Cyclic cocycles in the spectral action and one-loop corrections

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Outline

Noncommutative geometry

Taylor expansion of the spectral action

Cyclic cocycles

Loop corrections

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NC geometry: spectral triple $(\mathcal{A}, \mathcal{H}, D)$

• algebra $\mathcal{A}$	"coordinate algebra"
• self-adjoint operator D with $(i+D)^{-1}$ compact and $[D,a]$ bounded for $a \in \mathcal{A}$	"inverse fermion propagator"
• both acting on Hilbert space $\mathcal{H}$	"one-particle space"

Applications to gauge theories (e.g. $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$) :

- Gauge group $\mathcal{U}(\mathcal{A})$ of unitaries in $\mathcal{A}$ acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- Spectral invariant action functional [Chamseddine–Connes, 1996]:

$$\mathrm{Tr}\, f(D)$$

- More general: inner perturbations as gauge fields

$$D \mapsto D' = D + \sum_j a_j[D, b_j] \qquad (a_j, b_j \in \mathcal{A})$$

Applications to particle physics

We may consider asymptotic expansions of the form:

$$\mathrm{Tr}\, f((D+V)/\Lambda) \sim \sum_{k \le n} f_k \Lambda^k \alpha_k$$

for $V = \sum_j a_j [D, b_j]$, a suitable f, cutoff Λ and some n (dimension). The α_k are integral invariants of local polynomial functionals in the metric and in V.

Almost-commutative manifolds [Chamseddine–Connes–Marcolli 2007, vS] $(C(M, A_F), L^2(M,S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F)$:

$$\mathrm{Tr}\, f((D+V)/\Lambda) \sim f_4 \Lambda^4 \mathrm{Vol}(M) + f_2 \Lambda^2 \int R\sqrt{g} + f_0 \int R^2 \sqrt{g} + \cdots$$
$$+ \widetilde{f}_0 \int \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} - \widetilde{f}_2 \Lambda^2 \int |\phi|^2 + \widetilde{f}_0 \int |\phi|^4 + \cdots$$

Outline

Perturbative expansion of the spectral action

Instead, today we consider an expansion in powers of V of

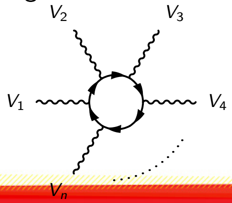
$$\operatorname{Tr} f(D+V) - \operatorname{Tr} f(D) = \sum_n \frac{1}{n} \langle V, V, \ldots, V \rangle$$

with [vS 2012, Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\langle V_1, V_2, \ldots, V_n \rangle = \frac{1}{2\pi i} \operatorname{Tr} \oint f'(z) V_1 (z-D)^{-1} \cdots V_n (z-D)^{-1}$$

We will depict it as a Feynman diagram:

$$\langle V_1, \ldots, V_n \rangle =$$



Cyclic permutations and the Ward identity

There are the following two properties of the bracket:

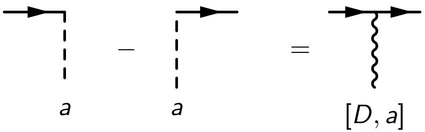
$$\langle V_1, \ldots, V_n \rangle = \langle V_2, \ldots, V_n, V_1 \rangle \qquad \text{(I)}$$

$$\langle aV_1, \ldots, , V_n \rangle - \langle V_1, \ldots, V_n a \rangle = \langle [D,a], V_1, \ldots, V_n \rangle \qquad \text{(II)}$$

Identity (II) is a type of 'Ward identity' as it boils down to

$$(z-D)^{-1}a - a(z-D)^{-1} = (z-D)^{-1}[D,a](z-D)^{-1}$$

We depict it as



Noncommutative integrals

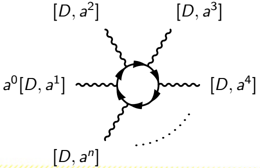
We may express multi-linear functionals on $\mathcal{A}$ as noncommutative integrals over universal differential forms:

- if φ_n is a $n+1$-linear functional on $\mathcal{A}$ so that $\varphi_n(a_0, a_1, \ldots, a_n) = 0$ if one of the $a_1, \ldots, a_n$ is a complex scalar, then we may write

$$\varphi_n(a_0, a_1, \ldots a_n) = \int_{\varphi_n} a_0 \delta a_1 \cdots \delta a_n.$$

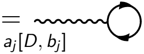
- We will be interested in the brackets $\langle V, \ldots, V \rangle$ as they appear in the perturbative expansion of the spectral action and define

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n] \rangle :=$$



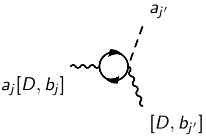
Brackets as noncommutative integrals

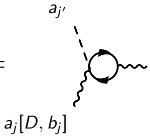
For one external edge we find with $V = a_j[D, b_j]$ and $A = a_j\delta(b_j)$:

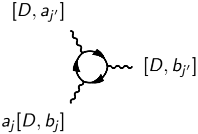
$\langle V \rangle =$ 

$= \int_{\phi_1} A,$

For two external edges, we apply the Ward identity and derive

$\langle V, V \rangle =$ 

$=$ 

$+$ 

$$= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A.$$

Brackets as noncommutative integrals

In conclusion, if $V = \sum_j a_j[D, b_j]$ is a gauge field with corresponding universal 1-form $A = \sum_j a_j \delta b_j$ we may write:

$$
\begin{aligned}
\langle V \rangle &= \int_{\phi_1} A, \\
\langle V, V \rangle &= \int_{\phi_2} A^2 + \int_{\phi_3} A\delta A, \\
\langle V, V, V \rangle &= \int_{\phi_3} A^3 + \int_{\phi_4} A\delta A A + \int_{\phi_5} A\delta A\delta A, \\
\langle V, V, V, V \rangle &= \int_{\phi_4} A^4 + \cdots
\end{aligned}
$$

Re-ordering the terms

We now introduce another multi-linear functional ψ_{2k-1} by setting

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2} \int_{\phi_{2k}} \delta\omega; \qquad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2} \int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2} \int_{\phi_2} (\delta A + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\frac{1}{2} \int_{\phi_3} A\delta A + \frac{1}{3} \int_{\phi_3} A^3 + \frac{1}{3} \int_{\phi_4} A\delta AA + \frac{1}{4} \int_{\phi_4} A^4$$
$$= \frac{1}{2} \int_{\psi_3} \left(A\delta A + \frac{2}{3} A^3 \right) + \frac{1}{4} \int_{\phi_4} (\delta A + A^2)^2$$

The perturbative expansion of the spectral action

Theorem
For a finitely-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and f in a suitable function class, there is the following absolutely convergent series expansion:

$$\mathrm{Tr}(f(D+V) - f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here the higher-dimensional *Chern–Simons forms* are given by

$$\mathrm{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt; \qquad F_t = t\delta A + t^2 A^2$$

The functionals ψ_{2k-1} and ϕ_{2k} turn out to define odd and even cyclic cocycles.

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Intermezzo: Hochschild and cyclic cocycles

- We define operators b and B_0 on multi-linear functionals by

$$\int_{b\phi_n} \omega \delta a = \int_{\phi_n} [\omega, a], \qquad \int_{B_0\phi_n} \omega = \int_{\phi_n} \delta\omega$$

and set $B = AB_0$ in terms of the operator A of cyclic anti-symmetrization.

- $b^2 = 0$ (Hochschild cohomology), $B^2 = 0$ and $bB + Bb = 0$ $\implies$ $(b+B)^2 = 0 \rightsquigarrow$ (even/odd) periodic cyclic cohomology
- An odd (b, B)-cocycle is given by a sequence $(\phi_1, \phi_3, \phi_5, \ldots)$, with

$$b\phi_{2k+1} + B\phi_{2k+3} = 0,$$

Similar for even (b, B)-cocycles.

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]\rangle$

The n-cochain ϕ_n has the following properties:

- $B_0\phi_n$ is invariant under cyclic permutations, so $B\phi_n = nB_0\phi_n$ for odd n and $B\phi_n = 0$ for even n.
- $b\phi_{2k-1} = \phi_{2k}$ so $b\phi_{2k} = 0$.

Proof ($k = 1$):

$$\int_{b\phi_1} a_0 \delta a_1 \delta a_2 = \langle a_0 a_1 [D, a_2]\rangle - \langle a_0[D, a_1 a_2]\rangle + \langle a_2 a_0[D, a_1]\rangle$$

$$= -\langle a_0[D, a_1]a_2\rangle + \langle a_2 a_0[D, a_1]\rangle = \langle a_0[D, a_1], [D, a_2]\rangle$$

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]\rangle$

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- $B_0\phi_n$ is invariant under cyclic permutations, so $B\phi_n = nB_0\phi_n$ for odd n and $B\phi_n = 0$ for even n.
- $b\phi_{2k-1} = \phi_{2k}$ so $b\phi_{2k} = 0$.
- $bB_0\phi_{2k} = 2\phi_{2k} - B_0\phi_{2k+1}$

Proof ($k = 1$):

$$\begin{aligned}\int_{bB_0\phi_2} a_0 \delta a_1 \delta a_2 &= \int_{B_0\phi_2} a_0 a_1 \delta a_2 - \int_{B_0\phi_2} a_0 \delta(a_1 a_2) + \int_{B_0\phi_2} a_2 a_0 \delta a_1 \\ &= \langle [D, a_0 a_1], [D, a_2]\rangle - \langle [D, a_0], [D, a_1 a_2]\rangle + \langle [D, a_2 a_0], [D, a_1]\rangle \\ &= \cdots \\ &= 2\langle a_0[D, a_1], [D, a_2]\rangle - \langle [D, a_0], [D, a_1], [D, a_2]\rangle\end{aligned}$$

Brackets and cyclic cocycles

We repeat: $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]\rangle$

The n-cochain ϕ_n has the following properties:

- $B_0\phi_n$ is invariant under cyclic permutations, so $B\phi_n = nB_0\phi_n$ for odd n and $B\phi_n = 0$ for even n.
- $b\phi_{2k-1} = \phi_{2k}$ so $b\phi_{2k} = 0$.
- $bB_0\phi_{2k} = 2\phi_{2k} - B_0\phi_{2k+1}$

Motivated by this we define

$$\psi_{2k-1} := \phi_{2k-1} - \tfrac{1}{2}B_0\phi_{2k} \qquad \Longrightarrow \qquad B\psi_{2k+1} = 2(2k+1)b\psi_{2k-1}.$$

Proposition (van Nuland–vS 2021)

1. *The sequence (ϕ_{2k}) is an even (b, B)-cocycle and each ϕ_{2k} defines an even Hochschild cocycle: $b\phi_{2k} = 0$.*
2. *The sequence $(\widetilde{\psi}_{2k-1} := (-1)^{k-1}\frac{(k-1)!}{(2k-1)!}\psi_{2k-1})$ is an odd (b, B)-cocycle.*

Gauge invariance

- For the Yang–Mills terms, it turns out that gauge invariance is a consequence of the fact that ϕ_{2k} are Hochschild cocycles:

$$\int_{\phi_{2k}} uF^k u^* = \int_{\phi_{2k}} F^k.$$

- Since the spectral action is a spectral invariant, it is in particular invariant under gauge transformations.
- This combines: also the Chern–Simons terms are gauge invariant:

$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(uAu^* + u\delta u^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \mathrm{cs}_{2k-1}(A)$$

Theorem (van Nuland–vS, 2021)

Let f be in a suitable function class. Then the sequence $(\widetilde{\psi}_{2k+1})$ defines an odd entire cocycle and the pairing of $(\widetilde{\psi}_{2k+1})$ with $K_1(\mathcal{A})$ is trivial

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Loop corrections to the spectral action

Goal: Compute the quantum partition function and analyze loop corrections to the spectral action

Concretely, we try to make sense of the partition function in the background field:

$$Z[J, V] = \int e^{-\operatorname{Tr} f(D+V+Q)+(J,V)} d[Q]$$

where

- $V = \sum_j a_j[D, b_j] \in \Omega^1_D(\mathcal{A})_{\text{s.a.}}$ is a background gauge field
- J is a source field in the dual space $\Omega^1_D(\mathcal{A})^*_{\text{s.a.}}$
- Q is a quantum field that is integrated over in the path integral

Loop corrections

In order to makes sense of the path integral we let Q be an arbitrary finite size hermitian matrix $Q = (Q_{kl})$ in an eigenbasis of D (eigenvalues: λ_k).

The quadratic and cubic terms become:

$$\frac{1}{2}\langle Q, Q\rangle = \frac{1}{2}\sum_{k,l} Q_{kl}Q_{lk}f'[\lambda_k, \lambda_l]$$

$$\frac{1}{3}\langle Q, Q, Q\rangle = \frac{1}{3}\sum_{k,l,m} Q_{kl}Q_{lm}Q_{mk}f'[\lambda_k, \lambda_l, \lambda_m]$$

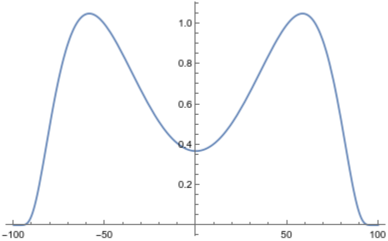
expressed in terms of divided differences:

$$f'[x, y] = \frac{f'(x) - f'(y)}{x - y} \qquad f'[x, y, z] = \frac{f'[x, y] - f'[y, z]}{x - z}$$

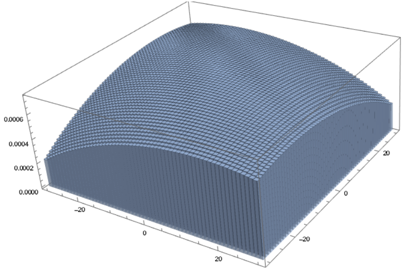
Gauge propagator

For suitable f' we may perform the Gaussian integration:

$$\overbrace{Q_{kl}Q}_{mn} = \frac{\int Q_{kl} Q_{mn} e^{-\frac{1}{2}\langle Q,Q\rangle} dQ}{\int e^{-\frac{1}{2}\langle Q,Q\rangle} dQ} = \delta_{kn}\delta_{lm} \frac{1}{f'[\lambda_k, \lambda_l]}$$



Example of a positive bump function f

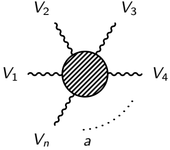


Divided difference $f'[m, n]$

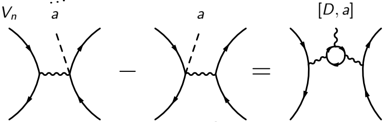
Quantum Ward identity

We thus have a gauge propagator, and may consider all 1PI diagrams:

$$\langle\!\langle V_1, \dots, V_n \rangle\!\rangle =$$



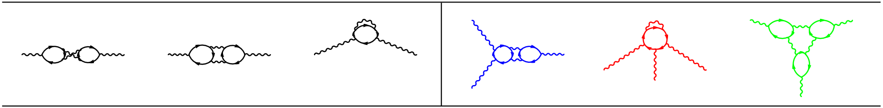
- There is a bosonic Ward identity:

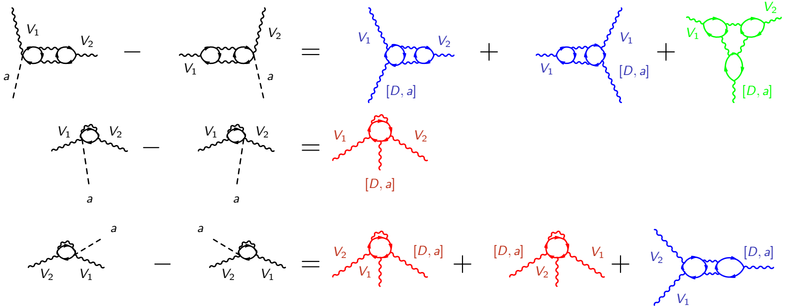


  which induces the following quantum Ward identity for the divergent one-loop contributions:

$$\langle\!\langle V_1, \dots, aV_j, \dots, V_n \rangle\!\rangle^{1L}_{\infty} - \langle\!\langle V_1, \dots, V_{j-1}a, \dots, V_n \rangle\!\rangle^{1L}_{\infty}$$
$$= \langle\!\langle V_1, \dots, V_{j-1}, [D, a], V_j, \dots, V_n \rangle\!\rangle^{1L}_{\infty}. \quad \text{(II')}$$

Proof of $\langle\langle aV_1, V_2\rangle\rangle^{1L}_{\infty} - \langle\langle V_1, V_2 a\rangle\rangle^{1L}_{\infty} = \langle\langle [D, a], V_1, V_2\rangle\rangle^{1L}_{\infty}$





One-loop renormalizable spectral action

As a consequence of the quantum Ward identity one can show that the divergent part of the 1L contribution $\langle\!\langle V, \ldots, V \rangle\!\rangle_\infty^{1L}$ has the same structure as $\langle V, \ldots, V \rangle$:

$$\sum_n \frac{1}{n} \langle\!\langle V, \ldots, V \rangle\!\rangle_\infty^{1L} = \sum_{k=1}^{\infty} \left(\int_{\widetilde{\psi}_{2k-1}} \mathrm{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\widetilde{\phi}_{2k}} F^k \right).$$

where $\widetilde{\phi}$ and $\widetilde{\psi}$ are the analogues of ϕ and ψ but now defined using the double bracket.

We conclude that the passage to the one-loop renormalized spectral action can be realized by a transformation in the space of noncommutative integrals, sending $\phi \mapsto \phi - \widetilde{\phi}$ and $\psi \mapsto \psi - \widetilde{\psi}$.

Thanks!