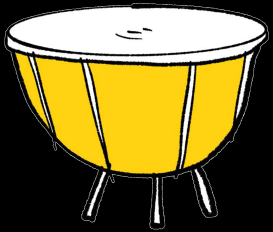
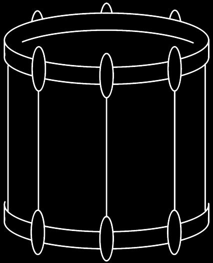


Geometry emerging from spectra

Walter van Suijlekom

Mark Kac in 1966

"Can one hear the shape of a drum"



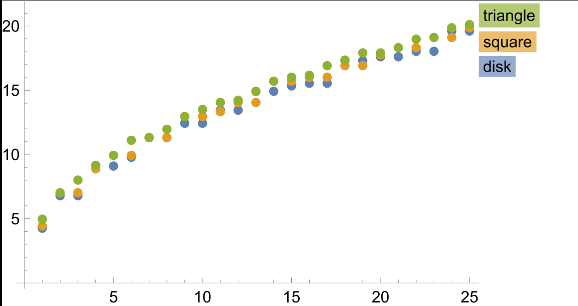
Lorentz in October 1910

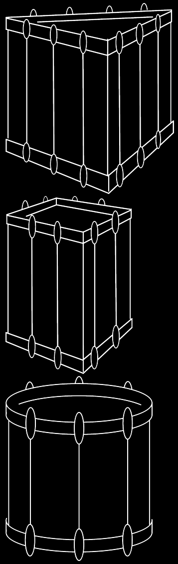
H.A. Lorentz by Jan Veth

Origins of spectral geometry:

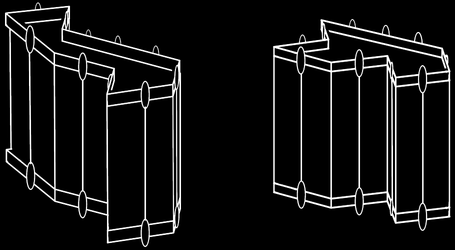
the high overtones behave inversely proportional to the volume.

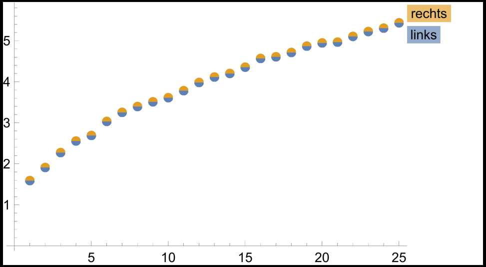
Weyl in February 1911





Isospectral drums!





… so the answer to Kac's question is **no**
and more information is needed…

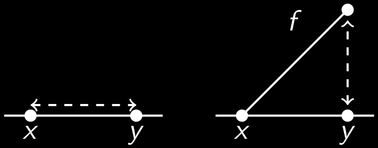
Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

- Distance $d(x, y)$ between two points is usually defined as
  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

- But it can also be defined as
  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$\boxed{d(x, y) = \sup_{\|[D_M, f]\| \leq 1} |\delta_x(f) - \delta_y(f)|}$$



Combination $(C(M), L^2(S_M), D_M)$ allows for reconstruction of geometry

Spectral truncations

joint with Connes

- More realistically, one should consider states on an **approximation** by projecting onto a frequency range around λ with bandwidth Λ:

- Our distance function still makes sense for (pure) states on such spectral truncations $P_\Lambda C(X) P_\Lambda$:

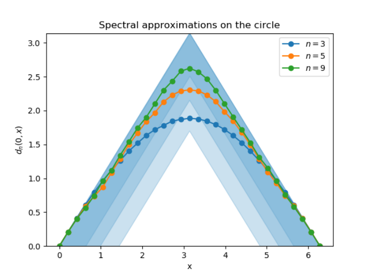
$$\boxed{d(\phi, \psi) = \sup_{\|[D,T]\| \leq 1} |\phi(T) - \psi(T)|}; \qquad T = P_\Lambda f P_\Lambda$$

Example: spectral truncation of the circle

- Eigenvectors of D_{S^1} are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** $P = P_n$ onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $C(S^1)^{(n)} := PC(S^1)P$ is an **operator system**
- Any $T = PfP$ in $C(S^1)^{(n)}$ can be written as a **Toeplitz matrix**

$$PfP \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

Distance function for spectral truncations of the circle



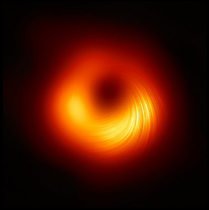
Proposition (vS21, Hekkelman 2021)

The sequence of state spaces $\{(\mathcal{S}(P_nC(S^1)P_n), d_n)\}$ *converges to* $(\mathcal{S}(C(S^1)), d_{S^1})$ *in Gromov–Hausdorff distance.*

And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori (Leimbach–vS23, RU) ...

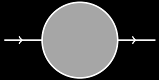
Cosmic emergence: from abstract simplicity to complex diversity

- these systems are **finite-dimensional** matrix systems (Toeplitz, band, etc.).
- fits conceptually with the idea that essentially all observations of geometry in (astro)physics are **spectral**



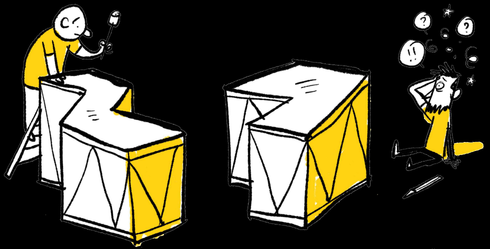
- more concretely: distance precision measurements (LIGO, ET) based on

$$\widetilde{D} = D -$$



Main challenges

1. develop mathematical techniques for finite-dimensional approximations of noncommutative (topological/metric) spaces syn. with UL, TUD
2. define (persistent) invariants of such approximations syn. with VU, UL
3. analyze ensembles of (fin-dim) spectral geometries syn. with prob.theory, THEP



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