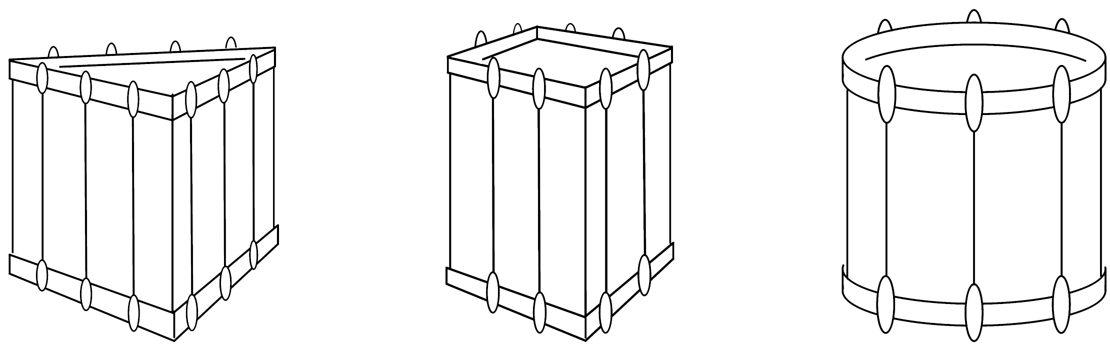
Drummed up for spectra in geometry

[Opgetrommeld voor spectra in de meetkunde]

Walter D. van Suijlekom

waltervs@math.ru.nl

October 12, 2022



Dear Rector, dear colleagues, dear family, friends and others present

You have just been musically welcomed —literally drummed up— by drumming group RADAC, as it was common in the 17th century to call people together in villages and towns. You have been able to hear, and perhaps even feel, the different sounds that drums of different sizes can produce. This will be the common thread of my story today: the *spectrum of vibrational frequencies* of some object —like a drum— clearly says something about its geometric shape. But does it also say everything? The math behind this is highly interesting, and that is what I would like to show you this afternoon, and let you hear. The second message I want to give you this afternoon is that mathematics is above all human work, and simply would not exist without the fruitful interaction between mathematicians.

Spectra and geometric shapes

Let us start with the word *spectrum*. In the mathematical literature, the term spectrum appears early in David Hilbert's work in 1906 to denote a collection of frequencies.[10] Over twenty years later, John von Neumann adopted the concept within the mathematical edifice of quantum mechanics.[21] Thanks to their work, we can now talk in the same way about, say, the auditory spectrum, consisting of vibrational frequencies of some object, or the visual spectrum, consisting of radiation frequencies of light.

However, it was my countryman Hendrik Lorentz who, as a physicist, posed the crucial mathematical question for the emergence of what we now call *spectral geometry*. I personally think this is a wonderful illustration of how research at the interface of mathematics and physics works, so I would like to take you back to the early 20th century.

By then, Lorentz was already a celebrated and highly respected physicist. He had received the Nobel Prize in 1902 and had been a professor at Leiden University for 25 years. Meanwhile, he was frequently invited for lectures and talks abroad: he went by steamer to the United States to give lectures there, and was thus away from home for months; he gave a lecture at the 1908 International Mathematics Congress in Rome, meanwhile enjoying Italian life —something I personally can understand very well. In October 1910, he was a visiting professor in Göttingen where he gave the so-called Wolfskehl lectures. These were no less than 6 lectures in 5 days, on varying topics.[14]

In his third lecture in Göttingen, Lorentz stated that the electromagnetic radiation spectrum of a 3-dimensional object would behave for high frequencies independently of the precise shape of that object, and is a multiple of its volume:

> *Hierbei entseht das mathematische Problem, zu beweisen, dass die Anzahl der genügend hohen Obertöne zwischen n und $n + dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist.*
>
> [*Here arises the mathematical problem to prove that the number of sufficiently high harmonics between n and $n + dn$ is independent of the shape of the envelope and only proportional to its volume.*]

Just a few months later, in late February 2011, the young mathematician Hermann Weyl answered the question in the affirmative, and of course with mathematical precision.[22] He was also working in Göttingen, and in response to Lorentz's question gave a clear proof, applicable even in higher dimensions. In fact, the foundation was then laid for spectral geometry: the discipline in mathematics that deals with the question of whether shapes are completely determined by their vibrational spectrum.

In the physics literature of the time, there is no trace of this answer, or, for example, any reference to Weyl's work. Lorentz does write about it a year later, this time no longer as a question but in affirmative form, ánd this time in French:

> *on démontre que le nombre des modes de vibrations possibles est proportionnel à V et indépendant de la forme du corps*[15]
>
> [*we show that the number of possible modes of vibration is proportional to V and independent on the shape of the body*]

This quote is from the fifth and last of his Michonis lectures given at the prestigious Collège de France in Paris in late November 1912. I must admit that in preparation for this lecture earlier this year I was red-eared reading the letter correspondence between

Figure 1: Announcement of Lorentz's Michonis lectures at the Collège de France in November 2012 [source: archives du Collège de France].

Lorentz and the institute —again, in beautiful French. Notice immediately, by the way, what a linguistic prodigy Lorentz is; something that had previously made him the ideal president of the famous Solvay Congress, with such participants as Albert Einstein, Marie Curie, Max Planck, Henri Poincar *et cetera*.

During that congress, in addition to his excellent scientific overview, Lorentz was able to put his language skills to use. After all, not all French participants spoke German, nor did the German participants speak English. [1]

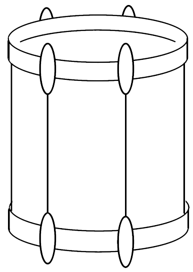
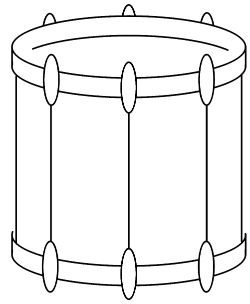
Because of the political unrest at the time, Lorentz does not return to Germany after this, and Göttingen also ceases to exist as the epicenter of mathematics. We must therefore wait some 50 years for further development of spectral geometry. In 1966, mathematician Mark Kac published a paper called "Can one hear the shape of a drum".[11] You can understand that this title strikes a chord with me not only as a mathematician, but also as a percussionist. We imagine the drum as a surface and consider all possible vibrational waves of this surface equal to zero at the edge. This is our vibrating drum: the sheet vibrates but the edge of the drum is fixed. The question now is: if we know all the vibrational frequencies of the surface, can we figure out the shape of the surface?

In fact, Kac's work provided a refinement of Lorentz's question, and Weyl's answer. For with it, from the vibrational spectrum at least the volume, or, in the case of the drum, the surface area could be ascertained. Note, by the way, that this is consistent with your intuition: large musical instruments (such as a double bass) generally make a lower sound

[1] Much of this background is drawn from the two extensive recently published bibliographies of Lorentz.[2,12]

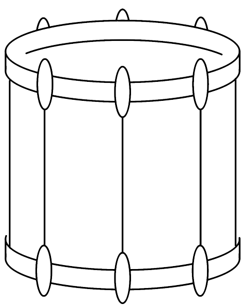
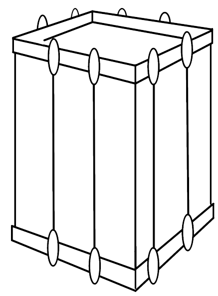
than small instruments (such as a violin).

Let me give an example: for the following two drums

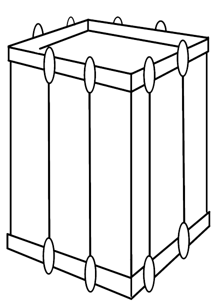
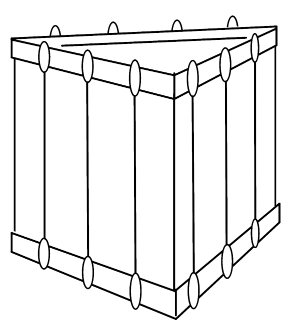
the first 25 vibrational frequencies are plotted in Figure 2a. Indeed, the small drum has higher frequencies than the large one, so it can be deduced from the sound which drum is being hit.

But Kac's question goes a step further and also looks at different shapes, such as a round and a square drum, in this case of equal surface area:

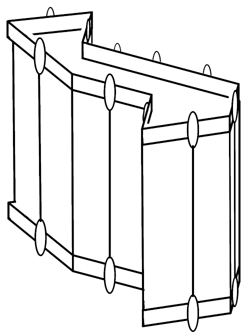
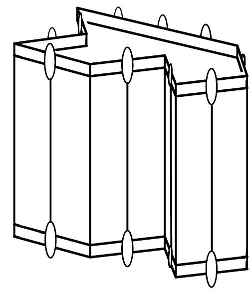
Again, the question is whether it is possible to hear the difference in shape between these two drums, regardless of their exact dimensions and thus only their shape. If we look at the first 25 vibrational frequencies (Figure 2b) we see that they differ from each other so the answer is again in the affirmative. What is also striking about this figure is that the frequencies for both drums roughly follow the same parabolic shape: this is a good illustration of Weyl's law: for sufficiently high overtones, they depend only on the volume —or in this case surface area—of the drum.

Let us make it a little more complicated and listen to a square and a triangular drum:

The first 25 vibrational frequencies are shown in Figure 2c and show that these drums make different sounds.

Finally, two drums with a somewhat more exotic shape, of equal area and circumference:

The first 25 vibrational frequencies are indicated in Figure 2d and the amazing result is that they have identical vibrational spectrum. So these two drums have different shapes but sound exactly the same.

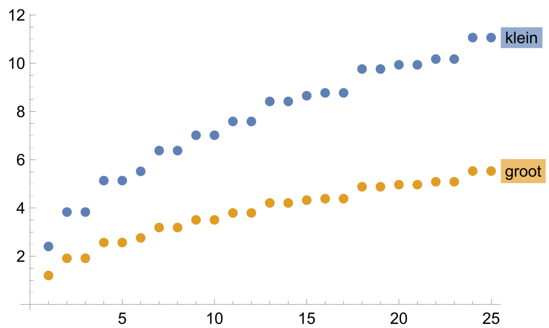
So our conclusion is that there is still one piece of the puzzle missing if we want to describe shapes completely using spectra. I will hint at that later, but first I want to gather further evidence for the use of spectra in describing shapes, and that is in physics. These two perspectives of mathematics and physics complement each other well, and is also a typical feature of my own research.

Spectra in physics

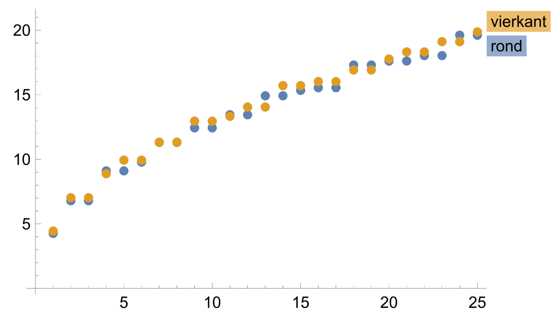
Of course, it was known well before Lorentz that spectra played a crucial role in the observation of composition and shape.

For the visual spectrum, we need to go back to Isaac Newton. Indeed, the term spectrum is introduced in physics by him to indicate how white light is composed of light of different frequencies, something he demonstrates as early as the 17th century using the prism.

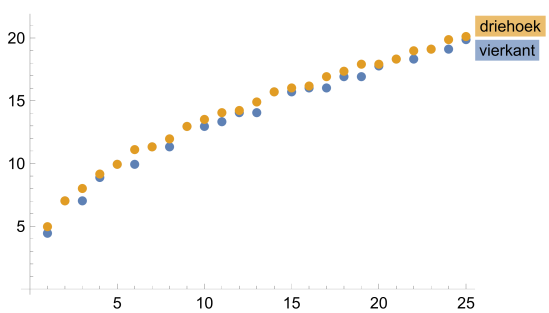
But, how can we now measure shapes using spectra? Let me start close to home: in fact, we often no longer measure distances ourselves with a classical tape measure. Instead, we use the spectrum of laser light:



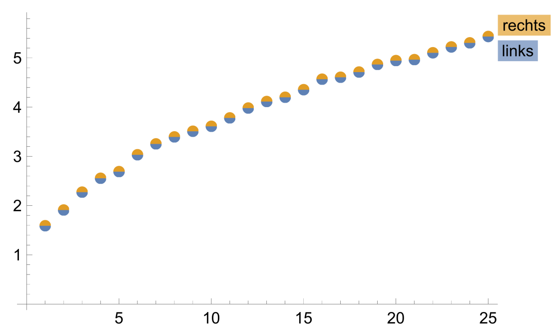
(a) The first 25 vibrational frequencies of two round drums of different cross-sections.



(b) The first 25 vibrational frequencies of a round and a square drum of equal area.



(c) The first 25 vibrational frequencies of a square and a triangular drum (isosceles and rectangular) of equal area.



(d) The first 25 identical vibrational frequencies of two different drums.

Figure 2: Comparison of the first 25 vibrational frequencies for different drums.

And maybe not in your living room, but then in a lab, we analyze the X-ray spectrum in an electron microscope, for example, and if we zoom in even further, observations in a particle accelerator such as at CERN are also based on resonance spectra. All with the goal of spectrally determining the composition, shape and possible velocity of an object or particle.

In astronomy, spectra also play a crucial role, simply because measurements can never be made *in situ* in distant galaxies. However, by looking closely at the visual spectrum of, say, a star or cluster, and in particular at the location of so-called absorption lines —small dark lines in the spectrum— we can determine the chemical composition of distant stars or clusters, and, for example, calculate their velocity (Figure 3). Through such observations, we determine the shape and curvature of the universe by spectrally observing the objects moving through it. Such observations have even led to the startling conclusion that the universe itself is expanding!

But we need not limit ourselves to the visual spectrum; the same is also true for observations using radio telescopes or with infrared telescopes. In the radio spectrum, we currently take pictures that map the surroundings of a black hole (Figure 5), and in the infrared spectrum, very recently the James Webb Telescope has captured the curvature of spacetime very insightfully in an image (Figure 4).

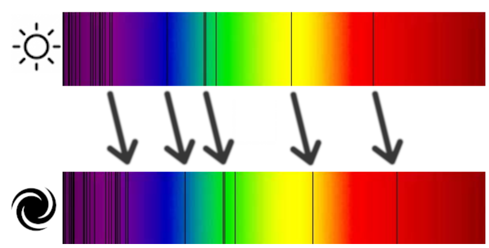


Figure 3: The absorption lines of the Sun (top) are redshifted in front of supercluster BAS11 (bottom), moving at 7% of the speed of light.

And to that can be added gravitational waves: ripples in the spacetime around us. The vibrational spectrum produced by those waves tells us something about the shape of black holes or neutron stars spiraling around each other as they merge into one (Figure 4).

In short, actually all measurements in physics are spectral in nature, so that therefore basically all knowledge of the physical world is determined by the analysis of spectra. This makes it even more crucial to have a mathematical theory that tells how spectra can be used in geometry.

Figure 4: Gravitational lensing as observed by the James Webb Telescope (zoom of cluster SMACS 0723) next to similarly distorted clocks in Salvador Dali's *La persistència de la memòria* (1931) which, by the way, the painter himself claimed was not inspired by Einstein's theory of relativity.

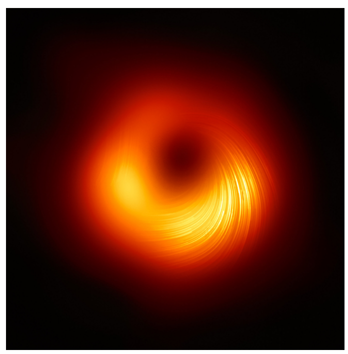


Figure 5: M87* seen by the Event Horizon Telescope (EHT); the lines show the polarization of light in the vicinity of the event horizon.[9]

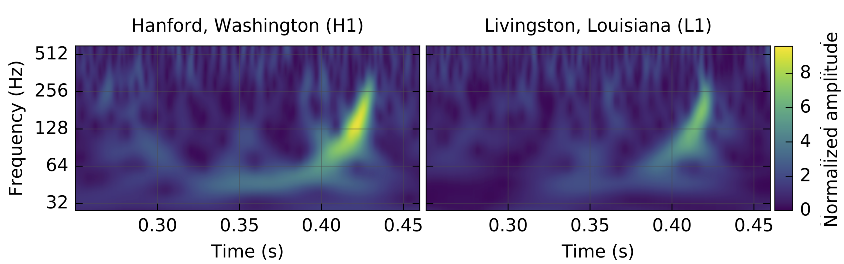


Figure 6: Gravitational waves as measured by LIGO (GW150914): a ripple of spacetime itself caused by the merging of two black holes.[1]

Historical prelude to noncommutative geometry

OK, so we know that all information about the shape of the universe, on small and large scales comes to us in the form of spectra. But how does that work mathematically? Is there a mathematical formalism to do geometry using spectra only? Especially if we consider that there are different objects that nevertheless have the same frequencies of vibration, like that last pair of drums considered before. The answer, of course, is yes —or else I would not be standing here—but for that we must first go back to the early 20th century. A little warning in advance is that the next part may be slightly more technical than the rest of my story. But no worries, with about ten minutes to go I promise there will be the necessary enlightenment.

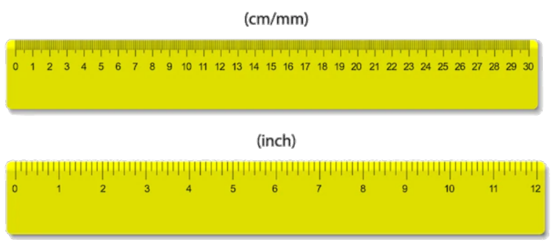


Figure 7: Choice of a gauge: centimeter versus inch.

In 1918, Hermann Weyl introduced the concept of *gauge symmetry*.[17–19,23,27] A good example of gauge is the choice between inches or centimeters, see Figure 7. Weyl's premise was that laws of nature would remain unchanged after adjusting the scale, in other words, after choosing a gauge. This hypothesis made it possible to give a geometric description of electromagnetism, and to use the symmetry for a derivation of the law of conservation of charge. Einstein, however, disagreed with Weyl and reacted sharply in an addendum to the article. Einstein's objection was that such a gauge symmetry could not be a good physical assumption because it would have the effect of making the location of, say, the absorption lines in the spectrum of an element depend on the origin of that atom. That is clearly not the case in nature!

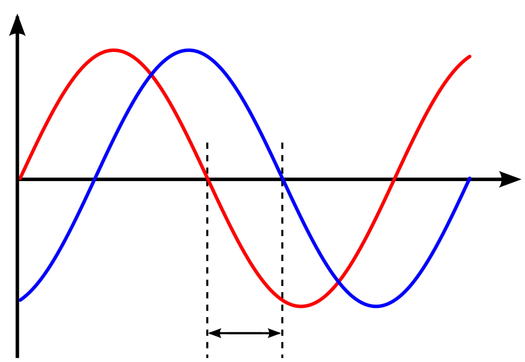


Figure 8: A phase shift leaves the vibrational frequency of a wave unchanged.

Weyl still tried to refute it in a response, but without conviction. However, just under a decade later, his idea found fertile ground in quantum physics.[8,13] The gauge symmetry of Weyl was to be replaced by a *phase* symmetry. Mathematically a small adjustment —a factor *i*—but with great consequences. Especially when we think back to our spectral approach to geometry using vibrational frequencies, we realize that a phase shift of a vibration leaves the frequency unchanged (Figure 8). And since we base our geometry on that spectrum, gauge symmetry thus takes on a new meaning that no doubt Einstein too would have agreed with. Weyl embraced this idea in 1929[24] and wrote about it himself[25] :

... durch die Quantentheorie, glaube ich, können wir mit großer Bestimtheit den Finger auf den Punkt legen, in welchem meine Theorie irrte: die Eichinvarianz verbindet die electromagnetische Potentiale nicht mit den g_{ij} der Gravitation, sondern mit den ψ_q der Materiefeldes. Das konnte ich freilich 1918 nicht wissen! Damals waren diese ψ nog völlig unbekant.

[*... by the quantum theory, I believe, with great certainty we can put the finger on the point, in which my theory was wrong: the gauge invariance connects the electromagnetic potentials not with the g_{ij} of gravity, but with the ψ_q of the matter field. Of course, I could not know that in 1918! At that time these ψ were still completely unknown.*]

Incidentally, Weyl's derivation of the law of conservation of charge is still valid, and actually an application of Noether's second Theorem.[3] Interestingly, Emmy Noether also published her work on this in 1918,[16] as did Weyl's first article. In it, she proves that there is a one-to-one relationship between symmetry and conserved quantities. When applied to Weyl's gauge symmetry, this gives the well-known law of charge conservation. Today, her Theorems are central to theoretical and mathematical physics, but at the time this recognition was far from that: not because of her qualities but because of her gender. Thus, her first attempt at *Habilitation* was rejected simply because she was not a male candidate. There is a well-known anecdote that Hilbert thereupon protested that he did not see why the gender of the candidate should matter; "after all, this was a university, not a bathhouse!" With no immediate result, and so from then on Noether lectured under Hilbert's name.[26]

A big step in the development of the principle of gauge symmetry is taken in the early 1950s. By Yang and Mills,[28] and independently by Utiyama,[20] electromagnetism was generalized, with so-called *non-abelian* gauge fields generalizing the electromagnetic field.[17] At first this seemed to be a theoretical exercise, but soon these theorems turned out to become the building blocks of particle physics. The nonabelian gauge fields turned out to be the force carriers of the weak and strong nuclear forces, just as the electromagnetic field can be seen as a carrier of the electromagnetic force. Well, instead of a single phase as in Weyl's work, nonabelian gauge fields give rise to a more general idea of phase.[27] I myself have always found it incredibly intriguing that abstract mathematics has so much to say about how nature behaves at the scale of elementary particles. Indeed, it attracted me to the interface between mathematics and physics.

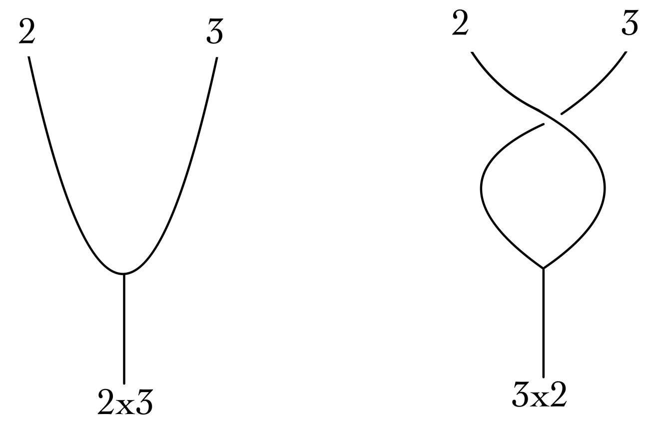
And we can even take another step, and this brings us to the core of my field of research. Indeed, the combination of the nonabelian gauge principle —and thus particle physics–with the spectral approach to geometry inevitably leads to **noncommutative geometry**. Indeed, Yang and Mills' gauge symmetry turns out to be realizable as symmetry of the vibrational spectrum of a noncommutative space: that is, a vibrating noncommutative drum. Thus we manage to describe geometrically not only gravity —with Einstein a consequence of the curvature of spacetime— but also the electromagnetic forces and the nuclear forces.

Figure 9: Alain Connes (r) next to Foucault's Pendulum in the Huygens Building, Radboud University [picture Bert Beelen, 2016].

Time to return to mathematics, because although I have now motivated it from a physics perspective, noncommutative geometry is a field deeply rooted in mathematics, as evidenced by this chair. The founder of this field is French mathematician Alain Connes. He won the Field's Medal in 1982 for his classification of von Neumann algebras —a concept that indeed goes back to the early days of quantum mechanics. He became a professor at the Collège de France soon after, more than 60 years after Lorentz's Michonis lectures there. In the 1980s and 1990s, he combined spectral geometry with spectra of possibly noncommuting coherences. It was precisely this combination that allowed him to answer Kac's old question in the affirmative: by "locally" listening to a drum, you can fully determine its shape.[4,5] It is like walking along the edge of the drums while listening to the sound, thus unraveling the global shape of the drum.

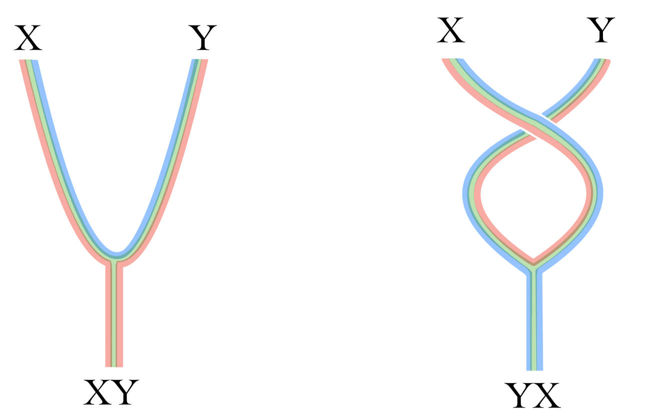
And so this spectral approach is not only applicable to drums of different shapes, or higher-dimensional geometric objects, it also applies to so-called non-commutative spaces. As mentioned, this provides applications in particle physics, but just to make sure you do not go home without having a small idea of what is noncommutative then, an example.

If you multiply numbers like 2 and 3 together, the order doesn't matter: 2×3 is the same as 3×2. My children already learn this commutative algebra in primary school, and of course there is nothing wrong with it at all. We could summarize this in a diagram as follows:



At the intersection, the numbers 2 and 3 are multiplied by each other, with the order first reversed in the diagram on the right. Since both diagrams are the same —the rotation on the right can easily be undone without changing the diagram— this illustrates commutative multiplication.

In the geometry I am concerned with, other rules of computation apply, so multiplication is not necessarily commutative. A diagrammatic representation of not-commuting is using strips of colored paper, as shown here:



The left clearly gives a different result than the right, and so this operation is not commutative. If the connoisseur recognizes matrix multiplication in this, it is perfectly correct because that is exactly what I am trying to illustrate here. In summary, then, our coordinates X and Y are not commutative:

$$XY \neq YX$$

Physically, by the way, the strips of paper represent a space of internal degrees of freedom —precisely the gauge degrees of freedom mentioned earlier. For example, red represents an electron, while blue is a neutrino, and the strip represents all their possible superpositions. If we choose coordinates that satisfy such computational rules, it is possible to 'bake in' the internal degrees of freedom in geometry. In line with Einstein's theory of relativity, curvature of such noncommutative space then gives rise to a geometric description of all four fundamental forces of nature: gravity, the electromagnetic force and the weak and strong nuclear forces.

Looking ahead to the next few years, I want to work toward an understanding of spectral geometry when only part of the spectrum is known. The current formalism of noncommutative geometry assumes knowledge of the full spectrum consisting of all vibrational frequencies. However, this is not very realistic: physically, it is clear that a complete vibrational spectrum is never available. A detector always has a certain bandwidth and resolution. And yet we think that with our spectroscopy we form a complete picture of reality, including curvature and distances in our own universe. Apparently, with our limited knowledge of the spectrum, we still approximate continuous reality. Think back to the drum again: even with our limited hearing, we manage just fine to determine the difference in size of the drums. Even if the range of hearing has diminished somewhat with advancing age.

I want to investigate the mathematics behind this: can we prove that with the knowledge of part of the spectrum we can converge to the original geometric form? To what extent is geometry in this way an *emergent* phenomenon, arising from a finite number of vibrations? Can we speak of geometry seen at a certain scale, or measured at a certain resolution?

Such scale-dependent geometry is at the same time a crucial preparation for formulating much-needed quantum theory in the mathematical edifice of noncommutative geometry. Indeed, at smaller scales, quantum effects will play a role in the propagation of elementary particles, and thus in the geometric formulation of particle physics. In fact, this can be seen as the holy grail of this field of application, and my field of research in particular.

I explained much of my research this afternoon using drums and with examples from physics. However, these are only metaphors for my research and my field. My day-to-day work does look a bit different, and without taking you far into it, I thought it would be good to give you a cursory glimpse of this: well-stocked chalkboards or just using (digital) pen and paper (Figure 10 and 11).

Mathematics as human work

Looking ahead to how I would like to do mathematics, I would like to bring the people behind it to the forefront. Above I have tried to emphasize the people behind all that mathematics and physics, and how important interaction with others was. I think this

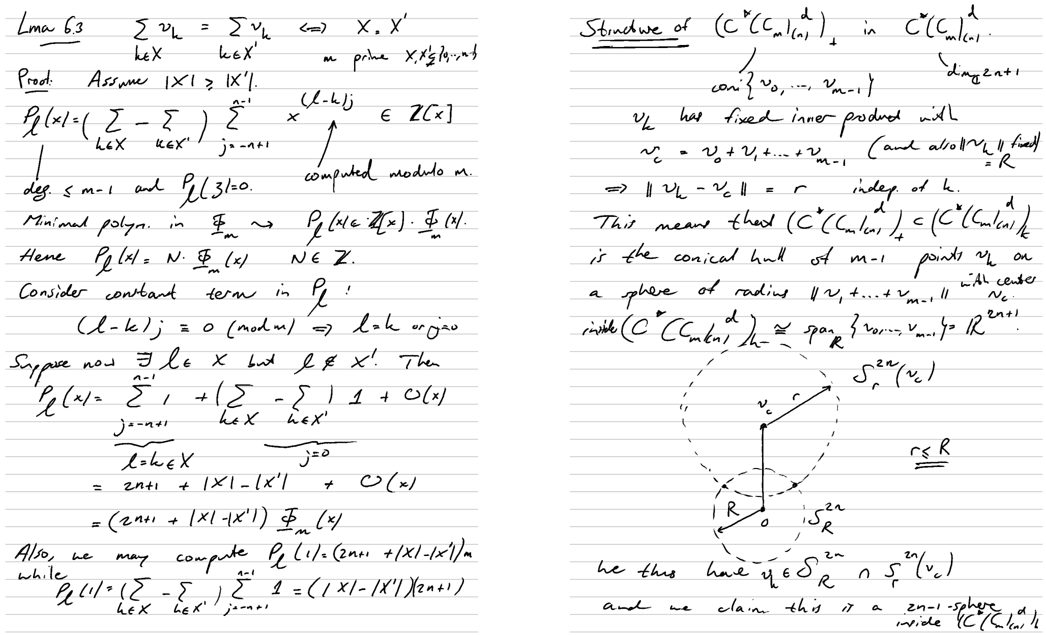


Figure 10: Two random pages from one of my (digital) notebooks.

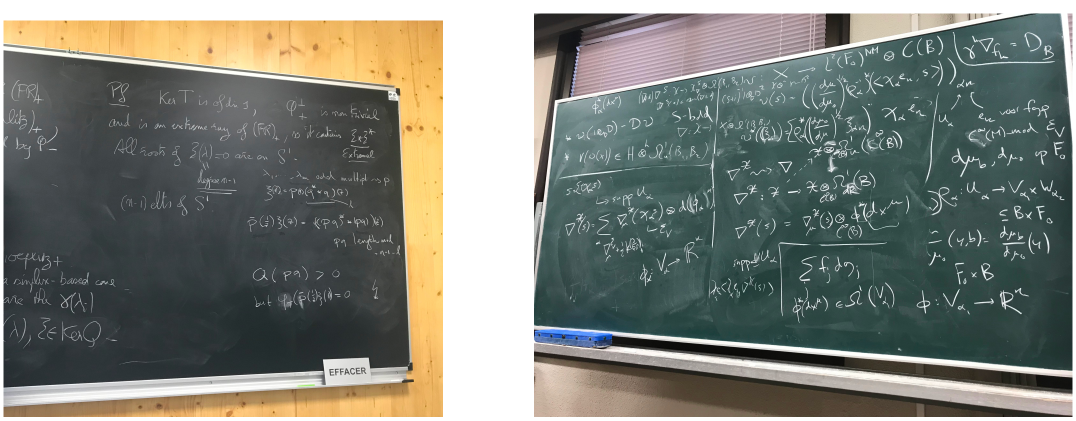


Figure 11: Two well-stocked chalkboards (at IH'ES and in Kyoto).

cohesion is an essential feature of science and especially mathematics. It creates a sustainable network of scientists, each of whom interacts with each other and, if possible or useful, between disciplines as well. And then there will also always be researchers who are closer to society to draw on the all-important more unbound research there for their applications. Recent developments in terms of such impact at NWO are positive, and I will continue to make a strong case for it in the coming years, also as a member of the Mathematics Round Table.

Something that has become painfully clear from my story, by the way, is that this seems to be primarily a male issue. This is of course completely unjustified but means that there is still a big battle to be fought. Again, math is people's work, not men's work! Fortunately, the situation at university among students has improved tremendously now, but with each subsequent step in the scientific career, more women drop out than men. And in doing so, we must also make sure that our policies are not being caught up by reality and should be inclusive in a broader sense.

But then, how do I envision my research group? I would like to compare it to climbing a mountain. There is of course a great risk here of falling into clichés, but from a Dutch perspective it is not so strange to dream of mountains when there are simply none nearby. In my research, I see myself walking on a mountain ridge, enjoying the view and all the beauty to be discovered while walking. Young people with different qualities and backgrounds join the journey; others leave to find a new path to another mountain peak. This is how I would like to see my research group continue to flourish in the coming years, in the longer-term pursuit of reaching the view from the summit.

So for me, interaction with other mathematicians and physicists takes center stage. Of course, it's great to have a mathematical insight, or to be able to capture a physical phenomenon in a formula. But sharing insight with a colleague or a student is surely what makes it really worthwhile.

This also brings me immediately to another point of a more philosophical nature, which is about how I see mathematics. For even though formulas, theorems, proofs *et cetera* constitute the language by which mathematicians understand each other, the core, as far as I am concerned, is sharing the underlying intuitive thinking. I see it as follows: on the one hand, the mathematician uses his logic as a basic tool; on the other hand, there is an immaterial reality of mathematical objects. Indeed, a mathematics based entirely on axioms and deduction —as Hilbert would have liked it to be–is doomed to fail, and this is because Gödel's Theorem proves the existence of assertions that are true but not provable. Such an assertion would then be an object in that immaterial reality, and is thus disconnected from the logic leading to the apparent paradox. Alain Connes articulates the unraveling of this immaterial beauty as follows[6,7]

> *Cette réalité dont je parle, du fait qu'elle n'est localisable ni dans l'espace ni dans le temps, donne, lorsqu'on a la chance d'en dévoiler une infime partie, une sensation de jouissance extraordinaire par le sentiment d'intemporalité qui s'en dégage.*
>
> [*This reality of which I speak, of the fact that it is localizable neither in space nor*

Figure 12: Violin concert by Benjamin Britten; duet between percussion and violin, end of first movement.

> *in time, gives, when one has the chance to reveal a tiny part of it, a sensation of extraordinary enjoyment by the feeling of timelyness which emerges from it.*]

To make this somewhat understandable to non-mathematicians, I will try to explain it using musical experience. In Figure 12 you see the sheet music of a passage from Benjamin Britten's Violin Concerto, at the end of the first movement. It is a beautiful interplay between violin and percussion, a piece that I had the opportunity to play myself some twenty years ago as a percussionist in the VU Orchestra, nota bene with Janine Jansen as soloist. The musical notation, tempo indications *et cetera* in music I would compare to formulas and proofs in mathematics. It makes musicians understand each other, but ultimately it is inferior to the underlying, more intuitive beauty that can be shared through that language. The beauty of mathematics has much in common with that musical beauty, and the way it is shared with each other during, say, an orchestral performance.

With my story this afternoon, I hope to have shown you at least a small piece of the beauty of my mathematics. I hope that, like a listener of a piece of music who may not know the complete notation, without knowledge of formulas or proofs, you have still been able to enjoy this spectral symphony of geometry.

Acknowledgements

Finally, a few words of thanks.

Although this is a chair in mathematics, let it be clear that my strong motivation for doing this mathematics comes from physics. That interest was sparked long ago, including in physics classes at the Jan Arentsz VWO. Many thanks to Mr. Zwagerman and the late Mr. Koch, whose daughter is present today.

During my physics studies in Amsterdam, my graduation supervisor was Gerard Bauerle. He introduced me to the field of noncommutative geometry, nota bene with a subject from string theory! I learned a lot from him in terms of content, of course, but I also got the first lessons on how to deal with a diverse audience from him.

After studying physics, I spent a year studying mathematics in Amsterdam, under the guidance of Klaas Landsman. Although of course I had secretly already been working on mathematics at the interface with physics, this was the formal beginning of my career as a mathematical physicist. The contact that I kept in the following years —when I was a PhD student in Italy —with Klaas was very nice, and allowed me, after another short stop in Bonn, to start as a postdoc in Klaas's group here in Nijmegen in 2007, to which he had moved in the meantime. I am very grateful to him for all the support in the years that followed, up to and including the current appointment.

By extension, this of course also applies to my colleagues in the mathematics department, within IMAPP and in the Faculty, up to and including the current deputy rector.

My PhD supervisors in Italy were Ludwik Dabrowski and Gianni Landi. I had the chance to be in Trieste as a PhD student in an incredibly inspiring time. In addition to the richness that the country has to offer, Trieste has been a very active center for noncommutative geometry, in particular for the "rolling of quantum spheres". So I'd like to thank both of them for their guidance and their support.

I have been working with Alain Connes for almost 10 years now, and this has been incredibly inspiring. His enthusiasm when we talk about a particular problem, behind a chalkboard for example, but also increasingly more often via zoom, is very infectious and I like to spread it further to for example my students.

Many thanks to all the friends I met while studying, in music or outside of it. From sharing an office during research internships to band rehearsals in small practice rooms, from preparing for half marathons to massive orchestra rehearsals.

I would like to thank my parents and sisters, for their support and endless patience in trying to understand not only my profession, but also this somewhat strange academic world, see also, for example, this entire entourage this afternoon. And although they are no longer among us, I am also grateful to my grandparents, and my grandfather Otto in particular: they played an important role in my development, motivating me to continue on my chosen path, while keeping an eye on the broader picture and continuing to develop that. Similarly, I would like to thank my in-laws: for them, too, that mathematics is a rather elusive reality, but that does not mean that they could not contribute ideas, about the title of this oration, for example. Thank you for that!

I have an incredibly sweet family around me: Daniël, Joris and Leonora, you make sure that Dad is back in the world after work at once! And we experience so many fun things together for my work, like this party today but also all the trips we have taken together to such beautiful places all over the world! Mathilde, thank you for the color you have given to the whole spectrum and for your endless support; for the beautiful things we have experienced together, right down to the music in which we have found each other.

I have spoken

References

[1] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* 116 (2016) 061102.

[2] F. Berends and D. van Delft. *Lorentz: Gevierd fysicus, geboren verzoener*. Prometheus, 2019.

[3] K. A. Brading. Which symmetry? Noether, Weyl, and conservation of electric charge. *Stud. Hist. Philos. Sci. B Stud. Hist. Philos. Modern Phys.* 33 (2002) 3–22.

[4] A. Connes. Gravity coupled with matter and the foundation of non-commutative geometry. *Commun. Math. Phys.* 182 (1996) 155–176.

[5] A. Connes. On the spectral characterization of manifolds. *J. Noncommut. Geom.* 7 (2013) 1–82.

[6] A. Connes, A. Lichnerowicz, and M. P. Schützenberger. *Triangle de pensées*. Editions Odile Jacob, 2000.

[7] A. Connes, A. Lichnerowicz, and M. P. Schützenberger. *Triangle of thoughts*. American Mathematical Society, Providence, RI, 2001. Translated from the 2000 French original by Jennifer Gage.

[8] V. Fock. Über die invariante Form der Wellen-und der Bewegungsgleichungen für einen geladenen Massenpunkt. *Z. Phys* 39 (1926) 226.

[9] C. Goddi et al. Polarimetric Properties of Event Horizon Telescope Targets from ALMA. *Astrophys. J. Lett.* 910 (2021) L14.

[10] D. Hilbert. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen (Vierte Mitteilung). *Gött. Nachr.* (1906) 157–209.

[11] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly* 73 (1966) 1–23.

[12] A. J. Kox. *Hendrik Antoon Lorentz. Natuurkundige (1853–1928)*. Balans, 2019.

[13] F. London. Quantummechanische Deutung der theorie von Weyl. *Z. Phys.* 42 (1927) 375.

[14] H. Lorentz. Alte un neu Fragen der Physik. *Phys. Zeitschrift.* 11 (1910) 1234.

[15] H. Lorentz. *Les théories statistiques en thermodynamique*. Teubner, Berlin, 1916. Dunoyer, Conférences faites au Collège de France en novembre 1912.

[16] E. Noether. Invariante variationsprobleme. *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen* (1918) 235–257.

[17] L. O'Raifeartaigh, editor. *The dawning of gauge theory*. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1997. Edited and with introductory essays by Lochlainn O'Raifeartaigh.

[18] N. Straumann. Early history of gauge theories and weak interactions. Invited talk at PSI Summer on Physics with Neutrinoes, Zuoz, Switzerland (August 4-10, 1996) [arXiv:hep-ph/9609230].

[19] N. Straumann. Hermann Weyl and the early history of gauge theories. In *Symmetries in algebra and number theory (SANT)*, pages 173–189. Universitätsverlag Göttingen, Göttingen, 2009.

[20] R. Utiyama. Invariant theoretical interpretation of interaction. *Phys. Rev.* 101 (1956) 1597.

[21] J. von Neumann. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. *Math. Ann.* 102 (1930) 49–131.

[22] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.* 71 (1912) 441–479.

[23] H. Weyl. Gravitation und Elektrizität. *Sitzungsber. Preuss. Akad. Wiss.* (1918) 465–480.

[24] H. Weyl. Gravitation and the electron. *PNAS* 15 (1929) 323–334.

[25] H. Weyl. 50 Jahre Relativitä tstheorie. *Naturwissenschaften* 38 (1930) 73–83.

[26] H. Weyl. Emmy noether 1882–1935. *Scripta Mathematica* 3 (1935) 201–220. (WGA 3:425-444).

[27] C. N. Yang. Hermann Weyl's contribution to physics. In *Hermann Weyl, 1885–1985*, pages 7–21. Eidgenössische Tech. Hochschule, Zürich, 1986.

[28] C.-N. Yang and R. L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.* 96 (1954) 191–195.