Second quantization of noncommutative spaces: emergence of the Standard Model and gravity

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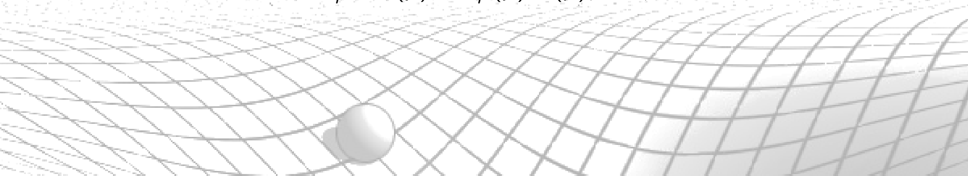
A fermion in spacetime

Minimal ingredients to describe a free fermion:

- coordinates on spacetime M:

$$x_\mu \cdot x_\nu(p) = x_\mu(p) x_\nu(p), etc.,$$



- propagation, described by Dirac operator $D_M = i\gamma^\mu \partial_\mu$

Noncommutative geometry

- Combination of coordinate algebra and operators is central to the noncommutative approach [Connes 1994], in terms of spectral triples:

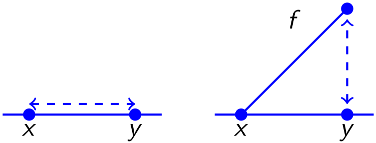
$$(\mathcal{A}, \mathcal{H}, D)$$

- The commutative case (Riemannian spin manifold M):
  - the algebra $C^\infty(M)$ of smooth functions on M
  - the Dirac operator D_M
  - both acting on Hilbert space $L^2(S_M)$ of square-integrable spinors.
- The noncommutative case:
  - an algebra $\mathcal{A}$
  - a (suitable) self-adjoint operator D
  - both acting on a Hilbert space $\mathcal{H}$

Reconstruction of geometry

- Reconstruction of M in the commutative case [Connes 1989]: $(C^\infty(M), L^2(\mathcal{S}_M), D_M)$:

$$d(x,y) = \sup_f \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$ (*e.g.* $[D_{\mathbb{R}}, f] = -i\frac{df}{dt}$)

Emerging bosons

Our fermionic starting point induces a bosonic theory:

- "Inner perturbations" by the coordinates [C 1996, CCS 2013]:

$$D_M \rightsquigarrow D_M + \sum_j a_j [D_M, a'_j]$$

  for functions a_j, a'_j depending on the coordinates x_μ.
- Then,

$$\sum_j a_j [D_M, a'_j] = A^\nu \gamma^\mu (\partial_\mu x^\nu) = A^\mu \gamma_\mu$$

  where A^μ is the electromagnetic 4-potential describing the photon.

Entering noncommutativity

Consider a finite space F, but with a *noncommutative* structure:

- Described by block diagonal matrices ("noncommutative coordinates")

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \dots, a_N$ are square matrices of size $n_1, n_2, \dots, n_N$.
- Hence we will consider the matrix algebra

$$A_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

  where $\mathbb{C}$ can be replaced by $\mathbb{R}$ or $\mathbb{H}$.
- A finite Dirac operator is given by a hermitian matrix.

Example: commutative two-point space

$$F = \ _1\bullet \qquad _2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(1,2) = \max \left\{ |\lambda_1 - \lambda_2| : \left\| \begin{pmatrix} 0 & \overline{c}(\lambda_2 - \lambda_1) \\ c(\lambda_1 - \lambda_2) & 0 \end{pmatrix} \right\| \le 1 \right\} = \frac{1}{|c|}$$

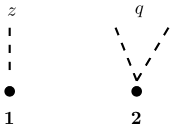
Example: noncommutative two-point space

Coordinates on F are elements in $\mathbb{C} \oplus \mathbb{H}$

- A complex number z
- A quaternion $q = q_0 + i q_k \sigma^k$; in terms of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It describes a two-point space, with internal structure:



Inner perturbations on nc two-point space

- 'Dirac operator' $D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- Inner perturbations:

$$D_F \rightsquigarrow D_F + \sum_j a_j [D_F, a_j'] = \begin{pmatrix} 0 & \overline{c\phi_1} & \overline{c\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- Distance between the two points is now $1/\sqrt{|c\phi_1|^2 + |c\phi_2|^2}$.
- We may call ϕ_1 and ϕ_2 the Higgs field.
- Indeed, the group of unitary block diagonal matrices is now $U(1) \times SU(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Almost-commutative spacetimes

We now combine mild matrix noncommutativity with spacetime:

- coordinates of the almost-commutative spacetime $M \times F$:

$$\boxed{\hat{x}^\mu(p) = (z^\mu(p), q^\mu(p))}$$

  as elements in $\mathbb{C} \oplus \mathbb{H}$ (for each μ and each point p of M)
- The combined Dirac operator becomes

$$\boxed{D_{M\times F} = D_M + \gamma_5 D_F}$$

  Note that $D^2_{M\times F} = D^2_M + D^2_F$, which will be useful later on.

Inner perturbations on $M \times F$

So, we describe $M \times F$ by:

$$\hat{x}^\mu = (z^\mu, q^\mu)\,; \qquad D_{M\times F} = D_M + \gamma_5 D_F$$

As before, we consider inner perturbations of $D_{M\times F}$ by $\hat{x}^\mu(p)$:

- The inner perturbations of D_F become scalar fields ϕ_1, ϕ_2.
- The inner perturbations of D_M become matrix-valued:

$$\sum_j a_j[D_M, a_j'] = a_\nu \gamma^\mu (\partial_\mu \hat{x}^\nu) =: A_\mu \gamma^\mu$$

with A_μ taking values in $\mathbb{C} \oplus \mathbb{H}$:

$$A_\mu = \begin{pmatrix} B_\mu & 0 & 0 \\ 0 & W_\mu^3 & W_\mu^+ \\ 0 & W_\mu^- & -W_\mu^3 \end{pmatrix}$$

corresponding to hypercharge and the W-bosons.

What do we have so far?

Noncommutative geometry allows for a description of the particle content of several models in particle physics (EW, SM, Pati–Salam, etc.):

- at the one-particle level, so essentially classical;
- so far, without a prescription on the dynamics of the fields.

We now resolve these questions by introducing a second quantization of spectral triples.

Second quantization of $(\mathcal{A}, \mathcal{H}, D)$

- The first step is to replace Hilbert space $\mathcal{H}$ by $\text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$, the complexified Clifford algebra of $\mathcal{H}_{\mathbb{R}}$.
- There is a one-parameter group of automorphisms σ_t on this Clifford algebra, associated to the operator $\exp(itD)$ (on $\mathcal{H}_{\mathbb{R}}$).
- We then have that for any $\beta > 0$ there exists a unique state $\varphi = \varphi_\beta$ on $\text{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$ that satisfies the KMS-condition at inverse temperature β:

$$\varphi(a\sigma_t(b))|_{t=i\beta} = \varphi(ba).$$

Proposition

If the operator $\exp(-\beta|D|)$ is of trace class, the state φ_β is of type I and the associated irreducible representation is given by the fermionic second quantization associated to the complex structure $I := i\,\mathrm{sign}\,D$ on $\mathcal{H}_{\mathbb{R}}$.

Fermionic second quantization

- Equip $\mathcal{H}_\mathbb{R}$ with complex structure, *e.g.* $I = i\,\mathrm{sign} D \rightsquigarrow$ Dirac sea e^{itD}
- $\mathrm{Cliff}_\mathbb{C}(\mathcal{H}_\mathbb{R})$ acts on the Fock space $\bigwedge \mathcal{H}_I$ via

$$\gamma_I(v) = a_I^*(v) + a_I(v); \qquad (v \in \mathcal{H}_\mathbb{R}).$$

Proposition (Chamseddine–Connes–vS, 2018)
(i) *The one-parameter group σ_t is implemented in the (physical) Fock representation by the one-parameter unitary group $\bigwedge \exp(it|D|)$:*

$$\gamma_I(\sigma_t(A)) = \bigwedge(e^{it|D|})\gamma_I(A) \bigwedge(e^{-it|D|}) \qquad A \in \mathrm{Cliff}_\mathbb{C}(\mathcal{H}_\mathbb{R}).$$

(ii) *If $\exp(-\beta|D|)$ is of trace class the state φ_β is of type I and is given by*

$$\varphi_\beta(A) = \mathcal{N}^{-1}\, \mathrm{Trace}\left(\bigwedge \exp(-\beta|D|)\gamma_I(A)\right) \qquad A \in \mathrm{Cliff}_\mathbb{C}(\mathcal{H}_\mathbb{R})$$

Gibbs states and entropy

- We thus have a density matrix

$$\rho_\beta = \mathcal{N}^{-1} \cdot \bigwedge(e^{-\beta|D|})$$

- Note that this is the Gibbs state for a Fermi gas on the (noncommutative) space that is described by $(\mathcal{A}, \mathcal{H}, D)$.

Theorem (Chamseddine–Connes–vS, 2018)
The (von Neumann) entropy,

$$S(\rho_\beta) = -\operatorname{Trace} \rho_\beta \log \rho_\beta,$$

of the above Gibbs state ρ_β is given by a spectral action Trace $h(\beta D)$ *for the function $h(x) = \mathcal{E}(e^{-x})$ where $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y (*i.e. *of size $1/(1+y)$ and $y/(1+y)$).*

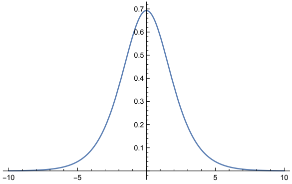
Analysis of the function h

- $\mathcal{E}(y)$ is the entropy of a partition of the unit interval in two intervals with size of ratio y:

$$\mathcal{E}(y) = -\operatorname{Trace} \rho_y \log \rho_y; \qquad \rho_y = \begin{pmatrix} \frac{1}{1+y} & 0 \\ 0 & \frac{y}{1+y} \end{pmatrix}.$$

- We have $\mathcal{E}(y) = \log(y+1) - \frac{y \log y}{y+1}$

$$h(x) = \mathcal{E}(e^{-x}) = \frac{x}{1+e^x} + \log(1+e^{-x})$$



and this is applied to the spectrum of βD.

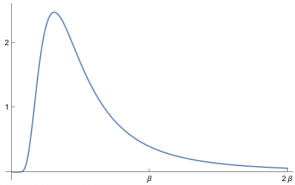
Entropy of two-point space

$$F = {}_1\bullet \qquad {}_2\bullet$$

- Distance $r := d(1,2) = 1/|c|$ in terms of $D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}$.
- For $r \to 0$ we have $S(\rho_\beta) = 0$;
- For $r \to \infty$ we have maximum entropy $S(\rho_\beta) = 2\log 2$;

Entropic force $F(r) = \beta^{-1}\partial_r S(\rho_\beta)$?

$$F(r) = \frac{\beta^3/2r^3}{\cosh^2(\beta/2r)}$$



Laplace transform and heat expansion

Proposition (Chamseddine–Connes–vS, 2018)
The function h is a Laplace transform:

$$h(x) = \int_0^\infty g(t) e^{-tx^2}$$

with

$$g(t) = \frac{-1}{8\sqrt{\pi} t^{5/2}} \sum_{n \in \mathbb{Z}} (-1)^n n^2 q^{n^2}; \qquad q = e^{-1/4t}.$$

This allows us to use heat asymptotics of $e^{-t\beta^2 D^2}$ to determine Trace $h(\beta D)$.

Asymptotic expansion of entropy

If Trace $e^{-tD^2} \sim \sum_k t^k b_k$ then

$$S(\rho_\beta) = \text{Trace}\, h(\beta D) \sim \sum_k \beta^{2k} \gamma(k) b_k \qquad \gamma(k) = \frac{1 - 2^{-2k}}{k} \pi^{-k} \xi(2k)$$

in terms of the Riemann ξ-function :

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

$\gamma(-1)$	$\gamma(-1/2)$	$\gamma(0)$	$\gamma(1/2)$	$\gamma(1)$	$\gamma(3/2)$
$\frac{9\zeta(3)}{2}$	$\frac{\pi^{3/2}}{3}$	$\log 2$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{8}$	$\frac{7\zeta(3)}{8\pi^{5/2}}$

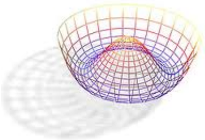
Entropy of the electroweak theory

Use $D^2_{M\times F} = D^2_M + D^2_F$ to compute (in 4d)

$$S(\rho_\beta) = \mathrm{Trace}\, h(\beta D_{M\times F}) \sim c_4\beta^{-4}\mathrm{Vol}(M) + c_2\beta^{-2}\int R\sqrt{g}$$
$$- c_0 \int C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \int c_0'\ \mathrm{Trace}\, F_{\mu\nu}F^{\mu\nu} - c_2'\beta^{-2}|\phi|^2 + c_0'|\phi|^4 + \cdots$$

We now recognize:

- (Higher-derivative) gravity
- The Yang–Mills term $F_{\mu\nu}F^{\mu\nu}$ for hypercharge and W-boson
- The Higgs potential $-\mu^2|\phi|^2 + \lambda|\phi|^4$



Beyond the SM with noncommutative geometry

- The matrix coordinates of the Standard Model in $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ arise naturally as a restriction of the following coordinates

$$\hat{x}^\mu(p) = (q_R^\mu(p), q_L^\mu(p), m^\mu(p)) \in \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

corresponding to a Pati–Salam unification:

$$U(1)_Y \times SU(2)_L \times SU(3) \to SU(2)_R \times SU(2)_L \times SU(4)$$

- The 96 fermionic degrees of freedom are structured as

$$\left(\begin{array}{cc|cc} \nu_R & u_{iR} & \nu_L & u_{iL} \\ e_R & d_{iR} & e_L & d_{iL} \end{array} \right) \qquad (i = 1, 2, 3)$$

- The finite Dirac operator is a 96×96-dimensional matrix containing Yukawa mass matrices, etc.

Inner perturbations

- Inner perturbations of D_M now give three gauge bosons:

$$W_R^\mu, \qquad W_L^\mu, \qquad V^\mu$$

  corresponding to $SU(2)_R \times SU(2)_L \times SU(4)$.
- For the inner perturbations of D_F we distinguish two cases, depending on the initial form of D_F:
  I The Standard Model $D_F = \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$
  II A more general D_F with zero $\overline{f}_L - f_L$-interactions.

Scalar sector of the spectral Pati–Salam model

I For a SM D_F, the resulting scalar fields are composite fields, expressed in scalar fields whose representations are:

	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\phi_{\dot{a}}^{b}$	2	2	1
$\Delta_{\dot{a}I}$	2	1	4
Σ_J^I	1	1	15

II For a more general finite Dirac operator, we have fundamental scalar fields:

particle	$SU(2)_R$	$SU(2)_L$	$SU(4)$
$\Sigma_{\dot{a}J}^{bJ}$	2	2	$1+15$
$H_{\dot{a}I\dot{b}J}$	3	1	10
	1	1	6

A dictionary and outlook

one-particle	second-quantized
$\mathcal{A}$	$\sigma_t^D \mapsto \sigma_t^{D_A}$
$\mathcal{H}$	$\mathrm{Cliff}_{\mathbb{C}}(\mathcal{H}_{\mathbb{R}})$
D	$\{\sigma_t^D\}_t$ arising from e^{itD}
spectral action	entropy of KMS

- Physical significance of this entropy: "entropic geometry"?
- Extension to type II?
- Quantization of inner perturbations