Noncommutative spaces at finite resolution

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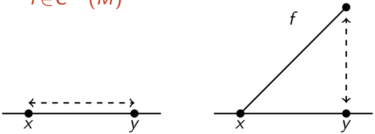
Motivation: hearing the shape of a drum

Mark Kac (1966):

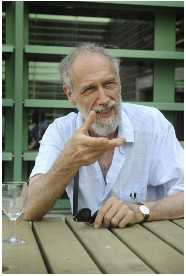
Riemannian (spin) geometry: (M, g) is not fully determined by spectrum of Δ_M (D_M).

- This is considerably improved by considering besides D_M also the C^*-algebra $C(M)$ of continuous functions on M [Connes 1989]
- In fact, the Riemannian distance function on M is equal to

$$d(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \|[D, f]\| \leq 1\}$$



Noncommutative geometry



So, if combined with the C^*-algebra $C(M)$, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), L^2(S_M), D_M) \longleftrightarrow (M, g)$$

Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

> *We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.*

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

The "usual" story: spectral triples

- a C^*-algebra A
- a self-adjoint operator D with (local) compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.
Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

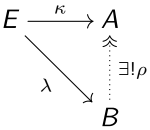
Isomorphisms are complete order isomorphisms

C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:



Shilov boundaries

There is a useful description of C^*-envelopes using Shilov ideals.

Definition

Let $\kappa : E \to A$ be a C^*-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The Shilov boundary ideal is the largest of such boundary ideals.

Proposition

Let $\kappa : E \to A$ be a C^*-extension. Then there exists a Shilov boundary ideal J and $C^*_{env}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1. Accordingly, its C^*-envelope is $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number prop(E) of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C^*_{env}(E)$ is a C^*-algebra.

Returning to harmonic functions in the disk we have $\mathrm{prop}(C_{\mathrm{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\min} F) = \max\{prop(E), prop(F)\}$$

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual E^d of a unital operator system is a unital operator system with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

  and similarly for the matrix order.
- Also, we have $(E^d)^d_+ \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

  pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

  and the other way around.

In the infinite-dimensional/non-unital case, this is more subtle (more later..).

Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.

The Shilov boundary of the operator system $C^*(\mathbb{Z})_{(n)}$ is S^1.
Consequently, the C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V\Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Farenick continues to exploit the duality by showing:

- every positive linear map of the $n \times n$ complex matrices is completely positive when restricted to the Toeplitz operator system.
- every unital isometry of the $n \times n$ Toeplitz matrices into the algebra of $n \times n$ complex matrices is a unitary similarity transformation.

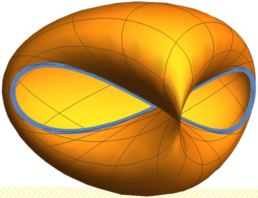
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

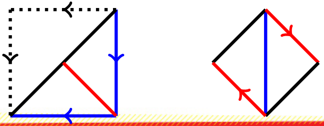
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is $\mathbb{T}^2/S_2$, given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!



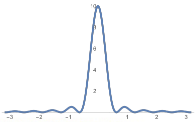
Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is an C^1-approximate order inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

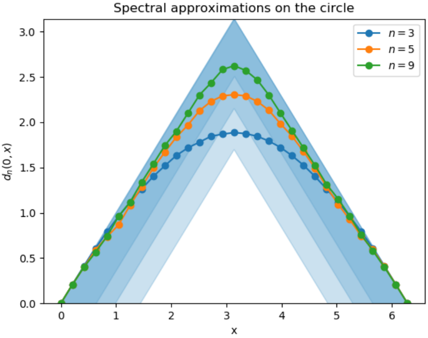
in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n:



- The fact that S_n is an C^1-approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where $\gamma_n \to 0$ as $n \to \infty$.
- Some (basic) Python simulations for point evaluation on S^1:



Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that bridges the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n.

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

. . . . . .

More on non-unital operator systems

Consider a matrix-ordered operator space $(E, \|\cdot\|)$.

- The noncommutative (nc) state space is defined for any n as

$$\mathcal{S}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| = 1, \phi \geq 0\}$$

not always convex
nor weakly $*$-compact

- The nc quasi-state space is defined for any n as

$$\widetilde{\mathcal{S}}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| \leq 1, \phi \geq 0\}$$

convex
and weakly $*$-compact

- The modified numerical radius $\nu_E : M_n(E) \to \mathbb{C}$ is defined as

$$\nu_E(x) = \sup \left\{ \left| \phi \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix} \right) \right| : \phi \in \widetilde{\mathcal{S}}_{2n}(E) \right\}.$$

Definition (Werner 2002)
A non-unital operator system is given by a matrix-ordered operator space for which $\nu_E(\cdot) = \|\cdot\|$.

Approximate order units

We now consider a particular class of non-unital operator systems.

Definition (Ng 1969)

Let E be a matrix-ordered $*$-vector space. An approximate order unit for E is an ordered net $\{e_\lambda\}_{\lambda \in \Lambda}$ of positive elements such that

> *for each $x^* = x \in E$ there exists a positive real number t and $\lambda \in \Lambda$ such that*
> $$-te_\lambda \leq x \leq te_\lambda.$$

In fact, if the approximate order unit is matrix-norm-defining in the sense that

$$\|x\| = \inf\left\{ t : \begin{pmatrix} te_\lambda^n & x \\ x^* & te_\lambda^n \end{pmatrix} \in M_{2n}(E)_+ \text{ for some } \lambda \in \Lambda \right\}$$

then E is a non-unital operator system [Karn 2005, Han 2010].

Assuming the existence of a matrix-norm-defining approximate order unit in E we may show familiar C^*-results such as

1. the nc state space $S_n(E)$ is convex

and if $E \subseteq A$ with a norm-defining approximate order unit for A contained in E we have that

2. any (pure) nc state on E can be extended to a (pure) state on A.
3. (Jordan decomposition) For each hermitian continuous linear functional $\phi : M_n(E) \to \mathbb{C}$ there exist positive linear functionals $\phi_+, \phi_- : M_n(E) \to \mathbb{C}$ such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$
4. we have an isometrical order isomorphism $M_n(A)^*_h / M_n(E)^\perp_h \to M_n(E)^*_h$

This also applies if we replace E and A by dense subspaces $\mathcal{E}$ and $\mathcal{A}$.

Operator systems, groupoids and bonds

Recall:

- Consider a locally compact groupoid G equipped with a (left invariant) Haar system $\nu = \{\nu_x\}$.
- The space $C_c(G)$ of compactly supported complex-valued continuous functions on G becomes a $*$-algebra with the convolution product and involution given by

$$f * g(\gamma) = \int_{G_x} f(\gamma\gamma_1^{-1})g(\gamma_1)d\nu_x(\gamma_1); \qquad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

  where $x = s(\gamma)$ for any $\gamma \in G$.
- $C_c(G)$ can be completed to the groupoid C^*-algebra $C^*(G)$

Definition

A bond is a triple (G, ν, Ω) consisting of a locally compact groupoid G, a Haar system $\nu = \{\nu_x\}$ and an open symmetric subset $\Omega \subseteq G$ containing the units $G^{(0)}$.

Proposition

Let (Ω, G, ν) be a bond. The closure of the subspace $C_c(\Omega) \subseteq C_c(G)$ in the C^*-algebra $C^*(G)$ is an operator system.

Example

1. *Consider $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ Fejér–Riesz operator system inside $C^*(\mathbb{Z})$.*
2. *Consider $\Omega_n = (-n, n) \subset C_m$ (so modulo m). The operator system consists of banded $m \times m$ circulant matrices of band width n.*

Thus, the ambient groupoid is crucial since these two operator systems are not even Morita equivalent.

3. *Given the set $X = \{1, \ldots, m\}$ consider a "band" $R_n \subset X \times X$ around the diagonal of width $n \rightsquigarrow$ banded $m \times m$ matrices of band width n.*

Operator systems associated to tolerance relations

- Suppose that X is a set and consider a relation $\mathcal{R} \subseteq X \times X$ on X that is reflexive, symmetric but not necessarily transitive.
- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R})$. Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

Example

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X and suppose that $\mathcal{R}$ generates the full equivalence class $X \times X$ (i.e. the graph corresponding to $\mathcal{R}$ is connected). Then

1. *the C^*-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X))$ and $prop(E(\mathcal{R})) = diam(\mathcal{R})$.*
2. *the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.*

Finite partial partitions of a metric measure space

A finite partial ϵ-partition of X is a finite collection $P = \{U_i\}$ of disjoint measurable sets $U_i \subseteq X$ such that $\mathrm{diam}(U_i) < \epsilon$; directed by refinement.

- The corresponding finite-dimensional algebra $\mathcal{A}_P$ with unit e_P is

$$\mathcal{A}_P = \left\{ \sum_{U,V \in P} a_{UV} |1_U\rangle\langle 1_V| : a_{UV} \in \mathbb{C} \right\} \cong \mathcal{K}(l^2(P))$$

- A tolerance relation $\mathcal{R}_\epsilon^P$ on the finite set P is given by

$$\mathcal{R}_\epsilon^P = \{U \times V \mid \quad U, V \in P, \quad U \times V \subseteq \mathcal{R}_\epsilon\} \subseteq P \times P$$

  and yields the operator system $E(\mathcal{R}_\epsilon^P)$.
- If $P \leq P'$ then $E(\mathcal{R}_\epsilon^P) \subseteq E(\mathcal{R}_\epsilon^{P'})$ and also $\mathcal{A}_P \subseteq \mathcal{A}_{P'}$.
- Approximate order unit $\{e_P\}_P$ of $\varinjlim \mathcal{A}_P$ is contained in $\varinjlim E(\mathcal{R}_\epsilon^P)$

Matrix order structure for finite partial partitions

Proposition

1. *The operator system* $E(\mathcal{R}_\epsilon^P)$ *is complete order-isomorphic to* $S_L(\mathcal{A}_P) \subseteq \mathcal{A}_P$, *in terms of Schur–Hadamard multiplication with the matrix*

$$L = (L_{ij}); \qquad L_{ij} = \begin{cases} 1 & \text{if } U_i \times U_j \subseteq \mathcal{R}_\epsilon \\ 0 & \text{if } U_i \times U_j \not\subseteq \mathcal{R}_\epsilon \end{cases}$$

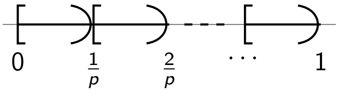
2. *The dual operator system* $E(\mathcal{R}_\epsilon^P)^d$ *of* $E(\mathcal{R}_\epsilon^P)$ *is linearly isomorphic to* $S_L(\mathcal{A}_P)$ *with cones of positive elements given by*

$$M_n(E(\mathcal{R}_\epsilon^P)^d)_+ = (S_L)_n(M_n(\mathcal{A}_P)_+)$$

3. *We have* $S_L((\mathcal{A}_P)_+) = S_L(\mathcal{A}_P)_+$ *if and only if* $\mathcal{R}_\epsilon^P$ *is an equivalence relation.*

Example: finite ϵ-partitions of the unit interval

Consider with $1/p < \epsilon$:



- The unital operator system $\mathcal{E}_P(\mathcal{R}_\epsilon)$ can be identified with the operator system $\mathcal{E}_{p,N}$ of $p \times p$ band matrices B of band width N, *i.e.*,

$$\mathcal{E}_{p,N} := \{B = (b_{ij}) \in M_p(\mathbb{C}) \mid \qquad b_{ij} = 0 \text{ if } |i-j| > N\}$$

- Since $\mathcal{E}_{p,N}\mathcal{E}_{p,N'} = \mathcal{E}_{p,N+N'}$ the propagation number of $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$ is equal to $\lceil p/N \rceil$.

Spaces at finite resolution

Proposition

Let X be a path metric measure space with a measure of full support.

1. $\mathcal{E}(\mathcal{R}_\epsilon) := \varinjlim E(\mathcal{R}_\epsilon^P)$ *is a dense subspace of* $E(\mathcal{R}_\epsilon)$
2. $\mathcal{A}_\epsilon := \varinjlim \mathcal{A}_P$ *is a dense* $*$*-subalgebra of the* C^**-algebra* $\mathcal{K}(L^2(X))$*;*
3. *there exists a matrix-norm-defining approximate order unit for* $\mathcal{A}_\epsilon$ *which is contained in* $\mathcal{E}(\mathcal{R}_\epsilon)$*.*

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then

1. $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ *and* $\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$
2. *The pure states of* $E(\mathcal{R}_\epsilon)$ *are given by vector states* $|\psi\rangle\langle\psi|$ *where the essential support of* $\psi \in L^2(X)$ *is* ϵ*-connected.*

Thanks!

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