

Cyclic cocycles in the spectral action and one-loop corrections

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(joint with Teun van Nuland)

NC geometry: spectral triples $(\mathcal{A}, \mathcal{H}, D)$

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|---|------------------------------|
| • * -algebra \mathcal{A} | “coordinate algebra” |
| • self-adjoint operator D with $(i + D)^{-1}$ compact and $[D, a]$ bounded for $a \in \mathcal{A}$ | “inverse fermion propagator” |
| • both acting on Hilbert space \mathcal{H} | “one-particle space” |

Applications to gauge theories:

- Gauge group $\mathcal{U}(\mathcal{A})$ of unitaries in \mathcal{A} acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- **Spectral** invariant **action** functional [Chamseddine–Connes, 1996]:

$$\text{Tr } f(D)$$

- More general: **inner perturbations** as gauge fields

$$D \mapsto D' = D + \sum_j a_j [D, b_j] \quad (a_j, b_j \in \mathcal{A})$$

Applications to particle physics

We may consider asymptotic expansions of the form:

$$\mathrm{Tr} f((D + V)/\Lambda) \sim \sum_{k \leq n} f_k \Lambda^k \alpha_k$$

for $V = \sum_j a_j [D, b_j]$, a suitable f , cutoff Λ and some n (dimension). The α_k are integral invariants of local polynomial functionals in the metric and in V .

Almost-commutative manifolds [Chamseddine–Connes–Marcolli 2007, vS] $(C(M, A_F), L^2(M, S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F)$:

$$\begin{aligned} \mathrm{Tr} f((D + V)/\Lambda) &\sim f_4 \Lambda^4 \mathrm{Vol}(M) + f_2 \Lambda^2 \int R \sqrt{g} + f_0 \int R^2 \sqrt{g} + \dots \\ &+ \tilde{f}_0 \int \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} - \tilde{f}_2 \Lambda^2 \int |\phi|^2 + \tilde{f}_0 \int |\phi|^4 + \dots \end{aligned}$$

Perturbative expansion of the spectral action

Instead, today we consider an expansion in powers of V of

$$\mathrm{Tr} f(D + V) - \mathrm{Tr} f(D) = \sum_n \frac{1}{n} \langle V, V, \dots, V \rangle$$

with [vS 2012, Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\langle V_1, V_2, \dots, V_n \rangle = \frac{1}{2\pi i} \mathrm{Tr} \oint f'(z) V_1(z - D)^{-1} \dots V_n(z - D)^{-1}$$

We will depict it as a Feynman diagram:

$$\langle V_1, \dots, V_n \rangle = \begin{array}{c} \begin{array}{c} V_2 \\ \diagup \\ \text{---} \end{array} \\ \begin{array}{c} V_3 \\ \diagup \\ \text{---} \end{array} \\ \begin{array}{c} V_1 \text{---} \text{---} \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \diagdown \\ V_n \end{array} \end{array}$$

Cyclic permutations and the Ward identity

There are the following two properties of the bracket:

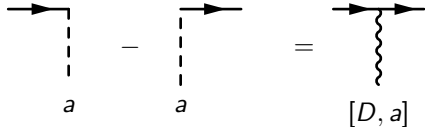
$$\langle V_1, \dots, V_n \rangle = \langle V_2, \dots, V_n, V_1 \rangle \quad (I)$$

$$\langle aV_1, \dots, V_n \rangle - \langle V_1, \dots, V_n a \rangle = \langle [D, a], V_1, \dots, V_n \rangle \quad (II)$$

Identity (II) is a type of 'Ward identity' as it boils down to

$$(z - D)^{-1} a - a(z - D)^{-1} = (z - D)^{-1} [D, a] (z - D)^{-1}$$

We depict it as



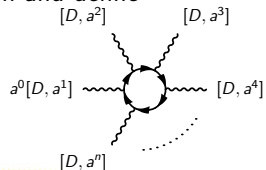
Noncommutative integrals

We may express multi-linear functionals on \mathcal{A} as noncommutative integrals over **universal differential forms**:

- if φ_n is a $n + 1$ -linear functional on \mathcal{A} so that $\varphi_n(a_0, a_1, \dots, a_n) = 0$ if one of the a_1, \dots, a_n is a complex scalar, then we may write

$$\varphi_n(a_0, a_1, \dots, a_n) = \int_{\varphi_n} a_0 \delta a_1 \cdots \delta a_n.$$

- We will be interested in the brackets $\langle V, \dots, V \rangle$ as they appear in the perturbative expansion of the spectral action and define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle :=$$


Brackets as noncommutative integrals

For **one external edge** we find with $V = a_j[D, b_j]$ and $A = a_j\delta(b_j)$:

$$\langle V \rangle = \frac{1}{a_j[D, b_j]} \text{ (circle with wavy line) } = \int_{\phi_1} A,$$

For **two external edges**, we apply the Ward identity and derive

$$\begin{aligned} \langle V, V \rangle &= \frac{1}{a_j[D, b_j]} \text{ (circle with wavy line, dashed line } a_{j'} \text{ and wavy line } [D, b_{j'}] \text{)} \\ &= \frac{1}{a_j[D, b_j]} \text{ (circle with wavy line } [D, b_{j'}] \text{)} + \frac{1}{a_j[D, b_j]} \text{ (circle with wavy line } [D, b_{j'}] \text{)} \\ &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A. \end{aligned}$$

Brackets as noncommutative integrals

In conclusion, if $V = \sum_j a_j [D, b_j]$ is a gauge field with corresponding universal 1-form $A = \sum_j a_j \delta b_j$ we may write:

$$\langle V \rangle = \int_{\phi_1} A,$$

$$\langle V, V \rangle = \int_{\phi_2} A^2 + \int_{\phi_3} A \delta A,$$

$$\langle V, V, V \rangle = \int_{\phi_3} A^3 + \int_{\phi_4} A \delta A A + \int_{\phi_5} A \delta A \delta A,$$

$$\langle V, V, V, V \rangle = \int_{\phi_4} A^4 + \dots$$

Re-ordering the terms

We now introduce another multi-linear functional ψ_{2k-1} by setting

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2} \int_{\phi_{2k}} \delta\omega; \quad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2} \int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2} \int_{\phi_2} (\delta A + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\begin{aligned} & \frac{1}{2} \int_{\phi_3} A\delta A + \frac{1}{3} \int_{\phi_3} A^3 + \frac{1}{3} \int_{\phi_4} A\delta A A + \frac{1}{4} \int_{\phi_4} A^4 \\ &= \frac{1}{2} \int_{\psi_3} \left(A\delta A + \frac{2}{3} A^3 \right) + \frac{1}{4} \int_{\phi_4} (\delta A + A^2)^2 \end{aligned}$$

The perturbative expansion of the spectral action

Theorem

For a finitely-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and f in a suitable function class, there is the following absolutely convergent series expansion:

$$\mathrm{Tr}(f(D + V) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{CS}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{CS}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt; \quad F_t = t\delta A + t^2 A^2$$

The functionals ψ_{2k-1} and ϕ_{2k} turn out to define odd and even cyclic cocycles.

Intermezzo: Hochschild and cyclic cocycles

- We define operators b and B_0 on multi-linear functionals by

$$\int_{b\phi_n} \omega \delta a = \int_{\phi_n} [\omega, a], \quad \int_{B_0\phi_n} \omega = \int_{\phi_n} \delta\omega$$

and set $B = AB_0$ in terms of the operator A of cyclic anti-symmetrization.

- $b^2 = 0$ (Hochschild cohomology), $B^2 = 0$ and $bB + Bb = 0$
 $\implies (b + B)^2 = 0 \rightsquigarrow$ (even/odd) **periodic cyclic cohomology**
- An **odd (b, B) -cocycle** is given by a sequence $(\phi_1, \phi_3, \phi_5, \dots)$, with

$$b\phi_{2k+1} + B\phi_{2k+3} = 0,$$

Similar for even (b, B) -cocycles.

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle$

The n -cochain ϕ_n has the following properties:

- $B_0 \phi_n$ is invariant under cyclic permutations, so $B \phi_n = n B_0 \phi_n$ for odd n and $B \phi_n = 0$ for even n .
- $b \phi_{2k-1} = \phi_{2k}$ so $b \phi_{2k} = 0$.

Proof (k = 1):

$$\begin{aligned} \int_{b\phi_1} a_0 \delta a_1 \delta a_2 &= \langle a_0 a_1 [D, a_2] \rangle - \langle a_0 [D, a_1 a_2] \rangle + \langle a_2 a_0 [D, a_1] \rangle \\ &= -\langle a_0 [D, a_1] a_2 \rangle + \langle a_2 a_0 [D, a_1] \rangle = \langle a_0 [D, a_1], [D, a_2] \rangle \end{aligned}$$

Brackets and cyclic cocycles

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- $b \phi_{2k-1} = \phi_{2k}$ so $b \phi_{2k} = 0$.
- $b B_0 \phi_{2k} = 2 \phi_{2k} - B_0 \phi_{2k+1}$

Proof ($k = 1$):

$$\begin{aligned} \int_{b B_0 \phi_2} a_0 \delta a_1 \delta a_2 &= \int_{B_0 \phi_2} a_0 a_1 \delta a_2 - \int_{B_0 \phi_2} a_0 \delta(a_1 a_2) + \int_{B_0 \phi_2} a_2 a_0 \delta a_1 \\ &= \langle [D, a_0 a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1 a_2] \rangle + \langle [D, a_2 a_0], [D, a_1] \rangle \\ &= \dots \\ &= 2 \langle a_0 [D, a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1], [D, a_2] \rangle \end{aligned}$$

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle$

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- $b \phi_{2k-1} = \phi_{2k}$ so $b \phi_{2k} = 0$.
- $b B_0 \phi_{2k} = 2 \phi_{2k} - B_0 \phi_{2k+1}$

Motivated by this we define

$$\psi_{2k-1} := \phi_{2k-1} - \frac{1}{2} B_0 \phi_{2k} \quad \Longrightarrow \quad B \psi_{2k+1} = 2(2k+1) b \psi_{2k-1}.$$

Proposition (van Nuland–vS 2021)

1. The sequence (ϕ_{2k}) is an *even* (b, B) -cocycle and each ϕ_{2k} defines an *even Hochschild cocycle*: $b \phi_{2k} = 0$.
2. The sequence $(\tilde{\psi}_{2k-1} := (-1)^{k-1} \frac{(k-1)!}{(2k-1)!} \psi_{2k-1})$ is an *odd* (b, B) -cocycle.

Gauge invariance

- For the **Yang–Mills terms**, it turns out that **gauge invariance** is a consequence of the fact that ϕ_{2k} are **Hochschild cocycles**:

$$\int_{\phi_{2k}} uF^k u^* = \int_{\phi_{2k}} F^k.$$

- Since the spectral action is a **spectral invariant**, it is in particular invariant under gauge transformations.
- This combines: also the **Chern–Simons terms** are **gauge invariant**:

$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \text{CS}_{2k-1}(uAu^* + u\delta u^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \text{CS}_{2k-1}(A)$$

Theorem (van Nuland–vS, 2021)

Let f be in a suitable function class. Then the sequence $(\tilde{\psi}_{2k+1})$ defines an **odd entire cocycle** and the pairing of $(\tilde{\psi}_{2k+1})$ with $K_1(\mathcal{A})$ is trivial

Loop corrections

Goal: Compute the quantum partition function and analyze loop corrections to the spectral action

Concretely, we try to make sense of the partition function in the **background field**:

$$Z[J, V] = \int e^{-\text{Tr} f(D+V+Q)+(J,V)} d[Q]$$

where

- $V = \sum_j a_j [D, b_j] \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$ is a **background gauge field**
- J is a **source field** in the dual space $\Omega_D^1(\mathcal{A})_{\text{s.a.}}^*$.
- Q is a **quantum field** that is integrated over in the path integral

Loop corrections

In order to make sense of the path integral we let Q be an **arbitrary finite size hermitian matrix** $Q = (Q_{kl})$ in an eigenbasis of D (eigenvalues: λ_k).

The quadratic and cubic terms become:

$$\frac{1}{2} \langle Q, Q \rangle = \frac{1}{2} \sum_{k,l} Q_{kl} Q_{lk} f'[\lambda_k, \lambda_l]$$

$$\frac{1}{3} \langle Q, Q, Q \rangle = \frac{1}{3} \sum_{k,l,m} Q_{kl} Q_{lm} Q_{mk} f'[\lambda_k, \lambda_l, \lambda_m]$$

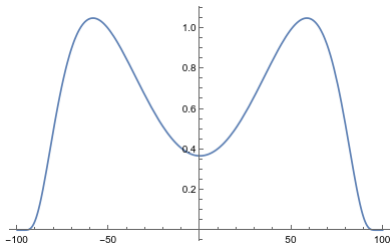
expressed in terms of divided differences:

$$f'[x, y] = \frac{f'(x) - f'(y)}{x - y} \quad f'[x, y, z] = \frac{f'[x, y] - f'[y, z]}{x - z}$$

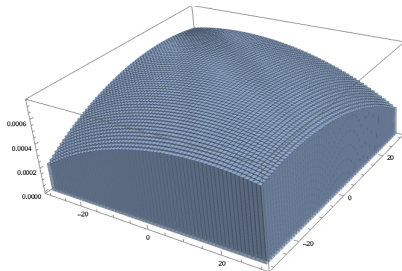
Gauge propagator

For suitable f' we may perform the Gaussian integration:

$$\overbrace{Q_{kl}Q_{mn}} = \frac{\int Q_{kl}Q_{mn}e^{-\frac{1}{2}\langle Q,Q \rangle}dQ}{\int e^{-\frac{1}{2}\langle Q,Q \rangle}dQ} = \delta_{kn}\delta_{lm} \frac{1}{f'[\lambda_k, \lambda_l]}$$



Example of a positive bump function f



Divided difference $f'[m, n]$

Quantum Ward identity

We thus have a gauge propagator, and may consider all 1PI diagrams:

$$\langle\langle V_1, \dots, V_n \rangle\rangle = \text{Diagram with a central shaded circle and external legs } V_1, V_2, V_3, V_4, V_n \text{ and a dashed line } a.$$

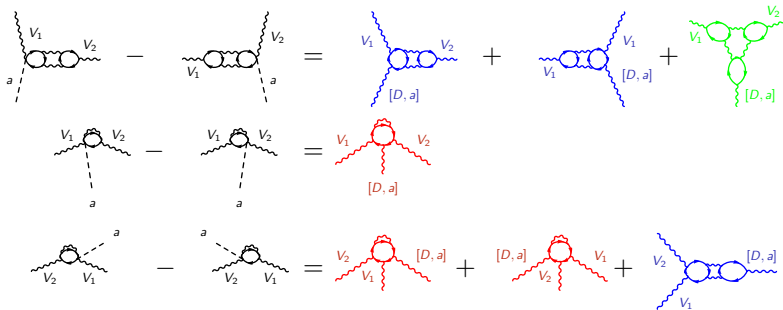
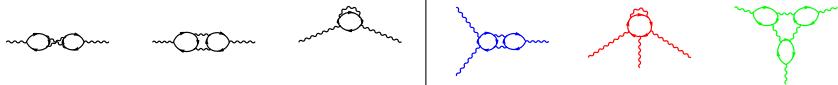
- There is a bosonic Ward identity:

$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3} \quad [D, a]$$

which induces the following **quantum Ward identity** for the divergent one-loop contributions:

$$\begin{aligned} \langle\langle V_1, \dots, aV_j, \dots, V_n \rangle\rangle_\infty^{1L} - \langle\langle V_1, \dots, V_{j-1}a, \dots, V_n \rangle\rangle_\infty^{1L} \\ = \langle\langle V_1, \dots, V_{j-1}, [D, a], V_j, \dots, V_n \rangle\rangle_\infty^{1L}. \quad (II') \end{aligned}$$

Proof of $\langle\langle aV_1, V_2 \rangle\rangle_{\infty}^{1L} - \langle\langle V_1, V_2a \rangle\rangle_{\infty}^{1L} = \langle\langle [D, a], V_1, V_2 \rangle\rangle_{\infty}^{1L}$



One-loop renormalizable spectral action

In general, as a consequence of the quantum Ward identity one can show that the divergent part of the 1L contribution $\langle\langle V, \dots, V \rangle\rangle_\infty^{1L}$ has the same structure as $\langle V, \dots, V \rangle$:

$$\sum_n \frac{1}{n} \langle\langle V, \dots, V \rangle\rangle_\infty^{1L} = \sum_{k=1}^{\infty} \left(\int_{\tilde{\psi}_{2k-1}} \text{CS}_{2k-1}(A) + \frac{1}{2k} \int_{\tilde{\phi}_{2k}} F^k \right).$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are the analogues of ϕ and ψ but now defined using the double bracket.

We conclude that the passage to the one-loop renormalized spectral action can be realized by a **transformation in the space of noncommutative integrals**, sending $\phi \mapsto \phi - \tilde{\phi}$ and $\psi \mapsto \psi - \tilde{\psi}$.

Thanks!