

Tolerance relations and operator systems

Walter van Suijlekom

Outline

Noncommutative geometry and operator systems

Spectral truncation of the circle

Gromov–Hausdorff convergence

Non-unital operator systems

Spaces at finite spatial resolution

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Spectral triples

More generally, we consider a triple (A, \mathcal{H}, D)

- a unital C^* -algebra A
- a self-adjoint operator D with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space \mathcal{H}

Generalized distance function:

- States are positive linear functionals $\phi : A \rightarrow \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of A :

$$d(\phi, \psi) = \sup_{a \in A} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of **all eigenvalues** of the Dirac operator.
- From a physical standpoint this is **not very realistic**: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for **doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution**.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and **based on [Connes–vS] (CMP, Szeged)**

Towards operator systems..

- (I) Given (A, \mathcal{H}, D) we project onto part of the spectrum of D :
- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
 - $D \mapsto PDP$, still a self-adjoint operator
 - $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, PAP is an operator system: $(PaP)^* = Pa^*P$.

- (II) Another approach would be to consider metric spaces up to a finite resolution :
- Consider integral operators associated to the tolerance relation R_ϵ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

Operator systems

Definition (Arveson 1969, Choi-Effros 1977)

An **operator system** is a $*$ -closed vector space E of bounded operators.

Unital: it contains the identity operator.

- E is **ordered**: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff

$$\langle \psi, T\psi \rangle \geq 0; \quad (\psi \in \mathcal{H}).$$

- in fact, E is **matrix ordered**: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on \mathcal{H}^n for any n .

Maps between operator systems E, F are **completely positive maps** in the sense that their extensions $M_n(E) \rightarrow M_n(F)$ are positive for all n .

Isomorphisms are **complete order isomorphisms**

C^* -envelope of a unital operator system

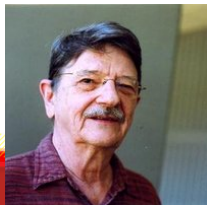
Arveson introduced the notion of C^* -envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.

Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A C^* -extension $\kappa : E \rightarrow A$ of a unital operator system E is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A C^* -envelope of a unital operator system is a C^* -extension $\kappa : E \rightarrow A$ with the following universal property:

$$\begin{array}{ccc} E & \xrightarrow{\kappa} & A \\ & \searrow \lambda & \uparrow \exists! \rho \\ & & B \end{array}$$



Shilov boundaries

There is a useful description of C^* -envelopes using Shilov ideals.

Definition

Let $\kappa : E \rightarrow A$ be a C^* -extension of an operator system. A **boundary ideal** is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \rightarrow A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The **Shilov boundary ideal** is the largest of such boundary ideals.

Proposition

Let $\kappa : E \rightarrow A$ be a C^* -extension. Then there exists a Shilov boundary ideal J and $C_{env}^*(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{\mathbb{D}})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is S^1 . Accordingly, its C^* -envelope is $C(S^1)$.

Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of $\leq n$ elements of E .

Definition

The *propagation number* $\text{prop}(E)$ of E is defined as the smallest integer n such that $\kappa(E)^{\circ n} \subseteq C_{\text{env}}^*(E)$ is a C^* -algebra.

Returning to harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\overline{\mathbb{D}})) = 1$.

Proposition

The propagation number is *invariant under complete order isomorphisms*, as well as *under stable=Morita equivalence* [EKT, 2019]:

$$\text{prop}(E) = \text{prop}(E \otimes_{\min} \mathcal{K})$$

More generally [Koot, 2021], we have

$$\text{prop}(E \otimes_{\min} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$

State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on E as **positive linear functionals of norm 1**.
- In the **finite-dimensional case**, the **dual** E^d of a unital operator system is a unital operator system with

$$E_+^d = \{ \phi \in E^d : \phi(T) \geq 0, \forall T \in E_+ \}$$

and similarly for the matrix order.

- Also, we have $(E^d)_+^d \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:
pure states on E \longleftrightarrow extreme rays in $(E^d)_+$
and the other way around.

In the infinite-dimensional/non-unital case, this is more subtle (more later..).

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Spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of D_{S^1} are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** $P = P_n$ onto $\text{span}_{\mathbb{C}}\{e_1, e_2, \dots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an **operator system**
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a **Toeplitz matrix**

$$PfP \sim (t_{k-l})_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any n).

Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on S^1 with a finite number of non-zero Fourier coefficients:

$$a = (\dots, 0, a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-2}, a_{n-1}, 0, \dots)$$

- an element a is positive iff $\sum_k a_k e^{ikx}$ is a positive function on S^1 .
- The C^* -envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$ (and $\text{prop} = \infty$).

Proposition

1. The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on S^1 .
2. The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).

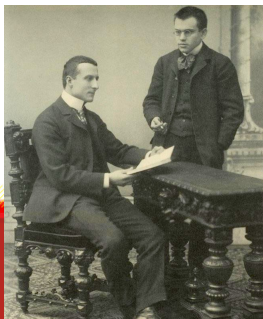
Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \rightarrow \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

Proposition

1. The **extreme rays** in $C(S^1)^{(n)}_+$ are $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ for any $\lambda \in S^1$.
2. The **pure state space** $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.



Curiosities on Toeplitz matrices

Theorem (Carathéodory)

Let T be an $n \times n$ Toeplitz matrix. Then $T \geq 0$ iff $T = V\Delta V^*$ with

$$\Delta = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}; \quad V = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

for some $d_1, \dots, d_n \geq 0$ and $\lambda_1, \dots, \lambda_n \in S^1$.

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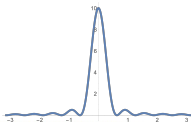
Convergence to the circle

[vS 2021] and [Hekkelman, 2021] study the Gromov–Hausdorff convergence of the state spaces $\mathcal{S}(C(S^1)^{(n)})$ with the distance function d_n to the circle.

- The map $R_n : C(S^1) \rightarrow C(S^1)^{(n)}$ given by compression with P_n allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a **C^1 -approximate order inverse** $S_n : C(S^1)^{(n)} \rightarrow C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \quad S_n(R_n(f)) = F_n * f$$

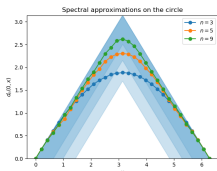
in terms of a Schur product with a matrix T_n and the convolution with the Fejér kernel F_n :



- The fact that S_n is a C^1 -approximate inverse of R_n implies that

$$d_{S^1}(\phi, \psi) - 2\gamma_n \leq d_n(\phi \circ S_n, \psi \circ S_n) \leq d_{S^1}(\phi, \psi)$$

where $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$.



- Using the maps R_n, S_n we can equip $\mathcal{S}(C(S^1)) \amalg \mathcal{S}(C(S^1)^{(n)})$ with a distance function that **bridges** the given distance functions on $\mathcal{S}(C(S^1))$ and $\mathcal{S}(C(S^1)^{(n)})$ within ϵ for large n .

Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(S(C(S^1)^{(n)}), d_n)\}$ converges to $(S(C(S^1)), d_{S^1})$ in Gromov–Hausdorff distance.

More general results on GH-convergence

Definition

Let $\{(\mathcal{E}_n, \mathcal{H}_n, D_n)\}_n$ be a sequence of operator system spectral triples and let $(\mathcal{E}, \mathcal{H}, D)$ be an operator system spectral triple. An **C^1 -approximate order isomorphism** for this set of data is given by linear maps $R_n : E \rightarrow E_n$ and $S_n : E_n \rightarrow E$ for any n such that the following three conditions hold:

1. R_n, S_n are positive, unital, contractive and Lipschitz-contractive
2. there exist sequences γ_n, γ'_n both converging to zero such that

$$\begin{aligned}\|S_n \circ R_n(a) - a\| &\leq \gamma_n \|[D, a]\|, \\ \|R_n \circ S_n(h) - h\| &\leq \gamma'_n \|[D_n, h]\|.\end{aligned}$$

Examples: Toeplitz and Fejér–Riesz [vS 2021], cubic truncations of \mathbb{T}^d [Berendschot 2019], fuzzy spheres [Rieffel 2000], quantum fuzzy spheres [Aguilar–Kaad–Kyed, 2021], Fourier truncations [Rieffel 2022], spectral truncations of tori [Leimbach 2022], ...

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Non-unital operator systems

Consider a matrix-ordered operator space $(E, \|\cdot\|)$.

- The noncommutative (nc) state space is defined for any n as

$$\mathcal{S}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| = 1, \phi \geq 0\}$$

not always convex
nor weakly *-compact

- The nc quasi-state space is defined for any n as

$$\tilde{\mathcal{S}}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| \leq 1, \phi \geq 0\}$$

convex
and weakly *-compact

- The modified numerical radius $\nu_E : M_n(E) \rightarrow \mathbb{C}$ is defined as

$$\nu_E(x) = \sup \left\{ \left| \phi \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \right| : \phi \in \tilde{\mathcal{S}}_{2n}(E) \right\}.$$

Definition

A non-unital operator system is given by a matrix-ordered operator space for which $\nu_E(\cdot) = \|\cdot\|$.

Approximate order units

We now consider a particular class of non-unital operator systems.

Definition (Ng 1969)

Let E be a matrix-ordered $*$ -vector space. An *approximate order unit* for E is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ with the following properties

1. $e_\lambda \in E_+$ for all $\lambda \in \Lambda$ and $e_\lambda \leq e_\mu$ whenever $\lambda \leq \mu$;
2. for each $x \in E_h$ there exists a positive real number t and $\lambda \in \Lambda$ such that

$$-te_\lambda \leq x \leq te_\lambda.$$

In fact, if the approximate order unit is *norm defining* in the sense that

$$\|x\| = \inf \left\{ t : \begin{pmatrix} te_\lambda^n & x \\ x^* & te_\lambda^n \end{pmatrix} \in M_{2n}(E)_+ \text{ for some } \lambda \in \Lambda \right\}$$

then E is a *non-unital operator system* [Karn 2005, Han 2010].

Assuming the existence of a **norm-defining approximate order unit** in E we may show familiar C^* -results such as

1. the nc state space $S_n(E)$ is **convex**

and if $E \subseteq A$ with a norm-defining approximate order unit **for A contained in E** we have that

2. any (pure) **state on E can be extended** to a (pure) state on A .
3. (**Jordan decomposition**) For each hermitian continuous linear functional $\phi : M_n(E) \rightarrow \mathbb{C}$ there exist positive linear functionals $\phi_+, \phi_- : M_n(E) \rightarrow \mathbb{C}$ such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$
4. we have an isometrical order isomorphism

$$M_n(A)_h^* / M_n(E)_h^\perp \rightarrow M_n(E)_h^*$$

This also applies if we replace E and A by dense subspaces \mathcal{E} and \mathcal{A} .

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Operator systems, groupoids and bonds

Definition (Connes-vS, 2021)

A *bond* is a triple (G, ν, Ω) consisting of a locally compact groupoid G , a Haar system $\nu = \{\nu_x\}$ and an open symmetric subset $\Omega \subseteq G$ containing the units $G^{(0)}$.

Proposition

Let (Ω, G, ν) be a bond. The closure of the subspace $C_c(\Omega) \subseteq C_c(G)$ in the C^* -algebra $C^*(G)$ is an operator system.

Example

1. Consider $\Omega_n = (-n, n) \subset \mathbb{Z} \rightsquigarrow$ *Fejér–Riesz operator system* inside $C^*(\mathbb{Z})$.
2. Consider $\Omega_n = (-n, n) \subset C_m$ (so modulo m). The operator system consists of *banded $m \times m$ circulant matrices* of band width n .
3. Given the set $X = \{1, \dots, m\}$ consider a “band” $R_n \subset X \times X$ around the diagonal of width $n \rightsquigarrow$ *banded $m \times m$ matrices* of band width n .

Operator systems associated to tolerance relations

- **Key motivating example:** a metric space (X, d) with the relation

$$\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$$

- If (X, μ) is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the **operator system** $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in \mathcal{R}_ϵ . Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$
- Closely related to the **Roe algebra** $C^*(X)$ associated to the coarse metric structure of (X, d) :

$$E(\mathcal{R}_\epsilon) \subseteq C^*(X)_\epsilon$$

with $C^*(X)_\epsilon$ the closure of locally compact operators with propagation $\leq \epsilon$.

Finite partial partitions of a metric measure space

A **finite partial ϵ -partition** of X is a finite collection $P = \{U_i\}$ of disjoint measurable sets $U \subseteq X$ such that $\text{diam}(U_i) < \epsilon$; directed by refinement.

- The corresponding finite-dimensional algebra \mathcal{A}_P with unit e_P is

$$\mathcal{A}_P = \left\{ \sum_{U, V \in P} a_{UV} |1_U\rangle\langle 1_V| : a_{UV} \in \mathbb{C} \right\} \cong \mathcal{K}(l^2(P))$$

- A **tolerance relation** \mathcal{R}_ϵ^P on the finite set P is given by

$$\mathcal{R}_\epsilon^P = \{U \times V \mid U, V \in P, \text{diam}(U \times V) < \epsilon\} \subseteq P \times P$$

and yields the (unital) **operator system** $E(\mathcal{R}_\epsilon^P) \subseteq \mathcal{A}_P$.

- If $P \leq P'$ then $E(\mathcal{R}_\epsilon^P) \subseteq E(\mathcal{R}_\epsilon^{P'})$ and also $\mathcal{A}_P \subseteq \mathcal{A}_{P'}$.
- Approximate order unit** $\{e_P\}_P$ of $\varinjlim \mathcal{A}_P$ is contained in $\varinjlim E(\mathcal{R}_\epsilon^P)$

Spaces at finite resolution [Connes-vS, 2021]

Proposition

Let X be a path metric measure space with a measure of full support.

1. $\mathcal{E}(\mathcal{R}_\epsilon) := \varinjlim E(\mathcal{R}_\epsilon^P)$ is a dense subspace of $E(\mathcal{R}_\epsilon)$
2. $\mathcal{A}_\epsilon := \varinjlim \mathcal{A}_P$ is a dense $*$ -subalgebra of the C^* -algebra $\mathcal{K}(L^2(X))$;
3. there exists a *norm-defining approximate order unit* for \mathcal{A}_ϵ which is contained in $\mathcal{E}(\mathcal{R}_\epsilon)$.

Proposition

Let X be a complete, locally compact path metric measure space with a measure of full support. Then $C_{\text{env}}^*(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$ and

$$\text{prop}(E(\mathcal{R}_\epsilon)) = \lceil \text{diam}(X)/\epsilon \rceil$$

The **pure states** of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is ϵ -connected.