# Tolerance relations and operator systems

Walter van Suijlekom

Radboud University

# Spectral triples

We consider a triple $(A, \mathcal{H}, D)$

- a unital $C^*$-algebra $A$
- a self-adjoint operator $D$ with compact resolvent and bounded commutators $[D, a]$ for $a \in \mathcal{A} \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:

- States are positive linear functionals $\phi : A \to \mathbb{C}$ of norm 1
- Pure states are extreme points of state space
- Distance function on state space of $A$:

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}$$

# Spectral data

- The mathematical reformulation of geometry in terms of spectral data (global analysis) requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.
- We aim for the underlying mathematical formalism for doing (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D'Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, Szeged)

# Towards operator systems..

(I) Given $(A, \mathcal{H}, D)$ we project onto part of the spectrum of $D$:

- $\mathcal{H} \mapsto P\mathcal{H}$, projection onto closed Hilbert subspace
- $D \mapsto PDP$, still a self-adjoint operator
- $A \mapsto PAP$, this is not an algebra any more (unless $P \in A$)

Instead, $PAP$ is an operator system: $(PaP)^* = Pa^*P$.

(II) Another approach would be to consider metric spaces up to a finite resolution :

- Consider integral operators associated to the tolerance relation $R_\epsilon$ given by $d(x, y) < \epsilon$

So first, some background on operator systems.

# Operator systems

***Definition (Arveson 1969, Choi-Effros 1977)***
*An operator system is a $*$-closed vector space $E$ of bounded operators.*
*Unital: it contains the identity operator.*

- $E$ is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
$$\langle \psi, T\psi \rangle \geq 0; \qquad (\psi \in \mathcal{H}).$$
- in fact, $E$ is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any $n$.

Maps between operator systems $E, F$ are completely positive maps in the sense that their extensions $M_n(E) \to M_n(F)$ are positive for all $n$.

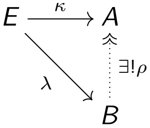
Isomorphisms are complete order isomorphisms

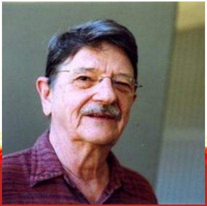
# $C^*$-envelope of a unital operator system

Arveson introduced the notion of $C^*$-envelope for unital operator systems in 1969, Hamana established existence and uniqueness in 1979.
Non-unital case: [Connes-vS 2020], [Kennedy–Kim–Manor 2021]

A $C^*$-extension $\kappa : E \to A$ of a unital operator system $E$ is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$.

A $C^*$-envelope of a unital operator system is a $C^*$-extension $\kappa : E \to A$ with the following universal property:



Example: operator system $C_{\text{harm}}(\overline{\mathbb{D}})$ of continuous harmonic functions with $C^*$-envelope $C(S^1)$.

## State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on $E$ as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual $E^d$ of a unital operator system is a unital operator system with

$$E^d_+ = \left\{\phi \in E^d : \phi(T) \geq 0, \forall T \in E_+\right\}$$

  and similarly for the matrix order (*cf.* work by Ng and Jia–Ng).
- It follows that we have the following useful correspondence:

$$\text{pure states on } E \longleftrightarrow \text{extreme rays in } (E^d)_+$$

  and the other way around.

In the infinite-dimensional/non-unital case, this is more subtle (more later)...

## Spectral truncation of the circle [Connes-vS, 2020]

- Eigenvectors of $D_{S^1}$ are Fourier modes $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- Orthogonal projection $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an operator system
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a Toeplitz matrix

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C^*_{\mathrm{env}}(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$

# Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_{(n)}$:

- functions on $S^1$ with a finite number of non-zero Fourier coefficients:

$$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element $a$ is positive iff $\sum_k a_k e^{ikx}$ is a positive function on $S^1$.
- The $C^*$-envelope of $C^*(\mathbb{Z})_{(n)}$ is given by $C^*(\mathbb{Z})$

***Proposition***

1. *The extreme rays in $(C^*(\mathbb{Z})_{(n)})_+$ are given by the elements $a = (a_k)$ for which the Laurent series $\sum_k a_k z^k$ has all its zeroes on $S^1$.*
2. *The pure states of $C^*(\mathbb{Z})_{(n)}$ are given by $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).*

# Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$
$$(T = (t_{k-l})_{k,l}, a = (a_k)) \mapsto \sum_k a_k t_{-k}$$

***Proposition***

1. *The extreme rays in* $C(S^1)^{(n)}_+$ *are* $\gamma(\lambda) = |f_\lambda\rangle\langle f_\lambda|$ *for any* $\lambda \in S^1$.
2. *The pure state space* $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n/S_n$.

# General results on GH-convergence

***Definition***

*Let $\{(\mathcal{E}_n, \mathcal{H}_n, D_n)\}_n$ be a sequence of operator system spectral triples and let $(\mathcal{E}, \mathcal{H}, D)$ be an operator system spectral triple. An $C^1$-approximate order isomorphism for this set of data is given by linear maps $R_n : E \to E_n$ and $S_n : E_n \to E$ for any $n$ such that the following three condition hold:*

1. *$R_n, S_n$ are positive, unital, contractive and Lipschitz-contractive*
2. *there exist sequences $\gamma_n, \gamma'_n$ both converging to zero such that*

$$\|S_n \circ R_n(a) - a\| \leq \gamma_n \|[D, a]\|,$$
$$\|R_n \circ S_n(h) - h\| \leq \gamma'_n \|[D_n, h]\|.$$

***Theorem***

*If $(R_n, S_n)$ is a $C^1$-approximate order isomorphism for $(\mathcal{E}_n, \mathcal{H}_n, D_n)$ and $(\mathcal{E}, \mathcal{H}, D)$, then the state spaces $(\mathcal{S}(E_n), d_{E_n})$ converge to $(\mathcal{S}(E), d_E)$ in Gromov–Hausdorff distance.*

# Spectral truncations and convergence to the circle

- The map $R_n : C(S^1) \to C(S^1)^{(n)}$ given by compression with $P_n$ allows to pull-back states from $C(S^1)^{(n)}$ to the circle
- There is a $C^1$-approximate order inverse $S_n : C(S^1)^{(n)} \to C(S^1)$:

$$R_n(S_n(T)) = T_n \odot T; \qquad S_n(R_n(f)) = F_n * f$$

in terms of a Schur product with a matrix $T_n$ and the convolution with the Fejér kernel $F_n$:

***Proposition (vS21, Hekkelman 2021)***

*The sequence of state spaces* $\{(\mathcal{S}(C(S^1)^{(n)}), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

Other examples: cubic truncations of $\mathbb{T}^d$ [Berendschot 2019], fuzzy spheres [Rieffel 2000], quantum spheres [Aguilar–Kaad–Kyed 2021], Fourier truncations [Rieffel 2022], spectral truncations of $\mathbb{T}^d$ [Leimbach], ...

## Non-unital operator systems

Consider a matrix-ordered operator space $(E, \|\cdot\|)$.

- The noncommutative (nc) state space is defined for any $n$ as

$$\mathcal{S}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| = 1, \phi \geq 0\}$$

not always convex
nor weakly $*$-compact

- The nc quasi-state space is defined for any $n$ as

$$\widetilde{\mathcal{S}}_n(E) := \{\phi \in M_n(E)^*, \|\phi\| \leq 1, \phi \geq 0\}$$

convex
and weakly $*$-compact

- The modified numerical radius $\nu_E : M_n(E) \to \mathbb{C}$ is defined as

$$\nu_E(x) = \sup\left\{\left|\phi\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)\right| : \phi \in \widetilde{\mathcal{S}}_{2n}(E)\right\}.$$

***Definition (Werner)***
*A non-unital operator system is given by a matrix-ordered operator space for which $\nu_E(\cdot) = \|\cdot\|$.*

# Approximate order units

We now consider a particular class of non-unital operator systems.

***Definition (Ng 1969)***

*Let $E$ be a matrix-ordered $*$-vector space. An approximate order unit for $E$ is a net $\{e_\lambda\}_{\lambda\in\Lambda}$ with the following properties*

1. $e_\lambda \in E_+$ *for all* $\lambda \in \Lambda$ *and* $e_\lambda \leq e_\mu$ *whenever* $\lambda \leq \mu$*;*
2. *for each* $x \in E_h$ *there exists a positive real number* $t$ *and* $\lambda \in \Lambda$ *such that*

$$-te_\lambda \leq x \leq te_\lambda.$$

In fact, if the approximate order unit is norm defining in the sense that

$$\|x\| = \inf\left\{t : \begin{pmatrix} te_\lambda^n & x \\ x^* & te_\lambda^n \end{pmatrix} \in M_{2n}(E)_+ \text{ for some } \lambda \in \Lambda\right\}$$

then $E$ is a non-unital operator system [Karn 2005, Han 2010].

Assuming the existence of a norm-defining approximate order unit in $E$ we may show familiar $C^*$-results such as

1. the nc state space $S_n(E)$ is convex

and if $E \subseteq A$ with a norm-defining approximate order unit for $A$ contained in $E$ we have that

2. any (pure) state on $E$ can be extended to a (pure) state on $A$.
3. (Jordan decomposition) For each hermitian continuous linear functional $\phi : E \to \mathbb{C}$ there exist positive linear functionals $\phi_+, \phi_- : E \to \mathbb{C}$ such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$
4. we have an isometrical order isomorphism $\boxed{A_h^*/E_h^\perp \to E_h^*}$

This also applies if we replace $E$ and $A$ by dense subspaces $\mathcal{E}$ and $\mathcal{A}$.

## Operator systems associated to tolerance relations

- Key motivating example: a metric space $(X, d)$ with the relation

$$\boxed{\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}}$$

- If $(X, \mu)$ is a measure space and $\mathcal{R}_\epsilon \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R}_\epsilon)$ as the closure of integral operators with support in $\mathcal{R}_\epsilon$. Note that $E(\mathcal{R}_\epsilon) \subseteq \mathcal{K}(L^2(X))$
- Closely related to the Roe algebra $C^*(X)$ associated to the coarse metric structure of $(X, d)$:

$$E(\mathcal{R}_\epsilon) \subseteq C^*(X)_\epsilon$$

with $C^*(X)_\epsilon$ the closure of locally compact operators with propagation $\leq \epsilon$.

# Finite partial partitions of a metric measure space

A finite partial $\epsilon$-partition of $X$ is a finite collection $P = \{U_i\}$ of disjoint measurable sets $U \subseteq X$ such that $\mathrm{diam}(U_i) < \epsilon$; directed by refinement.

- The corresponding finite-dimensional algebra $\mathcal{A}_P$ with unit $e_P$ is

$$\mathcal{A}_P = \left\{ \sum_{U,V \in P} a_{UV} |1_U\rangle\langle 1_V| : a_{UV} \in \mathbb{C} \right\} \cong \mathcal{K}(l^2(P))$$

- A tolerance relation $\mathcal{R}_\epsilon^P$ on the finite set $P$ is given by

$$\mathcal{R}_\epsilon^P = \{U \times V \mid \quad U, V \in P, \quad U \times V \subseteq \mathcal{R}_\epsilon\} \subseteq P \times P$$

  and yields the (unital) operator system $E(\mathcal{R}_\epsilon^P) \subseteq \mathcal{A}_P$.
- If $P \leq P'$ then $E(\mathcal{R}_\epsilon^P) \subseteq E(\mathcal{R}_\epsilon^{P'})$ and also $\mathcal{A}_P \subseteq \mathcal{A}_{P'}$.
- Approximate order unit $\{e_P\}_P$ of $\varinjlim A_P$ is contained in $\varinjlim E(\mathcal{R}_\epsilon^P)$

# Spaces at finite resolution [Connes-vS, 2021]

***Proposition***

*Let $X$ be a path metric measure space with a measure of full support.*

1. $\mathcal{E}(\mathcal{R}_\epsilon) := \varinjlim E(\mathcal{R}_\epsilon^P)$ *is a dense subspace of* $E(\mathcal{R}_\epsilon)$
2. $\mathcal{A}_\epsilon := \varinjlim \mathcal{A}_P$ *is a dense* $*$*-subalgebra of the* $C^*$*-algebra* $\mathcal{K}(L^2(X))$*;*
3. *there exists a norm-defining approximate order unit for* $\mathcal{A}_\epsilon$ *which is contained in* $\mathcal{E}(\mathcal{R}_\epsilon)$*.*

***Proposition***

*Let $X$ be a complete, locally compact path metric measure space with a measure of full support. Then* $C^*_{env}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$*.*
*The pure states of* $E(\mathcal{R}_\epsilon)$ *are given by vector states* $|\psi\rangle\langle\psi|$ *where the essential support of* $\psi \in L^2(X)$ *is* $\epsilon$*-connected.*