Cyclic cocycles in the spectral action and one-loop corrections

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Spectral triples $(\mathcal{A}, \mathcal{H}, D)$ and gauge theories

- Gauge group $\mathcal{U}(\mathcal{A})$ of unitaries in $\mathcal{A}$ acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- Spectral invariant action functional [Chamseddine–Connes, 1996]:

$$\mathrm{Tr}\, f(D) = \sum_k f(\lambda_k)$$

- More general: inner perturbations as gauge fields [Connes, 1996]

$$D \mapsto D + V; \qquad V = \sum_j a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}} \qquad (a_j, b_j \in \mathcal{A})$$

Asymptotic expansion of the spectral action

If $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple with simple dimension spectrum Sd, and $f(\sqrt{\cdot})$ a Laplace transform, then the spectral action is given asymptotically (as $\Lambda \to \infty$) by

$$\operatorname{Tr} f(D/\Lambda) \sim \sum_{\beta \in \mathrm{Sd}} f_\beta \Lambda^\beta \operatorname{res}_{z=\beta} \operatorname{Tr} |D|^{-z} + f(0)\zeta_D(0) + \mathcal{O}(\Lambda^{-1})$$

where $f_\beta := \int f(v) v^{\beta-1} dv$. [Connes–Marcolli 2007, vS 2015]

Example

Almost-commutative manifolds [Chamseddine–Connes–Marcolli 2007, vS]
$(C(M, A_F), L^2(M,S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F)$:

$$\operatorname{Tr} f((D+V)/\Lambda) \sim f_4 \Lambda^4 \mathrm{Vol}(M) + f_2 \Lambda^2 \int R\sqrt{g} + f_0 \int R^2 \sqrt{g} + \cdots$$
$$+ \widetilde{f_0} \int \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} - \widetilde{f_2} \Lambda^2 \int |\phi|^2 + \widetilde{f_0} \int |\phi|^4 + \cdots$$

Theorem (Chamseddine–Connes, 2006)
The scale-invariant part of the spectral action in dimension 4 is

$$\zeta_{D+V}(0)-\zeta_D(0) = -\fint VD^{-1}+\frac{1}{2}\fint(VD^{-1})^2-\frac{1}{3}\fint(VD^{-1})^3+\frac{1}{4}\fint(VD^{-1})^4$$

Moreover, if $\fint VD^{-1} = 0$ ('vanishing tadpole') we may write

$$\zeta_{D+V}(0)-\zeta_D(0) = -\frac{1}{2}\int_\psi (A\delta A + \frac{2}{3}A^3) + \frac{1}{4}\int_{\tau_0}(\delta A + A^2)^2$$

where $A = \sum_j a_j \delta(b_j)$ is the universal one-form for $V = \sum_j a_j[D, b_j]$.

- Recognize Chern–Simons form $\mathrm{cs}_3(A)$ and Yang–Mills form F^2
- ψ is a cyclic 3-cocycle and τ_0 is a Hochschild cocycle.
- The pairing $\langle \psi, u \rangle$ with K_1-group of $\mathcal{A}$ is trivial.

Goal: extend these expansions in powers of V and in A from the scale-invariant part to the full spectral action $\operatorname{Tr} f(D+V)$

Perturbative expansion of the spectral action

In [vS 2012] derived an expansion in powers of V of the form:

$$\operatorname{Tr} f(D+V) - \operatorname{Tr} f(D) = \sum_n \frac{1}{n} \langle V, V, \dots, V \rangle$$

where (for suitable f):

$$\langle V_1, V_2, \dots, V_n \rangle = \frac{1}{2\pi i} \operatorname{Tr} \oint f'(z) V_1 (z-D)^{-1} \cdots V_n (z-D)^{-1}$$

Such expansions exist for varying assumptions on f and D
[Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\langle V_1, V_2, \dots, V_n \rangle := \sum_{j=1}^{n} \operatorname{Tr}(T^D_{f^{[n]}}(V_j, \dots, V_n, V_1, \dots, V_{j-1})).$$

Cyclic permutations and a Ward identity

There are the following two properties of the bracket:

$$\langle V_1, \dots, V_n \rangle = \langle V_2, \dots, V_n, V_1 \rangle \tag{I}$$

$$\langle V_1, \dots, aV_j, \dots, V_n \rangle - \langle V_1, \dots, V_{j-1}a, \dots, V_n \rangle = \langle V_1, \dots, [D, a], \dots, V_n \rangle \tag{II}$$

Identity (II) boils down to the 'Ward identity'

$$(z - D)^{-1}a - a(z - D)^{-1} = (z - D)^{-1}[D, a](z - D)^{-1}$$

We associate the following multi-linear functionals ϕ_n on $\mathcal{A}$:

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \dots, [D, a_n] \rangle$$

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]\rangle$

The n-cochain ϕ_n has the following properties:

- $B_0\phi_n$ is invariant under cyclic permutations, so $B\phi_n = nB_0\phi_n$ for odd n and $B\phi_n = 0$ for even n.
- $b\phi_{2k-1} = \phi_{2k}$ so $b\phi_{2k} = 0$.

Proof ($k = 1$):

$$\begin{aligned}\int_{b\phi_1} a_0 \delta a_1 \delta a_2 &= \langle a_0 a_1 [D, a_2]\rangle - \langle a_0[D, a_1 a_2]\rangle + \langle a_2 a_0[D, a_1]\rangle \\ &= -\langle a_0[D, a_1]a_2\rangle + \langle a_2 a_0[D, a_1]\rangle = \langle a_0[D, a_1], [D, a_2]\rangle\end{aligned}$$

Brackets and cyclic cocycles

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- $b\phi_{2k-1} = \phi_{2k}$ so $b\phi_{2k} = 0$.
- $bB_0\phi_{2k} = 2\phi_{2k} - B_0\phi_{2k+1}$

Proof (k = 1):

$$\begin{aligned}
\int_{bB_0\phi_2} a_0 \delta a_1 \delta a_2 &= \int_{B_0\phi_2} a_0 a_1 \delta a_2 - \int_{B_0\phi_2} a_0 \delta (a_1 a_2) + \int_{B_0\phi_2} a_2 a_0 \delta a_1 \\
&= \langle [D, a_0 a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1 a_2] \rangle + \langle [D, a_2 a_0], [D, a_1] \rangle \\
&= \cdots \\
&= 2 \langle a_0 [D, a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1], [D, a_2] \rangle
\end{aligned}$$

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle$

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Motivated by this we define

$$\psi_{2k-1} := \phi_{2k-1} - \frac{1}{2}B_0\phi_{2k} \qquad \Longrightarrow \qquad B\psi_{2k+1} = 2(2k+1)b\psi_{2k-1}.$$

Proposition (van Nuland–vS 2021)

1. *The sequence (ϕ_{2k}) is an even (b, B)-cocycle and each ϕ_{2k} defines an even Hochschild cocycle: $b\phi_{2k} = 0$.*
2. *The sequence $(\widetilde{\psi}_{2k-1} := (-1)^{k-1}\frac{(k-1)!}{(2k-1)!}\psi_{2k-1})$ is an odd (b, B)-cocycle.*

Spectral action and universal forms

Reconsider the perturbative expansion $\sum_n \frac{1}{n}\langle V, \ldots V\rangle$ and recall that $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]\rangle$

- For $n = 1$ we find with $V = a_j[D, b_j]$ and $A = a_j\delta(b_j)$:

$$\langle V\rangle = \langle a_j[D, b_j]\rangle = \int_{\phi_1} A,$$

- For $n = 2$, we apply the Ward identity and derive

$$\begin{aligned} \langle V, V\rangle &= \langle a_j[D, b_j], a_{j'}[D, b_{j'}]\rangle \\ &= \langle a_j[D, b_j]a_{j'}, [D, b_{j'}]\rangle + \langle a_j[D, b_j], [D, a_{j'}], [D, b_{j'}]\rangle \\ &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A. \end{aligned}$$

Brackets as noncommutative integrals

In conclusion, if $V = \sum_j a_j[D, b_j]$ is a gauge field with corresponding universal 1-form $A = \sum_j a_j \delta b_j$ we may write:

$$
\begin{aligned}
\langle V \rangle &= \int_{\phi_1} A, \\
\langle V, V \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A, \\
\langle V, V, V \rangle &= \int_{\phi_3} A^3 + \int_{\phi_4} A\delta A A + \int_{\phi_5} A\delta A\delta A, \\
\langle V, V, V, V \rangle &= \int_{\phi_4} A^4 + \cdots
\end{aligned}
$$

Re-ordering the terms

Let us see how this can be expressed using the cochain ψ_{2k-1}, *i.e.*

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2}\int_{\phi_{2k}} \delta\omega; \qquad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2}\int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2}\int_{\phi_2} (\delta A + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\frac{1}{2}\int_{\phi_3} A\delta A + \frac{1}{3}\int_{\phi_3} A^3 + \frac{1}{3}\int_{\phi_4} A\delta AA + \frac{1}{4}\int_{\phi_4} A^4$$
$$= \frac{1}{2}\int_{\psi_3}\left(A\delta A + \frac{2}{3}A^3\right) + \frac{1}{4}\int_{\phi_4} (\delta A + A^2)^2$$

The perturbative expansion of the spectral action

Consider the following function class:

$$\mathcal{E}^{s,\gamma} := \left\{ f \in C^\infty : \ \exists C_f \geq 1 \text{ s.t. } \|\widehat{(fu^m)^{(n)}}\|_1 \leq C_f^{n+1} n!^\gamma \, \forall m \leq s, n \geq 0 \right\}$$

Theorem (van Nuland–vS, 2021)

For a s-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $f \in \mathcal{E}^{s,\gamma}$ with $\gamma \in (0,1)$, there is the following absolutely convergent series expansion:

$$\mathrm{Tr}(f(D+V) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here $V = \sum_j a_j [D, b_j]$, $A = \sum_j a_j \delta b_j$, and the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt; \qquad F_t = t\delta A + t^2 A^2$$

Gauge invariance

- For the Yang–Mills terms, it turns out that gauge invariance is a consequence of the fact that ϕ_{2k} are Hochschild cocycles, showing that:
$$\int_{\phi_{2k}} uF^k u^* = \int_{\phi_{2k}} F^k.$$
- Since the spectral action is a spectral invariant, it is in particular invariant under gauge transformations.
- This combines: also the Chern–Simons terms are gauge invariant:

$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(uAu^* + u\delta u^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A)$$

Pairing with K-theory

Theorem (van Nuland–vS, 2021)

Let $f \in \mathcal{E}^{s,\gamma}$ for $\gamma < 1$. Then:

1. *The sequence $(\widetilde{\psi}_{2k+1})$ defines an odd entire cocycle with respect to the Banach norm $\|a\|_1 = \|a\| + \|[D,a]\|$ on $\mathcal{A}$.*
2. *the pairing of $(\widetilde{\psi}_{2k+1})$ with $K_1(\mathcal{A})$ is trivial,* i.e. $(u, \widetilde{\psi}) = 0$

Proof.

1. Multiple operator integrals: $\exists C \geq 1$ such that for all n we have

$$\left\| T_{f^{[n]}}^{D,D,\dots,D}(V_1,\dots,V_n) \right\|_1 \leq \left(\frac{C^{n+1}}{n!^{1-\gamma}} \right) \|V_1\| \cdots \|V_n\| \left\|(D-i)^{-1}\right\|_s^s.$$

Apply this to $V_1 = a_0[D,a_1], V_2 = [D,a_2], \dots, V_n = [D,a_n]$ to find that $\exists C_\Sigma$ s.t. $|\widetilde{\psi}_{2k+1}(a_0,\dots a_{2k+1})| \leq C_\Sigma / k!$ for any bounded subset $\Sigma \subset \mathcal{A}$.

2. Take $A = 0$ and realize that $\sum_k \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(u\delta u^*) = (u, \widetilde{\psi}) = 0$ $\square$

Loop corrections

Goal: Compute the quantum partition function and analyze loop corrections to the spectral action

Concretely, we try to make sense of the partition function in the background field:

$$Z[J,V] = \int e^{-\operatorname{Tr} f(D+V+Q)+(J,V)} d[Q]$$

where

- $V = \sum_j a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$ is a background gauge field
- J is a source field in the dual space $\Omega^1_D(\mathcal{A})^*_{\text{s.a.}}$
- Q is a quantum field that is integrated over in the path integral

Loop corrections

In order to makes sense of the path integral we let Q be an arbitrary finite size hermitian matrix $Q = (Q_{kl})$ in an eigenbasis of D (eigenvalues: λ_k).

The quadratic term becomes:

$$\frac{1}{2}\langle Q, Q \rangle = \frac{1}{2} \sum_{k,l} Q_{kl} Q_{lk} f'[\lambda_k, \lambda_l]$$

and, more generally

$$\frac{1}{n}\langle Q, \cdots, Q \rangle = \frac{1}{n} \sum_{i_1, \ldots, i_n} Q_{i_1 i_2} Q_{i_2 i_3} \cdots Q_{i_n i_1} f'[\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_n}]$$

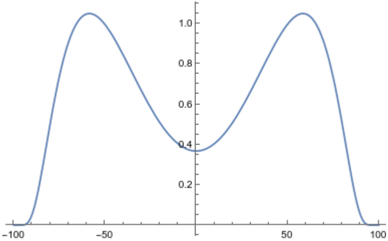
expressed in terms of divided differences:

$$f'[x, y] = \frac{f'(x) - f'(y)}{x - y}, \quad f'[x, y, z] = \frac{f'[x, y] - f'[y, z]}{x - z}, \quad \cdots$$

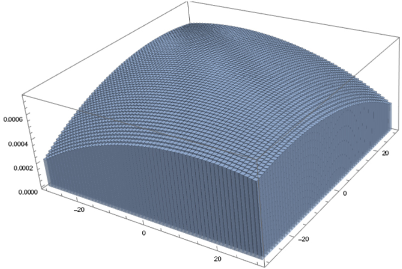
Gauge propagator

For suitable f' we may perform the Gaussian integration:

$$\overbrace{Q_{kl}Q}_{mn} = \frac{\int Q_{kl}Q_{mn}e^{-\frac{1}{2}\langle Q,Q\rangle}dQ}{\int e^{-\frac{1}{2}\langle Q,Q\rangle}dQ} = \delta_{kn}\delta_{lm}\frac{1}{f'[\lambda_k,\lambda_l]}$$



Example of a positive function f



Divided difference $f'[m,n]$

Feynman diagrams

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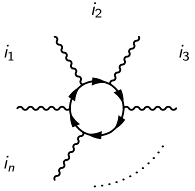

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$$f'[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}] = \frac{1}{2\pi i} \oint \frac{f'(z)}{(z-\lambda_{i_1})\cdots(z-\lambda_{i_n})} dz$$

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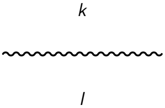

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$$\frac{1}{f'[\lambda_k, \lambda_l]}$$

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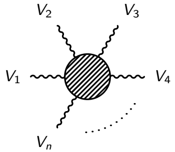
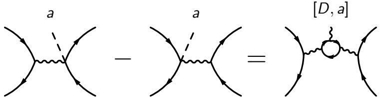
Quantum Ward identity

We thus have a gauge propagator, and may consider all one-particle irreducible diagrams:

$$\langle\!\langle V_1, \ldots, V_n \rangle\!\rangle =$$



- There is a bosonic Ward identity:



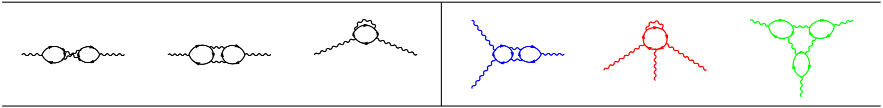
which is a consequence of the fact that

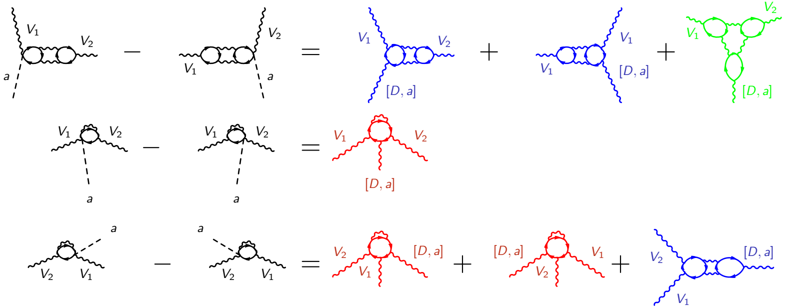
$$\frac{1}{f'[\lambda_k, \lambda_m] - f'[\lambda_l, \lambda_m]} = \frac{(\lambda_k - \lambda_l) f'[\lambda_k, \lambda_l, \lambda_m}{f'[\lambda_k, \lambda_m] f'[\lambda_l, \lambda_m]}$$

In any case, it induces the following quantum Ward identity for the divergent one-loop contributions:

$$\langle\!\langle V_1, \ldots, aV_j, \ldots, V_n \rangle\!\rangle_\infty^{1L} - \langle\!\langle V_1, \ldots, V_{j-1}a, \ldots, V_n \rangle\!\rangle_\infty^{1L} = \langle\!\langle V_1, \ldots, V_{j-1}, [D, a], V_j, \ldots, V_n \rangle\!\rangle_\infty^{1L}. \quad \text{(II')}$$

Proof of $\langle\langle aV_1, V_2 \rangle\rangle^{1L}_{\infty} - \langle\langle V_1, V_2 a \rangle\rangle^{1L}_{\infty} = \langle\langle [D, a], V_1, V_2 \rangle\rangle^{1L}_{\infty}$





One-loop renormalizable spectral action

In general, as a consequence of the quantum Ward identity one can show that the divergent part of the 1L contribution $\langle\langle V, \ldots, V \rangle\rangle_\infty^{1L}$ has the same structure as $\langle V, \ldots, V \rangle$:

$$\sum_n \frac{1}{n} \langle\langle V, \ldots, V \rangle\rangle_\infty^{1L} = \sum_{k=1}^{\infty} \left(\int_{\widetilde{\psi}_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\widetilde{\phi}_{2k}} F^k \right).$$

where the cyclic cocycles $\widetilde{\phi}$ and $\widetilde{\psi}$ are the analogues of ϕ and ψ but now defined using the double bracket.

We conclude that the passage to the one-loop renormalized spectral action can be realized by a transformation in the space of cyclic cocycles, sending $\phi \mapsto \phi - \widetilde{\phi}$ and $\psi \mapsto \psi - \widetilde{\psi}$.

Thanks!