Cyclic cocycles in the spectral action and one-loop corrections

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NC geometry: spectral triples $(\mathcal{A}, \mathcal{H}, D)$

• $*$-algebra $\mathcal{A}$	"coordinate algebra"
• self-adjoint operator D with $(i + D)^{-1}$ compact and $[D, a]$ bounded for $a \in \mathcal{A}$	"inverse fermion propagator"
• both acting on Hilbert space $\mathcal{H}$	"one-particle space"

Applications to gauge theories:

- Gauge group $\mathcal{U}(\mathcal{A})$ of unitaries in $\mathcal{A}$ acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- Spectral invariant action functional [Chamseddine–Connes, 1996]:

$$\mathrm{Tr}\, f(D)$$

- More general: inner perturbations as gauge fields

$$D \mapsto D' = D + \sum_j a_j[D, b_j] \qquad (a_j, b_j \in \mathcal{A})$$

Applications to particle physics

We may consider asymptotic expansions of the form:

$$\operatorname{Tr} f((D+V)/\Lambda) \sim \sum_{k \leq n} f_k \Lambda^k \alpha_k$$

for $V = \sum_j a_j[D, b_j]$, a suitable f, cutoff Λ and some n (dimension). The α_k are integral invariants of local polynomial functionals in the metric and in V.

Almost-commutative manifolds [Chamseddine–Connes–Marcolli 2007, vS] $(C(M, A_F), L^2(M, S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F)$:

$$\operatorname{Tr} f((D+V)/\Lambda) \sim f_4 \Lambda^4 \mathrm{Vol}(M) + f_2 \Lambda^2 \int R\sqrt{g} + f_0 \int R^2 \sqrt{g} + \cdots$$
$$+ \widetilde{f}_0 \int \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} - \widetilde{f}_2 \Lambda^2 \int |\phi|^2 + \widetilde{f}_0 \int |\phi|^4 + \cdots$$

Perturbative expansion of the spectral action

Instead, today we consider an expansion in powers of V of

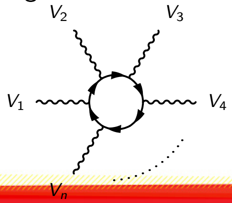
$$\operatorname{Tr} f(D+V) - \operatorname{Tr} f(D) = \sum_n \frac{1}{n} \langle V, V, \ldots, V \rangle$$

with [vS 2012, Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\langle V_1, V_2, \ldots, V_n \rangle = \frac{1}{2\pi i} \operatorname{Tr} \oint f'(z) V_1 (z-D)^{-1} \cdots V_n (z-D)^{-1}$$

We will depict it as a Feynman diagram:

$$\langle V_1, \ldots, V_n \rangle =$$



Cyclic permutations and the Ward identity

There are the following two properties of the bracket:

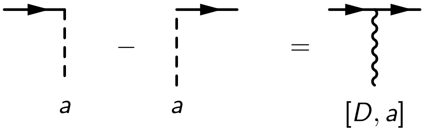
$$\langle V_1, \ldots, V_n \rangle = \langle V_2, \ldots, V_n, V_1 \rangle \qquad \text{(I)}$$

$$\langle aV_1, \ldots, , V_n \rangle - \langle V_1, \ldots, V_n a \rangle = \langle [D, a], V_1, \ldots, V_n \rangle \qquad \text{(II)}$$

Identity (II) is a type of 'Ward identity' as it boils down to

$$(z - D)^{-1} a - a(z - D)^{-1} = (z - D)^{-1} [D, a] (z - D)^{-1}$$

We depict it as



Noncommutative integrals

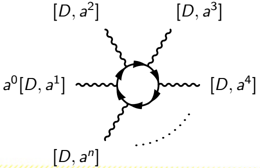
We may express multi-linear functionals on $\mathcal{A}$ as noncommutative integrals over universal differential forms:

- if φ_n is a $n+1$-linear functional on $\mathcal{A}$ so that $\varphi_n(a_0, a_1, \ldots, a_n) = 0$ if one of the $a_1, \ldots, a_n$ is a complex scalar, then we may write

$$\varphi_n(a_0, a_1, \ldots a_n) = \int_{\varphi_n} a_0 \delta a_1 \cdots \delta a_n.$$

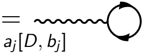
- We will be interested in the brackets $\langle V, \ldots, V \rangle$ as they appear in the perturbative expansion of the spectral action and define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle :=$$



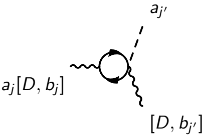
Brackets as noncommutative integrals

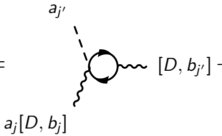
For one external edge we find with $V = a_j[D, b_j]$ and $A = a_j\delta(b_j)$:

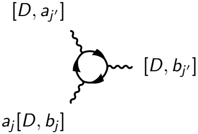
$\langle V \rangle =$ 

$= \int_{\phi_1} A,$

For two external edges, we apply the Ward identity and derive

$\langle V, V \rangle =$ 

$=$ 

$+$ 

$$= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A.$$

Brackets as noncommutative integrals

In conclusion, if $V = \sum_j a_j[D, b_j]$ is a gauge field with corresponding universal 1-form $A = \sum_j a_j \delta b_j$ we may write:

$$
\begin{aligned}
\langle V \rangle &= \int_{\phi_1} A, \\
\langle V, V \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A, \\
\langle V, V, V \rangle &= \int_{\phi_3} A^3 + \int_{\phi_4} A\delta A A + \int_{\phi_5} A\delta A\delta A, \\
\langle V, V, V, V \rangle &= \int_{\phi_4} A^4 + \cdots
\end{aligned}
$$

Re-ordering the terms

We now introduce another multi-linear functional ψ_{2k-1} by setting

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2} \int_{\phi_{2k}} \delta\omega; \qquad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2} \int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2} \int_{\phi_2} (\delta A + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\frac{1}{2} \int_{\phi_3} A\delta A + \frac{1}{3} \int_{\phi_3} A^3 + \frac{1}{3} \int_{\phi_4} A\delta AA + \frac{1}{4} \int_{\phi_4} A^4$$
$$= \frac{1}{2} \int_{\psi_3} \left(A\delta A + \frac{2}{3} A^3 \right) + \frac{1}{4} \int_{\phi_4} (\delta A + A^2)^2$$

The perturbative expansion of the spectral action

Theorem
For a finitely-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and f in a suitable function class, there is the following absolutely convergent series expansion:

$$\mathrm{Tr}(f(D+V) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt; \qquad F_t = t\delta A + t^2 A^2$$

The functionals ψ_{2k-1} and ϕ_{2k} turn out to define odd and even cyclic cocycles (more on this next week during conference at Fields)

Loop corrections

We adopt the background field method to analyze loop corrections to the perturbative expansion of the spectral action:

- $V \rightsquigarrow V + Q$ where $V = \sum_j a_j[D, b_j]$ as before but Q is allowed to be an arbitrary (finite size) hermitian matrix in an eigenbasis of D.
- The quadratic and cubic terms become:

$$\frac{1}{2}\langle Q, Q\rangle = \frac{1}{2}\sum_{k,l} Q_{kl} Q_{lk} f'[\lambda_k, \lambda_l]$$

$$\frac{1}{3}\langle Q, Q, Q\rangle = \frac{1}{3}\sum_{k,l,m} Q_{kl} Q_{lm} Q_{mk} f'[\lambda_k, \lambda_l, \lambda_m]$$

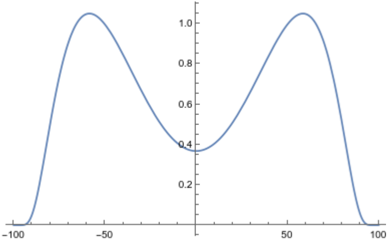
expressed in terms of divided differences:

$$f'[x,y] = \frac{f'(x) - f'(y)}{x - y} \qquad f'[x,y,z] = \frac{f'[x,y] - f'[y,z]}{x - z}$$

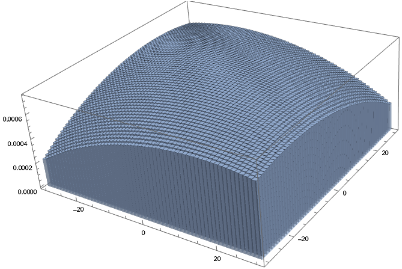
Gauge propagator

For suitable f' we may perform the Gaussian integration:

$$\overline{Q_{kl}Q}_{mn} = \frac{\int Q_{kl} Q_{mn} e^{-\frac{1}{2}\langle Q,Q\rangle} dQ}{\int e^{-\frac{1}{2}\langle Q,Q\rangle} dQ} = \delta_{kn}\delta_{lm}\frac{1}{f'[\lambda_k,\lambda_l]}$$



Example of a positive bump function f

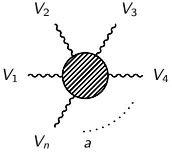


Divided difference $f'[m, n]$

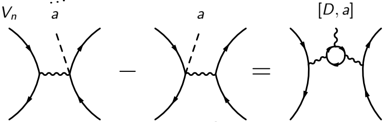
Quantum Ward identity

We thus have a gauge propagator, and may consider all 1PI diagrams:

$$\langle\!\langle V_1, \ldots, V_n \rangle\!\rangle =$$



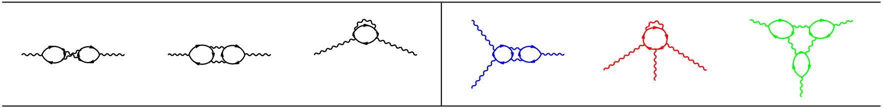
- There is a bosonic Ward identity:

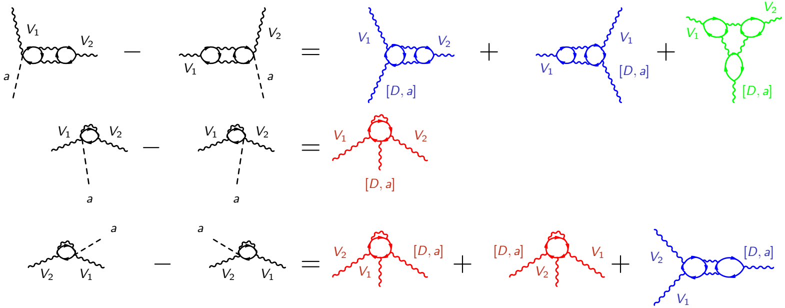


which induces the following quantum Ward identity for the divergent one-loop contributions:

$$\langle\!\langle V_1, \ldots, aV_j, \ldots, V_n \rangle\!\rangle_\infty^{1L} - \langle\!\langle V_1, \ldots, V_{j-1}a, \ldots, V_n \rangle\!\rangle_\infty^{1L}$$
$$= \langle\!\langle V_1, \ldots, V_{j-1}, [D, a], V_j, \ldots, V_n \rangle\!\rangle_\infty^{1L}. \quad \text{(II')}$$

Proof of $\langle\!\langle aV_1, V_2\rangle\!\rangle^{1L}_{\infty} - \langle\!\langle V_1, V_2 a\rangle\!\rangle^{1L}_{\infty} = \langle\!\langle [D,a], V_1, V_2\rangle\!\rangle^{1L}_{\infty}$





One-loop renormalizable spectral action

In general, as a consequence of the quantum Ward identity one can show that the divergent part of the 1L contribution $\langle\!\langle V, \ldots, V \rangle\!\rangle^{1L}_{\infty}$ has the same structure as $\langle V, \ldots, V \rangle$:

$$\sum_n \frac{1}{n} \langle\!\langle V, \ldots, V \rangle\!\rangle^{1L}_{\infty} = \sum_{k=1}^{\infty} \left(\int_{\widetilde{\psi}_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\widetilde{\phi}_{2k}} F^k \right).$$

where $\widetilde{\phi}$ and $\widetilde{\psi}$ are the analogues of ϕ and ψ but now defined using the double bracket.

We conclude that the passage to the one-loop renormalized spectral action can be realized by a transformation in the space of noncommutative integrals, sending $\phi \mapsto \phi - \widetilde{\phi}$ and $\psi \mapsto \psi - \widetilde{\psi}$.

Thanks!