

# Cyclic cocycles in the spectral action

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# Outline

Spectral triples and gauge theory

Taylor expansion of the spectral action

Cyclic cocycles

Loop corrections

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## NC geometry: spectral triple $(\mathcal{A}, \mathcal{H}, D)$ [C]

- |  |                              |
|--|------------------------------|
| • unital $*$ -algebra $\mathcal{A}$  | “coordinate algebra”         |
| • self-adjoint operator $D$ with $(i + D)^{-1}$ compact and $[D, a]$ bounded for $a \in \mathcal{A}$ | “inverse fermion propagator” |
| • both acting on Hilbert space $\mathcal{H}$   | “one-particle space”         |

$$D^{-1} = \bullet \longrightarrow \blacktriangleright \longrightarrow \bullet$$

- Distance function on state space of  $A = \overline{\mathcal{A}}$ :

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

- Eg. for Riemannian spin manifold  $(M, g)$  we have  $d(x, y) = d_g(x, y)$ .

## Spectral triples $(\mathcal{A}, \mathcal{H}, D)$ and gauge theories

Some historical motivation:

- in 1918 **Hermann Weyl** introduced the concept of *Maßstab-Invarianz* (later: Eich Invarianz or gauge invariance).
- it allowed for a geometrical description of electricity (and gravity) based on the transformation:

$$g \mapsto \lambda \cdot g, \quad V \mapsto V + d \log \lambda; \quad (\lambda \geq 0).$$

- **Einstein** objected as it would imply that the location of absorption lines of an element would depend on the history of the atom..
- About ten years later it was realized that the underlying principle for electrodynamics is **not scale, but phase-invariance**.
- With the **self-adjoint operator  $D$**  playing the role of the infinitesimal (inverse) **line element**, noncommutative geometry gives new meaning to Weyl's original idea, by transforming  $D \mapsto uDu^*$ .

## Spectral triples $(\mathcal{A}, \mathcal{H}, D)$ and gauge theories

- Gauge group  $\mathcal{U}(\mathcal{A})$  of unitaries in  $\mathcal{A}$  acting as

$$D \mapsto uDu^* = D + u[D, u^*]$$

- Spectral invariant action functional [Chamseddine–Connes, 1996]:

$$\mathrm{Tr} f(D) = \sum_k f(\lambda_k)$$

- More general: inner perturbations as gauge fields [Connes, 1996]

$$D \mapsto D + V; \quad V = \sum_j a_j [D, b_j] \in \Omega_D^1(\mathcal{A})_{\mathrm{s.a.}} \quad (a_j, b_j \in \mathcal{A})$$

so that à la Weyl:

$$V \mapsto uVu^* + u[D, u^*]$$

## Asymptotic expansion of the spectral action

If  $(\mathcal{A}, \mathcal{H}, D)$  is a **regular spectral triple** with **simple dimension spectrum**  $\text{Sd}$ , and  $f(\sqrt{\cdot})$  a Laplace transform, then the spectral action is given asymptotically (as  $\Lambda \rightarrow \infty$ ) by

$$\text{Tr} f(D/\Lambda) \sim \sum_{\beta \in \text{Sd}} f_{\beta} \Lambda^{\beta} \text{res}_{z=\beta} \text{Tr} |D|^{-z} + f(0) \zeta_D(0) + \mathcal{O}(\Lambda^{-1})$$

where  $f_{\beta} := \int f(v) v^{\beta-1} dv$ . [Connes–Marcolli 2007, vS 2015]

### **Example**

**Almost-commutative manifolds** [Chamseddine–Connes–Marcolli 2007, vS]  
 $(C(M, A_F), L^2(M, S) \otimes H_F, D_M \otimes 1 + \gamma_M \otimes D_F) :$

$$\begin{aligned} \text{Tr} f((D + V)/\Lambda) &\sim f_4 \Lambda^4 \text{Vol}(M) + f_2 \Lambda^2 \int R \sqrt{g} + f_0 \int R^2 \sqrt{g} + \dots \\ &+ \tilde{f}_0 \int \text{Tr} F_{\mu\nu} F^{\mu\nu} - \tilde{f}_2 \Lambda^2 \int |\phi|^2 + \tilde{f}_0 \int |\phi|^4 + \dots \end{aligned}$$

### **Theorem (Chamseddine–Connes, 2006)**

The *scale-invariant part* of the spectral action in dimension 4 (and with “vanishing tadpole”) is

$$\zeta_{D+V}(0) - \zeta_D(0) = -\frac{1}{2} \int_{\psi} (A\delta A + \frac{2}{3}A^3) + \frac{1}{4} \int_{\tau_0} (\delta A + A^2)^2$$

where  $A = \sum_j a_j \delta b_j$  is the universal one-form for  $V = \sum_j a_j [D, b_j]$ .

- Recognize **Chern–Simons** form  $cs_3(A)$  and **Yang–Mills** form  $F^2$
- $\psi$  is a **cyclic 3-cocycle** and  $\tau_0$  is a **Hochschild cocycle**.
- The pairing  $\langle \psi, u \rangle$  with  $K_1$ -group of  $\mathcal{A}$  is trivial.

*Goal: extend these expansions in powers of  $V$  and in  $A$  from the scale-invariant part to the full spectral action  $\text{Tr } f(D + V)$*



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## Perturbative expansion of the spectral action

In [vS 2012] derived an expansion in powers of  $V$  of the form:

$$\mathrm{Tr} f(D + V) - \mathrm{Tr} f(D) = \sum_n \frac{1}{n} \langle V, V, \dots, V \rangle$$

where (for suitable  $f$ ):

$$\langle V_1, V_2, \dots, V_n \rangle = \frac{1}{2\pi i} \mathrm{Tr} \oint f'(z) V_1(z - D)^{-1} \dots V_n(z - D)^{-1}$$

Such expansions exist for varying assumptions on  $f$  and  $D$   
[Skripka 2013, vNuland-Skripka 2021, vNuland-vS 2021]:

$$\langle V_1, V_2, \dots, V_n \rangle := \sum_{j=1}^n \mathrm{Tr}(T_{f^{[n]}}^D(V_j, \dots, V_n, V_1, \dots, V_{j-1})).$$

## Cyclic permutations and a Ward identity

There are the following two properties of the bracket:

$$\langle V_1, \dots, V_n \rangle = \langle V_2, \dots, V_n, V_1 \rangle \quad (\text{I})$$

$$\begin{aligned} \langle V_1, \dots, aV_j, \dots, V_n \rangle - \langle V_1, \dots, V_{j-1}a, \dots, V_n \rangle \\ = \langle V_1, \dots, [D, a], \dots, V_n \rangle \end{aligned} \quad (\text{II})$$

Identity (II) boils down to the 'Ward identity'

$$(z - D)^{-1}a - a(z - D)^{-1} = (z - D)^{-1}[D, a](z - D)^{-1}$$

The diagram illustrates the Ward identity in a graphical form. It shows the difference between two diagrams: a horizontal line with an arrow pointing right that ends at a vertical dashed line labeled 'a', and a vertical dashed line labeled 'a' that ends at a horizontal line with an arrow pointing right. This difference is equal to a horizontal line with an arrow pointing right that has a wavy vertical line labeled '[D, a]' attached to its center.

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## Brackets and cyclic cocycles

We define the following  $n$ -cochain  $\phi_n$  on  $\mathcal{A}$ :

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle$$

- $B_0 \phi_n$  is invariant under cyclic permutations, so  $B \phi_n = n B_0 \phi_n$  for odd  $n$  and  $B \phi_n = 0$  for even  $n$ .
- $b \phi_{2k-1} = \phi_{2k}$  so  $b \phi_{2k} = 0$ .

*Proof* ( $k = 1$ ):

$$\begin{aligned} \int_{b\phi_1} a_0 \delta a_1 \delta a_2 &= \langle a_0 a_1 [D, a_2] \rangle - \langle a_0 [D, a_1 a_2] \rangle + \langle a_2 a_0 [D, a_1] \rangle \\ &= -\langle a_0 [D, a_1] a_2 \rangle + \langle a_2 a_0 [D, a_1] \rangle = \langle a_0 [D, a_1], [D, a_2] \rangle \end{aligned}$$

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- $B_0 \phi_n$  is invariant under cyclic permutations, so  $B \phi_n = n B_0 \phi_n$  for odd  $n$  and  $B \phi_n = 0$  for even  $n$ .
- $b \phi_{2k-1} = \phi_{2k}$  so  $b \phi_{2k} = 0$ .
- $b B_0 \phi_{2k} = 2 \phi_{2k} - B_0 \phi_{2k+1}$

*Proof* ( $k = 1$ ):

$$\begin{aligned} \int_{b B_0 \phi_2} a_0 \delta a_1 \delta a_2 &= \int_{B_0 \phi_2} a_0 a_1 \delta a_2 - \int_{B_0 \phi_2} a_0 \delta(a_1 a_2) + \int_{B_0 \phi_2} a_2 a_0 \delta a_1 \\ &= \langle [D, a_0 a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1 a_2] \rangle + \langle [D, a_2 a_0], [D, a_1] \rangle \\ &= \dots \\ &= 2 \langle a_0 [D, a_1], [D, a_2] \rangle - \langle [D, a_0], [D, a_1], [D, a_2] \rangle \end{aligned}$$

## Brackets and cyclic cocycles

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- $B_0 \phi_n$  is invariant under cyclic permutations, so  $B\phi_n = nB_0 \phi_n$  for odd  $n$  and  $B\phi_n = 0$  for even  $n$ .
- $b\phi_{2k-1} = \phi_{2k}$  so  $b\phi_{2k} = 0$ .
- $bB_0 \phi_{2k} = 2\phi_{2k} - B_0 \phi_{2k+1}$

Motivated by this we define

$$\psi_{2k-1} := \phi_{2k-1} - \frac{1}{2} B_0 \phi_{2k} \quad \Longrightarrow \quad B\psi_{2k+1} = 2(2k+1)b\psi_{2k-1}.$$

So, with

$$\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle$$

we have with  $\psi_{2k-1} := \phi_{2k-1} - \frac{1}{2} B_0 \phi_{2k}$ :

**Proposition (van Nuland–vS 2021)**

1. The sequence  $(\phi_{2k})$  is an even  $(b, B)$ -cocycle and each  $\phi_{2k}$  defines an even Hochschild cocycle:  $b\phi_{2k} = 0$ .
2. The sequence  $(\tilde{\psi}_{2k-1} := (-1)^{k-1} \frac{(k-1)!}{(2k-1)!} \psi_{2k-1})$  is an odd  $(b, B)$ -cocycle.



## Spectral action and universal forms

Reconsider the perturbative expansion  $\sum_n \frac{1}{n} \langle V, \dots V \rangle$  and recall that  $\int_{\phi_n} a_0 \delta a_1 \cdots \delta a_n = \langle a_0 [D, a_1], [D, a_2], \dots, [D, a_n] \rangle$

- For  $n = 1$  we find with  $V = a_j [D, b_j]$  and  $A = a_j \delta(b_j)$ :

$$\langle V \rangle = \langle a_j [D, b_j] \rangle = \int_{\phi_1} A,$$

- For  $n = 2$ , we apply the Ward identity and derive

$$\begin{aligned} \langle V, V \rangle &= \langle a_j [D, b_j], a_{j'} [D, b_{j'}] \rangle \\ &= \langle a_j [D, b_j] a_{j'}, [D, b_{j'}] \rangle + \langle a_j [D, b_j], [D, a_{j'}], [D, b_{j'}] \rangle \\ &= \int_{\phi_2} A^2 + \int_{\phi_3} A \delta A. \end{aligned}$$

## Brackets as noncommutative integrals

In conclusion, if  $V = \sum_j a_j [D, b_j]$  is a gauge field with corresponding universal 1-form  $A = \sum_j a_j \delta b_j$  we may write:

$$\langle V \rangle = \int_{\phi_1} A,$$

$$\langle V, V \rangle = \int_{\phi_2} A^2 + \int_{\phi_3} A \delta A,$$

$$\langle V, V, V \rangle = \int_{\phi_3} A^3 + \int_{\phi_4} A \delta A A + \int_{\phi_5} A \delta A \delta A,$$

$$\langle V, V, V, V \rangle = \int_{\phi_4} A^4 + \dots$$

## Re-ordering the terms

Let us see how this can be expressed using the cochain  $\psi_{2k-1}$ , i.e.

$$\int_{\psi_{2k-1}} \omega = \int_{\phi_{2k-1}} \omega - \frac{1}{2} \int_{\phi_{2k}} \delta\omega; \quad \omega \in \Omega^{2k-1}(\mathcal{A})$$

For the first two terms, we have that

$$\int_{\phi_1} A + \frac{1}{2} \int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2} \int_{\phi_2} (\delta A + A^2)$$

while for the next we may apply the Ward identity, in combination with a noncommutative Stokes theorem to obtain

$$\begin{aligned} & \frac{1}{2} \int_{\phi_3} A\delta A + \frac{1}{3} \int_{\phi_3} A^3 + \frac{1}{3} \int_{\phi_4} A\delta A A + \frac{1}{4} \int_{\phi_4} A^4 \\ &= \frac{1}{2} \int_{\psi_3} \left( A\delta A + \frac{2}{3} A^3 \right) + \frac{1}{4} \int_{\phi_4} (\delta A + A^2)^2 \end{aligned}$$

## The perturbative expansion of the spectral action

Consider the following function class (with  $u(x) = x - i$ ):

$$\mathcal{E}^{s,\gamma} := \left\{ f \in C^\infty : \exists C_f \geq 1 \text{ s.t. } \|(\widehat{fu^m})^{(n)}\|_1 \leq C_f^{n+1} n!^\gamma \forall m \leq s, n \geq 0 \right\}$$

### **Theorem (van Nuland–vS, 2021)**

For a  $s$ -summable spectral triple  $(\mathcal{A}, \mathcal{H}, D)$  and  $f \in \mathcal{E}^{s,\gamma}$  with  $\gamma \in (0, 1)$ , there is the following absolutely convergent series expansion:

$$\text{Tr}(f(D + V) - f(D)) = \sum_{k=1}^{\infty} \left( \int_{\psi_{2k-1}} \text{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right)$$

Here  $V = \sum_j a_j [D, b_j]$ ,  $A = \sum_j a_j \delta b_j$ ,  $F = \delta A + A^2$ , and the higher-dimensional **Chern–Simons forms** are given by

$$\text{cs}_{2k-1}(A) := \int_0^1 A(F_t)^{k-1} dt; \quad F_t = t\delta A + t^2 A^2$$

## Gauge invariance

- For the **Yang–Mills terms**, it turns out that **gauge invariance** is a consequence of the fact that  $\phi_{2k}$  are **Hochschild cocycles**, showing that:

$$\int_{\phi_{2k}} u F^k u^* = \int_{\phi_{2k}} F^k.$$

- Since the spectral action is a **spectral invariant**, it is in particular invariant under gauge transformations.
- This combines: also the **Chern–Simons terms** are **gauge invariant**:

$$\sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \text{CS}_{2k-1}(uAu^* + u\delta u^*) = \sum_{k=1}^{\infty} \int_{\psi_{2k-1}} \text{CS}_{2k-1}(A)$$

## Pairing with $K$ -theory

### Theorem (van Nuland–vS, 2021)

Let  $f \in \mathcal{E}^{s,\gamma}$  for  $\gamma < 1$ . Then:

1. The sequence  $(\tilde{\psi}_{2k+1})$  defines an *odd entire cocycle* with respect to the Banach norm  $\|a\|_1 = \|a\| + \|[D, a]\|$  on  $\mathcal{A}$ .
2. the pairing of  $(\tilde{\psi}_{2k+1})$  with  $K_1(\mathcal{A})$  is trivial, i.e.  $(u, \tilde{\psi}) = 0$

### Proof.

1. Multiple operator integrals:  $\exists C \geq 1$  such that for all  $n$  we have

$$\left\| T_{f^{[n]}}^{D, D, \dots, D}(V_1, \dots, V_n) \right\|_1 \leq \left( \frac{C^{n+1}}{n!^{1-\gamma}} \right) \|V_1\| \cdots \|V_n\| \|(D - i)^{-1}\|_s^s.$$

Apply this to  $V_1 = a_0[D, a_1]$ ,  $V_2 = [D, a_2], \dots, V_n = [D, a_n]$  to find that  $\exists C_\Sigma$  s.t.  $|\tilde{\psi}_{2k+1}(a_0, \dots, a_{2k+1})| \leq C_\Sigma/k!$  for any bounded subset  $\Sigma \subset \mathcal{A}$ .

2. Take  $A = 0$  and realize that  $\sum_k \int_{\tilde{\psi}_{2k-1}} c_{S_{2k-1}}(u \delta u^*) = (u, \tilde{\psi}) = 0 \quad \square$

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## Loop corrections

*Goal: Compute the quantum partition function and analyze loop corrections to the spectral action*

Concretely, we try to make sense of the partition function in the **background field**:

$$Z[J, V] = \int e^{-\text{Tr} f(D+V+Q)+(J,V)} d[Q]$$

where

- $V = \sum_j a_j [D, b_j] \in \Omega_D^1(\mathcal{A})_{\text{s.a.}}$  is a **background gauge field**
- $J$  is a **source field** in the dual space  $\Omega_D^1(\mathcal{A})_{\text{s.a.}}^*$ .
- $Q$  is a **quantum field** that is integrated over in the path integral



## Loop corrections

In order to make sense of the path integral we let  $Q$  be an **arbitrary finite size hermitian matrix**  $Q = (Q_{kl})$  in an eigenbasis of  $D$  (eigenvalues:  $\lambda_k$ ).

The **quadratic term** becomes:

$$\frac{1}{2} \langle Q, Q \rangle = \frac{1}{2} \sum_{k,l} Q_{kl} Q_{lk} f'[\lambda_k, \lambda_l]$$

and, more generally

$$\frac{1}{n} \langle Q, \dots, Q \rangle = \frac{1}{n} \sum_{i_1, \dots, i_n} Q_{i_1 i_2} Q_{i_2 i_3} \dots Q_{i_n i_1} f'[\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}]$$

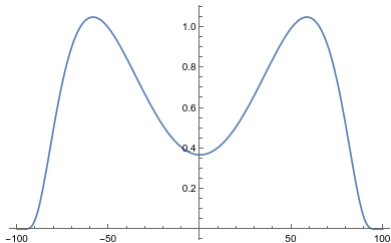
expressed in terms of divided differences:

$$f'[x, y] = \frac{f'(x) - f'(y)}{x - y}, \quad f'[x, y, z] = \frac{f'[x, y] - f'[y, z]}{x - z}, \quad \dots$$

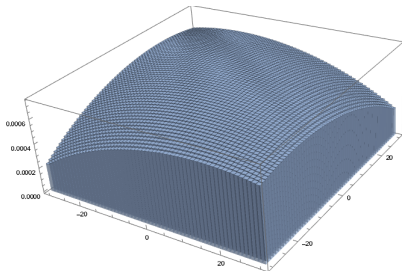
## Gauge propagator

For suitable  $f'$  we may perform the Gaussian integration:

$$\overbrace{Q_{kl} Q_{mn}} = \frac{\int Q_{kl} Q_{mn} e^{-\frac{1}{2}\langle Q, Q \rangle} dQ}{\int e^{-\frac{1}{2}\langle Q, Q \rangle} dQ} = \delta_{kn} \delta_{lm} \frac{1}{f'[\lambda_k, \lambda_l]}$$

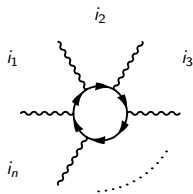


Example of a positive function  $f$

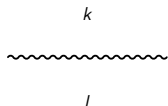


Divided difference  $f'[m, n]$

## Feynman diagrams



$$f'[\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}] = \frac{1}{2\pi i} \oint \frac{f'(z)}{(z-\lambda_{i_1}) \cdots (z-\lambda_{i_n})} dz$$



$$\frac{1}{f'[\lambda_k, \lambda_l]}$$

# Quantum Ward identity

We thus have a gauge propagator, and may consider all **one-particle irreducible diagrams**:

$$\langle\langle V_1, \dots, V_n \rangle\rangle = \text{Diagram with a central shaded circle and external legs } V_1, V_2, V_3, V_4, \dots, V_n$$

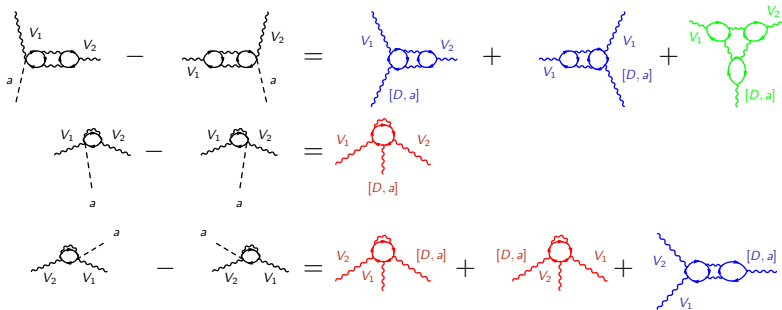
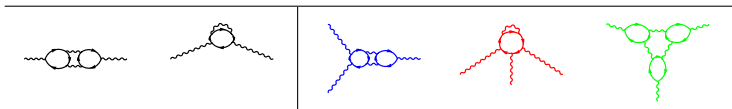
- There is a bosonic **Ward identity**:
 
$$\text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3} \quad [D, a]$$
 which is a consequence of the fact that

$$\frac{1}{f'[\lambda_k, \lambda_m] - f'[\lambda_l, \lambda_m]} = \frac{(\lambda_k - \lambda_l) f'[\lambda_k, \lambda_l, \lambda_m]}{f'[\lambda_k, \lambda_m] f'[\lambda_l, \lambda_m]}$$

In any case, it induces the following **quantum Ward identity** for the divergent one-loop contributions:

$$\begin{aligned} \langle\langle V_1, \dots, aV_j, \dots, V_n \rangle\rangle_{\infty}^{1L} - \langle\langle V_1, \dots, V_{j-1}a, \dots, V_n \rangle\rangle_{\infty}^{1L} \\ = \langle\langle V_1, \dots, V_{j-1}, [D, a], V_j, \dots, V_n \rangle\rangle_{\infty}^{1L}. \quad (II') \end{aligned}$$

# Proof of $\langle\langle aV_1, V_2 \rangle\rangle_{\infty}^{1L} - \langle\langle V_1, V_2a \rangle\rangle_{\infty}^{1L} = \langle\langle [D, a], V_1, V_2 \rangle\rangle_{\infty}^{1L}$



## One-loop renormalizable spectral action

In general, as a consequence of the quantum Ward identity one can show that the divergent part of the 1L contribution  $\langle\langle V, \dots, V \rangle\rangle_{\infty}^{1L}$  has the same structure as  $\langle V, \dots, V \rangle$ :

$$\sum_n \frac{1}{n} \langle\langle V, \dots, V \rangle\rangle_{\infty}^{1L} = \sum_{k=1}^{\infty} \left( \int_{\psi_{2k-1}^{1L}} \text{CS}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}^{1L}} F^k \right).$$

where the cyclic cocycles  $\phi^{1L}$  and  $\psi^{1L}$  are the analogues of  $\phi$  and  $\psi$  but now defined using the double bracket.

We conclude that the passage to the one-loop renormalized spectral action can be realized by a **transformation in the space of cyclic cocycles**, sending  $\phi \mapsto \phi - \phi^{1L}$  and  $\tilde{\psi} \mapsto \tilde{\psi} - \tilde{\psi}^{1L}$ .

*Thanks!*