Noncommutative spaces at finite resolution

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Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Spaces at finite spatial resolution

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The spectral approach to geometry

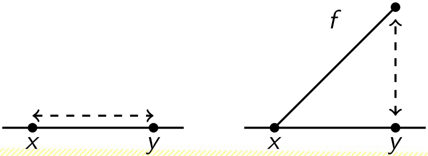
Given cpt Riemannian spin manifold (M, g) with spinor bundle S_M on M.

- the C^*-algebra $C(M)$
- the self-adjoint Dirac operator D_M
- both acting on Hilbert space $L^2(S_M)$

$\rightsquigarrow$ spectral triple: $(C(M), L^2(S_M), D_M)$

Reconstruction of distance function [Connes 1994]:

$$d(x, y) = \sup_{f \in C(M)} \{|f(x) - f(y)| : \|[D_M, f]\| \leq 1\}$$



Spectral triples

More generally, we consider a triple $(A, \mathcal{H}, D)$

- a C^*-algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A:

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

> *We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.*

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of D:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Choi-Effros 1977)
An operator system is a $*$-closed vector space E of bounded operators.
Unital: it contains the identity operator.

- E is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, E is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any n.

Maps between operator systems E, F are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all n.

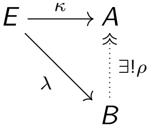
Isomorphisms are complete order isomorphisms

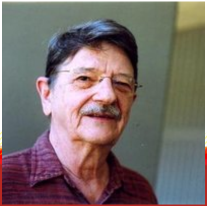
C^*-envelope of a unital operator system

Arveson introduced the notion of C^*-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^*-extension $\kappa : E \to A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^*-envelope of a unital operator system is a C^*-extension $\kappa : E \to A$ with the following universal property:





Shilov boundaries

There is a useful description of C^*-envelopes using Shilov ideals.

Definition

Let $\kappa : E \to A$ be a C^*-extension of an operator system. A boundary ideal is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The Shilov boundary ideal is the largest of such boundary ideals.

Proposition

Let $\kappa : E \to A$ be a C^*-extension. Then there exists a Shilov boundary ideal J and $C^*_{env}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\mathrm{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1. Accordingly, its C^*-envelope is $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E.

Definition

The propagation number prop(E) of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C^*_{env}(E)$ is a C^*-algebra.

Returning to harmonic functions in the disk we have $\mathrm{prop}(C_{\mathrm{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$prop(E) = prop(E \otimes_{min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$prop(E \otimes_{\min} F) = \max\{prop(E), prop(F)\}$$

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual E^d of a unital operator system is a unital operator system with

$$E^d_+ = \{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\}$$

  and similarly for the matrix order.
- Also, we have $(E^d)^d_+ \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:

  pure states on $E \longleftrightarrow$ extreme rays in $(E^d)_+$

  and the other way around.

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Spectral truncation of the circle: Toeplitz matrices

- Eigenvectors of D_{S^1} are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\text{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1.
- The C^*-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$.

Proposition

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ $(\lambda \in S^1)$.*

Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$

$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V \Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \qquad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \ldots, d_n \geq 0$ and $\lambda_1, \ldots, \lambda_n \in S^1$.

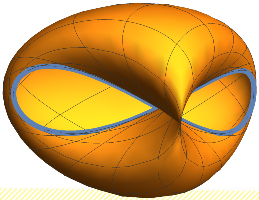
Spectral truncations of the circle ($n = 3$)

We consider $n = 3$ for which the Toeplitz matrices are of the form

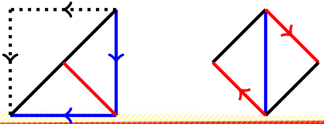
$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}$$

The pure state space is $\mathbb{T}^2/S_2$, given by vector states $|\xi\rangle\langle\xi|$ with

$$\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}$$



This is a Möbius strip!



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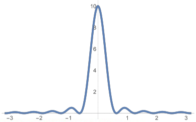
Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a C^1-approximate order inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

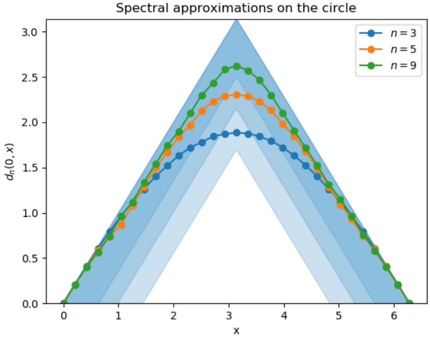
in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n:



- The fact that S_n is a C^1-approximate inverse of R_n allows one to prove

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where $\gamma_n \to 0$ as $n \to \infty$.
- Some (basic) Python simulations for point evaluation on S^1:



Gromov–Hausdorff convergence

Recall Gromov–Hausdorff distance between two metric spaces:

$$d_{\mathrm{GH}}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f : X \to Z, g : Y \to Z \text{ isometric}\}$$

and

$$d_H(X, Y) = \inf\{\epsilon \geq 0; X \subseteq Y_\epsilon, Y \subseteq X_\epsilon\}$$

- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that bridges the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n.

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

More general results on GH-convergence

Definition

Let $\{(\mathcal{E}_n, \mathcal{H}_n, D_n)\}_n$ be a sequence of operator system spectral triples and let $(\mathcal{E}, \mathcal{H}, D)$ be an operator system spectral triple. An C^1-approximate order isomorphism for this set of data is given by linear maps $R_n : E \to E_n$ and $S_n : E_n \to E$ for any n such that the following three condition hold:

1. *R_n, S_n are positive, unital, contractive and Lipschitz-contractive*
2. *there exist sequences γ_n, γ_n' both converging to zero such that*

$$\|S_n \circ R_n(a) - a\| \leq \gamma_n \|[D, a]\|,$$
$$\|R_n \circ S_n(h) - h\| \leq \gamma_n' \|[D_n, h]\|.$$

Examples: Toeplitz and Fejér–Riesz [vS 2021], cubic truncations of $\mathbb{T}^d$ [Berendschot 2019], fuzzy spheres [Rieffel 2000], quantum fuzzy spheres [Aguilar–Kaad–Kyed, 2021], Fourier truncations [Rieffel 2022], spectral truncations of tori [Leimbach 2022], ...

Outline

Operator systems, groupoids and bonds

Definition (Connes-vS, 2021)

A bond is a triple (G, ν, Ω) *consisting of a locally compact groupoid* G*, a Haar system* $\nu = \{\nu_x\}$ *and an open symmetric subset* $\Omega \subseteq G$ *containing the units* $G^{(0)}$*.*

Proposition

Let (Ω, G, ν) *be a bond. The closure of the subspace* $C_c(\Omega) \subseteq C_c(G)$ *in the* C^**-algebra* $C^*(G)$ *is an operator system.*

Example

1. *Consider* $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ *Fejér–Riesz operator system inside* $C^*(\mathbb{Z})$*.*
2. *Consider* $\Omega_n = (-n, n) \subset C_m$ *(so modulo* m*). The operator system consists of banded* $m \times m$ *circulant matrices of band width* n*.*
3. *Given the set* $X = \{1, \ldots, m\}$ *consider a "band"* $R_n \subset X \times X$ *around the diagonal of width* $n \rightsquigarrow$ *banded* $m \times m$ *matrices of band width* n*.*

Operator systems associated to tolerance relations

- Suppose that X is a set and consider a relation $\mathcal{R} \subseteq X \times X$ on X that is reflexive, symmetric but not necessarily transitive.
- Key motivating example: a metric space (X, d) with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If (X, μ) is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R})$ as the closure of integral operator with support in $\mathcal{R}$. Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

Tolerance relations on finite sets [Gielen–vS, 2022]

Let X be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on X and suppose that $\mathcal{R}$ generates the full equivalence class $X \times X$ (*i.e.* the graph corresponding to $\mathcal{R}$ is connected). Then

1. the C^*-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X)) \cong M_{|X|}(\mathbb{C})$ and $\mathrm{prop}(E(\mathcal{R})) = \mathrm{diam}(\mathcal{R})$.
2. If $\mathcal{R}$ is a chordal graph, then $E(\mathcal{R})^d \cong E(\mathcal{R})$ as a vector space, but with order structure given by being partially positive.
3. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle\langle v|$ for which the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.

Example

The operator systems of $p \times p$ band matrices with band width N.

1. *The propagation number of $\mathcal{E}_{p,N} \subseteq M_p(\mathbb{C})$ is equal to $\lceil p/N \rceil$.*
2. *The dual operator system consists of band matrices (with order given by partially positive).*

Spaces at finite resolution [Connes-vS, 2021]

Consider now a path metric measure space X with a measure of full support, and the following tolerance relation:

$$\boxed{\mathcal{R}_\epsilon := \{(x,y) \in X \times X : d(x,y) < \epsilon\}}$$

It gives rise to the operator system $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$.

Proposition

If X is a complete and locally compact path metric measure space X with a measure of full support, then

1. $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ *and* $\mathrm{prop}(E(\mathcal{R}_\epsilon)) = \lceil diam(X)/\epsilon \rceil$
2. *The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ-connected.*

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Outlook

- Spectral truncations: tori, compact Lie groups, etc.
- Bonds in groupoids: approximate order unit, duality, etc.
- Metric structure on state spaces for spaces at finite resolution
- Gromov–Hausdorff convergence, entropy
- General theory of spectral triples for operator systems
- ...

Thanks!