

# Noncommutative spaces at finite resolution

Walter van Suijlekom  
(joint with Alain Connes)

# Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

# Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

## The spectral approach to geometry

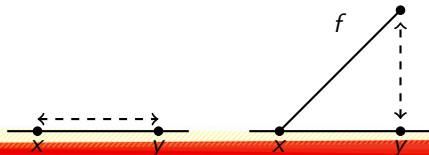
Given cpt Riemannian spin manifold  $(M, g)$  with spinor bundle  $S_M$  on  $M$ .

- the  $C^*$ -algebra  $C(M)$
- the self-adjoint Dirac operator  $D_M$
- both acting on Hilbert space  $L^2(S_M)$

$\rightsquigarrow$  spectral triple:  $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



## Spectral triples

More generally, we consider a triple  $(A, \mathcal{H}, D)$

- a  $C^*$ -algebra  $A$
- a self-adjoint operator  $D$  with compact resolvent and bounded commutators  $[D, a]$  for  $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space  $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals  $\phi : A \rightarrow \mathbb{C}$  of norm 1
- Pure states are extreme points of state space
- Distance function on state space of  $A$ :

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

## Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of **all eigenvalues** of the Dirac operator.
- From a physical standpoint this is **not very realistic**: detectors have limited energy ranges and resolution.

*We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.*

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

## Towards operator systems..

- (I) Given  $(A, \mathcal{H}, D)$  we project onto part of the spectrum of  $D$ :
- $\mathcal{H} \mapsto P\mathcal{H}$ , projection onto closed Hilbert subspace
  - $D \mapsto PDP$ , still a self-adjoint operator
  - $A \mapsto PAP$ , this is not an algebra any more (unless  $P \in A$ )

Instead,  $PAP$  is an operator system:  $(PaP)^* = Pa^*P$ .

- (II) Another approach would be to consider metric spaces up to a finite resolution :
- Consider integral operators associated to the tolerance relation  $R_\epsilon$  given by  $d(x, y) < \epsilon$

So first, some background on operator systems.

## Operator systems

### **Definition (Choi-Effros 1977)**

An **operator system** is a  $*$ -closed vector space  $E$  of bounded operators.

**Unital**: it contains the identity operator.

- $E$  is **ordered**: cone  $E_+ \subseteq E$  of positive operators, in the sense that  $T \in E_+$  iff

$$\langle \psi, T\psi \rangle \geq 0; \quad (\psi \in \mathcal{H}).$$

- in fact,  $E$  is **matrix ordered**: cones  $M_n(E)_+ \subseteq M_n(E)$  of positive operators on  $\mathcal{H}^n$  for any  $n$ .

Maps between operator systems  $E, F$  are **completely positive maps** in the sense that their extensions  $M_n(E) \rightarrow M_n(F)$  are positive for all  $n$ .

Isomorphisms are **complete order isomorphisms**

## $C^*$ -envelope of a unital operator system

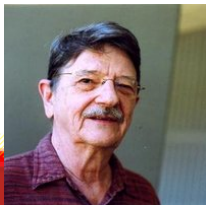
Arveson introduced the notion of  $C^*$ -envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.

Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A  $C^*$ -extension  $\kappa : E \rightarrow A$  of a unital operator system  $E$  is given by a complete order isomorphism onto  $\kappa(E) \subseteq A$  such that  $C^*(\kappa(E)) = A$ .

A  $C^*$ -envelope of a unital operator system is a  $C^*$ -extension  $\kappa : E \rightarrow A$  with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \exists! \rho \\ & & B \end{array}$$



## Shilov boundaries

There is a useful description of  $C^*$ -envelopes using Shilov ideals.

### **Definition**

Let  $\kappa : E \rightarrow A$  be a  $C^*$ -extension of an operator system. A **boundary ideal** is given by a closed 2-sided ideal  $I \subseteq A$  such that the quotient map  $q : A \rightarrow A/I$  is completely isometric on  $\kappa(E) \subseteq A$ .

The **Shilov boundary ideal** is the largest of such boundary ideals.

### **Proposition**

Let  $\kappa : E \rightarrow A$  be a  $C^*$ -extension. Then there exists a Shilov boundary ideal  $J$  and  $C_{env}^*(E) \cong A/J$ .

As an example consider the operator system of continuous harmonic functions  $C_{\text{harm}}(\overline{\mathbb{D}})$  on the closed disc. Then by the maximum modulus principle the Shilov boundary is  $S^1$ . Accordingly, its  $C^*$ -envelope is  $C(S^1)$ .

## Propagation number of an operator system

One lets  $E^{\circ n}$  be the norm closure of the linear span of products of  $\leq n$  elements of  $E$ .

### **Definition**

The *propagation number*  $\text{prop}(E)$  of  $E$  is defined as the smallest integer  $n$  such that  $\kappa(E)^{\circ n} \subseteq C_{\text{env}}^*(E)$  is a  $C^*$ -algebra.

Returning to harmonic functions in the disk we have  $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$ .

### **Proposition**

The propagation number is *invariant under complete order isomorphisms*, as well as *under stable=Morita equivalence* [EKT, 2019]:

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$\text{prop}(E \otimes_{\min} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$

## State spaces of operator systems

- The existence of a cone  $E_+ \subseteq E$  of positive elements allows to speak of states on  $E$  as **positive linear functionals of norm 1**.
- In the **finite-dimensional case**, the **dual**  $E^d$  of a unital operator system is a unital operator system with

$$E_+^d = \{ \phi \in E^d : \phi(T) \geq 0, \forall T \in E_+ \}$$

and similarly for the matrix order.

- Also, we have  $(E^d)_+^d \cong E_+$  as cones in  $(E^d)^d \cong E$ .
- It follows that we have the following useful correspondence:  
**pure states on  $E$   $\longleftrightarrow$  extreme rays in  $(E^d)_+$**   
and the other way around.

# Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

## Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of  $D_{S^1}$  are **Fourier modes**  $e_k(t) = e^{ikt}$  for  $k \in \mathbb{Z}$
- **Orthogonal projection**  $P = P_n$  onto  $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space  $C(S^1)^{(n)} := PC(S^1)P$  is an **operator system**
- Any  $T = PfP$  in  $C(S^1)^{(n)}$  can be written as a **Toeplitz matrix**

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have:  $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$  and  $\text{prop}(C(S^1)^{(n)}) = 2$  (for any  $n$ ).

## Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system  $C^*(\mathbb{Z})_{(n)}$ :

- functions on  $S^1$  with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element  $a$  is positive iff  $\sum_k a_k e^{ikx}$  is a positive function on  $S^1$ .
- The  $C^*$ -envelope of  $C^*(\mathbb{Z})_{(n)}$  is given by  $C^*(\mathbb{Z})$ .

### Proposition

1. The extreme rays in  $(C^*(\mathbb{Z})_{(n)})_+$  are given by the elements  $a = (a_k)$  for which the Laurent series  $\sum_k a_k z^k$  has all its zeroes on  $S^1$ .
2. The pure states of  $C^*(\mathbb{Z})_{(n)}$  are given by  $a \mapsto \sum_k a_k \lambda^k$  ( $\lambda \in S^1$ ).

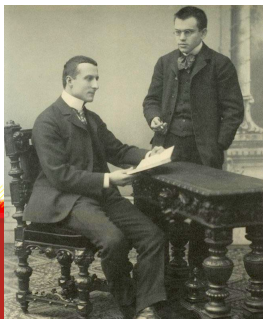
## Pure states on the Toeplitz matrices

Duality of  $C(S^1)^{(n)}$  and  $C^*(\mathbb{Z})_{(n)}$  [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \rightarrow \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

### Proposition

1. The **extreme rays** in  $C(S^1)^{(n)}$  are  $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$  for any  $\lambda \in S^1$ .
2. The **pure state space**  $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$ .



## Curiosities on Toeplitz matrices

### Theorem (Carathéodory)

Let  $T$  be an  $n \times n$  Toeplitz matrix. Then  $T \geq 0$  iff  $T = V\Delta V^*$  with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \quad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some  $d_1, \dots, d_n \geq 0$  and  $\lambda_1, \dots, \lambda_n \in S^1$ .

**Farenick** continues to exploit the duality by showing:

- every positive linear map of the  $n \times n$  complex matrices is completely positive when restricted to the Toeplitz operator system.
- every unital isometry of the  $n \times n$  Toeplitz matrices into the algebra of  $n \times n$  complex matrices is a unitary similarity transformation.

## Spectral truncations of the circle ( $n = 3$ )

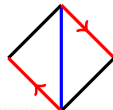
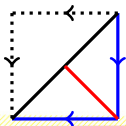
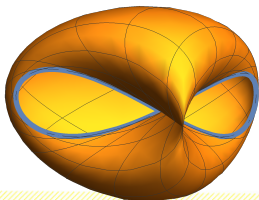
We consider  $n = 3$  for which the Toeplitz matrices are of the form

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is  $\mathbb{T}^2/S_2$ , given by vector states  $|\xi\rangle\langle\xi|$  with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$

This is a Möbius strip!



# Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

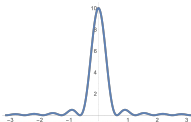
## Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces  $\mathcal{S}(C(S^1)^{(n)})$  with the distance function  $d_n$  to the circle.

- The map  $R_n : C(S^1) \rightarrow C(S^1)^{(n)}$  given by compression with  $P_n$  allows to pull-back states from  $C(S^1)^{(n)}$  to the circle
- There is a  **$C^1$ -approximate order inverse**  $S_n : C(S^1)^{(n)} \rightarrow C(S^1)$ :

$$R_n(S_n(T)) = T_n \odot T; \quad S_n(R_n(f)) = F_n * f$$

in terms of a Schur product with a matrix  $T_n$  and the convolution with the Fejér kernel  $F_n$ :

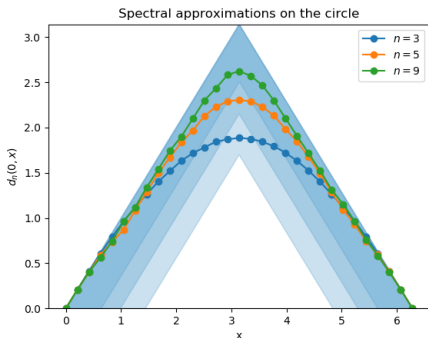


- The fact that  $S_n$  is a  $C^1$ -approximate inverse of  $R_n$  allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where  $\gamma_n \rightarrow 0$  as  $n \rightarrow \infty$ .

- Some (basic) Python simulations for point evaluation on  $S^1$ :



## Gromov–Hausdorff convergence

Recall **Gromov–Hausdorff distance** between two metric spaces:

$$d_{\text{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \rightarrow Z, g : Y \rightarrow Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps  $R_n, S_n$  we can equip  $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$  with a distance function that **bridges** the given distance functions on  $\mathcal{S}(C(S^1))$  and  $\mathcal{S}(C(S^1)^{(n)})$  within  $\epsilon$  for large  $n$ .

### ***Proposition (vS21, Hekkelman 2021)***

*The sequence of state spaces  $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$  converges to  $(\mathcal{S}(C(S^1)), d_{S^1})$  in Gromov–Hausdorff distance.*

# Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

## Operator systems, groupoids and bonds

Recall:

- Consider a **locally compact groupoid**  $G$  equipped with a (left invariant) Haar system  $\nu = \{\nu_x\}$ .
- The space  $C_c(G)$  of compactly supported complex-valued continuous functions on  $G$  becomes a  $*$ -algebra with the convolution product and involution given by

$$f * g(\gamma) = \int_{G_x} f(\gamma\gamma_1^{-1})g(\gamma_1)d\nu_x(\gamma_1); \quad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

where  $x = s(\gamma)$  for any  $\gamma \in G$ .

- $C_c(G)$  can be completed to the **groupoid  $C^*$ -algebra**  $C^*(G)$

## Definition

A **bond** is a triple  $(G, \nu, \Omega)$  consisting of a locally compact groupoid  $G$ , a Haar system  $\nu = \{\nu_x\}$  and an open symmetric subset  $\Omega \subseteq G$  containing the units  $G^{(0)}$ .

## Proposition

Let  $(\Omega, G, \nu)$  be a bond. The closure of the subspace  $C_c(\Omega) \subseteq C_c(G)$  in the  $C^*$ -algebra  $C^*(G)$  is an operator system.

## Example

1. Consider  $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$  **Fejér–Riesz operator system** inside  $C^*(\mathbb{Z})$ .
2. Consider  $\Omega_n = (-n, n) \subset C_m$  (so modulo  $m$ ). The operator system consists of **banded  $m \times m$  circulant matrices** of band width  $n$ .  
Thus, the ambient groupoid is crucial since these two operator systems are not even Morita equivalent.
3. Given the set  $X = \{1, \dots, m\}$  consider a “band”  $R_n \subset X \times X$  around the diagonal of width  $n \rightsquigarrow$  **banded  $m \times m$  matrices** of band width  $n$ .

## Operator systems associated to tolerance relations

- Suppose that  $X$  is a set and consider a relation  $\mathcal{R} \subseteq X \times X$  on  $X$  that is reflexive, symmetric but not necessarily transitive.
- Key motivating example: a metric space  $(X, d)$  with the relation

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

- If  $(X, \mu)$  is a measure space and  $\mathcal{R} \subseteq X \times X$  an open subset we obtain the operator system  $E(\mathcal{R})$  as the closure of integral operator with support in  $\mathcal{R}$ . Note that  $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

## Tolerance relations on finite sets [Gielen–vS, 2022]

Let  $X$  be a finite set and  $\mathcal{R} \subseteq X \times X$  a symmetric reflexive relation on  $X$  and suppose that  $\mathcal{R}$  generates the full equivalence class  $X \times X$  (i.e. the graph corresponding to  $\mathcal{R}$  is connected). Then

1. the  $C^*$ -envelope of  $E(\mathcal{R})$  is  $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$  and  $\text{prop}(E(\mathcal{R})) = \text{diam}(\mathcal{R})$ .
2. If  $\mathcal{R}$  is a chordal graph, then  $E(\mathcal{R})^d \cong E(\mathcal{R})$  as a vector space, but with order structure given by being **partially positive**.
3. the **pure states** of  $E(\mathcal{R})$  are given by vector states  $|v\rangle\langle v|$  for which the support of  $v \in \ell^2(X)$  is  $\mathcal{R}$ -connected.

### Example

The operator systems of  $p \times p$  **band matrices** with band width  $N$ .

1. The **propagation number** of  $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$  is equal to  $\lceil p/N \rceil$ .
2. The dual operator system consists of band matrices (with order given by **partially positive**).

## Spaces at finite resolution

Consider now a path metric measure space  $X$  with a measure of full support, and the following tolerance relation:

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

It gives rise to the operator system  $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$ .

### **Proposition**

*If  $X$  is a complete and locally compact path metric measure space  $X$  with a measure of full support, then*

1.  $C_{env}^*(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$  and  $\text{prop}(E(\mathcal{R}_\epsilon)) = \lceil \text{diam}(X)/\epsilon \rceil$
2. The **pure states** of  $E(\mathcal{R}_\epsilon)$  are given by vector states  $|\psi\rangle\langle\psi|$  where the essential support of  $\psi \in L^2(X)$  is  **$\epsilon$ -connected**.

## Outlook

- Spectral truncations: tori, compact Lie groups, etc.
- Bonds in groupoids: approximate order unit, duality, etc.
- Metric structure on state spaces for spaces at finite resolution
- Gromov–Hausdorff convergence, entropy
- General theory of spectral triples for operator systems
- ...

*Thanks!*