Higher Chern-Simons and Yang-Mills forms in the spectral action

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Noncommutative geometry: a spectral approach to geometry

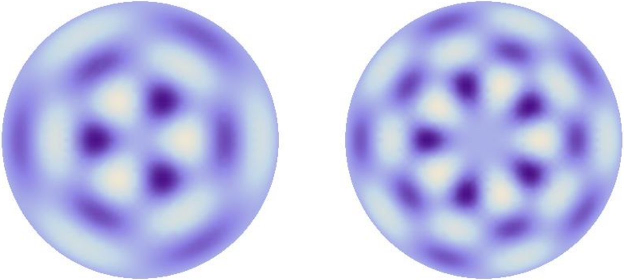
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

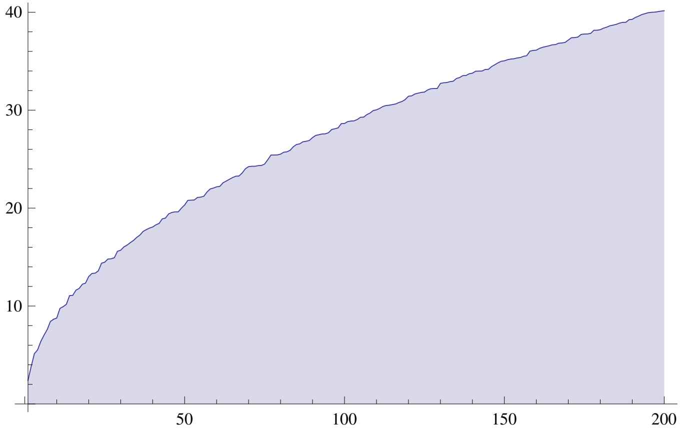
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

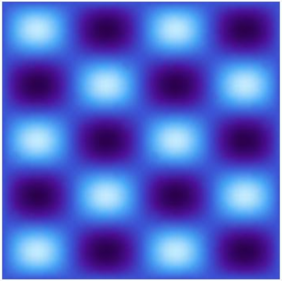
The disc

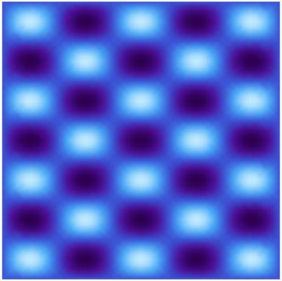


Wave numbers on the disc

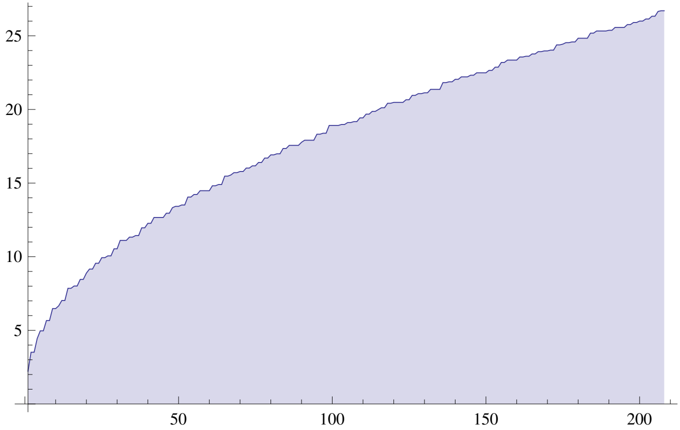


The square



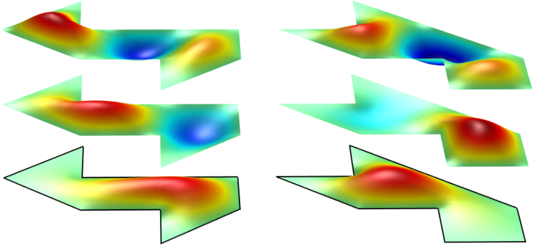


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

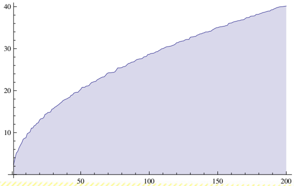
so the answer to Kac's question is no

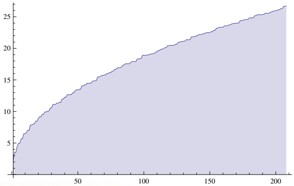
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension d of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \text{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):

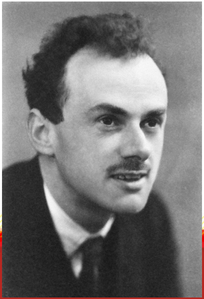




Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.



The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

  then $a^2 = b^2 = -1$ and $ab + ba = 0$
- The Dirac operator on the torus is

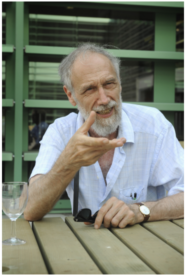
$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

  which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.

The 4-dimensional torus

- Consider the 4-torus $\mathbb{T}^4$ parametrized by t_1, t_2, t_3, t_4 and the Laplacian is
$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$
- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:
$$i^2 = j^2 = k^2 = ijk = -1.$$
- The Dirac operator on $\mathbb{T}^4$ is
$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$
- The relations $ij = -ji$, $ik = -ki$, *et cetera* imply that its square coincides with $\Delta_{\mathbb{T}^4}$.

Noncommutative geometry



If combined with the C^*-algebra $C(M)$, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), D_M) \longleftrightarrow (M, g)$$

The "usual" story

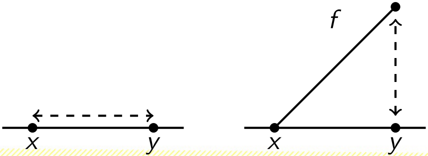
Given cpt Riemannian spin manifold (M, g) with spinor bundle S on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(M, S)$

$\rightsquigarrow$ spectral triple: $(C^\infty(M), L^2(M, S), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples $(\mathcal{A}, \mathcal{H}, D)$

- a C^*-algebra $\overline{\mathcal{A}}$
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A}$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Applications to gauge theories:

- Gauge group $\mathcal{U}(\mathcal{A})$ of unitaries in $\mathcal{A}$ acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- Spectral invariant action functional [Chamseddine–Connes, 1996]:

$$\operatorname{Trace} f(D)$$

- More general: inner perturbations as gauge fields

$$D \mapsto D' = D + \sum_j a_j[D, b_j] \qquad (a_j, b_j \in \mathcal{A})$$

Applications to particle physics

We consider asymptotic expansions of the form:

$$\mathrm{Trace}\, f((D+V)/\Lambda) \sim \sum_{k \leq n} f_k \Lambda^k \alpha_k$$

for a suitable f, cutoff Λ and some n (dimension).
The α_k are integral invariants of local polynomial functionals in the metric and in V.

Almost-commutative manifolds

$(C(M, A_F), L^2(M, S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F)$:

$$\mathrm{Trace}\, f((D+V)/\Lambda) \sim f_4 \Lambda^4 \mathrm{Vol}(M) + f_2 \Lambda^2 \int R\sqrt{g} + f_0 \int \mathrm{Trace}\, F_{\mu\nu} F^{\mu\nu} + \cdots$$

Perturbative expansion of the spectral action

Instead, we aim at an expansion of

$$\boxed{\mathrm{Trace}\, f(D+V)}$$

in powers of V and then to understand its structure as a gauge invariant action functional.

We will exploit the following trace formula [vS 2012, Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\mathrm{Trace}\, f(D+V) - \mathrm{Trace}\, f(D) = \sum_{n\geq 1} \frac{1}{n} \frac{1}{2\pi i}\, \mathrm{Trace} \oint f'(z) \left(V(z-D)^{-1} \right)^n$$

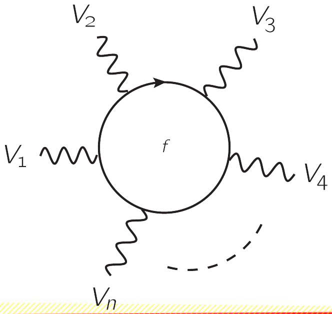
$$\mathrm{Trace}\, f(D+V) - \mathrm{Trace}\, f(D) = \sum_n \frac{1}{n} \langle V, V, \ldots, V \rangle$$

where we introduced brackets:

$$\langle V_1, V_2, \ldots, V_n \rangle = \frac{1}{2\pi i} \mathrm{Trace} \oint f'(z) \prod_j \left(V_j (z-D)^{-1} \right)$$

This can be depicted as a Feynman diagram:

$$\langle V_1, \ldots, V_n \rangle =$$



Ward identity

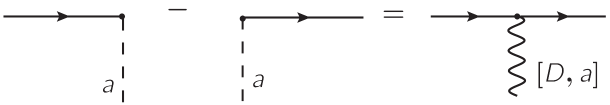
There is the following amusing property:

$$(z-D)^{-1}a - a(z-D)^{-1} = (z-D)^{-1}[D,a](z-D)^{-1}$$

which is a generalization of the usual Ward identity:

$$\frac{1}{z-\not{p}} - \frac{1}{z-(\not{p}+\not{q})} = \frac{1}{z-(\not{p}+\not{q})}\not{q}\frac{1}{z-\not{p}}$$

We depict it as



Universal differential forms

For any $*$-algebra $\mathcal{A}$ we may consider universal differential forms $\Omega^\bullet(\mathcal{A}) = \oplus_{n\in\mathbb{N}_0}\Omega^n(\mathcal{A})$.

- This is the universal differential graded algebra over $\mathcal{A} =: \Omega^0(\mathcal{A})$, endowed with differential d.
- A universal differential n-form ω is given by an expression of the form

$$\omega = \sum_j a_0^j da_1^j \cdots da_n^j,$$

  and its differential is

$$d\omega = \sum_j da_0^j da_1^j \cdots da_n^j.$$

- No commutation relations but we do have the Leibniz rule:

$$d(ab) = d(a)b + adb$$

Noncommutative integrals

The main reason for working with universal differential forms is that one may write arbitrary multi-linear functionals on $\mathcal{A}$ as noncommutative integrals.

- if φ_n is a $n+1$-linear functional on $\mathcal{A}$ with the property that $\varphi_n(a_0, a_1, \ldots, a_n) = 0$ if one of the $a_1, \ldots, a_n$ is a complex scalar, then we may write

$$\varphi_n(a_0, a_1, \ldots a_n) = \int_{\varphi_n} a_0 da_1 \cdots da_n.$$

- We will be searching for these types of "integrals"
- Let us connect to the spectral action functional.

Connes' differential forms

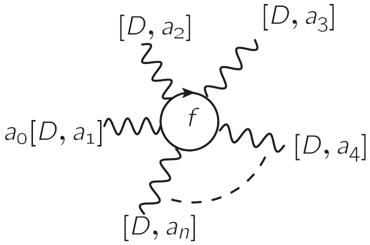
- If we are given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ then we write $\Omega^1_D(\mathcal{A}) = \{\sum_j a_j[D, b_j] : a_j, b_j \in \mathcal{A}\}$.
- A gauge field is a self-adjoint element $V = V^* \in \Omega^1_D(\mathcal{A})$, corresponding to a universal one-form $A = \sum_j a_j db_j$
- A gauge transformation acts as

$$A \mapsto uAu^* + udu^*; \qquad \text{or } V \mapsto uVu^* + u[D, u^*]$$

Brackets as noncommutative integrals

We now consider the brackets $\langle V, \dots, V \rangle$ as they appear in the perturbative expansion of the spectral action and express them in terms of the following noncommutative integrals:

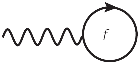
$$\int_{\phi_n} a_0 da_1 \cdots da_n =$$



Brackets as noncommutative integrals

For one external edge we find

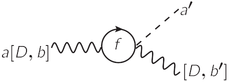
$$\langle a[D,b]\rangle = a[D,b]$$



$$= \int_{\phi_2} adb$$

For two external edges, we apply the Ward identity and derive

$$\langle a[D,b], a'[D,b']\rangle =$$



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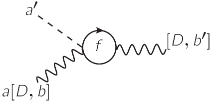

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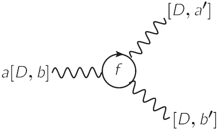

$$= \int_{\phi_2} adba'db' + \int_{\phi_3} adbda'db'$$

Brackets as noncommutative integrals

In conclusion, if $V = \sum_j a_j[D, b_j]$ is a gauge field with corresponding universal 1-form $A = \sum_j a_j db_j$ we may write:

$$
\begin{aligned}
\langle V \rangle &= \int_{\phi_1} A, \\
\langle V, V \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} AdA, \\
\langle V, V, V \rangle &= \int_{\phi_3} A^3 + \int_{\phi_4} AdAA + \int_{\phi_5} AdAdA, \\
\langle V, V, V, V \rangle &= \int_{\phi_4} A^4 + \cdots
\end{aligned}
$$

Noncommutative integrals

We now introduce another multi-linear functional ψ_{2k-1} by setting

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2}\int_{\phi_{2k}} d\omega; \qquad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2}\int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2}\int_{\phi_2} (dA + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\frac{1}{2}\int_{\phi_3} AdA + \frac{1}{3}\int_{\phi_3} A^3 + \frac{1}{3}\int_{\phi_4} AdAA + \frac{1}{4}\int_{\phi_4} A^4$$
$$= \frac{1}{2}\int_{\psi_3}\left(AdA + \frac{2}{3}A^3\right) + \frac{1}{4}\int_{\phi_4} (dA + A^2)^2$$

Chern–Simons and Yang–Mills forms

The systematics persist and we find with $F = dA + A^2$ for the curvature and Chern–Simons forms

$$\mathrm{cs}_1(A) = A; \qquad \mathrm{cs}_3(A) = \frac{1}{2}\left(AdA + \frac{2}{3}A^3\right);$$
$$\mathrm{cs}_5(A) = \frac{1}{3}\left(A(dA)^2 + \frac{3}{4}AdAA^2 + \frac{3}{4}A^3dA + \frac{3}{5}A^5\right),$$

that we have

$$\mathrm{Trace}\, f(D+V) - f(D) = \int_{\psi_1} \mathrm{cs}_1(A) + \frac{1}{2}\int_{\phi_2} F$$
$$+ \int_{\psi_3} \mathrm{cs}_3(A) + \frac{1}{4}\int_{\phi_4} F^2 + \int_{\psi_5} \mathrm{cs}_5(A) + \frac{1}{6}\int_{\phi_6} F^3 + \cdots$$

The perturbative expansion of the spectral action

Theorem
For a finitely-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and f in a suitable function class, there is the following absolutely convergent series expansion:

$$\mathrm{Trace}(f(D+V) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt,$$

where $F_t = tdA + t^2 A^2$ is the curvature of the gauge field $A_t = tA$.

Hochschild cocycles

The functionals ψ_{2k-1} and ϕ_{2k} turn out to define Hochschild and cyclic cocycles.

Let us recall what this means. On arbitrary multi-linear functionals φ the Hochschild boundary operator b is defined by

$$\int_{b\varphi} a_0 da_1 \cdots da_{n+1} = \int_{\varphi} a_0 a_1 da_2 \cdots da_{n+1}$$
$$+ \sum_{j=1}^{n} (-1)^j \int_{\varphi} a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_{n+1} + (-1)^{n+1} \int_{\varphi} a_{n+1} a_0 da_1 \cdots da_n,$$

for which one may check that $b^2 = 0$.

Hochschild cocycle means $b\varphi = 0$.

Hochschild cocycles in the spectral action

We claim that $b\phi_{2k-1} = \phi_{2k}$ for the functionals defined in terms of the brackets $\rightsquigarrow$ $b\phi_{2k} = 0$, making ϕ_{2k} Hochschild cocycles.

$$\int_{b\phi_1} a_0 da_1 da_2 = \langle a_0 a_1 [D, a_2] \rangle - \langle a_0 [D, a_1 a_2] \rangle + \langle a_2 a_0 [D, a_1] \rangle$$

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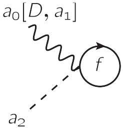

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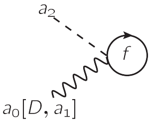

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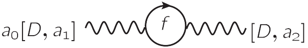


$$= \int_{\phi_2} a_0 da_1 da_2$$

Even and odd cyclic cocycles

There is an additional symmetry encoded by the loop diagrams, which can be translated in terms of the so-called B-boundary operator, defined in general by

$$\int_{B\varphi} a_0 da_1 \cdots da_n := \sum_{j=0}^{n} (-1)^{nj} \int_{\varphi} da_j da_{j+1} \cdots da_{j-1},$$

this time lowering the degree of φ by 1. Clearly, $B^2 = 0$ and one may even show that $(b + B)^2 = 0$.
We will then define odd cyclic cocycles as sequences of functionals

$$(\varphi_1, \varphi_3, \varphi_5, \ldots); \qquad \text{where } b\varphi_{2k+1} + B\varphi_{2k+3} = 0,$$

Similarly for even cyclic cocycles.

Even and odd cyclic cocycles in the spectral action

For the functionals ϕ_{2k} the representation as a 1-loop diagram shows that

$$B\phi_{2k} = 0$$

$\rightsquigarrow$ the sequence $\{\phi_{2k}\}$ that appears in the integrals for the Yang–Mills terms are not only Hochschild cocycles, they are even cyclic cocycles.

For the odd case one may show instead that the functionals ψ_{2k+1} that appear in the integrals for the Chern–Simons forms satisfy

$$B\psi_{2k+1} = 2(2k+1)b\psi_{2k-1}.$$

Hence, after a suitable normalization they give rise to odd cyclic cocycles.

Gauge invariance and pairing with K-theory

- For the Yang–Mills terms, it turns out that gauge invariance is a consequence of the fact that ϕ_{2k} are Hochschild cocycles, showing that:

$$\int_{\phi_{2k}} uF^k u^* = \int_{\phi_{2k}} F^k.$$

- Since the spectral action is a spectral invariant, it is in particular invariant under gauge transformations.
- This combines: also the Chern–Simons terms are gauge invariant:

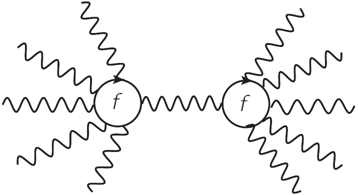
$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(uAu^* + udu^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A)$$

- By considering a pure gauge field, we find that the pairing of ψ with odd K-theory of $\mathcal{A}$ is trivial.

Outlook: towards an effective spectral action

- The Feynman diagrams for the brackets suggest to consider loop diagrams in the gauge potential V.
- We may apply the background field method in the functional integral and consider contractions of the form

$$\int_{\Omega^1_D(\mathcal{A})} \langle V_1, \ldots, V_n, V \rangle \langle V, W_1, \ldots, W_m \rangle e^{-\langle V, V \rangle} =$$

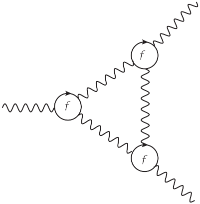


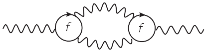
- For a toy model for which $\Omega^1_D(\mathcal{A}) \cong M_N(\mathbb{C})$ one arrives at a propagator (if it makes sense) of the form

$$G_{kl} = \frac{1}{f'[\lambda_k, \lambda_l]} = \frac{\lambda_k - \lambda_l}{f'(\lambda_k) - f'(\lambda_l)}$$

with $\{\lambda_k\}$ the spectrum of D.

- At that point, one may start analyzing the structure of a quantum effective spectral action functional with 1PI contributions of the type:





... tbc ...

Thanks!