Factorization of Dirac operators in unbounded KK-theory

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Outline

Motivation: unbounded KK-theory, curvature

Spectral triples and Morita equivalence

Riemannian submersions

Toric noncommutative manifolds

Immersions: spheres in Euclidean space

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Motivation for unbounded KK-theory

- Kasparov's bivariant K-theory: the backbone of Connes' noncommutative differential geometry
- Many constructions in differential geometry can be captured in *unbounded* KK-theory [BJ 1983]
- With the added value that geometry becomes visible:
  - gauge degrees of freedom (Particle Physics [CCvS])
  - curvature [Mesland–Rennie–vS].
- A key role played by the internal Kasparov product

$$S \otimes 1 + 1 \otimes_\nabla T$$

- We investigate a notion of curvature emerging from this unbounded refinement :

$$\begin{aligned} R_{(S,\nabla,T)} &:= (S \otimes 1 + 1 \otimes_\nabla T)^2 - (S^2 \otimes 1 + 1 \otimes_\nabla T^2) \\ &= [S \otimes 1, 1 \otimes_\nabla T] + (1 \otimes_\nabla T)^2 - 1 \otimes_\nabla T^2 \end{aligned}$$

- We work in three classes of examples:
  - Morita equivalences (applications to particle physics) [Chamseddine–Connes–Marcolli, vS]
  - Riemannian submersions [Kaad–vS, Mesland–Rennie–vS, ...]
  - Riemannian immersions: $S^n \hookrightarrow \mathbb{R}^{n+1}$ [vS–Verhoeven]

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Spectral triples and Morita equivalence

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and a (fgp) Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$:

- Hilbert space: $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$
- Choose (hermitian) connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$
- Define an operator $D' := 1 \otimes_\nabla D$ by

$$(1 \otimes_\nabla D)(\xi \otimes h) = (1 \otimes \pi_D)(\nabla(\xi))h + \xi \otimes (Dh).$$

where $\pi_D(a\delta(b)) = a[D, b]$.

Then $(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, 1 \otimes_\nabla D)$ is a spectral triple [C96] representing the internal Kasparov product of $[(\mathcal{E}, 0)]$ and $[(H, F_D)]$.

$$R_\nabla := 1 \otimes_\nabla D^2 - (1 \otimes_\nabla D)^2 = \pi_D(\nabla^2)$$

Morita self-equivalences

- Now take $\mathcal{A} = \mathcal{B}$ and $\mathcal{E} = \mathcal{A}$ as well, then $\mathcal{H}' \cong \mathcal{H}$ but connection $\nabla : \mathcal{A} \to \Omega^1(\mathcal{A})$ so that

$$1 \otimes_\nabla D \equiv D + \pi_D(\omega)$$

- The connection one-form is $\pi_D(\omega)^* = \pi_D(\omega) = \pi_D(\nabla(1))$, or,

$$\pi_D(\omega) = \sum_j a_j[D, b_j]; \qquad (a_j, b_j \in \mathcal{A})$$

- Curvature for Morita self-equivalences becomes

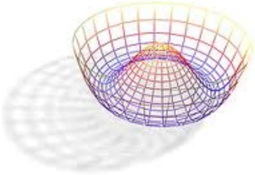
$$R_\nabla = \pi_D(\nabla^2) = -\pi_D(\delta\omega + \omega^2)$$

Two-point space

- Consider the spectral triple

$$\left(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}\oplus\mathbb{C}, D = \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}\right)$$

- Inner perturbations: $\pi_D(\omega) = \begin{pmatrix} 0 & c\phi \\ \overline{c}\overline{\phi} & 0 \end{pmatrix}$
- Curvature:



$$\begin{aligned} R_\nabla &= (1 \otimes_\nabla D^2) - (1 \otimes_\nabla D)^2 \\ &\equiv D^2 - (D + \pi_D(\omega))^2 = |c|^2(1 - |\phi + 1|^2)1_2 \end{aligned}$$

- In physical applications $\mathcal{A} = C^\infty(M, A_F)$ for some finite-dimensional matrix algebra A_F and ω is parametrized by gauge fields, scalar (Higgs) fields [CCvS 2013–].

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Riemannian submersions

Consider the following setup [Kaad–vS, 2016]:

- A Riemannian submersion

$$\pi : M \to B$$

  of compact spin^c manifolds M and B
- It is well-known from wrong-way functoriality [CS 1984] that

$$[M] = \pi! \widehat{\otimes}_{C(B)} [B] \tag{$*$}$$

- Can we write the Dirac operator on M as a tensor sum:

$$D_M = D_V \otimes \gamma + 1 \otimes_\nabla D_B + \widetilde{c}(\Omega)?$$

  for some unbounded KK-cycle $(C^\infty(M), X, D_V)$, representing the internal KK-product $(*)$, allowing for curvature defects Ω?

Recall: horizontal and vertical vector fields

- On $\mathscr{X}(M)$ one can introduce the direct sum connection

$$\nabla^{\oplus} = P_V \nabla^M P_V \oplus \pi^* \nabla^B,$$

- the second fundamental form:

$$S(X, Y, Z) := \left\langle \nabla^V_{(1-P)Z}(PX) - [(1-P)Z, PX], PY \right\rangle_M,$$

- the curvature of $\pi : M \to B$:

$$\Omega(X, Y, Z) := -\left\langle [(1-P)X, (1-P)Y], PZ \right\rangle_M$$

which combined yield a tensor $\omega \in \Omega^1(M) \otimes_{C^\infty(M)} \Omega^2(M)$

Proposition (Bismut, 1986)

The Levi-Civita connection ∇^M is related to the direct sum connection $\nabla^{\oplus}$ by the following formula

$$\langle \nabla^M_X Y, Z \rangle_M = \langle \nabla^{\oplus}_X Y, Z \rangle_M + \omega(X)(Y, Z).$$

Spin geometry and Clifford modules

Suppose that M and B are (even-dim) spin^c manifolds, so we have

$$\mathrm{Cl}(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_M), \qquad \mathrm{Cl}(B) \cong \mathrm{End}_{C^\infty(B)}(\mathscr{E}_B)$$

and hermitian Clifford connections $\nabla^{\mathscr{E}_M}$ and $\nabla^{\mathscr{E}_B}$.

- We define the horizontal spinor module:

$$\mathscr{E}_H := \mathscr{E}_B \otimes_{C^\infty(B)} C^\infty(M); \quad \mathrm{Cl}_H(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_H); \quad \nabla^{\mathscr{E}_H} := \pi^* \nabla^{\mathscr{E}_B}.$$

- We define the vertical spinor module:

$$\mathscr{E}_V := \mathscr{E}_H^* \otimes_{\mathrm{Cl}_H(M)} \mathscr{E}_M, \qquad \mathrm{Cl}_V(M) \cong \mathrm{End}_{C^\infty(M)}(\mathscr{E}_V)$$

$$\nabla_X^{\mathscr{E}_V} = 1 \otimes \nabla_X^{\mathscr{E}_H^*} + \nabla_X^{\mathscr{E}_M} \otimes 1 + \frac{1}{4} c(\omega(X)) \otimes 1,$$

- Finally,

$$\mathscr{E}_H \otimes_{C^\infty(M)} \mathscr{E}_V \cong \mathscr{E}_M, \text{ compatibly with } \mathrm{Cl}_H(M) \widehat{\otimes}_{C^\infty(M)} \mathrm{Cl}_V(M) \cong \mathrm{Cl}(M)$$

The vertical operator

- We define a C^*-correspondence X from $C(M)$ to $C(B)$ by completing $\mathscr{E}_V$ with respect to

$$\langle \phi_1, \phi_2 \rangle_X(b) := \int_{F_b} \langle \phi_1, \phi_2 \rangle_{\mathscr{E}_V} \, d\mu_{F_b}$$

- The following defines an odd self-adjoint unbounded operator in X

$$D_V = i \sum_{j=1}^{\dim(F)} c_V(e_j) \nabla^{\mathscr{E}_V}_{e_j}$$

where $\{e_j\}$ is a local orthonormal frame for $\mathscr{X}_V(M)$

Proposition (Kaad-vS, 2016)

The triple $(C^\infty(M), X, D_V)$ is an even unbounded Kasparov module from $C(M)$ to $C(B)$ with grading operator $\gamma_X : X \to X$.

The horizontal operator and the connection

- The Dirac operator $D_B : \mathrm{Dom}(D_B) \to L^2(\mathscr{E}_B)$ is locally

$$D_B = i \sum_{\alpha=1}^{\dim(B)} c(f_\alpha)\nabla^{\mathscr{E}_B}_{f_\alpha} : \mathscr{E}_B \to L^2(\mathscr{E}_B)$$

  Clearly $(C^\infty(B), L^2(\mathscr{E}_B), D_B; \gamma_B)$ is an (even) spectral triple
- The following defines a hermitian connection on X

$$\nabla^X_Z = \nabla^{\mathscr{E}_V}_{Z_H} + \frac{1}{2}k(Z_H)$$

  with $k = (\mathrm{Tr} \otimes 1)(S) \in \Omega^1(M)$ the mean curvature

  Lemma (Kaad–vS, 2016)

  The following local expression defines an odd symmetric unbounded operator in $X\widehat{\otimes}_{C(B)}L^2(\mathscr{E}_B)$:

$$(1 \otimes_\nabla D_B) := 1 \otimes D_B + i \sum_\alpha \nabla^X_{f_\alpha} \otimes c(f_\alpha)$$

The tensor sum

- The tensor sum we are after is given by

$$(D_V \times_\nabla D_B)_0 := D_V \otimes \gamma_B + 1 \otimes_\nabla D_B : \\ \mathrm{Dom}(D_V \times_\nabla D_B)_0 \to X \widehat{\otimes}_{C(B)} L^2(\mathscr{E}_B)$$

- The closure of this symmetric operator is denoted $D_V \times_\nabla D_B$.
Theorem (Kaad–vS, 2016)
(1) We have the identity

$$D_V \times_\nabla D_B = D_M - \frac{i}{8}\widetilde{c}(\Omega).$$

(2) The spectral triple $(C^\infty(M), L^2(\mathscr{E}_M), D_M)$ is the unbounded KK-product of $(C^\infty(M), X, D_V)$ with the spectral triple $(C^\infty(B), L^2(\mathscr{E}_B), D_B)$ up to the curvature term $-\frac{i}{8}\widetilde{c}(\Omega)$.

Curvature

- Consider curvature in this context [Mesland–Rennie–vS, 2019]

$$R_\nabla = 1 \otimes_\nabla D_B^2 - (1 \otimes_\nabla D_B)^2 = \pi_{D_B}(\nabla^2)$$

- One can show that

$$R_\nabla = c_H \circ \left((\nabla^{\mathscr{E}_V})^2 + dk\right)$$

in terms of horizontal Clifford multiplication c_H, the curvature of $\nabla^{\mathscr{E}_V}$ and the mean curvature k

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This extends to almost-regular fibrations of Riemannian spin^c manifolds.

- We consider the class of examples coming from actions of a torus $\mathbb{T}^n$ on a spin^c manifold $\overline{M}$ with principal stratum M and with $B = M/\mathbb{T}^n$ a spin^c manifold.
- Up to unitary isomorphism, there is a tensor sum factorization [Kaad–vS, 2019]:

$$D_{\overline{M}} = D_V \widehat{\otimes} \gamma + 1 \widehat{\otimes}_\nabla D_{M/\mathbb{T}^n} + \widetilde{c}(\Omega)$$

- Prototype: the four-sphere $\mathbb{S}^4$ with a $\mathbb{T}^2$-action (the starting point for the Connes–Landi four-sphere)

The four-sphere

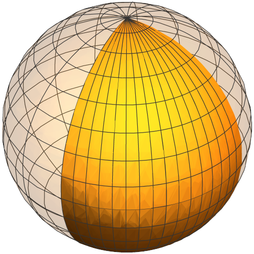
- Toroidal coordinates: $0 \leq \theta_1, \theta_2 < 2\pi$, $0 \leq \varphi \leq \pi/2$, $-\pi/2 \leq \psi \leq \pi/2$, and write

$$z_1 = e^{i\theta_1} \cos\varphi \cos\psi;$$
$$z_2 = e^{i\theta_2} \sin\varphi \cos\psi;$$
$$x = \sin\psi$$

- Action of $\mathbb{T}^2$ is by translating the θ_1, θ_2-coordinates

- The orbit space $\mathbb{S}^4/\mathbb{T}^2 \cong Q^2$ is a closed quadrant in the two-sphere, parametrized by

$$0 \leq \varphi \leq \pi/2, \qquad -\pi/2 \leq \psi \leq \pi/2$$



- Principal stratum $\mathbb{S}^4_0$ is a trivial $\mathbb{T}^2$-principal fiber bundle over the interior Q^2_0 of Q^2
- Moreover, $\pi : \mathbb{S}^4_0 \to Q^2_0$ is a (proper) Riemannian submersion for the metric on Q^2_0 induced by the round metric on $\mathbb{S}^2$

Spin geometry of $\mathbb{S}^4$

- The Dirac operator for the round metric is a $\mathbb{T}^2$-invariant selfadjoint operator

$$D_{\mathbb{S}^4} : \mathrm{Dom}(D_{\mathbb{S}^4}) \to L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$$

with local expression:

$$\begin{aligned} D_{\mathbb{S}^4} = {} & i\frac{1}{\cos\varphi\cos\psi}\gamma^1\frac{\partial}{\partial\theta_1} + i\frac{1}{\sin\varphi\cos\psi}\gamma^2\frac{\partial}{\partial\theta_2} \\ & + i\frac{1}{\cos\psi}\gamma^3\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi\right) + i\gamma^4\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi\right) \end{aligned}$$

- Since the 'subprincipal' stratum is of codimension two, it follows that $C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^4 \subset L^2(\mathbb{S}^4, \mathscr{E}_{\mathbb{S}^4})$ is a core for $D_{\mathbb{S}^4}$

Vertical operator

- We consider a C^*-correspondence X from $C_0(\mathbb{S}^4_0)$ to $C_0(Q^2_0)$, defined as the Hilbert C^*-completion of $C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^2$ wrt

$$\langle s, t \rangle_X = \int_{\mathbb{T}^2} \overline{s(\theta_1, \theta_2, \varphi, \psi)} \cdot t(\theta_1, \theta_2, \varphi, \psi) d\theta_1 d\theta_2 \cdot \sin\varphi \cos\varphi \cos^2\psi,$$

- There is a unitary isomorphism (induced by pointw. multipl.)

$$u : X \otimes_{C_0(Q^2_0)} L^2(Q^2_0) \otimes \mathbb{C}^2 \to L^2(\mathbb{S}^4_0) \otimes \mathbb{C}^4$$

- We define a symmetric operator $(D_V)_0 : C_c^\infty(\mathbb{S}^4_0) \otimes \mathbb{C}^2 \to X$:

$$(D_V)_0 = i \frac{1}{\cos\varphi \cos\psi} \sigma^1 \frac{\partial}{\partial\theta_1} + i \frac{1}{\sin\varphi \cos\psi} \sigma^2 \frac{\partial}{\partial\theta_2}.$$

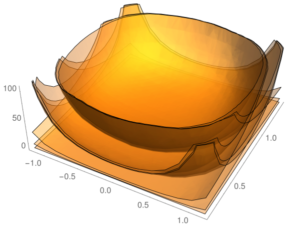
We denote its closure by $D_V : \mathrm{Dom}(D_V) \to X$.

- One can find families of eigenvectors $\Psi^{\pm}_{n_1 n_2}$ in $C^\infty(\mathbb{T}^2) \otimes \mathbb{C}^2$.

$$D_V \left(f \Psi^{\pm}_{n1, n_2} \right) = \pm \lambda_{n_1 n_2} \cdot f \Psi^{\pm}_{n_1 n_2}.$$

for any $f \in C_c^\infty(Q_0^2)$ where

$$\lambda_{n_1 n_2} = \sqrt{\frac{n_1^2}{\cos^2 \varphi \cos^2 \psi} + \frac{n_2^2}{\sin^2 \varphi \cos^2 \psi}}$$

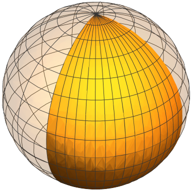


- Regularity, selfadjointness and compactness of the resolvent: D_V defines unbdd Kasparov module from $C_0(\mathbb{S}_0^4)$ to $C_0(Q_0^2)$

Horizontal operator

We define an operator $(D_{Q_0^2})_0 : C_c^\infty(Q_0^2) \otimes \mathbb{C}^2 \to L^2(Q_0^2) \otimes \mathbb{C}^2$ as the restriction of the Dirac operator on $\mathbb{S}^2$ to the open quadrant Q_0^2:

$$(D_{Q_0^2})_0 = i\frac{1}{\cos\psi}\sigma^1\frac{\partial}{\partial\varphi} + i\sigma^2\left(\frac{\partial}{\partial\psi} - \frac{1}{2}\tan\psi\right).$$



This is a symmetric operator. Its closure $D_{Q_0^2}$ defines a half-closed chain [Hil, 2010] from $C_0(Q_0^2)$ to $\mathbb{C}$

Tensor sum of D_V and $D_{Q_0^2}$

We lift the operators D_V and $D_{Q_0^2}$ to $X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2$.

- For D_V we take $D_V \widehat{\otimes} \gamma$ and have on $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ that

$$u(D_V \widehat{\otimes} \gamma)u^* = i\frac{1}{\cos\varphi\cos\psi}\gamma^1\frac{\partial}{\partial\theta_1} + i\frac{1}{\sin\varphi\cos\psi}\gamma^2\frac{\partial}{\partial\theta_2},$$

- For the operator $D_{Q_0^2}$, we first need a hermitian connection on X

$$\nabla_{\partial/\partial\varphi} = \frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi \qquad \nabla_{\partial/\partial\psi} = \frac{\partial}{\partial\psi} - \tan\psi$$

- The operator $1\widehat{\otimes}_\nabla D_{Q_0^2}$ then becomes

$$u(1\widehat{\otimes}_\nabla D_{Q_0^2})u^* = i\frac{1}{\cos\psi}\gamma^3\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot\varphi - \frac{1}{2}\tan\varphi\right) + i\gamma^4\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\tan\psi\right)$$

- The tensor sum we are after is given by

$$D_V \hat{\otimes} \gamma + 1 \hat{\otimes}_\nabla D_{Q_0^2}$$

- This is a symmetric operator and we denote its closure by

$$D_V \times_\nabla D_{Q_0^2} : \mathrm{Dom}(D_V \times_\nabla D_{Q_0^2}) \to X \otimes_{C_0(Q_0^2)} L^2(Q_0^2) \otimes \mathbb{C}^2.$$

- Since $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^2 \otimes_{C_c^\infty(Q_0^2)} C_c^\infty(Q_0^2) \otimes \mathbb{C}^2$ is a core for $D_V \times_\nabla D_{Q_0^2}$ which is mapped by u to the core $C_c^\infty(\mathbb{S}_0^4) \otimes \mathbb{C}^4$ of $D_{\mathbb{S}^4}$, it follows that

$$\boxed{u(D_V \times_\nabla D_{Q_0^2})u^* = D_{\mathbb{S}^4}}$$

  as an equality on $\mathrm{Dom}(D_{\mathbb{S}^4})$
- A connection and positivity property imply that this is an unbounded representative of the internal product

$$KK(C_0(\mathbb{S}_0^4), C_0(Q_0^2)) \otimes_{C_0(Q_0^2)} KK(C_0(Q_0^2), \mathbb{C}) \to KK(C_0(\mathbb{S}_0^4), \mathbb{C})$$

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Immersions in unbounded KK-theory: $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$

- Consider the embedding $\imath : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$.
- By [CS 1984] we know that

$$[\mathbb{S}^n] = \imath_! \otimes [\mathbb{R}^{n+1}]$$

  as classes in KK-theory.
- Let us consider the unbounded version, which starts by writing

$$D_{\mathbb{R}^{n+1}} = \gamma^1 \frac{1}{r} D_{\mathbb{S}^n} + \gamma^2 \left(i \frac{\partial}{\partial r} + \frac{n}{2r} \right)$$

- This is (a local expression for) an essentially self-adjoint operator with volume form $r^n dr \wedge d\sigma_{\mathbb{S}^n}$

The immersion KK-cycle

- We introduce the immersion module between $C(\mathbb{S}^n)$ and $C_0(\mathbb{R}^{n+1})$:

$$X = C_0(\mathbb{S}^n \times (1-\epsilon, 1+\epsilon))$$

equipped with $C_0(\mathbb{R}^{n+1})$-valued inner product given by

$$\langle \psi_1, \psi_2 \rangle_X (r, \theta) = \frac{1}{r^n} \overline{\psi_1(r,\theta)} \psi_2(r,\theta)$$

- Self-adjoint operator $S : \mathrm{Dom}(S) \to X$ defined as multiplication operator with some suitable $f \equiv f(r)$, such as

$$f(r) = \frac{\pi}{2\epsilon} \tan \frac{\pi(r-1)}{2\epsilon}$$

Proposition (vS–Verhoeven, 2019)
The triple $(C^\infty(\mathbb{S}^n), X, S)$ is an unbounded KK-cycle from $C(\mathbb{S}^n)$ to $C_0(\mathbb{R}^{n+1})$.

The internal Kasparov product

- We introduce a connection ∇ on X by

$$\nabla(\psi) = [D, \psi] - \frac{n}{2r} ic(dr)$$

- The tensor sum is given by

$$\begin{aligned} D_\times &= S \otimes 1 + 1 \otimes_\nabla D_{\mathbb{R}^{n+1}} \\ &= \gamma^1 \frac{1}{r} D_{\mathbb{S}^n} + \gamma^2 \left(i \frac{\partial}{\partial r} \right) + \gamma^3 f(r) \end{aligned}$$

- As an operator on $L^2((1-\epsilon, 1+\epsilon)) \otimes \mathbb{C}^2$ we have

$$T := i\gamma^2 \tfrac{\partial}{\partial r} + \gamma^3 f(r) = \begin{pmatrix} 0 & i\frac{\partial}{\partial r} + if(r) \\ i\frac{\partial}{\partial r} - if(r) & 0 \end{pmatrix}$$

for which we compute that index $T = 1$

We now arrive at the final result

Theorem

The triple $(C^\infty(\mathbb{S}^n), X \otimes_{C_0(\mathbb{R}^{n+1})} L^2(\mathscr{E}_{\mathbb{R}^{n+1}}), D_\times)$ *is an unbounded representative both of the internal Kasparov product of* $\imath_!$ *and* $[\mathbb{R}^{n+1}]$ *as well as of* $[\mathbb{S}^n]$ *and* $[T] = 1 \in KK(\mathbb{C}, \mathbb{C})$

The proof is a check of Kucerovsky's conditions

Interestingly, the curvature can also be computed. Since

$$\nabla = [D, \cdot] - \frac{n}{2r} ic(dr)$$

we find

$$R_\nabla = \frac{n^2}{4r^2}$$

which is (proportional to) the sectional curvature of the n-sphere...

Outlook

- New notion of curvature that arises in unbounded KK-theory, based on the difference of two natural symmetric operators that appear in the (unbounded version of the) internal Kasparov product
- This notion of curvature is in concordance with the applications to particle physics, Riemannian submersions, immersions of spheres into Euclidean spaces
- and much more to be explored!