

Spectral truncations in noncommutative geometry

Walter van Suijlekom

Spectral geometry: origins



H.A. Lorentz door Jan Veth

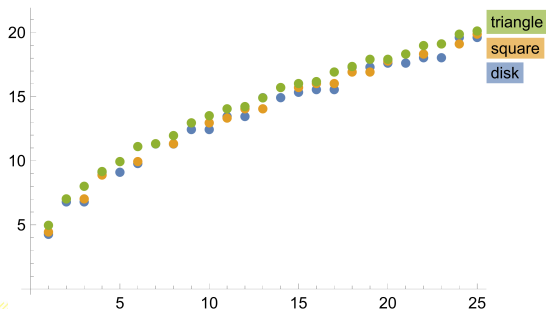
“Hierbei entsteht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n + dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist.”

“Here arises the mathematical problem of proving that the number of sufficiently high harmonics between n and $n + dn$ is independent of the shape of the envelope and proportional only to its volume.”

Weyl's Law

$$N(\Lambda) = \#\text{wave numbers} \leq \Lambda$$
$$\sim \frac{\Omega_d \text{Vol}(M)}{d(2\pi)^d} \Lambda^d$$

Evidence by the parabolic shapes ($\sqrt{\Lambda}$):



A spectral approach to geometry



"Can one hear the shape of a drum?" (Kac, 1966)

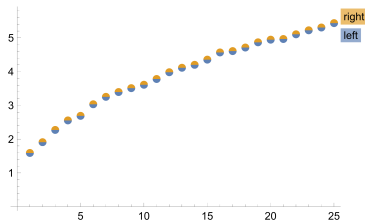
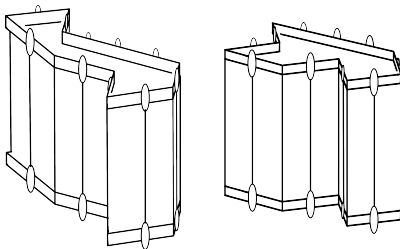
Or, more precisely, given a Riemannian manifold M , does the **spectrum of wave numbers** k in the **Helmholtz equation**

$$\Delta_M u = k^2 u$$

determine the **geometry of M** ?

Similarly, for a Riemannian spin manifold and Dirac operator D_M

Isospectral drums



so answer to Kac's question is **no**

Noncommutative geometry



If combined with the C^ -algebra $C(M)$, then the answer to Kac's question is affirmative.*

Connes' reconstruction theorem [2008]:

$$(C(M), L^2(S_M), D_M) \longleftrightarrow (M, g)$$

NC geometry: spectral triples (A, \mathcal{H}, D) [C]

- | | |
|--|------------------------------|
| • unital C^* -algebra A | “coordinate algebra” |
| • self-adjoint operator D with $(i+D)^{-1}$ compact and $[D, a]$ bounded for $a \in \mathcal{A} \subset A$ | “inverse fermion propagator” |
| • both acting on Hilbert space \mathcal{H} | “one-particle space” |

$$D^{-1} = \bullet \longrightarrow \blacktriangleright \longrightarrow \bullet$$

- Distance function on state space of $A = \overline{\mathcal{A}}$:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

- Eg. for Riemannian spin manifold (M, g) we consider $(C(M), L^2(S_M), D_M)$ for which we have $d(x, y) = d_g(x, y)$.

Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of **all eigenvalues** of the Dirac operator.
- From a physical standpoint this is **not very realistic**: detectors have limited energy ranges and resolution.

We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and **based on** [Connes–vS] (CMP, Szeged)

Towards operator systems..

- (I) Given (A, \mathcal{H}, D) we project onto part of the spectrum of D :
- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
 - $D \mapsto PDP$, still a self-adjoint operator
 - $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

- (II) Another approach would be to consider metric spaces up to a finite resolution :
- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Choi-Effros 1977)

An **operator system** is a $*$ -closed vector space E of bounded operators.

Unital: it contains the identity operator.

- E is **ordered**: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \quad (\psi \in \mathcal{H}).$$

- in fact, E is **matrix ordered**: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on \mathcal{H}^n for any n .

Maps between operator systems E, F are **completely positive maps** in the sense that their extensions $M_n(E) \rightarrow M_n(F)$ are positive for all n .

Isomorphisms are **complete order isomorphisms**

C^* -envelope of a unital operator system

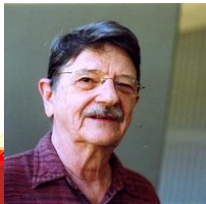
Arveson introduced the notion of C^* -envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.

Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^* -extension $\kappa : E \rightarrow A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^* -envelope of a unital operator system is a C^* -extension $\kappa : E \rightarrow A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \exists! \rho \\ & & B \end{array}$$



Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with C^* -envelope $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E .

Definition

The *propagation number* $\text{prop}(E)$ of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C_{\text{env}}^*(E)$ is a C^* -algebra.

Returning to harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition

The propagation number is *invariant under complete order isomorphisms*, as well as *under stable=Morita equivalence* [EKT, 2019]:

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$\text{prop}(E \otimes_{\min} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as **positive linear functionals of norm 1**.
- In the **finite-dimensional case**, the **dual** E^d of a unital operator system is a unital operator system with

$$E_+^d = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

and similarly for the matrix order.

- Also, we have $(E^d)_+^d \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:
pure states on E \longleftrightarrow extreme rays in $(E^d)_+$
and the other way around.

Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of D_{S^1} are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an **operator system**
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a **Toeplitz matrix**

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1 .
- The C^* -envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.

Proposition

1. The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1 .
2. The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).

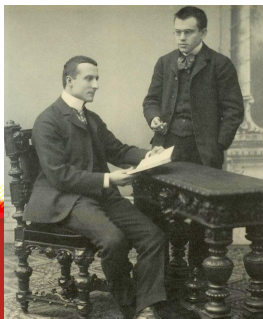
Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \rightarrow \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. The **extreme rays** in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.
2. The **pure state space** $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.



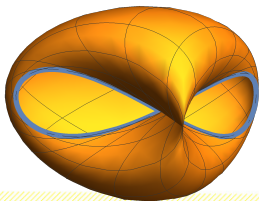
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

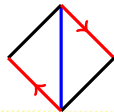
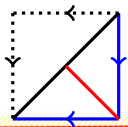
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is \mathbb{T}^2/S_2 , given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!



Finite Fourier transform and duality

- Fourier transform on the **cyclic group** maps $l^\infty(\mathbb{Z}/m\mathbb{Z})$ to $\mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ and vice versa, exchanging **pointwise and convolution** product.
- This can be phrased in terms of a **duality**:

$$\begin{aligned}\mathbb{C}[\mathbb{Z}/m\mathbb{Z}] \times l^\infty(\mathbb{Z}/m\mathbb{Z}) &\rightarrow \mathbb{C} \\ \langle c, g \rangle &\mapsto \sum_{k,l} c_l g(k) e^{2\pi i k l / m}\end{aligned}$$

compatibly with **positivity**.

- Thus we may consider the above duality for Toeplitz matrices as some sort of generalization of Fourier theory to operator systems.
- However, note that for finite Fourier theory the **symmetries are reduced** from S^1 to $\mathbb{Z}/m\mathbb{Z}$.

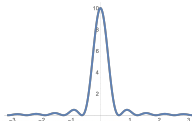
Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \rightarrow C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a **C^1 -approximate order inverse** $S_n : C(S^1)^{(n)} \rightarrow C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \quad S_n(R_n(f)) = F_n * f$$

in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n :

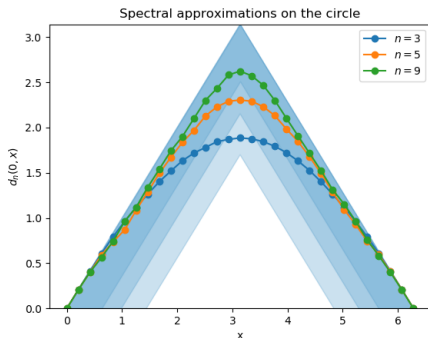


- The fact that S_n is a C^1 -approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$.

- Some (basic) Python simulations for point evaluation on S^1 :



Gromov–Hausdorff convergence

Recall **Gromov–Hausdorff distance** between two metric spaces:

$$d_{\text{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \rightarrow Z, g : Y \rightarrow Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that **bridges** the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n .

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ converges to $(\mathcal{S}(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

.....

Operator systems associated to tolerance relations [Connes–vS, 2021]

- Suppose that X is a set and consider a **relation** $\mathcal{R} \subseteq X \times X$ on X that is **reflexive, symmetric but not necessarily transitive**.
- **Key motivating example**: a metric space (X, d) with the relation

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

- If (X, μ) is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the **operator system** $E(\mathcal{R})$ as the closure of integral operator with support in \mathcal{R} . Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

Tolerance relations on finite sets [Gielen–vS, 2022]

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X and suppose that \mathcal{R} generates the full equivalence class $X \times X$ (i.e. the graph corresponding to \mathcal{R} is connected). Then

1. the \mathcal{C}^* -envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\text{prop}(E(\mathcal{R})) = \text{diam}(\mathcal{R})$.
2. If \mathcal{R} is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being **partially positive**.
3. the **pure states** of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is \mathcal{R} -connected.

Example

The operator systems of $p \times p$ **band matrices** with band width N .

1. The **propagation number** of $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$ is equal to $\lceil p/N \rceil$.
2. The dual operator system consists of band matrices (with order given by **partially positive**).

Spaces at finite resolution

Consider now a path metric measure space X with a measure of full support, and the following tolerance relation:

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

It gives rise to the operator system $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$.

Proposition

If X is a complete and locally compact path metric measure space X with a measure of full support, then

1. $C_{env}^*(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and $\text{prop}(E(\mathcal{R}_\epsilon)) = \lceil \text{diam}(X)/\epsilon \rceil$
2. The **pure states** of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is **ϵ -connected**.

Outlook

- Spectral truncations: tori, compact Lie groups, etc.
- Bonds in groupoids: approximate order unit, duality, etc.
- Metric structure on state spaces for spaces at finite spatial resolution
- Gromov–Hausdorff convergence, entropy
- General theory of spectral triples for operator systems
- ...

Thanks!