Radboud University Nijmegen

Faculty of Science

Cyclic Cocycles in the Spectral Action on Manifolds

Master Thesis in Mathematics

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2nd October 2023

Acknowledgements

In de eerste plaats wil ik mijn begeleider Walter van Suijlekom bedanken voor de supervisie van mijn scriptie. De vele gesprekken die ik met hem heb gevoerd hebben me op wiskundig vlak doen groeien en hebben bovendien het moreel hooggehouden. Daarnaast wil ik Teun van Nuland bedanken voor zijn interesse in en het meedenken over deze scriptie.

Ten slotte wil ik mijn 'partners in crime' bedanken, de studenten met wie ik een jaar lang in de masterkamer heb gezeten. Daan, Pepijn, Thijs, Safiere, Robin en Roy: bedankt voor de vele potjes tafeltennis, humor, en serieuze en minder serieuze gesprekken. Om die redenen mag ook Jaspers naam hier niet ontbreken, die regelmatig op bezoek kwam in de masterkamer om de voedselreserves aan te vullen.

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Chapter I

Introduction

In Yang–Mills theories, such as quantum field theory [21], one considers Lagrangians of the form $\mathcal{L} = \operatorname{tr} F^2$, with F the field strength tensor. In mathematical terms, such theories are described by a principal G-bundle $P \to M$, where G is a Lie group with Lie algebra $\mathfrak{g}$, and a choice of a connection one-form $\omega \in \Omega^1(P, \mathfrak{g})$. The curvature of ω is defined as

$$F \coloneqq \mathrm{d}\omega + \omega \wedge \omega \in \Omega^2(P, \mathfrak{g}). \tag{I.1}$$

Under a gauge transformation, the connection one-form (locally) transforms as [26]

$$\omega \mapsto g^{-1}\omega g + g^{-1}\,\mathrm{d}g, \tag{I.2}$$

where g is a map $U \to G$, with $U \subset M$. From this it follows that the curvature transforms as

$$F \mapsto g^{-1}Fg. \tag{I.3}$$

It follows from the cyclic property of the trace that the Yang–Mills action $\int_M \operatorname{tr} F^k$ is gauge-invariant, for all $k \in \mathbb{N}$. We wish to extend these notions from principal bundles to associative $*$-algebras.

To do this, one first needs to define a differential structure that arises from the algebra $\mathcal{A}$. This structure is called the *universal differential graded algebra* of $\mathcal{A}$, and is denoted by $(\Omega^\bullet\mathcal{A}, \mathrm{d})$. Here $\mathrm{d}\colon \Omega^k\mathcal{A} \to \Omega^{k+1}\mathcal{A}$ is an odd derivation, i.e. satisfies

$$\mathrm{d}(\omega_k\eta) = (\mathrm{d}\omega_k)\eta + (-1)^k\omega_k\,\mathrm{d}\eta,$$

for $\omega_k \in \Omega^k\mathcal{A}$ and $\eta \in \Omega^\bullet\mathcal{A}$. As substitution for the connection one-form on P, one takes a universal one-form $A \in \Omega^1\mathcal{A}$. Inspired by Eq. (I.1), one sets

$$F \coloneqq \mathrm{d}A + A^2 \in \Omega^2\mathcal{A}.$$

It will be shown later that A and F satisfy transformation laws similar to Eq. (I.2) and Eq. (I.3), respectively.

It now remains to find a suitable replacement for the Yang–Mills action on principal bundles. For this, we will for now consider linear maps

$$\phi\colon \Omega^k\mathcal{A} \to \mathbb{C};$$

in Chapter II, we will define such maps in slightly more generality. The space of such maps will be denoted $C^k(\mathcal{A})$. Between these spaces, we can define a coboundary map

$$b\colon C^k(\mathcal{A}) \to C^{k+1}(\mathcal{A}).$$

Maps $\phi \in C^{2k}(\mathcal{A})$ such that $b\phi = 0$ will be called *Hochschild cocycles.* A Hochschild cocycle $\phi \in C^{2k}(\mathcal{A})$ provides a good candidate for an action functional analogous to the Yang–Mills action functional on principal bundles, as it satisfies

$$\phi(uF^k u^*) = \phi(F^k),$$

for $u \in \mathcal{A}$ unitary.

However, the condition $b\phi = 0$ leaves us with a great many options for which ϕ to choose. The spectral action principle [4] provides a method to make them appear naturally. When the $*$-algebra is part of a spectral triple $(\mathcal{A}, H, D)$, one can define the spectral action

$$S(D) = \operatorname{tr} f(D/\Lambda),$$

that only depends on the spectrum of the Dirac operator D. Here f is a suitably nice function, needed to ensure finiteness of the action, and Λ is a cut-off parameter. In fact, if one chooses the spectral triple to be almost finite (i.e. the product of a canonical spectral triple on a Riemannian manifold and a finite spectral triple), one recovers the Yang–Mills action, coupled to Einstein plus Weyl gravity [4].

In [27], Van Nuland and Van Suijlekom have shown that under some conditions on the spectral triple and on f, the spectral action can be expanded as

$$\operatorname{tr}(f(D+V)) - \operatorname{tr}(f(D)) = \sum_{k=1}^{\infty} \left(\psi_{2k-1}(\mathrm{cs}_{2k-1}(A)) + \frac{1}{2k}\phi_{2k}(F^k) \right), \tag{I.4}$$

for $V = \sum_i a_i[D, b_i]$ a self-adjoint operator, with $a_i, b_i \in \mathcal{A}$. In the expansion (I.4), (ψ_{2k-1}) is an entire cyclic cocycle (up to a normalisation factor), and $\mathrm{cs}_{2k-1}(A)$ is a generalised Chern–Simons form; both will be defined later. Most importantly for now, the maps ϕ_{2k} are Hochschild cocycles.

The main goal of this thesis is to describe the maps ϕ_{2k} and ψ_{2k-1} for the case where $(\mathcal{A}, H, D) = (C^\infty(\mathbb{T}^d), L^2(\mathcal{S}_{\mathbb{T}^d}), D_{\mathbb{T}^d})$ is the canonical spectral triple on the d-dimensional torus $\mathbb{T}^d$. What we find, then, is that they are non-local expressions in the $(2k)$-forms $a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{2k}$, in the sense that they depend on all derivatives of each of the a_i.

I.1 Outline

In Chapter II, we introduce a total of four cohomology theories, but we start by defining the universal differential graded algebra $\Omega^\bullet\mathcal{A}$ of a unital algebra $\mathcal{A}$. Then, we introduce Hochschild cohomology. The Hochschild cocycles that are invariant under the action of the *cyclic permuter*, to be introduced in Section II.3 form a subcomplex of the Hochschild complex. The cohomology of this subcomplex is called *cyclic cohomology*, and is introduced in Section II.3. Moreover, we show that elements of the cyclic subcomplex, called cyclic cocycles, appear quite naturally as the maps that define an action in *odd* degrees, analogous to how Hochschild cocycles define an action in even degrees. For this, we also introduce generalised Chern–Simons forms. We then define *periodic cyclic cohomology* as a direct limit of even/odd cyclic cohomology groups. Finally, we define entire cyclic cohomology, which also takes topological aspects of the algebra into account.

Chapter III starts with a recap of spectral triples, and the canonical spectral triple of a compact manifold. Then, we introduce the spectral action, and sketch proofs of a few of the results of [27], to arrive at the expansion of the spectral action of Eq. (I.4).

In Chapter IV we return to the cohomology theories of Chapter II. We sketch a proof of the Connes–Hochschild–Konstant–Rosenberg Theorem [6], which states that the continuous Hochschild cohomology of degree k of $C^\infty(M)$ is isomorphic to the space of k-currents on M, which are continuous linear maps $\Omega^k(M) \to \mathbb{C}$. Then, we explicitly show how this follows from the condition $b\phi = 0$ for the case $M = \mathbb{T}^2$, under the assumption that ϕ is skewsymmetric. We finish the Chapter with a theorem and proof thereof that relates the periodic cyclic cohomology of $C^\infty(M)$ to the de Rham homology.

In Chapter V we briefly summarise the definition of the (operator-theoretic) K_1-functor. Then, in Section V.2, we prove that there is a pairing of the cyclic cohomology of $C^\infty(M)$ and $K_1(C(M))$. We then use this pairing to show that in case $M = \mathbb{T}^1$ is the unit circle, the class of the entire cyclic cocycle $(\tilde{\psi}_{2k-1})$ (which is a normalisation of the cochain (ψ_{2k-1}) that appears in Eq. (I.4)) is trivial.

Finally, in Chapter VI, we compute the maps ϕ_{2k} and ψ_{2k-1} on the d-dimensional torus. For this, we first describe the spectral triple $(C^\infty(\mathbb{T}^d), L^2(\mathcal{S}_{\mathbb{T}^d}), D_{\mathbb{T}^d})$. Then, in Section VI.2, we compute the multiple operator integrals for this spectral triple, which appear in the definition of the ϕ_{2k} and ψ_{2k-1}. From these computations, the expressions for ϕ_{2k} and ψ_{2k-1} follow quickly, and they are given in Corollaries VI.5 and VI.6.

Chapter II

Cohomology theories

Let M be a manifold, and consider the de Rham complex $(\Omega^\bullet(M), \mathrm{d}_{\mathrm{dR}})$. The exterior derivative of the product of two forms $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ satisfies

$$\mathrm{d}_{\mathrm{dR}}(\omega\eta) = (\mathrm{d}_{\mathrm{dR}}\omega)\eta + (-1)^k \omega\, \mathrm{d}_{\mathrm{dR}}\eta.$$

In Section II.1 we will generalise this notion and see how we can define a differential d on any unital algebra $\mathcal{A}$, forming a *universal differential graded algebra* $\Omega^\bullet\mathcal{A}$.

Let $C_k(\mathcal{A}) \coloneqq (\mathcal{A}^{\otimes(k+1)})^*$ be the space of linear maps $\mathcal{A}^{\otimes(k+1)} \to \mathbb{C}$, called *Hochschild k-cochains*. Demanding that an even Hochschild cochain satisfy a gauge invariance-like property, naturally gives rise to a coboundary map $b\colon C_k(\mathcal{A}) \to C_{k+1}(\mathcal{A})$; the cohomology of the complex $(C^\bullet(\mathcal{A}), b)$ is called *Hochschild cohomology*, and is explained in Section II.2. Moreover, every class in Hochschild cohomology can be represented by a map $\Omega^\bullet\mathcal{A} \to \mathbb{C}$. The subsequent Sections of this Chapter describe other cohomology theories, to wit cyclic cohomology, periodic cyclic cohomology and entire cyclic cohomology. These theories are defined by putting further restrictions on the Hochschild cochains, or on sequences thereof. We base this chapter on [2], to which we also refer for more details.

In Chapter III, we will rewrite the spectral action in terms of Hochschild cocycles, entire cyclic cocycles and universal differential forms.

II.1 Universal forms

Definition II.1. A *graded differential algebra* is a pair $(R^\bullet, \delta)$, where $R^\bullet = \oplus_{k=0}^\infty R^k$ is a complex associative algebra whose product is graded, and where $\delta\colon R^\bullet \to R^\bullet$ is a *differential*, i.e. a linear map of degree +1 such that $\delta^2 = 0$ and δ is an *odd derivation*:

$$\delta(\omega_k\eta) = (\delta\omega_k)\eta + (-1)^k\omega_k\delta\eta,$$

with $\omega_k \in R^k$ and $\eta \in R^\bullet$.

Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra, and consider the graded differential algebra $(\Omega^\bullet\mathcal{A}, \mathrm{d})$, defined as follows. Set $\overline{\mathcal{A}} \coloneqq \mathcal{A}/\mathbb{C}$, and define $\Omega^k\mathcal{A} \coloneqq \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes k}$. Denote the class in $\overline{\mathcal{A}}$ of an element $a \in \mathcal{A}$ by $\overline{a}$, and define

$$\begin{aligned} \mathrm{d}\colon \Omega^k\mathcal{A} &\to \Omega^{k+1}\mathcal{A}, \\ a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_k &\mapsto 1 \otimes \overline{a}_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_k. \end{aligned}$$

We define the product $\Omega^k\mathcal{A} \times \Omega^l\mathcal{A} \to \Omega^{k+l}\mathcal{A}$ in such a way that it forces d to be a derivation: for left multiplication by $a' \in \mathcal{A}$, we set inductively

$$a'(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k}) \coloneqq (a'a_0) \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k}. \tag{II.1}$$

For right multiplication, we set

$$\begin{aligned} (a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})a' &\coloneqq a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{(a_k a')} - (a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{k-1}})a_k \otimes \overline{a'} \\ &= (-1)^k a_0 a_1 \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'} \\ &\quad + \sum_{j=1}^{k-1} (-1)^{k-j} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_j a_{j+1}} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'} \\ &\quad + a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{k-1}} \otimes \overline{a_k a'}. \end{aligned} \tag{II.2}$$

Finally, we define

$$(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})(b_0 \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_l}) \coloneqq ((a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})b_0) \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_l}.$$

With this algebra structure on $\Omega^\bullet\mathcal{A}$, d is indeed an odd derivation. For example,

$$\begin{aligned} \mathrm{d}((a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})a') &= \sum_{j=0}^{k-1} (-1)^{k-j} 1 \otimes \overline{a_0} \otimes \cdots \otimes \overline{a_j a_{j+1}} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'} \\ &\quad + a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{k-1}} \otimes \overline{a_k a'}, \end{aligned}$$

which agrees with

$$\begin{aligned} &\mathrm{d}(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})a' + (-1)^k a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k})\,\mathrm{d}a' \\ &\quad = (-1)^{k+1} a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'} + \sum_{j=0}^{k-1} (-1)^{k-j} 1 \otimes \overline{a_0} \otimes \cdots \otimes \overline{a_j a_{j+1}} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'} \\ &\qquad + 1 \otimes \overline{a_0} \otimes \cdots \otimes \overline{a_{k-1}} \otimes \overline{a_k a'} + (-1)^k a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a'}, \end{aligned}$$

where we have used that $\overline{1} = \overline{0}$.

Note that $a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_k = a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_k$. This justifies the following Definition.

Definition II.2. We call $\Omega^\bullet\mathcal{A}$ the *universal graded differential algebra* of $\mathcal{A}$. An element $\omega_k \in \Omega^k\mathcal{A}$ is called a *universal k-form.*

The pair $(\Omega^\bullet, \mathrm{d})$ is universal in the following sense: given a graded differential algebra $(R^\bullet, \delta)$ and an algebra homomorphism $\psi\colon \mathcal{A} \to R^0$, there exists a lift of ψ, also denoted by ψ, intertwining the differentials:

$$\begin{array}{ccc} \Omega^k\mathcal{A} & \xrightarrow{\mathrm{d}} & \Omega^{k+1}\mathcal{A} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ R^k & \xrightarrow{\delta} & R^{k+1}. \end{array}$$

II.2 Hochschild cohomology

In Chapter I, we have seen how a principal bundle gives rise to a gauge-invariant functional on the space of connection one-forms. We want to extend these notions to the context of universal forms. Thus, let $\mathcal{A}$ be a unital algebra. Consider a universal 1-form $A \in \Omega^1\mathcal{A}$, and define its associated curvature as $F = \mathrm{d}A + A^2$. Under a transformation by a unitary $u \in \mathcal{A}$, we get

$$A \mapsto A_u := u(\mathrm{d} + A)u^* = u\,\mathrm{d}u^* + uAu^*. \tag{II.3}$$

It follows that $F_u = uFu^*$.

Now consider a linear functional ϕ on the space of universal $2k$-forms $\Omega^{2k}\mathcal{A}$. By analogy with the Yang-Mills functional on principal bundles, we want the action functional $A \mapsto \int_\phi F^k := \phi(F^k)$ to be gauge invariant, i.e. ϕ must satisfy

$$\int_\phi F^k = \int_\phi uF^k u^*. \tag{II.4}$$

A sufficient condition is that $\phi([\Omega^\bullet\mathcal{A}, \mathcal{A}]) = 0$. This is formalised in the notion of *Hochschild (co)cycles*, which we explain next. Although it is not immediately obvious, we will see towards the end of this Section how an even Hochschild cocycle induces a gauge-invariant functional on $\Omega^\bullet\mathcal{A}$.

Definition II.3. Let $\mathcal{A}$ be a unital algebra. The *Hochschild chain complex* $(C_\bullet(\mathcal{A}), b)$ is the complex of algebras *Hochschild k-chains* $C_k(\mathcal{A}) := \mathcal{A}^{\otimes(k+1)}$, with boundary maps $b\colon C_k(\mathcal{A}) \to C_{k-1}(\mathcal{A})$ given by

$$b(a_0 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1}(-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k + (-1)^k a_k a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

The homology of this complex is called *Hochschild homology.* We may dualise this chain complex to obtain the complex $(C^\bullet(\mathcal{A}), b)$ of *Hochschild cochains*, i.e. linear maps

$$\phi\colon \mathcal{A}^{\otimes(k+1)} \to \mathbb{C}.$$

The cohomology of this complex is called *Hochschild cohomology*, and is denoted by $H^\bullet(\mathcal{A}, \mathcal{A}^*)$ or $HH^\bullet(\mathcal{A})$.

To show that $b\colon C_k(\mathcal{A}) \to C_{k-1}(\mathcal{A})$ is indeed a boundary map, write it as $b = \sum_{i=0}^k (-1)^i \delta_i$, with

$$\delta_i(a_0 \otimes \cdots \otimes a_k) := \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k & \text{for } i = 0, \dots, k-1 \\ a_k a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} & \text{for } i = k. \end{cases}$$

One checks that $\delta_j\delta_i = \delta_i\delta_{j-1}$ if $i < j$, so that $b^2 = 0$ [19, Lemma 1.1.2].

Since the space of multilinear maps $\mathcal{A}^{k+1} \to \mathbb{C}$ is canonically bijective to the space of linear maps $\mathcal{A}^{\otimes(k+1)} \to \mathbb{C}$, we will frequently view a Hochshild cochain $\phi\colon \mathcal{A}^{\otimes(k+1)} \to \mathbb{C}$ as a multilinear map $\mathcal{A}^{k+1} \to \mathbb{C}$.

The following Theorem provides an alternative way to compute the Hochshild cohomology groups. It uses the the Ext-functor, which we briefly recall.

Let $\mathcal{A}$ be an algebra, with opposite algebra $\mathcal{A}^o$, and let $\mathcal{M}$ be a left $\mathcal{A} \otimes \mathcal{A}^o$-module. Let

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{P}_1 \longleftarrow \mathcal{P}_2 \longleftarrow \cdots \tag{II.5}$$

be a projective resolution of $\mathcal{M}$ as left $\mathcal{A}\otimes\mathcal{A}^o$-module. This means that the sequence (II.5) is exact, and that each module $\mathcal{P}_n$ is projective. Let $\mathcal{N}$ be another left $\mathcal{A} \otimes \mathcal{A}^o$-module. Then

$$\operatorname{Ext}^n_{\mathcal{A}\otimes\mathcal{A}^o}(\mathcal{M}, \mathcal{N}) \coloneqq \frac{\ker\left(\operatorname{Hom}(\mathcal{P}_n, \mathcal{N}) \to \operatorname{Hom}(\mathcal{P}_{n+1}, \mathcal{N})\right)}{\operatorname{im}\left(\operatorname{Hom}(\mathcal{P}_{n-1}, \mathcal{N}) \to \operatorname{Hom}(\mathcal{P}_n, \mathcal{N})\right)}$$

One can show that this definition of Ext is independent of the choice of projective resolution, up to isomorphism [1, Section 6.4].

Theorem II.4. *Let $\mathcal{A}$ be a unital algebra, and let $HH^\bullet(\mathcal{A})$ be its Hochschild cohomology groups. Then $HH^\bullet(\mathcal{A}) \cong \operatorname{Ext}^\bullet_{\mathcal{A}\otimes\mathcal{A}^o}(\mathcal{A}, \mathcal{A}^*)$, where we view $\mathcal{A}$ as a left $\mathcal{A} \otimes \mathcal{A}^o$-module via $(a \otimes b)c = acb$.*

Proof. Define the free left $\mathcal{A} \otimes \mathcal{A}^o$-modules $B_k \coloneqq \mathcal{A} \otimes \mathcal{A}^o \otimes \mathcal{A}^{\otimes k}$ generated by $\mathcal{A}^{\otimes k}$, and consider the *bar resolution*

$$0 \longleftarrow \mathcal{A} \xleftarrow[b']{} B_1(\mathcal{A}) \xleftarrow[b']{} B_2(\mathcal{A}) \xleftarrow[b']{} \cdots,$$

with boundary map

$$\begin{aligned} b'(x \otimes y \otimes a_1 \otimes \cdots \otimes a_k) \coloneqq{}& xa_1 \otimes y \otimes a_2 \otimes \cdots \otimes a_k \\ &+ \sum_{i=1}^{k-1} (-1)^i (x \otimes y \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k) \\ &+ (-1)^k (x \otimes a_k y \otimes a_1 \otimes \cdots \otimes a_{k-1}). \end{aligned}$$

One can show that $(B_\bullet(\mathcal{A}), b')$ constitutes a free resolution of $\mathcal{A}$ as a left $\mathcal{A} \otimes \mathcal{A}^o$-module. The isomorphism of cochain complexes

$$\operatorname{Hom}_{\mathcal{A}\otimes\mathcal{A}^o}(B_k(\mathcal{A}), \mathcal{A}^*) \to C^k(\mathcal{A}),$$

given by interpreting a map of left $\mathcal{A} \otimes \mathcal{A}^o$-modules as a map of $\mathcal{A}$-bimodules, then shows that Hochshild cohomology is an Ext-functor. For details, see [15, Section 3.2]. □

As a consequence of Theorem II.4, we can use any projective resolution of $\mathcal{A}$ to determine its Hochschild cohomology groups. In Chapter IV, we will use a topological adaptation of Theorem II.4 to relate the Hochschild cohomology groups of the algebra $C^\infty(M)$ of smooth functions on a manifold to the spaces of currents on M. First however, we show that every class in $HH^k(\mathcal{A})$ has a representative that can be interpreted as a map $\Omega^k\mathcal{A} \to \mathbb{C}$, which is a direct application of Theorem II.4.

Corollary II.5. *Let $[\phi] \in HH^n(\mathcal{A})$. Then we may assume that $[\phi]$ is represented by a cocycle ϕ such that $\phi(a_0 \otimes \cdots \otimes a_n) = 0$ if there is an $i > 0$ such that $a_i = 1$.*

Proof. We consider the sequence

$$0 \longleftarrow \mathcal{A} \xleftarrow{\bar{b}} B_1(\mathcal{A}) \xleftarrow{\bar{b}} B_2(\mathcal{A}) \xleftarrow{\bar{b}} \cdots \tag{II.6}$$

with $B_k(\mathcal{A}) \coloneqq \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \overline{\mathcal{A}}^{\otimes k}$ and

$$\begin{aligned}\bar{b}(a \otimes b \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k}) \coloneqq{}& aa_1 \otimes b \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_k} \\ &+ \sum_{i=1}^{k-1} (-1)^i (a \otimes b \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_j a_{j+1}} \otimes \cdots \otimes \overline{a_k}) \\ &+ (-1)^k (a \otimes a_k b \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{k-1}}).\end{aligned}$$

This complex is acyclic since we have a contracting chain homotopy

$$\begin{gathered} s\colon B_k(\mathcal{A}) \to B_{k+1}(\mathcal{A}), \\ a \otimes b \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k} \mapsto 1 \otimes b \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k} \otimes \overline{a}. \end{gathered}$$

Thus, Eq. (II.6) is a projective (even free, since $\overline{\mathcal{A}}^{\otimes k}$ is a vector space) resolution of $\mathcal{A}$ as a left $\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}$-module. The result then follows from the isomorphism

$$\alpha\colon \operatorname{Hom}_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}} \otimes \overline{\mathcal{A}}^{\otimes k}, \mathcal{A}^*) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes k}, \mathbb{C}) = \operatorname{Hom}(\Omega^k \mathcal{A}, \mathbb{C})$$

and the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}}(B_k(\mathcal{A}), \mathcal{A}^*) & \xrightarrow{\bar{b}} & \operatorname{Hom}_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}}(B_{k+1}(\mathcal{A}), \mathcal{A}^*) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ \operatorname{Hom}(\Omega^k \mathcal{A}, \mathbb{C}) & \xrightarrow{b} & \operatorname{Hom}(\Omega^{k+1} \mathcal{A}, \mathbb{C}) \end{array}$$

with $b\colon \Omega^{k+1}\mathcal{A} \to \Omega^k \mathcal{A}$ given by $\omega\, \mathrm{d}a \mapsto (-1)^k[\omega, a]$. □

Let $[\phi] \in HH^k(\mathcal{A})$ be represented by a Hochschild cocycle ϕ. The upshot of Corollary II.5 is that we may assume that ϕ is *normalised*, i.e. that $\phi(a_0, a_1, \ldots, a_k) = 0$ if $a_i = 1$ for some $i > 0$. We will often interpret a Hochschild cocycle ϕ as a map $\Omega^k \mathcal{A} \to \mathbb{C}$.

Returning to our motivation for studying Hochschild cocycles, we see that a normalised even Hochschild cocycle induces a map $\int_\phi\colon \Omega^{2k}\mathcal{A} \to \mathbb{C}$, via

$$a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{2k} \mapsto \int_\phi a_0\, \mathrm{d}a_1 \cdots \mathrm{d}a_{2k} \coloneqq \phi(a_0, a_1, \ldots, a_{2k}).$$

This map is well-defined since $\mathrm{d}1 = 0$. To see that it satisfies Eq. (II.4), we observe that the boundary map $b\colon C_{k+1}(\mathcal{A}) \to C_k(\mathcal{A})$ descends to a boundary map $b\colon \Omega^{k+1}\mathcal{A} \to \Omega^k \mathcal{A}$, given by $\omega\, \mathrm{d}a \mapsto (-1)^k[\omega, a]$, that also appears in the proof of Corollary II.5. One can see this by noting that $\Omega^k\mathcal{A}$ is the quotient of $C_k(\mathcal{A})$ by

$$\{a_0 \otimes \cdots \otimes a_k \mid a_i = 1 \text{ for some } i > 0\}$$

Thus, for a unitary $u \in \mathcal{A}$, we have

$$0 = \int_{b\phi} uF^k \, \mathrm{d}u^* = \int_\phi [uF^k, u^*],$$

if ϕ is a normalised $2k$-cocycle.

In the next Section, we will find a similar, but weaker, result for odd cocycles when evaluated on generalised Chern-Simons forms.

II.3 Cyclic cohomology

II.3.1 Chern-Simons forms

In odd dimensions, (powers of) the curvature two-form cannot be used to model physical processes. In this case, physicists often resort to Chern-Simons forms [29, 30], which in low dimensions are given by

$$\mathrm{cs}_1(\omega) = \omega; \quad \mathrm{cs}_3(\omega) = \frac{1}{2}\left(\omega \, \mathrm{d}\omega + \frac{2}{3}\omega^3\right),$$

for a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ on a principal fibre bundle. In higher dimensions, the *Chern-Simons* $(2k-1)$*-form* is defined as [23]

$$\mathrm{cs}_{2k-1}(\omega) \coloneqq \int_0^1 \omega(t \, \mathrm{d}\omega + t^2 \omega \wedge \omega)^{k-1} \, \mathrm{d}t,$$

or, in terms of a universal one-form $A \in \Omega^1(\mathcal{A})$

$$\mathrm{cs}_{2k-1}(A) \coloneqq \int_0^1 A(t \, \mathrm{d}A + t^2 A^2)^{k-1} \, \mathrm{d}t.$$

We have seen in the previous Section that the value of a normalised even cochain evaluated on powers of the curvature is gauge-invariant if it is a Hochschild cocycle. This is not the case for odd cochains evaluated on Chern–Simons forms. We need a stronger assumption, but even then the analogy is not perfect, as the gauge invariance holds only up to a pairing of the cochain with a unitary. In physics, this is resolved by imposing a constraint on the coupling parameter κ in the Chern–Simons action [10], since the path integral

$$\int \exp\left(i\frac{\kappa}{4\pi}\int_M \mathrm{tr}\,\frac{1}{2}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right)\right)\mathcal{D}[\omega]$$

must be well-defined. One can show that [10, Equation (60)]

$$\int_M \mathrm{tr}\,\frac{1}{2}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) \in 8\pi^2\mathbb{Z}$$

when ω is pure gauge, resulting in the requirement that κ lie in $(1/4\pi)\mathbb{Z}$ for the exponential in the integrand of the path integral to be well-defined.

Definition II.6. A Hochschild k-cochain ϕ is called *cyclic* if $\lambda\phi = \phi$, where the *cyclic permuter* $\lambda \colon C^k(\mathcal{A}) \to C^k(\mathcal{A})$ is defined as

$$\lambda\phi(a_0, \ldots, a_k) \coloneqq (-1)^k \phi(a_k, a_0, \ldots, a_{k-1}).$$

Next, we introduce the coboundary operator $b' \colon C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$, defined as

$$b'(a_0 \otimes \cdots \otimes a_k) \coloneqq \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k.$$

Note that the only difference between b and b' is the final term of $b(a_0 \otimes \cdots \otimes a_k)$, which is absent in $b'(a_0 \otimes \cdots \otimes a_k)$. Since $(1 - \lambda)b = b'(1 - \lambda)$ [2, Eq. 10.3], b maps cyclic cochains to cyclic cochains. This implies that the cyclic cocycles form a subcomplex of the ordinary Hochschild cocycles, which we will denote by $(C^\bullet_\lambda(\mathcal{A}), b)$.

Definition II.7. Let $(C^\bullet_\lambda(\mathcal{A}), b)$ be the complex of cyclic cocycles. We denote the cohomology of this complex by $HC^\bullet(\mathcal{A})$.

An equivalent and often convenient way to look at cyclic cocycles is to view them as *closed graded traces* over an algebra, which are maps $\Omega^\bullet\mathcal{A} \to \mathbb{C}$, satisfying the following properties.

Definition II.8. Let $(R^\bullet, \delta)$ be a differential graded algebra. A *closed graded trace* of dimension k is a linear map

$$\int \colon R^k \to \mathbb{C}$$

satisfying $\int \delta\omega = 0$ for all $\omega \in R^{k-1}$, and

$$\int \omega_i \omega_j = (-1)^{ij} \int \omega_j \omega_i$$

for all $\omega_i \in R^i$, $\omega_j \in R^j$, with $i + j = k$.

Given a unital algebra $\mathcal{A}$, an k-dimensional closed graded trace $\int \colon \Omega^k\mathcal{A} \to \mathbb{C}$ induces a cyclic k-cocycle $\phi \in Z^k_\lambda(\mathcal{A})$ via

$$\phi(a_0, \ldots, a_k) \coloneqq \int a_0 \, \mathrm{d}a_1 \cdots \mathrm{d}a_k.$$

In fact, this relation goes the other way as well, so that cyclic cocycles are equivalent to closed graded traces on $\Omega^\bullet\mathcal{A}$. This is the content of the following Lemma.

Lemma II.9 ([15])**.** *Let $\mathcal{A}$ be a unital algebra, and let ϕ be a cyclic k-cocycle on $\mathcal{A}$, that vanishes on $\mathbb{C} \oplus \mathcal{A}^k$. Define a linear map*

$$\int_\phi \colon \Omega^k\mathcal{A} \to \mathbb{C}$$
$$a_0 \, \mathrm{d}a_1 \cdots \mathrm{d}a_k \mapsto \phi(a_0, a_1, \ldots, a_k).$$

Then the map

$$\begin{aligned} Z^k_\lambda(\mathcal{A}) &\to \{\text{closed graded traces on } \Omega^k\mathcal{A}\} \\ \phi &\mapsto \int_\phi \end{aligned}$$

is a bijection.

Proof. We start by noting that $\int_\phi$ is well-defined due to the assumption that ϕ vanishes on $\mathbb{C} \oplus \mathcal{A}^k$ and ϕ is cyclic. Next, we show that $\int_\phi$ is a closed graded trace on $\Omega^k\mathcal{A}$, if $\phi \in Z^k_\lambda(\mathcal{A})$. Indeed,

$$\int_\phi \mathrm{d}a_1 \cdots \mathrm{d}a_k = \phi(1, a_1, \ldots, a_k) = 0,$$

since ϕ vanishes on $\mathbb{C} \oplus \mathcal{A}^k$. Furthermore, it suffices to verify the graded trace property only for multiplication of a k-form by a 0-form and of a $(k-1)$-form by a 1-form. The former follows directly from the fact that ϕ is in particular a *Hochschild* cocycle. For the latter, we use the cyclicity of ϕ:

$$\begin{aligned} \int_\phi a_0 \,\mathrm{d}a_1 \cdots \mathrm{d}a_{k-1}\,\mathrm{d}a_k &= \phi(a_0, \ldots, a_k) \\ &= (-1)^k \phi(a_k, a_0, \ldots, a_{k-1}) \\ &= (-1)^{k-1} \int_\phi \mathrm{d}a_k (a_0 \,\mathrm{d}a_1 \cdots \mathrm{d}a_{k-1}), \end{aligned}$$

since $\int_\phi$ is closed and $\mathrm{d}(a_k a_0) = \mathrm{d}a_k a_0 + a_k \,\mathrm{d}a_0$.

The inverse for the map $\phi \mapsto \int_\phi$ is given by

$$\int \mapsto \left[\phi_\int : (a_0, \ldots, a_k) \mapsto \int a_0 \,\mathrm{d}a_1 \cdots \mathrm{d}a_k\right].$$

By analogous computations, one shows that $\phi_\int$ is a cyclic cocycle. For details, see [15]. □

Remark. One can avoid the assumption that the cyclic cocycle vanishes on $\mathbb{C} \oplus \mathcal{A}^k$ by replacing the convention $\Omega^k\mathcal{A} = \mathcal{A} \oplus \overline{\mathcal{A}}^k$ by $\Omega^k\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^k$, where $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ is the unitisation of $\mathcal{A}$. This convention is adopted in [6, 8], and has the advantage that it works for non-unital algebras too.

Now, take a cyclic 1-cochain $\phi_1 \in C^1_\lambda(\mathcal{A})$. We evaluate $\phi_1(\mathrm{cs}_1(A^u)) = \phi_1(A^u)$:

$$\int_{\phi_1} A^u = \int_{\phi_1} (u \,\mathrm{d}u^* + uAu^*) = \int_{\phi_1} A + \int_{\phi_1} u \,\mathrm{d}u^*,$$

since ϕ_1 is Hochschild. One can show [8] that something similar holds for a cyclic 3-cocycle ϕ_3 evaluated on $\mathrm{cs}_3(A^u)$:

$$\int_{\phi_3} \mathrm{cs}_3(A^u) = \int_{\phi_3} \mathrm{cs}_3(A) + \frac{1}{3} \int_{\phi_3} u \,\mathrm{d}u^* \,\mathrm{d}u \,\mathrm{d}u^*. \tag{II.7}$$

In Chapter V we will see that the integrals $\int_{\phi_1} u\,\mathrm{d}u^*$ and $\int_{\phi_3} u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*$ only depend on the class of of the cyclic cocycle in $HC^\bullet(\mathcal{A})$ and on the class of the unitary in K-theory.

We expand the literature by showing that for a cyclic 5-cocycle $\phi_5 \in Z^5_\lambda(\mathcal{A})$, the integral $\int_{\phi_5} \mathrm{cs}_5(A)$ is gauge invariant up to a pure gauge term as well.

Proposition II.10. *For a cyclic cocycle $\phi_5 \in C^n(\mathcal{A})$, a universal 1-form $A \in \Omega^1\mathcal{A}$ and a unitary $u \in \mathcal{A}$ we have*

$$\int_{\phi_5} \mathrm{cs}_5(A^u) = \int_{\phi_5} \mathrm{cs}_5(A) + \frac{1}{30}\int_{\phi_5} u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*. \tag{II.8}$$

Proof. From the definition of Chern-Simons forms it follows that

$$\mathrm{cs}_5(A) = \frac{1}{3}A\,\mathrm{d}A\,\mathrm{d}A + \frac{1}{4}\left(A\,\mathrm{d}AA^2 + A^3\,\mathrm{d}A\right) + \frac{1}{5}A^5.$$

We write out each term of the gauge transformed $\mathrm{cs}_5(A^u)$, using $u\,\mathrm{d}u^* = -\,\mathrm{d}uu^*$.

$$\begin{aligned}
A^u\,\mathrm{d}A^u\,\mathrm{d}A^u = {} & uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* + uA\,\mathrm{d}Au^*\,\mathrm{d}uAu^* - uA^2\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& + uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* + u\,\mathrm{d}u^*\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* - \mathrm{d}u\,\mathrm{d}Au^*\,\mathrm{d}uAu^* \\
& + \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}uAu^* + u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& + uAu^*\,\mathrm{d}uA\,\mathrm{d}Au^* + uA\,\mathrm{d}A\,\mathrm{d}Au^* + uA^2u^*\,\mathrm{d}u\,\mathrm{d}Au^* \\
& + uA\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}Au^* + u\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}Au^* - \mathrm{d}u\,\mathrm{d}A\,\mathrm{d}Au^* \\
& + \mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}Au^* - \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}Au^* \\
& - uAu^*\,\mathrm{d}uA^2\,\mathrm{d}u^* - uA\,\mathrm{d}AA\,\mathrm{d}u^* - uA^2u^*\,\mathrm{d}uA\,\mathrm{d}u^* \\
& - uA\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* - u\,\mathrm{d}u^*\,\mathrm{d}uA^2\,\mathrm{d}u^* + \mathrm{d}u\,\mathrm{d}AA\,\mathrm{d}u^* \\
& - \mathrm{d}uAu^*\,\mathrm{d}uA\,\mathrm{d}u^* + \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* \\
& + uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* + uA\,\mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^* - uA^2\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& + uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* + u\,\mathrm{d}u^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* - \mathrm{d}u\,\mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& + \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* + u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*,
\end{aligned}$$

$$\begin{aligned}
A^u\,\mathrm{d}A^u(A^u)^2 = {} & uAu^*\,\mathrm{d}uA^3u^* + uA\,\mathrm{d}AA^2u^* + uA^2u^*\,\mathrm{d}uA^2u^* \\
& + uA\,\mathrm{d}u^*\,\mathrm{d}uA^2u^* + u\,\mathrm{d}u^*\,\mathrm{d}uA^3u^* - \mathrm{d}u\,\mathrm{d}AA^2u^* \\
& - \mathrm{d}uAu^*\,\mathrm{d}uA^2u^* - \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uA^2u^* - uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* \\
& - uA\,\mathrm{d}Au^*\,\mathrm{d}uAu^* + uA^2\,\mathrm{d}u^*\,\mathrm{d}uAu^* - uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& - u\,\mathrm{d}u^*\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* + \mathrm{d}u\,\mathrm{d}Au^*\,\mathrm{d}uAu^* - \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& - u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& + uAu^*\,\mathrm{d}uA^2\,\mathrm{d}u^* + uA\,\mathrm{d}AA\,\mathrm{d}u^* + uA^2u^*\,\mathrm{d}uA\,\mathrm{d}u^* \\
& + uA\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* + u\,\mathrm{d}u^*\,\mathrm{d}uA^2\,\mathrm{d}u^* - \mathrm{d}u\,\mathrm{d}AA\,\mathrm{d}u^* \\
& - \mathrm{d}uAu^*\,\mathrm{d}uA\,\mathrm{d}u^* - \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* - uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& - uA\,\mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^* + uA^2\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* - uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& - u\,\mathrm{d}u^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* + \mathrm{d}u\,\mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^* - \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& - u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*,
\end{aligned}$$

$$\begin{aligned}
(A^u)^5 = {} & uA^5u^* - uAu^*\,\mathrm{d}uA^3u^* - \mathrm{d}uA^4u^* - u\,\mathrm{d}u^*\,\mathrm{d}uA^3u^* \\
& - uA^2u^*\,\mathrm{d}uA^2u^* - uA\,\mathrm{d}u^*\,\mathrm{d}uA^2u^* + \mathrm{d}uAu^*\,\mathrm{d}uA^2u^* \\
& + \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uA^2u^* - uA^3u^*\,\mathrm{d}uAu^* + uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* \\
& + \mathrm{d}uA^2u^*\,\mathrm{d}uAu^* + u\,\mathrm{d}u\,\mathrm{d}uAu^*\,\mathrm{d}uAu^* - uA^2\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& + uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* + \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}uAu^* + u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uAu^* \\
& + uA^4\,\mathrm{d}u^* - uAu^*\,\mathrm{d}uA^2\,\mathrm{d}u^* - \mathrm{d}uA^3\,\mathrm{d}u^* - u\,\mathrm{d}u^*\,\mathrm{d}uA^2\,\mathrm{d}u^* \\
& - uA^2u^*\,\mathrm{d}uA\,\mathrm{d}u^* - uA\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* + \mathrm{d}uAu^*\,\mathrm{d}uA\,\mathrm{d}u^* \\
& + \mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}uA\,\mathrm{d}u^* - uA^3u^*\,\mathrm{d}u\,\mathrm{d}u^* + uAu^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& + \mathrm{d}uA^2u^*\,\mathrm{d}u\,\mathrm{d}u^* + u\,\mathrm{d}u^*\,\mathrm{d}uAu^*\,\mathrm{d}u\,\mathrm{d}u^* - uA^2\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* \\
& + uAu^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* + \mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* + u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*.
\end{aligned}$$

Next, we evaluate each of these terms under $\int_{\phi_5}$, which is a graded trace since ϕ_5 is a cyclic cocycle. Therefore, we can cyclically permute the universal forms in the 5-forms that appear in the expressions for $A^u\,\mathrm{d}A^u\,\mathrm{d}A^u$, $A^u\,\mathrm{d}A^u(A^u)^2$ and $(A^u)^5$. For the same reason, we have

$$\int_{\phi_5} A^u\,\mathrm{d}A^u(A^u)^2 = \int_{\phi_5} (A^u)^3\,\mathrm{d}A^u.$$

We find

$$\begin{aligned}
\int_{\phi_5} A^u\,\mathrm{d}A^u\,\mathrm{d}A^u = \int_{\phi_5} \big[& A\,\mathrm{d}A\,\mathrm{d}A + A^2\,\mathrm{d}Au^*\,\mathrm{d}u + \mathrm{d}AA^2u^*\,\mathrm{d}u - A^3\,\mathrm{d}u^*\,\mathrm{d}u \\
& + 3A^2u^*\,\mathrm{d}uAu^*\,\mathrm{d}u + 2A\,\mathrm{d}AAu^*\,\mathrm{d}u + 3A^2u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u \\
& + 3Au^*\,\mathrm{d}uA\,\mathrm{d}u^*\,\mathrm{d}u + 2\,\mathrm{d}AA\,\mathrm{d}u^*\,\mathrm{d}u + 2A\,\mathrm{d}A\,\mathrm{d}u^*\,\mathrm{d}u \\
& - \mathrm{d}A\,\mathrm{d}Au^*\,\mathrm{d}u + 5A\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u - 2\,\mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u \\
& + u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^* \big]
\end{aligned}$$

Since $\int_{\phi_5}$ is a graded trace and by the graded Leibniz rule for the differential, we can relate some of the terms of order 3 in A:

$$\begin{aligned}
0 &= \int_{\phi_5} \mathrm{d}\big(A^3u^*\,\mathrm{d}u\big) \\
&= \int_{\phi_5} \big[\mathrm{d}AA^2u^*\,\mathrm{d}u - A\,\mathrm{d}AAu^*\,\mathrm{d}u + A^2\,\mathrm{d}Au^*\,\mathrm{d}u - A^3\,\mathrm{d}u^*\,\mathrm{d}u\big],
\end{aligned}$$

so that

$$\int_{\phi_5} [\mathrm{d}AA^2u^*\,\mathrm{d}u - A^2\,\mathrm{d}A\,\mathrm{d}u^*\,\mathrm{d}u + A^3\,\mathrm{d}u^*\,\mathrm{d}u] = \int_{\phi_5} A\,\mathrm{d}AAu^*\,\mathrm{d}u.$$

Similarly we have for some of the terms of order 2 and 1 in A:

$$\begin{aligned}
\int_{\phi_5} \mathrm{d}AA\,\mathrm{d}u^*\,\mathrm{d}u &= \int_{\phi_5} A\,\mathrm{d}A\,\mathrm{d}u^*\,\mathrm{d}u = -\int_{\phi_5} \mathrm{d}A\,\mathrm{d}Au^*\,\mathrm{d}u, \\
\int_{\phi_5} \mathrm{d}Au^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u &= \int_{\phi_5} A\,\mathrm{d}u^*\,\mathrm{d}u\,\mathrm{d}u^*\,\mathrm{d}u.
\end{aligned}$$

Thus we can shorten $\int_{\phi_5} A^u \, \mathrm{d}A^u \, \mathrm{d}A^u$ to

$$\begin{aligned}\int_{\phi_5} A^u \, \mathrm{d}A^u \, \mathrm{d}A^u = \int_{\phi_5} & A \, \mathrm{d}A \, \mathrm{d}A + 3A \, \mathrm{d}A Au^* \, \mathrm{d}u + 3A^2 u^* \, \mathrm{d}u Au^* \, \mathrm{d}u \\ & + 3A^2 u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u + 3Au^* \, \mathrm{d}u A \, \mathrm{d}u^* \, \mathrm{d}u + 3A \, \mathrm{d}A \, \mathrm{d}u^* \, \mathrm{d}u \\ & + 3A \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u + u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^*.\end{aligned} \tag{II.9}$$

Analogously, we find for the other terms in $\int_{\phi_5} \mathrm{cs}_5(A^U)$

$$\begin{aligned}\int_{\phi_5} A^u \, \mathrm{d}A^u (A^u)^2 = \int_{\phi_5} & \big[A^3 \, \mathrm{d}A + 2A^4 u^* \, \mathrm{d}u + 2A^3 \, \mathrm{d}u^* \, \mathrm{d}u - 2A \, \mathrm{d}A Au^* \, \mathrm{d}u \\ & - 4A^2 u^* \, \mathrm{d}u Au^* \, \mathrm{d}u - 4A^2 u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \\ & - 4Au^* \, \mathrm{d}u A \, \mathrm{d}u^* \, \mathrm{d}u - 2A \, \mathrm{d}A \, \mathrm{d}u^* \, \mathrm{d}u \\ & - 4A \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u - u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^*\big],\end{aligned} \tag{II.10}$$

and

$$\begin{aligned}\int_{\phi_5} (A^u)^5 = \int_{\phi_5} & \big[A^5 - 5A^4 u^* \, \mathrm{d}u - 5A^3 \, \mathrm{d}u^* \, \mathrm{d}u + 5A^2 u^* \, \mathrm{d}u Au^* \, \mathrm{d}u \\ & + 5A^2 u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u + 5Au^* \, \mathrm{d}u A \, \mathrm{d}u^* \, \mathrm{d}u + 5A \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \\ & + u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^*\big].\end{aligned} \tag{II.11}$$

Combining Eqs. (II.9) to (II.11) we find that the only term that survives in the difference $\int_{\phi_5} (\mathrm{cs}_5(A^u) - \mathrm{cs}_5(A))$ is pure gauge, i.e. independent of A:

$$\begin{aligned}\int_{\phi_5} (\mathrm{cs}_5(A^u) - \mathrm{cs}_5(A)) &= \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) \int_{\phi_5} u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \\ &= \frac{1}{30} \int_{\phi_5} u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^* \, \mathrm{d}u \, \mathrm{d}u^*.\end{aligned}$$

□

Remark. In [7], Connes defines a pairing between $HC^k(\mathcal{A})$ and the algebraic K_1 group $K_1^{\mathrm{alg}}(\mathcal{A})$ of $\mathcal{A}$. If $\phi \in Z_\lambda^k(\mathcal{A})$ is a normalised cyclic cocycle and $u \in \mathcal{A}$ is invertible, it is given by

$$\langle [\phi]_\lambda, [u] \rangle = \frac{1}{\sqrt{2i}} 2^{-k} \Gamma\left(\frac{k}{2} + 1\right)^{-1} \phi(u^{-1}, u, \ldots, u^{-1}, u).$$

Under additional constraints, the normalisation of this pairing is specified up to a multiplicative constant, independent of k. Therefore, it makes sense to compare the quotient of these constants for $k = 3$ and $k = 5$ to the quotient of the constants $1/3$ and $1/30$ that appear in Eqs. (II.7) and (II.8), and observe that they coincide:

$$\frac{\sqrt{2i}^{-1} 2^{-3} \Gamma\left(\frac{3}{2} + 1\right)^{-1}}{\sqrt{2i}^{-1} 2^{-5} \Gamma\left(\frac{5}{2} + 1\right)^{-1}} = 10 = \frac{1/3}{1/30}.$$

Curiously, these same quotients for the cases $k = 1$ and $k = 3$ and for $k = 1$ and $k = 5$ both differ by a factor 2.

II.3.2 Connes' long exact sequence

Since the cyclic cocycles form a subcomplex of the Hochschild cocycles, we have a short exact sequence of complexes

$$0 \longrightarrow C_\lambda \hookrightarrow C \twoheadrightarrow C/C_\lambda \longrightarrow 0,$$

where we use C as a shorthand for $C(\mathcal{A})$. It induces a long exact sequence

$$\cdots \longrightarrow HC^k(\mathcal{A}) \longrightarrow HH^k(\mathcal{A}) \longrightarrow H^k(C/C_\lambda) \longrightarrow HC^{k+1}(\mathcal{A}) \longrightarrow \cdots \tag{II.12}$$

Lemma II.11. *Let $\mathcal{A}$ be a unital algebra. Then $H^k(C/C_\lambda) \cong HC^{k-1}(\mathcal{A})$.*

Proof. We consider the sequence

$$0 \longrightarrow C/C_\lambda \xrightarrow[1-\lambda]{} (C, b') \xrightarrow[N]{} C_\lambda \longrightarrow 0, \tag{II.13}$$

where $N \coloneqq \sum_{i=0}^{k} \lambda^i \colon C^k \to C^k$, and where

$$\begin{aligned} b' \colon C^k &\to C^{k+1} \\ \phi &\mapsto \left[b'\phi \colon a_0 \otimes \cdots \otimes a_k \mapsto \sum_{i=0}^{k-1} (-1)^i \phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k) \right]. \end{aligned}$$

The operators $1-\lambda$ and N are morphisms of complexes, since $N(1-\lambda) = (1-\lambda)N = 0$ and $bN = Nb'$, and $\ker(1-\lambda) \subseteq \operatorname{im} N$. In fact, $\ker N \subseteq \operatorname{im}(1-\lambda)$ holds as well, since [15, Section 3.7]

$$(1-\lambda) \sum_{i=0}^{k} (i+1)\lambda^i = N - (k+1)\mathrm{id}_{C^k}.$$

It follows that the sequence (II.13) is exact.

The maps $s \colon C^{k+1} \to C^k$ defined by

$$(s\phi)(a_0, \ldots, a_k) = \phi(1, a_0, \ldots, a_k)$$

satisfy $b's + sb' = \mathrm{id}_{C^k}$, i.e. they provide a contracting homotopy and thus the complex (C, b') is acyclic. Thus, the short exact sequence (II.13) induces a long exact sequence of the form

$$\cdots \longrightarrow H^k(C/C_\lambda) \longrightarrow 0 \longrightarrow HC^k(\mathcal{A}) \longrightarrow H^{k+1}(C/C_\lambda) \longrightarrow 0 \longrightarrow \cdots$$

Thus, the connecting morphism of the long exact sequence provides the isomorphism of cohomology groups $H^k(C/C_\lambda) \xrightarrow{\sim} HC^{k+1}(\mathcal{A})$. □

As a result, we have *Connes' long exact sequence*:

Theorem II.12. *The sequence*

$$\cdots \longrightarrow HC^k(\mathcal{A}) \xrightarrow[I]{} HH^k(\mathcal{A}) \xrightarrow[B]{} HC^{k-1}(\mathcal{A}) \xrightarrow[S]{} HC^{k+1}(\mathcal{A}) \longrightarrow \cdots \tag{II.14}$$

is exact. Here, the maps I, B and S are induced by the maps from the long exact sequence Eq. (II.12) *and the isomorphism from Lemma II.11.*

Corollary II.13. *Every class in $HC^k(\mathcal{A})$ can be represented by a normalised cyclic cocycle, i.e. a $\phi \in C^k_\lambda$ such that $\phi(a_0, \dots, a_k) = 0$ if $a_i = 1$ for some $i \geq 0$.*

Proof. See [19, Proposition 2.2.14]. □

II.3.3 The (b, B)-bicomplex

Definition II.14. Let $\mathcal{A}$ be a unital algebra. Define *cyclic skewsymmetriser* $N\colon C^k(\mathcal{A}) \to C^k(\mathcal{A})$ by $N := 1 + \lambda + \cdots + \lambda^k$, and define the map $s'\colon C^{k+1}(\mathcal{A}) \to C^k(\mathcal{A})$ by

$$(s'\phi)(a_0, \dots, a_k) := (-1)^k \phi(1, a_0, \dots, a_k).$$

Finally, the *Connes boundary map* $B\colon C^{k+1}(\mathcal{A}) \to C^k(\mathcal{A})$ is given by $B := Ns'(1 - \lambda)$.

It is no coincidence that we use the same notation for the Connes boundary map and the connecting homomorphism in Connes' long exact sequence. In fact, by a diagram chase, one shows the latter is exactly the map that is induced by the former [2]. We will use this fact in the computation of the cyclic cohomology groups of the algebra $C^\infty(M)$, in Theorem IV.10.

It follows from $(1 - \lambda)N = 0$ that $B^2 = 0$, so that B is indeed a boundary map. Moreover, one can verify that $bB + Bb = 0$ [2, Lemma 10.1], which implies that we can form the *Connes* or (b, B)*-bicomplex* [2, Definition 10.4].

Definition II.15. Define $BC^{pq} := C^{p-q}(\mathcal{A})$ for $p \geq q \geq 0$, and $BC^{pq} := 0$ otherwise. This yields the *Connes* bicomplex, whose vertical arrows are b and whose horizontal arrows are B:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\
C^3 & \xrightarrow{B} & C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
C^2 & \xrightarrow{B} & C^1 & \xrightarrow{B} & C^0 & & \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & & & \\
C^1 & \xrightarrow{B} & C^0 & & & & \\
\uparrow{\scriptstyle b} & & & & & & \\
C^0 & & & & & &
\end{array}$$

The cohomology of this complex is defined as the cohomology of the *total complex*

$$\mathrm{Tot}^k BC := C^k \oplus C^{k-2} \oplus \cdots \oplus C^{\#k},$$

where $\#k = 0$ or 1 according as k is even or odd.

In fact, the cohomology of the Connes bicomplex is isomorphic to the cyclic cohomology of $\mathcal{A}$:

Lemma II.16. *[14, Proposition 1.4.9] Define the map*

$$\begin{aligned} j\colon C^\bullet_\lambda(\mathcal{A}) &\to \mathrm{Tot}^\bullet BC(\mathcal{A}),\\ \phi_k &\mapsto (0,\dots,0,\phi_k). \end{aligned}$$

Then j induces an isomorphism $HC^\bullet(\mathcal{A}) \cong H^\bullet(\mathrm{Tot}BC(\mathcal{A}))$.

Proof. See [14]. ☐

II.4 Periodic cyclic cohomology

The periodicity map $S\colon HC^k(\mathcal{A}) \to HC^{k+2}(\mathcal{A})$ from Connes' long exact sequence gives two directed systems of abelian groups:

$$\cdots \xrightarrow{S} HC^{2k}(\mathcal{A}) \xrightarrow{S} HC^{2(k+1)}(\mathcal{A}) \xrightarrow{S} \cdots$$

$$\cdots \xrightarrow{S} HC^{2k+1}(\mathcal{A}) \xrightarrow{S} HC^{2(k+1)+1}(\mathcal{A}) \xrightarrow{S} \cdots$$

Definition II.17. The direct limits

$$HP^0(\mathcal{A}) \coloneqq \varinjlim HC^{2k}(\mathcal{A}) \text{ and } HP^1(\mathcal{A}) \coloneqq \varinjlim HC^{2k+1}(\mathcal{A})$$

are called the *even* and *odd periodic cyclic cohomology* of $\mathcal{A}$.

Thus, $HP^0(\mathcal{A})$ is the quotient of $\coprod_{k\in\mathbb{N}} HC^{2k}(\mathcal{A})$ by the relation $[\phi]_\lambda \sim [S\phi]_\lambda$, and similarly for $HP^1(\mathcal{A})$.

II.5 Entire cyclic cohomology

If we extend the (b,B)-bicomplex to allow for infinite (b,B)-cochains, the resulting cohomology is always zero [7]. However, if $\mathcal{A}$ is a unital Banach algebra, we can impose additional topological constraints on the cochains. This will yield an non-trivial cohomology theory. The norm on $\mathcal{A}$ allows us to define a norm on $C^k(\mathcal{A})$: for $\phi_k \in C^k(\mathcal{A})$, set

$$\|\phi_k\| \coloneqq \sup\{|\phi_k(a_0,\dots,a_k)| \mid \|a_j\| \le 1 \text{ for all } j\}.$$

Definition II.18. Let $\phi = (\phi_{2k})$ be a (possibly infinite) even cochain. We say ϕ is *entire* if the series

$$\sum_{k=0}^{\infty} \|\phi_{2k}\| \frac{z^k}{k!}$$

in the complex variable z, has infinite radius of convergence. Similarly, an odd cochain $\psi = (\psi_{2k+1})$ is called entire if the series

$$\sum_{k=0}^{\infty} \|\psi_{2k+1}\| \frac{z^k}{k!}$$

has infinite radius of convergence. We denote the space of all even, respectively odd, entire cochains by $CE^0(\mathcal{A})$, respectively $CE^1(\mathcal{A})$.

The boundary operator $b+B$ maps entire cochains to to entire cochains [14, Lemma 2.1.5], and thus we find the two-term complex

$$CE^0(\mathcal{A}) \underset{b+B}{\overset{b+B}{\rightleftarrows}} CE^1(\mathcal{A}).$$

The cohomology of this complex is called *entire cyclic cohomology*, and is denoted by $HE^0(\mathcal{A}) \oplus HE^1(\mathcal{A})$.

In fact, we need not assume that $\mathcal{A}$ is a Banach algebra: entire cyclic cohomology can be defined for locally convex algebras as well [7, Chapter IV.7., Remark 7.b]:

Definition II.19. Let $\mathcal{A}$ be a locally convex algebra. A cochain (ϕ_{2k}) (respectively (ϕ_{2k+1})) is called *entire* iff for any bounded subset $\Sigma \subseteq \mathcal{A}$ there exists a $C_\Sigma < \infty$ with

$$|\phi_{2k}(a_0, \ldots, a_{2k})| \leq \frac{C_\Sigma}{k!}$$

for all $a_j \in \Sigma$, $k \in \mathbb{N}$.

We will need this in Chapter V, where we will use the entire cyclic cohomology of the algebra of smooth functions on a manifold.

Chapter III

Spectral action

As we have already seen in Chapter I, many classical theories in theoretical physics are modelled by an action functional. Usually, these actions are described as integrals of tensor fields that live on smooth (pseudo-)Riemannian manifolds. Since (universal) forms on an algebra can be 'integrated' by pairing them with a linear functional, it is conceivable that we can generalise action functionals to the noncommutative setting. This Section is based on [28].

This generalisation is called the *spectral action*, and we will study it in this Chapter. First, we will briefly review the notion of a *spectral triple* from noncommutative geometry, and use it to define a differential graded algebra. In Section III.2.1, we will introduce the theory of multiple operator integrals, which we need to define the type of spectral action we will study.

III.1 Spectral triples

We summarize a few notions and results from noncommutative geometry, starting with the definition of a spectral triple. In Section III.1.1, we see how we can associate spectral triples to a certain class of manifolds, called spin^c manifolds, after which we construct a differential graded algebra that arises from a spectral triple, in Section III.1.2.

Definition III.1 ([28])**.** A *spectral triple* $(\mathcal{A}, H, D)$ consists of:

- a unital $*$-algebra $\mathcal{A}$, which is represented on the algebra $B(H)$ of bounded operators on
- a Hilbert space H, and
- a self-adjoint operator D with compact resolvent $(i + D)^{-1}$, and such that $[D, a]$ is a bounded operator for all $a \in \mathcal{A}$.

The operator D need not be bounded, so generally it is only defined on a dense subspace of H. In the following Section, we see how a certain class of compact Riemannian manifolds gives rise to a canonical spectral triple.

III.1.1 Canonical spectral triple of a spin^c manifold

We first review the definition of a Clifford algebra, adapted to our needs.

Definition III.2. Let V be a real or complex vector space with an inner product g. The *Clifford algebra* $\mathrm{Cl}(V, g)$ is the algebra generated by vectors $v \in V$ and with unit 1, subject to the relation $v^2 = g(v, v)1$. It is $\mathbb{Z}_2$-graded , with grading given by

$$\chi(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k.$$

This allows us to decompose the Clifford algebra $\mathrm{Cl}(V, g) := \mathrm{Cl}^0(V, g) \oplus \mathrm{Cl}^1(V, g)$ into an even and an odd part. In case $V = \mathbb{C}^n$ with the standard inner product g, we will write $\mathbb{C}\mathrm{l}_n := \mathrm{Cl}(\mathbb{C}^n, g)$.

Next, we extend this notion to that of a Clifford *bundle*, i.e. a smoothly varying collection of Clifford algebras over a smooth manifold.

Definition III.3. Let (M, g) be a Riemannian manifold. The *Clifford algebra bundle* is the bundle of Clifford algebras $\mathrm{Cl}(T_xM, g)$, with transition functions induced by those on TM. Usually, we will work with the complexified tangent bundle $T^{\mathbb{C}}M := \coprod_{x \in M} T_xM \otimes \mathbb{C}$ with a hermitian metric g. Its Clifford algebra bundle will be denoted by $\mathbb{C}\mathrm{l}(TM, g)$, or $\mathbb{C}\mathrm{l}(TM)$ if the metric is clear.

Locally, on a chart with coordinates $\{x^\mu\}$, the algebra of sections of $\mathbb{C}\mathrm{l}(TM)$ is generated by elements γ_μ, $\mu = 1, \ldots, \dim M$, subject to relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu},$$

with $g_{\mu\nu} = g(\partial/\partial x^\mu, \partial/\partial x^\nu)$.

The standard complex Clifford algebras can be computed: they are $\mathbb{C}\mathrm{l}_{2m} \cong M_{2^m}(\mathbb{C})$ and $\mathbb{C}\mathrm{l}_{2m+1} \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$ [28]. Thus, they can be represented irreducibly on the vector spaces $\mathbb{C}^{2^m}$ and $\mathbb{C}^{2^m} \oplus \mathbb{C}^{2^m}$. The differential-geometric analogue of this is an isomorphism of the Clifford bundle of a manifold M to the endomorphism bundle of a vector bundle over M.

Definition III.4. A Riemannian manifold M is called *spin^c* if there exists a vector bundle $\mathcal{S} \to M$ such that there is an Clifford algebra homomorphism

$$\begin{aligned} \mathbb{C}\mathrm{l}(TM) &\cong \mathrm{End}\,(\mathcal{S}) \;\text{ if } M \text{ is even-dimensional,} \\ \mathbb{C}\mathrm{l}^0(TM) &\cong \mathrm{End}\,(\mathcal{S}) \;\text{ if } M \text{ is odd-dimensional.} \end{aligned}$$

In this case, $\mathcal{S} \to M$ is called a *spinor bundle* of M, and sections of $\mathcal{S}$ are called *spinors.*

We will usually interpret the sections γ_μ that generate $\mathbb{C}\mathrm{l}(TM)$ as sections of $\mathrm{End}\,(\mathcal{S})$, and they will be called *gamma matrices.*

We are now in a position to define two of the three constituents of the canonical spectral triple on a compact manifold. First, the algebra is the algebra of smooth, complex-valued functions on M. Since M is compact, we can choose a Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$, which induces an inner product on the sections of $\mathcal{S}$ via

$$(\psi_1, \psi_2) := \int_M \langle \psi_1, \psi_2 \rangle(x) \sqrt{g} \, \mathrm{d}x.$$

The completion of $\Gamma(\mathcal{S})$ gives the Hilbert space $L^2(\mathcal{S})$ of square-integrable sections of $\mathcal{S}$. The representation of $C^\infty(M)$ on $L^2(\mathcal{S})$ is given by function-spinor multiplication, i.e.

$$f \mapsto [M_f \colon \psi \mapsto f\psi].$$

We will usually simply write f instead of M_f. It remains to construct the operator D, which we set out to do now. We will only define D locally, as this suffices for our purposes in Chapter VI.

Let M be a Riemannian spin^c manifold, and let ∇ be its Levi-Civita connection. The lift of this connection to the spinor bundle is called the *spin connection* on $\mathcal{S}$. Locally, it looks as follows. Let $(U, \{x^\mu\})$ be a chart of M, and fix a local orthonormal frame $\{E_1, \ldots, E_n\}$ over $T_U M$. Let $\tilde{\Gamma}^b_{\mu a}$ be the Christoffel symbols in this basis, defined through $\nabla E_a =: \tilde{\Gamma}^b_{\mu a} \, \mathrm{d}x^\mu \otimes E_b$. With respect to this basis, the Clifford relations take the form

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab}.$$

Then the *spin connection* $\nabla^{\mathcal{S}}$ is locally given by

$$\nabla^{\mathcal{S}}_\mu \psi = \left(\partial_\mu - \frac{1}{4} \tilde{\Gamma}^b_{\mu a} \gamma^a \gamma_b \right) \psi,$$

where we use the Einstein summation convention. By contracting the spin connection with the gamma matrices (a process that is called *Clifford multiplication*), we obtain a Dirac operator on $L^2(\mathcal{S})$.

Definition III.5. Let M be a spin^c manifold with spinor bundle $\mathcal{S}$. The Dirac operator $D_M \colon \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is locally given by

$$D_M \psi = -i\gamma^\mu \left(\partial_\mu - \frac{1}{4} \tilde{\Gamma}^a_{\mu b} \gamma^a \gamma_b \right) \psi.$$

Together with $C^\infty(M)$ and $L^2(\mathcal{S})$, D_M does indeed define a spectral triple:

Theorem III.6 ([28, Theorem 4.20])**.** *The operator D_M is self-adjoint on $L^2(\mathcal{S})$ with compact resolvent $(i + D_M)^{-1}$, and $[D_M, f]$ is bounded for all $f \in C^\infty(M)$.*

III.1.2 Connes' differential forms

Let $(\mathcal{A}, H, D)$ be a spectral triple. The map

$$\begin{aligned} [D, \cdot] \colon \mathcal{A} &\to B(H) \\ a &\mapsto [D, a] \end{aligned}$$

defines a derivation, and satisfies $[D,a]^* = -[D,a^*]$. Therefore, the map

$$\pi\colon \Omega^\bullet\mathcal{A} \to B(H)$$
$$a_0\,\mathrm{d}a_1\cdots\mathrm{d}a_k \mapsto a_0[D,a_1]\cdots[D,a_k]$$

is a $*$-homomorphism. However, π is *not* a homomorphism of differential graded algebras: if $\pi(\omega)=0$ for a universal form ω, then we do not necessarily have $\pi(\delta\omega)=0$. For example, let $\mathbb{T}^1 \subseteq \mathbb{C}$ be the unit circle with coordinate $\theta \in [0,2\pi)$ and consider the canonical spectral triple $(C^\infty(\mathbb{T}^1), L^2(\mathbb{T}^1), -i\,\mathrm{d}/\,\mathrm{d}\theta)$ (cf. Section VI.1). Let $e_n\colon [0,2\pi] \to \mathbb{C}$ be given by $\theta \mapsto e^{in\theta}$. Then, for $k \neq n \neq 0$,

$$\pi\left(e_n\,\mathrm{d}e_k - \frac{k}{n}e_k\,\mathrm{d}e_n\right) = e_n[D,e_k] - \frac{k}{n}e_k[D,e_n] = 0,$$

but

$$\pi\left(\mathrm{d}\left(e_n\,\mathrm{d}e_k - \frac{k}{n}e_k\,\mathrm{d}e_n\right)\right) = [D,e_n][D,e_k] - \frac{k}{n}[D,e_k][D,e_n] \neq 0.$$

Universal forms ω such that $\pi(\mathrm{d}\omega) \neq 0$ despite $\pi(\omega)=0$ are called *junk forms*.

Let $J_0 \coloneqq \oplus_{p=0}^\infty J_0^p$ be the graded two-sided ideal of $\Omega^\bullet\mathcal{A}$, with J_0^p given by

$$J_0^p \coloneqq \{\omega \in \Omega^p\mathcal{A} \mid \pi(\omega) = 0\}.$$

Then by [16, Prop. 35], $J \coloneqq J_0 + \mathrm{d}J_0$ is a graded two-sided ideal as well, so the quotient $\Omega^\bullet\mathcal{A}/J$ is well-defined.

Definition III.7 ([16, Def. 17])**.** The graded differential algebra of Connes' differential forms over the algebra $\mathcal{A}$ is given by

$$\Omega_D^\bullet\mathcal{A} \coloneqq \Omega^\bullet\mathcal{A}/J \cong \pi(\Omega^\bullet\mathcal{A})/\pi(\mathrm{d}J_0).$$

The differential $\Omega_D^k\mathcal{A} \to \Omega_D^{k+1}\mathcal{A}$ is given by

$$[\omega] \mapsto [\mathrm{d}\omega].$$

In degree 1, which will be most important to us, we have [16]

$$\Omega_D^1\mathcal{A} \cong \pi(\Omega^1\mathcal{A}) = \left\{\sum_j a_j[D,b_j] \mid a_j, b_j \in \mathcal{A}\right\}.$$

III.2 Spectral action

Let $(\mathcal{A},H,D)$ be a spectral triple, and let $f\colon \mathbb{R} \to \mathbb{C}$ be a suitably nice function (to be specified later, but one can think of a smooth function with compact support for now). One can define the *spectral action*

$$\mathrm{tr}(f(D)).$$

It is fluctuated by adding a self-adjoint Connes one-form $V = \sum_{j=1}^n a_j[D,b_j]$ to D. The variation of the spectral action is then given by $\mathrm{tr}(f(D+V)) - \mathrm{tr}(f(D))$. In the remainder of this Chapter, we will explain the following result from [27].

Theorem III.8. *Let $(\mathcal{A}, H, D)$ be an s-summable spectral triple and let $f \in \mathcal{E}^{s,\gamma}$ for $\gamma \in (0,1)$. The spectral action fluctuated by a self-adjoint Connes one-form $V := \pi_D(A) \in \Omega_D(\mathcal{A})_{\text{sa}}$ can be written as*

$$\operatorname{tr}(f(D+V)) - \operatorname{tr}(f(D)) = \sum_{k=1}^{\infty} \left(\int_{\psi_{2k-1}} \operatorname{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right).$$

Here ψ_{2k-1} and ϕ_{2k} are Hochschild cochains on the algebra $\mathcal{A}$. These are defined as multiple operator integrals, which we will study next.

III.2.1 Multiple operator integrals

Multiple operator integrals, which we will define below, appear naturally when one studies the perturbed spectral action $\operatorname{tr}(f(D+V))$. In this Section, we will summarise the results on multiple operator integrals from [27]. For small perturbations, one can use the Taylor expansion to write

$$\operatorname{tr}(f(D+V)) \sim \sum_{n=0}^{\infty} \operatorname{tr}\left(\frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(D+tV)|_{t=0} \right). \tag{III.1}$$

The derivatives of $t \mapsto f(D+tV)$ on the right-hand side can be rewritten as simplex integrals over operators, defined next. This definition makes use of the Fourier transform on $\mathbb{R}$, for which we adopt the following convention:

$$\widehat{f}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-ixt} \, \mathrm{d}x.$$

Definition III.9. Let D be an (unbounded) self-adjoint operator on a Hilbert space H and let $f \in C^n(\mathbb{R})$ be an n times differentiable function such that $\widehat{f^{(n)}} \in L^1(\mathbb{R})$. The multiple operator integral $T^D_{f^{[n]}} : B(H)^n \to B(H)$ is defined on $(V_1, \ldots, V_n) \in B(H)^n$ by

$$T^D_{f^{[n]}}(V_1, \ldots, V_n)y := \int_{\Delta_n} \int_{\mathbb{R}} e^{is_0 tD} V_1 e^{is_1 tD} \cdots V_n e^{is_n tD} y \widehat{f^{(n)}}(t) \, \mathrm{d}t \, \mathrm{d}\sigma(s_0, \ldots, s_n), \tag{III.2}$$

for $y \in H$. Here $\Delta_n := \{(s_0, \ldots, s_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum_{j=0}^n s_j = 1\}$ is de standard n-simplex, and σ is the standard measure on it.

Equation (6) in [27] draws the connection between multiple operator integrals and the spectral action:

$$\frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \operatorname{tr}(f(D+tV))|_{t=0} = \operatorname{tr}(T^D_{f^{[n]}}(V, \ldots, V)).$$

The multiple operator integral satisfies the following important continuity property, which is adapted from [27]. For this, recall that the Schatten s-class of H, denoted $\mathscr{S}^s$, is the space of compact operators on H whose sequences of singular values are s-summable [31].

Theorem III.10 ([27, Theorem 1.3.5])**.** *Let $s \in \mathbb{N}$, $\gamma \in (0,1]$, D self-adjoint in H with $(D-i)^{-1} \in \mathscr{S}^s$, $n \in \mathbb{N}_0$ and $f \in \mathcal{E}^{s,\gamma}$ (to be defined below). The map*

$$T^D_{f^{[k]}} \colon B(H)_1 \times \cdots \times B(H)_1 \to L^1(H)$$

is continuous, where $B(H)_1$ is the closed unit ball of $B(H)$, endowed with the strong operator topology, and $L^1(H)$ is the ideal of trace-class operators, endowed with the trace norm.

Next, we define the function classes and types of spectral triples that appear in Theorems III.8 and III.10.

Definition III.11. Let $s \in \mathbb{N}$. An *s-summable* spectral triple $(\mathcal{A}, H, D)$ consists of a separable Hilbert space H, a self-adjoint operator D in H and a unital $*$-algebra $\mathcal{A} \subset B(H)$, such that, for all $a \in \mathcal{A}$, $a\,\mathrm{dom}D \subseteq \mathrm{dom}D$ and $[D,a]$ extends to a bounded operator, and $(D-i)^{-1} \in \mathscr{S}^s$.

Let $\gamma \in (0,1]$, and $s \in \mathbb{N}_0$. Define the function space

$$\mathcal{E}^{s,\gamma} := \{f \in C^\infty(\mathbb{R}) \mid \exists C_f \geq 1\, \forall (m \leq s,\, n \in \mathbb{N}_0) : \|\widehat{(fu^m)^{(n)}}\|_1 \leq C_f^{n+1} n!^\gamma\},$$

where $u \colon \mathbb{R} \to \mathbb{C}$ is given by $x \mapsto x - i$.

Since the Taylor expansion of the spectral action (III.1) depends on all derivatives of f, it is natural to demand $f \in C^\infty(\mathbb{R})$. Furthermore, the bound on $\widehat{(fu^m)^{(n)}}$ is needed to ensure convergence of the Taylor expansion of the spectral action. Finally, s-summability of the spectral triple guarantees that the multiple operator integrals are continuous. This continuity also requires a bound on f, for which $f \in \mathcal{E}^{s,\gamma}$ suffices. In the remainder of this Chapter, we will assume that the spectral triple $(\mathcal{A}, H, D)$ is s-summable and that $f \in \mathcal{E}^{s,\gamma}$.

By summing over the rotational permutations of the arguments of the multiple operator integral, we obtain the following bracket

Definition III.12. Define $\langle \cdot, \ldots, \cdot \rangle \colon B(H)^n \to \mathbb{C}$ to be the multilinear function

$$\langle V_1, \ldots, V_n \rangle := \sum_{j=1}^{n} \mathrm{tr}(T^D_{f^{[n]}}(V_j, \ldots, V_1, V_n, \ldots, V_{j-1})). \tag{III.3}$$

In Chapter VI we will compute these brackets for the case where the spectral triple $(\mathcal{A}, H, D) = (C^\infty(\mathbb{T}^d), L^2(\mathcal{S}_{\mathbb{T}^d}), D_{\mathbb{T}^d})$ is the canonical spectral triple of a d-torus.

III.2.2 Hochschild cochains in the spectral action

The bracket satisfies the following two properties:

Lemma III.13 ([27, Lemma 3.3.3])**.** *For $V_1, \ldots, V_n \in B(H)$ and $a \in \mathcal{A}$ we have*

(i) $\langle V_1, \ldots, V_n \rangle = \langle V_n, V_1, \ldots, V_{n-1} \rangle$,

(ii) $\langle V_1, \ldots, aV_j, \ldots, V_n\rangle - \langle V_1, \ldots, V_{j-1}a, \ldots, V_n\rangle = \langle V_1, \ldots, V_{j-1}, [D, a], V_j, \ldots, V_n\rangle,$

where is is understood that for the edge case $j = 1$ we need to substitute n for $j - 1$ on the left-hand side.

Proof. The first property follows immediately from Definition III.12. The second property is first shown to hold for finite-rank operators $V_1, \ldots, V_n$, and then generalised to arbitrary bounded operators by using strong continuity of the multiple operator integral and the fact that the finite-rank operators lie strongly dense in $B(H)$. For details, see [27]. □

Definition III.14. Let $\phi_k \colon \mathcal{A}^{k+1} \to \mathbb{C}$ be the Hochschild cochain given by

$$\phi_n(a_0, \ldots, a_k) := \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_k]\rangle.$$

With the help of Lemma III.13, we will show that ϕ_k is a Hochschild cocycle for all $k \in \mathbb{N}$.

Proposition III.15. *For $k \in \mathbb{N}$, $b\phi_k = \phi_{k+1}$ if n is odd, and $b\phi_k = 0$ if k is even.*

Proof. Let $k \in \mathbb{N}$, and $a_0, \ldots, a_{k+1} \in \mathcal{A}$. Then by definition of ϕ_k, we have

$$b\phi_k(a_0, \ldots, a_{k+1}) = \sum_{i=0}^{k}(-i)^n \phi_k(a_0, \ldots, a_i a_{i+1}, \ldots, a_{k+1}) + (-1)^{k+1}\phi_k(a_{n+1}a_0, a_1 \ldots, a_k)$$

$$= \langle a_0 a_1[D, a_2], [D, a_3], \ldots, [D, a_{k+1}]\rangle + \sum_{i=1}^{k}(-1)^i \langle a_0[D, a_1], \ldots, [D, a_i a_{i+1}], \ldots, [D, a_{k+1}]\rangle$$
$$+ (-1)^{k+1}\langle a_{k+1}a_0[D, a_1], [D, a_2], \ldots, [D, a_k]\rangle.$$

Now we use $[D, a_i a_{i+1}] = [D, a_i]a_{i+1} + a_i[D, a_{i+1}]$ and an index shift to find

$$\begin{aligned}
b\phi_n(a_0, \ldots, a_{n+1}) = {} & \langle a_0 a_1[D, a_2], [D, a_3], \ldots, [D, a_{k+1}]\rangle \\
& + \sum_{i=1}^{k}\langle a_0[D, a_1], [D, a_2], \ldots, a_i[D, a_{i+1}], \ldots, [D, a_{k+1}]\rangle \\
& - \sum_{i=2}^{k+1}(-1)^i \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_{i-1}]a_i, \ldots, [D, a_{k+1}]\rangle \\
& + (-1)^{k+1}\langle a_{n+1}a_0[D, a_1], [D, a_2], \ldots, [D, a_k]\rangle \\
= {} & \langle a_0 a_1[D, a_2], [D, a_3], \ldots, [D, a_{k+1}]\rangle \\
& - \langle a_0 a_1[D, a_2], [D, a_3], \ldots, [D, a_{k+1}]\rangle \\
& + \sum_{i=2}^{k}(-1)^i \big(\langle a_0[D, a_1], [D, a_2], \ldots, a_i[D, a_{i+1}], \ldots, [D, a_{k+1}]\rangle \\
& \qquad \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_{i-1}]a_i, \ldots, [D, a_{k+1}]\rangle\big) \\
& + (-1)^k \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_n]a_{k+1}\rangle \\
& + (-1)^{k+1}\langle a_{k+1}a_0[D, a_1], [D, a_2], \ldots, [D, a_k]\rangle.
\end{aligned}$$

By the two properties from Lemma III.13, this gives

$$\begin{aligned} b\phi_k(a_0,\ldots,a_{k+1}) &= \sum_{i=2}^{k}(-i)^i\langle a_0[D,a_1],[D,a_2],\ldots,[D,a_{k+1}]\rangle \\ &\quad + (-1)^{k+1}\langle [D,a_{k+1}],a_0[D,a_1],\ldots,[D,a_1]\rangle. \\ &= \sum_{i=2}^{k+1}(-i)^i\langle a_0[D,a_1],[D,a_2],\ldots,[D,a_{k+1}]\rangle \end{aligned}$$

If k is even, the summation runs over an even number of terms and therefore vanishes. If k is odd, all terms but one cancel against each other, so that we arrive at $b\phi_k = \phi_{k+1}$. □

Let $k \in \mathbb{N}$, and define the operator $B_0 \colon C^{k+1}(\mathcal{A}) \to C^k(\mathcal{A})$ by $B_0 = s'(1-\lambda)$. In particular, if $\phi \in C^{k+1}(\mathcal{A})$ is normalised, we have

$$B_0\phi(a_0,\ldots,a_k) \coloneqq \phi(1,a_0,\ldots,a_k).$$

Comparing B_0 to Definition II.14, we see that $B = NB_0$.

It follows from the first property in Lemma III.13 that $B_0\phi_k$ is invariant under cyclic permutations, so that $B\phi_k = kB_0\phi_k$ for odd k and $B\phi_k = 0$ for even k. Thus, the sequence $(\phi_{2k})_k$ is a (b,B)-cocycle:

$$b\phi_{2k} + B\phi_{2k+2} = 0.$$

In fact, the odd cochains can be combined to form a (b,B)-cochain as well.

Proposition III.16. *For $k \in \mathbb{N}$, define*

$$\psi_{2k-1} \coloneqq \phi_{2k-1} - \frac{1}{2}B_0\phi_{2k}, \qquad \text{and} \qquad \tilde{\psi}_{2k-1} \coloneqq (-1)^{k-1}\frac{(k-1)!}{(2k-1)!}\psi_{2k-1}.$$

Then the sequence $(\tilde{\psi}_{2k-1})_k$ is an odd (b,B) cocycle.

Proof. Since the ϕ_k are normalised, we have $B_0^2\phi_k = 0$ for each k. Thus,

$$B\psi_{2k+1} = B\phi_{2k+1} - \frac{1}{2}NB_0^2\phi_{2k+2} = (2k+1)B_0\phi_{2k+1},$$

and

$$b\psi_{2k-1} = b\phi_{2k-1} - \frac{1}{2}bB_0\phi_{2k} = \phi_{2k} - \frac{1}{2}bB_0\phi_{2k},$$

as follows from Proposition III.15. We show that $bB_0\phi_{2k} = 2\phi_{2k} - B_0\phi_{2k+1}$ [27, Lemma 3.3.7]. For $a_0,\ldots,a_{2k} \in \mathcal{A}$,

$$\begin{aligned} bB_0\phi_{2k}(a_0,\ldots,a_{2k}) &= \sum_{j=0}^{2k-1}(-1)^j\langle [D,a_0],\ldots,[D,a_ja_{j+1}],\ldots,[D,a_{2k}]\rangle \\ &\quad + \langle [[D,a_{2k}a_0],[D,a_1],\ldots,[D,a_{2k_1}]\rangle. \end{aligned}$$

Then, by using $[D, a_j a_{j+1}] = [D, a_j]a_{j+1} + a_j[D, a_{j+1}]$ and an index shift, we find

$$\begin{aligned} bB_0\phi_{2k}(a_0, \ldots, a_{2k}) = {} & \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_{2k}]\rangle \\ & + \sum_{j=1}^{2k-1} (-1)^j \langle [D, a_0], \ldots, a_j[D, a_{j+1}], \ldots, [D, a_{2k}]\rangle \\ & - \sum_{j=1}^{2k-1} (-1)^j \langle [D, a_0], \ldots, [D, a_{j-1}]a_j, \ldots, [D, a_{2k}]\rangle \\ & - \langle [D, a_0], \ldots, [D, a_{2k-2}], [D, a_{2k-1}]a_{2k}\rangle \\ & + \langle a_{2k}[D, a_0], [D, a_1], \ldots, [D, a_{2k-1}]\rangle \\ & + \langle [D, a_{2k}]a_0, [D, a_1], \ldots, [D, a_{2k-1}]\rangle. \end{aligned}$$

By property (ii) from Lemma III.13, this equals

$$\begin{aligned} bB_0\phi_{2k}(a_0, \ldots, a_{2k}) = {} & \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_{2k}]\rangle \\ & + \sum_{j=1}^{2k-1} (-1)^j \langle [D, a_0], \ldots, [D, a_{2k}]\rangle \\ & + \langle [D, a_{2k}], [D, a_0], \ldots, [D, a_{2k-1}]\rangle \\ & + \langle [D, a_{2k}]a_0, [D, a_1], \ldots, [D, a_{2k-1}]\rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle [D, a_{2k}]a_0, [D, a_1], \ldots, [D, a_{2k-1}]\rangle = {} & \langle [D, a_{2k}], a_0[D, a_1], [D, a_2], \ldots, [D, a_{2k-1}]\rangle \\ & - \langle [D, a_{2k}], [D, a_0], \ldots, [D, a_{2k-1}]\rangle, \end{aligned}$$

we have

$$\begin{aligned} bB_0\phi_{2k}(a_0, \ldots, a_{2k}) &= \phi_{2k}(a_0, \ldots, a_{2k}) - \langle [D, a_0], \ldots, [D, a_{2k}]\rangle \\ &\quad + \langle [D, a_{2k}], [D, a_0], \ldots, [D, a_{2k-1}]\rangle \\ &\quad + \langle a_0[D, a_1], [D, a_2], \ldots, [D, a_{2k}]\rangle \\ &\quad - \langle [D, a_{2k}], [D, a_0], \ldots, [D, a_{2k-1}]\rangle \\ &= 2\phi_{2k}(a_0, \ldots, a_{2k}) - B_0\phi_{2k+1}(a_0, \ldots, a_{2k}), \end{aligned}$$

by property (i) from Lemma III.13. Thus,

$$\begin{aligned} B\tilde{\psi}_{2k+1} + b\tilde{\psi}_{2k-1} &= (-1)^k \frac{k!}{(2k+1)!} B\psi_{2k+1} + (-1)^{k-1} \frac{(k-1)!}{(2k-1)!} b\psi_{2k-1} \\ &= (-1)^k \frac{k!}{(2k)!} \left(1 - \frac{1}{2}\frac{2k}{k}\right) B_0\phi_{2k+1} \\ &= 0, \end{aligned}$$

which shows that $(\tilde{\psi}_{2k-1})$ is a (b, B)-cocycle. □

The statement of Proposition III.16 can be improved by taking topological aspects of $\mathcal{A}$ into account.

Proposition III.17 ([27, Lemma 3.5.2])**.** *Let $(\mathcal{A}, H, D)$ be a spectral triple. Equip $\mathcal{A}$ with the norm $\|a\|_D := \|a\| + \|[D, a]\|$, where $\|\cdot\|$ is the norm of $B(H)$. Then, for any bounded subset $\Sigma \subseteq \mathcal{A}$ there exists a C_Σ such that*

$$|\tilde{\psi}_{2k+1}(a_0, \ldots, a_{2k+1})| \leq \frac{C_\Sigma}{k!},$$

for all $a_j \in \Sigma$.

We recall recall from Definition II.19 that this shows that $(\tilde{\psi}_{2k-1})$ is an *entire* cyclic cocycle.

Chapter IV

De Rham Homology

In Chapter II, we have defined Hochschild cohomology for an arbitrary algebra $\mathcal{A}$. In the current Chapter, we will specialise to the case $\mathcal{A} = C^\infty(M)$, the algebra of smooth, complex-valued functions on a compact manifold M. The algebra $C^\infty(M)$ can be topologised in such a way that it becomes a *Fréchet algebra*, i.e. a locally convex algebra with complete and metrisable topology. Connes proved in [6] that with this topology, the Hochschild cohomology groups of *continuous* cochains on $C^\infty(M)$ are isomorphic to the spaces of currents on M. We will make this precise and sketch the proof of this result later in Theorem IV.4, but first we introduce currents in more detail. In Section IV.2 we illustrate Theorem IV.4 by showing that each continuous map $\phi\colon C^\infty(M) \to \mathbb{C}$ is cohomologous to a map that can be interpreted as a current, through exploiting $b\phi = 0$. Section IV.3 states a result that relates the continuous *cyclic* cohomology groups to the de Rham homology groups; we will use that result in Chapter V.

IV.1 The Connes–Hochschild–Kostant–Rosenberg Theorem

Definition IV.1. Let M be a compact manifold. A continuous linear map $\Omega^k(M) \to \mathbb{C}$ is called a *de Rham k-current* (on M). The space of k-currents, i.e. $\Omega^k(M)^*$, is denoted by $\Omega_k(M)$.

Here, a current $T\colon \Omega^k(M) \to \mathbb{C}$ is continuous if the following holds [9]: let $(\omega_n)_{n\in\mathbb{N}} \subseteq \Omega^k(M)$ be a sequence of k-forms with support contained in a single chart. If the derivative of each coefficient of ω_n uniformly tends to zero as $n \to \infty$, then $T(\omega_n) \to 0$.

A 0-current is the same as a distribution on M. Important examples of k-currents are k-dimensional submanifolds: if $N \subseteq M$ is a submanifold of dimension k, it defines a k-current $[N]$ via

$$[N](\omega) := \int_N \omega, \quad \omega \in \Omega^k(M).$$

Inspired by this example, we will often use the integral notation for currents: if $T \in \Omega_k(M)$, its value on $\omega \in \Omega^k(M)$ is denoted by $\int_T \omega$. Differentiation of currents is defined by generalising Stokes's theorem.

Definition IV.2. Let $T \in \Omega_k(M)$. Its *boundary* $\partial T \in \Omega_{k-1}(M)$ is defined via

$$\int_{\partial T} \omega := \int_T \mathrm{d}_{\mathrm{dR}}\omega.$$

Thus ∂ defines an operator $\Omega_k(M) \to \Omega_{k-1}(M)$ satisfying $\partial^2 = 0$. Therefore, the de Rham currents form a chain complex

$$0 \xleftarrow{\partial} \Omega_0(M) \xleftarrow{\partial} \Omega_1(M) \xleftarrow{\partial} \cdots.$$

The homology of this chain complex is called the *de Rham homology*, with homology groups denoted by $H_k^{\mathrm{dR}}(M)$.

As stated in the introduction of this Chapter, we wish to relate the Hochschild cohomology group of continuous cochains on $C^\infty(M)$ to the space of currents on M. For this, we topologise $C^\infty(M)$ as follows.

Definition IV.3. Let M be a compact manifold and let $C^\infty(M)$ be its algebra of smooth functions. Fix a finite set of generators $X_1, \ldots, X_k$ for the $C^\infty(M)$-module $\mathfrak{X}(M)$. The *Fréchet topology* on $C^\infty(M)$ is the topology generated by the seminorms

$$\|f\|_n = \sup\{|X_{i_1} \cdots X_{i_l} f| \mid 0 \leq l \leq n,\, i_1, \ldots, i_l \in \{1, \ldots, k\}\}.$$

Now, consider the chain complex of *continuous* Hochschild cochains on $C^\infty(M)$. We will denote the cohomology groups of this complex simply by $HH^\bullet(C^\infty(M))$, as there will be no risk of confusion with the non-continuous case. The following Theorem, proved by Connes, relates the cohomology groups $HH^\bullet(C^\infty(M))$ to the complex $\Omega_\bullet(M)$ of currents on M.

Theorem IV.4 (Connes–Hochschild–Kostant–Rosenberg [6])**.** *The map*

$$\begin{aligned} \alpha\colon HH^k(C^\infty(M)) &\longrightarrow \Omega_k(M) \\ [\phi] &\longmapsto C_\phi, \end{aligned}$$

with

$$\int_{C_\phi} a_0\, \mathrm{d}_{\mathrm{dR}} a_1 \wedge \cdots \wedge \mathrm{d}_{\mathrm{dR}} a_k := \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi \phi(a_{\pi(0)}, a_{\pi(1)}, \ldots, a_{\pi(k)})$$

the skewsymmetrisation of ϕ, is an isomorphism. Moreover, under this isomorphism, the map $(1/k)IB\colon HH^k(C^\infty(M)) \to HH^{k-1}(C^\infty(M))$ corresponds to the de Rham boundary $\partial\colon \Omega_k(M) \to \Omega_{k-1}(M)$.

We will only give a sketch of the proof of this Theorem. For details, one can consult [6] or [2, Section 8.5].

An important ingredient in the proof is that Theorem II.4 also holds in the topological case. To be precise, if $\mathcal{A}$ is a *topological* module over a complete locally convex algebra $\mathcal{B}$, then the Hochschild cohomology $HH^\bullet(\mathcal{A})$ coincides with the cohomology of the complex

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{M}^0, \mathcal{A}^*) \xrightarrow{b_1} \mathrm{Hom}_{\mathcal{B}}(\mathcal{M}^1, \mathcal{A}^*) \xrightarrow{b_2} \cdots,$$

where $(\mathcal{M}^\bullet, b)$ is a topological projective resolution of $\mathcal{A}$ over $\mathcal{B} := \mathcal{A}\hat{\otimes}\mathcal{A}^{\mathrm{o}}$, with $\hat{\otimes}$ the projective tensor product. In our case, with the algebra $\mathcal{A} = C^\infty(M)$, we have the isomorphisms $C^\infty(M)\hat{\otimes}C^\infty(M)^{\mathrm{o}} \cong C^\infty(M)\hat{\otimes}C^\infty(M) \cong C^\infty(M \times M)$, allowing us to view an $C^\infty(M)\hat{\otimes}C^\infty(M)^{\mathrm{o}}$-module as an $C^\infty(M \times M)$-module.

For now, we will assume that M has Euler characteristic $\chi(M)$ equal to zero. This assumption is necessary to construct a vector field $X \in \mathfrak{X}(M^2)$ that induces the contraction maps ι_X in the sequence (IV.1). After having shown that this sequence constitutes a topological projective resolution of $C^\infty(M)$, we will apply the topological version of Theorem II.4 to compute $HH^\bullet(C^\infty(M))$.

Consider then the sequence

$$\begin{array}{l} 0 \longrightarrow \Gamma^\infty(M^2, E_n) \xrightarrow{\iota_X} \cdots \xrightarrow{\iota_X} \Gamma^\infty(M^2, E_1) \\ \xrightarrow{\iota_X} C^\infty(M^2) \xrightarrow{\Delta^*} C^\infty(M) \longrightarrow 0, \end{array} \tag{IV.1}$$

The vector bundles $E_k \to M \times M$ defined as the pullback bundles under the projection $\mathrm{pr}_2 \colon M \times M \twoheadrightarrow M$ of the kth exterior power of the complexified cotangent bundle of M,

$$E_k := \mathrm{pr}_2^* \left(\bigwedge^k T^*_{\mathbb{C}}(M) \right).$$

The maps are given as follows: Δ^* is induced by the diagonal inclusion $\Delta \hookrightarrow M \to M \times M$, and ι_X is the contraction of a section of E_k with the section $X \in \Gamma^\infty(E_1^*)$, satisfying

a) $X_{(x,y)} = \exp_y^{-1}(x)$ for x, y close enough to the diagonal of $M \times M$, and where $\exp_y \colon T_y M \to M$ is the exponential map corresponding to a fixed Riemannian metric;

b) $X_{(x,y)} \neq 0$ if $x \neq y$.

Item a) defines X near the diagonal of $M \times M$. We extend X to the whole of $M \times M$ as follows. First, choose a bump function ψ that is 1 on a sufficiently small neighbourhood U of the diagonal. Moreover, choose a bump function ρ that is 1 on a neighbourhood of the diagonal and 0 outside U. Under the assumption that $\chi(M) = 0$, M admits a real nowhere vanishing vector field Y'. Setting $Y_{(x,y)} := i\rho(x,y)Y'_y$ yields a vector field on $M \times M$ that vanishes only on the diagonal of $M \times M$. Thus, we will replace X by

$$(x, y) \mapsto \psi(x,y)X_{(x,y)} + i(1 - \rho(x,y))Y'_y,$$

which defines a vector field that vanishes if and only if $x = y$.

It is clear that $\iota_X^2 = 0$, and the two properties listed above ensure that $\Delta^* \circ \iota_X = 0$. To show the sequence IV.1 is a topological projective resolution of $C^\infty(M)$, it remains to find a continuous contracting homotopy, showing that the sequence (IV.1) is exact. This is a technical affair, which we leave to [2].

Applying the functor $\mathrm{Hom}_{C^\infty(M^2)}(\cdot, C^\infty(M)^*)$ to the sequence (IV.1) induces a sequence whose cohomology is isomorphic to $HH^\bullet(C^\infty(M))$, by virtue of the topological version of Theorem II.4. Moreover, we have the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{C^\infty(M^2)}(\Gamma^\infty(M^2, E_k), C^\infty(M)^*) &\cong (\Gamma^\infty(M^2, E_k) \otimes_{C^\infty(M^2)} C^\infty(M))^* \\ &\cong \Omega^k_{\mathrm{dR}}(M)^* \\ &= \Omega_k(M).\end{aligned}$$

The first isomorphism follows from the tensor-hom adjunction, and the second one follows from

$$\Gamma^\infty(M^2, E_k) \otimes_{C^\infty(M^2)} C^\infty(M) \cong \Gamma^\infty(M, \Delta^* E^k) = \Omega^k_{\mathrm{dR}}(M),$$

which in turn is a direct application of the smooth version of [2, Proposition 2.12]. It remains to examine the boundary maps

$$\iota_X^* \colon \mathrm{Hom}_{C^\infty(M^2)}(\Gamma^\infty(M^2, E_k), C^\infty(M)^*) \to \mathrm{Hom}_{C^\infty(M^2)}(\Gamma^\infty(M^2, E_{k+1}), C^\infty(M)^*).$$

Since this is a map of $C^\infty(M^2)$-modules, and since X vanishes on the diagonal of M^2, ι_X^* is the zero map. Therefore, the cohomology groups of the complex

$$(\mathrm{Hom}_{C^\infty(M^2)}(\Gamma^\infty(M^2, E_\bullet), C^\infty(M)^*), \iota_X^*)$$

are simply the component modules themselves. In particular, this implies the isomorphism $HH^k(C^\infty(M)) \cong \Omega_k(M)$, at least in the case where the Euler characteristic $\chi(M)$ of M vanishes.

Let $\alpha\colon HH^k(C^\infty(M)) \to \Omega_k(M)$ be the map given by $[\phi] \mapsto C_\phi$, and let $\beta\colon \Omega_k(M) \to HH^k(C^\infty(M))$ be given by $C \mapsto \phi_C$, with

$$\phi_C(a_0, a_1, \ldots, a_k) := \int_C a_0 \, \mathrm{d}_{\mathrm{dR}} a_1 \wedge \cdots \wedge \mathrm{d}_{\mathrm{dR}} a_k,$$

for $a_0, \ldots, a_k \in C^\infty(M)$. It follows from the details of the proof of Theorem IV.4 that the maps α and β are well-defined morphisms, and that $\alpha \circ \beta = \mathrm{id}$, without assumptions on $\chi(M)$. If $\chi(M) = 0$, it also holds that $\beta \circ \alpha = \mathrm{id}$.

In case $\chi(M) \neq 0$, we consider $M \times \mathbb{T}^1$. The latter does have Euler characteristic zero, since $\chi(M \times \mathbb{T}^1) = \chi(M)\chi(\mathbb{T}^1) = 0$, by [25, Theorem 1, p. 481]. One can then show that the map

$$\begin{aligned} r\colon C^\infty(M \times \mathbb{T}^1) &\to C^\infty(M), \\ c &\mapsto c(\cdot, 1)\end{aligned}$$

induces a split injection $r^*\colon HH^k(C^\infty(M)) \to HH^k(C^\infty(M \times \mathbb{T}^1))$. Therefore, the map $\mathrm{id}_{HH^k(C^\infty(M\times\mathbb{T}^1))} = \beta \circ \alpha\colon HH^k(C^\infty(M \times \mathbb{T}^1)) \to HH^k(C^\infty(M \times \mathbb{T}^1))$ restricts to the identity map on $HH^k(C^\infty(M))$, so that the result of Theorem IV.4 holds for general M.

The statement about the correspondence between the boundary maps $(1/k)IB$ and ∂ follows from a straightforward computation that can be found in [2, Proposition 8.18].

IV.2 Computation of space of currents on the 2-torus

As an illustration of Theorem IV.4, we will use it to compute the space $\Omega_k(\mathbb{T}^2)$ of currents on the 2-torus. We will show that every class $[\phi] \in HH^k(C^\infty(\mathbb{T}^2))$ can be represented by a Hochschild cochain ϕ that satisfies

$$\phi(f_0, \ldots, f_k) = \phi(f_0 \partial_{\mu_1} f_1 \cdots \partial_{\mu_k} f_k, z^{\mu_1}, \ldots, z^{\mu_k}),$$

where we use the Einstein summation convention. In other words, we can interpret ϕ as a current

$$C_\phi \colon f_0 \, \mathrm{d_{dR}} f_1 \wedge \cdots \wedge \mathrm{d_{dR}} f_k = f_0 \partial_{\mu_1} f_1 \cdots \partial_{\mu_k} f_k \, \mathrm{d_{dR}} z^{\mu_1} \wedge \cdots \wedge \mathrm{d_{dR}} z^{\mu_k} \mapsto \phi(f_0, \ldots, f_k).$$

The proof of this statement also nicely illustrates the necessity of working with *continuous* Hochschild cochains in Theorem IV.4.

The de Rham complex of $\mathbb{T}^2$ is

$$\begin{aligned} &\Omega^0(\mathbb{T}^2) = C^\infty(\mathbb{T}^2) \\ &\Omega^1(\mathbb{T}^2) \cong C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2) \\ &\Omega^2(\mathbb{T}^2) \cong C^\infty(\mathbb{T}^2) \\ &\Omega^n(\mathbb{T}^2) \cong 0 \text{ for all } n > 2. \end{aligned} \tag{IV.2}$$

Proposition IV.5. *We have* $HH^0(C^\infty(\mathbb{T}^2)) \cong C^\infty(\mathbb{T}^2)^* =: \Omega_0(M)$.

Proof. This follows straight from the definition of the Hochschild cohomology groups, since

$$HH^0(C^\infty(\mathbb{T}^2)) := \ker\left(b \colon C^\infty(\mathbb{T}^2)^* \to (C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2))^*\right) = C^\infty(\mathbb{T}^2)^*.$$

The second equality is the result of $C^\infty(\mathbb{T}^2)$ being commutative. □

Proposition IV.6. *Let* $[\phi] \in HH^1(C^\infty(\mathbb{T}^2))$*, represented by a Hochschild cocycle*

$$\phi \colon C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2) \to \mathbb{C}.$$

Then ϕ *is well-defined as a multilinear functional. Furthermore, for any* $f, g \in C^\infty(\mathbb{T}^2)$*, we have*

$$\phi(f, g) = \phi(f \partial_z g, z) + \phi(f \partial_w g, w),$$

where $z, w \colon \mathbb{T}^2 \to \mathbb{T}^1$ *are the projections to the first and second coordinate, respectively. In other words, it suffices to know* $\phi(q, z)$ *and* $\phi(q, w)$ *for all* $q \in C^\infty(\mathbb{T}^2)$ *to determine* $\phi(f, g)$*. In particular, this implies that*

$$HH^1(C^\infty(\mathbb{T}^2)) \cong (C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2))^* \cong C^\infty(\mathbb{T}^2)^* \oplus C^\infty(\mathbb{T}^2)^*,$$

with the first isomorphism given by

$$[\phi] \mapsto ((f, g) \mapsto (\phi(f \partial_z g, z), \phi(f \partial_w g, w))).$$

Proof. We have already seen in the proof of Proposition IV.5 that

$$b\colon C^\infty(\mathbb{T}^2)^* \to (C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2))^*$$

is the zero map. Hence, $\phi\colon C^\infty(\mathbb{T}^2) \oplus C^\infty(\mathbb{T}^2) \to \mathbb{C}$ is well defined. For the second part of the claim of Proposition IV.6, we note that for a representative ϕ of $[\phi] \in HH^1(C^\infty(\mathbb{T}^2))$, we have

$$\phi(fg, h) - \phi(f, gh) + \phi(fh, g) = 0 \tag{IV.3}$$

for all $f, g, h \in C^\infty(\mathbb{T}^2)$. Take $g = z^{n-1}$ and $h = z$, for $n \in \mathbb{N}$. Then $\phi(f, z^n) = \phi(fz^{n-1}, z) + \phi(fz, z^{n-1})$. Continuing inductively, follows that

$$\phi(f, z^n) = n\phi(fz^{n-1}, z). \tag{IV.4}$$

Similarly, $\phi(f, \bar{z}^n) = n\phi(f\bar{z}^{n-1}, \bar{z})$. By the continuous analogue of Corollary II.5, which follows from [6, Lemma 43], we may assume that ϕ is normalised, i.e. $\phi(f, 1) = 0$ for all $f \in C^\infty(\mathbb{T}^2)$. Therefore, we may again apply Eq. (IV.3) (with $g = z$ and $h = \bar{z} = z^{-1}$, and f replaced by fz^{-1}) to find

$$\phi(f, z^{-1}) = -\phi(fz^{-2}, z).$$

More generally, we have

$$\phi(f, z^{-n}) = n\phi(fz^{-n+1}, z^{-1}) = -n\phi(fz^{-n-1}, z). \tag{IV.5}$$

Therefore, Eq. (IV.4) is true for all $n \in \mathbb{Z}$. Of course, this Equation also holds with z replaced by w.

This allows us to reduce the more general case to a sum of terms that have become tangible by the previous paragraphs:

$$\phi(f, z^n w^m) = \phi(fz^n, w^m) + \phi(fw^m, z^n) = m\phi(fz^n w^{m-1}, w) + n\phi(fz^{n-1}w^m, z), \tag{IV.6}$$

for $n, m \in \mathbb{Z}$.

Now note that any $g \in C^\infty(\mathbb{T}^2)$ admits a Fourier decomposition given by $g(z, w) = \sum_{n,m\in\mathbb{Z}} a_{nm} z^n w^m$, with coefficients $a_{nm} \in \mathbb{C}$. Since ϕ is continuous with respect to the Fréchet topology, we may take out the sums, apply the result of Eq. (IV.6), and reinstate the sums to find

$$\begin{aligned}
\phi(f, g) &= \sum_{n,m\in\mathbb{Z}} a_{nm}\phi(f, z^n w^m) \\
&= \sum_{n,m\in\mathbb{Z}} a_{nm}\left(m\phi(fz^n w^{m-1}, w) + n\phi(fz^{n-1}w^m, z)\right) \\
&= \phi(f\partial_z g, z) + \phi(f\partial_w g, w),
\end{aligned} \tag{IV.7}$$

which proves the claim. □

It is worth remarking that so far, we have not relied on Theorem IV.4 to compute the Hochschild cohmology groups $HH^0(C^\infty(\mathbb{T}^2))$ and $HH^1(C^\infty(\mathbb{T}^2))$. However, in higher degrees the use of the Theorem is inevitable. More specifically, we need it to show that for compact manifolds M, the antisymmetrisation map $\mathbb{A}_k\colon HH^k(C^\infty(M)) \to HH^k(C^\infty(M))$, induced by

$$(\mathbb{A}_k\phi)(f_0,\dots,f_k) \coloneqq \frac{1}{k!}\sum_{\tau\in S_k}(-1)^\tau\phi(f_0,f_{\tau(1)},\dots,f_{\tau(k)})$$

is an isomorphism.

The next Proposition is of course a direct consequence of Theorem IV.4, and in its proof heavily relies on $\mathbb{A}_k$ being an isomorphism. Nevertheless, it is a nice demonstration of the use of the property $b\phi = 0$ to arrive at the desired conclusion.

Proposition IV.7. *let* $[\phi] \in HH^2(C^\infty(\mathbb{T}^2))$. *Then* $\mathbb{A}_2\phi\colon C^\infty(\mathbb{T}^2)\oplus C^\infty(\mathbb{T}^2)\oplus C^\infty(\mathbb{T}^2) \to \mathbb{C}$ *is a well-defined multilinear functional. Furthermore, for any* $f, g, h \in C^\infty(\mathbb{T}^2)$, *we have*

$$\phi(f,g,h) = \phi(f(\partial_z g\partial_w h - \partial_w h\partial_z g), z, w),$$

i.e. it suffices to know $\phi(q, z, w)$ *for all* $q \in C^\infty(\mathbb{T}^2)$ *to determine* $\phi(f, g, h)$. *In particular, this implies that*

$$HH^2(C^\infty(\mathbb{T}^2)) \cong C^\infty(\mathbb{T}^2)^*.$$

Proof. Let ϕ represent the class $[\phi] \in HH^2(C^\infty(\mathbb{T}))$. We may and will again assume that ϕ is normalised. The map $\mathbb{A}_2\phi$ is well-defined since $\mathbb{A}_2 b\psi = 0$ for any coboundary $b\psi$. Moreover, we know that $\mathbb{A}_2$ is an isomorphism in cohomology, so that we may as well assume that $\phi = \mathbb{A}_2\phi$. Since $b\phi = 0$, we have for all $f, g, h, k \in C^\infty(\mathbb{T}^2)$ that

$$\phi(fg,h,k) - \phi(f,gh,k) + \phi(f,g,hk) - \phi(fk,g,h) = 0. \tag{IV.8}$$

This equation will play a central role in the remainder of this proof, which we will divide into two parts.

Claim IV.8. $\phi(f, z^n, z^m) = 0$ *for all* $n, m \in \mathbb{Z}$ *and* $f \in C^\infty(\mathbb{T}^2)$.

Proof. We start with $n \in \mathbb{N}$ and $m = 1$. By antisymmetry of ϕ, we know the Claim to be true for $n = 1$. Now take $g = k = z$, $h = z^{n-1}$, and $f \in C^\infty(\mathbb{T}^2)$ arbitrary. Then Eq. (IV.8) reads

$$\phi(fz,z^{n-1},z) - \phi(f,z^n,z) + \phi(f,z,z^n) - \phi(fz,z,z^{n-1}) = 0.$$

Since ϕ is antisymmetric, this is equivalent to $\phi(f, z^n, z) = \phi(fz, z^{n-1}, z)$. By induction, it follows that $\phi(f, z^n, z) = 0$.

Therefore, for arbitrary positive m and n, Eq. (IV.8) allows us to write

$$\begin{aligned}\phi(f,z^n,z^m) &= \phi(f,z^{n-1}z,z^m)\\ &= \phi(fz^{n-1},z,z^m) + \phi(f,z^{n-1},z^{m+1}) - \phi(fz^m,z^{n-1},z)\\ &= \phi(f,z^{n-1},z^{m+1}).\end{aligned}$$

Thus, we can lower the first exponent at the cost of increasing the second one. Doing so $n-1$ times, we find

$$\phi(f, z^n, z^m) = \phi(f, z, z^{m+n-1}) = 0.$$

Upon replacing z by $\bar{z} = z^{-1}$, it becomes clear that $\phi(f, z^n, z^m) = 0$ also when m and n are both negative.

It remains to consider the case where the exponents have opposite sign. Consider Eq. (IV.8) with $g = k = \bar{z}$, $h = z$, and f replaced by fz. Then Eq. (IV.8) reads

$$\phi(f, z, \bar{z}) - \phi(fz, 1, \bar{z}) + \phi(fz, \bar{z}, 1) - \phi(f, \bar{z}, z) = 0,$$

so that $\phi(f, z, \bar{z}) = 0$, since ϕ is normalised and antisymmetric. Hence, for $m > 0$,

$$\begin{aligned}\phi(f, \bar{z}, z^m) &= \phi(f\bar{z}z, \bar{z}, z^m) \\ &= \phi(f\bar{z}, 1, z^m) - \phi(f\bar{z}, z, z^m\bar{z}) + \phi(fz^m, z, \bar{z}) \\ &= \phi(f\bar{z}, z, z^{m-1}) \\ &= 0.\end{aligned}$$

Now, let $n, m > 0$. Using Eq. (IV.8) once more, we find

$$\begin{aligned}\phi(f, z^n, \bar{z}^m) &= \phi(f, z^{n-1}z, \bar{z}^m) \\ &= \phi(fz^{n-1}, z, \bar{z}^m) + \phi(f, z^{n-1}, \bar{z}^m z) - \phi(f\bar{z}^m, z^{n-1}, z).\end{aligned}$$

We have already shown that $\phi(fz^{n-1}, z, \bar{z}^m) = \phi(f\bar{z}^m, z^{n-1}, z) = 0$. Thus, $\phi(f, z^n, \bar{z}^m) = \phi(f, z^{n-1}, \bar{z}^{m-1})$, so that we can simultaneously lower n and m by 1. It therefore follows by induction that $\phi(f, z^n, \bar{z}^m) = 0$. All in all, we conclude that $\phi(f, z^n, z^m) = 0$ for all $n, m \in \mathbb{Z}$, and $f \in C^\infty(\mathbb{T}^2)$. Of course, this argument can be copied to show the same result with z replaced by w. □

Claim IV.9. *For all $n, m, k, l \in \mathbb{Z}$, and $f \in C^\infty(\mathbb{T}^2)$, we have $\phi(f, z^n w^m, z^k w^l) = (nl - mk)\phi(fz^{n+k-1}w^{m+l-1}, z, w)$.*

Proof. We start by showing that for all $f, q \in C^\infty(\mathbb{T}^2)$, $n \in \mathbb{N}$, we have $\phi(f, z^n, q) = n\phi(fz^{n-1}, z, q)$. By induction, suppose there exists a $k \leq n$ such that for all $m \leq k$, we have

$$\phi(f, z^n, q) = m\phi(fz^{n-1}, z, q) + \phi(f, z^{n-m}, z^m q). \tag{IV.9}$$

Clearly, this holds true for $m = 0$. Then, for $m = k$, we have

$$\phi(f, z^{n-k}, z^k q) = \phi(fz^{n-k-1}, z, z^k q) + \phi(f, z^{n-k-1}, z^{k+1}q) - \phi(fz^k q, z^{n-k-1}, z), \tag{IV.10}$$

by Eq. (IV.8). The final term of the equation above vanishes, by Claim IV.8. For the term $\phi(fz^{n-k-1}, z, z^k q)$, we find

$$\begin{aligned}\phi(fz^{n-k-1}, z, z^k q) &= \phi(fz^{n-k-1}, z, z^{k-1}qz) \\ &= -\phi(fz^{n-k}, z^{k-1}q, z) + \phi(fz^{n-k-1}, z^k q, z) + \phi(fz^{n-k}, z, z^{k-1}q).\end{aligned}$$

Since ϕ is antisymmetric, this implies $\phi(fz^{n-k-1}, z, z^k q) = \phi(fz^{n-k}, z, z^{k-1}q)$. Inductively repeating this argument gives us

$$\phi(fz^{n-k-1}, z, z^k q) = \phi(fz^{n-1}, z, q). \tag{IV.11}$$

Combining Eqs. (IV.9) to (IV.11), we find

$$\begin{aligned}\phi(f, z^n, q) &= k\phi(fz^{n-1}, z, q) + \phi(f, z^{n-k}, z^k q)\\ &= k\phi(fz^{n-1}, z, q) + \phi(fz^{n-k-1}, z, z^k q) + \phi(f, z^{n-k-1}, z^{k+1}q)\\ &= k\phi(fz^{n-1}, z, q) + \phi(fz^{n-1}, z, q) + \phi(f, z^{n-k-1}, z^{k+1}q)\\ &= (k+1)\phi(fz^{n-1}, z, q) + \phi(f, z^{n-k-1}, z^{k+1}q),\end{aligned}$$

completing the induction process.

Most importantly, for $m = n$, Eq. (IV.9) simply reads

$$\phi(f, z^n, q) = n\phi(fz^{n-1}, z, q).$$

This result can be extended to negative n. Write $m = -n > 0$. Then

$$\phi(f, z^n, q) = \phi(f, \bar{z}^m, q) = m\phi(f\bar{z}^{m-1}, \bar{z}, q).$$

We use Eq. (IV.8) to write

$$\phi(f\bar{z}^m z, \bar{z}, q) = \phi(f\bar{z}^m, 1, q) - \phi(f\bar{z}^m, z, \bar{z}q) + \phi(fq\bar{z}^m, z, \bar{z}) = -\phi(f\bar{z}^m, z, \bar{z}q),$$

where the last equality follows from normalisation of ϕ and Claim IV.8. Now,

$$\phi(f\bar{z}^m, z, q\bar{z}) = -\phi(f\bar{z}^{m-1}, q, \bar{z}) + \phi(f\bar{z}^m, zq, \bar{z}) + \phi(f\bar{z}^{m+1}, z, q).$$

By Eq. (IV.8) we have

$$\phi(f\bar{z}^m, zq, \bar{z}) = \phi(f\bar{z}^{m-1}, q, \bar{z}) - \phi(f\bar{z}^{m-1}, \bar{z}, q) + \phi(f\bar{z}^m, \bar{z}, zq).$$

By antisymmetry of ϕ, this implies $\phi(f\bar{z}^{m-1}, q, \bar{z}) = \phi(f\bar{z}^m, zq, \bar{z})$. Thus,

$$\phi(f, z^n, q) = m\phi(f\bar{z}^{m-1}, \bar{z}, q) = -m\phi(f\bar{z}^{m+1}, z, q) = n\phi(fz^{n-1}, z, q). \tag{IV.12}$$

Now our hard work is finally going to pay off. Fix $n, m, k, l \in \mathbb{Z}$, and $f \in C^\infty(\mathbb{T}^2)$. Then,

$$\begin{aligned}\phi(f, z^n w^m, z^k w^l) &= -\phi(fz^n w^m, z^k, w^l) + \phi(f, z^{n+k}w^m, w^l) + \phi(fw^l, z^n w^m, z^k)\\ &= -\phi(fz^n w^m, z^k, w^l)\\ &\qquad + \phi(fz^{n+k}, w^m, w^l) + \phi(f, z^{n+k}, w^{m+l}) - \phi(fw^l, z^{n+k}, w^m)\\ &\qquad + \phi(fw^{l+m}, z^n, z^k) + \phi(fw^l, w^m, z^{n+k}) - \phi(fw^l z^k, w^m, z^n).\end{aligned}$$

For both the first and the second equality, we have used Eq. (IV.8). To each term we can apply our earlier found result (IV.12):

$$\begin{aligned}\phi(f, z^n w^m, z^k w^l) &= (-kl + (n+k)(m+l) - (n+k)m\\ &\qquad - m(n+k) + nm)\phi(fz^{n+k-1}w^{m+l-1}, z, w)\\ &= (nl - mk)\phi(fz^{n+k-1}w^{m+l-1}, z, w),\end{aligned}$$

which finishes the proof of Claim IV.9. □

Using the continuity of ϕ in the same way as in the proof of Proposition IV.6 yields the desired result. □

IV.3 Cyclic cohomology in terms of de Rham homology

The Connes–Hochschild–Kostant–Rosenberg Theorem allows us to compute the cyclic cohomology groups of $C^\infty(M)$, with M a compact manifold.

Theorem IV.10 ([2, Theorem 10.5]). *For each $k \in \mathbb{N}$, there is a canonical isomorphism*

$$HC^k(C^\infty(M)) \cong Z_k^{\mathrm{dR}} \oplus H_{k-2}^{\mathrm{dR}}(M) \oplus H_{k-4}^{\mathrm{dR}}(M) \oplus \cdots \oplus H_{\#k}^{\mathrm{dR}}(M),$$

where Z_k^{dR} is the space of closed k-currents on M, and $\#k = 0$ or 1 according as k is even or odd.

Since the proof of this result provides an illustrative example of the use of the Connes long exact sequence of Theorem II.12, we include it in this thesis. The proof follows [2].

Proof. Consider the class $[\phi]_\lambda \in HC^k(C^\infty(M))$ of a cyclic cocycle ϕ. Recall the maps I, B and S from Theorem II.12, with B given by $Ns'(1-\lambda)$ (Definition II.14), so $B\phi = 0$. Since the operation $(1/k)IB$ corresponds, under Theorem IV.4, to taking the boundary of a current, the current C_ϕ that is induced by ϕ (viewed as a Hochschild cocycle) is closed. For this reason, the antisymmetrisation $\mathbb{A}_k\phi$ of ϕ is a cyclic cocycle as well:

$$\begin{aligned}(\lambda\mathbb{A}_k\phi)(a_0, a_1, \ldots, a_k) &= (-1)^k(\mathbb{A}_k\phi)(a_k, a_0, \ldots, a_{k-1}) \\ &= (-1)^k \int_{C_\phi} a_k \,\mathrm{d_{dR}}a_0 \wedge \cdots \wedge \mathrm{d_{dR}}a_{k-1} \\ &= (-1)^k \int_{C_\phi} (\mathrm{d_{dR}}(a_0a_k) - a_0\,\mathrm{d_{dR}}a_k) \wedge \mathrm{d_{dR}}a_1 \wedge \cdots \wedge \mathrm{d_{dR}}a_{k-1},\end{aligned}$$

as follows from the Leibniz rule. Permuting $\mathrm{d_{dR}}a_k$ past the $(k-1)$-form $\mathrm{d_{dR}}a_1 \wedge \cdots \wedge \mathrm{d_{dR}}a_{k-1}$ yields

$$\begin{aligned}(\lambda\mathbb{A}_k\phi)(a_0, a_1, \ldots, a_k) &= -(-1)^k(-1)^{k-1} \int_{C_\phi} a_0\,\mathrm{d_{dR}}a_1 \wedge \cdots \wedge \mathrm{d_{dR}}a_k \\ &\qquad + (-1)^k \int_{C_\phi} \mathrm{d_{dR}}(a_0a_k) \wedge \mathrm{d_{dR}}a_1 \wedge \cdots \wedge \mathrm{d_{dR}}a_{k-1} \\ &= \phi(a_0, a_1, \ldots, a_k) \\ &\qquad + \int_{\partial C_\phi} a_0a_k\,\mathrm{d_{dR}}a_1 \wedge \cdots \wedge \mathrm{d_{dR}}a_{k-1} \\ &= \phi(a_0, a_1, \ldots, a_k),\end{aligned}$$

since $\partial C_\phi = 0$.

We know from Theorem IV.4 that $[\phi] = [\mathbb{A}_k\phi] \in HH^k(C^\infty(M))$, so that $[\phi - \mathbb{A}_k\phi]_\lambda \in \ker I = \operatorname{im} S$. Thus, there exists a $\psi_{k-2} \in Z_\lambda^{k-2}$ such that $S[\psi_{k-2}]_\lambda = [\phi - \mathbb{A}_k\phi]_\lambda$. The cyclic cocycle ψ_{k-2} is determined modulo $\ker S = \operatorname{im} B$. As a result, the corresponding current $C_{\psi_{k-2}}$ is determined up to a de Rham boundary, implying that the class $[C_{\psi_{k-2}}] \in H_{k-2}^{\mathrm{dR}}(M)$ depends uniquely on $[\phi]_\lambda$.

Now, we can apply the same treatment to ψ_{k-2}. Again, $\mathbb{A}_{k-2}\psi_{k-2} \in Z_\lambda^{k-2}$ and $I([\psi_{k-2} - \mathbb{A}_{k-2}\psi_{k-2}]_\lambda) = 0$, so we can find a cocycle $\psi_{k-4} \in Z_\lambda^{k-4}$ such that $S[\psi_{k-4}]_\lambda = [\psi_{k-2} - \mathbb{A}_{k-2}\psi_{k-2}]_\lambda$. Repeating this process produces cyclic cocycles $\psi_{k-2j} \in Z_\lambda^{k-2j}$ satisfying $S[\psi_{k-2j}]_\lambda = [\psi_{k-2(j-1)} - \mathbb{A}_{k-2(j-1)}\psi_{k-2(j-1)}]_\lambda$, with $j = 0, \ldots, \lfloor k/2 \rfloor$. Write $\phi_k \coloneqq \mathbb{A}_k\phi$, and $\phi_{k-2j} \coloneqq \mathbb{A}_{k-2j}\psi_{k-2j}$ for $j \geq 1$. We claim then that

$$\phi = \sum_{0 \leq 2j \leq k} S^j \phi_{k-2j} \mod B_\lambda^k. \tag{IV.13}$$

Since $\mathbb{A}_0$ and $\mathbb{A}_1$ are the respective identity maps, this claim is easy to verify if $k = 0$ or $k = 1$. For higher k, we have for $j > 0$

$$S^j \phi_{k-2j} = S^j \mathbb{A}_{k-2j}\psi_{k-2j} = S^j \psi_{k-2j} - S^{j+1}\psi_{k-2(j+1)} \mod B_\lambda^k,$$

and

$$\phi_k = \phi - S\psi_{k-2} \mod B_\lambda^k,$$

so that the sum in Eq. (IV.13) telescopes to $\phi \mod B_\lambda^k$.

Therefore, the map $HC^k(C^\infty(M)) \to H_k^{\mathrm{dR}}(M)$ given by

$$[\phi]_\lambda = [\phi_k]_\lambda + [S\phi_{k-2}]_\lambda + \cdots + [S^{\lfloor k/2 \rfloor}\phi_{\# k}]_\lambda \mapsto C_\phi + [C_{\phi_{k-2}}] + \cdots + [C_{\phi_{\# k}}]$$

is well-defined. Its inverse is induced by the map sending a $(k-2j)$-current Γ_{k-2j} to the cyclic cocycle ϕ_{k-2j} given by

$$\phi_{k-2j}(a_0, a_1, \ldots, a_{k-2j}) \coloneqq \int_{\Gamma_{k-2j}} a_0 \, \mathrm{d}_{\mathrm{dR}} a_1 \wedge \cdots \wedge \mathrm{d}_{\mathrm{dR}} a_{k-2j}. \qquad \square$$

Recall from Definition II.17 that the even/odd periodic cyclic cohomology groups are defined as the direct limits of the even/odd cyclic cohomology groups. Since $\Omega^k(M) = 0$ if $k > \dim M$, Theorem IV.10 shows that the cyclic cohomology groups stabilise, implying the following result.

Corollary IV.11 ([2, Theorem 10.6]). *On a compact manifold M, we have the isomorphisms*

$$HP^0(C^\infty(M)) \cong H_{\mathrm{even}}^{\mathrm{dR}}(M), \quad HP^1(C^\infty(M)) \cong H_{\mathrm{odd}}^{\mathrm{dR}}(M).$$

Chapter V

K-theory

K-theory provides yet another functorial way to extract information from C^*-algebras. In particular, to any C^*-algebra $\mathcal{A}$, we can associate two abelian groups, $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. As the K_1-functor suffices for our purposes, we will ignore K_0, and we will neither touch on deeper results in K-theory, such as the index map and the six-term exact sequence. All we will eventually need is that there is a pairing $HE^1(\mathcal{A}) \times K_1(\mathcal{A}) \to \mathbb{C}$. For more details, one can consult [24], on which we base large parts of this chapter.

V.1 The K_1-functor

Definition V.1. Let $\mathcal{A}$ be a unital C^*-algebra. We will write $\mathcal{U}(\mathcal{A})$ for its group of unitary elements. Moreover, define

$$\mathcal{U}_n(\mathcal{A}) \coloneqq \mathcal{U}(M_n(\mathcal{A})), \quad \text{and} \quad \mathcal{U}_\infty(\mathcal{A}) \coloneqq \bigcup_{n=1}^{\infty} \mathcal{U}_n(\mathcal{A}).$$

Define a binary operation $\oplus$ on $\mathcal{U}_\infty(\mathcal{A})$, given by

$$u \oplus v \coloneqq \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(\mathcal{A}), \quad u \in \mathcal{U}_n(\mathcal{A}),\, v \in \mathcal{U}_m(\mathcal{A}).$$

With the operation $\oplus$, $\mathcal{U}_\infty(\mathcal{A})$ is a commutative semigroup. By dividing out an equivalence relation, we will turn it into an abelian group. This equivalence relation is defined in Definition V.3 and is an extension of the notion of homotopic paths in $\mathcal{U}_n(\mathcal{A})$.

Definition V.2. Let $u, v \in \mathcal{U}_n(\mathcal{A})$. Then u and v are called *homotopic*, denoted $u \sim_{\mathrm{h}} v$, if there exists a continuous path $f\colon [0,1] \to \mathcal{U}_n(\mathcal{A})$, such that $f(0) = u$ and $f(1) = v$.

The topology on $\mathcal{U}_n(\mathcal{A}) \subseteq M_n(\mathcal{A})$ is the subspace topology, where $M_n(\mathcal{A})$ is topologised as follows: one chooses an injective $*$-homomorphism $\phi\colon \mathcal{A} \to B(H)$, and extends ϕ to a map $\phi_n\colon M_n(\mathcal{A}) \to B(H^n)$ via

$$\phi_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \coloneqq \begin{pmatrix} \phi(a_{11})\xi_1 + \cdots + \phi(a_{1n})\xi_n \\ \vdots \\ \phi(a_{n1})\xi_1 + \cdots + \phi(a_{nn})\xi_n \end{pmatrix}.$$

One can then define a norm on $M_n(\mathcal{A})$ by $\|a\| = \|\phi_n(a)\|$, for $a \in M_n(\mathcal{A})$; this norm is independent of the choice of ϕ.

Since two unitaries in $\mathcal{U}_\infty(\mathcal{A})$ need not have the same dimension, we extend the homotopy relation to allow for comparing matrices of different sizes.

Definition V.3. Let $u \in \mathcal{U}_n(\mathcal{A})$, $v \in \mathcal{U}_m(\mathcal{A})$. Then u *is equivalent to* v, denoted $u \sim_1 v$, if there exists a $k \geq \max\{n, m\}$ such that $u \oplus 1_{k-n} \sim_{\mathrm{h}} v \oplus 1_{k-m}$ in $\mathcal{U}_k(\mathcal{A})$, where 1_r is the identity matrix in $M_r(\mathcal{A})$, the space of $r \times r$ matrices with coefficients in $\mathcal{A}$.

The following properties of $\sim_1$ follow easily from its definition.

Lemma V.4. *Let $\mathcal{A}$ be a unital C^*-algebra. Then*

(i) $\sim_1$ *is an equivalence relation on* $\mathcal{U}_\infty(\mathcal{A})$,

(ii) $u \sim_1 u \oplus 1_n$ *for all* $u \in \mathcal{U}_\infty(\mathcal{A})$ *and* $n \in \mathbb{N}$,

(iii) $u \oplus v \sim_1 v \oplus u$ *for all* $u, v \in \mathcal{U}_\infty(\mathcal{A})$,

(iv) for $u, v, u', v' \in \mathcal{U}_\infty(\mathcal{A})$ *such that* $u \sim_1 u'$ *and* $v \sim_1 v'$, *we have* $u \oplus v \sim_1 u' \oplus v'$,

(v) if $u, v \in \mathcal{U}_n(\mathcal{A})$ *for some* n, *then* $uv \sim_1 vu \sim_1 u \oplus v$,

(vi) $(u \oplus v) \oplus w \sim_1 u \oplus (v \oplus w)$ *for all* $u, v, w, \in \mathcal{U}_\infty(\mathcal{A})$.

Proof. See [24, Lemma 8.1.2]. □

Definition V.5. Let $\mathcal{A}$ be a unital C^*-algebra. Its K_1*-group* is defined as $K_1(\mathcal{A}) := \mathcal{U}_\infty(\mathcal{A})/\sim_1$, with binary operation given by $[u] + [v] := [u \oplus v]$, with $[u]$ denoting the equivalence class of $u \in \mathcal{U}_\infty(\mathcal{A})$.

$(K_1(\mathcal{A}), +)$ is indeed an abelian group: it follows from Lemma V.4 that $+$ is well-defined, commutative and associative; its identity element is $[1_n]$ for any $n \in \mathbb{N}$, and $-[u] = [u^*]$.

Example V.6 ([24])**.** The K_1-group of the continuous functions on the circle is generated by the class represented by the function $e_1 \colon \mathbb{T}^1 \to \mathbb{C}$, given by $z \mapsto z$. Explicitly, the isomorphism is given by the following sequence:

$$K_1(C(\mathbb{T}^1)) \xrightarrow[\Delta]{\sim} \mathcal{U}(C(\mathbb{T}^1))/\mathcal{U}_0(C(\mathbb{T}^1)) \xrightarrow[w]{\sim} \mathbb{Z}.$$

Here Δ is the map induced by the determinant map det, which is for an arbitrary unital C^*-algebra $\mathcal{A}$ defined by

$$\begin{gathered} \det \colon M_k(\mathcal{A}) \to \mathcal{U}(\mathcal{A})/\mathcal{U}_0(\mathcal{A}), \\ \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \mapsto \sum_{\sigma \in S_k} (-1)^\sigma \prod_{j=1}^{k} a_{j\sigma(j)}, \end{gathered}$$

and $\mathcal{U}_0(C(\mathbb{T}^1))$ is the connected component of the identity. One can show that det descends to a well-defined group homomorphism $\Delta\colon K_1(\mathcal{A}) \to \mathcal{U}(\mathcal{A})/\mathcal{U}_0(\mathcal{A})$, which is an isomorphism if $\mathcal{A} = C(\mathbb{T}^1)$ [24]. The map w is called the *winding number*, and is induced by the map

$$\mathcal{U}(C(\mathbb{T}^1)) = C(\mathbb{T}^1, \mathbb{T}^1) \to \mathbb{Z}$$
$$u \mapsto \frac{1}{2\pi i}\int_{\mathbb{T}^1} u^* \,\mathrm{d}u.$$

Again, one can show that this descends to an isomorphism $\mathcal{U}(C(\mathbb{T}^1))/\mathcal{U}_0(C(\mathbb{T}^1)) \to \mathbb{Z}$ [17].

V.2 Pairing of cyclic cohomology with K-theory on manifolds

Let $\mathcal{A}$ be a unital C^*-algebra, and let $\psi \in C^n(\mathcal{A}, \mathcal{A}^*)$ be a Hochschild cochain. Define $\psi^q\colon (M_q(\mathcal{A}))^{n+1} \to \mathbb{C}$ by linear extension of

$$\operatorname{tr}\#\psi\colon (\mu_0 \otimes a_0, \ldots, \mu_n \otimes a_n) \longmapsto \operatorname{tr}(\mu_0 \cdots \mu_n)\psi(a_0, \ldots, a_n),$$

for $a_0, \ldots, a_n \in \mathcal{A}$ and $\mu_0, \ldots, \mu_n \in M_q(\mathbb{C})$.

Now, suppose $\psi = (\psi_{2k+1})$ is an entire entire cocycle, and suppose $u \in \mathcal{U}_q(\mathcal{A})$. Then we define the pairing

$$\langle \psi, u\rangle := (2\pi i)^{-1/2}\sum_{k=0}^{\infty}(-1)^k k!\psi^q_{2k+1}\left(u^*, u, \ldots, u^*, u\right).$$

In [7] Connes shows that his pairing descends to a pairing $HE^1(\mathcal{A}) \times K_1^{\mathrm{alg}}(\mathcal{A}) \to \mathbb{C}$, where $K_1^{\mathrm{alg}}(\mathcal{A})$ is the *algebraic* K_1-group of $\mathcal{A}$. $K_1^{\mathrm{alg}}(\mathcal{A})$ is defined as the quotient of $\mathrm{GL}_\infty(\mathcal{A})$ by it commutator subgroup, where $\mathrm{GL}_\infty(\mathcal{A})$ is the inductive limit of the groups $\mathrm{GL}_n(\mathcal{A})$ of invertible $n \times n$ matrices with coefficients in $\mathcal{A}$, under the embedding maps

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

There exists a comparison map $K_1^{\mathrm{alg}}(\mathcal{A}) \to K_1(\mathcal{A})$ between the algebraic and operator K_1-group. However, in general, this map is not an isomorphism. For example, $K_1^{\mathrm{alg}}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, but $K_1(\mathbb{C}) = 0$ [2, 24].

Since the algebraic K_1-groups are difficult to compute, we would like to define an analogous pairing, with the algebraic $K_1^{\mathrm{alg}}(\mathcal{A})$ replaced by its operator analogue $K_1(\mathcal{A})$. Moreover, we want to use this pairing to deduce properties of the entire cyclic cocycle $(\tilde{\psi}) \in HE^1(C^\infty(M))$, for M a compact manifold. This raises a further problem: $C^\infty(M)$ is not a C^*-algebra, thus we cannot use the toolkit from Section V.1 to define $K_1(C^\infty(M))$. There are two ways around this problem: first, one can extend the domain of definition of K_1 to Fréchet algebras, as is done in [5]. Since $C^\infty(M)$ is a dense subalgebra of $C(M)$ that is closed under the holomorphic functional calculus [2], and since the Fréchet topology on

$C^\infty(M)$ is finer than the norm topology on $C(M)$, one can in fact show that $K_1(C^\infty(M)) \cong K_1(C(M))$ [5, Corollary 3.9]. We will not follow this route any further, to avoid having to define K_1 for Fréchet algebras. Instead, we will simply replace $K_1(C^\infty(M))$ in the pairing by $K_1(C(M))$. This gives the following Proposition.

Proposition V.7. *Let M be a compact manifold. The pairing*

$$\begin{aligned}\langle\cdot,\cdot\rangle\colon HC^{2k+1}(C^\infty(M)) \times K_1(C(M)) &\to \mathbb{C}\\ ([\psi],[u]) &\mapsto \psi^q\left(u^*,u,\ldots,u^*,u\right),\end{aligned}$$

where $u \in \mathcal{U}_q(C(M))$ is a smooth *representative of the class $[u] \in K_1(C(M))$, is well-defined.*

Lemma V.8. *Each class $[u] \in K^1(C(M))$ has a smooth representative.*

Proof. This follows from Whitney's Approximation Theorem [18, Theorem 6.26]. Namely, let $u \in \mathcal{U}_k(C(M))$ be a continuous representative of $[u]$. We can view u as a map of smooth manifolds $u\colon M \to \mathrm{U}_k$, where U_k is the Lie group of unitary $k \times k$ matrices. Whitney's Approximation Theorem then tells us that there is a homotopy $H\colon [0,1]\times M \to \mathrm{U}_k$ from u to a smooth map $\tilde{u}\colon M \to \mathrm{U}_k$. Viewing H as a map $\tilde{H}\colon [0,1] \to \mathcal{U}_k(C(M))$ provides a path from u to $\tilde{u}$. We can show that $\tilde{H}$ is continuous by using continuity of H, compactness of M, and by noting that the topology on $\mathcal{U}_k(C(M)) = C(M,\mathrm{U}_k)$ is equivalent to the compact-open topology, and using [20, Theorem 46.11]. This shows that $[u]$ is represented by the smooth unitary $\tilde{u} \in \mathcal{U}_k(C^\infty(M))$. □

Proof of Proposition V.7. To show that the quantity $\psi(u^*,u,\ldots,u^*,u)$ does not depend on the class of u in the K-theory of $C(M)$, we again use Whitney's Approximation Theorem. Let $u,v \in \mathcal{U}_k(C^\infty(M))$ be two equivalent smooth representatives, so that there exists a path $f\colon [0,1] \to \mathcal{U}_k(C(M))$ satisfying $f(0) = u$ and $f(1) = v$. Again by [20, Theorem 46.11], we view f as a continuous map

$$\begin{aligned}\tilde{f}\colon [0,1]\times M &\to \mathrm{U}_k\\ (t,x) &\mapsto f(t)(x).\end{aligned}$$

Whitney's Approximation Theorem tells us that f is homotopic relative to $\{0,1\}\times M$ to a smooth map $\tilde{g}\colon [0,1]\times M \to \mathrm{U}_k$; thus, the map

$$\begin{aligned}g\colon [0,1] &\to \mathcal{U}_k(C^\infty(M))\\ t &\mapsto g_t := \tilde{g}(t,\cdot)\end{aligned}$$

provides a smooth path from u to v. Now consider

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi(g_t^*,g_t,\ldots,g_t^*,g_t) = \psi((g_t^*)',g_t,\ldots,g_t^*,g_t) + \psi(g_t^*,g_t',\ldots,g_t^*,g_t)\\ + \cdots + \psi(g_t^*,g_t,\ldots,g_t^*,g_t').\end{aligned} \tag{V.1}$$

We show that this derivative is zero, implying that $\psi(g_t^*,g_t,\ldots,g_t^*,g_t)$ is constant in t, so that

$$\psi(u^*,u,\ldots,u^*,u) = \psi(v^*,v,\ldots,v^*,v).$$

Since ψ is a cocycle, we have

$$\begin{aligned}0 &= b\psi(g_t^*, g_t' g_t^*, g_t, g_t^*, \dots, g_t^*, g_t)\\ &= \psi(g_t^* g_t'' g_t^*, g_t, \dots, g_t^*, g_t) - \psi(g_t^*, g_t', \dots, g_t^*, g_t)\\ &\quad + \sum_{j=2}^{2k+1} (-1)^j \psi(g_t^*, g_t' g_t^*, \dots, G_t, \dots, g_t^*, g_t) + \psi(G_t, g_t' g_t^*, \dots, g_t^*, g_t),\end{aligned}$$

where $G_t = g_t g_t^* = 1$ or $G_t = g_t^* g_t = 1$, depending on its position in ψ. With $(g_t^*)' = -g_t^* g_t' g_t^*$, and the normalisation of ψ, it follows that

$$\psi((g_t^*)', g_t, \dots, g_t^*, g_t) = -\psi(g_t^*, g_t', \dots, g_t^*, g_t).$$

The other terms in Eq. (V.1) cancel similarly, so that indeed $\frac{\mathrm{d}}{\mathrm{d}t}\psi(g_t^*, g_t, \dots, g_t^*, g_t) = 0$ and the pairing is independent of the choice of smooth representative of a class in $K_1(C(M))$.

Finally, we show that this pairing yields 0 if ψ is a coboundary, i.e. $\psi = b\phi$ for some $\phi \in C_\lambda^{2k}(\mathcal{A})$ which we may assume to be normalised, by Corollary II.13. Let $u = (u_{ij}) \in \mathcal{U}_k(C^\infty(M))$ be a unitary matrix with smooth coefficient functions; denote the components of it conjugate transpose by $v = u^* = (v_{ij})$. Let e_{ij} denote the matrix in $M_k(\mathbb{C})$ which has value 1 in the (i,j)th entry, and 0 everywhere else. Then $u = \sum_{i,j} u_{ij} \otimes e_{ij}$ and $u^{-1} = \sum_{i,j} v_{ij} \otimes e_{ij}$. It follows that

$$\begin{aligned}\langle \psi, u\rangle &= \psi^k\left(u^*, u, u^*, u, \dots, u^*, u\right)\\ &= \sum_{i_1,j_1} \cdots \sum_{i_{2k+2},j_{2k+2}} \operatorname{tr}\left(e_{i_1 j_1} \cdots e_{i_{2k+2} j_{2k+2}}\right) \psi\left(v_{i_1 j_1}, u_{i_2 j_2}, \dots, v_{i_{2k+1} j_{2k+1}}, u_{i_{2k+2} j_{2k+2}}\right)\\ &= \sum_{i_1,j_1} \cdots \sum_{i_{2k+2},j_{2k+2}} \operatorname{tr}\left(\delta_{j_1 i_2}\delta_{j_2 i_3} \cdots \delta_{j_{2k+1} i_{2k+2}} e_{i_1 j_{2k+2}}\right) \psi\big(v_{i_1 j_1}, u_{i_2 j_2}, \dots,\\ &\qquad\qquad v_{i_{2k+1} j_{2k+1}}, u_{i_{2k+2} j_{2k+2}}\big)\\ &= \sum_{i_1,\dots,i_{2k+2}} \psi\left(v_{i_1 i_2}, u_{i_2 i_3}, v_{i_3 i_4}, \dots, u_{i_{2k+2} i_1}\right),\end{aligned}$$

since $e_{ij} e_{i'j'} = \delta_{ji'} e_{ij'}$. Then, as $\psi = b\phi$,

$$\begin{aligned}\langle \psi, u\rangle = \sum_{i_1,\dots,i_{2k+2}} \Bigg(&\sum_{l=0}^{k-1} \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \dots, v_{i_{2l+1} i_{2l+2}} u_{i_{2l+2} i_{2l+3}}, \dots, u_{i_{2k+2} i_1}\right)\\ &+ \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \dots, u_{i_{2k} i_{2k+1}}, v_{i_{2k+1} i_{2k+2}} u_{i_{2k+2} i_1}\right)\\ &- \sum_{l=0}^{k-1} \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \dots, u_{i_{2l+2} i_{2l+3}} v_{i_{2l+3} i_{2l+4}}, \dots, u_{i_{2k+2} i_1}\right)\\ &- \phi\left(u_{i_{2k+2} i_1} v_{i_1 i_2}, u_{i_2 i_3}, \dots, v_{i_{2k+1} i_{2k+2}}\right)\Bigg).\end{aligned}$$

Since u and v are inverse to each other, we have $\sum_{i_{2l+2}} v_{i_{2l+1} i_{2l+2}} u_{i_{2l+2} i_{2l+3}} = \delta_{i_{2l+1} i_{2l+3}}$.

Therefore,

$$\begin{aligned}\langle \psi, u\rangle = &\sum_{l=0}^{k-1} \sum_{i_1,\ldots,\hat{i}_{2l+2},\ldots,i_{2k+2}} \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \ldots, \delta_{i_{2l+1} i_{2l+3}}, \ldots, u_{i_{2k+2} i_1}\right) \\ &+ \sum_{i_1,\ldots,i_{2k+1}} \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \ldots, u_{i_{2k} i_{2k+1}}, \delta_{i_{2k+1} i_1}\right) \\ &- \sum_{l=0}^{k-1} \sum_{i_1,\ldots,\hat{i}_{2l+3},\ldots,i_{2k+2}} \phi\left(v_{i_1 i_2}, u_{i_2 i_3}, \ldots, \delta_{i_{2l+2} i_{2l+4}}, \ldots, u_{i_{2k+2} i_1}\right) \\ &- \sum_{i_2,\ldots,i_{2k+2}} \phi\left(\delta_{i_{2k+2} i_2}, u_{i_2 i_3}, \ldots, v_{i_{2k+1} i_{2k+2}}\right) \\ = &\ 0,\end{aligned}$$

as ϕ is normalised and cyclic, and where $\hat{i}_{2l+2}$ means that the variable i_{2l+2} is left out of the summation. □

V.3 Pairing of $(\tilde{\psi})$ with K-theory

Since the spectal action functional is a spectral invariant [28, Section 7.1], it is also invariant under unitary transformations

$$D + V \mapsto u(D + V)u^*,$$

for a unitary $u \in \mathcal{A}$. This gauge invariance will be used to derive that for the algebra $\mathcal{A} = C^\infty(\mathbb{T}^1)$, the entire cyclic cocycle $(\tilde{\psi})$ from Proposition III.16 is in fact a coboundary. Before we can prove this, we show that the pairing from Proposition V.7 induces a pairing of the entire cyclic cohomology of $C^\infty(M))$ (Definition II.19) and the K-theory of $C(M)$ (Definition V.5).

Theorem V.9. *The pairing*

$$\begin{aligned}&\langle \cdot, \cdot\rangle_{C^*} \equiv \langle \cdot, \cdot\rangle \colon HE^1(C^\infty(M)) \times K_1(C(M)) \to \mathbb{C} \\ &\qquad ([\psi], [u]) \mapsto (2\pi i)^{-1/2} \sum_{k=0}^{\infty} (-1)^k k! \psi^q_{2k+1}(u^*, u, \ldots, u^*, u)\end{aligned}$$

is well-defined.

Proof. We start by recalling the algebraic analogue

$$\begin{aligned}&\langle \cdot, \cdot\rangle_{\mathrm{alg}} \colon HE^1(\mathcal{A}) \times K_1^{\mathrm{alg}}(\mathcal{A}) \to \mathbb{C} \\ &\qquad ([\psi], [u]_{\mathrm{alg}}) \mapsto (2\pi i)^{-1/2} \sum_{k=0}^{\infty} (-1)^k k! \psi^q_{2k+1}(u^*, u, \ldots, u^*, u),\end{aligned}$$

which is defined in [7, Corollary IV.7.27], for any locally convex algebra $\mathcal{A}$. Since every representative u of a class $[u] \in K_1(C(M))$ also represents a class $[u]_{\text{alg}} \in K_1^{\text{alg}}(C(M))$, we have $\langle[\psi],[u]\rangle_{C^*} = \langle[\psi],[u]_{\text{alg}}\rangle_{\text{alg}}$ is finite and independent of the choice of representative of $[\psi] \in HE^1(C^\infty(M))$. Moreover, as ψ_{2k+1} is a cyclic cocycle for each $k \in \mathbb{N}$, the value of

$$\psi^q_{2k+1}(u^*, u, \dots, u^*, u)$$

is independent of the choice of representative u of $[u] \in K_1(C(M))$, by virtue of Proposition V.7. □

We should point out that in the proof of Proposition V.7, we have used the definition of cyclic cocycles in terms of the cyclic skewsymmetriser (cf. Definition II.6). However, in the proof of Theorem V.9, we view a cyclic cocycle as a finite (b, B)-cocycle. The switch between these two viewpoints is enabled by Lemma II.16.

From now on, we will work exclusively with the pairing $\langle\cdot,\cdot\rangle_{C^*}$, and therefore drop the subscript from the notation.

Let $\tilde{\psi}$ be the entire cyclic cocycle from Proposition III.16. Proposition III.17 states that $\tilde{\psi}$ is indeed entire, *but in the topology induced by the norm* $\|\cdot\|_D$, not necessarily in the Fréchet topology. Recall that the Fréchet topology is generated by family of the seminorms $\{\|\cdot\|_n\}$, see Definition IV.3. Therefore, any Fréchet-bounded subset in $C^\infty(\mathbb{T}^1)$ is also $\|\cdot\|_D$-bounded, since for any $f \in C^\infty(\mathbb{T}^1)$ we have

$$\|f\|_D = \|f\|_\infty + \|if'\|_\infty \leq \|f\|_0 + C\|f\|_1,$$

for some $C > 0$, depending only on the choice of generators of $\mathfrak{X}(\mathbb{T}^1)$. Thus, $\tilde{\psi} \in HE^1(C^\infty(\mathbb{T}^1))$ is also entire when $C^\infty(\mathbb{T}^1)$ is endowed with the Fréchet topology.

Theorem V.10. *The class* $[(\tilde{\psi})] \in HE^1(C^\infty(\mathbb{T}^1))$ *is trivial.*

We will prove this Theorem by showing that the pairing $\langle\cdot,\cdot\rangle$ is non-degenerate if $M = \mathbb{T}^1$. For this, we will need to compute $HE^1(C^\infty(\mathbb{T}^1))$. This computation will be treated as a black box, for which we will use results from [22], [9] and [3].

Proof. Since the spectral action depends only on the spectrum of its argument, it is invariant under gauge transformations by a unitary, i.e. $\operatorname{tr}(f(D)) = \operatorname{tr}(f(uDu^*))$. At the level of universal forms, gauge transformations take the form $\mathrm{d} + A \mapsto \mathrm{d} + A^u$, with $A^u = u\,\mathrm{d}u^* + uAu^*$. This follows from minimal coupling: for $\omega \in \Omega^\bullet\mathcal{A}$, we must have

$$u(\mathrm{d}+A)u^*\omega = u\,\mathrm{d}(u^*)\omega + uu^*\,\mathrm{d}\omega + uAu^*\omega = (\mathrm{d} + u\,\mathrm{d}u^* + uAu^*)\omega = (\mathrm{d} + A^u)\omega.$$

Each Yang-Mills term in the spectral action is gauge invariant as well, since each ϕ_{2k} is a Hochschild cocycle. It follows from Theorem III.8 that the totality of the Chern–Simons terms must also be gauge invariant:

$$\sum_{k=0}^{\infty} \int_{\psi_{2k+1}} \mathrm{cs}_{2k+1}(A^u) = \sum_{k=0}^{\infty} \int_{\psi_{2k+1}} \mathrm{cs}_{2k+1}(A). \tag{V.2}$$

However, the individual Chern–Simons terms are not gauge invariant. This implies that if $A = u^* \,\mathrm{d}u$ is pure gauge, its associated Chern–Simons forms do not vanish, contrary to its curvature. It therefore follows from the gauge invariance of the spectral action that

$$\sum_{k=0}^{\infty} \int_{\psi_{2k+1}} \mathrm{cs}_{2k+1}\left(u^* \,\mathrm{d}u\right) = 0.$$

From a straightforward calculation, we know that for $u \in \mathcal{U}_q(C^\infty(M))$,

$$\mathrm{cs}_{2k+1}\left(u^* \,\mathrm{d}u\right) = \frac{k!^2}{(2k+1)!} u^* \,\mathrm{d}u \left(\mathrm{d}u^* \,\mathrm{d}u\right)^k,$$

so that

$$\int_{\psi^q_{2k+1}} \mathrm{cs}_{2k+1}(u^* \,\mathrm{d}u) = \frac{k!^2}{(2k+1)!} \psi^q_{2k+1}(u^*, u, \ldots, u^*, u).$$

Substituting this into Eq. (V.2), together with the definition of $(\tilde{\psi})$, we find

$$(2\pi i)^{-1/2} \sum_{k=0}^{\infty} (-1)^k k! \tilde{\psi}^q_{2k+1}(u^*, u, \ldots, u^*, u) = 0. \tag{V.3}$$

The quantity on the left-hand side is, by definition, the value of the pairing $\langle [\tilde{\psi}], [u] \rangle$. The following two Lemmas will help us to analyse this pairing further.

Lemma V.11. *Let M be a compact manifold. Then*

$$HE^1(C^\infty(M)) \cong HP^1(C^\infty(M)).$$

The isomorphism is induced by the natural forgetful transformations between continuous periodic and entire cyclic cohomology.

Proof. See [22, Theorem 6.2]. □

Lemma V.12. *Let M be a compact manifold. Then for all $k = 0, \ldots, n := \dim M$ we have the Poincaré-like duality*

$$H_k^{\mathrm{dR}}(M) \cong H^{n-k}_{\mathrm{dR}}(M).$$

Proof. This follows from [9, Theorem 16] and [3, Theorem 3]. □

If $M = \mathbb{T}^1$, Lemma V.11 and Corollary IV.11 tell us that that

$$HE^1(C^\infty(\mathbb{T}^1)) \cong HP^1(C^\infty(\mathbb{T}^1)) \cong H_1^{\mathrm{dR}}(\mathbb{T}^1),$$

and Lemma V.12 tells us that $H_1^{\mathrm{dR}}(\mathbb{T}^1) \cong \mathbb{C}$. A generator for $H_1^{\mathrm{dR}}(\mathbb{T}^1)$ is given by the class of the integration operator

$$\int : \Omega^1(\mathbb{T}^1) \to \mathbb{C},$$

sending $\omega \in \Omega^1(\mathbb{T}^1)$ to $\int_{\mathbb{T}^1} \omega$. Thus, these two lemmas combined show that the entire cyclic cocycle $(\tilde{\psi})$ is (b, B)-cohomologous to a cyclic cocycle $\tilde{\zeta}_1 \in Z^1_\lambda(C^\infty(\mathbb{T}^1))$. In turn, we may view the class $[\tilde{\zeta}_1] \in HC^1(C^\infty(\mathbb{T}^1))$ as a multiple of $[\int] \in H_1^{\mathrm{dR}}(\mathbb{T}^1)$, say, $[\tilde{\zeta}_1] = c[\int]$, for $c \in \mathbb{C}$. Now, take the unitary $u = e_k$, for any $k \in \mathbb{Z}$. Then

$$0 = \langle [\tilde{\psi}], [u] \rangle = \langle [\tilde{\zeta}_1], [u] \rangle = c \int_{\mathbb{T}^1} e_{-k} \, \mathrm{d}e_k = ck2\pi i.$$

This must hold for all $k \in \mathbb{Z}$, implying that $c = 0$. Thus, $[\tilde{\zeta}_1] = 0 \in H_1^{\mathrm{dR}}(\mathbb{T}^1)$. Together with Lemma V.11, this yields $[\tilde{\psi}] = 0 \in HE^1(C^\infty(\mathbb{T}^1))$, i.e. the entire cyclic cocycle $(\tilde{\psi})$ is exact. $\square$

Chapter VI

Computation of cocycles in spectral action on tori

Recall that each ϕ_k from Definition III.14 is a multilinear functional on $\mathcal{A}^{k+1}$. In view of the Connes–Hochschild–Kostant–Rosenberg Theorem (Theorem IV.4), it seems plausible that for $\mathcal{A} = C^\infty(M)$ (with M a compact manifold), each ϕ_k is (an infinite linear combination of) integrals of derivatives of $k+1$ functions on M. In this Chapter, we prove that this is indeed the case when M is the d-torus, and write down a concrete expression for each ϕ_k.

VI.1 Spin geometry on the d-torus

Let $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ be the d-torus, with coordinates $(\theta^1, \ldots, \theta^d)$ ranging from 0 to 2π. Let $\{\partial_\mu \coloneqq \partial/\partial\theta^\mu\}$ be the induced frame of the tangent bundle. Denote the (flat) metric on $\mathbb{T}^d$ by g; note that $g(\partial_\mu, \partial_\nu) = \delta_{\mu\nu}$. Since $\mathbb{T}^d$ is a Lie group, the tangent bundle $T\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{R}^d$ is globally trivialisable. Therefore, so is the (complex) Clifford algebra bundle $\mathbb{C}\mathrm{l}(T\mathbb{T}^d)$. In fact,

$$\mathbb{C}\mathrm{l}(T\mathbb{T}^d) \cong \begin{cases} \mathbb{T}^d \times \mathbb{C}\mathrm{l}_d & \text{if } d \text{ is even,} \\ \mathbb{T}^d \times \mathbb{C}\mathrm{l}_d^0 & \text{if } d \text{ is odd.} \end{cases}$$

From [28] we know that $\mathbb{C}\mathrm{l}_d = M_{2^{d/2}}(\mathbb{C})$ if d is even, and $\mathbb{C}\mathrm{l}_d^0 = M_{2^{(d-1)/2}}(\mathbb{C})$ if d is odd. Therefore, the spinor bundle on $\mathbb{T}^d$ is

$$\mathcal{S}_{\mathbb{T}^d} = \mathbb{T}^d \times \mathbb{C}^{2^{\lfloor d/2 \rfloor}}.$$

Consequently, the Hilbert space of square-integrable spinors on $\mathbb{T}^d$ is the tensor product

$$L^2(\mathcal{S}_{\mathbb{T}^d}) = L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor d/2 \rfloor}}.$$

The Dirac operator $D \coloneqq D_{\mathbb{T}^d}$ associated with the spin connection on $\mathbb{T}^d$ acts on the dense subspace $C^\infty(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor d/2 \rfloor}}$. The metric on $\mathbb{T}^d$ is constant, so its Christoffel symbols are all zero and the Dirac operator in $(\theta^1, \ldots, \theta^d)$-coordinates is given by

$$D = -i\partial_\mu \otimes \gamma^\mu,$$

where the set of gamma matrices $\{\gamma^\mu \mid \mu = 1, \ldots, d\} \subseteq \operatorname{End}\left(\mathbb{C}^{2^{\lfloor d/2 \rfloor}}\right)$ generates the (even part of) the Clifford algebra $\mathbb{C}\mathrm{l}_d$.

It follows from the Lichnerowicz formula [28, Theorem 4.21] for a flat metric that the square of D is the Laplacian $\Delta^{\mathcal{S}}$ of the spin connection. This Laplacian has domain $\Gamma(\mathcal{S})$, on which is it positive and self-adjoint. On $\mathbb{T}^d$, it is simply given by

$$\Delta^{\mathcal{S}} = -\sum_{\mu=1}^{d} (\partial_\mu)^2 \,.$$

The eigenvalues of $\Delta^{\mathcal{S}}$ are $\|\mathbf{n}\|^2$ with $\mathbf{n} \in \mathbb{Z}^d$, and it has an orthonormal basis of eigenvectors given by $\{e_{\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$, where

$$e_{\mathbf{n}}(\theta) \coloneqq \frac{1}{(2\pi)^{d/2}} \exp\left(i \sum_{\mu=1}^{d} n_\mu \theta_\mu\right).$$

To find the eigenvectors of the Dirac operator, we inductively construct an explicit representation of the gamma matrices.

Lemma VI.1 ([13]). *Let $d \geq 1$, and let $\{\gamma_d^\mu \mid \mu = 1, \ldots, d\} \subseteq \operatorname{End}\left(\mathbb{C}^{2^{\lfloor d/2 \rfloor}}\right)$ be the matrices that are defined as follows. For $d = 1$, we have $\gamma_1^1 = 1$. For $d \geq 2$ and even, we inductively set*

$$\gamma_d^\mu = \begin{cases} \begin{pmatrix} 0 & -i\gamma_{d-1}^\mu \\ i\gamma_{d-1}^\mu & 0 \end{pmatrix} & \text{if } \mu < d, \\ \begin{pmatrix} 0 & \mathrm{id}_{2^{\lfloor d/2 \rfloor -1}} \\ \mathrm{id}_{2^{\lfloor d/2 \rfloor -1}} & 0 \end{pmatrix} & \text{if } \mu = d, \end{cases}$$

where $\mathrm{id}_{2^{\lfloor d/2 \rfloor -1}} \in M_{2^{\lfloor d/2 \rfloor -1}}$ is the identity matrix. For $d \geq 3$ and odd, we set

$$\gamma_d^\mu = \begin{cases} \gamma_{d-1}^\mu & \text{if } \mu < d, \\ \begin{pmatrix} \mathrm{id}_{2^{\lfloor d/2 \rfloor -1}} & 0 \\ 0 & -\mathrm{id}_{2^{\lfloor d/2 \rfloor -1}} \end{pmatrix} & \text{if } \mu = d. \end{cases}$$

Then, for each $d \in \mathbb{N}$, the matrices $\{\gamma_d^\mu \mid \mu = 1, \ldots, d\}$ define a representation of the Clifford algebra $\mathbb{C}\mathrm{l}_d$ (when d is even) and $\mathbb{C}\mathrm{l}_d^0$ (when d is odd).

Proof. It is easy to verify that these matrices are self-adjoint and satisfy the Clifford relation

$$\gamma_d^\mu \gamma_d^\nu + \gamma_d^\nu \gamma_d^\mu = 2\delta^{\mu\nu}.$$

Their linear independence follows by an inductive argument. □

With this result at hand, we can compute the eigenvectors of D.

Proposition VI.2. *Let $d \geq 2$. For each $\mathbf{n} \in \mathbb{Z}^d$, D has $2^{\lfloor d/2 \rfloor -1}$ eigenvectors with eigenvalue $\lambda_{\mathbf{n}} := \sqrt{\|\mathbf{n}\|}$ and $2^{\lfloor d/2 \rfloor -1}$ eigenvectors with eigenvalue $-\lambda_{\mathbf{n}}$. Concretely, the eigenvectors with eigenvalue $\pm\lambda_{\mathbf{n}}$ are given by pure tensors of the form $e_{\mathbf{n}} \otimes v^{\pm}_{\mathbf{n},j}$, with $v^{\pm}_{\mathbf{n},j} \in \mathbb{C}^{2^{\lfloor d/2 \rfloor}}$, and where $j = 1, \ldots, 2^{\lfloor d/2 \rfloor -1}$. The $v^{\pm}_{\mathbf{n},j}$ are linearly independent for a fixed choice of $\pm$ and $\mathbf{n}$.*

Proof. Let $\mathbf{n} \in \mathbb{Z}^d$, let $v \in \mathbb{C}^{2^{\lfloor d/2 \rfloor -1}}$, and suppose that d is even. By a direct computation, it follows that

$$\begin{aligned} D(e_{\mathbf{n}} \otimes v) &= e_{\mathbf{n}} \otimes (n_\mu \gamma^\mu v) \\ &= e_{\mathbf{n}} \otimes \begin{pmatrix} 0 & -i\sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} \\ i\sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} & 0 \end{pmatrix} v. \end{aligned}$$

The Proposition is shown if we show that the space of solutions of the equation

$$D(e_{\mathbf{n}} \otimes v) = \epsilon \lambda_{\mathbf{n}} e_{\mathbf{n}} \otimes v$$

is $2^{\lfloor d/2 \rfloor -1}$-dimensional, both for $\epsilon = +1$ and $\epsilon = -1$. This is equivalent to showing that the spaces of solutions of

$$\begin{pmatrix} 0 & -i\sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} \\ i\sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} & 0 \end{pmatrix} v = \epsilon \lambda_{\mathbf{n}} v \tag{VI.1}$$

are $2^{\lfloor d/2 \rfloor -1}$-dimensional. We show this statement first for the positive eigenvalues. Write $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, with $v_1, v_2 \in \mathbb{C}^{2^{\lfloor d/2 \rfloor -1}}$, so that Equation (VI.1) is rewritten as the system

$$\left(-i \sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} \right) v_2 = \lambda_n v_1, \tag{VI.2}$$

$$\left(i \sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} \right) v_1 = \lambda_n v_2. \tag{VI.3}$$

The matrix $\left(-i\sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id}\right)$ is invertible, with inverse

$$\frac{1}{\lambda_n^2} \left(i \sum_{j=1}^{d-1} n_j \gamma^j_{d-1} + n_d \mathrm{id} \right),$$

which follows from the Clifford relations. Therefore, Equations (VI.2) and (VI.3) are equivalent, and since the gamma matrices are linearly independent, the dimension of the space of solutions of Equation (VI.1) is $2^{\lfloor d/2 \rfloor -1}$. Denote an orthonormal basis for this space by $\{v^+_{\mathbf{n},1}, \ldots, v^+_{\mathbf{n},2^{\lfloor d/2 \rfloor -1}}\}$. The argument for the eigenvalue $-\lambda_n$ is analogous, and yields an orthonormal basis $\{v^-_{\mathbf{n},1}, \ldots, v^-_{\mathbf{n},2^{\lfloor d/2 \rfloor -1}}\}$.

Now suppose $d \geq 3$ is odd. After repeating the first steps as when d is even, we see that we must replace Equation (VI.1) by

$$\begin{pmatrix} n_d \mathrm{id} & -i\sum_{j=1}^{d-2} n_j \gamma_{d-2}^j + n_{d-1}\mathrm{id} \\ i\sum_{j=1}^{d-2} n_j \gamma_{d-2}^j + n_{d-1}\mathrm{id} & -n_d \mathrm{id} \end{pmatrix} v = \lambda_n v.$$

Via similar arguments as when d is even, this system of $2^{\lfloor d/2 \rfloor}$ equations can be reduced to a system of $2^{\lfloor d/2 \rfloor - 1}$ linearly independent equations. □

This Proposition has two important consequences. First, it guarantees diagonalisability of D, which will be a crucial ingredient of our derivation of an explicit expression of the cochains ϕ_n and ψ_n. Secondly, it implies the following Proposition, which tells us that Theorem III.8 applies to the canonical spectral triple on the d-torus, that we have just constructed.

Proposition VI.3. *The spectral triple $(C^\infty(\mathbb{T}^d), L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor d/2 \rfloor}}, D)$ is s-summable for all $s > d$.*

Proof. Since each eigenvalue $\lambda_{\mathbf{n}}$ of D has multiplicity $2^{\lfloor d/2 \rfloor}$, we have

$$\|(D-i)^{-s}\|_s^s = 2^{\lfloor d/2 \rfloor} \sum_{\mathbf{n} \in \mathbb{Z}^d} \left(\frac{1}{\|\mathbf{n}\| - i} \right)^s < \infty,$$

for all $s > d$. □

VI.2 Computation of the bracket

Now that we know the spin geometry geometry of the d-torus, we can derive an expression for the bracket

$$\langle \cdot, \ldots, \cdot \rangle \colon \Omega_D^1(C^\infty(\mathbb{T}^d))^k \to \mathbb{C}$$

of Definition III.12. In the next Section, we will use this expression to compute the Hochschild cochains ϕ_k and ψ_{2k+1}.

Since we keep the dimension d fixed, we will omit from the notation in γ_d^μ, i.e. $\gamma^\mu = \gamma_d^\mu$.

Theorem VI.4. *Let $k \in \mathbb{N}$, and let $V_1, \ldots, V_k \in \Omega_D^1(C^\infty(\mathbb{T}^d))$. Write $V_r = \gamma^\mu V_\mu^r$, with $V_\mu^r \in C^\infty(\mathbb{T}^d)$. Then*

$$\begin{aligned} \langle V_1, \ldots, V_k \rangle = \sum_{\substack{\mathbf{n}_1, \ldots, \mathbf{n}_k \in \mathbb{Z}^d \\ \epsilon_1, \ldots, \epsilon_k = \pm}} \sum_{j_1, \ldots, j_k = 1}^{2^{\lfloor d/2 \rfloor - 1}} \int_{\Delta_{k-1}} & f^{(k)}(\epsilon_1 s_1 \lambda_{\mathbf{n}_1} + \cdots + \epsilon_k s_k \lambda_{\mathbf{n}_k}) \, \mathrm{d}\sigma \\ & \langle v_{\mathbf{n}_k, j_k}^{\epsilon_k} | \gamma^{\mu_1} | v_{\mathbf{n}_1, j_1}^{\epsilon_1} \rangle \langle v_{\mathbf{n}_1, j_1}^{\epsilon_1} | \gamma^{\mu_2} | v_{\mathbf{n}_2, j_2}^{\epsilon_2} \rangle \cdots \langle v_{\mathbf{n}_{k-1}, j_{k-1}}^{\epsilon_{k-1}} | \gamma^{\mu_k} | v_{\mathbf{n}_k, j_k}^{\epsilon_k} \rangle \\ & \widehat{V_{\mu_1}^1}(\mathbf{n}_k - \mathbf{n}_1) \widehat{V_{\mu_2}^2}(\mathbf{n}_1 - \mathbf{n}_2) \cdots \widehat{V_{\mu_k}^k}(\mathbf{n}_{k-1} - \mathbf{n}_k). \end{aligned}$$

where

$$\widehat{V_\mu^r}(\mathbf{n}) := \int_{\mathbb{T}^d} V_\mu^r(\theta) e_{-\mathbf{n}}(\theta) \, \mathrm{d}\theta$$

is the $\mathbf{n}$*-Fourier coefficient of* V_μ^r.

Proof. Let $V_1, \dots, V_k \in \Omega^1_D(C^\infty(\mathbb{T}^d))$. From Proposition VI.2 we know that the set

$$\{\chi^\epsilon_{\mathbf{n},j} \coloneqq e_{\mathbf{n}} \otimes v^\epsilon_{\mathbf{n},j} \mid \mathbf{n} \in \mathbb{Z}^d,\ \epsilon = \pm,\ j = 1, \dots, 2^{\lfloor d/2 \rfloor - 1}\}$$

is a complete orthonormal set of eigenvectors of D. Therefore, we can diagonalise D as

$$D = \sum_{\substack{\mathbf{n} \in \mathbb{Z} \\ \epsilon = \pm}} \sum_{j=1}^{2^{\lfloor d/2 \rfloor - 1}} \epsilon \lambda_{\mathbf{n}} |\chi^\epsilon_{\mathbf{n},j}\rangle\langle\chi^\epsilon_{\mathbf{n},j}|.$$

Then by the continuous functional calculus for self-adjoint operators, it follows that

$$e^{istD} = \sum_{\substack{\mathbf{n} \in \mathbb{Z} \\ \epsilon = \pm}} \sum_{j=1}^{2^{\lfloor d/2 \rfloor - 1}} e^{\epsilon i s t \lambda_{\mathbf{n}}} |\chi^\epsilon_{\mathbf{n},j}\rangle\langle\chi^\epsilon_{\mathbf{n},j}|.$$

Inserting this into the expression for $\operatorname{tr} T^D_{f^{[k]}}(V_1, \dots, V_k)$ gives

$$\begin{aligned}\operatorname{tr} T^D_{f^{[k]}}(V_1, \dots, V_k) = \operatorname{tr} \int_{\Delta_k} \int_{\mathbb{R}} \sum_{\substack{\mathbf{n}_0, \dots, \mathbf{n}_k \in \mathbb{Z}^d \\ \epsilon_0, \dots, \epsilon_k = \pm}} \sum_{j_0, \dots, j_k = 1}^{2^{\lfloor d/2 \rfloor - 1}} \exp(it(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \dots + \epsilon_k s_k \lambda_{\mathbf{n}_k})) \\ |\chi^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle\langle\chi^{\epsilon_0}_{\mathbf{n}_0,j_0}|V_1|\chi^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \cdots \langle\chi^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|V_k|\chi^{\epsilon_k}_{\mathbf{n}_k,j_k}\rangle\langle\chi^{\epsilon_k}_{\mathbf{n}_k,j_k}|\widehat{f^{(k)}}(t)\,\mathrm{d}t\,\mathrm{d}\sigma.\end{aligned}$$

We wish to examine the matrix elements $\langle\chi^\epsilon_{\mathbf{n},j}|V_r|\chi^{\epsilon'}_{\mathbf{n}',j'}\rangle$ more closely. By Proposition VI.2, each eigenvector is a pure tensor, i.e. $\chi^\epsilon_{\mathbf{n},j} = e_{\mathbf{n}} \otimes v^\epsilon_{\mathbf{n},j}$. Therefore, the matrix element is a sum (over μ) of products of two matrix elements — one coming from $\mathbb{C}^{2^{\lfloor d/2 \rfloor}}$, and one coming from $L^2(\mathbb{T}^d)$:

$$\begin{aligned}\langle\chi^\epsilon_{\mathbf{n},j}|V_r|\chi^{\epsilon'}_{\mathbf{n}',j'}\rangle &= \langle v^\epsilon_{\mathbf{n},j}|\gamma^\mu|v^{\epsilon'}_{\mathbf{n}',j'}\rangle \int_{\mathbb{T}^d} e_{-\mathbf{n}}(\theta) V^r_\mu(\theta) e_{\mathbf{n}'}(\theta)\,\mathrm{d}\theta \\ &= \langle v^\epsilon_{\mathbf{n},j}|\gamma^\mu|v^{\epsilon'}_{\mathbf{n}',j'}\rangle \widehat{V^r_\mu}(\mathbf{n} - \mathbf{n}').\end{aligned}$$

This gives

$$\begin{aligned}\operatorname{tr} T^D_{f^{[k]}}(V_1, \dots, V_k) = \operatorname{tr} \int_{\Delta_k} \int_{\mathbb{R}} \sum_{\substack{\mathbf{n}_0, \dots, \mathbf{n}_k \in \mathbb{Z}^d \\ \epsilon_0, \dots, \epsilon_k = \pm}} \sum_{j_0, \dots, j_k = 1}^{2^{\lfloor d/2 \rfloor - 1}} \exp(it(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \dots + \epsilon_k s_k \lambda_{\mathbf{n}_k})) \\ \langle v^{\epsilon_0}_{\mathbf{n}_0,j_0}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \cdots \langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_k}_{\mathbf{n}_k,j_k}\rangle \\ \widehat{V^1_{\mu_1}}(\mathbf{n}_0 - \mathbf{n}_1) \cdots \widehat{V^k_{\mu_k}}(\mathbf{n}_{k-1} - \mathbf{n}_k)|\chi^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle\langle\chi^{\epsilon_k}_{\mathbf{n}_k,j_k}|\widehat{f^{(k)}}(t)\,\mathrm{d}t\,\mathrm{d}\sigma.\end{aligned}$$

To shorten the notation, we define

$$G^{\epsilon_0, \dots, \epsilon_k}_{\substack{\mathbf{n}_0, \dots, \mathbf{n}_k \\ j_0, \dots, j_k}} \coloneqq \langle v^{\epsilon_0}_{\mathbf{n}_0,j_0}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \cdots \langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_k}_{\mathbf{n}_k,j_k}\rangle \widehat{V^1_{\mu_1}}(\mathbf{n}_0 - \mathbf{n}_1) \cdots \widehat{V^k_{\mu_k}}(\mathbf{n}_{k-1} - \mathbf{n}_k).$$

Since the eigenvectors $\{\chi^\eta_{\mathbf{p},t} \mid \mathbf{p} \in \mathbb{Z}^d, t \in \{1,\ldots,2^{\lfloor d/2 \rfloor - 1}\}, \eta = \pm\}$ form an orthonormal basis of $L^2(\mathcal{S}_{\mathbb{T}^d}) = L^2(\mathbb{T}^d) \otimes \mathbb{C}^{2^{\lfloor d/2 \rfloor}}$, the operator trace can be computed by sandwiching the operator between $\langle \chi^\eta_{\mathbf{p},t}|$ and $|\chi^\eta_{\mathbf{p},t}\rangle$, and summing over $\mathbf{p}$, t and η:

$$
\begin{aligned}
&\operatorname{tr} T^D_{f^{[k]}}(V_1,\ldots,V_k) \\
&= \sum_{\substack{\mathbf{p}\in\mathbb{Z}^d\\ \eta=\pm}} \sum_{t=1}^{2^{\lfloor d/2 \rfloor-1}} \langle \chi^\eta_{\mathbf{p},t}| \int_{\Delta_k} \int_{\mathbb{R}} \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_k=\pm}} \sum_{j_0,\ldots,j_k=1}^{2^{\lfloor d/2 \rfloor-1}} \\
&\qquad \exp(it(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \cdots + \epsilon_k s_k \lambda_{\mathbf{n}_k})) G^{\epsilon_0,\ldots,\epsilon_k}_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\\ j_0,\ldots,j_k}} |\chi^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle \langle \chi^{\epsilon_k}_{\mathbf{n}_k,j_k}|\chi^\eta_{\mathbf{p},t}\rangle \widehat{f^{(k)}}(t)\,\mathrm{d}t\,\mathrm{d}\sigma \\
&= \sum_{\substack{\mathbf{p}\in\mathbb{Z}^d\\ \eta=\pm}} \sum_{t=1}^{2^{\lfloor d/2 \rfloor-1}} \int_{\mathbb{T}^d} \int_{\Delta_k} \int_{\mathbb{R}} \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_k=\pm}} \sum_{j_0,\ldots,j_k=1}^{2^{\lfloor d/2 \rfloor-1}} \exp(it(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \cdots + \epsilon_k s_k \lambda_{\mathbf{n}_k})) \\
&\qquad G^{\epsilon_0,\ldots,\epsilon_k}_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\\ j_0,\ldots,j_k}} e_{\mathbf{n}_0-\mathbf{p}}(\theta) \langle v^\eta_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle \delta_{\mathbf{n}_k,\mathbf{p}} \delta_{j_k,t} \delta^{\epsilon_k,\eta} \widehat{f^{(k)}}(t)\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}\theta
\end{aligned} \tag{VI.4}
$$

where we have used that $\langle \chi^{\epsilon_k}_{\mathbf{n}_k,j_k}|\chi^\eta_{\mathbf{p},t}\rangle = \delta_{\mathbf{n}_k,\mathbf{p}}\delta_{j_k,t}\delta^{\epsilon_k,\eta}$. To proceed, we want to invert the Fourier transform on $f^{(k)}$, and replace the integral over the quantity $e_{\mathbf{n}_0-\mathbf{p}}(\theta)\langle v^\eta_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle$ by a triplet of delta functions. To do so, we need to be able to interchange the integral over $\mathbb{R}$ and the second pair of sums in Eq. (VI.4). This is possible in view of the inequality

$$
|\langle v^{\epsilon_{l-1}}_{\mathbf{n}_{l-1},j_{l-1}}|\gamma^{\mu_l}|v^{\epsilon_l}_{\mathbf{n}_l,j_l}\rangle| \leq 1,
$$

which follows from the Cauchy-Schwartz inequality, the normalisation of the $v^{\epsilon_l}_{\mathbf{n}_l,j_l}$, and the fact that the eigenvalues of γ^{μ_l} are ±1. Furthermore, each V^r_μ is smooth, so its Fourier coefficients are absolutely summable [11]. Thus, for fixed $\mathbf{p}$, η and t,

$$
\begin{aligned}
&\int_{\mathbb{T}^d} \int_{\Delta_k} \int_{\mathbb{R}} \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_k=\pm}} \sum_{j_0,\ldots,j_k=1}^{2^{\lfloor d/2 \rfloor-1}} |\exp(it(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \cdots + \epsilon_k s_k \lambda_{\mathbf{n}_k})) \\
&\qquad G^{\epsilon_0,\ldots,\epsilon_k}_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_k\\ j_0,\ldots,j_k}} e_{\mathbf{n}_0-\mathbf{p}}(\theta) \langle v^\eta_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle \delta_{\mathbf{n}_k,\mathbf{p}} \delta_{j_k,t} \delta^{\epsilon_k,\eta} \widehat{f^{(k)}}(t)|\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}\theta < \infty.
\end{aligned}
$$

Therefore, we can apply Fubini's theorem to conclude that we may interchange the integrals over $\mathbb{T}^d$, Δ_k and $\mathbb{R}$, and the sums $\mathbf{n}_0,\ldots,\mathbf{n}_k$, $\epsilon_0,\ldots,\epsilon_k$ and $j_0,\ldots,j_k$ in the expression (VI.4) for $\langle V_1,\ldots,V_k\rangle_m$. Doing so and inverting the Fourier transform yields

$$
\begin{aligned}
&\operatorname{tr} T^D_{f^{[k]}}(V_1,\ldots,V_k) \\
&= \sum_{\substack{\mathbf{p}\in\mathbb{Z}^d\\ \eta=\pm}} \sum_{t=1}^{2^{\lfloor d/2 \rfloor-1}} \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1}\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_{k-1}=\pm}} \sum_{j_0,\ldots,j_{k-1}=1}^{2^{\lfloor d/2 \rfloor-1}} G^{\epsilon_0,\ldots,\epsilon_{k-1},\eta}_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1},\mathbf{p}\\ j_0,\ldots,j_{k-1},t}} \\
&\qquad \int_{\Delta_k} f^{(k)}(\epsilon_0 s_0 \lambda_{\mathbf{n}_0} + \cdots + \epsilon_{k-1} s_{k-1} \lambda_{\mathbf{n}_{k-1}} + \eta s_k \lambda_{\mathbf{p}})\,\mathrm{d}\sigma \int_{\mathbb{T}^d} e_{\mathbf{n}_0-\mathbf{p}}(\theta)\,\mathrm{d}\theta \langle v^\eta_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle.
\end{aligned} \tag{VI.5}
$$

We recognise the quantity $\int_{\mathbb{T}^d} e_{\mathbf{n}_0-\mathbf{p}}(\theta)\,\mathrm{d}\theta\langle v^{\eta}_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle$ on the last line as an inner product of orthonormal basis spinors:

$$\int_{\mathbb{T}^d} e_{\mathbf{n}_0-\mathbf{p}}(\theta)\,\mathrm{d}\theta\langle v^{\eta}_{\mathbf{p},t}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle = \langle \chi^{\eta}_{\mathbf{p},t}|\chi^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle = \delta_{\mathbf{n}_0,\mathbf{p}}\delta_{j_0,t}\delta^{\epsilon_0,\eta}.$$

Hence the first pair of sums in Eq. (VI.5) only contributes to the expression when $\mathbf{p}=\mathbf{n}_0$, $t=j_0$ and $\eta=\epsilon_0$. Therefore Eq. (VI.5) simplifies to

$$\begin{aligned}
&\operatorname{tr} T^{D}_{f^{[k]}}(V_1,\ldots,V_k)\\
&= \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1}\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_{k-1}=\pm}} \sum_{j_0,\ldots,j_{k-1}=1}^{2^{\lfloor d/2\rfloor-1}} G^{\epsilon_0,\ldots,\epsilon_{k-1},\epsilon_0}_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1},\mathbf{n}_0\\ j_0,\ldots,j_{k-1},j_0}}\\
&\qquad \int_{\Delta_k} f^{(k)}(\epsilon_0 s_0\lambda_{\mathbf{n}_0}+\cdots+\epsilon_{k-1}s_{k-1}\lambda_{\mathbf{n}_{k-1}}+\epsilon_0 s_k\lambda_{\mathbf{n}_0})\,\mathrm{d}\sigma\\
&= \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1}\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_{k-1}=\pm}} \sum_{j_0,\ldots,j_{k-1}=1}^{2^{\lfloor d/2\rfloor-1}} \int_{\Delta_k} f^{(k)}(\epsilon_0 s_0\lambda_{\mathbf{n}_0}+\cdots+\epsilon_{k-1}s_{k-1}\lambda_{\mathbf{n}_{k-1}}+\epsilon_0 s_k\lambda_{\mathbf{n}_0})\,\mathrm{d}\sigma\\
&\qquad \langle v^{\epsilon_0}_{\mathbf{n}_0,j_0}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle\cdots\langle v^{\epsilon_{k-2}}_{\mathbf{n}_{k-2},j_{k-2}}|\gamma^{\mu_{k-1}}|v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}\rangle\langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle\\
&\qquad \widehat{V^1_{\mu_1}}(\mathbf{n}_0-\mathbf{n}_1)\cdots\widehat{V^{k-1}_{\mu_{k-1}}}(\mathbf{n}_{k-2}-\mathbf{n}_{k-1})\widehat{V^k_{\mu_k}}(\mathbf{n}_{k-1}-\mathbf{n}_0).
\end{aligned}$$

To obtain the expression for the whole bracket $\langle V_1,\ldots,V_k\rangle$, we sum over the rotational permutations of the Connes one-forms $V_1,\ldots,V_k$:

$$\begin{aligned}
&\langle V_1,\ldots,V_k\rangle\\
&= \sum_{\substack{\mathbf{n}_0,\ldots,\mathbf{n}_{k-1}\in\mathbb{Z}^d\\ \epsilon_0,\ldots,\epsilon_{k-1}=\pm}} \sum_{j_0,\ldots,j_{k-1}=1}^{2^{\lfloor d/2\rfloor-1}} \sum_{\tau\in R_k}\\
&\qquad \int_{\Delta_k} f^{(k)}(\epsilon_0 s_0\lambda_{\mathbf{n}_0}+\cdots+\epsilon_{k-1}s_{k-1}\lambda_{\mathbf{n}_{k-1}}+\epsilon_0 s_k\lambda_{\mathbf{n}_0})\,\mathrm{d}\sigma\\
&\qquad \langle v^{\epsilon_0}_{\mathbf{n}_0,j_0}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle\cdots\langle v^{\epsilon_{k-2}}_{\mathbf{n}_{k-2},j_{k-2}}|\gamma^{\mu_{k-1}}|v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}\rangle\langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle\\
&\qquad \widehat{V^{\tau(1)}_{\mu_1}}(\mathbf{n}_0-\mathbf{n}_1)\cdots\widehat{V^{\tau(k-1)}_{\mu_{k-1}}}(\mathbf{n}_{k-2}-\mathbf{n}_{k-1})\widehat{V^{\tau(k)}_{\mu_k}}(\mathbf{n}_{k-1}-\mathbf{n}_0),
\end{aligned}$$

where R_k is the group of rotational permutations of $\{1,\ldots,k\}$. Upon relabelling the indices, we find the following, interpreting R_k now as the group of rotational permutations of $\{0,\dots,k-1\}$:

$$\begin{aligned}&\langle V_1,\dots,V_k\rangle\\&=\sum_{\substack{\mathbf{n}_0,\dots,\mathbf{n}_{k-1}\in\mathbb{Z}^d\\ \epsilon_0,\dots,\epsilon_{k-1}=\pm}}\sum_{j_0,\dots,j_{k-1}=1}^{2^{\lfloor d/2\rfloor-1}}\sum_{\tau\in R_k}\\&\quad\int_{\Delta_k} f^{(k)}(\epsilon_{\tau(0)}s_0\lambda_{\mathbf{n}_{\tau(0)}}+\cdots+\epsilon_{\tau(k-1)}s_{k-1}\lambda_{\mathbf{n}_{\tau(k-1)}}+\epsilon_{\tau(0)}s_k\lambda_{\mathbf{n}_{\tau(0)}})\,\mathrm{d}\sigma\\&\quad\langle v^{\epsilon_0}_{\mathbf{n}_0,j_0}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle\cdots\langle v^{\epsilon_{k-2}}_{\mathbf{n}_{k-2},j_{k-2}}|\gamma^{\mu_{k-1}}|v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}\rangle\langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_0}_{\mathbf{n}_0,j_0}\rangle\\&\quad\widehat{V^1_{\mu_1}}(\mathbf{n}_0-\mathbf{n}_1)\cdots\widehat{V^{k-1}_{\mu_{k-1}}}(\mathbf{n}_{k-2}-\mathbf{n}_{k-1})\widehat{V^k_{\mu_k}}(\mathbf{n}_{k-1}-\mathbf{n}_0).\end{aligned}$$

By [12], the symmetrised sum over the simplex integrals can be replaced by a single simplex integral, over Δ_{k-1}:

$$\begin{aligned}&\sum_{\tau\in R_k}\int_{\Delta_k} f^{(k)}(\epsilon_{\tau(0)}s_0\lambda_{\mathbf{n}_{\tau(0)}}+\cdots+\epsilon_{\tau(k-1)}s_{k-1}\lambda_{\mathbf{n}_{\tau(k-1)}}+\epsilon_{\tau(0)}s_k\lambda_{\mathbf{n}_{\tau(0)}})\,\mathrm{d}\sigma\\&\qquad=\int_{\Delta_{k-1}} f^{(k)}(\epsilon_0 s_0\lambda_{\mathbf{n}_0}+\cdots+\epsilon_{k-1}s_{k-1}\lambda_{\mathbf{n}_{k-1}})\,\mathrm{d}\sigma,\end{aligned}$$

so that after another relabelling $\mathbf{n}_0$, ϵ_0 and j_0 to $\mathbf{n}_k$, ϵ_k and j_k respectively, and relabelling the simplex coordinates $s_0,\dots,s_{k-1}$ to $s_1,\dots,s_k$, we find

$$\begin{aligned}\langle V_1,\dots,V_k\rangle=&\sum_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_k\in\mathbb{Z}^d\\ \epsilon_1,\dots,\epsilon_k=\pm}}\sum_{j_1,\dots,j_k=1}^{2^{\lfloor d/2\rfloor-1}}\int_{\Delta_{k-1}} f^{(k)}(\epsilon_1 s_1\lambda_{\mathbf{n}_1}+\cdots+\epsilon_k s_k\lambda_{\mathbf{n}_k})\,\mathrm{d}\sigma\\&\langle v^{\epsilon_k}_{\mathbf{n}_k,j_k}|\gamma^{\mu_1}|v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle\langle v^{\epsilon_1}_{\mathbf{n}_1,j_1}|\gamma^{\mu_2}|v^{\epsilon_2}_{\mathbf{n}_2,j_2}\rangle\cdots\langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}}|\gamma^{\mu_k}|v^{\epsilon_k}_{\mathbf{n}_k,j_k}\rangle\\&\widehat{V^1_{\mu_1}}(\mathbf{n}_k-\mathbf{n}_1)\widehat{V^2_{\mu_2}}(\mathbf{n}_1-\mathbf{n}_2)\cdots\widehat{V^k_{\mu_k}}(\mathbf{n}_{k-1}-\mathbf{n}_k),\end{aligned}\tag{VI.6}$$

which completes the proof. □

Remark. We specialise to the circle, so that the matrix elements $\langle v^{\epsilon_{l-1}}_{\mathbf{n}_{l-1},j_{l-1}}|\gamma^{\mu_l}|v^{\epsilon_l}_{\mathbf{n}_l,j_l}\rangle$ are all equal to 1. For any choice of $n_1,\dots,n_k\in\mathbb{Z}$, fix $n,m_2,\dots,m_k\in\mathbb{Z}$ such that $n_1=n$ and $n_j=n_{j-1}+m_j$, for $j=2,\dots,k$. Then

$$\begin{aligned}&\int_{\Delta_{k-1}} f^{(k)}(s_1n_1+\cdots+s_kn_k)\,\mathrm{d}\sigma\,\widehat{V^1}(n_k-n_1)\widehat{V^2}(n_1-n_2)\cdots\widehat{V^k}(n_{k-1}-n_k)\\&=\int_{\Delta_{k-1}} f^{(k)}\left(n+\sum_{j=2}^k s_j\sum_{i=2}^j m_i\right)\mathrm{d}\sigma\,\widehat{V^1}\left(\sum_{i=2}^j m_i\right)\widehat{V^2}(-m_2)\cdots\widehat{V^k}(-m_k).\end{aligned}$$

Assuming that f admits a series expansion with infinite radius of convergence, we can expand $f^{(k)}$ around n:

$$f^{(k)}\left(n+\sum_{j=2}^k s_j\sum_{i=2}^j m_i\right)=\sum_{l=0}^\infty\frac{f^{(k+l)}(n)}{l!}\left(\sum_{j=2}^k s_j\sum_{i=2}^j m_i\right)^l.$$

Such a choice of f is possible, as for example the function $x \mapsto e^{-x^2}$ is in $\mathcal{E}^{s,1/2}$ for any $s \in \mathbb{N}_0$ [27].

To prevent the notation from getting cluttered, we now fix $k = 2$ and write $m = m_2$. Nevertheless, the following results can be generalised to arbitrary k. Then,

$$\langle V^1, V^2 \rangle = \sum_{n,m \in \mathbb{Z}} \widehat{V^1}(m) \widehat{V^2}(-m) \int_{\Delta_1} f^{(2)}(n + s_2 m) \, \mathrm{d}\sigma,$$

and termwise computing the simplex integral yields

$$\begin{aligned} \int_{\Delta_1} f^{(2)}(n + s_1 m) \, \mathrm{d}\sigma &= \int_{\Delta_1} \sum_{l=0}^{\infty} \frac{f^{(l+2)}(n)}{l!} s_2^l m^l \, \mathrm{d}\sigma \\ &= \sqrt{2} \sum_{l=0}^{\infty} \frac{f^{(l+2)}(n)}{(l+1)!} m^l. \end{aligned}$$

Now, recall that $(im)\widehat{V}(m) = \widehat{\partial_\theta V}(m)$, so that

$$\langle V^1, V^2 \rangle = \sqrt{2} \sum_{n,m \in \mathbb{Z}} \sum_{l=0}^{\infty} \frac{f^{(l+2)}(n)}{(l+1)!} (-i)^l \widehat{\partial_\theta^l V^1}(m) \widehat{V^2}(-m).$$

Now, suppose V_1 and V_2 are real-valued. This is a realistic assumption, since we encountered the bracket $\langle V, \ldots, V \rangle$ in the expansion of the perturbed spectral action $\mathrm{tr}(f(D+V))$, with V self-adjoint. Then we can apply Parseval's identity [11] to replace the sum over m by an integral over $\mathbb{T}^1$:

$$\begin{aligned} \langle V^1, V^2 \rangle &= \sqrt{2} \sum_{n \in \mathbb{Z}} \sum_{l=0}^{\infty} \frac{f^{(l+2)}(n)}{(l+1)!} (-i)^l \int_{\mathbb{T}^1} (\partial_\theta^l V^1) V^2 \\ &= \sqrt{2} \sum_{l=0}^{\infty} \frac{\mathrm{tr}(f^{(l+2)}(D))}{(l+1)!} (-i)^l \int_{\mathbb{T}^1} (\partial_\theta^l V^1) V^2 \end{aligned}$$

Thus, the bracket $\langle V_1, V_2 \rangle$ can be seen as an infinite linear combination of integrals of derivatives of V_1 and V_2.

VI.3 Computation of the Hochschild cochains

The concrete expression for the bracket of the previous Section allows us to describe the Hochschild cocycles ϕ_k. Having done so, we turn to the cochains ψ_{2k+1}, which we will also cast in a more tangible form with the help of the expressions for the ϕ_k.

Corollary VI.5. *Let $k \in \mathbb{N}$, and let $g_0, \dots, g_k \in C^\infty(\mathbb{T}^d)$. Then $\phi_k(g_0, \dots, g_k)$ is given by*

$$\phi_k(g_0,\dots,g_k) = (-i)^{k+1} \sum_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_k \in \mathbb{Z}^d \\ \epsilon_1,\dots,\epsilon_k = \pm}} \sum_{j_1,\dots,j_k=1}^{2^{\lfloor d/2 \rfloor -1}} \int_{\Delta_{k-1}} f^{(k)}(\epsilon_1 s_1 \lambda_{\mathbf{n}_1} + \cdots + \epsilon_k s_k \lambda_{\mathbf{n}_k})\, \mathrm{d}\sigma$$
$$\langle v^{\epsilon_k}_{\mathbf{n}_k,j_k} | \gamma^{\mu_1} | v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \langle v^{\epsilon_1}_{\mathbf{n}_1,j_1} | \gamma^{\mu_2} | v^{\epsilon_2}_{\mathbf{n}_2,j_2}\rangle \cdots \langle v^{\epsilon_{k-1}}_{\mathbf{n}_{k-1},j_{k-1}} | \gamma^{\mu_k} | v^{\epsilon_k}_{\mathbf{n}_k,j_k}\rangle$$
$$\left(\left(\widehat{g_0 \partial_{\mu_1} g_1}(\mathbf{n}_k - \mathbf{n}_1) \right) \left(\widehat{\partial_{\mu_2} g_2}(\mathbf{n}_1 - \mathbf{n}_2) \right) \cdots \left(\widehat{\partial_{\mu_k} g_k}(\mathbf{n}_{k-1} - \mathbf{n}_k) \right) \right).$$

Proof. This follows from Theorem VI.4, by taking $V_1 = g_0[D, g_1] = -i\gamma^\mu(g_0 \partial_\mu g_1)$ and $V_j = [D, g_j] = -i\gamma^\mu \partial_\mu g_j$ for $j \geq 2$. □

It follows immediately that

$$B_0\phi_{2k}(g_0,\dots,g_{2k-1}) \coloneqq \phi_{2k}(1, g_0, \dots, g_{2k-1})$$
$$= (-i)^{2k+1} \sum_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_{2k} \in \mathbb{Z}^d \\ \epsilon_1,\dots,\epsilon_{2k} = \pm}} \sum_{j_1,\dots,j_{2k}=1}^{2^{\lfloor d/2 \rfloor -1}} \int_{\Delta_{2k-1}} f^{(2k)}(\epsilon_1 s_1 \lambda_{\mathbf{n}_1} + \cdots + \epsilon_{2k} s_{2k} \lambda_{\mathbf{n}_{2k}})\, \mathrm{d}\sigma$$
$$\langle v^{\epsilon_{2k}}_{\mathbf{n}_{2k},j_{2k}} | \gamma^{\mu_1} | v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \langle v^{\epsilon_1}_{\mathbf{n}_1,j_1} | \gamma^{\mu_2} | v^{\epsilon_2}_{\mathbf{n}_2,j_2}\rangle \cdots \langle v^{\epsilon_{2k-1}}_{\mathbf{n}_{2k-1},j_{2k-1}} | \gamma^{\mu_{2k}} | v^{\epsilon_{2k}}_{\mathbf{n}_{2k},j_{2k}}\rangle$$
$$\left(\left(\widehat{\partial_{\mu_1} g_0}(\mathbf{n}_k - \mathbf{n}_1) \right) \left(\widehat{\partial_{\mu_2} g_1}(\mathbf{n}_1 - \mathbf{n}_2) \right) \cdots \left(\widehat{\partial_{\mu_{2k}} g_{2k-1}}(\mathbf{n}_{2k-1} - \mathbf{n}_{2k}) \right) \right).$$

Combining this with the expression for ϕ_{2k-1}, we derive an expression for ψ_{2k-1}:

Corollary VI.6.

$$\psi_{2k-1}(g_0,\dots,g_{2k-1}) \coloneqq \phi_{2k-1}(g_0,\dots,g_{2k-1}) - \frac{1}{2} B_0\phi_{2k}(g_0,\dots,g_{2k-1})$$
$$= (-i)^{2k} \sum_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_{2k-1} \in \mathbb{Z}^d \\ \epsilon_1,\dots,\epsilon_{2k-1} = \pm}} \sum_{j_1,\dots,j_{2k-1}=1}^{2^{\lfloor d/2 \rfloor -1}} \int_{\Delta_{2k-2}} f^{(2k-1)}(\epsilon_1 s_1 \lambda_{\mathbf{n}_1} + \cdots + \epsilon_{2k-1} s_{2k-1} \lambda_{\mathbf{n}_{2k-1}})\, \mathrm{d}\sigma$$
$$\langle v^{\epsilon_{2k-1}}_{\mathbf{n}_{2k-1},j_{2k-1}} | \gamma^{\mu_1} | v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \langle v^{\epsilon_1}_{\mathbf{n}_1,j_1} | \gamma^{\mu_2} | v^{\epsilon_2}_{\mathbf{n}_2,j_2}\rangle \cdots \langle v^{\epsilon_{2k-2}}_{\mathbf{n}_{2k-2},j_{2k-2}} | \gamma^{\mu_{2k-1}} | v^{\epsilon_2 k-1}_{\mathbf{n}_{2k-1},j_{2k-1}}\rangle$$
$$\left(\left(\widehat{g_0 \partial_{\mu_1} g_1}(\mathbf{n}_{2k-1} - \mathbf{n}_1) \right) \left(\widehat{\partial_{\mu_2} g_2}(\mathbf{n}_1 - \mathbf{n}_2) \right) \cdots \left(\widehat{\partial_{\mu_{2k-1}} g_{2k-1}}(\mathbf{n}_{2k-2} - \mathbf{n}_{2k-1}) \right) \right)$$
$$- \frac{(-i)^{2k+1}}{2} \sum_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_{2k} \in \mathbb{Z}^d \\ \epsilon_1,\dots,\epsilon_{2k} = \pm}} \sum_{j_1,\dots,j_{2k}=1}^{2^{\lfloor d/2 \rfloor -1}} \int_{\Delta_{2k-1}} f^{(2k)}(\epsilon_1 s_1 \lambda_{\mathbf{n}_1} + \cdots + \epsilon_{2k} s_{2k} \lambda_{\mathbf{n}_{2k}})\, \mathrm{d}\sigma$$
$$\langle v^{\epsilon_{2k}}_{\mathbf{n}_{2k},j_{2k}} | \gamma^{\mu_1} | v^{\epsilon_1}_{\mathbf{n}_1,j_1}\rangle \langle v^{\epsilon_1}_{\mathbf{n}_1,j_1} | \gamma^{\mu_2} | v^{\epsilon_2}_{\mathbf{n}_2,j_2}\rangle \cdots \langle v^{\epsilon_{2k-1}}_{\mathbf{n}_{2k-1},j_{2k-1}} | \gamma^{\mu_{2k}} | v^{\epsilon_{2k}}_{\mathbf{n}_{2k},j_{2k}}\rangle$$
$$\left(\left(\widehat{\partial_{\mu_1} g_0}(\mathbf{n}_k - \mathbf{n}_1) \right) \left(\widehat{\partial_{\mu_2} g_1}(\mathbf{n}_1 - \mathbf{n}_2) \right) \cdots \left(\widehat{\partial_{\mu_{2k}} g_{2k-1}}(\mathbf{n}_{2k-1} - \mathbf{n}_{2k}) \right) \right).$$

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