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MASTER'S THESIS

Truncated Geometry on the Circle

Evert-Jan M. Hekkelman

Supervisor

Dr. W.D. van Suijlekom

Second reader

Dr. P. Hochs

Abstract

A. Connes and W. D. van Suijlekom introduced a theoretical framework to deal with truncations of spectral triples [14], and made a detailed study of such truncations of the canonical spectral triple of the circle. Van Suijlekom later showed that the state spaces of the truncated circle converge as metric spaces in Gromov-Hausdorff sense to the state space of the whole spectral triple, in the limit of taking larger and larger projections to truncate with [41]. In this thesis, we continue this work by showing that the Gromov-Hausdorff limit of the pure state spaces of the truncated circle is also the state space of the whole spectral triple, and show explicitly that it cannot be the *pure* state space of the circle. Furthermore, we observe that work by T. Loring and H. Schulz-Baldes on the spectral localizer [26] can be applied directly to calculate the index of the Connes-Moscovici index theorem on a spectral triple using only data from a truncation of the spectral triple, and we give a self-contained proof of this fact for the canonical spectral triple of the circle.

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Chapter 1

Introduction

In many ways, noncommutative geometry [10] exemplifies the spirit of modern mathematics. At its heart, it is about the translation of geometric data into algebraic and analytical data, and a subsequent generalisation that goes beyond geometry. Such projects are mathematics at its best, combining different fields to create a new perspective on old structures.

In order to be more precise, consider how a locally compact Hausdorff space X can be recovered via Gelfand duality from its continuous complex-valued functions vanishing at infinity, which forms a commutative C^*-algebra $C_0(X)$. The whole topological structure of X can be read off from algebraic and analytical properties of $C_0(X)$ [42, Chapter 1.11]. Below is a small selection of examples of this correspondence:

X	$C_0(X)$
compact	unital
compactification	unitization
homeomorphisms	$*$-isomorphisms
open subsets	closed ideals
connected components	projections

Remarkably, any commutative C^*-algebra A in turn gives rise to a locally compact Hausdorff space $\Omega(A)$ such that $A = C_0(\Omega(A))$ [31, Chapter 2.1]. This duality between locally compact Hausdorff spaces and commutative C^*-algebras gives some motivation to call the study of not necessarily commutative C^*-algebras *noncommutative topology.* Noncommutative geometry is a project in the same vein, and as the name suggests, it covers not only topology, but geometry as well. It is an extension of Gelfand duality that also translates a Riemannian spin^c-manifold's (smooth) structure, Riemannian metric and spinor bundle into algebraic and analytical data and vice versa.

For compact Riemannian spin^c-manifolds M with spinor bundle $S \to M$ and Dirac-operator D_M associated to the Levi-Civita connection lifted to the spinor bundle, the data that captures the whole structure comes in the form of a triple: the *canonical spectral triple* [11]

$$(C^\infty(M), L^2(S), D_M).$$

In general, a spectral triple

$$(\mathcal{A}, H, D)$$

consists of a typically unital $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space H, and a self-adjoint (possibly unbounded) operator D on H with compact resolvent, with the condition that $[D, a]$ is bounded for each a in $\mathcal{A}$. Like in noncommutative topology, not only does every compact Riemannian spin^c-manifold give rise to a spectral triple with commutative $\mathcal{A}$, but we can reconstruct the manifold with its entire structure from its canonical spectral triple as well [12]. This proof dates from 2008, although the statement was already formulated in 1996 [11]. Precisely because of this correspondence, it is justifiable to call the study of (general) spectral triples *noncommutative geometry*.

A perspective on this correspondence that may appeal more to the imagination has to do with the question whether one can 'hear the shape of a drum'. In 1966, Mark Kac published an article "Can one hear the shape of a drum?" [24], which sparked a wave of research into geometric reconstructions using spectral data. Mathematically, the frequencies that a drum produces when struck can be interpreted as the eigenvalues of the Laplace operator on a domain shaped like the drum in the Euclidean plane $\mathbb{R}^2$. More generally, for a compact Riemannian manifold (M, g), one can define the *spectrum* of M as the sequence of eigenvalues of the Laplace-Beltrami operator Δ on M. If M has a boundary, one can consider Dirichlet or Neumann boundary conditions, resulting in a Dirichlet spectrum and a Neumann spectrum. Kac's question could therefore be formulated as: given two isospectral domains in the Euclidean plane, i.e. domains with equal Dirichlet or Neumann spectra, must these domains then be isometric? Or more generally, if two manifolds are isospectral, does that imply that the manifolds are isometric as well?

The more general question was in fact answered already by the time Kac published his article, as non-isometric Riemannian manifolds were found with equal spectra by John Milnor in 1964 [30]. In the more specific case concerning domains in the Euclidean plane, a definitive answer took three decades to find. Also here the answer was negative, published by Gordon, Webb and Wolpert in 1992 [22]. These authors found an example of a pair of simply-connected domains in $\mathbb{R}^2$ whose Neumann spectrum and Dirichlet spectrum both coincide, see Figure 1.1.

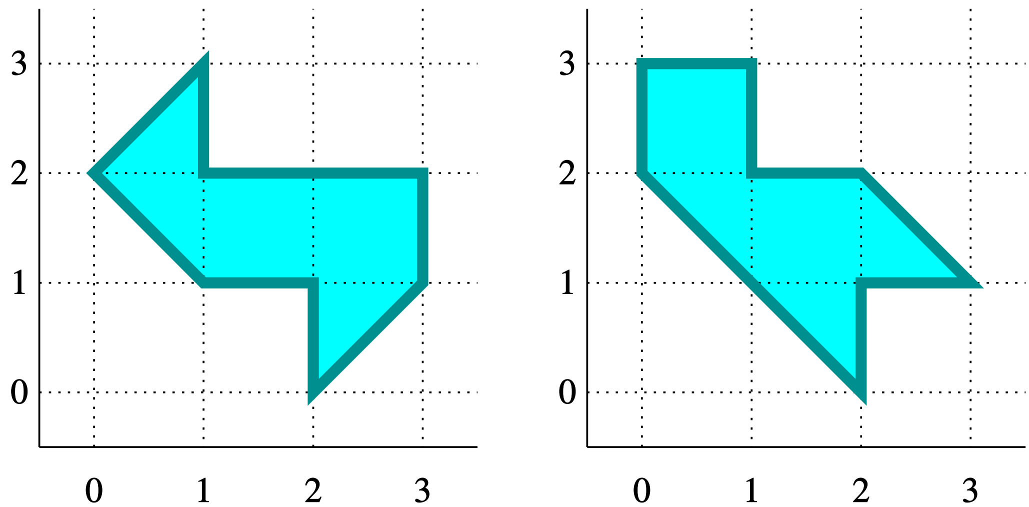


Figure 1.1: Two domains in $\mathbb{R}^2$ with equal Neumann spectrum and Dirichlet spectrum [22]. Figure is in public domain.

So, while a direct reconstruction of a Riemannian manifold (M, g) from the spectrum of the Laplace-Beltrami operator has been shown to be impossible, the reconstruction theorem in noncommutative geometry can be interpreted as showing what spectral data *is* enough to perform a reconstruction of (M, g). Not the Laplace-Beltrami operator should be considered, but its square root the Dirac operator (although these spectra are essentially the same), in combination with the 'coordinate functions' $C^\infty(M)$. Calling $C^\infty(M)$ the 'coordinate functions' in this context is suggestive of how the reconstruction works, as we need to construct local charts to recover M as a manifold and we can distill local coordinates from $C^\infty(M)$. With this extra information, Kac's question can be answered positively.

This approach to geometry is not just of interest to the mathematician with an appetite for abstract generalisations. To the contrary, noncommutative geometry has enjoyed many interactions with other fields from its very beginning. Notably, there are intimate connections with physics, from the quantum Hall effect [10, Chapter 4.6] to particle physics [40]. A noncommutative Standard Model has been constructed which describes the space-time we live in as a spectral triple, with an action that gives General Relativity coupled with the Lagrangian of the standard model [11, 40].

In this context of physics, it makes sense to think about the limitations of reality. While for Riemannian spin^c-manifolds the whole spectrum of the Dirac operator is needed for the reconstruction of its structure, in practice we will only be able to do measurements up to a certain energy level. For a spectral triple $(\mathcal{A}, H, D)$ this would mean having a projection P in $B(H)$ projecting onto a finite part of the spectrum of D, and only being able to access $(P\mathcal{A}P, PH, PD)$. A meaningful question to ask, is what structure and what invariants of $(\mathcal{A}, H, D)$ can be recovered in a truncation $(P\mathcal{A}P, PH, PD)$.

A careful study of such truncations can be found in [17], further developed in [14]. This last paper, by A. Connes and W. van Suijlekom, is the foundation upon which this thesis is built. There, a theoretical framework is constructed for the truncation of spectral triples, after which the truncated canonical spectral triple of the circle is studied in detail. In this thesis we will continue this study of the truncated circle.

There are two new results presented in this thesis. In **Chapter 4**, it is shown that the pure state space of the truncated canonical spectral triple of the circle converges to the state space of the circle in Gromov-Hausdorff sense, when taking the limit of larger and larger projections to truncate with. At first sight, it is somewhat surprising that these pure state spaces do not converge to the *pure* state space of the circle, which we explore in detail. Secondly, in **Chapter 5** we observe that work by H. Schulz-Baldes and T. Loring on the spectral localizer [26, 27] can be directly applied to compute the index as appearing in the Connes-Moscovici Theorem using only data that is available in the truncation of a spectral triple. We give a self-contained proof of this fact in the case of the circle, i.e. a proof that the winding number of a unitary function $u \in C^\infty(S^1)$ can be calculated using only data from the truncated spectral triple of the circle if the truncating projection has a large enough rank.

In preparation for these topics, we cover some preliminaries on operator theory, noncommutative geometry and the Gromov-Hausdorff metric in **Chapter 2**, and treat the general theory of truncating spectral triples, along with a study of this in the context of the spectral triple of the circle, in **Chapter 3**.

Chapter 2

Preliminaries

2.1 Operator Theory

Since this thesis is about noncommutative geometry, a brief recap of some operator theory is in order. We will assume that the reader is familiar with this subject, however.

In general, in this thesis H will denote a Hilbert space, $B(H)$ the bounded operators on H and $K(H)$ the compact operators on H. The space $B(H)$ forms the ultimate example of a C^*-algebra, and an important tool in operator theory is *representing* a C^*-algebra by mapping it to $B(H)$ for some Hilbert space H [31, Section 3.4].

Definition 2.1.1. A *representation* of a C^*-algebra A is a pair (H, φ) where H is a Hilbert space and $\varphi : A \to B(H)$ is a $*$-homomorphism. The representation is called *faithful* if φ is injective.

Any C^*-algebra has a faithful representation via the GNS-construction [31, Theorem 3.4.1]. This makes it possible to view every C^*-algebra as a C^*-subalgebra of $B(H)$. In an implicit way, this leads us to the concept of states on a C^*-algebra, as the GNS-construction is based on states.

Definition 2.1.2. Let A be a C^*-algebra. A *state* τ is a positive linear functional $\tau : A \to \mathbb{C}$ with $\|\tau\| = 1$. The set of states on A is denoted $\mathcal{S}(A)$.

In the light of seeing C^*-algebras as noncommutative topology as explained in the introduction, states are the noncommutative analogue of probability measures. Indeed, in the case that A is a commutative C^*-algebra – and hence $A \cong C_0(X)$ for some locally compact Hausdorff space X – $\mathcal{S}(A)$ is isomorphic to the space of probability measures on X [15, Example VIII.5.13].

By the Banach-Alaoglu Theorem [15, Theorem V.3.1], the set of states on a C^*-algebra is weak*-compact. Hence, by the Krein-Milman Theorem [15, Theorem V.7.4], $\mathcal{S}(A)$ is the convex hull of its extreme points. These extreme points are the *pure states* on A.

Definition 2.1.3. A state on a C^*-algebra is called *pure* if it is an extreme point of $\mathcal{S}(A)$. The set of pure states is denoted $\mathcal{P}(A)$.

Proposition 2.1.4. *Any state on the matrix algebra $M_n(\mathbb{C})$ is of the form $T \mapsto \operatorname{Tr}(\rho T)$ for a positive $\rho \in M_n(\mathbb{C})$ with $\operatorname{Tr}\rho = 1$. The pure states are all of the form $T \mapsto \langle v, Tv\rangle$ for some unit vector $v \in \mathbb{C}^n$.*

Proof. The first claim is an incarnation of the fact that the trace-class operators are the dual of the space of compact operators on any Hilbert space [31, Section 4.2]. For the second claim, note that v is a cyclic vector for the representation $(\mathbb{C}^n, \mathrm{id})$, hence by [31, Theorem 5.1.7] the GNSrepresentation corresponding to $T \mapsto \langle v, Tv\rangle$ is unitarily equivalent to the representation $(\mathbb{C}^n, \mathrm{id})$, which implies that it is a pure state by [31, Theorem 5.1.6]. □

Some authors prefer to define a pure state in another (equivalent) way, which is put forward in the upcoming proposition. First recall the following useful property of positive linear functionals on a C^*-algebra.

Lemma 2.1.5. *Let A be a C^*-algebra, and $\tau : A \to \mathbb{C}$ a positive linear functional. Then*

$$\|\tau\| = \lim_{\lambda} \tau(u_\lambda)$$

for all approximate units $(u_\lambda)_{\lambda\in\Lambda}$ of A. In particular, if A is unital $\|\tau\| = \tau(1)$.

Proof. See [31, Theorem 3.3.3]. □

Proposition 2.1.6. *Let A be a C^*-algebra, and $\tau \in \mathcal{S}(A)$ a state. Then the following are equivalent:*

1. *The functional τ is a pure state;*
2. *Whenever ρ is a positive linear functional such that $\rho(a) \leq \tau(a)$ for all positive $a \in A$, then there exists a $t \in [0, 1]$ such that $\rho = t\tau$;*
3. *Whenever ρ is a state such that $\rho(a) \leq \tau(a)$ for all positive $a \in A$, then $\rho = \tau$.*

Proof. The proof is an adaptation of [31, Theorem 5.1.8]. Suppose that τ is a pure state, let ρ be a positive linear functional such that $\rho(a) \leq \tau(a)$ for all positive $a \in A$. Assume that ρ is not equal to 0 or τ. Then also $\tau - \rho$ is a nonzero positive linear functional, and

$$\tau = \rho + (\tau - \rho) = \|\rho\| \frac{\rho}{\|\rho\|} + \|\tau - \rho\| \frac{\tau - \rho}{\|\tau - \rho\|}.$$

Note that $\frac{\rho}{\|\rho\|}$ and $\frac{\tau-\rho}{\|\tau-\rho\|}$ are now states, and necessarily $\|\rho\| + \|\tau - \rho\| = \|\tau\| = 1$ due to Lemma 2.1.5. Hence τ is the convex combination of $\frac{\rho}{\|\rho\|}$ and $\frac{\tau-\rho}{\|\tau-\rho\|}$, and since τ is an extreme point this is a contradiction. This proves 1 implies 2.

To see that 2 implies 1, suppose that 2 holds for the state τ. If there exist positive linear functionals ρ_1, ρ_2 such that $\tau = t\rho_1 + (1 - t)\rho_2$ for $t \in (0, 1)$, then we have in particular that $\rho_1(a) \leq \tau(a)$ for all positive $a \in A$. Hence there exists $t' \in [0, 1]$ such that $t\rho_1 = t'\tau$. Observe that

$$1 = \|\tau\| = t\,\|\rho_1\| + (1 - t)\,\|\rho_2\|$$

according to Lemma 2.1.5, which forces $1 = \|\rho_1\| = \|\rho_2\|$. But since $t = \|t\rho_1\| = \|t'\tau\| = t'$, we can then conclude that $\rho_1 = \tau$. We see that indeed τ is an extreme point of $\mathcal{S}(A)$.

The equivalence of 2 and 3 is immediate from the observation that $\mathcal{S}(A)$ spans the positive linear functionals on A. □

As we will see in Chapter 3, by truncating a spectral triple we will be performing an operation on a C^*-algebra which results in something that has less structure. The right framework will be that of *operator systems*, which do not have an algebra structure and need not be topologically closed. Similar to how any C^*-algebra can be seen as a C^*-subalgebra of $B(H)$, these operator systems can be defined in an abstract manner without reference to any ambient space, but also as a concrete set in $B(H)$. We will take the concrete description as the definition of operator systems.

Definition 2.1.7. An *operator system* E is a subspace of $B(H)$ for some Hilbert space H, such that $E^* = E$ and $1 \in E$.

The equivalence of this simple definition and its abstract counterpart is quite spectacular, called the Choi-Effros Theorem [9]. We will not be needing much more of this theory than this definition, a review of operator systems and their applications to truncated geometry can be found in [14].

What will be important in this thesis, is that on such operator systems the concept of a (pure) state space is still perfectly definable.

Definition 2.1.8. The cone of positive elements in E is defined as $E_+ := E \cap B(H)_+$.

Definition 2.1.9. Let E be an operator system. A *state* τ on E is a positive linear functional $\tau : E \to \mathbb{C}$ such that $\tau(1) = 1$. The set of states on E is denoted $\mathcal{S}(E)$. The set of extreme points of $\mathcal{S}(E)$ is likewise referred to as the set of pure states, denoted $\mathcal{P}(E)$.

Note that this indeed generalises the definition of a (pure) state on a unital C^*-algebra via Lemma 2.1.5. Furthermore, the Banach-Alaoglu Theorem [15, Theorem V.3.1] still applies, so we also have that $\mathcal{S}(E)$ is the convex hull of $\mathcal{P}(E)$ in the setting of operator systems by the Krein-Milman Theorem [15, Theorem V.7.4].

2.2 Noncommutative Geometry

Today, noncommutative geometry has developed into a diverse mathematical field of its own right. Since the publication Connes' landmark book on noncommutative geometry in 1994 [10] the areas of research in this field have only expanded. One of the central aspects of noncommutative geometry is the spectral triple, and much (but certainly not all) research in this field focuses on this concept. Interestingly, the name 'spectral triple' does not yet appear in the aforementioned book. Instead, Connes talks about unbounded K-cycles over involutive algebra. In any case, we will use the definitions as in [10] but with modern terminology, starting with the main event: spectral triples [10, Definition 4.2.11].

Definition 2.2.1. A *spectral triple* $(\mathcal{A}, H, D)$ consists of a unital $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space H, together with an unbounded self-adjoint operator D such that its resolvent $(D+i)^{-1}$ is compact and $[D, a]$ is bounded for each $a \in \mathcal{A}$. We will generally use the notation $A = \overline{\mathcal{A}}$ for what is then a C^*-algebra.

Example 2.2.2. Let M be a Riemannian manifold, with spinor bundle $S \to M$ and Dirac operator D_M associated to the Levi-Civita connection, lifted to the spinor bundle. Then the *canonical spectral triple* associated to M is the triple [11]

$$(C^\infty(M), L^2(S), D_M).$$

For details on how this Dirac operator is defined, see [19, Section 3.4]. What will be the main subject of interest in this thesis is the canonical spectral triple of the circle,

$$\left(C^\infty(S^1), L^2(S^1), -i\frac{d}{dx}\right).$$

With some extra conditions, one can recover a manifold M with its whole smooth structure from its canonical spectral triple [12]. This reconstruction theorem is quite involved, but one aspect is of interest to this thesis. By Gelfand duality it is no challenge to recover M as a topological space from $C^\infty(M)$ as the space of characters $\Omega(C(M)) = \Omega(\overline{C^\infty(M)})$. The Riemannian distance on $M \cong \Omega(C(M))$ is then encoded in its canonical spectral triple in the following way:

$$d(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}.$$

This observation led Connes to define a distance on the state space $\mathcal{S}(A)$ for general spectral triples $(\mathcal{A}, H, D)$ in an analogous manner [10, Chapter 6].

Definition 2.2.3. Let $(\mathcal{A}, H, D)$ be a spectral triple, then the *Lipschitz seminorm* on $\mathcal{A}$ is defined as $\|[D, a]\|$, which we will sometimes denote $\|a\|_1$ for brevity.

Definition 2.2.4. Let $(\mathcal{A}, H, D)$ be a spectral triple. Then the Connes distance between $\varphi, \psi \in \mathcal{S}(A)$ is defined as

$$d(\varphi.\psi) = \sup_{a \in \mathcal{A}} \{|\varphi(a) - \psi(a)| : \|a\|_1 \leq 1\}.$$

As noted earlier, in the case that A is a commutative C^*-algebra, the state space $\mathcal{S}(A)$ is isomorphic to a space of probability measures on some locally compact space X. In that situation, the Connes distance is equal to the *Monge-Kantorovich metric* [33]. A survey that explores this relation and more on the topic of metrics in noncommutative geometry has been written by P. Martinetti [29]. As we will focus on the canonical spectral triple of the circle in this thesis, for which A is indeed commutative, in that context we will refer to this distance formula as the Monge-Kantorovich metric.

An important class of examples of spectral triples are those for which the Connes distance induces the same topology as the weak*-topology. These are called *metric spectral triples*, following terminology as in [25]. This concept is intimately related to work by M. Rieffel on compact quantum metric spaces, as a spectral triple $(\mathcal{A}, H, D)$ is metric exactly when the pair $(\mathcal{A}, \|\cdot\|_1)$ is something he dubbed a *compact quantum metric space* [34].

2.3 The Gromov-Hausdorff Metric

Metric spaces are of course ubiquitous in mathematics, and as we saw in the previous section, the state space on C^*-algebras and operator systems can be turned into metric spaces by the Connes distance formula. As alluded to in the introduction, we will want to compare these metric spaces with each other in this thesis. In an amusing twist, it is possible to construct a metric on a 'space' of metric spaces (up to isometry), so the metric spaces themselves form something resembling a metric space. The quotation marks are vital as this subject should be handled with care, set theoretical paradoxes are right around the corner. The exposition in this chapter leans heavily on *A Course in Metric Geometry* [5] by D. Burago, I. Burago and S. Ivanov. This book is recommended for the reader who wishes to learn more about metric spaces of metric spaces than needed for this thesis.

As remarked by these authors, the actual distance between metric spaces does not matter so much. The real purpose and power of the Gromov-Hausdorff distance is the notion of convergence that it induces, as this makes it possible to meaningfully talk about convergence of sequences of metric spaces. Secondly, there are a myriad of different constructions inducing such a topology, and these will generally be different. There exist concepts of uniform convergence, Lipschitz convergence, and Gromov-Hausdorff convergence for example. One advantage of the Gromov-Hausdorff metric is that the distance between two compact metric spaces will always be finite. Furthermore, there exist many techniques and criteria to determine Gromov-Hausdorff convergence, which makes it a versatile tool in studying compact metric spaces. An extension to non-compact metric spaces also exists [5, Chapter 8], but this requires a more delicate approach and will not be needed in this thesis.

As a very brief synopsis, the starting point for defining the Gromov-Hausdorff metric is the Hausdorff metric which compares subsets within a metric space. After embedding two metric spaces in a third metric space, the Hausdorff distance can be calculated between these embedded metric spaces – which clearly depends on the embedding and choice of the ambient space. Taking the infimum of all such possible embeddings then yields the Gromov-Hausdorff metric.

For a subset S in a metric space, denote the r-neighbourhood of S by $U_r(S)$, i.e.

$$U_r(S) = \bigcup_{x\in S} B_r(x).$$

Definition 2.3.1. Let A and B be subsets of a metric space. The *Hausdorff distance* between A and B, denoted $d_H(A, B)$, is defined as

$$d_H(A, B) = \inf\{r > 0 : A \subseteq U_r(B) \text{ and } B \subseteq U_r(A)\}.$$

Definition 2.3.2. Let X and Y be metric spaces. The *Gromov-Hausdorff distance* between X and Y, denoted $d_{GH}(X, Y)$, is defined as the infimum of all $r > 0$ such that there exists a metric space Z with subsets $X', Y' \subseteq Z$ isometric to X and Y respectively with $d_H(X', Y') < r$.

Note that in this definition, the isometries from X and Y to X' and Y' must be considered with respect to the metrics of X' and Y' taken as the restriction of the metric on Z, not with their induced intrinsic metrics. For example, if X is a sphere with its standard Riemannian metric, one cannot take $Z = \mathbb{R}^3$ with $X' = S^2$, as X and X' would then not be isometric.

We would like to state some results about this distance on the collection of compact metric spaces up to isometry. As noted in the introduction of this section, this collection is not a set. But following the book [5, Chapter 7], we will abusively refer to this as a space. We are justified in doing so, as all statements made about this 'space' of metric spaces can be reformulated in terms of its elements – calling it a space is simply a way of shortening formulations.

Perhaps the most pressing question at this point, is if the Gromov-Hausdorff distance actually defines a (finite) metric on the space of isometryc classes of compact metric spaces. A sketch of a proof of this will be included at the end of this section, but before we come to that we will need a few other definitions and results. First of all, it is possible to simplify Definition 2.3.2 so a less ridiculously large class of ambient spaces Z needs to be considered. See also [5, Remark 7.3.12].

Proposition 2.3.3. *Let X and Y be metric spaces. Consider all (pseudo)metrics d on the disjoint union $X \sqcup Y$ such that $d|_{X\times X} = d_X$ and $d|_{Y\times Y} = d_Y$. Then*

$$\begin{aligned} d_{GH}(X, Y) &= \inf_{d \text{ metric on } X\sqcup Y} \{d_H(X, Y)\} \\ &= \inf_{d \text{ pseudometric on } X\sqcup Y} \{d_H(X, Y)\}, \end{aligned}$$

considering X, Y subsets of $(X \sqcup Y, d)$.

Proof. It is immediate that

$$d_{GH}(X, Y) \leq \inf_{d \text{ metric on } X\sqcup Y} \{d_H(X, Y)\}.$$

For the other way around, consider an ambient metric space Z and fix isometries $f : X \to X' \subseteq Z$ and $g : Y \to Y' \subseteq Z$ with $d_H(X', Y') = r$. Then take $\delta > 0$ and define a metric on $X \sqcup Y$ by $d|_{X\times X} = d_X$, $d|_{Y\times Y} = d_Y$, and for $x \in X$, $y \in Y$

$$d(x, y) = d_Z(f(x), g(y)) + \delta.$$

The δ is needed to ensure that $d(x, y) = 0$ implies that $x = y$, as it may happen that $X' \cap Y' \neq \varnothing$. In the ambient space $X \sqcup Y$, we then have $d_H(X, Y) \leq r + \delta$. Hence

$$\inf_{d \text{ metric on } X\sqcup Y} \{d_H(X, Y) : X, Y \subseteq X \sqcup Y\} \leq r.$$

Taking the infimum over all ambient spaces Z and isometries f and g then yields the inequality

$$\inf_{d \text{ metric on } X\sqcup Y} \{d_H(X, Y) : X, Y \subseteq X \sqcup Y\} \leq d_{GH}(X, Y).$$

Let us now think about the infimum over all pseudometrics on $X \sqcup Y$. Again one inequality is free, as

$$\inf_{d \text{ pseudometric on } X \sqcup Y} \{d_H(X,Y)\} \leq \inf_{d \text{ metric on } X \sqcup Y} \{d_H(X,Y)\}.$$

Finally, for any pseudometric d on $X \sqcup Y$ we can define the metric space $Z = (X \sqcup Y)/d$. This can only reduce the Hausdorff distance between X and Y, hence

$$d_{GH}(X,Y) \leq \inf_{d \text{ pseudometric on } X \sqcup Y} \{d_H(X,Y)\}.$$

This concludes the proof. □

Next, there are a number of useful reformulations of the Gromov-Hausdorff metric. Calculating the infimum over all embeddings of two given metric spaces into ambient spaces is a difficult thing, even if we can restrict ourselves to $X \sqcup Y$ as the base set of the ambient metric space. Instead, we would want to directly compare X and Y in some manner.

Definition 2.3.4. Let X and Y be two sets. A *correspondence* between X and Y is a set $\mathfrak{R} \subseteq X \times Y$ such that for every $x \in X$ there exists at least one $y \in Y$ such that $(x,y) \in \mathfrak{R}$ and similarly for every $y \in Y$ there exists an $x \in X$ such that $(x,y) \in \mathfrak{R}$.

Example 2.3.5. If $f : X \to Y$ is a surjective map, then

$$\mathfrak{R} = \{(x, f(x)) : x \in X\}$$

defines a correspondence. However, not all correspondences arise in this way, as it is allowed that an $x \in X$ corresponds with more than one $y \in Y$.

Definition 2.3.6. Let $\mathfrak{R}$ be a correspondence between metric spaces X and Y. The *distortion* of $\mathfrak{R}$ is defined by

$$\operatorname{dis} \mathfrak{R} = \sup\{|d_X(x,x') - d_Y(y,y')| : (x,y),(x',y') \in \mathfrak{R}\}.$$

Comparing X and Y in this direct manner gives an equivalent approach of the Gromov-Hausdorff distance, namely [5, Theorem 7.3.25].

Theorem 2.3.7. *For any two metric spaces X and Y*

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\mathfrak{R}} (\operatorname{dis} \mathfrak{R}),$$

where the infimum is taken over all correspondences $\mathfrak{R}$ between X and Y.

Proof. The details can be found at [5, Theorem 7.3.25]. In summary, if X and Y are subspaces of some metric space Z and $d_H(X,Y) < r$, then

$$\mathfrak{R} = \{(x,y) \in X \times Y : d_Z(x,y) < r\}$$

defines a correspondence with distortion less than $2r$. Likewise, if $\mathfrak{R}$ is a correspondence with distortion $2r$, then one can define a metric on $X \sqcup Y$ by setting $d|_{X \times X} = d_X$, $d|_{Y \times Y} = d_Y$ and for $x \in X$, $y \in Y$

$$d(x,y) = \inf\{d_X(x,x') + r + d_Y(y',y) : (x',y') \in \mathfrak{R}\}.$$

This then implies that $d_{GH}(X,Y) \leq r$. □

This theorem has a corollary that will be particularly useful in Chapter 4 of this thesis. Specifically, it leads to a technique to show convergence in the Gromov-Hausdorff topology.

Definition 2.3.8. Let X be a metric space and $\varepsilon > 0$. A set $S \subseteq X$ is called an *ε-net* if $\operatorname{dist}(x,S) \leq \varepsilon$ for every $x \in X$.

Definition 2.3.9. Let X and Y be metric spaces and $f : X \to Y$ an arbitrary map. The *distortion* of f is defined by

$$\operatorname{dis} f = \sup_{x_1, x_2 \in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|.$$

Definition 2.3.10. Let X and Y be metric spaces and $\varepsilon > 0$. A (possibly non-continuous) map $f : X \to Y$ is called an *ε-isometry* if $\operatorname{dis} f \leq \varepsilon$ and $f(X)$ is an ε-net in Y.

Corollary 2.3.11. *Let X and Y be metric spaces and $\varepsilon > 0$. Then*

1. *If $d_{GH}(X, Y) < \varepsilon$, then there exists a 2ε-isometry from X to Y.*
2. *If there exists an ε-isometry from X to Y, then $d_{GH}(X, Y) < 2\varepsilon$.*

Proof. A proof of this corollary can be found at [5, Corollary 7.3.28]. The idea of the proof is that any correspondence $\mathfrak{R} \subseteq X \times Y$ gives rise to a function $f : X \to Y$ by choosing an $f(x) \in Y$ for all $x \in X$ such that $(x, f(x)) \in \mathfrak{R}$. If $\operatorname{dis} \mathfrak{R} < \varepsilon$, this function is a 2ε-isometry. In the other direction, for an ε-isometry one can construct the correspondence

$$\mathfrak{R} = \{(x, y) \in X \times Y : d(y, f(x)) \leq \varepsilon\},$$

which then has a distortion less than 3ε, so Theorem 2.3.7 implies $d_{GH}(X, Y) \leq \frac{3}{2}\varepsilon < 2\varepsilon$. □

Finally, we can now sketch a proof that d_{GH} defines a metric.

Proposition 2.3.12. *The Gromov-Hausdorff distance defines a finite metric on the space of isometry classes of compact metric spaces.*

Proof. The distance is well-defined on this space of isometry classes as for isometric metric spaces the Gromov-Hausdorff distance is zero.

- For all compact metric spaces X and Y, $d_{GH}(X, Y)$ is finite. Indeed, if X and Y are compact, $\operatorname{diam}(X)$ and $\operatorname{diam}(Y)$ are finite. Put a metric d on $X \sqcup Y$ by $d|_{X \times X} = d_X$, $d|_{Y \times Y} = d_Y$ and $d(x, y) = C$. Choosing
$$C \geq \frac{1}{2} \max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}$$
ensures that the triangle inequality holds. If we then consider X and Y as subsets in $X \sqcup Y$, clearly $d_H(X, Y) = C$. Hence,
$$d_{GH}(X, Y) \leq C < \infty.$$
- If $d_{GH}(X, Y) = 0$, then X and Y are isometric. The proof of this can be found in [5, Theorem 7.3.30]. The idea is to use Corollary 2.3.11 and compactness of X and Y to construct distance preserving functions $f : X \to Y$ and $g : Y \to X$. It follows that $f \circ g$ is a distance preserving map from Y to Y, which must be surjective due to Y being compact. In particular, f is then also surjective, which means we have found an isometry $f : X \to Y$.
- Symmetry of d_{GH} follows from the symmetry of d_H.
- The triangle inequality is a routine exercise, see [5, Proposition 7.3.16].

□

Chapter 3

Truncated Geometry

3.1 General Theory

Given a spectral triple $(\mathcal{A}, H, D)$, the most natural way to study truncations is by use of a projection P onto a finite number of eigenvectors of D. Various aspects of how the metric on the spectral triple interacts with such a projection have been studied in [17], but the insight that the triple $(P\mathcal{A}P, PH, PD)$ is a natural approach of studying truncations is due to A. Connes and W. van Suijlekom [14]. This is not obvious, as such a triple is not a spectral triple anymore, $P\mathcal{A}P$ is not even an algebra if P is not an element of $\mathcal{A}$; it is only an operator system. The theoretical framework for studying such objects is given in their paper, dubbing such not-quite spectral triples *operator system spectral triples.*

Definition 3.1.1. An *operator system spectral triple* is a triple $(\mathcal{E}, H, D)$ where $\mathcal{E}$ is a dense subspace of an operator system E in $B(H)$, H is a Hilbert space and D is a self-adjoint operator in H with compact resolvent, such that $[D, T]$ is a bounded operator for all $T \in \mathcal{E}$.

Note that triples of the form $(P\mathcal{A}P, PH, PD)$ are then indeed operator system spectral triples.

It is not always clear what projection P to consider. Recall that for a spectral triple $(\mathcal{A}, H, D)$, the Dirac operator D is self-adjoint and has compact resolvent. In particular, this means that the spectrum of D is real, discrete, and has no finite accumulation points. Hence if we take a projection P_ρ onto the low-lying spectrum of D, defined by $P_\rho := \chi_{[-\rho,\rho]}(D)$, the Hilbert space $P_\rho H$ is finite-dimensional. Projections of this type are therefore a natural option. This is far from the only possibility, as one can also take for example $\chi_{[0,\rho]}(D)$ as we will do in Chapter 4, and already for the canonical spectral triple of the torus non-trivial other options exist [3].

Although such operator system spectral triples do not posses an algebra structure and are therefore more limited than their algebraically better equipped counterparts, the theory of these triples is still rich. As the concept of a state space is still perfectly well defined for operator systems, we can define a metric on the state space $\mathcal{S}(\mathcal{E})$ of an operator system spectral triple $(\mathcal{E}, H, D)$ via the Connes distance formula

$$d(\varphi, \psi) = \sup_{x \in \mathcal{E}} \{|\varphi(x) - \psi(x)| : \|[D, x]\| \leq 1\}.$$

When the operator system $\mathcal{E} = \mathcal{A}$ is a $*$-algebra as well, this reduces to the usual Connes distance formula on the state space of the C^*-algebra $A = \overline{\mathcal{A}}$ [14].

If we are in the situation that we have a spectral triple $(\mathcal{A}, H, D)$, we can cook up a sequence of operator system spectral triples by considering larger and larger projections P_n and employ this theoretical framework to see what structures on $(\mathcal{A}, H, D)$ can be recovered in the limit. A useful tool in studying the state spaces of a sequence of operator system spectral triples is put forward in [41] by W. van Suijlekom. In that paper, he gives a sufficient condition for the Gromov-Hausdorff convergence of such state spaces, namely the existence of so-called C^1-approximate order isomorphisms.

Definition 3.1.2. Let $\{(\mathcal{E}_n, H_n, D_n)\}_n$ be a sequence of operator system spectral triples and let $(\mathcal{E}, H, D)$ also be an operator system spectral triple. A C^1-*approximate order isomorphism* for this set of data is given by linear maps $R_n : E \to E_n$ and $S_n : E_n \to E$ such that

1. the maps R_n and S_n are positive, unital maps;
2. there exist sequences γ_n, γ_n' both converging to zero such that

$$\begin{aligned} \|S_n \circ R_n(a) - a\| &\leq \gamma_n \|a\|_1 , \\ \|R_n \circ S_n(h) - h\| &\leq \gamma_n' \|h\|_1 ; \end{aligned}$$

3. the maps R_n and S_n are contractive with respect to both the operator norms and Lipschitz semi-norms (recall that the Lipschitz seminorm is defined as $\|a\|_1 = \|[D, a]\|$).

When such a collection of maps exists, we can pull back states between E_n and E because R_n and S_n are all positive unital maps:

$$\begin{aligned} R_n^* : \mathcal{S}(E_n) &\to \mathcal{S}(E) & \qquad S_n^* : \mathcal{S}(E) &\to \mathcal{S}(E_n) \\ \varphi_n &\mapsto \varphi_n \circ R_n; & \varphi &\mapsto \varphi \circ S_n. \end{aligned}$$

Even better, these maps directly induce Gromov-Hausdorff convergence of the state spaces $\mathcal{S}(E_n)$ to $\mathcal{S}(E)$ via [41, Proposition 4] and [41, Theorem 5]. To keep this thesis as self-contained as possible, we have included these below.

Proposition 3.1.3. *If (R_n, S_n) is a C^1-approximate order isomorphism for $\{(\mathcal{E}_n, H_n, D_n)\}_n$ and $(\mathcal{E}, H, D)$, then*

1. *For all $\varphi_n, \psi_n \in \mathcal{S}(E_n)$ we have*

$$d_E(R_n^*\varphi_n, R_n^*\psi_n) \leq d_{E_n}(\varphi_n, \psi_n) \leq d_E(R_n^*\varphi_n, R_n^*\psi_n) + 2\gamma_n'.$$

2. *For all $\varphi, \psi \in \mathcal{S}(E)$ we have*

$$d_{E_n}(S_n^*\varphi, S_n^*\psi) \leq d_E(\varphi, \psi) \leq d_{E_n}(S_n^*\varphi, S_n^*\psi) + 2\gamma_n.$$

Proof. For the proof of this proposition we refer to [41, Proposition 4]. □

For the theorem below, which is [41, Theorem 5], we do not refer to the proof as it can be found there because this contains a slight mistake, which we will correct.

Theorem 3.1.4. *Suppose $\{(\mathcal{E}_n, H_n, D_n)\}_n$ and $(\mathcal{E}, H, D)$ are operator system spectral triples such that the topologies on $\mathcal{S}(E_n)$ and $\mathcal{S}(E)$ defined by the metrics d_{E_n} and d_E, respectively, agree with the weak-$*$ topologies on them. If (R_n, S_n) is a C^1-approximate order isomorphism for this set of data, then the state spaces $(\mathcal{S}(E_n), d_{E_n})$ converge to $(\mathcal{S}(E), d_E)$ in Gromov–Hausdorff distance.*

Proof. The Gromov-Hausdorff metric in its usual form is only defined for compact spaces. We therefore have to put some topological requirement on $\mathcal{S}(E_n)$ and $\mathcal{S}(E)$ to ensure that these are compact. In the weak $*$-topology they are compact due to the Banach-Alaoglu Theorem [15, Theorem V.3.1], so if the weak $*$-topology coincides with the topology induced by the metric for all these state spaces, we can rightfully consider the Gromov-Hausdorff distance between them.

We employ the strategy of correspondences to prove Gromov-Hausdorff convergence, via Theorem 2.3.7. Define the correspondences $\mathfrak{R}_n \subseteq \mathcal{S}(E_n) \times \mathcal{S}(E)$ by

$$\mathfrak{R}_n = \{(\varphi_n, R_n^*\varphi_n) : \varphi_n \in \mathcal{S}(E_n)\} \cup \{(S_n^*\varphi, \varphi) : \varphi \in \mathcal{S}(E)\}.$$

One can now simply calculate

$$|d_n(S_n^*\varphi, S_n^*\psi) - d(\varphi, \psi)| \leq 2\gamma_n;$$
$$|d_n(\varphi_n, \psi_n) - d(R_n^*\varphi_n, R_n^*\psi_n)| \leq 2\gamma_n',$$

having used Proposition 3.1.3 for both inequalities, and the somewhat more difficult cross terms

$$\begin{aligned}|d_n(S_n^*\varphi, \psi_n) - d(\varphi, R_n^*(\psi_n))| &\leq |d_n(S_n^*\varphi, \psi_n) - d(R_n^*(S_n^*(\varphi)), R_n^*(\psi_n))| \\ &\quad + |d(R_n^*(S_n^*(\varphi)), R_n^*(\psi_n)) - d(\varphi, R_n^*(\psi_n))| \\ &\leq 2\gamma_n' + d(\varphi, R_n^*(S_n^*(\varphi)) \\ &= 2\gamma_n' + \sup_{x \in \mathcal{E}}\{|\varphi(x) - \varphi(S_n(R_n(x)))| : \|x\|_1 \leq 1\} \\ &\leq 2\gamma_n' + \gamma_n,\end{aligned}$$

which also follows from Proposition 3.1.3 in combination with the definition of C^1-approximate order isomorphisms, Definition 3.1.2. Using this, we can estimate

$$\operatorname{dis}\mathfrak{R}_n \leq 2\gamma_n + 2\gamma_n',$$

which converges to zero. □

3.2 The Truncated Circle

The canonical spectral triple of the circle is a useful toy model to study spectral truncations. Here we can see what kind of data of the original spectral triple can be recovered from its truncation, which gives guidance for what to expect in more general cases. The canonical spectral triple of the circle is of the form

$$\left(C^\infty(S^1), L^2(S^1), -i\frac{d}{dx}\right),$$

both the algebra and Hilbert space of this triple are spanned by the eigenvectors $\{e_n(t) = e^{int}\}_{n\in\mathbb{Z}}$ which form an orthonormal basis in $L^2(S^1)$.

We have to consider what projection we will truncate the spectral triple with. Projecting onto $\operatorname{span}_{\mathbb{C}}\{e_{-n}, \ldots, e_n\}$ is a natural option, which we will use in Chapter 5. However, as we will explain, it is more general to project onto $\operatorname{span}_{\mathbb{C}}\{e_1, \cdots, e_n\}$, denote this projection P_n. Consider what the space $P_nC^\infty(S^1)P_n$ looks like. For any function $f \in C^\infty(S^1)$ we can write the action of P_nfP_n on the finite dimensional Hilbert space $\operatorname{span}_{\mathbb{C}}\{e_k\}_{k=1}^n$ as the $n \times n$ matrix

$$P_nfP_n \sim \begin{pmatrix} \hat{f}(0) & \hat{f}(-1) & \hat{f}(-2) & \cdots & \hat{f}(-n+1) \\ \hat{f}(1) & \hat{f}(0) & \hat{f}(-1) & \cdots & \hat{f}(-n+2) \\ \hat{f}(2) & \hat{f}(1) & \hat{f}(0) & \cdots & \hat{f}(-n+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{f}(n-1) & \hat{f}(n-2) & \hat{f}(n-3) & \cdots & \hat{f}(0) \end{pmatrix},$$

which is a *Toeplitz matrix*. The operator system $C(S^1)^{(n)} := P_nC(S^1)P_n = P_nC^\infty(S^1)P_n$ is called the *Toeplitz operator system*. If we denote the projection onto $\operatorname{span}_{\mathbb{C}}\{e_{-n}, \ldots, e_n\}$ by Q_n and the projection onto $\operatorname{span}_{\mathbb{C}}\{e_1, \ldots e_{2n+1}\}$ by Q_n', then we can quickly see that $Q_nC(S^1)Q_n \cong Q_n'C(S^1)Q_n'$ and also $Q_nL^2(S^1) \cong Q_n'L^2(S^1)$. The only difference between defining the truncated operator system via Q_n or Q_n' is that $Q_n'D = Q_nD + n + 1$. Adding such a constant makes very little difference, as it drops out of every commutator bracket. This argument shows why it is more general to project onto $\operatorname{span}_{\mathbb{C}}\{e_1, \cdots, e_n\}$, and so we will define P_n as the projection onto this subspace for the rest of this chapter and Chapter 4.

3.2.1 State Space

The state space on the truncated circle possesses a very rich structure, which has been studied in-depth in [14]. Of particular interest to this thesis, W. van Suijlekom proved the Gromov-Hausdorff convergence of $\mathcal{S}(C(S^1)^{(n)})$ to $\mathcal{S}(C(S^1))$ [41, Proposition 13] as a direct application of Theorem 3.1.4, using the C^1-approximate order isomorphism-paradigm. The existence of such maps will also be used extensively in dealing with the pure state space in Chapter 4. One half of the pairs of maps (R_n, S_n) that form this C^1-approximate order isomorphism is quite simple.

Lemma 3.2.1. *The map*

$$\begin{aligned} R_n : C(S^1) &\to C(S^1)^{(n)} \\ f &\mapsto P_n f P_n \end{aligned}$$

is positive, unital and contractive with respect to the operator norm and Lipschitz semi-norm.

Proof. Since P commutes with D, we have that

$$\begin{aligned} \|[P_n D, P_n f P_n]\| &= \|P_n D P_n f P_n - P_n f P_n D\| \\ &= \|P_n [D, f] P_n\| \\ &\leq \|[D, f]\| \end{aligned}$$

and so R_n is contractive with respect to the Lipschitz semi-norm. All other claims are elementary. □

We have to find a counterpart that is 'close to an inverse' in order to satisfy Definition 3.1.2. The right candidate is the following, as demonstrated in [41, Proposition 8 and Lemma 11], reproduced below.

Proposition 3.2.2. *Define the map*

$$\begin{aligned} S_n : C(S^1)^{(n)} &\to C(S^1) \\ T \quad &\mapsto \big(x \mapsto \operatorname{Tr}(|\psi\rangle \langle\psi| \, \alpha_x(T))\big), \end{aligned}$$

where α_x is the natural action of S^1 on $C(S^1)^{(n)}$ and $|\psi\rangle = \frac{1}{\sqrt{n}}(e_1 + \cdots + e_n) \in P_n L^2(S^1)$. Then S_n is the adjoint of R_n when equipping $C(S^1)$ with the L^2 norm and $C(S^1)^{(n)}$ with the normalised Hilbert-Schmidt norm, i.e.

$$\langle g, S_n(T)\rangle_{L^2(S^1)} = \frac{1}{n} \operatorname{Tr}\left((R_n(g))^* T\right).$$

Furthermore, S_n is positive, unital and contractive with respect to the operator norm and Lipschitz semi-norm, and

$$S_n(R_n(f))(x) = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \hat{f}(k) e^{ikt} = (F_n * f)(x),$$

where F_n is the Fejér kernel.

Proof. For the proof, we refer to [41]. □

Proposition 3.2.3. *The maps (R_n, S_n) as defined above form a C^1-approximate order isomorphism for the operator systems $\{(C(S^1)^{(n)}, P_n L^2(S^1), P_n D)\}_n$ and $(C^\infty(S^1), L^2(S^1), D)$. Hence the state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}_n$ converge to $(\mathcal{S}(C(S^1)), d)$ in Gromov-Hausdorff distance.*

Proof. See [41]. □

An extension of these results to tori can be found in the master's thesis [3].

3.2.2 Pure State Space

Chapter 4 of this thesis will focus on what happens to the pure state spaces of the truncated circle in the limit, taking projections of larger and larger rank. Traditionally, pure states correspond to 'points'. Indeed, on an algebra of continuous functions on a compact Hausdorff space X, pure states correspond to points of X, see [15, Example VIII.5.13] and the proof of [15, Theorem VIII.8.1]. Similarly, on an algebra of bounded operators on a Hilbert space H, pure states correspond to rays in H by [31, Theorem 5.1.6 and Theorem 5.1.7].

The description of the pure states on $C(S^1)^{(n)}$ is given in [14], in an elegant manner involving a duality of the Toeplitz operator system with another operator system those authors dub the Fejér-Riesz operator system. In this thesis, we will use a more direct approach.

A very useful ingredient for this, is the following decomposition theorem dating from 1911 proven by C. Caratheodory and L. Fejér [7]. We introduce the notation

$$f_z = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & z & z^2 & \cdots & z^{n-1} \end{pmatrix} \in \mathbb{C}^n,$$

which is a column of a so-called Vandermonde matrix.

Theorem 3.2.4. *Any positive Toeplitz matrix $T \in C(S^1)^{(n)}$ of rank $r \leq n-1$ can be uniquely decomposed as $T = \sum_{k=1}^{r} d_k \, |f_{\lambda_k}\rangle \langle f_{\lambda_k}|$ for $d_1, \ldots, d_r > 0$ and $\lambda_1, \ldots, \lambda_r \in S^1$. This is called the Vandermonde decomposition. If the rank of T is n, this decomposition is still possible but not unique.*

We omit the proof and refer to [43]. While we use this classical theorem to classify the pure states on the Toeplitz operator system, it can also be derived by the aforementioned operator system duality, see [14, Theorem 4.14].

Proposition 3.2.5. *A state on $C(S^1)^{(n)}$ is pure if and only if it is a vector state $\varphi_\xi : T \mapsto \langle \xi, T\xi \rangle$ for the unit vector $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1}) \in \mathbb{C}^n$ such that the polynomial $P_\xi(z) := \sum_k \xi_k z^{n-k-1}$ has all its zeroes on S^1.*

Proof. In order for φ_ξ to even define a state, ξ has to be a unit vector. Suppose that $\xi \in \mathbb{C}^n$ is a unit vector such that the polynomial P_ξ as all its zeroes on S^1. To see that the vector state φ_ξ defines a *pure* state, we claim that it suffices to show that for any $\omega \in \mathbb{C}^n$ with

$$\langle \omega, T\omega \rangle \leq \langle \xi, T\xi \rangle \quad \forall\, 0 \leq T \in C(S^1)^{(n)}$$

we have that $\omega \in \mathbb{C}\xi$. By the Hahn-Banach Theorem [15, Theorem III.6.2], any positive linear functional on the Toeplitz matrices can be extended to a positive linear functional on $M_n(\mathbb{C})$, which are spanned by vector states as seen in Proposition 2.1.4. Hence, also all positive linear functionals on $C(S^1)^{(n)}$ are spanned by vector states, so if the above holds we can conclude that φ_ξ is pure due to Proposition 2.1.6.

By the Vandermonde decomposition of Toeplitz matrices, we can write any positive $T \in C(S^1)^{(n)}$ in the form

$$\sum_{k=1}^{n-1} d_k \, |f_{\lambda_k}\rangle \langle f_{\lambda_k}|$$

with $d_1, \ldots, d_{n-1} \geq 0$. Observe that on S^1, we have $\overline{\lambda} = \lambda^{-1}$ and so

$$\langle f_\lambda, \omega \rangle = \sum_{k=0}^{n-1} \omega_k \lambda^{-k} = \lambda^{-n+1} P_\omega(\lambda).$$

Therefore,

$$\langle \omega, T\omega \rangle = \sum_{k=1}^{n-1} d_k \, |\langle f_{\lambda_k}, \omega \rangle|^2 = \sum_{k=1}^{n-1} d_k \, |P_\omega(\lambda_k)|^2 ,$$

and we see that $\langle\omega, T\omega\rangle \leq \langle\xi, T\xi\rangle$ for all positive Toeplitz matrices T if and only if $|P_\omega(\lambda)|^2 \leq |P_\xi(\lambda)|^2$ for all $\lambda \in S^1$.

Suppose we have an $\omega \in \mathbb{C}^n$ with $|P_\omega(\lambda)|^2 \leq |P_\xi(\lambda)|^2$ for all $\lambda \in S^1$. Immediately we see that the polynomial P_ω has the same roots on S^1 as the polynomial P_ξ, but also necessarily with the same multiplicities by considering the asymptotics of P_ξ around its roots. Hence P_ω has exactly the same roots with multiplicities as P_ξ, from which it follows immediately that $P_\omega \in \mathbb{C}P_\xi$, and thus $\omega \in \mathbb{C}\xi$.

On the other hand, suppose that $\xi \in \mathbb{C}^n$ is a vector such that the polynomial P_ξ has roots $\lambda_1, \cdots, \lambda_n$ (counted with multiplicities), of which say λ_n is not an element of S^1. Then $|z - \lambda_n|$ attains a minimum on S^1 which is strictly greater than zero, say $\delta > 0$. Choose some $\lambda \in S^1$ that is not equal to one of $\lambda_1, \cdots \lambda_n$, and note that $|z - \lambda| \leq 2$ on S^1. Then

$$\begin{aligned}
|P_\xi(z)|^2 &= \big|c \prod_{k=1}^{n} (z - \lambda_k)\big|^2 \\
&\geq \delta^2 \big|c \prod_{k=1}^{n-1} (z - \lambda_k)\big|^2 \\
&\geq \frac{\delta^2}{4} \big|c(z - \lambda) \prod_{k=1}^{n-1} (z - \lambda_k)\big|^2,
\end{aligned}$$

so the polynomial $\frac{\delta}{2} c(z - \lambda) \prod_{k=1}^{n-1}(z - \lambda_k) = \frac{\delta}{2}(z - \lambda)P_\xi/(z - \lambda_n)$ corresponds to some $\omega \in \mathbb{C}^n$ with the property that

$$\langle\omega, T\omega\rangle \leq \langle\xi, T\xi\rangle$$

for all positive $T \in C(S^1)^{(n)}$. Clearly, P_ω is not a scalar multiple of P_ξ so $\omega \notin \mathbb{C}\xi$, and hence the vector state φ_ξ is not pure. □

Corollary 3.2.6. *The pure states of $C(S^1)^{(n)}$ are classified by the functions $\widehat{\xi^* * \xi} \in C(S^1)$ where $\xi \in \mathbb{C}^n$ is a unit vector such that the polynomial $\sum_{k=0}^{n-1} \xi_k z^{n-k-1}$ has all its zeroes on S^1. Concretely, any pure state is of the form $P_n f P_n \mapsto \int_{S^1} f \ \widehat{\xi^* * \xi} \ d\lambda$.*

Proof. According to Proposition 3.2.5, the pure states on $C(S^1)^{(n)}$ are given by $T \mapsto \langle\xi, T\xi\rangle$, where the unit vector $\xi \in \mathbb{C}^n$ is such that the polynomial $\sum_k \xi_k z^{n-k-1}$ has all its roots on S^1. A short calculation gives that

$$\langle\xi, P_n f P_n \xi\rangle = \sum_{|j| \leq n-1} (\xi^* * \xi)_j \ \hat{f}(-j),$$

and thus we can use the Plancherel Theorem [37, Theorem 9.13] to conclude that

$$\langle\xi, P_n f P_n \xi\rangle = \int_{S^1} f \ \widehat{\xi^* * \xi} \ d\lambda.$$ □

This corollary essentially characterises the pure states τ_n of $C(S^1)^{(n)}$ by the Radon-Nikodym derivative of $R_n^*(\tau_n)$ as a state (i.e. probability measure, see Section 2.1) on $C(S^1)$. This works since the Fourier basis $\{e_n\}_{n \in \mathbb{Z}}$ forms an orthonormal basis of $C(S^1)$, hence for a function $g \in \mathrm{span}_{\mathbb{C}}\{e_1, \ldots e_n\}$ there is some ambivalence in considering a state $\tau : P_n f P_n \mapsto \int_{S^1} f g d\lambda$ on $C(S^1)^{(n)}$ or its pullback $R_n^* \tau : f \mapsto \int_{S^1} f g d\lambda$. We will exploit this, although responsibly in order to prevent confusion.

Notation. When considering a pure state τ_n on $C(S^1)^{(n)}$, we will somewhat abusively refer to the Radon-Nikodym derivative of $R_n^* \tau_n$, with respect to the normalised Haar-measure $d\lambda$ on S^1, as $\frac{d\tau_n}{d\lambda}$ because this function uniquely defines the pure state τ_n. In the other way around, if f is a function of the form such that it defines a pure state on $C(S^1)^{(n)}$, we will denote that pure state τ_f. In summary, $\frac{d\tau_f}{d\lambda} = f$.

Proposition 3.2.7. *If τ_n is a pure state on $C(S^1)^{(n)}$, then*

$$\frac{d\tau_n}{d\lambda}(t) = \widehat{\xi^* * \xi}(t) = c\prod_{j=1}^{n-1}(2 - 2\cos(t - \theta_j)),$$

where $e^{i\theta_j}$ are the roots of the polynomial $\sum_k \xi_k z^{n-k-1}$ and $c \in \mathbb{R}$ is a scaling factor such that $\frac{d\tau_n}{d\lambda}$ integrates to 1. Likewise, any function of this form defines a pure state.

Proof. According to Corollary 3.2.6, any pure state τ_n on $C(S^1)^{(n)}$ corresponds to a function of the form $\frac{d\tau_n}{d\lambda} = \widehat{\xi^* * \xi}$. Since the polynomial $P_\xi(z) = \sum_{k=0}^{n-1} \xi_k z^{n-k}$ has all its zeroes on S^1, it is of the form $P_\xi(z) = c\prod_{j=1}^{n-1}(z - e^{i\theta_j})$. Observe that $\widehat{\xi^* * \xi} = |\hat{\xi}|^2$, and $\hat{\xi}$ is simply the polynomial P_ξ restricted to S^1. Hence,

$$\begin{aligned}
\widehat{\xi^* * \xi}(t) &= |\hat{\xi}|^2(t) \\
&= \left|P_\xi(e^{it})\right|^2 \\
&= c\prod_{j=1}^{n-1}\left|e^{it} - e^{i\theta_j}\right|^2 \\
&= c\prod_{j=1}^{n-1}(2 - 2\cos(t - \theta_j)).
\end{aligned}$$

From this calculation we can also see that $c\prod_{j=1}^{n-1}(2 - 2\cos(t - \theta_j))$ must integrate to 1, since $\|\hat{\xi}\|_2 = \|\xi\| = 1$.

For the other way around, the above proves that any polynomial of the form $c\prod_{j=1}^{n-1}(2-2\cos(t-\theta_j))$ that integrates to 1 is equal to $\widehat{\xi^* * \xi}(t)$ for some $\xi \in \mathbb{C}^n$ such that the polynomial $\sum_k \xi_k z^{n-k-1}$ has all its zeroes on S^1 and $\|\xi\| = 1$. According to Corollary 3.2.6, this indeed defines a pure state on $C(S^1)^{(n)}$. □

Corollary 3.2.8. *If $f \in C(S^1)$ defines a pure state τ_f on $C(S^1)^{(n)}$, then rotations of this function also define pure states.*

Proof. This follows immediately from the form of f proven in Proposition 3.2.7. □

Chapter 4

Convergence of the Pure State Space of the Truncated Circle

As we have seen, the state spaces of the truncated circle converge to the state space of the circle (Proposition 3.2.3), and we know what the pure states on the truncated circle look like. This might give one the courage to try to prove that the pure state spaces $(\mathcal{P}(C(S^1)^{(n)}), d_n)$ converge to $(\mathcal{P}(C(S^1)), d) \cong (S^1, d)$, but this turns out to not be the case.

4.1 No convergence to S^1

The strategy we will employ is that we will give a lower bound for the distortion of any map from $\mathcal{P}(C(S^1)^{(n)})$ to S^1, which gives a lower bound for the Gromov-Hausdorff distance between these spaces according to Corollary 2.3.11. This will be achieved by showing that it is inevitable that there exist $\tau_{g_n}, \tau_{f_n} \in \mathcal{P}(C(S^1)^{(n)})$ such that they are both mapped to points close to each other in S^1, but with their relative distance $d_n(\tau_{f_n}, \tau_{g_n})$ large.

Lemma 4.1.1. *The Fejér-kernel rotated by $\lambda = e^{i\theta}$*

$$f_n^\lambda(x) = \sum_{|k| \leq n-1} \left(1 - \frac{|k|}{n}\right) e^{ik(\theta - x)},$$

defines a pure state on $C(S^1)^{(n)}$ in the sense of Corollary 3.2.6.

Proof. Take the polynomial $\sum_{k=0}^{n-1} z^k$. Observe that

$$(z-1)\sum_{k=0}^{n-1} z^k = z^n - 1,$$

and hence the roots of $\sum_{k=0}^{n-1} z^k$ are precisely the nth roots of unity with the exception of 1 itself. The coefficients of this polynomial form the (normalised) vector $\xi = \frac{1}{\sqrt{n}}(1, ..., 1) \in \mathbb{C}^n$, and $\widehat{\xi^* * \xi}$ defines a pure state. A simple calculation gives that

$$\xi^* * \xi(k) = \sum_j \bar{\xi}_j \xi_{j+k} = 1 - \frac{|k|}{n},$$

hence $\sum_{|k| \leq n-1} \left(1 - \frac{|k|}{n} e^{ikx}\right) = \sum_{|k| \leq n-1} \left(1 - \frac{|k|}{n} e^{-ikx}\right)$ defines a pure state on $C(S^1)^{(n)}$. As noted in Corollary 3.2.8, rotations of this pure state are then also pure states. $\square$

Compare these pure states $\tau_{f_n^\lambda}$ to the states on $C(S^1)$ denoted $\Psi^\sharp_{x,N}$ in [17, Section 5.4]. The relation between these is that $R_n^*(\tau_{f_n^\lambda}) = \Psi^\sharp_{\lambda,n}$. Note that, as those authors rightfully comment, the states $R_n^*(\tau_{f_n^\lambda})$ are not pure on $C(S^1)$. For the next lemma, note the similarity with [17, Proposition 5.11]. The difference is that in this thesis we are talking about the *intrinsic* distance on the truncated circle, so we have to add a a small step to move between the intrinsic distance and the distance on the whole spectral triple.

Lemma 4.1.2. *Take points $\lambda = e^{i\theta}$ and $\mu = e^{i\varphi}$ on S^1, and states $\tau_{f_n^\lambda}, \tau_{f_n^\mu} \in \mathcal{P}(C(S^1)^{(n)})$ defined by Fejér kernels like in Lemma 4.1.1. Then*

$$\lim_{n\to\infty} d_n(\tau_{f_n^\lambda}, \tau_{f_n^\mu}) = d(\lambda, \mu).$$

Proof. By rotation invariance of S^1, we can take $\theta, \varphi \in [0, 2\pi)$ such that $d(\lambda, \mu) = \frac{1}{2\pi}(\varphi - \theta) \leq 1/2$ without loss of generality.

The distance $d_n(\tau_{f_n^\lambda}, \tau_{f_n^\mu})$ is hard to calculate, whereas the distance formula on $\mathcal{S}(C(S^1))$ is much better understood. In fact, $d(R_n^*(\tau_{f_n^\lambda}), R_n^*(\tau_{f_n^\mu}))$ can be calculated with help of the paper [6]. This is helpful since the distance formula on $\mathcal{S}(C(S^1)^{(n)})$ agrees with the distance formula on $\mathcal{S}(C(S^1))$ in the limit (see Proposition 3.2.3). Define

$$\begin{aligned} \alpha_n :& [0, 2\pi] \to \mathbb{R} \\ & t \mapsto \frac{1}{2\pi}\int_0^t f_n^\lambda(x)dx - \frac{1}{2\pi}\int_0^t f_n^\mu(x)dx \end{aligned}$$

and

$$a_\alpha = \sup\{t \in \mathbb{R} : \lambda(\{x \in [0, 2\pi] : \alpha_n(x) \geq t\}) > 1/2\},$$

then $d(R_n^*(\tau_{f_n^\lambda}), R_n^*(\tau_{f_n^\mu})) = \frac{1}{2\pi}\int_0^{2\pi} |\alpha_n(t) - a_\alpha|\, dt$. As n grows to infinity, $\frac{1}{2\pi}\int_0^t f_n^\lambda(x)dx$ converges uniformly to $H(t-\theta)$ outside $t = \theta$, where H is the Heaviside step-function. Thus α_n converges uniformly to $H(t-\theta) - H(t-\varphi) = \chi_{[\theta,\varphi]}(t)$ outside $t = \theta$ and $t = \varphi$. Hence by elementary analysis a_α converges to 0. An illustration of the function α_n can be found in Figure 4.1.

Finally we can calculate that as n grows to infinity,

$$\begin{aligned} d(R_n^*(\tau_{f_n^\lambda}), R_n^*(\tau_{f_n^\mu})) &= \frac{1}{2\pi}\int_0^{2\pi} |\alpha_n(t) - a_\alpha|\, dt \\ &\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \chi_{[\theta,\varphi]}(t)dt \\ &= \frac{1}{2\pi}(\varphi - \theta). \end{aligned}$$

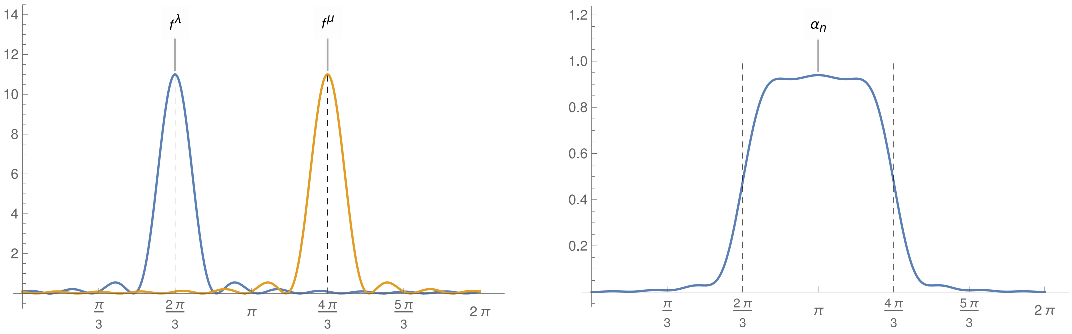
Therefore

$$\lim_{n\to\infty} d(R_n^*(\tau_{f_n^\lambda}), R_n^*(\tau_{f_n^\mu})) = d(\lambda, \mu),$$

thus according to Proposition 3.2.3 also

$$\lim_{n\to\infty} d_n(\tau_{f_n^\lambda}, \tau_{f_n^\mu}) = d(\lambda, \mu). \qquad \square$$

The lemma above indicates that the distance between pure states $\tau_{f_n^\lambda}$ corresponding to these Fejér kernels asymptotically agrees with the distance between the points they are centered around. That means that we can recover $\mathcal{P}(C(S^1)) \cong S^1$ in the limit with only this type of pure states. In other words, the truncated circle has enough pure states to recover the whole circle. See also [17, Proposition 5.12], with again the added subtlety that *we* are talking about the intrinsic distance on the truncated circle, although in the limit this is the same as the distance on the whole spectral triple after a pullback.



(a) Here the functions f_n^λ and f_n^μ are depicted with $\lambda = 2\pi/3$ and $\mu = 4\pi/3$ for $n = 10$.

(b) A plot of the function $\alpha_n(t) = \frac{1}{2\pi}\int_0^t (f_n^\lambda(x) - f_n^\mu(x))dx$.

Figure 4.1: An illustration of the function α_n as it appears in the proof of Lemma 4.1.2.

Proposition 4.1.3. *Define the subsets $\mathcal{F}_n \subset \mathcal{P}(C(S^1)^{(n)})$ by*

$$\mathcal{F}_n := \{\tau_{f_n^\lambda} : \lambda \in S^1\}.$$

Then the sequence of metric spaces $(\mathcal{F}_n, d_n)$ converges to (S^1, d) in Gromov-Hausdorff sense.

Proof. Define the sets

$$\mathfrak{R}_n = \{(\tau_{f_n^\lambda}, \lambda) : \lambda \in S^1\} \subset \mathcal{F}_n \times S^1.$$

Because the elements of $\mathcal{F}_n$ are labeled by S^1, the projections of $\mathfrak{R}_n$ onto the first and second coordinate are both surjective, making these sets correspondences (Definition 2.3.4). The functions

$$\begin{aligned} S^1 \times S^1 &\to \mathbb{R}_{\geq 0} \\ (\lambda, \mu) &\mapsto \left| d_n(\tau_{f_n^\lambda}, \tau_{f_n^\mu}) - d(\lambda, \mu) \right| \end{aligned}$$

are continuous, and monotonically decreasing to 0 according to Lemma 4.1.2. By Dini's Theorem [36, Theorem 7.13], this means that they uniformly converge to 0 and hence this gives immediately that

$$\lim_{n\to 0} \operatorname{dist} \mathfrak{R}_n = 0.$$

Therefore we can conclude, using Theorem 2.3.7, that

$$\lim_{n\to\infty} d_{GH}(\mathcal{F}_n, S^1) = 0. \qquad \square$$

Note that this result is also comparable to a result proven by L. Glaser and A. Stern [20], which asserts that the pure state space of any spectral triple is the Gromov-Hausdorff limit of 'localised' (not necessarily pure) states on the truncated spectral triple.

However, the pure state space of the truncated circle is *too large* to converge to just (the pure states on) the circle. This claim will be made more precise in the form of the next proposition, for which we will need the following lemma.

Lemma 4.1.4. *Let $0 < \varepsilon < 1/6$, and let $f : S^1 \to S^1$ be a map with $\operatorname{dis} f < \varepsilon$. Then $f(S^1)$ is an ε-net in S^1.*

Proof. Suppose that f is a map with $\operatorname{dis} f < \varepsilon < 1/6$, but that there exists some point $y_1 \in S^1$ with $\overline{B_\varepsilon(y_1)} \cap f(S^1) = \varnothing$. This can happen only if there also exist points $\rho_1, \rho_2 \in S^1$ such that one of the two arcs connecting $f(\rho_1)$ and $f(\rho_2)$ has a length strictly greater than 2ε and does not intersect $f(S^1)$. Without loss of generality, we can assume that y_1 is exactly in between $f(\rho_1)$ and $f(\rho_2)$. See Figure 4.2 for an illustration.

There are now two cases.

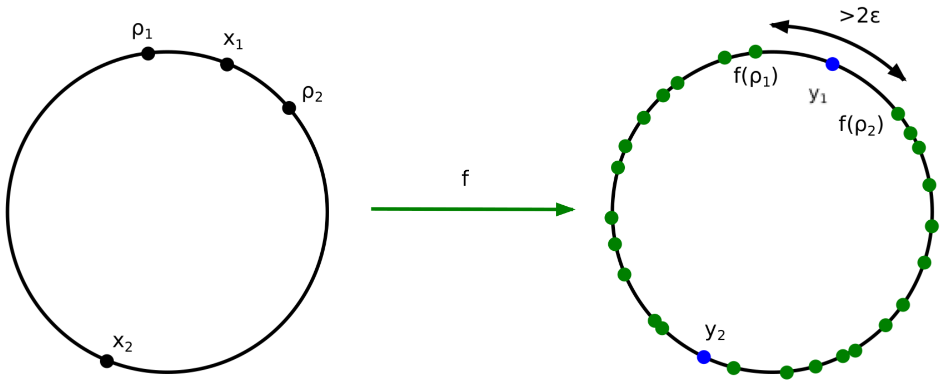


Figure 4.2: An illustration of the situation in the proof of Lemma 4.1.4.

1. First there is the pathological situation where $d(f(\rho_1), f(\rho_2)) \leq 2\varepsilon$. This means that $f(S^1)$ is contained in the *other* arc connecting $f(\rho_1)$ and $f(\rho_2)$, which necessarily has a diameter less than 2ε. But then there are two points $x_1, x_2 \in S^1$ that are antipodes – with relative distance $1/2$ – that are both mapped into a set with diameter less than 2ε. This is a contradiction if $\varepsilon < 1/6$ because $\operatorname{dis} f < \varepsilon$.

2. We are left with the situation $d(f(\rho_1), f(\rho_2)) > 2\varepsilon$. Consider the (antipodal) points $x_1, x_2 \in S^1$ that lie exactly in between ρ_1 and ρ_2, i.e. the points such that $d(x_i, \rho_1) = d(x_i, \rho_2)$ for $i = 1, 2$. Likewise, y_1 lies in between $f(\rho_1)$ and $f(\rho_2)$; denote its antipode by y_2. Then since $\operatorname{dis} f < \varepsilon$, the distance of $f(x_1)$ to $f(\rho_1)$ differs at most 2ε from the distance of $f(x_1)$ to $f(\rho_2)$. Because $d(f(\rho_1), f(\rho_2)) > 2\varepsilon$, the set of points whose distances to $f(\rho_1)$ and $f(\rho_2)$ differ by at most 2ε is $B_\varepsilon(y_1) \cup B_\varepsilon(y_2)$. Hence $f(x_1) \in B_\varepsilon(y_1) \cup B_\varepsilon(y_2)$, and by the same argument also $f(x_2) \in B_\varepsilon(y_1) \cup B_\varepsilon(y_2)$. But $B_\varepsilon(y_1) \cap f(S^1) = \varnothing$, so we see that the antipodes x_1 and x_2 both have to be mapped into the same set with diameter 2ε. Again, this is a contradiction.

We therefore see that no such points ρ_1 and ρ_2 can exist, hence $f(S^1)$ must form an ε-net in S^1. □

Proposition 4.1.5. *The sequence of metric spaces $(\mathcal{P}(C(S^1)^{(n)}), d_n)$ does not converge to (S^1, d) in the sense of Gromov-Hausdorff convergence.*

Proof. Suppose that $(\mathcal{P}(C(S^1)^{(n)}), d_n)$ does converge to (S^1, d). According to Corollary 2.3.11, we should then be able to find γ_n-isometries

$$\psi_n : \mathcal{P}(C(S^1)^{(n)}) \to S^1$$

with $\lim_{n\to\infty} \gamma_n = 0$.

For each n, take the polynomial $\sum_k \xi_k z^{n-k-1} = \frac{1}{\sqrt{2}}(1 - z^{n-1})$, i.e. $\xi = \frac{1}{\sqrt{2}}(-1, 0, ..., 0, 1)$. Then the function

$$g_n(t) := \widehat{\xi^* * \xi}(t) = \frac{1}{2}(-e^{-i(n-1)t} + 2 - e^{i(n-1)t}) = 1 - \cos((n-1)t)$$

defines a pure state τ_{g_n} on $C(S^1)^{(n)}$ in the manner of Proposition 3.2.5. This pure state will be mapped to some $\mu_n \in S^1$ by the ε-isometry ψ_n.

Note that the restriction of ψ_n to the Fejér-kernels $\mathcal{F}_n$ is a map from S^1 to S^1 with distortion less than γ_n. Hence due to Lemma 4.1.4, the set $\psi_n(\mathcal{F}_n)$ forms a γ_n-net in S^1. This means that we can choose a pure (Fejér kernel) state $\tau_{f_n^{\lambda_n}} \in \mathcal{P}(C(S^1)^{(n)})$ such that it is mapped to some $\rho_n \in S^1$

with $d(\rho_n, \mu_n) < \gamma_n$. We then have that

$$\operatorname{dis} f \geq \left| d_n(\tau_{g_n}, \tau_{f_n^{\lambda_n}}) - d(\rho_n, \mu_n) \right| \geq d_n(\tau_{g_n}, \tau_{f_n^{\lambda_n}}) - \gamma_n.$$

However, as we will show now, the distance between τ_{g_n} and $\tau_{f_n^{\lambda_n}}$ converges to $\frac{1}{4}$ as $n \to \infty$ regardless of the points λ_n, which gives a contradiction.

Let us simply carry out the calculation. First of all, the distance formula on the truncated circle is difficult to compute directly. Since the distance formulas on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ agree in the limit (Proposition 3.2.3), we can pass to the easier distance formula by noting that

$$\lim_{n\to\infty} d_n(\tau_{f_n^{\lambda_n}}, \tau_{g_n}) = \lim_{n\to\infty} d(R_n^*(\tau_{f_n^{\lambda_n}}), R_n^*(\tau_{g_n})).$$

There is one more complication, as it is hard to calculate $\lim_{n\to\infty} d(R_n^*(\tau_{f_n^{\lambda_n}}), R_n^*(\tau_{g_n}))$ directly since the points λ_n may not all be the same. However, it is possible to calculate $\lim_{n\to\infty} d(R_n^*(\tau_{f_n^{\lambda}}), R_n^*(\tau_{g_n}))$ for λ kept constant. Since it turns out these limits are all the same, $\lim_{n\to\infty} d(R_n^*(\tau_{f_n^{\lambda_n}}), R_n^*(\tau_{g_n}))$ then must equal that value as well by compactness of S^1 and Dini's Theorem [36, Theorem 7.13].

Once more, we will use the paper [6] to calculate the distance $d(R_n^*(\tau_{f_n^{\lambda}}), R_n^*(\tau_{g_n}))$. Recall that

$$\begin{aligned} \alpha_n :&[0, 2\pi] \to \mathbb{R} \\ t \mapsto& \frac{1}{2\pi}\int_0^t f_n^{\lambda}(x)dx - \frac{1}{2\pi}\int_0^t g_n(x)dx \end{aligned}$$

and

$$a_\alpha = \sup\{t \in \mathbb{R} : \lambda(\{x \in [0, 2\pi] : \alpha_n(x) \geq t\}) > 1/2\},$$

so that $d(R_n^*(\tau_{f_n^{\lambda}}), R_n^*(\tau_{g_n})) = \frac{1}{2\pi}\int_0^{2\pi} |\alpha_n(t) - a_\alpha|\, dt$. As before, as n grows to infinity, $\frac{1}{2\pi}\int_0^t f_n^{\lambda}(x)dx$ converges uniformly to $H(t-\theta)$ outside $t = \theta$. Next,

$$\frac{1}{2\pi}\int_0^t g_n(x)dx = \frac{1}{2\pi}\left[1 - \frac{1}{n-1}\sin((n-1)x)\right]_0^t = \frac{t}{2\pi} - \frac{1}{2\pi(n-1)}\sin((n-1)t),$$

which converges uniformly to $\frac{t}{2\pi}$ as n grows to infinity. Thus $\lim_{n\to\infty} \alpha_n(t) = H(t-\theta) - \frac{t}{2\pi}$. This also makes it clear that $a_\alpha = \frac{1}{2} - \frac{\theta}{2\pi}$ by elementary analysis. An illustration of these functions and calculations can be found in Figure 4.3.

Finally, the result is a simple computation:

$$\begin{aligned} d(R_n^*(\tau_{f_n^{\lambda}}), R_n^*(\tau_{g_n})) &= \frac{1}{2\pi}\int_0^{2\pi} |\alpha_n(t) - a_\alpha|\, dt \\ &\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left| H(t-\theta) - \frac{t}{2\pi} - \left(\frac{1}{2} - \frac{\theta}{2\pi}\right)\right| dt \\ &= \frac{1}{4}. \end{aligned}$$

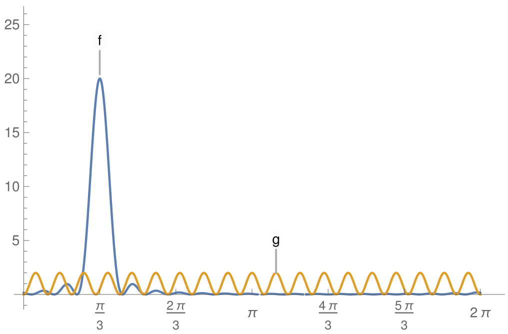
In conclusion, we have shown that for any γ_n-isometry $\psi_n : \mathcal{P}(C(S^1)^{(n)}) \to S^1$

$$\operatorname{dis} \psi_n \geq d_n(\tau_{f_n^{\lambda_n}}, \tau_{g_n}) - \gamma_n,$$

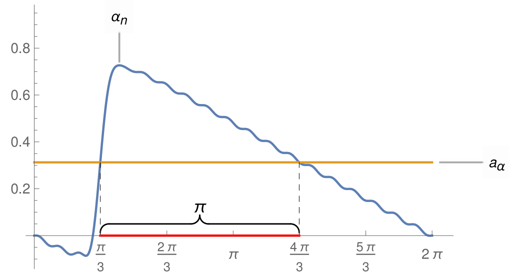
while at the same time

$$\lim_{n\to\infty} d_n(\tau_{f_n^{\lambda_n}}, \tau_{g_n}) = \frac{1}{4}.$$

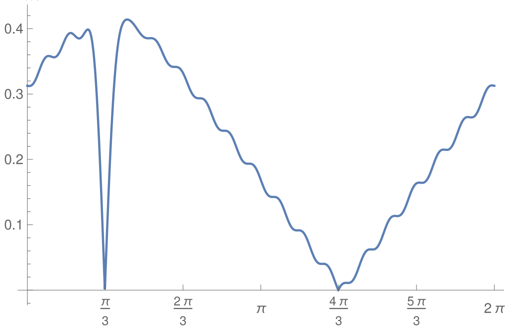
Hence we can infer that $(\mathcal{P}(C(S^1)^{(n)}), d_n)$ cannot converge to (S^1, d) as metric spaces in the sense of Gromov-Hausdorff convergence. □



(a) Here the functions f_n^λ and g_n depicted with $k\frac{2\pi}{m} = \pi/3$ for $n = 20$.



(b) An indication of how a_α is determined.



(c) A plot of $|\alpha_n - a_\alpha|$. Three distinct regions can be distinguished: the interval between 0 and $\pi/3$, between $\pi/3$ and $4\pi/3$ and finally between $4\pi/3$ to 2π.

Figure 4.3: An illustration of the calculations performed in Proposition 4.1.5.

4.2 Convergence to $\mathcal{S}(C(S^1))$

Somewhat unexpectedly, $\mathcal{P}(C(S^1)^{(n)})$ does not converge to S^1 as metric spaces in the Gromov-Hausdorff sense. The problem seemed to be, in fact, that the spaces $\mathcal{P}(C(S^1)^{(n)})$ were too large to converge to S^1. It may seem bold, but we now propose $\mathcal{S}(C(S^1))$ as the limit.

Immediately we have a suitable candidate for ε-isometries between these spaces, in order to prove Gromov-Hausdorff convergence by the strategy of using Corollary 2.3.11. Consider

$$R_n^* : \mathcal{P}(C(S^1)^{(n)}) \to \mathcal{S}(C(S^1))$$

by pullback with the compression P_n as introduced in Section 3.2.1. Then $\operatorname{dis} R_n^* \xrightarrow{n\to\infty} 0$ since the distance formula on $\mathcal{S}(C(S^1)^{(n)})$ agrees with the distance formula on $\mathcal{S}(C(S^1))$ in the limit, as we have also used before (Proposition 3.2.3). If we can now establish that $R_n^*(\mathcal{P}(C(S^1)^{(n)}))$ is an ε-net if we choose n large enough, we will have shown that $\mathcal{S}(C(S^1))$ is the Gromov-Hausdorff limit of $\mathcal{P}(C(S^1)^{(n)})$ as metric spaces. First, we will prove that all states on the circle can be approximated by the pullback of pure states in $\mathcal{P}(C(S^1)^{(n)})$. Next, we check that this can be done uniformly in n.

4.2.1 Approximating states

In this subsection, an elaborate scheme will be carried out to prove that any state on the circle can be approximated by the pullbacks of pure states on the truncated circle. We will do this in three steps of increasing difficulty as follows:

A state ψ on $C(S^1)$

$\downarrow$ Can be approximated by

A state $\sum_{i=1}^m t_i \mathrm{ev}_{\lambda_i}$ on $C(S^1)$, which is a convex combination of evaluations at the m-roots of unity

$\downarrow$ Can be approximated by

A state

$$\sum_{i=1}^m \frac{\frac{d\varphi}{d\lambda}(\lambda_i)}{\sum_{j=1}^m \frac{d\varphi}{d\lambda}(\lambda_j)} \mathrm{ev}_{\lambda_j}$$

on $C(S^1)$, where λ_j are the m-roots of unity and φ is a pure state on $C(S^1)^{(n)}$

$\downarrow$ Can be approximated by

A state $R_n^*\chi$ on $C(S^1)$ where χ is a pure state on $C(S^1)^{(n)}$

Recall that a pure state φ on $C(S^1)^{(n)}$ is uniquely characterised by the Radon-Nikodym derivative with respect to the normalised Haar-measure on S^1 of $R_n^*\varphi$, which is a state (i.e. a probability measure) on $C(S^1)$, and that we denote this Radon-Nikodym derivative $\frac{d\varphi}{d\lambda}$ instead of $\frac{dR_n^*\varphi}{d\lambda}$ to ease notation. See also Section 3.2.

The first step in the outlined scheme is by far the easiest, as the set

$$\left\{ \sum_{i=1}^m t_i \mathrm{ev}_{\lambda_i} \ : \ m \in \mathbb{N},\ 0 \le t_i \le 1, \sum_{i=1}^m t_i = 1, \lambda_i^m = 1 \right\},$$

i.e. convex combinations of evaluations at the roots of unity, is dense in $\mathcal{S}(C(S^1))$ with respect to the weak *-topology. Since S^1 is a compact metric space, the weak *-topology on $\mathcal{S}(C(S^1))$ agrees with the topology induced by the Monge-Kantorovich metric [33].

The second step of the scheme can be done in a single lemma.

Lemma 4.2.1. *Let $\lambda_1, ..., \lambda_m$ be the solutions of $\lambda^m = 1$ (the m-roots of unity), and take any state of the form $\sum_{i=1}^m t_i ev_{\lambda_i}$ with $\sum_{i=1}^m t_i = 1$ and $t_i \geq 0$ for all i. Then for every $l \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > 0$ we can find $\varphi \in \mathcal{P}(C(S^1)^{(m+1+l)})$ such that on $C(S^1)$*

$$d\left(\sum_{i=1}^m t_i ev_{\lambda_i}, \sum_{j=1}^m \frac{\frac{d\varphi}{d\lambda}(\lambda_j)}{\sum_{i=1}^m \frac{d\varphi}{d\lambda}(\lambda_i)} ev_{\lambda_j}\right) < \varepsilon.$$

Proof. Let us first prove the case $l = 0$. Take the pure state φ_N on $C(S^1)^{m+1}$ defined by $\frac{d\varphi_N}{d\lambda}(t) = c\prod_{j=1}^m (1 - \cos(t - \lambda_j + \varepsilon_j))$, with $\varepsilon_i = \sqrt{\frac{2t_i}{N}}$. This indeed corresponds to a pure state in $\mathcal{P}(C(S^1)^{(m+1)})$ according to Proposition 3.2.7. To emphasise, $\frac{d\varphi_N}{d\lambda}$ indicates the Radon-Nikodym derivative of $R^*_{(m+1)}\varphi_N$ as a probability measure on $C(S^1)$ with respect to the normalised Haar-measure $d\lambda$ on S^1, we mean no usual derivative of a function. Consider what happens at the points λ_i if we take N large, i.e. ε_i small.

Note that the derivative of $1 - \cos(t)$ is at most 1 everywhere, so we have for all the factors of φ_N that as $N \to \infty$

$$\begin{aligned} 1 - \cos(\lambda_i - \lambda_j + \varepsilon_j) &= 1 - \cos(\lambda_i - \lambda_j) + \mathcal{O}(\varepsilon_i) \\ &= 1 - \cos(\lambda_i - \lambda_j) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \end{aligned}$$

For the factor with $\lambda_i = \lambda_j$ note that by Taylor expansion $1 - \cos(\varepsilon_i)$ expands as

$$\begin{aligned} 1 - \cos(\varepsilon_i) &= \frac{\varepsilon_i^2}{2} + \mathcal{O}(\varepsilon_i^3) \\ &= \frac{t_i}{N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right). \end{aligned}$$

Combined, this gives that

$$\begin{aligned} \frac{d\varphi_N}{d\lambda}(\lambda_i) &= c\left(\frac{t_i}{N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right)\right)\left(\prod_{j\neq i}\left((1 - \cos(\lambda_i - \lambda_j)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right)\right) \\ &= c\frac{t_i}{N}\prod_{j\neq i}(1 - \cos(\lambda_i - \lambda_j)) + \mathcal{O}\left(\frac{1}{N^{3/2}}\right). \end{aligned}$$

Notice that $c\prod_{j\neq i}(1 - \cos(\lambda_i - \lambda_j))$ has the same value for all λ_i by symmetry. Hence if we pass these values to the projective space we end up with the ratio

$$\begin{aligned} \left[\frac{d\varphi_N}{d\lambda}(\lambda_1) : \cdots : \frac{d\varphi_N}{d\lambda}(\lambda_m)\right] &= \left[\frac{t_1}{N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right) : \cdots : \frac{t_m}{N} + \mathcal{O}\left(\frac{1}{N^{3/2}}\right)\right] \\ &= \left[t_1 + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) : \cdots : t_m + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right]. \end{aligned}$$

As the topology on $\mathbb{R}P^{m-1}$ is the quotient topology induced by the topology on $\mathbb{R}^m$, we see that the ratio above converges to $[t_1 : \cdots : t_m]$ in $\mathbb{R}P^{m-1}$. This proves that we can find $\varphi_N \in \mathcal{P}(C(S^1)^{(m+1)})$ such that $\left[\frac{d\varphi_N}{d\lambda}(\lambda_1) : \cdots : \frac{d\varphi_N}{d\lambda}(\lambda_m)\right]$ is arbitrarily close to $[t_1 : \cdots : t_m]$ in the metric that metrizes the topology of the projective space $\mathbb{R}P^{m-1}$. We can characterise the topological space $\mathbb{R}P^{m-1}$ as

$\widehat{S^m}/\sim$ with $\widehat{S^m} = \{x \in \mathbb{R}^m : \|x\|_1 = 1\}$ the unit sphere with respect to the L^1-norm, inheriting the subspace topology from $\mathbb{R}^m$, and the equivalence relation $\sim$ identifying x with $-x$. Since $\frac{1}{\sum_{j=1}^m \frac{d\varphi_N}{d\lambda}(\lambda_j)}\left(\frac{d\varphi_N}{d\lambda}(\lambda_1), \ldots, \frac{d\varphi_N}{d\lambda}(\lambda_m)\right)$ and $(t_1, \ldots, t_m)$ are already living on $\widehat{S^m}$ – even in the same quadrant – the vectors $\frac{1}{\sum_{j=1}^m \frac{d\varphi_N}{d\lambda}(\lambda_j)}\left(\frac{d\varphi_N}{d\lambda}(\lambda_1), \ldots, \frac{d\varphi_N}{d\lambda}(\lambda_m)\right)$ then also converge to $(t_1, \ldots, t_m)$ in $\mathbb{R}^m$ as $N \to \infty$. It is immediate that therefore that the states $\frac{1}{\sum_{j=1}^m \frac{d\varphi_N}{d\lambda}(\lambda_j)} \sum_{j=1}^m \frac{d\varphi_N}{d\lambda}(\lambda_j)\mathrm{ev}_{\lambda_j}$ converge to $\sum_{j=1}^m t_j \mathrm{ev}_{\lambda_j}$ in the weak*-topology. Again we can use that the Monge-Kantorovich metric induces the weak*-topology to conclude that we can choose N such that

$$d\left(\sum_{i=1}^m t_i \mathrm{ev}_{\lambda_i}, \sum_{j=1}^m \frac{\frac{d\varphi_N}{d\lambda}(\lambda_j)}{\sum_{i=1}^m \frac{d\varphi_N}{d\lambda}(\lambda_i)} \mathrm{ev}_{\lambda_j}\right) < \varepsilon.$$

The cases $l \geq 1$ follow more or less immediately. If we choose some point μ on the circle that is not equal to any of the λ_j, we can guarantee that $1 - \cos(t - \mu)$ has no roots in the points $\lambda_1, \ldots, \lambda_m$. For any $l \in \mathbb{N}$, we can take $\varphi_N \in \mathcal{P}(C(S^1)^{(m)})$ such that the ratio $[\frac{d\varphi_N}{d\lambda}(\lambda_1) : \cdots : \frac{d\varphi_N}{d\lambda}(\lambda_m)]$ is arbitrarily close to

$$\left[\frac{t_1}{(1 - \cos(\lambda_1 - \mu))^l} : \cdots : \frac{t_m}{(1 - \cos(\lambda_m - \mu))^l}\right]$$

by the argument for the case $l = 0$ above. Then $\frac{d\varphi_N}{d\lambda}(1 - \cos(t - \mu))^l$ defines the pure state (up to scaling) in $\mathcal{P}(C(S^1)^{(m+1+l)})$ that satisfies the statement in the lemma. $\square$

For the third and final step in our scheme, we need to prove that the states of this type of convex combination can be approximated by the pullback of some pure state on the truncated circle. To accomplish that, we will need the following propositions.

Proposition 4.2.2. *Let $K \subseteq X$ be some compact subset of $\mathbb{R}^n$ and let $f \in C(K)$ be a positive function attaining its maximum in the unique point x_0. Then the sequence of linear functionals $(\tau_n)_{n\in\mathbb{N}}$ defined by $\tau_n : g \mapsto \int_K \frac{f^n}{\|f^n\|_1} g dx$, converges to ev_{x_0} in the weak*-topology on $C(K)^*$.*

Proof. Denote the maximum of f by M. For every $\varepsilon > 0$, $f^{-1}(M-\varepsilon, M]$ is an open neighbourhood of x_0, denote this by U_ε. Outside this neighbourhood $\frac{f^n}{\|f^n\|_1} \xrightarrow{n\to\infty} 0$ uniformly, since

$$\|f^n\|_1 \geq \int_{U_{\varepsilon/2}} f^n dx > |U_{\varepsilon/2}| (M - \varepsilon/2)^n,$$

and so for $x \notin U_\varepsilon$

$$\frac{f^n}{\|f^n\|_1}(x) \leq \frac{1}{|U_{\varepsilon/2}|}\left(\frac{M-\varepsilon}{M-\varepsilon/2}\right)^n.$$

Therefore,

$$\begin{aligned}
|\tau_n(g) - \mathrm{ev}_{x_0}(g)| &= \left|\int_K \frac{f^n}{\|f^n\|_1} g dx - \mathrm{ev}_{x_0}(g)\right| \\
&= \left|\int_K \frac{f^n}{\|f^n\|_1}(g - g(x_0))dx\right| \\
&\leq \int_{K-U_\varepsilon} \frac{f^n}{\|f^n\|_1} |g - g(x_0)| \, dx + \left|\int_{U_\varepsilon} \frac{f^n}{\|f^n\|_1}(g - g(x_0))dx\right| \\
&\leq \underbrace{\int_{K-U_\varepsilon} \frac{f^n}{\|f^n\|_1} |g - g(x_0)| \, dx}_{\xrightarrow{n\to\infty} 0} + \underbrace{\sup_{x\in U_\varepsilon} |g(x) - g(x_0)|}_{\xrightarrow{\varepsilon\to 0} 0} \underbrace{\int_{U_\varepsilon} \frac{f^n}{\|f^n\|_1} dx}_{\leq 1}.
\end{aligned}$$

Since the second term becomes small as $\varepsilon \to 0$ independent of n, we see that this converges to 0 indeed. $\square$

Proposition 4.2.3. *The convex combinations* $\sum_{j=1}^{m} \frac{1}{m} ev_{\lambda_j}$*, where* λ_j *are the solutions of* $\lambda^m = 1$*, are weak* **-limits in* $\mathcal{S}(C(S^1))$ *of sequences* $(R^*_{n(m+1)}\tau_n)_{n\in\mathbb{N}}$ *with* $\tau_n \in \mathcal{P}(C(S^1)^{(n(m+1))})$.

Proof. Consider the function $g_m(t) = 1 - \cos(mt)$. We have already seen that this function defines a pure state τ_{g_m} on $C(S^1)^{(m+1)}$. Likewise, due to Proposition 3.2.7, the function $\frac{(g_m)^n}{\|(g_m)^n\|}$ defines a pure state $\tau_n := \tau_{\frac{(g_m)^n}{\|(g_m)^n\|_1}}$ on $C(S^1)^{(n(m+1))}$.

Note that g_m reaches its maximum in the m points λ_j, denote the roots in between by μ_j. By symmetry of g_m, $\left\|(g_m\chi_{[\mu_j,\mu_{j+1}]})^n\right\|_1 = \frac{1}{m}\|(g_m)^n\|$. Hence

$$\frac{(g_m)^n}{\|(g_m)^n\|_1} = \frac{1}{m}\sum_{j=1}^{m} \frac{(g_m\chi_{[\mu_j,\mu_{j+1}]})^n}{\left\|(g_m\chi_{[\mu_j,\mu_{j+1}]})^n\right\|_1},$$

and by applying Proposition 4.2.2 on all these terms we conclude that

$$R^*_{n(m+1)}\tau_n \xrightarrow{w^*} \sum_{j=1}^{m} \frac{1}{m}\mathrm{ev}_{\lambda_j}. \qquad \square$$

Therefore, convex combinations of this type can indeed be approximated by pure states of the truncated circle. We will now use one more trick, which is to multiply the convergent sequence of pure states of the proposition above with the Radon-Nikodym derivative of another pure state, which results in a new sequence of pure states that converges to what we need.

Proposition 4.2.4. *Let* $\lambda_1, ..., \lambda_m$ *be the solutions of* $\lambda^m = 1$*, and take any* $\varphi \in \mathcal{P}(C(S^1)^{(k)})$ *such that* $\frac{d\varphi}{d\lambda}(\lambda_j) \neq 0$ *for at least one* j*. Then*

$$\sum_{j=1}^{m} \frac{\frac{d\varphi}{d\lambda}(\lambda_j)}{\sum_{i=1}^{m}\frac{d\varphi}{d\lambda}(\lambda_i)} ev_{\lambda_j}$$

is the weak **-limit of a sequence* $(R^*_{k+n(m+1)}\chi_n)_{n\in\mathbb{N}}$ *with* $\chi_n \in \mathcal{P}(C(S^1)^{(k+n(m+1))})$.

Proof. Consider the space of linear functionals $C(S^1)^*$. Observe that on $C(S^1)^*$:

1. The map

$$\begin{aligned} M_f^* : C(S^1)^* &\to C(S^1)^* \\ \tau &\mapsto \tau \circ M_f, \end{aligned}$$

   where M_f indicates multiplication by f, is weak*-continuous. This is trivial, since if $\tau_n \xrightarrow{w^*} \tau$, then by definition $\tau_n(fg) \to \tau(fg)$ for all $g \in C(S^1)$ so $M_f^*\tau_n \xrightarrow{w^*} M_f^*\tau$.

2. If $\tau_n \xrightarrow{w^*} \tau$, then by definition also $\tau_n(1) \to \tau(1)$. As scalar multiplication is weak*-continuous

$$\frac{\tau_n}{\tau_n(1)} \xrightarrow{w^*} \frac{\tau}{\tau(1)},$$

   provided that these scalars are nonzero.

Take $\varphi \in \mathcal{P}(C(S^1)^{(k)})$. For this proof, we will ease some notation by denoting the linear functional $g \mapsto \int_{S^1} fg\, d\lambda$ on $C(S^1)$ simply by f. As seen in the proof of Proposition 4.2.3, if we define $g_m(t) = 1 - \cos(mt)$ have that

$$\frac{(g_m)^n}{\|(g_m)^n\|} \xrightarrow{w^*} \sum_{j=1}^{m} \frac{1}{m}\mathrm{ev}_{\lambda_j}.$$

If we apply observation 1 on this sequence with $M^*_{\frac{d\varphi}{d\lambda}}$, we get that

$$\frac{(g_m)^n\frac{d\varphi}{d\lambda}}{\|(g_m)^n\|_1} \xrightarrow{w^*} \sum_{j=1}^{m} \frac{\frac{d\varphi}{d\lambda}(\lambda_j)}{m}\mathrm{ev}_{\lambda_j}.$$

When $\frac{d\varphi}{d\lambda}(\lambda_j) \neq 0$ for at least one j, all these are nonzero positive linear functionals on $C(S^1)$ so due to Lemma 2.1.5 evaluating these functionals at 1 gives a nonzero scalar.

By observation 2, we can therefore conclude that

$$\frac{(g_m)^n \frac{d\varphi}{d\lambda}}{\left\|(g_m)^n \frac{d\varphi}{d\lambda}\right\|_1} \xrightarrow{w^*} \sum_{j=1}^{m} \frac{\frac{d\varphi}{d\lambda}(\lambda_j)}{\sum_{i=1}^{m} \frac{d\varphi}{d\lambda}(\lambda_i)} \mathrm{ev}_{\lambda_j}.$$

Finally, the functional $f \mapsto \int_{S^1} f \frac{(g_m)^n \frac{d\varphi}{d\lambda}}{\|(g_m)^n \frac{d\varphi}{d\lambda}\|_1} \, d\lambda$ is exactly $R^*_{k+(n(m+1))}\chi_n$ for χ_n the pure state on $C(S^1)^{(k+(n(m+1)))}$, defined via Proposition 3.2.7 by

$$\frac{d\chi_n}{d\lambda} = \frac{(g_m)^n \frac{d\varphi}{d\lambda}}{\left\|(g_m)^n \frac{d\varphi}{d\lambda}\right\|_1}.$$

□

We have completed all the steps that were described in the beginning of this subsection. Combined, we can prove the following proposition.

Proposition 4.2.5. *Given any state $\psi \in \mathcal{S}(C(S^1))$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we can find a pure state $\chi_n \in \mathcal{P}(C(S^1)^{(n)})$ with $d(\psi, R_n^*(\chi_n)) < \varepsilon$, where d is the Monge-Kantorovich metric on $\mathcal{S}(C(S^1))$.*

Proof. The proof of this proposition is nothing but the execution of the steps as in the flowchart on page 26. Take $\psi \in \mathcal{S}(C(S^1))$ and $\varepsilon > 0$. The set

$$\left\{ \sum_{i=1}^{n} t_i \mathrm{ev}_{\lambda_i} \;:\; 0 \leq t_i \leq 1, \sum_{i=1}^{n} t_i = 1, \lambda_i^n = 1 \right\}$$

is dense in $\mathcal{S}(C(S^1))$ with respect to the weak *-topology which agrees with the Monge-Kantorovich metric. Hence, we can find a convex combination $\tau = \sum_{i=1}^{m} t_i \mathrm{ev}_{\lambda_i}$ such that $d(\tau, \psi) < \varepsilon$.

Using Lemma 4.2.1, we can choose a (non-pure) state on $C(S^1)$

$$\rho = \sum_{j=1}^{m} \frac{\frac{d\varphi}{d\lambda}(\lambda_j)}{\sum_{i=1}^{m} \frac{d\varphi}{d\lambda}(\lambda_i)} \mathrm{ev}_{\lambda_j}$$

with φ a pure state in $\mathcal{P}(C(S^1)^{(m+1)})$ such that $d(\rho, \tau) < \varepsilon$.

According to Lemma 4.2.4 we can find a sequence $(\chi_n)_{n\in\mathbb{N}}$ with $\chi_n \in \mathcal{P}(C(S^1)^{(m+n(m+1))})$ such that $R^*_{m+n(m+1)}(\chi_n)$ converges to ρ in the Monge-Kantorovich metric. We can thus find $M \in \mathbb{N}$ such that for $n \geq M$ we have that $d(R^*_{m+n(m+1)}(\chi_n), \rho) < \varepsilon$ and hence

$$d(\psi, R^*_{m+n(m+1)}(\chi_n)) \leq d(\psi, \tau) + d(\tau, \rho) + d(\rho, R^*_{m+n(m+1)}((\chi_n)) < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

However, note that $\chi_n \in \mathcal{P}(C(S^1)^{(m+n(m+1))})$ and therefore it remains to be proven that in the intermediate pure state spaces there also exist states that are close to ψ. Recall that by using Proposition 4.2.1 we can also find $\varphi \in \mathcal{P}(C(S^1)^{(m+1+l)})$ such that $d(\rho, \tau) < \varepsilon$, for $l \in \mathbb{Z}_{\geq 0}$. The argument above can then be repeated to find a sequence $\chi'_n \in \mathcal{P}(C(S^1)^{(l+m+n(m+1))})$ converging to ψ. Choosing $l = 0, ..., m$ fills the gaps. As these result in a finite number of interlacing, separate sequences, we can combine these into a sequence $(\chi_n)_{n\in\mathbb{N}}$ with $\chi_n \in \mathcal{P}(C(S^1)^{(n)})$ so that for each $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that for $n \geq N$ $d(\psi, \chi_n) < \varepsilon$. □

4.2.2 Uniformity

The result of the previous section means we can approximate all elements in $\mathcal{S}(C(S^1))$ by elements in $\mathcal{P}(C(S^1)^{(n)})$. In order to show Gromov-Hausdorff convergence, it remains to be shown that this approximation can be done uniformly so that $\mathcal{P}(C(S^1)^{(n)})$ forms an ε-net in $\mathcal{S}(C(S^1))$. A simple argument suffices, since $\mathcal{S}(C(S^1))$ is weak* compact.

Proposition 4.2.6. *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $R_n^*(\mathcal{P}(C(S^1)^{(n)}))$ forms an ε-net in $\mathcal{S}(C(S^1))$.*

Proof. By the Banach-Alaoglu Theorem [15, Theorem V.3.1], the unit ball of $C(S^1)^*$ is weak*-compact. The set $\mathcal{S}(C(S^1))$, as a weak*-closed subset of the unit ball, is then compact as well. Hence, given $\varepsilon > 0$, we can find a finite number of $\psi_1, ..., \psi_m \in \mathcal{S}(C(S^1))$ such that the balls $B_\varepsilon(\psi_i)$ cover $\mathcal{S}(C(S^1))$.

According to Proposition 4.2.5, for each ψ_i we can find $N_i \in \mathbb{N}$ such that for $n \geq N_i$ there exists a pure state $\varphi_i \in \mathcal{P}(C(S^1)^{(n)})$ such that $d(\psi_i, R_n^*(\varphi_i)) < \varepsilon$. Thus we have that for any $\chi \in B_\varepsilon(\psi_i)$ and $n \geq N_i$,

$$\operatorname{dist}(\chi, R_n^*(\mathcal{P}(C(S^1)^{(n)}))) \leq d(\chi, \psi_i) + \operatorname{dist}(\psi_i, R_n^*(\mathcal{P}(C(S^1)^{(n)}))) < \varepsilon + \varepsilon = 2\varepsilon.$$

Now take any $\chi \in \mathcal{S}(C(S^1))$. Because

$$\mathcal{S}(C(S^1)) \subseteq \bigcup_{i=1}^{m} B_\varepsilon(\psi_i),$$

χ must be an element of the ball $B_\varepsilon(\psi_i)$ for some i. For $n \geq N := \max_i N_i$ we have, by the calculation above, that

$$\operatorname{dist}(\chi, R_n^*(\mathcal{P}(C(S^1)^{(n)}))) < \varepsilon.$$

Hence $R_n^*(\mathcal{P}(C(S^1)^{(n)}))$ forms an ε-net in $\mathcal{S}(C(S^1))$ for $n \geq N$. □

This concludes our project of proving the Gromov-Hausdorff convergence of $\mathcal{P}(C(S^1)^{(n)})$ to $\mathcal{S}(C(S^1))$ as metric spaces.

Theorem 4.2.7. *The metric spaces $\mathcal{P}(C(S^1)^{(n)})$ converge to the metric space $\mathcal{S}(C(S^1))$ in Gromov-Hausdorff convergence.*

Proof. According to Proposition 4.2.6, for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $R_n^*(\mathcal{P}(C(S^1)^{(n)}))$ is an ε-net. Moreover,

$$\operatorname{dis} R_n^* \xrightarrow{n\to\infty} 0,$$

since the distance formula on $\mathcal{S}(C(S^1)^{(n)})$ agrees with the distance formula on $\mathcal{S}(C(S^1))$ in the limit (Proposition 3.2.3). Hence for each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$ the map R_n^* is an ε-isometry.

By using Corollary 2.3.11 we can then conclude that

$$\lim_{n\to\infty} d_{GH}(\mathcal{P}(C(S^1)^{(n)}), \mathcal{S}(C(S^1))) = 0,$$

in other words, the metric spaces $\mathcal{P}(C(S^1)^{(n)})$ converge to the metric space $\mathcal{S}(C(S^1))$ in the Gromov-Hausdorff sense. □

4.2.3 Geometric Interpretation

As remarked in Section 3.2.2, we are interested in the state spaces of the truncated circle, because these traditionally correspond to 'points'. This makes it interesting to contemplate the geometric shape of these spaces and their convergence to $\mathcal{S}(C(S^1))$.

As proven in [14, Proposition 4.8], there exists a geometric interpretation of the truncated pure state spaces of the circle which is the following proposition.

Proposition 4.2.8. *The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$ is the quotient of the n-torus by the symmetric group on n objects.*

Proof. In Proposition 3.2.5 we have already characterised pure states in $\mathcal{P}(C(S^1)^{(n+1)})$ as vectors $\xi \in \mathbb{C}^{n+1}$ such that the polynomial $\sum_k \xi_k z^{n-k}$ has all its zeroes on S^1. Denote these roots by $\lambda_1, ..., \lambda_n$ (with multiplicities). Using Vieta's formulas, we can then express ξ in elementary symmetric polynomials in the λ_k's

$$\xi = \begin{pmatrix} 1 \\ \sum_k \lambda_k \\ \sum_{k<l} \lambda_k \lambda_l \\ \vdots \\ \lambda_1 \cdots \lambda_n \end{pmatrix}.$$

This gives the identification $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$. □

The result in this section therefore gives the Gromov-Hausdorff convergence of the spaces $\mathbb{T}^n/S_n$ – equipped with a metric via the Connes distance formula – to the state space on the circle $\mathcal{S}(C(S^1))$. A visualisation of the spaces $\mathbb{T}^n/S_n$ becomes problematic as the dimension rises, but at the very least we can try to understand the first spaces in the sequence.

The space $\mathcal{P}(C(S^1)^{(2)})$ is homeomorphic to $\mathbb{T}^1/S_1 \cong S^1$. Indeed, pure states φ correspond with vectors

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{ix} \end{pmatrix},$$

with $x \in S^1$. The metric on the state space of $C(S^1)^{(2)}$ is defined by the formula

$$d(\varphi, \psi) = \sup\{|\varphi(A) - \psi(A)| : \|[D, A]\| \leq 1\},$$

where only self-adjoint A need to be considered [14]. Such Toeplitz matrices look like

$$A = \begin{pmatrix} a_0 & a_{-1} \\ \overline{a_{-1}} & a_0 \end{pmatrix}.$$

In order to compute the distance, it will suffice to take $a_0 = 0$, since the diagonal automatically commutes with D and also $\varphi(I_2) = \psi(I_2) = 1$, where I_2 is the identity for 2x2 matrices. Therefore, we need only consider Toeplitz matrices of the form

$$A = \begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix}.$$

Now, the Dirac operator takes the form

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$[D,A] = \begin{pmatrix} 0 & -z \\ \overline{z} & 0 \end{pmatrix}.$$

A quick calculation then gives that $\|[D,A]\| = |z|$. Therefore, if we have pure states φ, ψ corresponding to vectors

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{ix} \end{pmatrix}, \qquad \zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{iy} \end{pmatrix},$$

we can first calculate

$$\begin{aligned} \langle \xi, A\xi \rangle &= \frac{1}{2} \begin{pmatrix} 1 \\ e^{-ix} \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{ix} \end{pmatrix} \\ &= \frac{1}{2}(ze^{ix} + \overline{z}e^{-ix}) \\ &= \mathrm{Re}(ze^{ix}), \end{aligned}$$

so that

$$\begin{aligned} d(\varphi,\psi) &= \sup\{|\varphi(A) - \psi(A)| : \|[D,A]\| \leq 1\} \\ &= \sup\{|\langle \xi, A\xi \rangle - \langle \zeta, A\zeta \rangle| : \|[D,A]\| \leq 1\} \\ &= \sup\{\left|\mathrm{Re}(ze^{ix}) - \mathrm{Re}(ze^{iy})\right| : |z| \leq 1\} \\ &= \sup\{\left|\mathrm{Re}(ze^{ix}) - \mathrm{Re}(ze^{iy})\right| : |z| = 1\} \\ &= \sup\{|\cos(x+t) - \cos(y+t)| : t \in [0, 2\pi]\}. \end{aligned}$$

We can rewrite

$$\cos(x+t) - \cos(y+t) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(t + \frac{x+y}{2}\right),$$

and therefore

$$\begin{aligned} d(\varphi,\psi) &= \sup\{|\cos(x+t) - \cos(y+t)| : t \in [0, 2\pi]\} \\ &= 2\sin\left(\frac{x-y}{2}\right). \end{aligned}$$

This is exactly the chord distance between two points on the circle. An obvious visualisation of the space $\mathcal{P}(C(S^1)^{(2)})$ would therefore just be the circle, declaring the distance between two points to be the chord distance. If we consider the pure states to be 'points' of some space, an internal visualisation of the metric would be more instructive. For that end, see Figure 4.4.

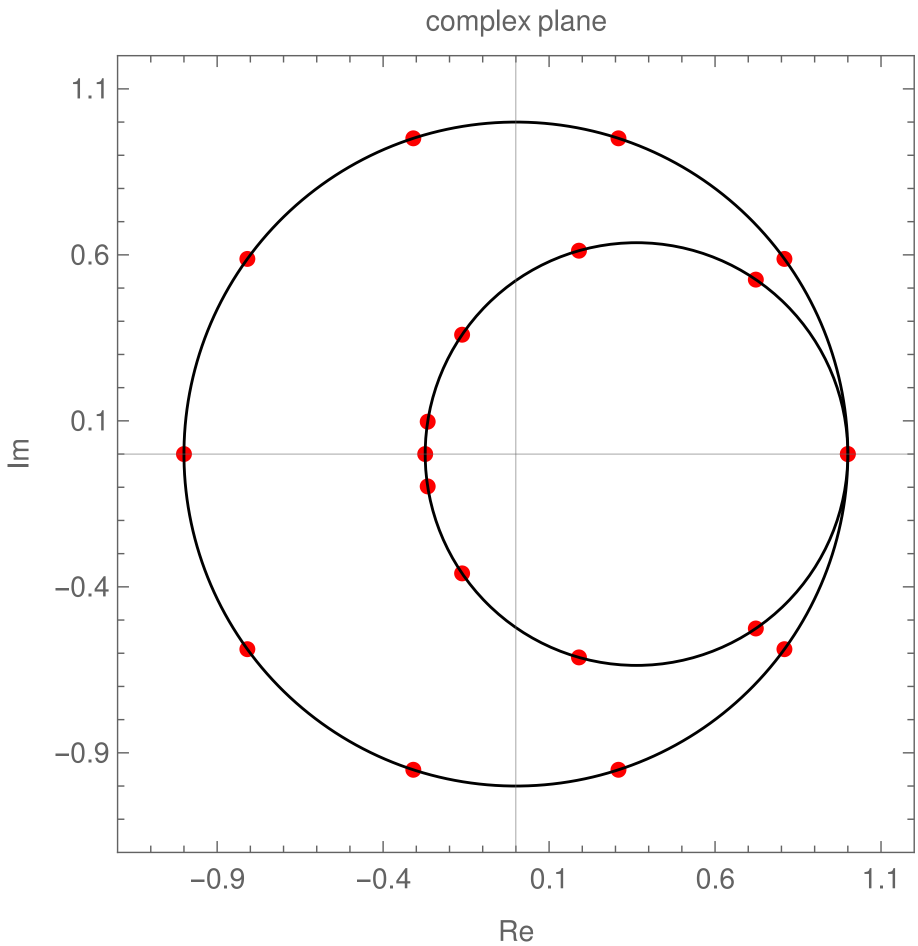


Figure 4.4: A visualisation of the metric space $\mathcal{P}(C(S^1)^{(2)})$. The larger circle indicates the space without consideration for the metric. The smaller circle gives an indication of the metric space as 'seen' from the point 1. The red dots are plotted such that the arc length on the smaller circle between 1 and a dot is the same as the chord length between 1 and the corresponding dot on the larger circle.

Chapter 5

Index Theory

5.1 General Theory

Generations of mathematicians have been captivated by what is fundamentally the question of how analytical properties of a manifold are determined or related to its topology. When analytical and topological paths meet, when an analytical calculation and a topological calculation give the same answer, there must be a deep connection to uncover. Projects in this vein have led to some of the most deep and beautiful theorems mathematics has to offer. Famously, the Atiyah-Singer Theorem from 1963 [2] is a milestone in the development of this theory, which generalises previous theorems of this type [19, Chapter 3.5, Chapter 4.2]: from the Gauss-Bonnet Theorem for surfaces in $\mathbb{R}^3$ proven in 1848 [4], its modern formulation for two-dimensional manifolds from 1888 [16], to its further generalisation to the Gauss-Bonnet-Chern Theorem for even-dimensional Riemannian manifolds in 1944 [8], not to mention the Riemann-Roch Theorem for Riemann surfaces (1865) [35]. Even to this day the programme continues. The noncommutative analogue dates from 1995, when A. Connes and H. Moscovici generalized the Atiyah-Singer Theorem further, broadening its scope to spectral triples [13].

In the Atiyah-Singer Theorem and Connes-Moscovici Theorem, the analytical data comes in the form of the *Fredholm index* of a certain operator. For an operator $A : H \to H'$ on Hilbert spaces H and H', it is defined as

$$\begin{aligned}\operatorname{ind} A &= \dim\ker A - \dim\ker A^* \\ &= \dim\ker A - \dim(\operatorname{ran} A)^{\perp}.\end{aligned}$$

This is inherently only an interesting number in infinite dimensions, since one finds by the Rank-Nullity Theorem (see your favourite basic Linear Algebra book) that in finite dimensions for any such operator A automatically $\operatorname{ind} A = \dim H - \dim H'$. However, in infinite dimension this turns out to be an incredibly powerful invariant with respect to compact perturbations, which has far-reaching applications as indicated by its fundamental use in the aforementioned theorems.

For the definitions of the Fredholm index, we will follow "A Course in Functional Analysis" by J. Conway [15, Chapter XI].

Definition 5.1.1. Let H be a Hilbert space and let A be a bounded linear operator $A : H \to H$. The operator A is called a *Fredholm operator* if its image in the quotient algebra (Calkin algebra) $B(H)/K(H)$ is invertible. The set of all Fredholm operators on H is denoted $\mathcal{F}(H) \subseteq B(H)$.

Theorem 5.1.2. *Let H be a Hilbert space. An operator $A \in B(H)$ is Fredholm if and only if* $\operatorname{ran} A$ *is closed and both* $\ker A$ *and* $\ker A^*$ *are finite-dimensional.*

The proof of this theorem depends on quite a lot of machinery, involving spectral theory of compact operators and the Fredholm Alternative. A complete proof can be found in [15, Chapter VII, Chapter XI]. Due to this proposition, we can define the Fredholm index of a Fredholm operator.

Definition 5.1.3. Let H be a Hilbert space and $A \in B(H)$ a Fredholm operator. The *Fredholm index* of A is defined as

$$\operatorname{ind} A = \dim \ker A - \dim \ker A^*.$$

Some basic results for the Fredholm index are the following.

Theorem 5.1.4. *Let H be a Hilbert space and $A, B \in B(H)$ be a Fredholm operators.*

1. $\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B$*;*
2. *If $K \in B(H)$ is compact, then* $\operatorname{ind} A + K = \operatorname{ind} A$*;*
3. *The map* $\operatorname{ind} : \mathcal{F}(H) \to \mathbb{Z}$ *is continuous;*
4. *A and B belong to the same connected component of $\mathcal{F}(H)$ if and only if* $\operatorname{ind} A = \operatorname{ind} B$*;*
5. *$\mathcal{F}(H)$ is an open subset of $B(H)$.*

Proof. Again, the proof of these statements require results in functional analysis that go beyond the scope of this thesis, so we refer to [15, Section XI.5]. □

Note that because $\mathcal{F}(H)$ is an open subset of the Banach space $B(H)$, its connected components are path-connected. In other words, whenever two operators have the same index, there exists a path with constant index connecting the two. This is tangentially related to the strategy that will be employed in Section 5.4.

5.2 Index Theory on Truncated Spectral Triples

Let us briefly return to the Connes-Moscovici index formula. There are two such formulae, for even and odd spectral triples. The even case gives a formula for the index of the twisted Dirac operator pDp, where $p \in \mathcal{A}$ is a projection. In the odd case, which is also the case of the circle, the index that is to be considered is the index of PuP [13] for a unitary $u \in \mathcal{A}$. Here P is the Hardy projection, which is the projection onto the positive spectrum of D, i.e. $P = \chi(D \geq 0)$ or equivalently $P = \frac{1}{2}(1 + \operatorname{sign} D)$. In this chapter, we will focus on the case of odd spectral triples.

It would be hopeful if we could calculate this index from within a truncation of a spectral triple, as that would indicate that some of this structure is retained by the truncation. More precisely, we would need to calculate the index of PuP from within the truncated triple $(\mathcal{A}_\rho, H_\rho, D_\rho)$, using only the finite Toeplitz matrix $P_\rho u P_\rho$ – where we now take P_ρ to be the projection $\chi_{[-\rho,\rho]}(D)$. To emphasise, this is a different projection than what we have considered for the circle in the previous chapters, as we previously took a projection onto $\operatorname{span}_{\mathbb{C}}\{e_1, \dots, e_n\}$. Using the projection $\chi_{[-\rho,\rho]}(D)$, which in the circle would mean a projection onto $\operatorname{span}_{\mathbb{C}}\{e_{-n}, \dots, e_n\}$, is more convenient for this chapter, as we will make extensive use of the sign of eigenvalues and their asymmetry around 0. This only changes things on a superficial level, see the remarks in Section 3.2.

The index of PuP is a number that is only nontrivial when considering PuP as an operator on an infinite-dimensional Hilbert space, as noted in the opening paragraphs of this chapter. Calculating it from the finite-dimensional matrix $P_\rho u P_\rho$ therefore seems like a daunting task. However, in the articles [26] and [27] by T. Loring and H. Schulz-Baldes, machinery is produced that readily makes this computation possible. Specifically, they show the index of PuP to be equal to the signature of a finite-dimensional matrix which they dub the *spectral localizer*.

First, let us introduce some notation. Given a spectral triple $(\mathcal{A}, H, D)$ and $u \in \mathcal{A}$, denote

$$D' = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix},$$

which are then considered operators on $H \oplus H$. Furthermore, denote $T_\rho = P_\rho T P_\rho$ for any $T \in B(H)$ for slight ease of notation, and define

$$D'_\rho = \begin{pmatrix} D_\rho & 0 \\ 0 & -D_\rho \end{pmatrix},$$

$$H_\rho = \begin{pmatrix} 0 & u_\rho \\ u^*_\rho & 0 \end{pmatrix},$$

which are operators on $P_\rho H \oplus P_\rho H$. This Hilbert space is finite dimensional because D has compact resolvent.

Definition 5.2.1. Let $(\mathcal{A}, H, D)$ be an odd spectral triple and consider an element $u \in \mathcal{A}$. Then the *spectral localizer* $L_{\kappa,\rho}$ associated to u is a finite-dimensional matrix with $\kappa \geq 0$, $n \in \mathbb{N}$, defined as

$$L_{\kappa,\rho} = \kappa D'_\rho + H_\rho = \begin{pmatrix} \kappa D_\rho & u_\rho \\ u^*_\rho & -\kappa D_\rho \end{pmatrix}.$$

Remarkably, by choosing κ and ρ with care, the spectral asymmetry of the spectral localizer indicates the index of PuP. Specifically, half the signature of $L_{\kappa,\rho}$ turns out to equal this index, where the signature of a self-adjoint invertible matrix means the number of positive eigenvalues of $L_{\kappa,\rho}$ minus the number of negative eigenvalues. This approach to solving the problem might feel as a deus ex machina, and without explanation this is not illuminating. Let us therefore give a brief explanation of why the spectrum of the spectral localizer is asymmetric, and why this asymmetry should be equal to the index of an operator. Of course, for details of the machinations of the spectral localizer, the best places to look are the original articles [26] and [27]. Furthermore, the explanation here is inspired by a seminar held by H. Schulz-Baldes on March 24 in the Global Noncommutative Geometry Seminar [38].

The spectrum of $\left(\begin{smallmatrix} \kappa D & \lambda u \\ \lambda u^* & -\kappa D \end{smallmatrix}\right)$ for $\lambda = 0$ is some subset of $\mathbb{R}$ consisting of isolated points without accumulation points. If we 'dial up' λ from 0 to 1 we are adding the mass term $\left(\begin{smallmatrix} 0 & u \\ u^* & 0 \end{smallmatrix}\right)$. This causes the spectrum to shift which might generate an asymmetry in the spectrum. Moreover, we can expect that the only eigenvalues that will have crossed 0 are 'small', hence there is some cutoff ρ so that the spectral asymmetry in $\left(\begin{smallmatrix} \kappa D & u \\ u^* & -\kappa D \end{smallmatrix}\right)$ can in fact already be detected by the finite-dimensional $L_{\kappa,\rho}$. The last ingredient is that this asymmetry equals twice the index of PuP. The easiest way to prove this in full generality is via the formula

$$\operatorname{ind} PuP = \operatorname{Sf}(u^* D u, D),$$

where Sf (the spectral flow) denotes the number of negative eigenvalues that become positive on the path from u^*Du to D minus the number of positive eigenvalues that become negative. This formula dates from the 1970's due to M.F. Atiyah, V.K. Pathodi, I.M. Singer and G. Lusztig [1, Section 7].

This is all comes together in [27, Theorem 1].

Theorem 5.2.2. *Let $(\mathcal{A}, H, D)$ be an odd spectral triple, and let $u \in \mathcal{A}$ be invertible with gap $g = \left\|u^{-1}\right\|^{-1} > 0$. Define*

$$\kappa_0 = \frac{g^3}{12 \|u\| \, \|[D, u]\|}.$$

Suppose that κ and ρ are such that

$$\frac{2g}{\kappa_0} \leq \frac{2g}{\kappa} < \rho.$$

Then the matrix $L_{\kappa,\rho}$ satisfies the bound

$$(L_{\kappa,\rho})^2 \geq \frac{g^2}{4}\mathbb{1}.$$

In particular, $L_{\kappa,\rho}$ is invertible and therefore has a well-defined signature, which satisfies

$$\operatorname{ind} PuP = \frac{1}{2}\operatorname{Sig}(L_{\kappa,\rho}).$$

Proof. See [27]. □

This theorem states exactly that if we have an odd spectral triple $(\mathcal{A}, H, D)$ and we truncate it with P_ρ to an operator system spectral triple $(\mathcal{A}_\rho, H_\rho, D_\rho)$, then we can calculate the index of PuP for any unitary $u \in \mathcal{A}$ in the truncated system $(\mathcal{A}_\rho, H_\rho, D_\rho)$ if ρ is large enough.

Covering both the proofs of the Connes-Moscovici theorem and the above theorem would blow up this thesis to monstrous proportions. Instead, staying in the spirit of the previous chapter, we will produce a self-contained proof of Theorem 5.2.2 in the specific case when $(\mathcal{A}, H, D)$ is the canonical spectral triple of the circle in Section 5.4, but first treat regular index theory on the circle in Section 5.3.

5.3 Index Theory on the Circle

For the specific case of the circle, we are looking for the index of PuP where $u \in C^\infty(S^1)$ is unitary and P takes the form

$$Pe_n = \begin{cases} e_n & \text{if } n \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

on the orthonormal basis $e_n(t) = e^{int}$ of $L^2(S^1)$, which are all eigenfunctions of D. The multiplication operator PuP on $PL^2(S^1)$ is also called a *Toeplitz operator*. The Connes-Moscovici index theorem for the circle simply gives that the index of the Toeplitz operator PuP is equal to minus the *winding number* of u [23], which is an example of how the admittedly quite abstract concept of a Fredholm index on an infinite-dimensional Hilbert space can have a remarkably simple geometric rendition. For the sake of keeping this thesis self-contained, we will sketch a proof of this connection here, using the book "C^*-algebras and Operator Theory" by G. J. Murphy [31, Chapter 3.5].

Definition 5.3.1. If φ is an invertible function in $C(S^1)$, then the *winding number* of φ is denoted and defined as

$$\operatorname{win}\varphi = \frac{1}{2\pi i}\oint_\varphi \frac{dz}{z},$$

where $\oint_\varphi$ should be interpreted as the complex integral over the curve $\varphi(S^1) \subseteq \mathbb{C}$. By standard complex analysis, this is always an integer [18, Chapter 22.5].

For C^1 functions, this can be easily rewritten to

$$\operatorname{win}\varphi = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\varphi'(t)}{\varphi(t)}dt,$$

or even more concise, for unitary $u \in C^1(S^1)$

$$\operatorname{win} u = \int_{S^1} u^* du.$$

To get started with the proof that this is connected to the index of PuP, we claim that multiplying Toeplitz operators can be done naively modulo compact operators.

Lemma 5.3.2. *Let $f, g \in C(S^1)$, and consider the Toeplitz operators PfP and PgP on $H = PL^2(S^1)$. Then $(PfP)(PgP) + K(H) = PfgP + K(H)$.*

Proof. In the special case that $f = z^n$ and $g = z^m$ it is an easy calculation that $(PfP)(PgP) - PfgP$ is a finite-rank operator. By density of the trigonometric polynomials in $C(S^1)$ (Stone-Weierstrass), one can complete the proof [31, Lemma 3.5.9]. $\square$

Next, it is useful to characterise which Toeplitz operators are Fredholm operators. We do this by the following proposition, which is an amalgamation of [31, Theorem 3.5.8], [31, Theorem 3.5.11] and [31, Corollary 3.5.12].

Proposition 5.3.3. *Let $f \in C(S^1)$. Then PfP is a Fredholm operator on $H = PL^2(S^1)$ if and only if f vanishes nowhere.*

Proof. Consider the C^*-algebra $\mathcal{A}$ generated by the Toeplitz operators PfP for $f \in C(S^1)$. We claim that the map

$$\begin{aligned} \psi : C(S^1) &\to \mathcal{A}/K(H) \\ f &\mapsto PfP + K(H) \end{aligned}$$

is a $*$-isomorphism. Linearity and preservation of the involution is clear. Multiplicity follows from Lemma 5.3.2, hence ψ is a $*$-homomorphism. Surjectivity follows from the definition of $\mathcal{A}$. Injectivity requires more work.

Suppose that PfP is a compact operator. Consider the operator $Pe_1P \in B(H)$, which is the right-shift

$$\begin{aligned} Pe_1P : H &\to H \\ e_n &\mapsto e_{n+1}; \end{aligned}$$

$$((Pe_1P)^*)^n (e_m) = \begin{cases} e_{m-n} & \text{if } m \geq n \\ 0 & \text{if } m < n, \end{cases}$$

and therefore it is easy to see that for all finite-rank operators $v \in B(H)$, $\lim_{n\to\infty} ((Pe_1P)^*)^n v = 0$. It follows that for all compact operators $v \in K(H)$ we have $\lim_{n\to\infty} ((Pe_1P)^*)^n v = 0$. Observe that for $m \geq 0$

$$\begin{aligned} (Pe_1P)^*(PfP)(Pe_1P)e_m &= Pe_{-1}Pfe_{m+1} \\ &= Pe_{-1}P\left(\sum_n \hat{f}(n)e_{n+m+1}\right) \\ &= Pe_{-1}\left(\sum_{n+m+1\geq 0} \hat{f}(n)e_{n+m+1}\right) \\ &= P\left(\sum_{n+m+1\geq 0} \hat{f}(n)e_{n+m}\right) \\ &= \sum_{n+m\geq 0} \hat{f}(n)e_{n+m} \\ &= (PfP)e_m, \end{aligned}$$

i.e. $(Pe_1P)^*(PfP)(Pe_1P) = PfP$. We can therefore deduce that if PfP is compact,

$$\|PfP\| = \|((Pe_1P)^*)^m (PfP)(Pe_1P)^m\| \leq \|((Pe_1P)^*)^m PfP\| \xrightarrow{m\to\infty} 0,$$

so $PfP = 0$. This can only be the case if $f = 0$. We see that indeed ψ is a bijective $*$-homomorphism, i.e. a $*$-isomorphism.

Due to this isomorphism, PfP is invertible modulo compact operators if and only if f is invertible in $C(S^1)$, in other words, PfP is Fredholm if and only if f vanishes nowhere. $\square$

To determine the index of PfP it will suffice to know the index of Pe_mP, which is easy to calculate.

Lemma 5.3.4. *Given $e_m = e^{imt} \in C(S^1)$ for $m \in \mathbb{Z}$, the index of Pe_mP as an operator on $H = PL^2(S^1)$ is equal to $-m$.*

Proof. With respect to the basis $\{e_n : n \geq 0\}$ of H, multiplying by e_m is simply a shift by m. Its adjoint $e_m^* = e_{-m}$ is then a shift in the other direction. A quick look at the kernels of Pe_mP and $Pe_{-m}P$ immediately gives the result. $\square$

One final ingredient we will need is that we can rewrite any non-vanishing function on S^1 in a convenient way.

Proposition 5.3.5. *If φ is an invertible function in $C(S^1)$, then there exists a unique integer $n \in \mathbb{Z}$ such that $\varphi = e_n e^\gamma$ for some $\gamma \in C(S^1)$.*

Proof. The possibility of writing $\varphi = e_n e^\gamma$ can be shown by reducing to the simple case of $\varphi = z - \lambda$, see [31, Lemma 3.5.14] for details. To prove the uniqueness of n, it suffices if we can show that $e_n = e^\gamma$ for $\gamma \in C(S^1)$ implies that $n = 0$. So, suppose $e_n = e^\gamma$ for $\gamma \in C(S^1)$. Note that $Pe^{t\gamma}P$ is a Fredholm operator on $PL^2(S^1)$ for all $t \in [0, 1]$ according to Proposition 5.3.3. Hence the map

$$\begin{aligned} \alpha : [0, 1] &\to \mathbb{Z} \\ t &\mapsto \operatorname{ind} Pe^{t\gamma}P \end{aligned}$$

is constant according to Theorem 5.1.4. We then see by using Lemma 5.3.4 that

$$-n = \operatorname{ind} Pe_nP = \alpha(1) = \alpha(0) = \operatorname{ind} 1 = 0.$$ $\square$

The unique integer from Proposition 5.3.5 is actually the winding number. Indeed, if $f = e_n e^\gamma$ is invertible, then

$$\operatorname{win} f = \frac{1}{2\pi i} \oint_{e_n e^\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \oint_{e_n} \frac{dz}{z} = n.$$

Finally, we can now prove the index theorem for Toeplitz operators. This theorem first appeared in 1957 due to Israel Gohberg and Mark Krein [21], although a less recognisable version was already proven in 1920 by Fritz Noether [32]. We will follow the proof in [31, Theorem 3.5.15].

Theorem 5.3.6. *Let $f \in C(S^1)$ be invertible, and let $H = PL^2(S^1)$. Then the Toeplitz operator PfP on H satisfies*

$$\operatorname{ind} PfP = -\operatorname{win} f.$$

Proof. Due to Proposition 5.3.5, we know that $f = e_n e^\gamma$ for some $n \in \mathbb{Z}$ and $\gamma \in C(S^1)$, where n is the winding number of f. Using Lemma 5.3.2 and Theorem 5.1.4, it follows that

$$\begin{aligned} \operatorname{ind} PfP &= \operatorname{ind} Pe^\gamma e_n P \\ &= \operatorname{ind}(Pe^\gamma P)(Pe_nP) \\ &= \operatorname{ind}(Pe^\gamma P) + \operatorname{ind}(Pe_nP) \\ &= \operatorname{ind}(Pe^\gamma P) - n. \end{aligned}$$

All that is left to show is that $\operatorname{ind}(Pe^{\gamma}P) = 0$ for all $\gamma \in C(S^1)$. In fact, $Pe^{\gamma}P$ is always invertible, which can be established by first considering $\gamma = \sum_{|k|\leq N} \hat{\gamma}(k)e_k$ and using that such trigonometric polynomials are dense in $C(S^1)$ (for details, see [31, Theorem 3.5.15]). Invertible operators have index 0, hence $\operatorname{ind}(Pe^{\gamma}P) = 0$ which finishes the proof. □

5.4 Index Theory on the Truncated Circle

In the previous section, we have seen that the index of the operator PuP with $u \in C^\infty(S^1)$ unitary is equal to the winding number of u, a formula which depends on the geometric and topological structure of S^1. As demonstrated in Section 5.2, this winding number can then be calculated from within a truncation of the spectral triple of the circle, using the spectral localizer. Here we will give a self-contained proof of this fact. Still, we do not claim originality of the ideas behind the results presented in this section.

Recall that the spectral localizer is defined for $u \in C^\infty(S^1)$ as the finite matrix

$$L_{\kappa,\rho} = \begin{pmatrix} \kappa D_\rho & u_\rho \\ u_\rho^* & -\kappa D_\rho \end{pmatrix}.$$

In the case of the circle, we will choose ρ to be an integer, which makes $L_{\kappa,n}$ a $(2n+1)\times(2n+1)$ matrix. We will prove that the asymmetry of the spectral localizer can detect the index of PuP by first proving it for the unitary functions $e_m(t) = e^{imt}$ via an explicit calculation. In the general case we will then show that any unitary u can be connected via a path to an e_m with equal winding number, along which no spectral flow occurs for the spectral localizer. Hence the result follows for u as well.

Lemma 5.4.1. *Given $e_m(t) = e^{imt}$, $n \geq |m|$ and $\kappa < \frac{2}{|m|}$, the spectral localizer $L_{\kappa,n}$ satisfies*

$$\operatorname{ind} Pe_mP = \frac{1}{2}\operatorname{Sig}(L_{\kappa,n}).$$

Proof. According to Lemma 5.3.4, $\operatorname{ind} Pe_mP = -m$. We will now explicitly calculate the eigenvalues of $L_{\kappa,n}$. Let us assume that $m \geq 0$, so that $P_ne_mP_n$ is a lower-triangular $(2n+1)\times(2n+1)$ matrix of the form

$$P_ne_mP_n = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{2n+1-m} & 0 \end{array}\right),$$

then $L_{\kappa,n}$ takes the form

$$L_{\kappa,n} = \left(\begin{array}{c|c|c|c} \begin{smallmatrix} -\kappa n & & \\ & \ddots & \\ & & \kappa(-n+m-1) \end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & \begin{smallmatrix} \kappa(-n+m) & & \\ & \ddots & \\ & & \kappa n \end{smallmatrix} & I_{2n+1-m} & 0 \\ \hline 0 & I_{2n+1-m} & \begin{smallmatrix} \kappa n & & \\ & \ddots & \\ & & \kappa(-n+m) \end{smallmatrix} & 0 \\ \hline 0 & 0 & 0 & \begin{smallmatrix} \kappa(-n+m-1) & & \\ & \ddots & \\ & & -\kappa n \end{smallmatrix} \end{array}\right).$$

To find the eigenvalues, let us calculate $\det(L_{\kappa,n} - \lambda I_{2(2n+1)})$. As a first step, observe that

$$\det(L_{\kappa,n} - \lambda I_{2(2n+1)}) = \left(\prod_{j=n-m+1}^{n} (\kappa j + \lambda)^2\right)\det(M),$$

where

$$M = \left(\begin{array}{ccc|ccc} \kappa(-n+m)-\lambda & & & & & \\ & \ddots & & & I_{2n+1-m} & \\ & & \kappa n-\lambda & & & \\ \hline & & & \kappa n-\lambda & & \\ & I_{2n+1-m} & & & \ddots & \\ & & & & & \kappa(-n+m)-\lambda \end{array}\right).$$

For block matrices

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $CD = DC$, we have that $\det(T) = \det(AD - BC)$ [39]. In our case, $C = I_{2n+1-m}$ which commutes with anything, and hence

$$\begin{aligned} \det(M) &= \det\left(\begin{pmatrix} \kappa(-n+m)-\lambda & & \\ & \ddots & \\ & & \kappa n-\lambda \end{pmatrix}\begin{pmatrix} \kappa n-\lambda & & \\ & \ddots & \\ & & \kappa(-n+m)-\lambda \end{pmatrix} - I_{2n+1-m}\right) \\ &= \det\begin{pmatrix} (\kappa(-n+m)-\lambda)(\kappa n-\lambda)-1 & & \\ & \ddots & \\ & & (\kappa n-\lambda)(\kappa(-n+m)-\lambda)-1 \end{pmatrix} \\ &= \prod_{j=-n+m}^{n} ((\kappa j - \lambda)(\kappa(m-j) - \lambda) - 1). \end{aligned}$$

All in all, the eigenvalues of $L_{\kappa,n}$ are exactly the roots of the polynomial

$$\det(L_{\kappa,n} - \lambda I_{2(2n+1)}) = \left(\prod_{j=n-m+1}^{n} (\kappa j + \lambda)^2\right)\left(\prod_{j=-n+m}^{n} ((\kappa j - \lambda)(\kappa(m-j) - \lambda) - 1)\right).$$

If $n \geq m$, the roots of $\prod_{j=n-m+1}^{n}(\kappa j + \lambda)^2$ are all strictly negative. Let us now show that the roots of the second part of the polynomial come in pairs that are negative and positive, if κ is chosen small enough. That would mean that $\frac{1}{2}\operatorname{Sig}(L_{\kappa,n}) = -m$.

The roots of the second term satisfy

$$\begin{aligned} 0 &= (\kappa j - \lambda)(\kappa(m-j) - \lambda) - 1 \\ &= \lambda^2 - \kappa m\lambda + j\kappa^2 m - j^2\kappa^2, \end{aligned}$$

i.e.

$$\lambda = \frac{\kappa m}{2}\left(1 \pm \sqrt{1 + 4\left(\frac{1}{\kappa^2 m^2} + \frac{j^2}{m^2} - \frac{j}{m}\right)}\right),$$

for some $j = -n+m, \ldots, n$. These are indeed pairs of positive and negative eigenvalues, as long as

$$\begin{gathered} \sqrt{1 + 4\left(\frac{1}{\kappa^2 m^2} + \frac{j^2}{m^2} - \frac{j}{m}\right)} > 1 \quad \forall j \in \{-n+m, \ldots, n\} \\ \Longleftrightarrow \\ 1 + \kappa^2(j^2 - jm) > 0 \quad \forall j \in \{-n+m, \ldots, n\} \\ \Longleftrightarrow \\ \kappa < \frac{2}{m}. \end{gathered}$$

In summary, for $m \geq 0$, taking $n \geq m$ and $\kappa < \frac{2}{m}$ ensures that

$$\frac{1}{2}\operatorname{Sig}(L_{\kappa,n}) = -m.$$

For $m \leq 0$, the argument above can be replicated with few changes, so indeed for $n \geq |m|$ and $\kappa < \frac{2}{|m|}$,

$$\frac{1}{2}\operatorname{Sig}(L_{\kappa,n}) = -m. \qquad \square$$

We now want to connect general unitaries $u \in C^\infty(S^1)$ to elementary unitaries e_m without spectral flow occuring in the spectral localizers along this path. The following two lemmas are essential, and are inspired by [27], although we take different functions G_n and achieve tighter bound.

Lemma 5.4.2. *For every $n \geq 0$, there exists a function $G_n : \mathbb{R} \to [0,1]$ with support in $[-n,n]$, such that for $u \in C^\infty(S^1)$ unitary and $n \geq 2/\kappa$,*

$$\kappa^2 D^2 \geq 1 - G_n(D)^2; \tag{5.1}$$

$$\|[G_n(D),u]\| \leq \frac{2}{\sqrt{3}}\frac{1}{n}\|[D,u]\| \approx \frac{1.15}{n}\|[D,u]\|. \tag{5.2}$$

Proof. First note that regardless of how we choose G_n,

$$\begin{aligned}[G_n(D),e_k]e_l &= G_n(D)e_{k+l} - e_k G_n(D)e_l \\ &= (G_n(k+l) - G_n(l))e_{k+l}.\end{aligned}$$

Furthermore, if G_n is Lipschitz, we will have for arbitrary unitary $u = \sum \hat{u}_k e_k$ and $\psi = \sum_l \psi_l e_l$,

$$\begin{aligned}\|[G_n(D),u]\psi\|^2 &= \left\|\left[G_n(D),\sum_k \hat{u}_k e_k\right]\sum_l \psi_l e_l\right\|^2 \\ &= \left\|\sum_k\sum_l \hat{u}_k\psi_l[G_n(D),e_k]e_l\right\|^2 \\ &= \left\|\sum_k\sum_l \hat{u}_k\psi_l(G_n(k+l)-G_n(l))e_{k+l}\right\|^2 \\ &= \left\|\sum_k\sum_j \hat{u}_k\psi_{j-k}(G_n(j)-G_n(j-k))e_j\right\|^2 \\ &= \sum_j\left|\sum_k \hat{u}_k\psi_{j-k}(G_n(j)-G_n(j-k))\right|^2 \\ &\leq \|G_n\|^2_{\mathrm{Lip}}\sum_j\sum_k |k\hat{u}_k\psi_{j-k}|^2 \\ &\leq \|G_n\|^2_{\mathrm{Lip}}\left(\sum_j|\psi_j|^2\right)\left(\sum_k|k\hat{u}_k|^2\right) \\ &\leq \|G_n\|^2_{\mathrm{Lip}}\|[D,u]\|^2\|\psi\|^2,\end{aligned}$$

where we have used Young's convolution inequality. This leads to the estimate

$$\|[G_n(D),u]\| \leq \|G_n\|_{\mathrm{Lip}}\|[D,u]\|.$$

Now the trick is to find $G_n : \mathbb{R} \to [0,1]$ with support in $[-n,n]$ and with minimal slope such that

$$\kappa^2 D^2 \geq 1 - G_n(D)^2.$$

As the right-hand side only depends on n, we impose the relation $n \geq 2/\kappa$. The justification for this can be found in Lemma 5.4.1. There, we have calculated exactly what κ and n suffice if we take $u = e_m$, and the optimal choices of $\kappa = 2/|m|$ and $n = |m|$ results in $n \geq 2/\kappa$. Taking this as a general restriction ensures that we do not lock ourselves out of an optimal bound for n and κ.

We therefore have to find G_n such that

$$\frac{4}{n^2} D^2 \geq 1 - G_n(D)^2.$$

This is effectively only a restriction for G_n on $[-\frac{n}{2}, \frac{n}{2}]$, as for greater spectral parameters the left-hand side already is greater than 1. On this interval, we can rewrite the expression to

$$G_n(x) \geq \sqrt{1 - \frac{4x^2}{n^2}}.$$

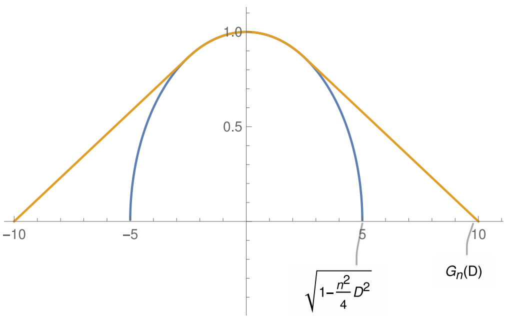
It is now a question of elementary analysis to find a function of minimal slope that satisfies this constraint, see also Figure 5.1.

This variational problem does not uniquely determine G_n, but the following is an example.

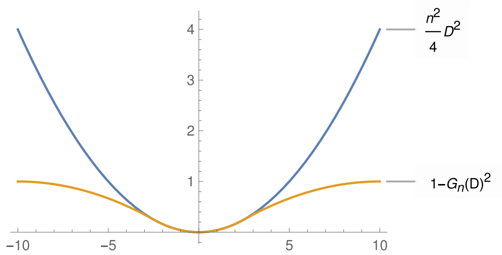
$$G_n(x) = \begin{cases} \sqrt{1 - \frac{4x^2}{n^2}} & \text{if } |x| \leq \frac{n}{4} \\ \frac{2}{\sqrt{3}}\left(1 - \frac{|x|}{n}\right) & \text{if } \frac{n}{4} \leq |x| \leq n \\ 0 & \text{if } n \leq |x| \,. \end{cases}$$

This function satisfies $\|G_n\|_{\mathrm{Lip}} = \frac{2}{\sqrt{3}} \frac{1}{n}$ so we can conclude that G_n satisfies the requirements of this lemma with the bound

$$\|[G_n(D), u]\| \leq \frac{2}{\sqrt{3}} \frac{1}{n} \|[D, u]\| \,. \qquad \square$$



(a) Here the functions $G_n(D)$ and $\sqrt{1 - \frac{n^2}{4} D^2}$ are depicted with $n = 10$.



(b) Here the functions $1 - G_n(D)^2$ and $\frac{n^2}{4} D^2$ are depicted with $n = 10$.

Figure 5.1: An illustration of the functions $G_n(D)$ and $\frac{n^2}{4} D_n^2$ and the relations between them as in the proof of Lemma 5.4.2.

Lemma 5.4.3. *For any unitary $u \in C^\infty(S^1)$, taking*

$$\kappa < \frac{2\sqrt{3} - 3}{\|[D, u]\|} \approx \frac{0.464}{\|[D, u]\|}$$

and $n \geq 2/\kappa$ ensures that

$$(L_{\kappa,n})^2 > 0.$$

Proof. Since $L_{\kappa,n} = \kappa D'_n + H_n$, clearly

$$(L_{\kappa,n}) = \kappa^2 D_n'^2 + H_n^2 + \kappa(D'_n H_n + H_n D'_n).$$

The third term is easiest to handle, as

$$\|D'_n H_n + H_n D'_n\| = \left\| \begin{pmatrix} 0 & [D,u] \\ [D,u]^* & 0 \end{pmatrix} \right\| \leq \|[D,u]\| \, .$$

Denote $\pi_n = P_n \oplus P_n$. Then, due to Lemma 5.4.2, we can take the function G_n as defined there so that

$$\kappa^2 D_n'^2 \leq \pi_n - G_n(D')^2.$$

For the H_n^2 term we have to do some work.

$$\begin{aligned} H_n^2 &= \pi_n H \pi_n H \pi_n \\ &\geq \pi_n H G_n(D')^2 H \pi_n \\ &= \pi_n G_n(D') H^2 G_n(D') \pi_n + \pi_n [G_n(D') H, [G_n(D'), H]] \pi_n \\ &= G_n(D')^2 + \pi_n [G_n(D') H, [G_n(D'), H]] \pi_n. \end{aligned}$$

Combined, we get that

$$(L_{\kappa,n})^2 \geq \pi_n + \pi_n [G_n(D') H, [G_n(D'), H]] \pi_n + \kappa(D'_n H_n + H_n D'_n).$$

To conclude, note that we can now estimate the last two terms by

$$\begin{aligned} \|[G_n(D')H, [G_n(D'), H]] + \kappa(D'_n H_n + H_n D'_n)\| &\leq 2 \|G_n(D')\| \, \|H\| \, \|[G_n(D'), H]\| + \kappa \, \|[D,u]\| \\ &\leq \left(\frac{4}{n\sqrt{3}} + \kappa \right) \|[D,u]\| \\ &\leq \left(1 + \frac{2}{\sqrt{3}} \right) \kappa \, \|[D,u]\| \, . \end{aligned}$$

Therefore, taking

$$\kappa < \frac{1}{\left(1 + \frac{2}{\sqrt{3}}\right) \|[D,u]\|} = \frac{2\sqrt{3} - 3}{\|[D,u]\|}$$

ensures that $(L_{\kappa,n})^2 > 0$. □

With these lemmas in hand it is an easy task to complete the proof of the main result of this chapter.

Proposition 5.4.4. *For a unitary $u \in C^\infty(S^1)$ with κ, n satisfying*

$$\kappa < \frac{2\sqrt{3} - 3}{\|[D,u]\|}, \tag{5.3}$$

$$\frac{2}{\kappa} \leq n, \tag{5.4}$$

the spectral asymmetry of the spectral localizer gives the index of PuP via

$$\frac{1}{2} \operatorname{Sig}(L_{\kappa,n}) = \operatorname{ind} PuP.$$

Proof. Any smooth unitary function can be written as $u = e^{i\gamma(t)}$, with $\gamma : [0, 2\pi] \to \mathbb{R}$ smooth. Without loss of generality, we can take $\gamma(0) = 0$. The winding number m of u is then equal to $\gamma(2\pi)$. Furthermore, $\|[D,u]\| = \|\gamma'(t)\|_\infty \geq m$. It is then also clear that we can define a homotopy connecting u and e_m by $u_s = e^{i(smt+(1-s)\gamma(t))}$, so that $u_0 = u$, and $u_1 = e_m$, and all u_s in between are also unitary with winding number m. Then

$$\begin{aligned}
\|[D, u_s]\| &= \left\| e^{i(smt+(1-s)\gamma(t))} \right\|_\infty \\
&= \|sm + (1-s)\gamma'(t)\|_\infty \\
&\leq sm + (1-s)\,\|\gamma(t)\|_\infty \\
&\leq \|\gamma(t)\|_\infty \\
&= \|[D, u]\|\,,
\end{aligned}$$

so by continuously assigning a spectral localizer $L_{\kappa,n,s}$ to U_s for every s, we know that if κ, n are chosen such that the bounds (5.3) and (5.4) hold, Lemma 5.4.3 holds for every $L_{\kappa,n,s}$. This proves the existence of a spectral gap around 0 along the whole path, and hence

$$\operatorname{Sig}(L_{\kappa,n,0}) = \operatorname{Sig}(L_{\kappa,n,1}).$$

Note that we can now use Lemma 5.4.1 as the bounds (5.3) and (5.4) are stronger requirements than what is needed for that lemma. Indeed, with our choice of κ and n,

$$\begin{aligned}
\kappa &< \frac{2\sqrt{3}-3}{\|[D,u]\|} \leq \frac{2\sqrt{3}-3}{|m|} < \frac{2}{|m|} \\
n &\geq \frac{2}{\kappa} \geq (4\sqrt{3}-6)\,|m| > |m|\,.
\end{aligned}$$

We conclude that

$$\frac{1}{2}\operatorname{Sig}(L_{\kappa,n}) = \operatorname{ind} PuP. \qquad \square$$

Chapter 6

Conclusion

The protagonist of this thesis was the circle, which is as simple as spectral triples get. This is a shaky foundation to build conjectures on for truncations of general spectral triples. Even though we realise that, we would like to take a moment to look ahead at possible directions for further research.

6.1 State Spaces

The results in Chapter 4 on the convergence of the pure state spaces of the truncated circle at the very least show that it is foolhardy to expect a general theorem that asserts the convergence of the pure state space of a truncated spectral triple to the pure state space of the original triple. Philosophically, it is interesting to contemplate what this means for the interpretation of pure states as 'points' when dealing with a truncated spectral triple. Already for the circle, this paradigm would mean that what we would then see as the underlying topological space of the truncated system converges to the infinite-dimensional geometric object $\mathcal{S}(C(S^1))$ instead of the far friendlier one-dimensional S^1. Instead of looking at pure states, for canonical spectral triples L. Glaser and A. Stern already showed that one can reconstruct the metric structure of a Riemannian manifold by 'localised' states on its truncated canonical spectral triple. Their result shows that the state space of truncated spectral triples is rich enough for such a reconstruction. The case of the circle might indicate that the *pure* state spaces are also rich enough for such a reconstruction, but this will have to be explored.

More generally, before looking into the fate of *pure* state spaces of truncated triples, one should first study the whole state space. As we have used for the circle, convergence of the truncated state spaces to the original state space follows immediately from the existence of C^1-approximate order isomorphisms [41]. The existence of such maps has been shown in the case of tori as well [3], but in more general spectral triples constructing these seems like the first challenge.

After establishing such maps for a more general class of spectral triples $(\mathcal{A}, H, D)$ and their truncations $(\mathcal{A}_n, H_n, D_n)$, it not only follows that $\mathcal{S}(A_n) \to \mathcal{S}(A)$, but one would then also be halfway proving that $\mathcal{P}(A_n) \to \mathcal{S}(A)$. As we have used in this thesis, we would get for free that the distortion of the maps $R_n^* : \mathcal{P}(A_n) \to \mathcal{S}(A)$ converges to 0 by definition of a C^1-approximate order isomorphism (Definition 3.1.2). For showing (or refuting) Gromov-Hausdorff convergence of $\mathcal{P}(A_n)$ to $\mathcal{S}(A)$, one should show that every state ψ in $\mathcal{S}(A)$ can be approximated by pullbacks of pure states $R_n^*(\varphi_n)$ for φ_n in $\mathcal{P}(A_n)$. By compactness of $\mathcal{S}(A)$ we can then conclude that $R_n^*(\mathcal{P}(A_n))$ forms an ε-net in $\mathcal{S}(A)$ (see Proposition 4.2.6) which would make R_n^* ε-isometries and Gromov-Hausdorff convergence of $\mathcal{P}(A_n)$ to $\mathcal{S}(A)$ would follow.

In summary, the two challenges we foresee for generalising the results of Chapter 4 are showing the existence of C^1-approximate order isomorphisms for a more general class of spectral triples and their truncations, and subsequently the approximation of any state $\psi \in \mathcal{S}(A)$ by pullbacks $R_n^* g_n$ of pure states $g_n \in \mathcal{P}(A_n)$.

6.2 Index Theory

The result on index theory discussed in Chapter 5 is already quite general, the strategy of the spectral localizer as put forward in [27] works for all odd spectral triples. A generalisation to even spectral triples is an obvious next step, which has already been taken in [28] by the same two authors, T. Loring and H. Schulz-Baldes. Hence also for even spectral triples, the index in the Connes-Moscovici index theorem can be computed from a spectral truncation if the truncation has large enough rank.

What could be improved still are the bounds on κ and ρ that guarantee that the spectral localizer contains enough information to distill the index. Certainly in the case of the circle we have provided stronger bounds, but even these do not seem to be optimal as indicated by some numerical experimentation. For practical purposes it is interesting to search for ways to sharpen these bounds, preferably in the most general case. Conceptually though, the results of T. Loring and H. Schulz-Baldes convincingly show that the act of truncating spectral triples in the framework by A. Connes and W. van Suijlekom conserves essential invariants of the spectral triple, which is a strong argument in favour of further study of such truncations.

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