# Noncommutative spaces at finite resolution

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# The spectral approach to geometry

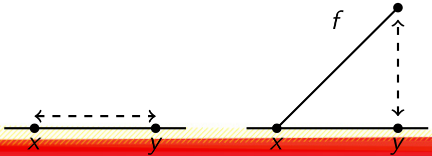
Given cpt Riemannian spin manifold $(M, g)$ with spinor bundle $S_M$ on $M$.

- the $C^*$-algebra $C(M)$
- the self-adjoint Dirac operator $D_M$
- both acting on Hilbert space $L^2(S_M)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x,y) = \sup_{f \in C(M)} \{ |f(x) - f(y)| : \|[D_M, f]\| \leq 1 \}$$



# Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a $C^*$-algebra $A$
- a self-adjoint operator $D$ with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of $A$:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

# Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

*We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.*

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

# Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of $D$:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, $PAP$ is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation $R_\epsilon$ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

# Operator systems

***Definition (Choi-Effros 1977)***
*An operator system is a $*$-closed vector space $E$ of bounded operators.*
*Unital: it contains the identity operator.*

- $E$ is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, $E$ is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any $n$.

Maps between operator systems $E, F$ are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all $n$.

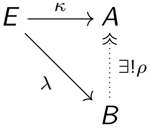
Isomorphisms are complete order isomorphisms

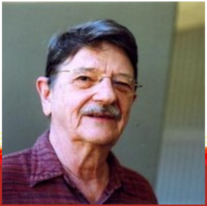
# $C^*$-envelope of a unital operator system

Arveson introduced the notion of $C^*$-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A $C^*$-extension $\kappa : E \to A$ of a unital operator system $E$ is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A $C^*$-envelope of a unital operator system is a $C^*$-extension $\kappa : E \to A$ with the following universal property:



# Shilov boundaries

There is a useful description of $C^*$-envelopes using Shilov ideals.

***Definition***

*Let $\kappa : E \to A$ be a $C^*$-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.*

*The Shilov boundary ideal is the largest of such boundary ideals.*

***Proposition***

*Let $\kappa : E \to A$ be a $C^*$-extension. Then there exists a Shilov boundary ideal $J$ and $C^*_{env}(E) \cong A/J$.*

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is $S^1$. Accordingly, its $C^*$-envelope is $C(S^1)$.

# Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of $E$.

***Definition***

*The propagation number prop$(E)$ of $E$ is defined as the smallest integer $n$ such that $\kappa(E)^{\circ n} \subseteq C^*_{env}(E)$ is a $C^*$-algebra.*

Returning to harmonic functions in the disk we have $\mathrm{prop}(C_{\mathrm{harm}}(\overline{\mathbb{D}})) = 1$.

***Proposition***

*The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:*

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

*More generally [Koot, 2021], we have*

$$prop(E \otimes_{\mathrm{min}} F) = \max\{prop(E), prop(F)\}$$

## State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on $E$ as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual $E^d$ of a unital operator system is a unital operator system with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

and similarly for the matrix order.
- Also, we have $(E^d)^d_+ \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

and the other way around.

# Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of $D_{S^1}$ are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\mathrm{prop}(C(S^1)^{(n)}) = 2$ (for any $n$).

# Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on $S^1$ with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element $a$ is positive iff $\sum_k a_k e^{ikx}$ is a positive function on $S^1$.
- The $C^*$-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.

***Proposition***

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on $S^1$.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$.*

# Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

***Proposition***

1. *The extreme rays in $C(S^1)_+^{(n)}$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.*
2. *The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.*

## Operator systems, groupoids and bonds

Recall:

- Consider a locally compact groupoid $G$ equipped with a (left invariant) Haar system $\nu = \{\nu_x\}$.
- The space $C_c(G)$ of compactly supported complex-valued continuous functions on $G$ becomes a $*$-algebra with the convolution product and involution given by

$$f * g(\gamma) = \int_{G_x} f(\gamma\gamma_1^{-1}) g(\gamma_1) d\nu_x(\gamma_1); \qquad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

  where $x = s(\gamma)$ for any $\gamma \in G$.
- $C_c(G)$ can be completed to the groupoid $C^*$-algebra $C^*(G)$

**Definition**

*A bond is a triple* $(G,\nu,\Omega)$ *consisting of a locally compact groupoid* $G$, *a Haar system* $\nu = \{\nu_x\}$ *and an open symmetric subset* $\Omega \subseteq G$ *containing the units* $G^{(0)}$.

**Proposition**

*Let* $(\Omega, G, \nu)$ *be a bond. The closure of the subspace* $C_c(\Omega) \subseteq C_c(G)$ *in the* $C^*$*-algebra* $C^*(G)$ *is an operator system.*

**Example**

1. *Consider* $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ *Fejér–Riesz operator system inside* $C^*(\mathbb{Z})$.
2. *Consider* $\Omega_n = (-n, n) \subset C_m$ *(so modulo* $m$*). The operator system consists of banded* $m \times m$ *circulant matrices of band width* $n$.

*Thus, the ambient groupoid is crucial since these two operator systems are not even Morita equivalent.*

3. *Given the set* $X = \{1, \dots, m\}$ *consider a "band"* $R_n \subset X \times X$ *around the diagonal of width* $n \rightsquigarrow$ *banded* $m \times m$ *matrices of band width* $n$.

# Operator systems associated to tolerance relations

- Suppose that $X$ is a set and consider a relation $\mathcal{R} \subseteq X \times X$ on $X$ that is reflexive, symmetric but not necessarily transitive.
- Key motivating example: a metric space $(X, d)$ with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If $(X, \mu)$ is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R})$ as the closure of integral operator with support in $\mathcal{R}$. Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

# Tolerance relations on finite sets [Gielen–vS, 2022]

Let $X$ be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on $X$ and suppose that $\mathcal{R}$ generates the full equivalence class $X \times X$ (*i.e.* the graph corresponding to $\mathcal{R}$ is connected). Then

1. the $C^*$-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\text{prop}(E(\mathcal{R})) = \text{diam}(\mathcal{R})$.
2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

***Example***

*The operator systems of $p \times p$ band matrices with band width $N$.*

1. *The propagation number of $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$ is equal to $\lceil p/N \rceil$.*
2. *The dual operator system consists of band matrices (with order given by partially positive).*

# Spaces at finite resolution

Consider now a path metric measure space $X$ with a measure of full support, and the following tolerance relation:

$$\boxed{\mathcal{R}_\epsilon := \{(x,y) \in X \times X : d(x,y) < \epsilon\}}$$

It gives rise to the operator system $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$.

***Proposition***

*If $X$ is a complete and locally compact path metric measure space $X$ with a measure of full support, then*

1. $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ *and* $\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$
2. *The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is $\epsilon$-connected.*

# Outlook

- Spectral truncations: tori, compact Lie groups, etc.
- Bonds in groupoids: approximate order unit, duality, etc.
- Metric structure on state spaces for spaces at finite resolution
- Gromov–Hausdorff convergence, entropy
- General theory of spectral triples for operator systems
- ...

Thanks!