $g(y_i) = 0$. Given arbitrary $n \geq 2$, assume that the claim holds for $n-1$. Let $x_1, \ldots, x_n$ be linearly independent over $\ker f$, and $\sum_{i=1}^n f(x_i) \otimes g(y_i) = 0$. Then either the $f(x_i)$ or the $g(y_i)$ are linearly independent; the former is impossible since no linear combination of x_i can lie in $\ker f$. So without loss of generality, write $g(y_m) = \sum_{i=1}^{m-1} \lambda_i g(y_i)$. Then

$$\sum_{i=1}^{m} f(x_i) \otimes g(y_i) = \sum_{i=1}^{m-1} f(x_i + \lambda_i x_m) \otimes g(y_i).$$

Note, however, that the $x_i + \lambda_i x_m$ are still linearly independent over $\ker f$, and so by the induction hypothesis we have $g(y_i) = 0$ for $1 \leq i \leq m-1$, and therefore also $g(y_m) = 0$.

By Lemma 2.1 in [15] every $z \in X_1 \odot Y_1$ can be represented as $z = \sum_{i=1}^m x_i \otimes y_i + \sum_{i=1}^n x'_i \otimes y'_i$ with x_i linearly independent over $\ker f$ and $x'_i \in \ker f$. But then if $f \odot g(z) = 0$ we have

$$0 = f \odot g(z) = \sum_{i=1}^{m} f(x_i) \otimes g(y_i)$$

so that by the above claim we see that $g(y_i) = 0$. So we have found the desired expansion of z. ☐

If W is finite-dimensional, with basis $e_1, \ldots e_m$, then any element in $V \odot W$ is of the form

$$\sum_{j=1}^{n} v_j \otimes w_j = \sum_{j=1}^{n} \sum_{i=1}^{m} (w_j)_i v_j \otimes e_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (w_j)_i v_j \right) \otimes e_i$$

$$:= \sum_{i=1}^{n} v'_i \otimes e_i.$$

So in some sense the elements of $V \odot W$ are vectors in W with coefficients in V. In particular, we have that we can identify $V \odot M_n \cong M_n(V)$ through

$$\sum_{k,l=1}^{n} v_{kl} \otimes e_{kl} \leftrightarrow (v_{kl})_{k,l=1}^{n}$$

where e_{kl} are the canonical basis elements for M_n, i.e. matrices with a 1 in the (k, l) spot, and zeroes everywhere else.

If V and W are Hilbert spaces, then there is a natural inner product on $V \odot W$, namely given on simple tensors by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle := \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle$$

It is easily verified that this is indeed an inner product, and we shall denote the completion of $V \odot W$ with respect to this norm as $V \otimes W$, which is then again a Hilbert space.

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