Noncommutative spaces at finite resolution

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A spectral approach to geometry

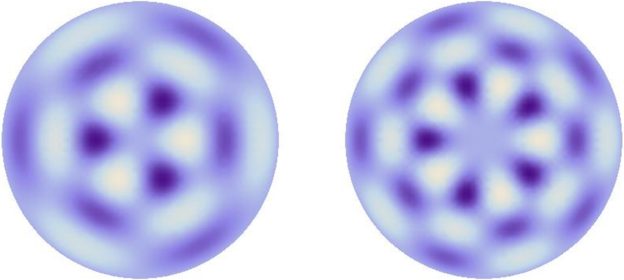
"Can one hear the shape of a drum?" (Kac, 1966)

Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

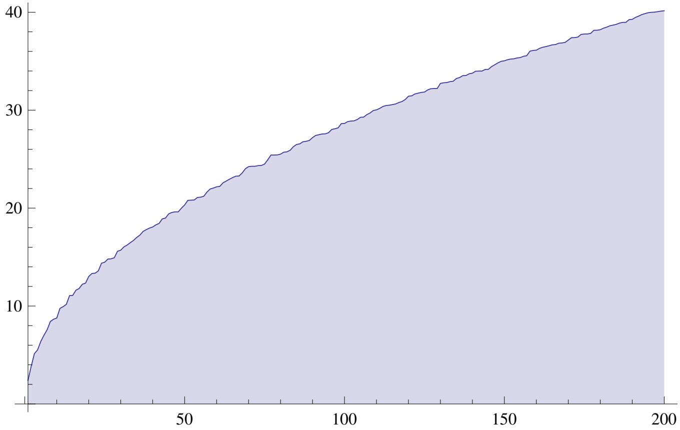
$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

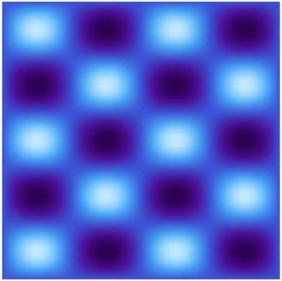
The disc

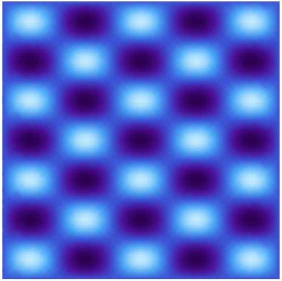


Wave numbers on the disc

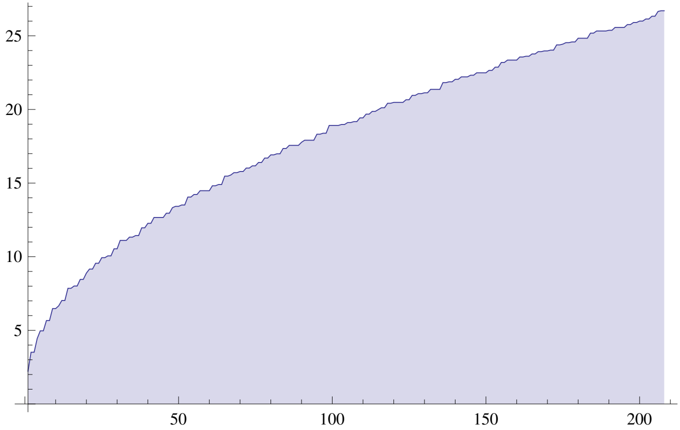


The square



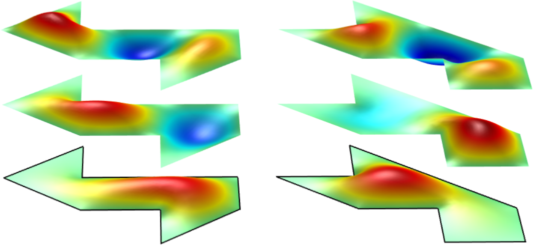


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

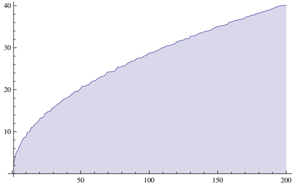
so the answer to Kac's question is no

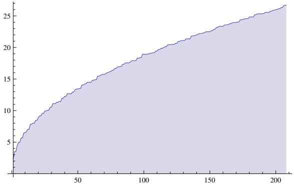
Weyl's estimate

Nevertheless, certain information can be extracted from the spectrum:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \text{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):

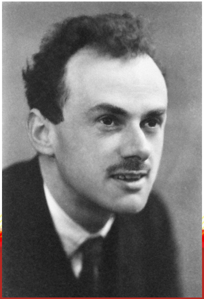




Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.
- Let us give some examples.



The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

with square $\Delta_{\mathbb{S}^1}$.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- At first sight it seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered the possibility that a and b be complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

  then $a^2 = b^2 = -\mathbb{I}$ and $ab + ba = 0$
- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

  which satisfies $(D_{\mathbb{T}^2})^2 = (-\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2})\mathbb{I}$.

The 4-dimensional torus

- The 4-torus $\mathbb{T}^4$ can be parametrized by t_1, t_2, t_3, t_4 with Laplacian:

$$\Delta_{\mathbb{T}^4} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial t_4^2}.$$

- The search for a differential operator that squares to $\Delta_{\mathbb{T}^4}$ again involves matrices, but we also need quaternions:

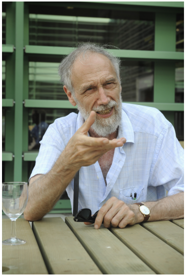
$$i^2 = j^2 = k^2 = ijk = -1.$$

- The Dirac operator on $\mathbb{T}^4$ is

$$D_{\mathbb{T}^4} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} + j\frac{\partial}{\partial t_3} + k\frac{\partial}{\partial t_4} & 0 \end{pmatrix}$$

- The relations $ij = -ji$, $ik = -ki$, $\ldots$ imply that $D_{\mathbb{T}^4}$ squares to $\Delta_{\mathbb{T}^4}$.

Noncommutative geometry



If combined with the C^*-algebra $C(M)$, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), L^2(S_M), D_M) \longleftrightarrow (M, g)$$

The “usual” story

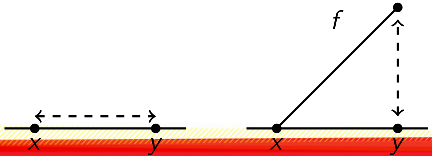
Given cpt Riemannian spin manifold (M, g) with spinor bundle S_M on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(S_M)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a C^*-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.
Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

Isomorphisms are complete order isomorphisms

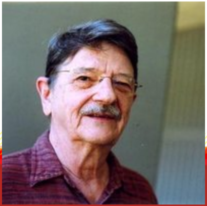
C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:

$$\begin{tikzcd} E \arrow[r, "\kappa"] \arrow[dr, "\lambda"'] & A \\ & B \arrow[u, dotted, two heads, "\exists! \rho"'] \end{tikzcd}$$



Example: operator system $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^*-envelope $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number prop(E) of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C^*_{env}(E)$ is a C^*-algebra.

Returning to harmonic functions in the disk we have $\mathrm{prop}(C_{\mathrm{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual E^d of a unital operator system is a unital operator system with

$$E_+^d = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

  and similarly for the matrix order.
- Also, we have $(E^d)_+^d \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

$$\text{pure states on } E \longleftrightarrow \text{extreme rays in } (E^d)_+$$

  and the other way around.

Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\operatorname{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\operatorname{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

Farenick continues to exploit the duality by showing:

- every positive linear map of the $n \times n$ complex matrices is completely positive when restricted to the Toeplitz operator system.
- every unital isometry of the $n \times n$ Toeplitz matrices into the algebra of $n \times n$ complex matrices is a unitary similarity transformation.

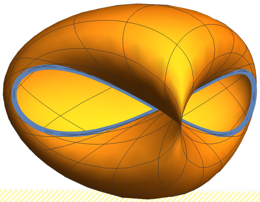
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

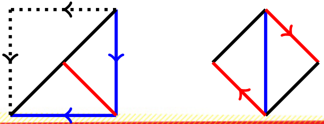
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is $\mathbb{T}^2/S_2$, given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!



Finite Fourier transform and duality

- Fourier transform on the cyclic group maps $l^\infty(\mathbb{Z}/m\mathbb{Z})$ to $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ and vice versa, exchanging pointwise and convolution product.
- This can be phrased in terms of a duality:

$$\begin{aligned} \mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \times l^\infty(\mathbb{Z}/m\mathbb{Z}) &\to \mathbb{C} \\ \langle c, g \rangle &\mapsto \sum_{k,l} c_l g(k) e^{2\pi i k l/m} \end{aligned}$$

  compatibly with positivity.
- Thus we may consider the above duality for Toeplitz matrices as some sort of generalization of Fourier theory to operator systems.
- However, note that for finite Fourier theory the symmetries are reduced from S^1 to $\mathbb{Z}/m\mathbb{Z}$.

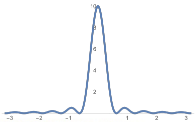
Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a C^1-approximate order inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

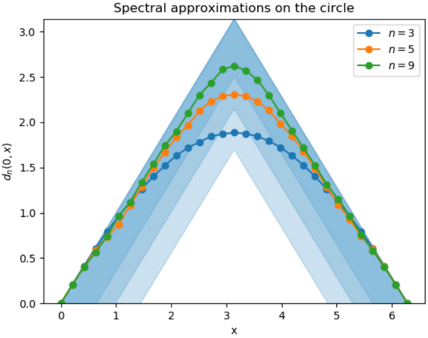
in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n:



- The fact that S_n is a C^1-approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi,\psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi,\psi)$$

where $\gamma_n \to 0$ as $n \to \infty$.
- Some (basic) Python simulations for point evaluation on S^1:



Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that bridges the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n.

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

. . . . . .

Operator systems associated to tolerance relations

- Suppose that X is a set and consider a relation $\mathcal{R} \subseteq X \times X$ on X that is reflexive, symmetric but not necessarily transitive.
- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R})$ as the closure of integral operator with support in $\mathcal{R}$. Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

Tolerance relations on finite sets [Gielen–vS, 2022]

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X and suppose that $\mathcal{R}$ generates the full equivalence class $X \times X$ (*i.e.* the graph corresponding to $\mathcal{R}$ is connected). Then

1. the C^*-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\mathrm{prop}(E(\mathcal{R})) = \mathrm{diam}(\mathcal{R})$.
2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

Example

The operator systems of $p \times p$ band matrices with band width N.

1. *The propagation number of $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$ is equal to $\lceil p/N \rceil$.*
2. *The dual operator system consists of band matrices (with order given by partially positive).*

Spaces at finite resolution

Consider now a path metric measure space X with a measure of full support, and the following tolerance relation:

$$\boxed{\mathcal{R}_\epsilon := \{(x,y) \in X \times X : d(x,y) < \epsilon\}}$$

It gives rise to the operator system $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$.

Proposition

If X is a complete and locally compact path metric measure space X with a measure of full support, then

1. $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ *and* $\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$
2. *The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.*

Outlook

- Spectral truncations: tori, compact Lie groups, etc.
- Bonds in groupoids: approximate order unit, duality, etc.
- Metric structure on state spaces for spaces at finite resolution
- Gromov–Hausdorff convergence, entropy
- General theory of spectral triples for operator systems
- ...

Thanks!